Conjugate gradient for frictional contact

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PROCEEDINGS

CONTACT MECHANICS
INTERNATIONAL SYMPOSIUM

OCTOBER 7–9, 1992
EPFL, LAUSANNE, SWITZERLAND

Edited by Alain Curnier

PRESSES POLYTECHNIQUES ET UNIVERSITAIRES ROMANDES

Proc. Contact Mechanics Int. Symp., Edt. A. Curnier, PPUR, 423-432, 1992

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1. INTRODUCTION

In Raous et al [1], an algorithm for solving unilateral contact problem with Coulomb friction was proposed. It is based on the coupling of a fixed point method on the sliding limit with a minimization problem under constraints corresponding to a Tresca friction problem.

With a view to solving this minimization problem under constraints, we previously used projection techniques associated with Gauss-Seidel procedure accelerated by means of either overrelaxation [1] or Aitken's procedure [2].

To be able to use the conjugate gradient method, a special projection procedure has to be introduced to preserve the conjugate property between the directions, and the differentiability has to be obtained by regularizing the friction term.

We present here a variant of the methods proposed by May [3], Dilintas et al [4], Jeusette-Sonzogni [5], and Marks-Salamon [6] for the frictionless problem, and we extend the method to the frictional case.

Preconditioning is essential to improve the efficiency of the conjugate gradient method. Four preconditioning procedures are tested. Two of them are specific to the friction problem and take the regularization term into account ; this is very important because of the connection between the regularization parameters and the conditioning of the problem.

2. THE PROBLEM

A unilateral contact problem with Coulomb friction can be split into a fixed point problem where the sliding limit g is the unknown, and a minimization problem under constraints of a non differentiable functional (see Raous et al[1]).

We are dealing here with quasi static problems and this formulation has been developed by using an incremental form. The problem to be solved at each step t_k is therefore very similar to the static one. At each time t_k, we have to solve:

Problem 1: Find g fixed point of the application

$$g \mapsto \mu . \left| F_N(u_g) \right| \tag{1}$$

with $u_g = u^{k-1} + \Delta u_g^k$, where Δu_g^k is solution of problem 2

Problem 2: Find $\Delta u_g^k \in \mathbb{K}^k$ such that:

$$J(\Delta u_g^k) \le J(v) \quad , \quad \forall v \in \mathbb{K}^k \tag{2}$$

$$\text{with} \quad J(v) = \frac{1}{2} a(v,v) - (R^{k-1} + \Delta f^k, v) + j(v) \tag{3}$$

where:

$\mathbb{K}^k = \left\{ \Delta u^k \ / \ (u^{k-1} + \Delta u^k)_N \le 0 \quad \text{on } \Gamma_3 \right\}$, the convex $\mathbb{K}^k$ is depending on k-1, it is variable but remains given at each step k ,

$a(v,v) = \int_\Omega \sigma . e \, dx$,

$(R^{k-1} + \Delta f^k, v) = \int_\Omega \Delta \varphi_1^k . v \, dx + \int_{\Gamma_2} \Delta \varphi_2^k . v \, dl + \int_{\Gamma_3} R^{k-1} . v \, dl$,

$j(v) = \int_{\Gamma_3} g . |v_T| \, dl$,

u displacements ,
σ the stresses ,
$\Delta \varphi_1^k$ increment of volumic forces ,
$\Delta \varphi_2^k$ increment of surfacic forces ,
R^{k-1} frictional forces at the previous step (equilibrium residu) ,
μ friction coefficient ,
F_N normal component of the contact force ,
Γ_2 part of the boundary submitted to a given load φ_2 ,
Γ_3 part of the boundary where contact can occur ,
v_T tangential velocity of a contact point .

