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# Local existence for systems of conservation laws with partial diffusion.

Jean-Paul ADOGBO & Raphaël Danchin

## Abstract

This paper is dedicated to the study of the local existence theory of the Cauchy problem for symmetric hyperbolic partially diffusive systems (also known as hyperbolic-parabolic system) in dimension  $d \geq 1$ . The system under consideration is a coupling between a symmetric hyperbolic system and a parabolic system. We address the question of well-posedness for large data having critical Besov regularity. This improves the analysis of Serre [20] and Kawashima [15]. Our results allow for initial data whose components have different regularities and we enlarge the class of the components experiencing the diffusion to  $H^s$ , with  $s > d/2$  (instead of  $s > d/2 + 1$  in Serre's work and  $s > d/2 + 2$  in Kawashima's one).

Our results rely on Gårding's inequality, composition estimates and product laws. As an example, we consider the Navier-Stokes-Fourier equations.

## 1 Introduction

Many physical phenomena may be modelled by first order hyperbolic equations with degenerate dissipative or diffusive terms. This is the case for example in gas dynamics, where the mass is conserved during the evolution, but the momentum balance includes a diffusion (viscosity) or damping (relaxation) term, or, in numerical simulations, of conservation laws by relaxation schemes. In this paper we consider systems of the form

$$\partial_t u + \sum_{\alpha=1}^d \partial_\alpha L^\alpha(u) = \text{Div} (B(u) \nabla u) \stackrel{\text{def}}{=} \sum_{\alpha, \beta=1}^d \partial_\alpha (B^{\alpha\beta}(u) \partial_\beta u), \quad (1)$$

in which  $u : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{U}$  is the unknown. The phase space  $\mathbb{U}$  is an open convex subset of  $\mathbb{R}^n$ . The nonlinearities are encoded in the smooth functions

$$L^\alpha : \mathbb{U} \rightarrow \mathcal{M}_{n \times d}, \quad B^{\alpha\beta} : \mathbb{U} \rightarrow \mathcal{M}_n.$$

Among the systems having the form (1) are the Navier-Stokes-Fourier equations, magneto-hydrodynamics equations and electromagnetism equations [15, chap. 6], supercritical fluid models with chemical reactions [12], Baer-Nunziato system [5], etc. In each case, diffusion (e.g. thermal conduction or viscosity) acts on some components of the unknown, while other components remain unaffected. The Navier-Stokes-Fourier equations will be addressed at the end of the paper.

It is well known since the works of A. Majda in [18] and D. Serre in [19] (see also [3, chapter 10]) that general systems of conservation laws (that is (1) with  $B(u) = 0$  for all  $u \in \mathbb{U}$ ) which are *Friedrichs-symmetrizable* supplemented with smooth decaying data admit local-in-time strong solutions. These solutions may develop singularities (shock waves) in finite time even if the initial data are small perturbations of a constant solution.

An important issue is to find as weak as possible conditions ensuring the existence of local or global solutions, to describe their long time behavior and, where applicable, to study the convergence to some limit system.

In his PhD thesis [15] (that is recognized to be the seminal work on symmetric hyperbolic partially diffusive systems) Kawashima proved the local existence for general data belonging to  $H^s(\mathbb{R}^d)$  with  $s > d/2 + 2$  and exhibited a sufficient condition for global well-posedness in the case of small data belonging to  $H^s(\mathbb{R}^d)$  with  $s > d/2 + 3$ . This condition is now known as the *Kawashima-Shizuta condition*. It will be discussed in a companion paper, the present work being dedicated to the local well-posedness theory for, possibly, large data.

A bit later, Serre in [21] justified the requirements made by Kawashima in his thesis, using only *natural assumptions*: entropy-dissipative (see Definition 1.2 below) and the constancy of the range of the symbol  $B(\xi; u)$ , see assumption **A** below. Moreover, in [20], the same author provided the normal form for (1) close to that used by S. Kawashima and Y. Shizuta [16] which allowed him to enlarge the class of admissible initial data to  $H^s(\mathbb{R}^d)$  with  $s > 1 + d/2$  (instead of  $2 + d/2$  in Kawashima's PhD thesis [15]), a result which in turn will be improved in the present paper.

