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# A non linear optimal control problem related to a road de-icing device: Analysis and Numerical experiments.

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## Abstract

In order to design a road de-icing device by heating, we consider in a two dimensional setting the optimal control of an advection-diffusion equation with a nonlinear boundary condition of the Stefan-Boltzmann type. The problem models the heating of a road during a winter period to keep positive its surface temperature above a given threshold. The heating device is performed through the circulation of a coolant in a porous layer of the road. We prove the well-posedness of the nonlinear optimal control problem, subjected to unilateral constraints, set up a gradient based algorithm then discuss some numerical results associated with real data obtained from experimental measurements. The study, initially developed in a one dimensional simpler setting in [1], aims to quantify the minimal energy to be provided to keep the road surface without frost or snow.

**Keywords:** Optimal control, Non linear advection-diffusion equation, Unilateral constraint, Road heating, Energy, De-icing

# 1 Introduction

## 1.1 Context

Winter road operations are an important issue in many countries. Although road de-icing agents remains the most widely used solution, road heating systems can be used at certain critical points of the road network (slopes with trucks traffic, bridges, access to hospital emergency services, etc.). Different types of heating systems are deployed: electromagnetic induction heating thanks to steel wool fiber, electric heating, infrared lamps above the road surface, circulation of a heat transfer fluid in pipes inserted in the road [2–4]. ”The use of pavement as a source of heat to recover energy and melt snow is a technique used in several countries and real systems have been deployed all around the world e.g. in Europe [5], Japan [6] and United States [7]. The pavement heating systems using pipes in which a heat transfer fluid circulates need power between  $320 \text{ W/m}^2$  and  $600 \text{ W/m}^2$ , which is less than that required by electric heating ( $500 \text{ W/m}^2$  and  $750 \text{ W/m}^2$ ) [8, 9]. This new type of heating system is used for pavements, bridges, pedestrian walkways, airport platforms and heliports [10, 11]. For the winter period, energy quantities to maintain positive pavement temperature and to thaw the road depend on the climate, and more precisely on the parameters: wind speed, outside temperature, presence of snow or not and intensity of snowfall. Climate data vary from country to country and from year to year. The determination of the energy quantities is therefore relative. According to the international literature review, the heating power ranges between  $160$  and  $1200 \text{ W/m}^2$  [11–13]. According to studies of bituminous hydraulic systems manufacturers using tubes, it can be concluded that the performance of these systems depends on the parameters of flow rate, pipe spacing, depth and tube diameters. For the road heating, several models for snow melting systems using pipes have been developed to treat the complexity of surface snow conditions [14, 15]. They predict the transient surface conditions and temperatures including the extent of snow cover. The model developed in this paper does not take in consideration the presence of snow cover. One time snow fall, it will be instantly melt by surface heat flux since we adapt the temperature of the injected fluid to keep the road surface temperature above  $0^\circ\text{C}$ . Simulation model and experiments have shown high performance of such hydraulic systems using tubes to defreeze road surface using renewable energy but tubes are subjected to mechanical constraint caused by traffic circulation. In this work we are interested in a road heat exchanger obtained by the gravitational circulation of a heat transfer fluid in a porous pavement without the use of pipes inserted in the road [16, 17]. More precisely the pavement structure studied is composed of three asphalt layers where the central one is a highly porous draining

asphalt through which circulates a fluid (water) via gravitational flow under slant effect. We refer to [1, Figure 2, page 776] for a scheme of the demonstrator.

Very few studies are conducted for this kind of device [18, 19]. In a recent study [20], the authors initiated modelling works for such devices. An instrumented demonstrator has been developed by Cerema (Centre for expertise and engineering on risks, urban and country planning, environment and mobility) in order to study how to build such a road and to develop thermo-hydraulic models validated from experimental measurements [20]. This demonstrator is referenced in [21] as one of the 50 worldwide large-scale demonstrators of hydronic asphalt pavement systems. It is an experimental roadway constructed in 2014 at Egletons, France. The road with 50 m in length and 4 m in width, with a transversal slope around 2% with no longitudinal slope, is composed of three layers: a wearing course layer of semi-phaneritic asphalt concrete 0.06 m thick; a bonding course layer of 0/14 porous asphalt 0.08 m thick; and a base layer of asphalt concrete with a thickness of 0.05 m (see Figure 1).

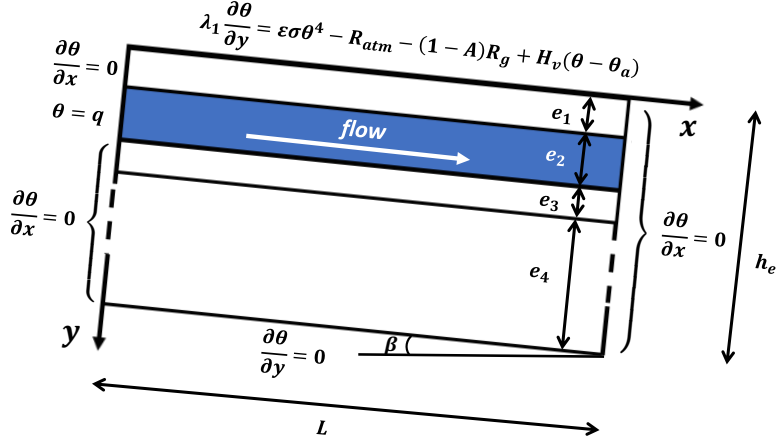


**Fig. 1** The Egletons demonstrator.

Whatever the technique used (electrical heating, pipes, porous asphalt), the design of the devices needs to quantify the minimal energy to be provided to keep the road surface without frost or snow, for a given winter meteorological scenario. This optimal control based approach is not addressed in the literature. We can mention [1] in which the authors proposed a method to optimize the heating power needed by a road to maintain its surface temperature above  $0^{\circ}\text{C}$  during winter. This previous work considered a 1D road thermic model with a punctual heating source inserted in the road. This model is well representative of a road equipped with an electrical heating. We extend in this paper the modeling to a heating device mentioned above and the inlet temperature of the heating coolant is now the control variable instead of the heating source considered for the 1D case.

## 1.2 2D diffusion-advection model

We consider the 2D model of road as described in Figure 2, which corresponds to the heating device of [1, Figure 2, page 776]. As represented in Figure 2, space variables are  $x$  along the sub-horizontal transversal axis of the pavement with slant angle  $\beta$  and



**Fig. 2** Schema of pavement structure with its limit conditions ( $q$  and  $\theta_a$  are respectively the injection temperature of the fluid and the air temperature).

$y$  along the upwards sub-vertical axis, perpendicular to  $x$ . The road is assumed to have no longitudinal slant and to be infinite in its third dimension.  $h_e$  and  $L$  denote the height of the road structure and its length respectively. The temperature of the road is denoted by  $\theta(x, y, t)$ , the hydraulic load by  $H(x, y, t)$  and the hydraulic conductivity by  $K(x, y, \theta)$ . We are then interested in the following diffusion-convection system: for all  $(x, y, t)$  in  $(0, L) \times (0, h_e) \times (0, T)$ ,

$$\begin{cases} c(x, y) \partial_t \theta(x, y, t) - \operatorname{div}(k(x, y) \nabla \theta(x, y, t)) - \operatorname{div}(K \theta(x, y, t) \nabla H(x, y, \theta)) = 0, \\ \operatorname{div}(K \nabla H(x, y, t)) = 0, \end{cases} \quad (1)$$

where  $c$  and  $k$  are positive functions. The first equation of (1) corresponds to the energy conservation and the second one to the mass conservation and the Darcy's law. We consider homogeneous Neumann conditions for the temperature  $\theta$  except for the upstream condition of porous asphalt layer ( $x = 0$ ,  $e_1 \leq y \leq e_1 + e_2$ ) and the road surface condition ( $y = 0$ ). We then have:

$$\begin{cases} \partial_x \theta(0, y, t) = 0, y \in (0, e_1) \cup (e_1 + e_2, h_e), t \in (0, T) \\ \theta(0, y, t) = q(t), y \in (e_1, e_1 + e_2), t \in (0, T) \\ \partial_x \theta(L, y, t) = 0, y \in (0, h_e), t \in (0, T) \\ \partial_y \theta(x, h_e, t) = 0, x \in (0, L), t \in (0, T) \\ k(0) \partial_y \theta(x, 0, t) = \sigma \varepsilon(t) \theta^4(x, 0, t) - f_1(t) + f_2(t) \theta(x, 0, t), x \in (0, L), t \in (0, T) \end{cases} \quad (2)$$

where  $f_1$  and  $f_2$  are defined by:

$$f_1(t) := (1 - A(t))R_g(t) + R_{atm}(t) + H_v(t)\theta_a(t) - L_f I(t), \quad f_2(t) := H_v(t). \quad (3)$$

We introduce the following notations :

- $c(x, y)$  : volumic heat capacity of the road material at the point  $(x, y)$  ( $J.K^{-1}.m^{-3}$ ),
- $k(x, y)$  : thermal conductivity of the road material at the point  $(x, y)$  ( $W.K^{-1}.m^{-1}$ ),
- $\varepsilon, A$  : emissivity and albedo of the road surface,
- $\sigma$  : Stefan-Boltzmann constant ( $5.67 \times 10^{-8} W/m^2K^4$ ),
- $R_{atm}, R_g$  : atmospheric and global radiation ( $W/m^2$ ),
- $\theta_a$  : air temperature ( $K$ ),  $H_v$  : convection heat transfer coefficient ( $W/m^2K$ ),
- $I$  : snowrate ( $mm.s^{-1}$ ),  $L_f$  : latent heat of fusion of the ice per kg ( $J.kg^{-1}$ ).

Boundary conditions for the hydraulic load  $H$  are only given on the boundary of  $(0, L) \times (e_1, e_1 + e_2)$  since it is assumed that  $K = 0$  outside of  $(0, L) \times (e_1, e_1 + e_2)$  (the fluid circulates in the second road layer). We impose homogeneous Neumann condition (hydraulic tightness) except for the upstream and downstream boundaries:

$$\begin{cases} H(0, y, t) = H_1(t), H(L, y, t) = 0, & y \in (e_1, e_1 + e_2), t \in (0, T), \\ \partial_y H(x, e_1, t) = \partial_y H(x, e_1 + e_2, t) = 0, & x \in (0, L), t \in (0, T), \end{cases} \quad (4)$$

where  $H_1$  represents the hydraulic load imposed upstream of fluid circulating in porous draining asphalt layer. We assume a saturated fluid circulation with a constant hydraulic conductivity  $K$  in the second layer: the second equation of (1) leads to  $\Delta H = 0$  and then:

$$\forall (x, y, t) \in (0, L) \times (e_1, e_1 + e_2) \times (0, T), \partial_x H(x, y, t) = -\frac{H_1(t)}{L}, \partial_y H(x, y, t) = 0. \quad (5)$$

For physical reasons, a saturated circulation implies that for all  $t \in (0, T)$ ,  $H_1(t) \geq \beta L$  for some  $\beta > 0$ . Eventually, introducing the following notations:

$$\begin{cases} \Omega = (0, L) \times (0, h_e), & \Sigma_b = (0, L) \times \{0\}, \\ \Sigma_c = \{0\} \times (e_1, e_1 + e_2), & \Sigma_d = \{L\} \times (e_1, e_1 + e_2), & \Sigma_f = (0, L) \times \{h_e\}, \end{cases} \quad (6)$$

the temperature  $\theta = \theta(x, y, t)$  in the four layers road occupying the domain  $\Omega$  satisfies the following boundary value problem:

$$\begin{cases} c(x, y) \partial_t \theta - \operatorname{div}(k(x, y) \nabla \theta) + K \frac{H_1(t)}{L} 1_{(e_1, e_1 + e_2)}(y) \partial_x \theta = 0, & \Omega \times (0, T), \\ k \nabla \theta \cdot \nu = 0, & (\partial \Omega \setminus (\Sigma_b \cup \Sigma_c)) \times (0, T), \\ \theta = q, & \Sigma_c \times (0, T), \\ k(x, y) \partial_y \theta = \sigma \varepsilon(t) \theta^4 - f_1(t) + f_2(t) \theta, & \Sigma_b \times (0, T), \\ \theta = \theta_0, & \Omega \times \{0\}. \end{cases} \quad (7)$$

$q$  is the temperature of the injected fluid inside the road through the part  $\Sigma_c$ . The initial temperature  $\theta_0$  is defined as the solution of the following boundary value problem

$$\begin{cases} -\operatorname{div}(k(x, y)\nabla\theta_0) = 0, & \Omega, \\ k\nabla\theta_0 \cdot \nu = 0, & \partial\Omega \setminus (\Sigma_b \cup \Sigma_c \cup \Sigma_f), \\ \theta_0 = q_0, & \Sigma_c, \\ k(x, y)\partial_y\theta_0 = \sigma\varepsilon(0)\theta_0^4 - f_1(0) + f_2(0)\theta_0, & \Sigma_b, \\ \theta_0 = \theta_s, & \Sigma_f, \end{cases} \quad (8)$$

where the soil temperature  $\theta_s$  is given and assumed time-independent.

### 1.3 The optimal control problem

Our goal is to find the optimal inlet temperature  $q(\cdot)$  over the time interval  $(0, T)$  corresponding to a minimal consumption of heating energy and allowing to maintain the surface temperature  $\theta(x, 0, \cdot)$  above a minimal prescribed temperature  $\underline{\theta}$ . We assume that the temperature  $q$  of the injected fluid on  $\Sigma_c$  is positive and constant with respect to the variable  $(x, y)$ . We then define the heating energy  $E_h$  as the energy loss by the coolant between its entrance to the road, at  $x = 0$  and its exit at  $x = L$ , that is:

$$E_h = C_f \int_0^T v(t) \int_{e_1}^{e_1+e_2} (q(t) - \theta(L, y, t))^+ dy dt \quad (9)$$

where  $v(t) := \frac{KH_1(t)}{L}$  for all  $t \geq 0$  and  $C_f$  is the volumetric heat capacity of water.  $a^+ := \max(a, 0)$  denotes the positive part of any real  $a$ . We consider the positive part to calculate the spent heating energy without taking into account a possible energy retrieving by the fluid.

The corresponding optimal control problem is the following one :

$$\begin{cases} \inf_{q \in H^1(0, T)} J_\alpha(q) & \text{where } J_\alpha(q) := \frac{1}{2}E_h^2(q) + \frac{\alpha}{2}\|q_t\|_{L^2(0, T)}^2 \\ \text{subjected to} & q(0) = q_0 > 0, \quad q > 0 \text{ in } t \in [0, T], \\ & \theta \geq \underline{\theta} \text{ on } \Sigma_b \times (0, T), \quad \theta = \theta(q) \text{ solves (12);} \end{cases} \quad (10)$$

$\alpha \geq 0$  is a regularizing parameter to ensure the control  $q$  to be regular enough which is of interest in practice. We also emphasize that the initial value  $q_0$  of the control  $q$  is related to the initial temperature  $\theta_0$  solution of (8).

In Section 2, we perform the mathematical analysis of the direct problems (7) and (8). The well-posedness is first obtained for the intermediate systems for the which nonlinear term  $\theta^4$  is replaced by  $|\theta|^3\theta$ . Monotonicity and fixed point arguments are employed (see Lemma 1 and Theorem 1). Then, assuming that the inlet temperature

$q$  is positive, we prove a maximum principle and get that the former solution is indeed positive so that the systems coincide (see Theorem 2). We also prove that the map  $q \rightarrow \theta$  is Lipschitz continuous (see Theorem 3), a regularity property used for the analysis of the optimal control problem (10). Section 3 is devoted to the study the control problem. We first prove that the the functional  $J$  is strictly convex (see Lemma 5) and then that the problem (10) admits a unique solution (see Theorem 4). Eventually, we prove that a penalized/regularized approximation  $J_{\alpha, \epsilon, r}$  defined in (40) of the cost  $J_\alpha$  has a directional derivative (see Proposition 8) leading to a minimizing sequence  $(q^k)_{k \in \mathbb{N}}$  defined in (44) in term of an adjoint problem. Section 4 discusses some numerical experiments while Section 5 concludes and provides several perspectives.

To our knowledge, few works have addressed optimal control problem involving nonlinear boundary condition. We mention [22] where an optimal control problem for the heat equation with a nonlinear boundary condition of the form  $\partial_\nu u = g(u, f)$  is studied; the function  $g$  is assumed locally Lipschitz continuous, strictly monotone increasing in  $u$  and decreasing in  $f$ . Existence and uniqueness of weak solution are obtained with the Leray-Schauder theorem and maximum principles. We also mention [23] where a one dimensional nonlinear boundary control problem occurring in heat conduction with a boundary condition governed by the Stefan-Boltzmann law is studied; the law (similar to the one considered in [1]) writes:  $\partial_x y(t, 1) = -y^4(t, 1) + u(t)$ ,  $t > 0$  where the control  $u$  is assumed in  $[0, 1]$  for all  $t > 0$ . The existence and uniqueness of positives solutions is proved as well as optimal controls (so as to minimize the state  $y$  at time  $T > 0$ ). In this simple 1D setting, integral representations of the solution using Green's function are employed. More recently, we mention the works [24, 25] for some related studies.