A particular approximation of problem 2, which is based on the use of P1 finite elements, is given in [7]: we project both v and j(v) into the finite element space discretization. In this case, existence and uniqueness, when the friction coefficient is small, have been proved in [7]. The main problem to be solved can then be written:

Problem 3: Find $\bar{x} \in \mathbb{K}$ such that:

$$J(\bar{x}) \leq J(x) \quad , \quad \forall x \in \mathbb{K} \tag{4}$$

$$\text{with} \quad J(x) = \tfrac{1}{2} x^T A x - F^T x + G^T |x_t| \ , \tag{5}$$

where:

A is the classical finite element matrix (dimension N) ,

F is the loading vector ,

G is the sliding limit vector ,

N is the number of degrees of freedom ,

$x_t \in \mathbb{R}^m$ where m is the number of contact nodes ,

$\mathbb{K} = \prod_{i=1}^{N} K_i$ with $K_i = \mathbb{R}^-$ if "i" is the number of the normal component of the displacement of a contact node and $K_i = \mathbb{R}$ if not.

3. UNILATERAL CONTACT

The constraints for the minimization are the conditions of non penetration of the solid into the obstacle. They are characterized by the convex $\mathbb{K}$. Using P1 elements for the discretization, the constraints are separate due to the positiveness of the shape functions, and the convex $\mathbb{K}$ is written as the product of $\mathbb{R}$ or $\mathbb{R}^-$ sets depending on the degree of freedom concerned. A projection onto $\mathbb{K}$ is therefore simply the product of projections onto $\mathbb{R}^-$.

With the conjugate gradient method it is not possible to use a simple projection as has been done previously with the Gauss-Seidel method (see [1]) because the conjugation property between two successive directions would then be broken. To preserve the qualities of the conjugate gradient, we will search for the solution in the subspace $\mathbb{R}^{N-k}$, where k is the number of saturated constraints. This means that whenever a penetration occurs, the system will lose a degree of freedom and the new directions and the conjugation will be defined on the boundary of the convex (related to the active constraints). The convergence in a finite number of iterations is therefore preserved in each subspace, where the conjugations are written. The condition of compressive normal force is naturally tested and a constraint can be relaxed if necessary.

There exist various ways to carrying out this process. May [3] introduces evolutive boundary conditions into the matrix itself. The conjugation is always written with the actual matrix ; but these changes are costly. Dilintas et al [4] and Jeusette-Sonzogni [5] project the descent directions. This means that for a calculated descent direction which would give penetration, the solution is stopped on the boundary of the convex and will keep going in the tangential direction, i.e. on the boundary of the convex. We use this process but we introduce a different algorithm to check if the normal force is negative or zero and to relax the constraint if necessary.

The main steps of the algorithm are as follows:

- regular gradient conjugate iteration and computation of the optimal coefficient for the descent ρ_{opt},
- for all the contact nodes, in the case of penetration, computation of the maximum value ρ_i of the descent coefficient which ensures that the solution remains in the convex,
- the final descent coefficient is the minimum between ρ_{opt} and ρ_i ,
- when a new contact is obtained, we take the normal component of the direction in question to be zero in what follows. We project therefore the direction and avoid dealing with a matrix characterizing the projection. Then, we initiate the next conjugate gradient iteration (on the subspace $\mathbb{R}^{N-1}$) with a regular gradient direction,
- the dual condition on the normal force F_N that should be either negative or zero has to be checked. If the condition is not fullfilled by a node in contact (for the others F_N=0), we have to relax the constraint (u_N=0) on this node and to use a non projected direction again. Our strategy differs of those used by Dilintas et al [4] and Jeusette-Sonzogni [5]. A diagonal process is developped: the condition on F_N is checked after partial convergence of the conjugate gradient, i.e. when an accuracy ε_1 is obtained. The value of ε_1 is taken to be $10^2.\varepsilon_0$ where ε_0 is the require final accuracy ($\varepsilon_0 = 0{,}5.10^{-5}$).