The notion of entropy is not new. In fact Godunov [13], Friedrichs and Lax [11] introduced the entropy for hyperbolic conservation laws, that is, (1) with  $B = 0$ . As regards the concept of entropy dissipation, we refer to [22], [17], [4] and the references therein.

The normal form is the rewriting of (1) as a so-called symmetric hyperbolic (partially) diffusive system in which one component of the solution may be regarded as solution of a hyperbolic equation with source term, while the second component satisfies a parabolic equation and is thus likely to be smoothed out instantaneously. As an example, we shall consider the Navier-Stokes-Fourier system, where the density satisfies a transport equation while the equations satisfied by the velocity and the temperature are parabolic (see section 4).

The question now is whether we can reduce the regularity of the component that experiences the diffusion. In fact, since the work of R. Danchin [7, 9, 10] concerning the local and global well-posedness for the Navier-Stokes-Fourier system, it is known that the degree of regularity of the initial density is higher than that of the initial velocity, which, in turn, is higher than that of the initial temperature.

The present paper aims at investigating this well-posedness issue for a class of general hyperbolic-parabolic systems.

## 1.1 Reformulation of the system

In this section, we specify the structure of viscous systems of conservation laws that are entropy-dissipative, in spirit of the work of D. Serre in [21].

First, we assume that the first-order system of conservation laws ((1) with  $B = 0$ ) admits a strong convex entropy  $\eta$  of flux  $q$ . More precisely, we assume that there exists a pair  $(\eta, q)$  defined on  $\mathbb{U}$  such that if  $u$  satisfies

$$\partial_t u + \text{Div}L(u) = 0$$

then

$$\partial_t \eta(u) + \text{div}q(u) = 0.$$

Here, strong convexity means that  $D^2\eta(u)$  is positive definite for every state  $u \in \mathbb{U}$ . In order to pursue our reformulation, we need the following two definitions.

**Definition 1.1.** We say that (1) is strongly entropy-dissipative if it formally implies the following inequality for all  $u \in \mathbb{U}$ ,  $X_1, \dots, X_d \in \mathbb{R}^n$ :

$$\sum_{\alpha, \beta=1}^d D^2\eta(u) \left( X_\alpha, B^{\alpha\beta}(u) X_\beta \right) \geq \omega(u) \sum_{\alpha=1}^d \left| \sum_{\beta=1}^d B^{\alpha\beta}(u) X_\beta \right|^2,$$

where  $\omega(u)$  is strictly positive and continuous and  $D^2\eta(u)(\cdot, \cdot)$  denotes the inner product related to  $D^2\eta(u)$ .

Let us define the partial and total symbol as follows: for all  $\xi \in \mathbb{R}^d$ ,  $u \in \mathbb{U}$

$$B^\alpha(\xi, u) \stackrel{\text{def}}{=} \sum_{\beta=1}^d B^{\alpha\beta}(u) \xi_\beta \quad \text{and} \quad B(\xi, u) \stackrel{\text{def}}{=} \sum_{\alpha, \beta=1}^d \xi_\alpha \xi_\beta B^{\alpha\beta}(u). \quad (2)$$

**Definition 1.2.** We say that (1) is entropy-dissipative if for all  $\xi \in \mathbb{R}^d$ ,  $u \in \mathbb{U}$  and  $X \in \mathbb{R}^n$ ,

$$D^2\eta(u) (X, B(\xi, u) X) \geq \omega(u) \sum_{\alpha=1}^d |B^\alpha(\xi, u) X|^2$$

where  $\omega(u)$  is strictly positive and continuous.

Obviously Definition 1.1 implies Definition 1.2.

The last ingredient that we need to introduce the normal form of the viscous systems of conservation laws that are entropy-dissipative is the following.