## 2 Analysis of the boundary value problems (7)-(8)

We define the spaces  $V = \{v \in H^1(\Omega), v = 0 \text{ on } \Sigma_c\}$  and  $V_1 = \{v \in H^1(\Omega), v = 0 \text{ on } \Sigma_c \cup \Sigma_f\}$  both endowed with the scalar product and the norm of  $H^1(\Omega)$ .

In the sequel we assume the following regularity properties on the data:

$$\begin{cases} k \in L^\infty(\Omega), \inf_{\Omega} k(x, y) = k_0 > 0, \partial_y k \in L^2(\Omega), \sigma \geq 0, \\ c \in L^\infty(\Omega), \inf_{\Omega} c(x, y) = c_0 > 0, \quad H_1 \in L^\infty(0, T), H_1 \geq 0, \\ \varepsilon \in W^{1, \infty}(0, T), \inf_{(0, T)} \varepsilon = \varepsilon_0 > 0, \\ f_1 \in H^1(0, T), f_1 \geq 0, f_2 \in W^{1, \infty}(0, T), f_2 \geq 0, \\ q \in H^1(0, T), q(0) = q_0 > 0. \end{cases} \quad (11)$$

The analysis of (7) and (8) relies on a monotonicity argument: we first replace in (7) the nonlinear term  $\theta^4$  by  $|\theta|^3\theta$  and in (8) the nonlinear term  $\theta_0^4$  by  $|\theta_0|^3\theta_0$  and then show that if  $q$  is non-negative, then  $\theta_0$  and  $\theta$  are non negative as well on  $\Sigma_b$ .



## 2.1 Intermediates problems and monotonicity argument

We consider the following boundary problem: find  $\theta$  solution of

$$\begin{cases} c(x, y)\partial_t\theta - \operatorname{div}(k(x, y)\nabla\theta) + v(t)1_{(e_1, e_1+e_2)}(y)\partial_x\theta = 0, & \Omega \times (0, T), \\ k\nabla\theta \cdot \nu = 0, & (\partial\Omega \setminus (\Sigma_b \cup \Sigma_c)) \times (0, T), \\ \theta = q, & \Sigma_c \times (0, T), \\ k(x, y)\partial_y\theta = \sigma\varepsilon(t)|\theta|^3\theta - f_1(t) + f_2(t)\theta, & \Sigma_b \times (0, T), \\ \theta = \theta_0, & \Omega \times \{0\} \end{cases} \quad (12)$$

where  $\theta_0$  is, for any real  $\theta_s \geq 0$ , solution of

$$\begin{cases} -\operatorname{div}(k(x, y)\nabla\theta_0) = 0, & \Omega, \\ k\nabla\theta_0 \cdot \nu = 0, & \partial\Omega \setminus (\Sigma_b \cup \Sigma_c \cup \Sigma_f), \\ \theta_0 = q_0, & \Sigma_c, \\ k(x, y)\partial_y\theta_0 = \sigma\varepsilon(0)|\theta_0|^3\theta_0 - f_1(0) + f_2(0)\theta_0, & \Sigma_b, \\ \theta_0 = \theta_s, & \Sigma_f. \end{cases} \quad (13)$$

**Lemma 1.** *For any  $\theta_s \geq 0$  and  $q_0 \geq 0$  there exists  $\theta_0 \in H^1(\Omega)$  unique solution of system (13). This solution satisfies  $\theta_0 \geq 0$  a.e. in  $\Omega$ . More precisely, the following properties hold :*

- *Let any  $l \geq 0$  small enough such that  $\sigma\varepsilon(0)l^4 + f_2(0)l - f_1(0) \leq 0$ . If  $q_0 \geq l$  and  $\theta_s \geq l$ , then  $\theta_0 \geq l$  a.e. in  $\Omega$ .*
- *Let any  $l > 0$  large enough such that  $\sigma\varepsilon(0)l^4 + f_2(0)l - f_1(0) \geq 0$ . If  $q_0 \leq l$  and  $\theta_s \leq l$ , then  $\theta_0 \leq l$  a.e. in  $\Omega$ .*

*Proof. Existence of a solution.* By linearization and fixed point method. Let  $\check{\theta}_0 \in H^1(\Omega)$ . We are looking for  $\tilde{\theta}_0 \in H^1(\Omega)$  solution of

$$\begin{cases} -\operatorname{div}(k(x, y)\nabla\tilde{\theta}_0) = 0, & \Omega, \\ k\nabla\tilde{\theta}_0 \cdot \nu = 0, & \partial\Omega \setminus (\Sigma_b \cup \Sigma_c \cup \Sigma_f), \\ \tilde{\theta}_0 = q_0, & \Sigma_c, \\ k(x, y)\partial_y\tilde{\theta}_0 = \sigma\varepsilon(0)|\check{\theta}_0|^3\tilde{\theta}_0 - f_1(0) + f_2(0)\tilde{\theta}_0, & \Sigma_b, \\ \tilde{\theta}_0 = \theta_s, & \Sigma_f. \end{cases} \quad (14)$$

Let  $g : [0, h_e] \rightarrow [0, 1]$  be a smooth cut-off function with support in  $[e_1 - \eta, e_1 + e_2 + \eta]$ ,  $g = 1$  on  $(e_1, e_1 + e_2)$  and  $h : [0, h_e] \rightarrow [0, 1]$  a smooth cut-off function with support in  $[h_e - \eta, h_e]$  for some  $\eta > 0$  small enough,  $h = 1$  on  $(h_e - \frac{\eta}{2}, h_e)$ . In order to get homogeneous Dirichlet boundary conditions, we introduce the variable  $\tilde{\theta}_0 = \underline{\theta}_0 + g(y)q_0 + \theta_s h(y)$ . Then, since  $g(y)q_0 + \theta_s h(y) = 0$  on a neighbourhood of

$\Sigma_b$ ,  $\underline{\theta}_0 = \tilde{\theta}_0$  on a neighbourhood of  $\Sigma_b$ , and therefore  $\tilde{\theta}_0 \in H^1(\Omega)$  is solution of (14) if and only if  $\underline{\theta}_0 \in H^1(\Omega)$  is solution of

$$\begin{cases} -\operatorname{div}(k(x, y)\nabla\underline{\theta}_0) = \partial_y \left( k(x, y)(g(y)q_0 + \theta_s h(y))_y \right), & \Omega, \\ k\nabla\underline{\theta}_0 \cdot \nu = 0, & \partial\Omega \setminus (\Sigma_b \cup \Sigma_c \cup \Sigma_f), \\ \underline{\theta}_0 = 0, & \Sigma_c, \\ k(x, y)\partial_y\underline{\theta}_0 = \sigma\varepsilon(0)|\hat{\theta}_0|^3\underline{\theta}_0 - f_1(0) + f_2(0)\underline{\theta}_0, & \Sigma_b, \\ \underline{\theta}_0 = 0, & \Sigma_f \end{cases} \quad (15)$$

where  $\hat{\theta}_0 := \tilde{\theta}_0 - g(y)q_0 - \theta_s h(y) \in V_1$ , that, for all  $v \in V_1$ , is solution of

$$\int_{\Omega} k\nabla\underline{\theta}_0 \cdot \nabla v + \int_{\Sigma_b} \left( \sigma\varepsilon(0)|\hat{\theta}_0|^3\underline{\theta}_0 - f_1(0) + f_2(0)\underline{\theta}_0 \right) v \, dx = - \int_{\Omega} k(x, y)(g(y)q_0 + \theta_s h(y))_y \partial_y v. \quad (16)$$

Let then  $a : V_1 \times V_1 \rightarrow \mathbb{R}$ ,  $(u, v) \mapsto \int_{\Omega} k\nabla u \cdot \nabla v + \int_{\Sigma_b} \left( \sigma\varepsilon(0)|\hat{\theta}_0|^3 + f_2(0) \right) uv \, dx$ . Since  $f_2(0) \geq 0$ ,  $\varepsilon(0) \geq \varepsilon_0 > 0$ ,  $\sigma \geq 0$  and  $k \geq k_0 > 0$ , we get that

$$\begin{aligned} |a(u, v)| &\leq \|k\|_{L^\infty(\Omega)} \|u\|_{V_1} \|v\|_{V_1} + \sigma\varepsilon(0) \|\hat{\theta}_0\|_{L^5(\Sigma_b)}^3 \|u\|_{L^5(\Sigma_b)} \|v\|_{L^5(\Sigma_b)} + f_2(0) \|u\|_{L^2(\Sigma_b)} \|v\|_{L^2(\Sigma_b)} \\ &\leq C(\|k\|_{L^\infty(\Omega)} + \sigma\varepsilon(0) \|\hat{\theta}_0\|_{L^5(\Sigma_b)}^3 + f_2(0)) \|u\|_{V_1} \|v\|_{V_1}, \quad \forall (u, v) \in V_1 \times V_1 \end{aligned}$$

using since  $\Omega \subset \mathbb{R}^2$  that the injection  $\gamma_0(V_1) \hookrightarrow L^5(\partial\Omega)$ , where  $\gamma_0 : u \mapsto u|_{\partial\Omega}$ , is continuous. Similarly, we get that  $a(u, u) \geq k_0 \|u\|_{V_1}^2$  for all  $u \in V_1$ . Consequently, the Lax Milgram theorem implies the existence of  $\underline{\theta}_0 \in V_1$  unique solution of (16), which is the weak solution of (15) with the estimate

$$\|\underline{\theta}_0\|_{V_1} \leq \frac{1}{k_0} \left( \|k(x, y)(g(y)q_0 + \theta_s h(y))_y\|_{L^2(\Omega)} + h_\varepsilon^{1/2} |f_1(0)| \right). \quad (17)$$

We define  $V_{4/5} = \{v \in H^{4/5}(\Omega), v = 0 \text{ on } \Sigma_c \cup \Sigma_f\}$ . Then we have, since  $V_{4/5} \hookrightarrow H^{4/5}(\Omega)$ , (see [26, Théorème 9.4, page 47])

$$\gamma_0(V_{4/5}) \hookrightarrow H^{4/5-1/2}(\partial\Omega) = H^{3/10}(\partial\Omega) \hookrightarrow L^5(\partial\Omega).$$

In the sequel, we always denote  $\gamma_0(u) = u$  and therefore  $\gamma_0(V_{4/5}) = V_{4/5}$ .

Let  $\Lambda : V_{4/5} \rightarrow V_{4/5}$ ,  $\hat{\theta}_0 \mapsto \underline{\theta}_0$  where  $\underline{\theta}_0 \in V_1$  is the unique solution of (16). Then  $\Lambda$  is continuous. Indeed, let  $(\hat{\theta}_0^n)_{n \in \mathbb{N}}$  be a sequence of  $V_{4/5}$  and  $\hat{\theta}_0 \in V_{4/5}$  such that  $\hat{\theta}_0^n \rightarrow \hat{\theta}_0$  in  $V_{4/5}$ . Let us prove that  $\underline{\theta}_0^n = \Lambda(\hat{\theta}_0^n) \rightarrow \underline{\theta}_0 = \Lambda(\hat{\theta}_0)$  in  $V_{4/5}$ . From (17),  $(\underline{\theta}_0^n)_{n \in \mathbb{N}}$  is a bounded sequence of  $V_1$  and thus there exists a subsequence, still denoted by  $(\underline{\theta}_0^n)_{n \in \mathbb{N}}$  and  $\underline{\theta}_0 \in V_1$  such that  $\underline{\theta}_0^n \rightarrow \underline{\theta}_0$  weakly in  $V_1$ . Since  $V_1 \hookrightarrow L^5(\Sigma_b)$  is compact,  $\underline{\theta}_0^n \rightarrow \underline{\theta}_0$

in  $L^5(\Sigma_b)$ , and passing to the limit in (16) we obtain, since  $\widehat{\theta}_0^n \rightarrow \widehat{\theta}_0$  in  $L^5(\Sigma_b)$  that  $\underline{\theta}_0$  satisfies, for all  $v \in V_1$

$$\int_{\Omega} k \nabla \underline{\theta}_0 \cdot \nabla v + \int_{\Sigma_b} \left( \sigma \varepsilon(0) |\widehat{\theta}_0|^3 \underline{\theta}_0 - f_1(0) + f_2(0) \underline{\theta}_0 \right) v \, dx = - \int_{\Omega} k(x, y) (g(y) q_0 + \theta_s h(y))_y \partial_y v$$

that is  $\underline{\theta}_0 = \Lambda(\widehat{\theta}_0)$ . This solution being unique, the whole sequence  $(\underline{\theta}_0^n)_{n \in \mathbb{N}}$  weakly converges to  $\underline{\theta}_0$  in  $V_1$  and since  $V_1 \hookrightarrow V_{4/5}$  is compact,  $\underline{\theta}_0^n = \Lambda(\widehat{\theta}_0^n) \rightarrow \underline{\theta}_0 = \Lambda(\widehat{\theta}_0)$  in  $V_{4/5}$ . Thus  $\Lambda$  is continuous. Moreover,  $\Lambda(V_1) \subset K_1 = \{u \in V_1, \|u\|_{V_1} \leq \frac{1}{k_0} (\|k(x, y) (g(y) q_0 + \theta_s h(y))_y\|_{L^2(\Omega)} + h_e^{1/2} |f_1(0)|)\}$  is a compact convex subset of  $V_{4/5}$ . thus, from the Schauder fixed point theorem, there exists  $\widehat{\theta}_0 \in V_1$  solution of

$$\begin{aligned} \int_{\Omega} k \nabla \widehat{\theta}_0 \cdot \nabla v + \int_{\Sigma_b} \left( \sigma \varepsilon(0) |\widehat{\theta}_0|^3 \widehat{\theta}_0 - f_1(0) + f_2(0) \widehat{\theta}_0 \right) v \, dx \\ = - \int_{\Omega} k(x, y) (g(y) q_0 + \theta_s h(y))_y \partial_y v \quad \forall v \in V_1 \end{aligned} \quad (18)$$

and thus  $\theta_0 = \widehat{\theta}_0 + g(y) q_0 + \theta_s h(y) \in H^1(\Omega)$  is a solution of (13).

**Uniqueness of the solution of (13).** Let  $\theta_0^1, \theta_0^2 \in H^1(\Omega)$  be two solutions of (13). Then  $\theta_0 = \theta_0^1 - \theta_0^2$  is solution of

$$\begin{cases} -\operatorname{div}(k(x, y) \nabla \theta_0) = 0, & \Omega, \\ k(x, y) \nabla \theta_0 \cdot \nu = 0, & \partial\Omega \setminus (\Sigma_b \cup \Sigma_c \cup \Sigma_f), \\ \theta_0 = 0, & \Sigma_c \cup \Sigma_f, \\ k(x, y) \partial_y \theta_0 = \sigma \varepsilon(0) (|\theta_0^1|^3 \theta_0^1 - |\theta_0^2|^3 \theta_0^2) + f_2(0) \theta_0, & \Sigma_b. \end{cases} \quad (19)$$

Multiplying (19)-1 by  $\theta_0$  and integrating by part, we get

$$\int_{\Omega} k |\nabla \theta_0|^2 + \int_{\Sigma_b} \sigma \varepsilon(0) (|\theta_0^1|^3 \theta_0^1 - |\theta_0^2|^3 \theta_0^2) \theta_0 + f_2(0) |\theta_0|^2 = 0.$$

Since for all  $(a, b) \in \mathbb{R}^2$ ,  $(|a|^3 a - |b|^3 b)(a - b) \geq 0$ , the term of the left-hand side  $(|\theta_0^1|^3 \theta_0^1 - |\theta_0^2|^3 \theta_0^2)(\theta_0^1 - \theta_0^2) \geq 0$  and thus  $\int_{\Omega} k |\nabla \theta_0|^2 + \int_{\Sigma_b} f_2(0) |\theta_0|^2 \leq 0$ . Therefore, since  $f_2(0) \geq 0$ ,  $k > 0$  and  $\theta_0 \in V_1$ , we get that  $\theta_0 = 0$ , that is  $\theta_0^1 = \theta_0^2$ .