4. THE FRICTION

4.1 Regularization

To obtain the differentiability, the absolute value of the friction term (6) has to be regularized.

$$(\sum_{i=1}^{m} g_i \cdot \left| x^k_{num(i)} \right|) \qquad (6)$$

Three classical methods of regularization have been tested and their efficiency has been compared:

- regularization REG1 (square root)

$$\varphi_\varepsilon(x)=\sqrt{x^2+\varepsilon^2} \quad , \quad \varphi'_\varepsilon(x)=\frac{x}{\sqrt{x^2+\varepsilon^2}} \ , \tag{7}$$

- regularization REG2 (hyperbolic tangent)

$$\varphi_\varepsilon(x)=\varepsilon.\mathrm{Ln}(\mathrm{ch}(\tfrac{x}{\varepsilon})) \quad , \quad \varphi'_\varepsilon(x)=\tanh(\tfrac{x}{\varepsilon}) \ , \tag{8}$$

- regularization REG3 (piecewise polynomial)

$$\varphi_\varepsilon(x)=\begin{cases} -x-\frac{\varepsilon}{2} & \text{if } x<-\varepsilon \\ \frac{x^2}{2\varepsilon} & \text{if } -\varepsilon\le x\le\varepsilon \\ x-\frac{\varepsilon}{2} & \text{if } x>\varepsilon \end{cases} \quad , \quad \varphi'_\varepsilon(x)=\begin{cases} -1 & \text{if } x<-\varepsilon \\ \frac{x}{\varepsilon} & \text{if } -\varepsilon\le x\le\varepsilon \\ +1 & \text{if } x>\varepsilon \end{cases} . \tag{9}$$

4.2 Conjugate gradient for the non linear case

Problem 4: Find $\bar{x}\in\mathbb{K}$ such that

$$J_\varepsilon(\bar{x})\le J_\varepsilon(x) \quad , \quad \forall x\in\mathbb{K} \tag{10}$$

$$\text{with} \quad J_\varepsilon(x)=\tfrac{1}{2}x^TAx-F^Tx+\sum_{i=1}^{m} g_i.\varphi_\varepsilon(x_{num(i)}) \ . \tag{11}$$

Conjugation between the successive descent directions v^k is written with the Hessian J''_ε of (11).

To avoid having to compute the Hessian, we approximate J''_ε with a Taylor development of $J'_\varepsilon(x^k)=r^k$, taking all the other derived functions to be small and the displacements between two iterations to be very close. This gives us :

$$v^{k+1}=P.r^{k+1}+\gamma^k.v^k \quad \text{with} \quad \gamma^k=\frac{<r^{k+1}-r^k,P.r^{k+1}>}{<r^k,P.r^k>} \tag{12}$$

The optimal descent coefficient is computed by writing:

$$<J'_\varepsilon(x^k+\rho^k.v^k),v^k>=0 \tag{13}$$

This non linear problem can be solved in two different ways:

- an approximation using the Taylor development of $J'_\varepsilon(x^{k+1})$:

$$\rho^k = \frac{< r^k . v^k >}{< J''_\varepsilon (x^k).v^k, v^k >} \tag{14}$$

- a few iterations of an iterative method like the false position to avoid computing J''_ε. The prescribed accuracy of the determination of ρ^k is 10^{-4} and the number of iterations required is about twenty.

The two procedures have been tested. The first is preferred because it does not need to iterate and we obtain nearly the same number of gradient iterations. Computing time is therefore shorter.

5. PRECONDITIONING

It is well known that the convergence can be considerably improved by applying a suitable preconditioning procedure which consists of working on the matrix $C^{-1/2}AC^{-1/2}$. This is integrated into the algorithm and needs only a change of variables (r^k is replaced by $C^{-1/2}.r^k$, v^k by $C^{1/2}.v^k$ and x^k by $C^{-1/2}.x^k$) and an extra resolution of (15).

$$C.z^{k+1} = r^{k+1} \tag{15}$$

Four preconditioning procedures (i.e. four choices of C) have been tested:

- the simple diagonal, C=D=diag(A),
- SSOR or Evans Preconditioning :

$$C = \frac{1}{\omega(2-\omega)}.[D+\omega E].D^{-1}.[D+\omega E]^t \tag{16}$$

where D=diag(A), E is the lower triangular part of A and ω a relaxation parameter (in fact, we avoid computing an optimum ω, we use ω=1 ; and the preconditioning is a Gauss-Seidel one in this case),

- the two previous preconditioning applied to the Hessian $J''_\varepsilon(x^k)$ and taking into account the non linear term. If I denotes the set of the tangential components of the contact nodes, we have:

$$\begin{cases} \left(J''_\varepsilon(x^k)\right)_{ii} = A_{ii} + g_i \cdot \varphi''_\varepsilon(x^k_{num(i)}) \quad , \quad \text{if } i \in I \\ \left(J''_\varepsilon(x^k)\right)_{ij} = A_{ij} \quad , \quad \text{if } i \neq j \quad \text{or} \quad \text{if } i = j \notin I \end{cases} \tag{17}$$

The non linear part is located on the diagonal and involves only a few terms. We apply the preconditioning (either the simple diagonal one or the SSOR one) to the Hessian $J''_\varepsilon(x^k)$ instead of the matrix A . In this case, the preconditioning is connected to the regularization parameter ε. The efficiency of this procedure will be demonstrated.

6. NUMERICAL RESULTS

6.1 Examples

The compression of a block on a plane (fig. 1) and a dovetail assembly (fig.2) will be considered. Details of the characteristics and the loadings can be found in [1]. Different meshes will be used.

It should be note that the solutions are obtained here with a given sliding limit g corresponding to the last iteration of the fixed point on g.

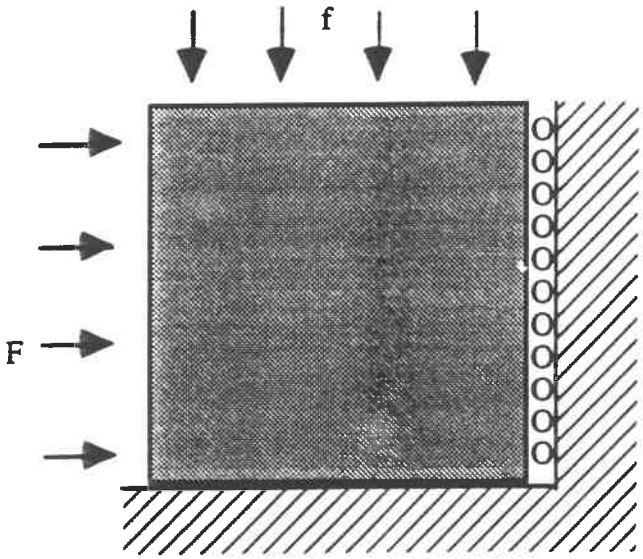

Fig. 1 compression of a block on a plane

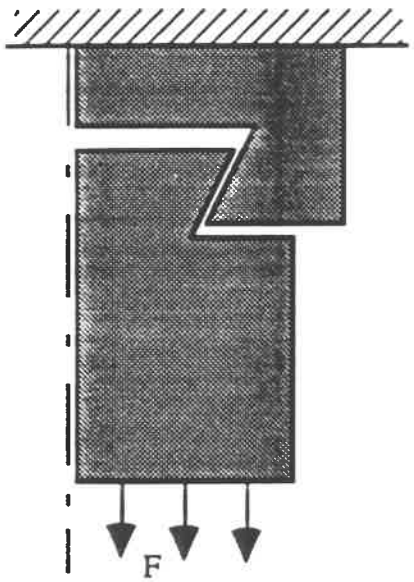

Fig. 2 dovetail assembly

6.2 Comparison between the various projections of the descent direction

(Table 3) and (table 4) give the number of iterations involved in the various methods of projecting of the descent direction:

- GC1 the Jeusette-Sonzogni method,

- GC2 the Dilintas method,
- GC3 our method.