**Assumption A:** The range of the total symbol  $B(\xi, u)$  depends on neither  $\xi \in \mathbb{R}^d \setminus \{0\}$  nor the state  $u \in \mathbb{U}$ . More precisely, the rank of the symbol  $B(\xi, u)$  is precisely  $n - n_1$ , for some  $0 \leq n_1 \leq n$ . In other words, the range of  $B(\xi, u)$  is isomorphic to  $\{0\} \times \mathbb{R}^{n-n_1}$ .

Without loss of generality we may always assume that the  $n_1$  first rows of  $B(\xi, u)$  are null, up to a linear change of coordinates, so that the system contains  $n_1$  conservation laws. A typical illustration is the conservation of mass in gas dynamics.

The following result helps us in our endeavour of constructing the normal form for (1) (see [21] for the proof).

**Theorem 1.1** ([20] Theorem 1.1). Assume that the system (1) is entropy dissipative in the sense of Definition 1.2, that the total symbol satisfies Assumption A and that the  $n_1$  first rows are first-order conservation laws, i.e.

$$\partial_t u_i + \text{div} (L_i(u)) = 0 \quad \text{for all } i = 1, \dots, n_1.$$

Let the dual variables  $z_{n_1+1}, \dots, z_n$  be defined by

$$z_j = \frac{\partial \eta}{\partial u_j}.$$

Then the map

$$u = \begin{pmatrix} v \\ w \end{pmatrix} \Leftrightarrow U := \begin{pmatrix} v \\ z \end{pmatrix},$$

is a global diffeomorphism from  $\mathbb{U}$  onto its image  $\mathcal{U}$ , where  $v = (u, \dots, u_{n_1})^T$ . The viscous flux  $b(u) \nabla_x u$  can be written as  $Z(U) \nabla_x z$ . The tensor  $Z$  is uniquely defined and the operator  $Z(U) \nabla_x$  is strongly elliptic:

$$\sum_{\alpha, \beta=1}^d \sum_{i, j \geq p+1} \xi_\alpha \lambda_i \xi_\beta \lambda_j Z_{ij}^{\alpha\beta}(U) \geq c_1(U) |\xi|^2 |\lambda|^2, \quad \forall \xi \in \mathbb{R}^d, \lambda \in \mathbb{R}^{n-p}, \quad (3)$$

for some positive continuous function  $c_1(U)$ . Furthermore, if the system (1) is strongly entropy dissipative, then the tensor  $Z$  satisfies:

$$\langle Z(U)F, F \rangle \stackrel{\text{def}}{=} \sum_{\alpha, \beta=1}^d \sum_{j \geq p+1} F_{i\alpha} F_{j\beta} Z_{ij}^{\alpha\beta}(U) \geq c_0(U) \|Z(U)F\|^2, \quad \forall F \in \mathcal{M}_{(n-p) \times d}(\mathbb{R}), \quad (4)$$

for some positive continuous function  $c_0(U)$ .

## 1.2 Normal form

We are in position to give the normal form of (1). With the new variables, the system (1) becomes

$$\partial_t U + \sum_{\alpha} A^{\alpha}(U) \partial_{\alpha} U = \begin{pmatrix} 0 \\ D_{ww}^2 \eta \sum_{\alpha, \beta} \partial_{\alpha} (Z(U)^{\alpha\beta} \partial_{\beta} z) \end{pmatrix} \quad (5)$$

where  $A^{\alpha}(U) = (dU) dL^{\alpha} (dU)^{-1}$  and  $Z^{\alpha\beta}(U) = B^{\alpha\beta}(u) (dU)^{-1}$ .