**Proof of  $\theta_0 \geq l$  a.e. in  $\Omega$  if moreover  $\theta_s \geq l$ ,  $q_0 \geq l$  for any  $l$  such that  $\sigma\varepsilon(0)l^4 + f_2(0)l - f_1(0) \leq 0$ .** Let  $\bar{\theta}_0 = \theta_0 - l$ . Then  $\bar{\theta}_0$  is solution of

$$\begin{cases} -\operatorname{div}(k(x, y)\nabla\bar{\theta}_0) = 0, & \Omega, \\ k(x, y)\nabla\bar{\theta}_0 \cdot \nu = 0, & \partial\Omega \setminus (\Sigma_b \cup \Sigma_c \cup \Sigma_f), \\ \bar{\theta}_0 = q_0 - l, & \Sigma_c, \\ k(x, y)\partial_y\bar{\theta}_0 = \sigma\varepsilon(0)|\theta_0|^3\theta_0 - f_1(0) + f_2(0)\theta_0, & \Sigma_b, \\ \bar{\theta}_0 = \theta_s - l, & \Sigma_f. \end{cases} \quad (20)$$

Let  $\bar{\theta}_0^-$  be the negative part of  $\bar{\theta}_0$ . Then  $\bar{\theta}_0^- \in H^1(\Omega)$  and since  $\bar{\theta}_0 = \theta_s - l \geq 0$  on  $\Sigma_f$  and  $\bar{\theta}_0 = q_0 - l \geq 0$  on  $\Sigma_c$ , we get  $\bar{\theta}_0^- = 0$  on  $\Sigma_c \cup \Sigma_f$ . Multiplying (20) by  $\bar{\theta}_0^-$ , we get

$$\begin{aligned} \int_{\Omega} k\nabla\bar{\theta}_0^- \cdot \nabla\bar{\theta}_0^- &= \int_{\Omega} k|\nabla\bar{\theta}_0^-|^2 = \int_{\Sigma_b} (f_1(0) - \sigma\varepsilon(0)|\theta_0|^3\theta_0 - f_2(0)\theta_0)\bar{\theta}_0^- \\ &\leq \int_{\Sigma_b} (f_1(0) - \sigma\varepsilon(0)l^3l - f_2(0)l)\bar{\theta}_0^- \end{aligned}$$

since  $\bar{\theta}_0^- = 0$  if  $\theta_0 \geq l$ , from which we get, since  $f_1(0) - \sigma\varepsilon(0)l^3l - f_2(0)l \geq 0$  that  $\int_{\Omega} k|\nabla\bar{\theta}_0^-|^2 \leq 0$  and thus  $\bar{\theta}_0^- = 0$  (since  $k > 0$  and  $\bar{\theta}_0^- \in V_1$ ), that is  $\theta_0 \geq l$ .

The other statements are obtained in a similar way.  $\square$

**Theorem 1.** *Let  $q \in H^1(0, T)$ ,  $q(0) = q_0$  and  $\theta_0 \in H^1(\Omega)$  be the unique solution of (13) associated with  $q_0$  given in Lemma 1. Then there exists  $\theta \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$  unique solution of system (12).*

*Proof.* As in the elliptic case, in order to get a homogeneous Dirichlet condition on  $\Sigma_c$ , we perform the change of variable  $\bar{\theta} := \theta - g(y)q(t)$  where  $g : [0, h_e] \rightarrow [0, 1]$  is a smooth cut-off function with support in  $[e_1 - \eta, e_1 + e_2 + \eta]$  for some  $\eta > 0$  small enough,  $g = 1$  on  $(e_1, e_1 + e_2)$ .  $\bar{\theta}$  solves

$$\begin{cases} c(x, y)\partial_t\bar{\theta} - \operatorname{div}(k(x, y)\nabla\bar{\theta}) + v(t)1_{(e_1, e_1+e_2)}(y)\partial_x\bar{\theta} \\ \quad = -c(x, y)g(y)q_t + q\partial_y(k(x, y)g_y), & \Omega \times (0, T), \\ k\nabla\bar{\theta} \cdot \nu = 0, & \partial\Omega \setminus (\Sigma_b \cup \Sigma_c) \times (0, T), \\ \bar{\theta} = 0, & \Sigma_c \times (0, T), \\ k(x, y)\partial_y\bar{\theta} = \sigma\varepsilon(t)|\bar{\theta}|^3\bar{\theta} - f_1(t) + f_2(t)\bar{\theta}, & \Sigma_b \times (0, T), \\ \bar{\theta}(0) = \theta_0 - g(y)q_0 = \bar{\theta}_0, & \Omega \times \{0\}. \end{cases} \quad (21)$$

The change of variable does not affect the condition on  $\Sigma_b$  since  $g = 0$  on  $\Sigma_b$ .

The existence of  $\bar{\theta} \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$  is obtained for instance using the Galerkin method and monotonicity arguments (see [25]). From the regularity of  $g(y)q(t)$  we deduce that  $\theta = \bar{\theta} + g(y)q(t) \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$  is solution of (12). In the rest of the proof, we provide only some estimates for  $\bar{\theta}$ .

**Estimate in  $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ .**

Since the term

$$\operatorname{div}(k(x, y)\nabla\bar{\theta}) = c(x, y)\partial_t\bar{\theta} + v(t)1_{(e_1, e_1+e_2)}(y)\partial_x\bar{\theta} - c(x, y)g(y)q_t + q\partial_y(k(x, y)g_y)$$

is in  $L^2((0, T) \times \Omega)$ , multiplying (21-1) by  $\bar{\theta}$  and integrating over  $\Omega \times (0, t)$ , we get :

$$\begin{aligned} & \frac{1}{2}\|\sqrt{c}\bar{\theta}(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega k|\nabla\bar{\theta}|^2 + \frac{1}{2} \int_0^t \int_{\Sigma_d} v|\bar{\theta}|^2 + \int_0^t \int_{\Sigma_b} f_2|\bar{\theta}|^2 + \int_0^t \int_{\Sigma_b} \sigma\varepsilon(t)|\bar{\theta}|^5 = \\ & \frac{1}{2}\|\sqrt{c}\bar{\theta}_0\|_{L^2(\Omega)}^2 + \int_0^t \int_{\Sigma_b} f_1\bar{\theta} - \int_0^t \int_\Omega c(x, y)g(y)q_t\bar{\theta} + \int_0^t \int_\Omega q\partial_y(k(x, y)g_y)\bar{\theta}. \end{aligned}$$

But

$$\left| \int_0^t \int_{\Sigma_b} f_1(t)\bar{\theta} \right| \leq L^{1/2}\|f_1\|_{L^2(0, T)} \left( \int_0^t \int_{\Sigma_b} \bar{\theta}^2 \right)^{1/2} \leq \frac{\alpha L}{k_0}\|f_1\|_{L^2(0, T)}^2 + \frac{1}{4} \int_0^t \int_\Omega k|\nabla\bar{\theta}|^2$$

where  $\sqrt{\alpha}$  corresponds to the embedding constant from  $H^1(\Omega)$  into  $L^2(\Sigma_b)$ ,

$$\begin{aligned} \left| \int_0^t \int_\Omega c(x, y)g(y)q_t\bar{\theta} \right| & \leq \|cg\|_{L^2(\Omega)}\|q_t\|_{L^2(0, T)}\|\bar{\theta}\|_{L^2((0, t)\times\Omega)} \\ & \leq \frac{\beta}{k_0}\|cg\|_{L^2(\Omega)}^2\|q_t\|_{L^2(0, T)}^2 + \frac{1}{4} \int_0^t \int_\Omega k|\nabla\bar{\theta}|^2 \end{aligned}$$

where  $\sqrt{\beta}$  corresponds to the Poincaré constant ( $\|\bar{\theta}\|_{L^2((0, t)\times\Omega)} \leq \sqrt{\beta}\|\nabla\bar{\theta}\|_{L^2((0, t)\times\Omega^2)}$ ), and

$$\begin{aligned} \left| \int_0^t \int_\Omega q\partial_y(k(x, y)g_y)\bar{\theta} \right| & \leq \|\partial_y(k(x, y)g_y)\|_{L^2(\Omega)}\|q\|_{L^2(0, T)}\|\bar{\theta}\|_{L^2((0, t)\times\Omega)} \\ & \leq \frac{\beta}{k_0}\|\partial_y(k(x, y)g_y)\|_{L^2(\Omega)}^2\|q\|_{L^2(0, T)}^2 + \frac{1}{4} \int_0^t \int_\Omega k|\nabla\bar{\theta}|^2 \end{aligned}$$

which gives, for some constant  $C > 0$

$$\begin{aligned} & \|\sqrt{c}\bar{\theta}\|_{L^\infty(0, T; L^2(\Omega))}^2 + \int_0^T \int_\Omega k|\nabla\bar{\theta}|^2 + \int_0^T \int_{\Sigma_b} \sigma\varepsilon(t)|\bar{\theta}|^5 \leq \|\sqrt{c}(\theta_0 - g(y)q_0)\|_{L^2(\Omega)}^2 \\ & + \frac{C}{k_0}\|f_1\|_{L^2(0, T)}^2 + \frac{C}{k_0}\|cg\|_{L^2(\Omega)}^2\|q_t\|_{L^2(0, T)}^2 + \frac{C}{k_0}\|\partial_y(k(x, y)g_y)\|_{L^2(\Omega)}^2\|q\|_{L^2(0, T)}^2. \end{aligned} \quad (22)$$

**Estimate in**  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ .

We also have, multiplying (21)-1 by  $\partial_t \bar{\theta}$  and integrating over  $\Omega \times (0, t)$  :

$$\begin{aligned} & \int_0^t \int_\Omega c |\partial_t \bar{\theta}|^2 + \frac{1}{2} \int_\Omega k |\nabla \bar{\theta}|^2(t) + \int_0^t v \int_{e_1}^{L+e_2} \partial_x \bar{\theta} \partial_t \bar{\theta} - \int_0^t \int_{\Sigma_b} f_1 \partial_t \bar{\theta} + \frac{1}{2} \int_0^t \int_{\Sigma_b} f_2 \partial_t |\bar{\theta}|^2 \\ & + \int_0^t \int_{\Sigma_b} \sigma \varepsilon |\bar{\theta}|^3 \partial_t \bar{\theta} = - \int_0^t \int_\Omega c(x, y) g(y) q_t \partial_t \bar{\theta} + \int_0^t \int_\Omega q \partial_y (k(x, y) g_y) \partial_t \bar{\theta} + \frac{1}{2} \int_\Omega k |\nabla \bar{\theta}|^2(0). \end{aligned}$$

But, a.e in  $t \in (0, T)$

$$\begin{aligned} & \int_0^t \int_{\Sigma_b} f_2(\tau) \partial_t |\bar{\theta}|^2 = \int_{\Sigma_b} f_2(t) |\bar{\theta}|^2(t) - \int_{\Sigma_b} f_2(0) |\bar{\theta}_0|^2 - \int_0^t \int_{\Sigma_b} (f_2)_t(\tau) |\bar{\theta}|^2, \\ & \int_{\Sigma_b} f_2(0) |\bar{\theta}_0|^2 \leq \|f_2\|_{L^\infty(0, T)} \int_{\Sigma_b} |\bar{\theta}_0|^2 \leq \frac{\alpha}{k_0} \|f_2\|_{L^\infty(0, T)} \int_{\Sigma_b} k |\nabla \bar{\theta}_0|^2 \end{aligned}$$

and

$$\begin{aligned} \left| \int_0^t \int_{\Sigma_b} (f_2)_t(\tau) |\bar{\theta}|^2 \right| & \leq \|(f_2)_t\|_{L^\infty(0, T)} \int_0^t \int_{\Sigma_b} |\bar{\theta}|^2 \leq \alpha \|(f_2)_t\|_{L^\infty(0, T)} \int_0^t \int_\Omega |\nabla \bar{\theta}|^2 \\ & \leq \frac{\alpha}{k_0} \|(f_2)_t\|_{L^\infty(0, T)} \int_0^t \int_\Omega k |\nabla \bar{\theta}|^2. \end{aligned}$$

We write

$$\int_0^t \int_{\Sigma_b} f_1(\tau) \partial_t \bar{\theta} = \int_{\Sigma_b} f_1(t) \bar{\theta} - \int_{\Sigma_b} f_1(0) \bar{\theta}_0 - \int_0^t \int_{\Sigma_b} (f_1)_t(\tau) \bar{\theta}$$

and

$$\begin{aligned} \left| \int_{\Sigma_b} f_1(0) \bar{\theta}_0 \right| & \leq L^{1/2} \|f_1\|_{L^\infty(0, T)} \|\bar{\theta}_0\|_{L^2(\Sigma_b)} \leq \frac{\alpha L}{4k_0} \|f_1\|_{L^\infty(0, T)}^2 + \int_\Omega k |\nabla \bar{\theta}_0|^2, \\ \left| \int_{\Sigma_b} f_1(t) \bar{\theta} \right| & \leq \|f_1\|_{L^\infty(0, T)} \int_{\Sigma_b} |\bar{\theta}| \leq \sqrt{\alpha} L^{1/2} \|f_1\|_{L^\infty(0, T)} \left( \int_\Omega |\nabla \bar{\theta}|^2 \right)^{1/2} \\ & \leq \frac{\alpha L}{k_0} \|f_1\|_{L^\infty(0, T)}^2 + \frac{1}{4} \int_\Omega k |\nabla \bar{\theta}|^2. \end{aligned}$$

Similarly,

$$\left| \int_0^t \int_{\Sigma_b} (f_1)_t(\tau) \bar{\theta} \right| \leq L \frac{\alpha}{4k_0} \|(f_1)_t\|_{L^2(0, T)}^2 + \int_0^t \int_\Omega k |\nabla \bar{\theta}|^2.$$

We also have

$$\begin{aligned}\int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) |\bar{\theta}|^3 \bar{\theta} \partial_t \bar{\theta} &= \frac{1}{5} \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) \partial_t |\bar{\theta}|^5 \\ &= \frac{1}{5} \int_{\Sigma_b} \sigma \varepsilon(t) |\bar{\theta}|^5(t) - \frac{1}{5} \int_{\Sigma_b} \sigma \varepsilon(0) |\bar{\theta}_0|^5 - \frac{1}{5} \int_0^t \int_{\Sigma_b} \sigma \varepsilon_t(\tau) |\bar{\theta}|^5\end{aligned}$$

and

$$\left| \int_0^t \int_{\Sigma_b} \sigma \varepsilon_t(\tau) |\bar{\theta}|^5 \right| \leq \frac{1}{\varepsilon_0} \|\varepsilon_t\|_{L^\infty(0,T)} \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) |\bar{\theta}|^5.$$

Eventually, we get

$$\begin{aligned}\left| \int_0^t v \int_{e_1}^{e_1+e_2} \partial_x \bar{\theta} \partial_t \bar{\theta} \right| &\leq \frac{K}{L} \frac{\|v\|_{L^\infty(0,T)}}{k_0 c_0} \int_0^t \int_{\Omega} k |\nabla \bar{\theta}|^2 + \frac{1}{4} \int_0^t \int_{\Omega} c |\partial_t \bar{\theta}|^2, \\ \left| \int_0^t \int_{\Omega} c(x,y) g(y) q_t \partial_t \bar{\theta} \right| &\leq \|\sqrt{c}g\|_{L^2(\Omega)} \|q_t\|_{L^2(0,T)} \|\sqrt{c} \partial_t \bar{\theta}\|_{L^2((0,t) \times \Omega)} \\ &\leq \|\sqrt{c}g\|_{L^2(\Omega)}^2 \|q_t\|_{L^2(0,T)}^2 + \frac{1}{4} \int_0^t \int_{\Omega} c |\partial_t \bar{\theta}|^2\end{aligned}$$

and

$$\begin{aligned}\left| \int_0^t \int_{\Omega} q \partial_y(k(x,y)g_y) \partial_t \bar{\theta} \right| &\leq \|\partial_y(k(x,y)g_y)\|_{L^2(\Omega)} \|q\|_{L^2(0,T)} \|\partial_t \bar{\theta}\|_{L^2((0,t) \times \Omega)} \\ &\leq \frac{1}{c_0} \|\partial_y(k(x,y)g_y)\|_{L^2(\Omega)}^2 \|q\|_{L^2(0,T)}^2 + \frac{1}{4} \int_0^t \int_{\Omega} c |\partial_t \bar{\theta}|^2.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{4} \int_0^t \int_{\Omega} c |\partial_t \bar{\theta}|^2 + \frac{1}{4} \int_{\Omega} k |\nabla \bar{\theta}|^2(t) + \frac{1}{2} \int_{\Sigma_b} f_2(t) |\bar{\theta}|^2 + \frac{1}{5} \int_{\Sigma_b} \sigma \varepsilon(t) |\bar{\theta}|^5 \\ &\leq \left( \frac{K}{L} \frac{\|v\|_{L^\infty(0,T)}}{k_0 c_0} + \frac{\alpha}{2k_0} \|(f_2)_t\|_{L^\infty(0,T)} + 1 \right) \int_0^t \int_{\Omega} k |\nabla \bar{\theta}|^2 + \frac{1}{5\varepsilon_0} \|\varepsilon_t\|_{L^\infty(0,T)} \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) |\bar{\theta}|^5 \\ &\quad + \frac{5\alpha L}{4k_0} \|f_1\|_{L^\infty(0,T)}^2 + \frac{\alpha L}{4k_0} \|(f_1)_t\|_{L^2(0,T)}^2 + \left( \frac{\alpha}{2k_0} \|f_2\|_{L^\infty(0,T)} + \frac{3}{2} \right) \int_{\Omega} k |\nabla \bar{\theta}_0|^2 \\ &\quad + \frac{1}{5} \sigma \varepsilon(0) \int_{\Sigma_b} |\bar{\theta}_0|^5 + \|\sqrt{c}g\|_{L^2(\Omega)}^2 \|q_t\|_{L^2(0,T)}^2 + \frac{1}{c_0} \|\partial_y(k(x,y)g_y)\|_{L^2(\Omega)}^2 \|q\|_{L^2(0,T)}^2\end{aligned}\tag{23}$$

leading for all  $t > 0$ , since  $H^1(0, T) \hookrightarrow L^\infty(0, T)$  and  $\bar{\theta} = \theta - g(y)q(t)$  to

$$\begin{aligned} \int_0^t \int_\Omega c |\partial_t \theta|^2 + \int_\Omega k |\nabla \theta|^2(t) + \int_{\Sigma_b} f_2(t) |\theta|^2 + \int_{\Sigma_b} \sigma \varepsilon(t) |\theta|^5 \\ \leq C \left( \|\sqrt{k} \nabla \bar{\theta}_0\|_{L^2(\Omega)}^2 + \int_{\Sigma_b} |\bar{\theta}_0|^5 + \|f_1\|_{H^1(0, T)}^2 + \|q\|_{H^1(0, T)}^2 \right) \end{aligned} \quad (24)$$

for some constant  $C > 0$  which depends on the norm of the data  $f_2, \varepsilon, v$ , etc.