Results with the overrelaxation method with projection previously published in [1] are also given (they need an extra determination of the optimum relaxation coefficient which is not taken into account here).

(Table 3) is related to example 1 with a discretization of 366 degrees of freedom and 20 nodes of contact, (table 4) deals with example 2 with 2234 degrees of freedom and 25 nodes of contact ; the SSOR preconditioning is used on the Hessian and ε is equal to 10^{-8}.

Methods Applied:	Iterations for :		
	REG1	REG2	REG3
GC 1	102	71	93
GC 2	98	93	90
GC 3	50	49	47
SORP (ω opti=1,58)	225		

Table 3 example 1 (366 degrees of freedom)

Methods Applied:	Iterations for :		
	REG1	REG2	REG3
GC1	289	227	257
GC3	241	223	220
SORP (ω opti=1,96)	776		

Table 4 example 2 (2234 degrees of freedom)

Line GC3 in both tables shows the savings in the number of iterations obtained. And as predicted, it is much more efficient than the overrelaxation with projection.

6.3 Comparison between the various regularization procedures

In all tables, REG1, REG2 and REG3 relate to the regularization procedures presented in (§4.1).

The three methods of regularization are fairly equivalent, but since the polynomial one is slightly more efficient and easier to implement, it was the one we preferred.

6.4 Comparison between the preconditioning procedures

The influence of the preconditioning is shown on (table 5) and (table 6). Here it is very important to include the non linear part in the preconditioning, because of the presence of non sliding nodes among those in contact with the obstacle, which gives the regularization terms (equivalent to $1/\varepsilon$) more weight than the diagonal terms of A.

The computational CPU times given in (table 5), (table 6) and (table 7) were obtained on a VAX 6310.

A Morse storage of the matrix widely improves the computational time. It has been used here instead of a skyline one.

Methods applied	GC3 with a preconditioning on A		GC3 with a preconditioning on J"		SORP ω opti
	Diagonal	SSOR	Diagonal	SSOR	
Iterations	115	49	113	47	225
Time	30s	28s	30s	27s	30s

Table 5 example 1 with 366 degrees of freedom (computed with REG1 , the approximate computing of μ^k, $\varepsilon = 10^{-8}$ and partial convergence at 10^{-3})

Methods applied	GC3 with a preconditioning on A		GC3 with a preconditioning on J"		SORP ω opti
	Diagonal	SSOR	Diagonal	SSOR	
Iterations	676	305	584	277	3646
Time	12mn22s	9mn48s	11mn20s	9mn20s	26mn19s

Table 6 example 2 with 4506 degrees of freedom (computed with REG2 , the approximate computing of μ^k, $\varepsilon = 10^{-8}$ and partial convergence at $0{,}5.10^{-3}$)

7. CONCLUSION

Various methods and various preconditioning procedures were tested. The most satisfactory results were obtained using the following strategy:

- to deal with the unilateral contact, using the variant of Dilintas' method and checking relaxation of some of the saturated constraints if necessary

after a partial convergence ($\varepsilon=10^{-3}$ or $0{,}5.10^{-3}$) of the algorithm (final convergence $\varepsilon=0.5.10^{-5}$),

- to deal with the friction term, using the piecewise polynomial or the hyperbolic tangent regularization procedure with $\varepsilon=10^{-7}$ or $\varepsilon=10^{-8}$ (accuracy of the solution has been checked by making comparisons with solutions obtained by means of non regularized methods).
- for the preconditioning procedure, using Evans' one (SSOR with $\omega=1$) applied to the Hessian including the non linear part (polynomial regularization makes it easy to calculate $\varphi''_\varepsilon(x)$).

(Table 7) shows efficiency of this strategy on example 2 with a structured mesh with 4506 degrees of freedom, of which 50 are involved in the contact.

Method	Iterations	CPU time
GC3 ($\alpha=1$)	277	9mn20s
SORP (ω opti=1,86)	3646	26mn19s

Table 7 example 2 (4506 degrees of freedom)

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