**Uniqueness of  $\theta \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$  solution of (12).**

Let  $\theta_1, \theta_2 \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$  be two solutions of (12). Then  $\theta = \theta_1 - \theta_2 \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$  is solution of

$$\begin{cases} c(x, y) \partial_t \theta - \operatorname{div}(k(x, y) \nabla \theta) + v(t) 1_{(e_1, e_1 + e_2)}(y) \partial_x \theta = 0, & \Omega \times (0, T), \\ k \nabla \theta \cdot \nu = 0, & \partial \Omega \setminus (\Sigma_b \cup \Sigma_c) \times (0, T), \\ \theta = 0, & \Sigma_c \times (0, T), \\ k(x, y) \partial_y \theta = \sigma \varepsilon(t) (|\theta_1|^3 \theta_1 - |\theta_2|^3 \theta_2) + f_2(t) \theta, & \Sigma_b \times (0, T), \\ \theta = 0, & \Omega \times \{0\}. \end{cases} \quad (25)$$

Since  $\operatorname{div}(k(x, y) \nabla \theta) = c(x, y) \partial_t \theta + K \frac{H_1(t)}{L} 1_{(e_1, e_1 + e_2)}(y) \theta_x \in L^2((0, T) \times \Omega)$  we have for all  $t \in (0, T)$ , multiplying (25-1) by  $\theta$  and integrating over  $\Omega \times (0, t)$ :

$$\begin{aligned} \frac{1}{2} \|\sqrt{c} \theta(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega k |\nabla \theta|^2 + \int_0^t \int_{\Sigma_a} v(\tau) |\theta|^2 + \int_0^t \int_{\Sigma_b} f_2(\tau) |\theta|^2 \\ + \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) (|\theta_1|^3 \theta_1 - |\theta_2|^3 \theta_2) (\theta_1 - \theta_2) = 0. \end{aligned}$$

But for all  $(a, b) \in \mathbb{R}^2$ ,  $(|a|^3 a - |b|^3 b)(a - b) \geq 0$  and thus, since  $f_2 \geq 0$ ,  $\varepsilon \geq 0$ ,  $c > 0$  and  $k > 0$ ,  $\theta = 0$  that is  $\theta_1 = \theta_2$ .  $\square$

## 2.2 Maximum principle

In the following, we denote

$$H_+^1(0, T) := \{q \in H^1(0, T), q(0) = q_0, q(t) \geq 0 \quad \forall t \in (0, T)\}. \quad (26)$$

**Theorem 2.** For any  $q \in H_+^1(0, T)$ , let  $\theta$  be the solution of (12) given in Theorem 1. Then  $\theta(\cdot, t) \geq 0$  for all  $t \in (0, T)$  a.e. in  $\Omega$ . More precisely, we have :

- Let any  $l > 0$  large enough such that  $q_0 \leq l, \theta_s \leq l$  and  $\sigma \varepsilon(t) l^4 + f_2(t) l - f_1(t) \geq 0$  for all  $t \in [0, T]$ . If  $q(t) \leq l$  in  $(0, T)$ , then  $\theta(\cdot, t) \leq l$  for all  $t \in [0, T]$  a.e. in  $\Omega$ .
- Let any  $l \geq 0$  small enough such that  $q_0 \geq l, \theta_s \geq l$  and  $\sigma \varepsilon(t) l^4 + f_2(t) l - f_1(t) \leq 0$  for all  $t \in [0, T]$ . If  $q(t) \geq l$  in  $(0, T)$ , then  $\theta(\cdot, t) \geq l$  for all  $t \in [0, T]$  a.e. in  $\Omega$ .



*Proof.* Let  $\theta^-$  be the negative part of  $\theta$ . Then  $\theta^- \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$  and since  $\theta = q \geq 0$  on  $\Sigma_c$ , we get  $\theta^- = 0$  on  $\Sigma_c$ . Multiplying (12) by  $\theta^-$ , we get

$$\int_{\Omega} c \partial_t \theta \theta^- + \int_{\Omega} k \nabla \theta \cdot \nabla \theta^- + \int_{e_1}^{e_1+e_2} v(t) \partial_x \theta \theta^- + \int_{\Sigma_b} (\sigma \varepsilon(t) |\theta|^3 \theta - f_1(t) + f_2(t) \theta) \theta^- dx = 0$$

from which we get, a.e. in  $(0, T)$ ,

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{c} \theta^-\|_{L^2(\Omega)}^2 + \int_{\Omega} k |\nabla \theta^-|^2 + \int_{\Sigma_d} v(t) (\theta^-)^2 + \int_{\Sigma_b} (\sigma \varepsilon(t) |\theta^3| \theta + f_2(t) \theta) \theta^- = \int_{\Sigma_b} f_1(t) \theta^-$$

and then, for all  $t \in (0, T)$

$$\begin{aligned} & \frac{1}{2} \|\sqrt{c} \theta^-(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \int_{\Omega} k |\nabla \theta^-|^2 + \int_0^t \int_{\Sigma_d} v(\theta^-)^2 \\ & + \int_0^t \int_{\Sigma_b} (\sigma \varepsilon(\tau) |\theta^3| (\theta^-)^2 + f_2(\tau) (\theta^-)^2) = \int_0^t \int_{\Sigma_b} f_1(\tau) \theta^- + \frac{1}{2} \|\sqrt{c} \theta^-(\cdot, 0)\|_{L^2(\Omega)}^2. \end{aligned}$$

Using that  $f_1 \geq 0, f_2 \geq 0, v \geq 0$  in  $(0, T)$ ,  $c > 0, k > 0$  and according to the first part of Lemma 1 that  $\theta_0 \geq 0$ , it follows that the negative part of the solution  $\theta$  vanishes a.e. in  $\Omega$  for all  $t \in (0, T)$ . The second statement is obtained in a similar way (we refer to the proof of Lemma 1).  $\square$

**Remark 1.** The last item in Theorem 2 shows that the system has a minimal positive temperature as soon as the control  $q$  is large enough.

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  the real function be defined by  $f(r) = ar^4 + br - c$  with  $a, b, c > 0$ . It is easily seen that the unique root of  $f$  satisfies  $\min\left(\left(\frac{c}{2a}\right)^{1/4}, \frac{c}{2b}\right) \leq R < \frac{c}{b}$ . Recalling that the function  $\varepsilon, f_1$  and  $f_2$  are uniformly positives and continuous in  $[0, T]$ , we consider

$$g(t) := \min\left(\left(\frac{f_1(t)}{2\sigma\varepsilon(t)}\right)^{1/4}, \frac{f_1(t)}{2f_2(t)}\right), \quad t \in [0, T]$$

and define  $G := \min_{t \in [0, T]} g(t) > 0$ . The second item of Theorem 2 implies that if  $\theta_s \geq g(0)$  and  $q(t) \geq g(t)$  in  $[0, T]$  then  $\theta(\cdot, t) \geq g(t)$  for all  $t \in [0, T]$  a.e. in  $\Omega$ . This leads to a minimal temperature depending on the time variable and also to a simple command law : we refer to [1, section 5.5] for some related discussions in a 1D situation.

Eventually, we emphasize that the previous analysis does not prove that an arbitrarily large temperature can be achieved as soon as the injected temperature is arbitrarily large: this property, physically intuitive, has been observed in our numerical experiments.

**Theorem 3.** The map  $\mathcal{T} : q \mapsto \theta$  is Lipschitz continuous from  $H_+^1(0, T)$  onto  $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ . More precisely, let  $\theta_i$  be the solution of (12) associated with the data  $(\theta_0, q_i)$  given in Theorem 1 and let  $\theta = \theta_1 - \theta_2, q = q_1 - q_2$  and

$\bar{\theta} = \theta - g(y)q(t)$  where  $g : [0, h_e] \rightarrow [0, 1]$  is a smooth cut-off function with support in  $[e_1 - \eta, e_1 + e_2 + \eta]$  for some  $\eta > 0$  small enough,  $g = 1$  on  $(e_1, e_1 + e_2)$ . Then, for all  $t \geq 0$  and  $p = 0$  or  $p \geq 2$ :

$$\begin{aligned} \|\bar{\theta}\|_{L^\infty(0,T;L^{p+2}(\Omega))}^{p+2} &+ \int_0^t \int_\Omega k |\nabla \bar{\theta}|^2 |\bar{\theta}|^p + \frac{p}{4} \int_0^t \int_\Omega k |\bar{\theta}|^{p-2} |\nabla |\bar{\theta}|^2|^2 + \int_0^t \int_{\Sigma_d} v(\tau) |\bar{\theta}|^{p+2} \\ &+ \int_0^t \int_{\Sigma_b} f_2(\tau) |\bar{\theta}|^{p+2} + \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) |\bar{\theta}|^{p+5} \\ &\leq C (\|v\|_{L^\infty(0,T)} + \|cg\|_{L^{p+2}(\Omega)}^{p+2} + \|\partial_y(k(x,y)g_y)\|_{L^{p+2}(\Omega)}^{p+2}) \|q\|_{H^1(0,T)}^{p+2} \end{aligned} \quad (27)$$

with the convention that  $p \int_0^t \int_\Omega k |\bar{\theta}|^{p-2} |\nabla |\bar{\theta}|^2|^2 = 0$  if  $p = 0$ .

Moreover, if  $q_n \in H_+^1(0, T) \rightarrow q$  in  $H^1(0, T)$  as  $n \rightarrow \infty$  then  $\mathcal{T}(q_n) = \theta_n \xrightarrow{*} \mathcal{T}(q) = \theta$  in  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ .

*Proof.* We have that  $\theta = \theta_1 - \theta_2 \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$  is solution of

$$\begin{cases} c(x, y) \partial_t \theta - \operatorname{div}(k(x, y) \nabla \theta) + v(t) 1_{(e_1, e_1 + e_2)}(y) \partial_x \theta = 0, & \Omega \times (0, T), \\ k \nabla \theta \cdot \nu = 0, & (\partial\Omega \setminus (\Sigma_b \cup \Sigma_c)) \times (0, T), \\ \theta = q, & \Sigma_c \times (0, T), \\ k(x, y) \partial_y \theta = \sigma \varepsilon(t) (\theta_1^4 - \theta_2^4) + f_2(t) \theta, & \Sigma_b \times (0, T), \\ \theta = 0, & \Omega \times \{0\}. \end{cases} \quad (28)$$

As in the proof of Theorem 1,  $\bar{\theta} := \theta - g(y)q(t)$  solves

$$\begin{cases} c(x, y) \partial_t \bar{\theta} - \operatorname{div}(k(x, y) \nabla \bar{\theta}) + v(t) 1_{(e_1, e_1 + e_2)}(y) \partial_x \bar{\theta} \\ \quad = -c(x, y) g(y) q_t + q \partial_y (k(x, y) g_y), & \Omega \times (0, T), \\ k \nabla \bar{\theta} \cdot \nu = 0, & (\partial\Omega \setminus (\Sigma_b \cup \Sigma_c)) \times (0, T), \\ \bar{\theta} = 0, & \Sigma_c \times (0, T), \\ k(x, y) \partial_y \bar{\theta} = \sigma \varepsilon(t) (\theta_1^4 - \theta_2^4) + f_2(t) \bar{\theta}, & \Sigma_b \times (0, T), \\ \bar{\theta}(0) = 0, & \Omega \times \{0\}. \end{cases} \quad (29)$$

Since  $\operatorname{div}(k(x, y)\nabla\bar{\theta}) \in L^2((0, T) \times \Omega)$ , multiplying (29-1) by  $\bar{\theta}|\bar{\theta}|^p$  ( $p = 0$  or  $p \geq 2$ ) we get for all  $t \in (0, T)$ , integrating over  $\Omega \times (0, t)$ , that

$$\begin{aligned} & \frac{1}{p+2} \int_{\Omega} |\bar{\theta}|^{p+2}(t) + \int_0^t \int_{\Omega} k \nabla \bar{\theta} \cdot \nabla (\bar{\theta} |\bar{\theta}|^p) + \frac{1}{p+2} \int_0^t \int_{\Sigma_d} v(\tau) L |\bar{\theta}|^{p+2} + \int_0^t \int_{\Sigma_b} f_2(\tau) |\bar{\theta}|^{p+2} \\ & + \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) (\theta_1^4 - \theta_2^4) \bar{\theta} |\bar{\theta}|^p \\ & = \frac{1}{p+2} \int_0^t \int_{e_1}^{e_1+e_2} v(\tau) |q|^{p+2} + \int_0^t \int_{\Omega} (-c(x, y)g(y)q_t + q\partial_y(k(x, y)g_y)) \bar{\theta} |\bar{\theta}|^p. \end{aligned} \quad (30)$$

But, if  $p \geq 2$  :

$$\int_0^t \int_{\Omega} k \nabla \bar{\theta} \cdot \nabla (\bar{\theta} |\bar{\theta}|^p) = \int_0^t \int_{\Omega} k |\nabla \bar{\theta}|^2 |\bar{\theta}|^p + \frac{p}{4} \int_0^t \int_{\Omega} k |\bar{\theta}|^{p-2} |\nabla |\bar{\theta}|^2|^2.$$

Since  $\theta = \bar{\theta}$  on  $\Sigma_b$ , the last term of the left-hand side is on  $\Sigma_b$  :

$$(\theta_1^4 - \theta_2^4) \bar{\theta} = (\theta_1^4 - \theta_2^4)(\theta_1 - \theta_2) = (\theta_1 - \theta_2)^2 (\theta_1^2 + \theta_2^2)(\theta_1 + \theta_2)$$

and since  $\theta_1 \geq 0$  and  $\theta_2 \geq 0$ , we infer that

$$\theta_1 + \theta_2 \geq |\theta_1 - \theta_2|, \quad \theta_1^2 + \theta_2^2 \geq |\theta_1^2 - \theta_2^2| = |\theta_1 - \theta_2|(\theta_1 + \theta_2) \geq |\theta_1 - \theta_2|^2.$$

Consequently,  $(\theta_1^4 - \theta_2^4) \bar{\theta} |\bar{\theta}|^p \geq |\theta_1 - \theta_2|^{p+5} = |\theta|^{p+5}$  and thus, since  $\theta = \bar{\theta}$  on  $\Sigma_b$ ,

$$\int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) (\theta_1^4 - \theta_2^4) \bar{\theta} |\bar{\theta}|^p \geq \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) |\bar{\theta}|^{p+5}.$$

On the other hand,

$$0 \leq \int_0^t \int_{e_1}^{e_1+e_2} v(\tau) |q|^{p+2} \leq e_2 \|v\|_{L^\infty(0, T)} \|q\|_{L^{p+2}(0, T)}^{p+2} \leq C \|v\|_{L^\infty(0, T)} \|q\|_{H^1(0, T)}^{p+2}$$

and

$$\begin{aligned} & \left| \int_0^t \int_{\Omega} (-c(x, y)g(y)q_t + q\partial_y(k(x, y)g_y)) \bar{\theta} |\bar{\theta}|^p \right| \\ & \leq \|\bar{\theta}\|_{L^\infty((0, T); L^{p+2}(\Omega))}^{p+1} \int_0^t (|c| \|g\|_{L^{p+2}(\Omega)} + |q| \|\partial_y(k(x, y)g_y)\|_{L^{p+2}(\Omega)}) \\ & \leq t^{1/2} \|\bar{\theta}\|_{L^\infty((0, T); L^{p+2}(\Omega))}^{p+1} (\|cg\|_{L^{p+2}(\Omega)} + \|\partial_y(k(x, y)g_y)\|_{L^{p+2}(\Omega)}) \|q\|_{H^1(0, T)} \\ & \leq \epsilon \|\bar{\theta}\|_{L^\infty((0, T); L^{p+2}(\Omega))}^{p+2} + C(\epsilon) (\|cg\|_{L^{p+2}(\Omega)}^{p+2} + \|\partial_y(k(x, y)g_y)\|_{L^{p+2}(\Omega)}^{p+2}) \|q\|_{H^1(0, T)}^{p+2}. \end{aligned}$$

Thus, for any  $\epsilon$  small enough, estimate (30) leads to (27).

For  $p = 0$ , remembering that  $\bar{\theta} := \theta - g(y)q(t)$ , we then obtain :

$$\begin{aligned} & \|\sqrt{c}\theta\|_{L^\infty((0,T);L^2(\Omega))} + \|\sqrt{k}\nabla\theta\|_{L^2((0,T);L^2(\Omega)^2)} + \|\sqrt{v}\theta\|_{L^2((0,T);L^2(\Sigma_d))} + \|\sqrt{f_2}\theta\|_{L^2((0,T);L^2(\Sigma_d))} \\ & + \|(\sigma\varepsilon)^{\frac{1}{5}}\theta\|_{L^5((0,T);L^5(\Sigma_b))} \leq C\|q\|_{H^1(0,T)} \end{aligned} \quad (31)$$

for some constant  $C > 0$  which depends on the norm of the data  $f_2, \varepsilon, v$ , etc, that is  $\mathcal{T}$  is Lipschitz continuous from  $H_+^1(0, T)$  onto  $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ .

Let now  $(q_n)_{n \in \mathbb{N}}$  be a sequence of  $H_+^1(0, T)$  and  $q \in H_+^1(0, T)$  such that  $q_n \rightarrow q$  in  $H^1(0, T)$  and  $\theta_n$  and  $\theta$ ,  $n \in \mathbb{N}$  being the solutions of (12) associated respectively to  $q_n$  and  $q$ , given in Theorem 1. From (22) and (24) we deduce, since  $(q_n)_{n \in \mathbb{N}}$  is bounded in  $H^1(0, T)$  that  $(\theta_n)_{n \in \mathbb{N}}$  is bounded in  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ . Thus there exists a subsequence  $(\theta_n)_{n \in \mathbb{N}}$  and  $\tilde{\theta}$  such that  $\theta_n \overset{*}{\rightharpoonup} \tilde{\theta}$  in  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ . Since  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \hookrightarrow L^5((0, T) \times \Sigma)$  is compact,  $\tilde{\theta}$  is clearly solution of (12) associated with  $q$ , by uniqueness,  $\tilde{\theta} = \theta$  and all the sequence  $(\theta_n)_{n \in \mathbb{N}}$  weakly\* converges to  $\theta$  in  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ .  $\square$

### 3 Analysis of the optimal control problem (10)

#### 3.1 Well-posedness

Recalling that  $\underline{\theta}$  is the minimal prescribed temperature on  $\Sigma_b$ , we denote by

$$\mathcal{C} = \{q \in H_+^1(0, T) \mid \mathcal{T}(q) = \theta \geq \underline{\theta} \text{ a.e. on } \Sigma_b \times (0, T)\}.$$

As a direct consequence of Theorem 2, we have

**Lemma 2.** *If  $\underline{\theta}$  satisfies  $q_0 \geq \underline{\theta} \geq 0$  a.e. in  $\Omega$  and  $f_1(t) - f_2(t)\underline{\theta} - \sigma\varepsilon(t)\underline{\theta}^4(t) \geq 0$  for all  $t \in (0, T)$ , then the set  $\mathcal{C}$  is non empty.*

In the sequel, we will assume that  $\underline{\theta} \geq 0$  is small enough. In particular, in view of Remark 1, if  $\theta_s \geq G$ , then  $\mathcal{C}$  is not empty for any  $\underline{\theta} \leq G$ , achieved with a control  $q$  such that  $q(t) \geq G$  in  $[0, T]$ .

Moreover, the solution  $\theta$  enjoys a concavity property with respect to the control  $q$ .

**Lemma 3.** *Assume  $q, \bar{q} \in H_+^1(0, T)$ . For any  $\lambda \in (0, 1)$ , let  $\theta_\lambda$  be the solution of (12) associated with the control function  $\lambda q + (1 - \lambda)\bar{q}$  and let  $[\theta] := \lambda\theta_q + (1 - \lambda)\theta_{\bar{q}}$  where  $\theta_q$  solves (12) associated with the control  $q$ . Then  $[\theta] \leq \theta_\lambda$  on  $\Omega \times (0, T)$ .*

*Proof.* Theorem 2 implies that for all  $t \in [0, T]$ ,  $\theta_q(t) \geq 0$ ,  $\theta_{\bar{q}}(t) \geq 0$  and  $\theta_\lambda(t) \geq 0$ . From (12),  $W := \theta_\lambda - [\theta]$  solves, a.e.  $t \in (0, T)$

$$\int_{\Omega} cW_t(t)\phi + \int_{\Omega} k\nabla W(t) \cdot \nabla\phi - \int_{\Sigma} k\nabla W(t) \cdot \nu\phi + v(t) \int_{e_1}^{e_1+e_2} \partial_x W\phi = 0, \quad \forall \phi \in V.$$

Taking  $\phi = W^-(t)$ , integrating over  $(0, t)$  and using that  $W(0) = 0$ , we get

$$\begin{aligned} \frac{1}{2} \int_{\Omega} c|W^-(t)|^2 + \int_0^t \int_{\Omega} k|\nabla W^-|^2 - \int_0^t \int_{\Sigma} k\nabla W \cdot \nu W^- \\ + \int_0^t v(\tau) \int_{\Omega} 1_{(e_1, e_1+e_2)}(y) \partial_x W W^- = 0, \quad a.e. t \in (0, T). \end{aligned} \quad (32)$$

But since  $H_1 \geq 0$  and  $W = 0$  on  $\Sigma_c$ , we can write that

$$\begin{aligned} \int_0^t v(\tau) L \int_{\Omega} 1_{(e_1, e_1+e_2)}(y) \partial_x W W^- &= \frac{1}{2} \int_0^t v(\tau) \int_{\Omega} 1_{(e_1, e_1+e_2)}(y) \partial_x |W^-|^2 \\ &= \frac{1}{2} \int_0^t \int_{\Sigma_d} v(\tau) |W^-|^2(\cdot, y, \tau) dy d\tau \geq 0. \end{aligned}$$

On the other hand, since  $\theta_{\lambda} \geq 0$ ,  $\theta_q \geq 0$  and  $\theta_{\bar{q}} \geq 0$ ,

$$\begin{aligned} - \int_0^t \int_{\Sigma} k\nabla W \cdot \nu W^- &= - \int_0^t \int_{\Sigma_b} k\nabla W \cdot \nu W^- \\ &= \int_0^t \int_{\Sigma_b} \sigma\varepsilon(\tau) (\theta_{\lambda}^4 - \lambda\theta_q^4 - (1-\lambda)\theta_{\bar{q}}^4) W^- + \int_0^t \int_{\Sigma_b} f_2(\tau) |W^-|^2. \end{aligned}$$

The convexity of  $x \rightarrow x^4$  leads to  $[\theta]^4(x, 0, s) \leq \lambda\theta_q^4(x, 0, s) + (1-\lambda)\theta_{\bar{q}}^4(x, 0, s)$ , then to

$$\theta_{\lambda}^4(x, 0, s) - [\theta]^4(x, 0, s) \geq \theta_{\lambda}^4(x, 0, s) - \lambda\theta_q^4(x, 0, s) - (1-\lambda)\theta_{\bar{q}}^4(x, 0, s).$$

Thus, since  $W^- \leq 0$ , we get

$$\int_0^t \int_{\Sigma_b} \sigma\varepsilon(\tau) (\theta_{\lambda}^4 - \lambda\theta_q^4 - (1-\lambda)\theta_{\bar{q}}^4) W^- \geq \int_0^t \int_{\Sigma_b} \sigma\varepsilon(\tau) (\theta_{\lambda}^4 - [\theta]^4) W^-.$$

But, the right-hand side is exactly  $(\alpha^4 - \beta^4)(\alpha - \beta)^- = ((\alpha - \beta)^-)^2 (\alpha^2 + \beta^2)(\alpha + \beta)$  with  $\alpha = \theta_{\lambda}(x, 0, s) \geq 0$  and  $\beta = [\theta] = \lambda\theta_q(x, 0, s) + (1-\lambda)\theta_{\bar{q}}(x, 0, s) \geq 0$  leading to  $\int_0^t \int_{\Sigma_b} \sigma\varepsilon(\tau) (\theta_{\lambda}^4 - \lambda\theta_q^4 - (1-\lambda)\theta_{\bar{q}}^4) W^- \geq 0$ . Since  $f_2 \geq 0$ , we also have  $\int_0^t \int_{\Sigma_b} f_2(\tau) |W^-|^2 \geq 0$ . Consequently, (32) gives a.e.  $t \in (0, T)$  :

$$\frac{1}{2} \int_{\Omega} c|W^-(t)|^2 + \int_0^t \int_{\Omega} k|\nabla W^-|^2 + \int_0^t \int_{\Sigma_b} f_2(\tau) |W^-|^2 + \frac{1}{2} \int_0^t \int_{\Sigma_d} v(\tau) |W^-|^2(\cdot, y, \tau) dy d\tau \leq 0$$

that is, since  $c > 0$ ,  $k > 0$  and  $v \geq 0$ ,  $W^- = 0$  on  $\Omega \times (0, T)$ . Thus  $\theta_{\lambda} \geq [\theta]$  on  $\Omega \times (0, T)$ .  $\square$

**Lemma 4.** *Assume the hypotheses of Lemma 2.  $\mathcal{C}$  is a closed convex set of  $H^1(0, T)$ .*

*Proof.* Let  $q, \bar{q} \in \mathcal{C}$  and check that for any  $\lambda \in (0, 1)$ ,  $\lambda q + (1 - \lambda)\bar{q}$  belongs to  $\mathcal{C}$ .  $q \in \mathcal{C}$  implies that for all  $t \in [0, T]$ ,  $\theta_q(t) \geq \underline{\theta} \geq 0$  where  $\theta_q$  solves (12) associated with the source  $q$ . Similarly,  $\theta_{\bar{q}}(t) \geq \underline{\theta} \geq 0$ . Let  $\theta_\lambda$  be the solution associated with the control function  $\lambda q + (1 - \lambda)\bar{q}$  and let  $[\theta] := \lambda\theta_q + (1 - \lambda)\theta_{\bar{q}}$ . From Lemma 3 we have  $\theta_\lambda \geq [\theta]$  on  $\Omega \times (0, T)$  and the convexity of  $\mathcal{C}$  follows.  $\square$

**Lemma 5.** *Assume the hypotheses of Lemma 2. For any  $\alpha > 0$ , the functional  $J_\alpha$  defined in (10) is strictly convex on  $H_+^1(0, T)$ .*

*Proof.* We decompose  $J_\alpha$  as follows  $J_\alpha(q) = J_1(q) + J_{\alpha,2}(q)$  with

$$J_1(q) = \frac{1}{2} \left( C_f \int_0^T v(t) \int_{e_1}^{e_1+e_2} (q(t) - \theta(L, y, t))^+ dy dt \right)^2, \quad J_{\alpha,2}(q) = \frac{1}{2} \alpha \int_0^T q_t^2 dt.$$

$J_{\alpha,2}$  is strictly convex. Let us prove that  $J_1$  is convex.

For any  $q, \bar{q} \in H_+^1(0, T)$  and  $\lambda \in (0, 1)$ , let  $\theta_\lambda$  be the solution associated to the control function  $\lambda q + (1 - \lambda)\bar{q}$  and let  $[\theta] := \lambda\theta_q + (1 - \lambda)\theta_{\bar{q}}$ . We have previously proved that  $\theta_\lambda \geq [\theta]$  on  $\{L\} \times (e_1, e_1 + e_2) \times (0, T)$  and thus

$$\lambda q + (1 - \lambda)\bar{q} - \theta_\lambda \leq \lambda q + (1 - \lambda)\bar{q} - [\theta] \leq \lambda(q - \theta_q) + (1 - \lambda)(\bar{q} - \theta_{\bar{q}})$$

which gives

$$\begin{aligned} (\lambda q + (1 - \lambda)\bar{q} - \theta_\lambda)^+ &\leq (\lambda(q - \theta_q) + (1 - \lambda)(\bar{q} - \theta_{\bar{q}}))^+ \\ &\leq \lambda(q - \theta_q)^+ + (1 - \lambda)(\bar{q} - \theta_{\bar{q}})^+. \end{aligned}$$

Thus, since for all  $t \in [0, T]$ ,  $v(t) > 0$ :

$$\begin{aligned} 0 &\leq \int_0^T v \int_{e_1}^{e_1+e_2} (\lambda q + (1 - \lambda)\bar{q} - \theta_\lambda(L, y, t))^+ dy dt \\ &\leq \lambda \int_0^T v \int_{e_1}^{e_1+e_2} (q(t) - \theta_q(L, y, t))^+ dy dt + (1 - \lambda) \int_0^T v \int_{e_1}^{e_1+e_2} (\bar{q}(t) - \theta_{\bar{q}}(L, y, t))^+ dy dt \end{aligned}$$

which implies that  $J_1$  is convex since  $x \mapsto x^2$  is convex.  $\square$

**Lemma 6.** *Assume the hypotheses of Lemma 2.  $J_\alpha$  is continuous and weakly lower semi-continuous on  $H_+^1(0, T)$ .*

*Proof.*  $J_\alpha$  is continuous on  $H_+^1(0, T)$ . Let  $(q_n)_{n \in \mathbb{N}}$  a sequence of  $H_+^1(0, T)$  and  $q \in H_+^1(0, T)$  such that  $q_n \rightarrow q$  in  $H^1(0, T)$  and  $\theta_n$  and  $\theta$ ,  $n \in \mathbb{N}$  being the solution of (12) associated respectively to  $q_n$  and  $q$ , given in Theorem 1. Then  $J_2(q_n) \rightarrow J_2(q)$ .

Moreover from Theorem 3,  $\theta_n \rightarrow \theta$  in  $L^2(0, T; H^1(\Omega))$  and thus  $\theta_n \rightarrow \theta$  in  $L^2(0, T; L^2(\Sigma))$ . Since for all  $a \in \mathbb{R}$ ,  $a^+ = \frac{1}{2}(|a| + a)$ , for all  $q \in \mathcal{C}$ :

$$\begin{aligned}
& \int_0^T v(t) \int_{e_1}^{e_1+e_2} (q(t) - \theta(L, y, t))^+ dy dt \\
&= \frac{1}{2} \int_0^T v(t) \int_{e_1}^{e_1+e_2} |q(t) - \theta(L, y, t)| dy dt + \frac{1}{2} \int_0^T v(t) \int_{e_1}^{e_1+e_2} (q(t) - \theta(L, y, t)) dy dt
\end{aligned}$$

and thus, since  $v \in L^\infty(0, T)$ ,  $J_1(q_n) \rightarrow J_1(q)$ . So  $J_\alpha(q_n) \rightarrow J_\alpha(q)$ , that is  $J_\alpha$  is continuous on  $H_+^1(0, T)$ .

$J_\alpha$  is weakly lower semi-continuous on  $H_+^1(0, T)$ .  $J_{\alpha,2}$  is clearly weakly lower semi-continuous on  $H_+^1(0, T)$ .

Let  $(q_n)_{n \in \mathbb{N}}$  a sequence of  $H_+^1(0, T)$  and  $q \in H_+^1(0, T)$  such that  $q_n \rightharpoonup q$  in  $H^1(0, T)$  and  $\theta_n$  and  $\theta$ ,  $n \in \mathbb{N}$  being the solutions of (12) associated respectively to  $q_n$  and  $q$ , given in Theorem 1. Then from Theorem 3,  $(\theta_n)_{n \in \mathbb{N}}$  weakly\* converge to  $\theta$  in  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ .

On the other hand, the imbedding  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \hookrightarrow L^2(0, T, L^2(\Sigma))$  is compact, thus  $\theta_n \rightarrow \theta$  in  $L^2(0, T, L^2(\Sigma))$ . Therefore:

$$\int_0^T v(t) \int_{e_1}^{e_1+e_2} (q_n(t) - \theta_n(L, y, t))^+ dy dt \rightarrow \int_0^T v(t) \int_{e_1}^{e_1+e_2} (q(t) - \theta(L, y, t))^+ dy dt$$

and thus  $J_1(q_n) \rightarrow J_1(q)$ .

Therefore  $J_\alpha = J_1 + J_{\alpha,2}$  is weakly lower semi-continuous on  $H_+^1(0, T)$ .  $\square$

**Theorem 4.** Assume the hypotheses of Lemma 2. For any  $\alpha > 0$ , the optimal problem

$$\inf_{q \in \mathcal{C}} J_\alpha(q) \tag{33}$$

admits a unique solution.

*Proof.*  $J_\alpha$  is strictly convex, weakly lower semi-continuous on  $\mathcal{C}$ . Since  $\mathcal{C}$  is closed and convex,  $\mathcal{C}$  is weakly closed. Since  $\lim_{\|q\|_{H^1(0, T)} \rightarrow +\infty} \|q_t\|_{L^2(0, T)}^2 = +\infty$  we have  $\lim_{\|q\|_{H^1(0, T)} \rightarrow +\infty} J_\alpha(q) = +\infty$ . Therefore,  $H^1(0, T)$  being an Hilbert space,  $J_\alpha$  admits a unique global minimum in  $\mathcal{C}$ .  $\square$

In practice, as in [1], the condition  $\theta \geq \underline{\theta}$  on  $\Sigma_b$  in  $\mathcal{C}$  is taken into account through a penalty method. For any parameter  $\epsilon > 0$ , we introduce the penalized problem:

$$(\mathcal{P}_{\alpha, \epsilon}) : \inf_{q \in H_+^1(0, T)} J_{\alpha, \epsilon}(q) \tag{34}$$

with

$$J_{\alpha, \epsilon}(q) := J_\alpha(q) + \frac{\epsilon^{-1}}{2} \|(\theta - \underline{\theta})^-\|_{L^2((0, T) \times \Sigma_b)}^2, \tag{35}$$

$a^- = \min(a, 0)$  denotes the negative part of any real  $a$ .

**Theorem 5.** For any  $\epsilon > 0$ , Problem  $(\mathcal{P}_{\alpha, \epsilon})$  admits a unique solution  $q_\epsilon$ .

*Proof.* For all  $q \in H_+^1(0, T)$ , let  $J_{\alpha, \epsilon}(q) = J_\alpha(q) + J_3^\epsilon(q)$  with  $J_3^\epsilon(q) = \frac{\epsilon^{-1}}{2} \|(\theta(0, \cdot) - \underline{\theta})^-\|_{L^2((0, T) \times \Sigma_b)}^2$ .

$J_{\alpha, \epsilon}$  is strictly convex on  $H_+^1(0, T)$ . It suffices to check that  $J_3^\epsilon$  is convex.

Let  $q, \bar{q} \in H_+^1(0, T)$ ,  $\theta_\lambda$  be the solution associated with the control function  $\lambda q + (1 - \lambda)\bar{q}$  and let  $[\theta] := \lambda\theta_q + (1 - \lambda)\theta_{\bar{q}}$ . From Lemma 3  $\theta_\lambda \geq [\theta]$  on  $(0, T) \times \Sigma_b$  and thus

$$\theta_\lambda - \underline{\theta} \geq [\theta] - \underline{\theta} = \lambda(\theta_q - \underline{\theta}) + (1 - \lambda)(\theta_{\bar{q}} - \underline{\theta})$$

which gives

$$-(\theta_\lambda - \underline{\theta})^- \leq -(\lambda(\theta_q - \underline{\theta}) + (1 - \lambda)(\theta_{\bar{q}} - \underline{\theta}))^- \leq -\lambda(\theta_q - \underline{\theta})^- - (1 - \lambda)(\theta_{\bar{q}} - \underline{\theta})^-.$$

The convexity of  $x \rightarrow x^2$  leads to the convexity of  $J_3^\epsilon$ .

$J_{\alpha, \epsilon}$  is weakly lower semi-continuous on  $H_+^1(0, T)$ . It suffices to check that  $J_3^\epsilon$  is weakly lower semi-continuous on  $H_+^1(0, T)$ .

Let  $(q_n)_{n \in \mathbb{N}}$  be a sequence of  $H_+^1(0, T)$  and  $q \in H_+^1(0, T)$  such that  $q_n \rightharpoonup q$  in  $H^1(0, T)$ ; let then  $\theta_n = \mathcal{T}(q_n)$   $n \in \mathbb{N}$  and  $\theta = \mathcal{T}(q)$  be the solutions of (12) given in Theorem 1 associated respectively with  $q_n$  and  $q$ . Then we deduce from Theorem 3 that the sequence  $(\theta_n)_{n \in \mathbb{N}}$  weakly  $\star$  converges to  $\theta$  in  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$ .

On the other hand, the embedding  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \hookrightarrow L^2(0, T; L^2(\Sigma))$  is compact, thus  $\theta_n \rightarrow \theta$  in  $L^2(0, T; L^2(\Sigma))$ . Therefore  $J_3^\epsilon(q_n) \rightarrow J_3^\epsilon(q)$  and thus  $J_{\alpha, \epsilon} = J_1 + J_{\alpha, 2} + J_3^\epsilon$  is weakly lower semi-continuous on  $H_+^1(0, T)$ .

Moreover,  $\lim_{\|q\|_{H^1(0, T)} \rightarrow +\infty} J_{\alpha, \epsilon}(q) = +\infty$  since  $\lim_{\|q\|_{H^1(0, T)} \rightarrow +\infty} J_\alpha(q) = +\infty$ .

Eventually, since the set  $H_+^1(0, T)$  is a non empty closed convex set of  $H^1(0, T)$ , we conclude that the problem  $(\mathcal{P}_{\alpha, \epsilon})$  admits a unique solution  $q_\epsilon$ .  $\square$

**Theorem 6.** *The unique solution  $q_\epsilon$  of Problem  $(\mathcal{P}_{\alpha, \epsilon})$  strongly converges in  $H^1(0, T)$  as  $\epsilon \rightarrow 0$  to the unique solution  $q$  of (33).*

*Proof.* If  $q \in \mathcal{C}$  is the unique solution of (33), then  $q \in H_+^1(0, T)$  and thus, for all  $\epsilon > 0$ ,  $J_{\alpha, \epsilon}(q_\epsilon) \leq J_{\alpha, \epsilon}(q) = J_\alpha(q)$ . Therefore,  $J_{\alpha, 2}(q_\epsilon) = \frac{\alpha}{2} \|(q_\epsilon)_t\|_{L^2(0; T)}^2 \leq J_\alpha(q)$  and since  $q_\epsilon(0) = q_0$ ,  $(q_\epsilon)_\epsilon$  is bounded in  $H^1(0, T)$ . Thus there exists a subsequence  $(q_{\epsilon_n})_{n \in \mathbb{N}}$  and  $\bar{q} \in H_+^1(0, T)$  such that  $q_{\epsilon_n} \rightharpoonup \bar{q}$  in  $H^1(0, T)$ . Let us prove that  $\bar{q} \in \mathcal{C}$  and  $\bar{q} = q$ .

$\bar{q} \in \mathcal{C}$ . It suffices to prove that  $\theta \geq \underline{\theta}$  a.e. on  $\Sigma_b \times (0, T)$ . Since  $q_{\epsilon_n} \rightharpoonup \bar{q}$  in  $H^1(0, T)$ , we deduce from Theorem 3 that  $\mathcal{T}(q_{\epsilon_n}) = \theta_{q_{\epsilon_n}} \xrightarrow{\star} \mathcal{T}(q) = \theta$  in  $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$  and thus that  $\theta_{q_{\epsilon_n}} \rightarrow \theta$  in  $L^2(0, T; L^2(\Sigma))$ . Moreover, since for all  $n \in \mathbb{N}$ ,  $J_3^{\epsilon_n}(q_{\epsilon_n}) = \frac{\epsilon_n^{-1}}{2} \|(\theta_{q_{\epsilon_n}} - \underline{\theta})^-\|_{L^2((0, T) \times \Sigma_b)}^2 \leq J_{\alpha, \epsilon_n}(q_{\epsilon_n}) \leq J_\alpha(q)$ ,  $(\frac{\epsilon_n^{-1}}{2} \|(\theta_{q_{\epsilon_n}} - \underline{\theta})^-\|_{L^2((0, T) \times \Sigma_b)}^2)_{n \in \mathbb{N}}$  is bounded in  $L^2(0, T; L^2(\Sigma))$ , and thus  $\|(\theta_{q_{\epsilon_n}} - \underline{\theta})^-\|_{L^2((0, T) \times \Sigma_b)}^2 \rightarrow 0$ , which gives  $\|(\theta - \underline{\theta})^-\|_{L^2((0, T) \times \Sigma_b)}^2 = 0$  that is  $\theta \geq \underline{\theta}$  a.e. on  $\Sigma_b \times (0, T)$ . Thus  $\bar{q} \in \mathcal{C}$ .



$\bar{q} = q$ . Since for all  $n \in \mathbb{N}$ ,  $J_\alpha(q_{\epsilon_n}) \leq J_{\alpha, \epsilon_n}(q_{\epsilon_n}) \leq J_{\alpha, \epsilon_n}(q) = J_\alpha(q)$  and  $J_\alpha$  is lower semi-continuous on  $H_+^1(0, T)$ , we have  $J_\alpha(\bar{q}) \leq \liminf_{n \rightarrow +\infty} J_\alpha(q_{\epsilon_n}) \leq J_\alpha(q)$ . Thus, since  $\bar{q} \in \mathcal{C}$ ,  $J_\alpha(q) = J_\alpha(\bar{q})$  and uniqueness argument gives  $q = \bar{q}$ .

Since  $\theta_{q_{\epsilon_n}} \rightarrow \theta$  in  $L^2(0, T; L^2(\Sigma))$  and  $q_{\epsilon_n} \rightarrow \bar{q}$  in  $L^2(0, T)$ ,  $\lim_{n \rightarrow +\infty} J_1(q_{\epsilon_n}) = J_1(q)$  and thus  $\lim_{n \rightarrow +\infty} J_{\alpha, 2}(q_{\epsilon_n}) = \lim_{n \rightarrow +\infty} \|(q_{\epsilon_n})_t\|_2^2 = J_{\alpha, 2}(q) = \|q_t\|_2^2$ . Thus  $q_{\epsilon_n}$  converges in  $H^1(0, T)$  to  $q$ .

Therefore the whole sequence  $(q_\epsilon)_\epsilon$  converges in  $H^1(0, T)$  toward  $q$ . Moreover, in view of Theorem 3, this strong convergence implies the convergence of the corresponding solution  $\theta_\epsilon = \mathcal{T}(q_\epsilon)$  to the solution  $\theta = \mathcal{T}(q)$  in  $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ .  $\square$

### 3.2 Regularization and differentiability of the cost

**Proposition 7.** *For any  $q \in H_+^1(0, T)$  such that  $q > 0$  and any  $\bar{q} \in H^1(0, T)$  such that  $\bar{q}(0) = 0$ , the map  $\mathcal{T}$  is differentiable at the point  $q$  in the direction  $\bar{q}$ .*

*Proof.* For any  $q \in H_+^1(0, T)$  such that  $q > 0$  and any  $\bar{q} \in H^1(0, T)$ ,  $\bar{q}(0) = 0$ , there exists  $\eta_0$  such that, for all  $\eta \in \mathbb{R}$ ,  $|\eta| < \eta_0$ ,  $q + \eta\bar{q} \in H_+^1(0, T)$ . We expand the solution  $\theta_{q+\eta\bar{q}} = \mathcal{T}(q + \eta\bar{q})$  associated with the control function  $q + \eta\bar{q}$  as follows

$$\mathcal{T}(q + \eta\bar{q}) = \mathcal{T}(q) + \eta\bar{\theta} + \eta^2\theta_\eta$$

where  $\bar{\theta}$  and  $\theta_\eta$  respectively solves

$$\begin{cases} c(x, y)\partial_t\bar{\theta} - \operatorname{div}(k(x, y)\nabla\bar{\theta}) + v(t)1_{(e_1, e_1+e_2)}(y)\partial_x\bar{\theta} = 0, & \Omega \times (0, T), \\ k\nabla\bar{\theta} \cdot \nu = 0, & (\partial\Omega \setminus (\Sigma_b \cup \Sigma_c)) \times (0, T), \\ \bar{\theta} = \bar{q}, & \Sigma_c \times (0, T), \\ k(x, y)\partial_y\bar{\theta} = 4\sigma\varepsilon(t)\theta_q^3\bar{\theta} + f_2(t)\bar{\theta}, & \Sigma_b \times (0, T), \\ \bar{\theta} = 0, & \Omega \times \{0\} \end{cases} \quad (36)$$

and

$$\begin{cases} c(x, y)\partial_t\theta_\eta - \operatorname{div}(k(x, y)\nabla\theta_\eta) + v(t)1_{(e_1, e_1+e_2)}(y)\partial_x\theta_\eta = 0, & \Omega \times (0, T), \\ k\nabla\theta_\eta \cdot \nu = 0, & (\partial\Omega \setminus (\Sigma_b \cup \Sigma_c)) \times (0, T), \\ \theta_\eta = 0, & \Sigma_c \times (0, T), \\ k(x, y)\partial_y\theta_\eta = \sigma\varepsilon(t)f_\eta(t) + f_2(t)\theta_\eta, & \Sigma_b \times (0, T), \\ \theta_\eta = 0, & \Omega \times \{0\} \end{cases} \quad (37)$$

with  $\theta_q := \mathcal{T}(q)$  and  $f_\eta(t) = 4\theta_q^3\theta_\eta + 6\theta_q^2(\bar{\theta} + \eta\theta_\eta)^2 + 4\eta\theta_q(\bar{\theta} + \eta\theta_\eta)^3 + \eta^2(\bar{\theta} + \eta\theta_\eta)^4$ .

Proceeding as in the proof of Theorem 1, we get that the linear system (36) is well-posed: the unique solution  $\bar{\theta} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$  satisfies the estimate

$$\|\sqrt{c}\bar{\theta}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \int_0^T \int_\Omega k |\nabla \bar{\theta}|^2 + \int_0^T \int_{\Sigma_b} \sigma \varepsilon(t) |\theta_q|^3 \bar{\theta}^2 + \int_0^T \int_{\Sigma_b} f_2(t) \bar{\theta}^2 \leq C \|\bar{q}\|_{H^1(0,T)}^2. \quad (38)$$

This implies that the linear application  $\bar{q} \rightarrow D\mathcal{T}(q) \cdot \bar{q} := \bar{\theta}$  from  $\{q \in H^1(0, T), q(0) = 0\}$  into  $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$  is continuous.

In order to get the announced result, it is sufficient to show that the solution  $\theta_\eta$  is uniformly bounded with respect to  $\eta$  in  $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ . First, from Theorem 3 with  $p = 2$ , we deduce that

$$\begin{aligned} & \|c^{\frac{1}{4}}(\bar{\theta} + \eta\theta_\eta)\|_{L^\infty((0,T);L^4(\Omega))} + \|\sqrt{k}\nabla|\bar{\theta} + \eta\theta_\eta|^2\|_{L^2((0,T);L^2(\Omega)^2)}^{\frac{1}{2}} + \|v^{\frac{1}{4}}(\bar{\theta} + \eta\theta_\eta)\|_{L^4((0,T);L^4(\Sigma_d))} \\ & + \|f_2^{\frac{1}{4}}(\bar{\theta} + \eta\theta_\eta)\|_{L^4((0,T);L^4(\Sigma_d))} + \eta^{\frac{3}{4}}\|(\sigma\varepsilon)^{\frac{1}{7}}(\bar{\theta} + \eta\theta_\eta)\|_{L^{\frac{7}{4}}((0,T);L^7(\Sigma_b))} \leq C\|\bar{q}\|_{H^1(0,T)} \end{aligned} \quad (39)$$

for some constant  $C > 0$  independent of  $\eta$ . Then, multiplying the system (37) by  $\theta_\eta$  and integrating by parts yields, with  $\theta_\eta(\cdot, 0) = 0$ , to, for all  $t \in (0, T)$

$$\begin{aligned} & \frac{1}{2}\|\sqrt{c}\theta_\eta(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega k |\nabla \theta_\eta|^2 + \int_0^t \int_{\Sigma_d} v(\tau) |\theta_\eta|^2 + \int_0^t \int_{\Sigma_b} f_2(\tau) |\theta_\eta|^2 + \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) 4\theta_q^3 \theta_\eta^2 \\ & = - \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) \left( 6\theta_q^2 (\bar{\theta} + \eta\theta_\eta)^2 + 4\eta\theta_q (\bar{\theta} + \eta\theta_\eta)^3 + \eta^2 (\bar{\theta} + \eta\theta_\eta)^4 \right) \theta_\eta. \end{aligned}$$

Using  $\int_0^t \int_{\Sigma_b} |\bar{\theta} + \eta\theta_\eta|^4 \leq C \int_0^t \int_\Omega |\nabla |\bar{\theta} + \eta\theta_\eta|^2|^2$  and that if  $u \in H^1(\Omega) \cap L^\infty(\Omega)$ , then  $u \in L^\infty(\partial\Omega)$  and  $\|u\|_{L^\infty(\partial\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ , we get that, for any  $\epsilon > 0$

$$\begin{aligned} & \left| \int_0^t \int_{\Sigma_b} 6\sigma \varepsilon(\tau) \theta_q^2 (\bar{\theta} + \eta\theta_\eta)^2 \theta_\eta \right| \leq 6\|\theta_q\|_{L^\infty((0,T)\times\Sigma_b)}^{1/2} \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) (\bar{\theta} + \eta\theta_\eta)^2 \theta_q^{3/2} |\theta_\eta| \\ & \leq 6\|\theta_q\|_{L^\infty((0,T)\times\Omega)}^{1/2} \left( \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) (\bar{\theta} + \eta\theta_\eta)^4 \right)^{1/2} \left( \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) \theta_q^3 \theta_\eta^2 \right)^{1/2} \\ & \leq C\|\theta_q\|_{L^\infty((0,T)\times\Omega)} \epsilon^{-1} \int_0^t \int_{\Sigma_b} (\bar{\theta} + \eta\theta_\eta)^4 + \epsilon \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) \theta_q^3 \theta_\eta^2 \\ & \leq C(\epsilon, q) \|\bar{q}\|_{H^1(0,T)}^4 + \epsilon \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) \theta_q^3 \theta_\eta^2. \end{aligned}$$

Similarly,

$$\begin{aligned}
& \left| \int_0^t \int_{\Sigma_b} 4\eta \sigma \varepsilon(\tau) \theta_q (\bar{\theta} + \eta \theta_\eta)^3 \theta_\eta \right| \\
& \leq 4\eta \|\theta_q\|_{L^\infty((0,T) \times \Sigma_b)} \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) (\bar{\theta} + \eta \theta_\eta)^{\frac{7}{3}} (\bar{\theta} + \eta \theta_\eta)^{\frac{2}{3}} |\theta_\eta| \\
& \leq C \|\theta_q\|_{L^\infty((0,T) \times \Omega)} \left( \eta^3 \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) (\bar{\theta} + \eta \theta_\eta)^7 \right)^{\frac{1}{3}} \left( \int_0^t \int_{\Sigma_b} (\bar{\theta} + \eta \theta_\eta)^4 \right)^{\frac{1}{6}} \left( \int_0^t \int_{\Sigma_b} \theta_\eta^2 \right)^{1/2} \\
& \leq C(\epsilon, q) \left( \eta^3 \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) (\bar{\theta} + \eta \theta_\eta)^7 \right)^{\frac{2}{3}} \left( \int_0^t \int_{\Sigma_b} (\bar{\theta} + \eta \theta_\eta)^4 \right)^{\frac{1}{3}} + \epsilon \int_0^T \int_{\Sigma_b} k |\nabla \theta_\eta|^2 \\
& \leq C(\epsilon, q) \|\bar{q}\|_{H^1(0,T)}^4 + \epsilon \int_0^T \int_{\Sigma_b} k |\nabla \theta_\eta|^2.
\end{aligned}$$

We also have

$$\begin{aligned}
\left| \int_0^t \int_{\Sigma_b} \eta^2 \sigma \varepsilon(\tau) (\bar{\theta} + \eta \theta_\eta)^4 \theta_\eta \right| &= \left| \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) (\bar{\theta} + \eta \theta_\eta)^2 (\theta_{q+\eta\bar{q}} - \theta_q)^2 \theta_\eta \right| \\
&\leq \|\theta_{q+\eta\bar{q}} - \theta_q\|_{L^\infty((0,T) \times \Sigma_b)}^2 \left| \int_0^t \int_{\Sigma_b} \sigma \varepsilon(\tau) (\bar{\theta} + \eta \theta_\eta)^2 \theta_\eta \right| \\
&\leq C(\epsilon, q, \bar{q}, \eta_0) \|\bar{q}\|_{H^1(0,T)}^4 + \epsilon \int_0^T \int_{\Sigma_b} k |\nabla \theta_\eta|^2.
\end{aligned}$$

It follows, for  $\epsilon = \frac{1}{4}$ , that

$$\begin{aligned}
\|\sqrt{c} \theta_\eta(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \int_{\Omega} k |\nabla \theta_\eta|^2 + \int_0^t \int_{\Sigma_d} v(s) |\theta_\eta|^2 + \int_0^t \int_{\Sigma_b} f_2(s) |\theta_\eta|^2 + \int_0^t \int_{\Sigma_b} \sigma \varepsilon(s) 4\theta_q^3 \theta_\eta^2 \\
\leq C(q, \bar{q}, \eta_0),
\end{aligned}$$

implying the uniform boundedness of the solution  $\theta_\eta$  in  $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$  for  $\eta$  small enough.  $\square$

In order to apply a gradient method, we consider a regularization of  $J_{\alpha, \epsilon}$ , and introduce for any  $r > 1$  the functional

$$J_{\alpha, \epsilon, r}(q) := \frac{1}{2r} C_f^2 a_r(q, \theta)^2 + \frac{\alpha}{2} \int_0^T \dot{q}_t^2(t) dt + \frac{\epsilon^{-1}}{2} \left\| (\theta(\cdot, 0, \cdot) - \underline{\theta}) \right\|_{L^2((0,T) \times \Sigma_b)}^2, \quad (40)$$

with  $a_r(q, \theta) := \int_0^T v(t) \int_{\Sigma_d} g_r(q(t) - \theta) dy dt$  and  $g_r(s) := (s^+)^r$  for all  $s$ .

**Proposition 8.** *For any  $q \in H_+^1(0, T)$  such that  $q > 0$  and any  $\bar{q} \in H^1(0, T)$  such that  $\bar{q}(0) = 0$ , the functional  $J_{\alpha, \epsilon, r}$  is differentiable at the point  $q$  in the direction  $\bar{q}$ ,*

and the directional derivative  $DJ_{\alpha,\epsilon,r}(q) \cdot \bar{q} = \lim_{\eta \rightarrow 0} \frac{J_{\alpha,\epsilon,r}(q+\eta\bar{q}) - J_{\alpha,\epsilon,r}(q)}{\eta}$  is given by

$$DJ_{\alpha,\epsilon,r}(q) \cdot \bar{q} = C_f^2 a_r(q, \theta) \int_0^T v(t) \int_{\Sigma_d} ((q(t) - \theta)^+)^{r-1} (\bar{q}(t) - \bar{\theta}) + \alpha \int_0^T q_t \bar{q}_t + \epsilon^{-1} \int_0^T \int_{\Sigma_b} (\theta - \underline{\theta})^- \bar{\theta} \quad (41)$$

where  $\bar{\theta}$  solves (36).

*Proof.* The directional differentiability of  $J_{\alpha,\epsilon,r}$  follows for Proposition 7, the fact that  $s \rightarrow ((s - \underline{\theta})^-)^2$  and  $s \rightarrow g_r(s)$  are  $\mathcal{C}^1(\mathbb{R})$  for any  $r > 1$  and  $g'(s) = r(s^+)^{r-1}$ .  $\square$

### 3.3 Descent direction of the cost

Before to set up a minimizing sequence for the cost, we reformulate the derivative  $DJ_{\alpha,\epsilon,r}(q)$  into a more explicit form in term of an adjoint solution. Preliminary, we consider the following simplified cost (still denoted by  $J_{\alpha,\epsilon,r}$ )

$$J_{\alpha,\epsilon,r}(q) = \frac{1}{r} \int_0^T \int_{\Sigma_d} ((q(t) - \theta(s, t))^+)^r ds dt + \frac{\alpha}{2} \int_0^T q_t^2(t) dt + \frac{\epsilon^{-1}}{2} \left\| (\theta(\cdot, 0, \cdot) - \underline{\theta})^- \right\|_{L^2((0, T) \times \Sigma_b)}^2 \quad (42)$$

for which (i) we consider a unit constant in the original first term, (ii) we assume that the hydraulic load  $H_1$  (and so  $v = KH_1/L$ ) is time-independent and (iii) we remove the square in the original first term.

We then introduce the following linear problem : find  $p$  solution

$$\begin{cases} -c(x, y) \partial_t p - \operatorname{div}(k(x, y) \nabla p) - K \frac{H_1(t)}{L} 1_{(e_1, e_1 + e_2)}(y) \partial_x p = 0, & \Omega \times (0, T), \\ k \nabla p \cdot \nu = 0, & (\partial\Omega \setminus (\Sigma_b \cup \Sigma_c \cup \Sigma_d)) \times (0, T), \\ k(x, y) \partial_x p = ((-\bar{\theta})^+)^{r-1}, & \Sigma_d \times (0, T), \\ p = 0, & \Sigma_c \times (0, T), \\ k(x, y) \partial_y p = (4\sigma\varepsilon(t)\bar{\theta}^3 + f_2)p - \epsilon^{-1}(\bar{\theta} - \underline{\theta})^-, & \Sigma_b \times (0, T), \\ p = 0, & \Omega \times \{T\} \end{cases} \quad (43)$$

where  $\bar{\theta} = \theta - g(y)q(t)$  is the solution of (21). Proceeding as in Section 2 and using that  $\bar{\theta}$  belongs to  $H^1(0, T; L^2(0, T)) \cap L^\infty(0, T; H^1(\Omega))$  (see Theorem 1), we get that problem (43) admits a unique solution in  $L^2(0, T; H^1(\Omega))$ .  $p$  is the adjoint variable associated with  $\bar{\theta}$ .

Proceeding for instance as in the proof of [1, Theorem 4.5]), we then get the following expression for the derivative of the penalized/regularized cost.

**Proposition 9.** For any  $q \in H_+^1(0, T)$  such that  $q > 0$ , let  $\bar{\theta}$  be the unique corresponding solution in  $H^1(0, T; L^2(\Omega))$  of (21). Let then  $p$  be the unique corresponding solution of (43). For any  $\bar{q} \in H^1(0, T)$  such that  $\bar{q}(0) = 0$ , the derivative of  $J_{\alpha, \epsilon, r}$  at  $q$  in the direction  $\bar{q}$  is given by

$$DJ_{\alpha, \epsilon, r}(q) \cdot \bar{q} = \alpha \int_0^T q_t \bar{q}_t + \int_0^T \int_{\Omega} (c(x, y)g(y)\bar{q}_t p + \bar{q}k(x, y)g_y \partial_y p).$$

As in [1], this expression of the derivative allows to define a minimizing sequence for  $J_{\alpha, \epsilon, r}$ . More precisely, for any initial function  $q^0 \in H_+^1(0, T)$  with  $q^0(0) = q_0$  and  $\eta > 0$  small enough, we define the sequence  $(q^k)_{k \in \mathbb{N}}$  by:

$$q^{k+1} = P_{\mathbb{R}^+}(q^k - \eta \bar{q}^k), \quad k \geq 0 \quad (44)$$

where  $\bar{q}^k \in H^1(0, T)$  such that  $\bar{q}^k(0) = 0$  solves  $(\bar{q}^k, \bar{q})_{H^1(0, T)} = DJ_{\alpha, \epsilon, r}(q^k) \cdot \bar{q}$  for all  $\bar{q} \in H^1(0, T)$  and where  $P_{\mathbb{R}^+}$  denotes the projection operator on  $\mathbb{R}^+$ .

## 4 Numerical experiments

### 4.1 Experimental data and initial condition

We use real data obtained from measurements on the French highway A75 in Cantal (1100 meter altitude) from october 2009 to march 2010. Measurements are made each hour and allow to compute the time functions  $f_1$  and  $f_2$  defined in (3). We refer to [1, Figure 4, page 792] where the functions  $f_1$  and  $f_2$  are depicted.

As in [1], we use the following numerical values identified for the Egletons demonstrator road [17]:

$$\begin{aligned} e_1 &= 0.06, e_2 = 0.08, e_3 = 0.05, e_4 = 14.81, h_e = 15, L = 4 \\ c_1 &= 2144309, c_2 = 1769723, c_3 = 2676728, c_4 = 1947505 \\ k_1 &= 2.34, k_2 = 1.56, k_3 = 1.76, k_4 = 2.08 \\ K &= 2.2 \times 10^{-2}, \sigma = 5.67 \times 10^{-8}, \varepsilon = 0.92, \theta_s = 288.15. \end{aligned} \quad (45)$$

The reals  $c_i$  and  $k_i$  ( $1 \leq i \leq 4$ ) denote the constant values of the function  $c(x, y)$  and  $k(x, y)$  in each layers. Moreover we define the initial condition, a priori not determined by experiments, as the solution of the stationary 1D-model:

$$\begin{cases} -(k(y)\theta_{0,y}(y))_y = 0, & y \in (0, h_e), \\ -k(0)\theta_{0,y}(0) = f_1(0) - f_2(0)\theta_0(0) - \sigma\varepsilon(0)\theta_0(0)^4, & \theta_0(h_e) = \theta_s. \end{cases} \quad (46)$$

The resolution of (46) leads to a continuous function, affine on each layers, and increasing from the value  $\theta_0(y = 0) \approx 273.15 + 6.29$  Kelvin at the road surface to  $\theta_0(y = h_e) = \theta_s$  Kelvin.

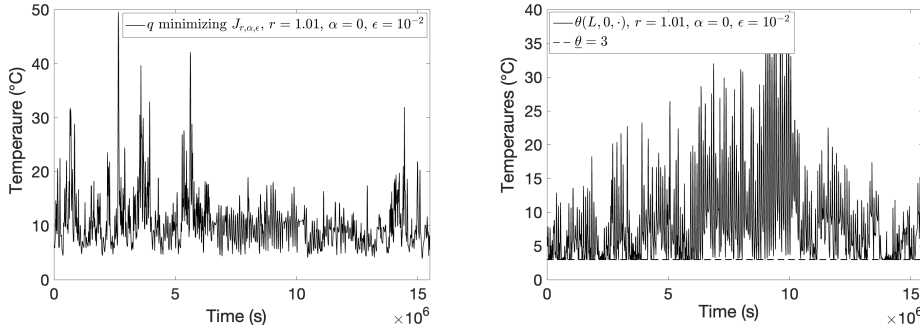
## 4.2 Numerical experiments

We discuss some numerical experiments related to the extremal optimal problem (34). The direct (7) and the adjoint problem (43) are discretized using finite differences schemes. The time discretization parameter  $\Delta t$  is equal to 7200 s which corresponds  $4.65 \times 10^{-4} T$  with  $T=4304$  h (6 months). The spatial discretization is not uniform and is determined by the method described in [27] and used in [20].

The descent algorithm is initialized with a time-independent initial control  $q_0 = 10^\circ\text{C}$  and stop when the  $k^{\text{th}}$  iterate  $q_k$  satisfies

$$\frac{|J_{\alpha,\epsilon,r}(q_k) - J_{\alpha,\epsilon,r}(q_{k-1})|}{J_{\alpha,\epsilon,r}(q_0)} \leq 10^{-10}.$$

We start formally with the case  $\alpha = 0$  (corresponding to controls in  $L^2(0, T)$ ) then consider increasing values of  $\alpha$  (the limiting case “ $\alpha = \infty$ ” leads to constant controls). Tables 1-2 collect some results with respect to  $\epsilon$  in  $\{10^{-2}, 10^{-1}\}$  for  $r = 1.01$  and  $r = 1.1$  leading to a constraint  $\|(\theta(\cdot, 0, \cdot) - \underline{\theta})^-\|_{L^2((0, T) \times \Sigma_b)}$  less than  $10^{-1}$ . Figure 3 depicts the optimal control  $q$  and the associated surface temperature  $\theta(L, 0, \cdot)$  for  $r = 1.01$ ,  $\alpha = 0$  and  $\epsilon = 10^{-2}$ .



**Fig. 3** The optimal control  $q$  (left) and the associated surface temperature  $\theta(L, 0, \cdot)$  (right) for  $\underline{\theta} = 3$ ,  $r = 1.01$ ,  $\alpha = 0$  and  $\epsilon = 10^{-2}$ .

In order to lighten the notations, we put:

$$\begin{aligned} \Delta_2 &= \|(\theta(\cdot, 0, \cdot) - \underline{\theta})^-\|_{L^2((0, T) \times \Sigma_b)}, \Delta_\infty = \|(\theta(\cdot, 0, \cdot) - \underline{\theta})^-\|_{L^\infty((0, T) \times \Sigma_b)}, [^\circ\text{C}] \\ \|q\|_\infty &= \|q\|_{L^\infty((0, T))}, [^\circ\text{C}], \|P_h(q)\|_\infty = \|P_h(q)\|_{L^\infty((0, T))}, [\text{KW.m}^{-2}] \end{aligned}$$

where  $P_h$  denotes the power heating defined by  $P_h(t; q) = v C_f \int_{e_1}^{e_1+e_2} (q(t) - \theta(L, y, t))^+ dy$  for all  $t \in [0, T]$ . In the sequel, the energy  $E_h(q)$  is expressed in  $\text{KWh.m}^{-2}$ . As expected more  $\epsilon$  is high, more the constraint on the surface temperature is relaxed:  $\Delta_2$  is a non-decreasing function w.r.t.  $\epsilon$  as  $\Delta_\infty$  is. As

observed in lines 5 and 6 of Tables 1-2, the maximum inlet fluid temperature  $\|q\|_\infty$  and the maximal power  $\|P_h(q)\|_\infty$  are very sensitive to the small values of  $\epsilon$ : divide the constraints of lines 3 and 4 by 2 is very low sensitive on  $\|q\|_\infty$  and  $\|P_h(q)\|_\infty$  for  $\epsilon > 3.33 \times 10^{-2}$  but leads to a doubling of these latter variables for  $\epsilon < 3.33 \times 10^{-2}$ . By comparing Table 1 and Table 2, we observe higher values for all variables for  $r = 1.01$  in comparison with those corresponding to  $r = 1.1$ . On the other hand, the

**Table 1** Numerical norms of the optimal control minimizing (42) with  $\alpha = 0$ ,  $r = 1.1$ ,  $\underline{\theta} = 3$  and with respect to  $\epsilon \in \{10^{-2}, 10^{-1}\}$ .

$\epsilon$	$10^{-2}$	$2 \times 10^{-2}$	$3.33 \times 10^{-2}$	$5 \times 10^{-2}$	$10^{-1}$
$E_h(q)$	$2.41 \times 10^2$	$2.23 \times 10^2$	$2.19 \times 10^2$	$2.18 \times 10^2$	$2.17 \times 10^2$
$\Delta_2$	$9.98 \times 10^0$	$1.62 \times 10^1$	$2.33 \times 10^1$	$3.23 \times 10^1$	$5.56 \times 10^1$
$\Delta_\infty$	$7.22 \times 10^{-2}$	$1.09 \times 10^{-1}$	$1.34 \times 10^{-1}$	$1.85 \times 10^{-1}$	$2.61 \times 10^{-1}$
$\ q\ _\infty$	79.9	45.2	36.1	36.0	35.9
$\ P_h(q)\ _\infty$	1.55	0.87	0.73	0.73	0.73
# iterates	185	118	110	98	96

**Table 2** Norms of the optimal control minimizing (42) with  $\alpha = 0$ ,  $r = 1.01$ ,  $\underline{\theta} = 3$  and with respect to  $\epsilon \in \{10^{-2}, 10^{-1}\}$ .

$\epsilon$	$10^{-2}$	$2 \times 10^{-2}$	$3.33 \times 10^{-2}$	$5 \times 10^{-2}$	$10^{-1}$
$E_h(q)$	$2.57 \times 10^2$	$2.25 \times 10^2$	$2.19 \times 10^2$	$2.18 \times 10^2$	$2.17 \times 10^2$
$\Delta_2$	$9.67 \times 10^0$	$1.57 \times 10^1$	$2.08 \times 10^1$	$2.93 \times 10^1$	$5.16 \times 10^1$
$\Delta_\infty$	$6.08 \times 10^{-2}$	$1.05 \times 10^{-1}$	$1.20 \times 10^{-1}$	$1.60 \times 10^{-1}$	$2.38 \times 10^{-1}$
$\ q\ _\infty$	90.9	49.5	36.7	36.6	36.4
$\ P_h(q)\ _\infty$	1.76	0.96	0.75	0.75	0.75
# iterates	166	104	116	120	81

limiting case “ $\alpha = +\infty$ ” displays different behaviors. We give in Tables 3-4 similar results to those of Tables 1-2. We fit the values of  $\epsilon$  in order to have the same values of  $\Delta_2$  for the cases  $\alpha = 0$  and “ $\alpha = +\infty$ ”.

**Table 3** Norms of the optimal control minimizing (42) with “ $\alpha = +\infty$ ”,  $r = 1.1$ ,  $\underline{\theta} = 3$  and with respect to the parameter  $\epsilon \in \{3.45 \times 10^{-3}, 8.67 \times 10^{-3}\}$ .

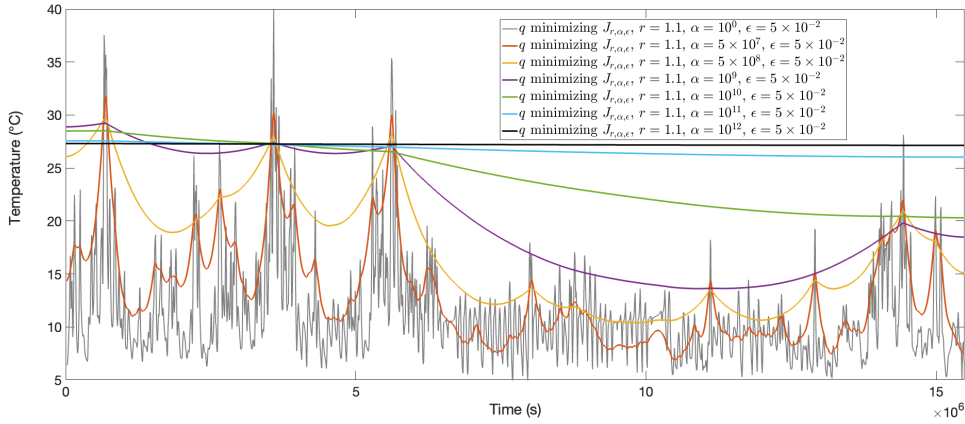
$\epsilon$	$3.45 \times 10^{-3}$	$3.76 \times 10^{-3}$	$4.30 \times 10^{-3}$	$5.21 \times 10^{-3}$	$8.67 \times 10^{-3}$
$E_h(q)$	$7.82 \times 10^2$	$7.82 \times 10^2$	$7.80 \times 10^2$	$7.75 \times 10^2$	$7.60 \times 10^2$
$\Delta_2$	$9.98 \times 10^0$	$1.62 \times 10^1$	$2.33 \times 10^1$	$3.23 \times 10^1$	$5.56 \times 10^1$
$\Delta_\infty$	$5.05 \times 10^{-1}$	$5.20 \times 10^{-1}$	$5.45 \times 10^{-1}$	$5.83 \times 10^{-1}$	$6.96 \times 10^{-1}$
$\ q\ _\infty$	29.95	29.89	29.79	29.65	29.24
$\ P_h(q)\ _\infty$	0.78	0.77	0.77	0.77	0.75
# iterates	12	12	12	12	12

**Table 4** Norms of the optimal control with  $\alpha = +\infty$ ,  $r = 1.01$ ,  $\underline{\theta} = 3$  and with respect to  $\epsilon \in \{3.10^{-3}, 8.89^{-3}\}$ .

$\epsilon$	$3.10 \times 10^{-3}$	$3.49 \times 10^{-3}$	$3.96 \times 10^{-3}$	$4.99 \times 10^{-3}$	$8.89 \times 10^{-3}$
$E_h(q)$	$7.90 \times 10^2$	$7.87 \times 10^2$	$7.85 \times 10^2$	$7.80 \times 10^2$	$7.65 \times 10^2$
$\Delta_2$	$9.67 \times 10^0$	$1.57 \times 10^1$	$2.08 \times 10^1$	$2.93 \times 10^1$	$5.16 \times 10^1$
$\Delta_\infty$	$4.62 \times 10^{-1}$	$4.82 \times 10^{-1}$	$5.04 \times 10^{-1}$	$5.47 \times 10^{-1}$	$6.70 \times 10^{-1}$
$\ q\ _\infty$	31.31	31.21	31.10	30.89	30.30
$\ P_h(q)\ _\infty$	0.78	0.78	0.78	0.77	0.76
# iterates	12	12	12	12	12

Comparing Table 1 and Table 3, or Table 2 and Table 4, the energy  $E_h$ , the maximal inlet temperature  $\|q\|_\infty$  and the maximal power  $\|P_h(q)\|_\infty$  are almost non-varying functions w.r.t.  $\epsilon$ . Note that the energy is approximately three times greater than in the  $L^2$  case and the maximal inlet temperature is much smaller ( $6^\circ\text{C}$  or much higher in the case of stricter compliance with the surface temperature constraint). We can observe that a constant control has less effect on  $\Delta_\infty$  than a  $L^2$  control, what is illustrated in Figure 3: the  $L^2$  control acts locally.

We illustrate the  $H_+^1(0, T)$  minimization in Figure 4 and in Table 5 for  $\epsilon = 5.10^{-2}$ ,  $r = 1.1$ ,  $\underline{\theta} = 3$  and  $\alpha$  between  $10^0$  and  $10^{13}$ .



**Fig. 4** The optimal control  $q \in H^1(0, T)$  for  $\epsilon = 5.10^{-2}$ ,  $r = 1.1$ ,  $\underline{\theta} = 3$  and  $\alpha \in [10^0, 10^{12}]$ .

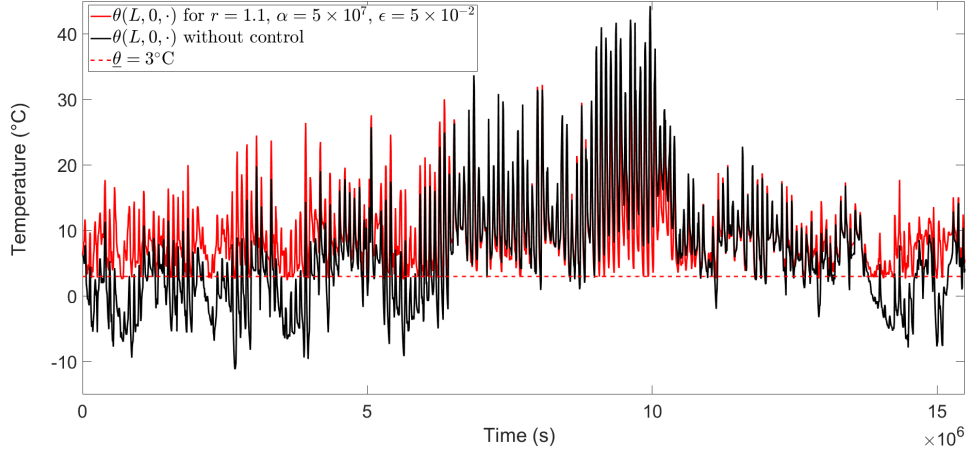
We see in Figure 4 the effect of  $\alpha$  in the cost function for regularizing the control  $q$ . The control for  $\alpha = 10^0$  has a shape very close to the one of the  $L^2$ -control: a local acting to ensure a road surface temperature above  $\underline{\theta}$  in the  $L^2$  sense. When  $\alpha$  increases, the control  $q$  tends to the constant control analyzed previously (case  $\alpha = +\infty$ ). The choice of  $\alpha$  is very depending on the heating strategy. From Table 5, the regularization of the control has an impact on the  $\Delta_2$  and  $\Delta_\infty$  criteria: they are approximately multiplied by five when  $\alpha$  increases from  $10^0$  to  $10^{12}$ . If one wants to keep  $\Delta_2$  or  $\Delta_\infty$



**Table 5** Norms of the optimal control with  $\epsilon = 5.10^{-2}$ ,  $r = 1.1$ ,  $\underline{\theta} = 3$  and  $\alpha \in [10^0, 10^{12}]$ .

$\alpha$	$10^0$	$5 \times 10^7$	$10^9$	$10^{10}$	$10^{11}$	$10^{12}$
$E_h(q)$	$2.29 \times 10^2$	$3.32 \times 10^2$	$4.52 \times 10^2$	$5.50 \times 10^2$	$6.47 \times 10^2$	$7.30 \times 10^2$
$\Delta_2$	$3.57 \times 10^1$	$6.57 \times 10^1$	$1.08 \times 10^2$	$1.40 \times 10^2$	$1.57 \times 10^2$	$1.73 \times 10^2$
$\Delta_\infty$	$2.05 \times 10^{-1}$	$5.85 \times 10^{-1}$	$8.72 \times 10^{-1}$	$9.80 \times 10^{-1}$	$1.15 \times 10^0$	$1.19 \times 10^0$
$\ q\ _\infty$	39.94	31.76	29.75	29.22	28.52	27.56
$\ P_h(q)\ _\infty$	0.76	0.49	0.72	0.82	0.80	0.77
# iterates	91	862	793	755	422	281

close to target values,  $\alpha$  and  $\epsilon$  have to be adapted. An interesting observation concerns the power  $\|P_h(q)\|_\infty$ : the control for  $\alpha = 5 \times 10^7$  (red curve in Figure 4) allows to achieve a minimal spent power to heat the road. Figure 5 depicts the evolution in time of the road surface temperature  $\theta(L, 0, \cdot)$  for an optimal control  $q \in H^1_+(0, T)$  with  $\epsilon = 5.10^{-2}$ ,  $r = 1.1$ ,  $\underline{\theta} = 3$  and  $\alpha = 10^9$  (red). The black line corresponds to the uncontrolled case for which a homogeneous Neumann condition is imposed on  $\Sigma_c$ .



**Fig. 5** The road surface temperature  $\theta(L, 0, \cdot)$  ( $x = L$ ) for an optimal control  $q \in H^1(0, T)$  with  $\epsilon = 5.10^{-2}$ ,  $r = 1.1$ ,  $\underline{\theta} = 3$  and  $\alpha = 10^9$  (red) and the road surface temperature  $\theta(L, 0, \cdot)$  ( $x = L$ ) without control (black).

## 5 Conclusion

We have proposed an original approach to approximate the minimal energy for heating a road in order to keep its surface frost free. The heat exchanger is based on the circulation of water in a porous bonding layer of the road. A transient 2D validated diffusion-advection model including a nonlinear Stefan-Boltzmann law is used to describe the heat exchanger and we look for an optimal inlet fluid temperature

to maintain the surface temperature above a threshold value. Theoretical results have established the well-posedness of the corresponding direct and optimal control problem.

Different heating strategies have been considered numerically by the use of several norms. The  $L^2$  minimization leads to sparse controls with  $L^\infty$ -norm higher than the  $H^1$  minimization. It appears that the total energy needed to keep the road surface temperature over  $3^\circ\text{C}$  during a winter with snow is between 200 kWh and 700 kWh per  $\text{m}^2$  of road depending on the choice of the  $H^1$ -regularization parameter. Moreover, the  $L^\infty$ -norm of the optimal inlet temperature  $q$  ranges in  $30\text{-}90^\circ\text{C}$ : more the control is regular less is the temperature. The values of energy and power are relevant with those found in the literature review.

For further works, as done in a 1D setting considered in [1, Section 5.4], it would be interesting to consider bang-bang controls taking only two values and allowing then to bound the control. Another perspective is to develop a methodology by optimal control for an energy retrieving application. Indeed, during the summer season, road surfaces can reach high temperatures of the order of  $70^\circ\text{C}$ . These temperatures are harmful and cause environmental problems such as urban heat island and structural damage of roads due to the hardening and rutting produced by thermal cycles. The bituminous solar collectors are efficient active systems thanks to their ability to recover solar energy that can be used for various applications. In order to maximize the retrieved energy, the inlet fluid temperature could be optimized thanks to a methodology similar to the one proposed in this study for road heating.

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