The system (5) can be symmetrized by the diagonal by block, symmetric, positive definite matrix

$$S^0(U) = \begin{pmatrix} D_{vv}^2 \eta - D_{vw}^2 \eta (D_{ww}^2 \eta)^{-1} D_{wv}^2 & 0 \\ 0 & (D_{ww}^2)^{-1} \end{pmatrix}. \quad (6)$$

More precisely, setting  $S^{\alpha}(U) \stackrel{\text{def}}{=} S^0(U) A^{\alpha}(U)$ , the system (5) is equivalent to

$$S^0(U) \partial_t U + \sum_{\alpha} S^{\alpha}(U) \partial_{\alpha} U = \sum_{\alpha, \beta} \partial_{\alpha} (Y^{\alpha\beta}(U) \partial_{\beta} U). \quad (7)$$

In the sequel, we shall consider the following general system

$$S^0(U) \partial_t U + \sum_{\alpha} S^{\alpha}(U) \partial_{\alpha} U = \sum_{\alpha, \beta} \partial_{\alpha} (Y^{\alpha\beta}(U) \partial_{\beta} U) + f(U, \nabla U), \quad (8)$$

where  $f$  and the coefficients of the system (8) satisfy

### Assumptions B:

1. The matrix  $S^0(U)$  is symmetric, positive definite for every  $U \in \mathcal{U}$ , and has the form

$$S^0(U) = \begin{pmatrix} S_{11}^0(U) & 0 \\ 0 & S_{22}^0(U) \end{pmatrix} \quad \text{with } S_{11}^0(U) \in \mathcal{M}_{n_1}(\mathbb{R}) \quad \text{and } S_{22}^0(U) \in \mathcal{M}_{n-n_1}(\mathbb{R}).$$

2. The matrices  $S^{\alpha}(U)$  are symmetric, for every  $U \in \mathcal{U}$  and  $\alpha = 1, \dots, d$ .
3. For every  $U \in \mathcal{U}$  and  $\alpha, \beta = 1, \dots, d$ , the matrices  $Y^{\alpha\beta}(U)$  have the following form

$$Y^{\alpha\beta}(U) = \begin{pmatrix} 0_{n_1} & 0 \\ 0 & Z^{\alpha\beta}(U) \end{pmatrix} \quad \text{with } Z^{\alpha\beta}(U) \in \mathcal{M}_{n-n_1}(\mathbb{R}). \quad (9)$$

Moreover inequality (4) holds.

4. There exists some  $\bar{U} \in \mathcal{U}$  such that  $f$  satisfies  $f(\bar{U}, \cdot) = 0$  and may be written

$$f(U, \nabla U) \stackrel{\text{def}}{=} \begin{pmatrix} f^1(U, \nabla U) \\ f^2(U, \nabla U) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} f^1(U) \\ f^{21}(U) + f^{22}(U, \nabla U^1) + f^{23}(U, \nabla U^2) \end{pmatrix} \quad (10)$$

where  $f^1$ ,  $f^{21}$  and  $f^{22}$  are some smooth functions satisfying  $f^1(\bar{U}) = 0$ ,  $f^{21}(\bar{U}) = 0$ ,  $f^{22}(\bar{U}, 0) = f^{23}(\bar{U}, 0) = 0$ . The function  $f^{23}$  is quadratic in terms of  $\nabla U$  (i.e.  $f^{23}$  is a finite combination of terms of type  $v(U) \nabla U \otimes \nabla U$ , where  $v$  is a smooth function).

**Remark 1.1.** *The assumptions that have to be made on  $f$  depend on the regularity framework. For instance in [15], since the solution is more regular, there no structural restriction on  $f$ . Our result (Theorems 1.2) is still valid if we allow  $f^2$  to satisfy:  $f^2 = f^2(U, \nabla U)$  is at most quadratic in  $\nabla V^2$ , meaning that the third derivative with respect to  $\nabla V^2$  is zero.*

Let us set  $V \stackrel{\text{def}}{=} U - \bar{U}$ . Then  $V$  is solution of

$$S^0(U)\partial_t V + \sum_{\alpha} S^{\alpha}(U)\partial_{\alpha} V = \sum_{\alpha, \beta} \partial_{\alpha}(Y^{\alpha\beta}(U)\partial_{\beta} V) + f(U, \nabla U) \quad (11)$$

which can be rewritten as follows:

$$\begin{cases} S_{11}^0(U)\partial_t V^1 + \sum_{\alpha=1}^d (S_{11}^{\alpha}(U)\partial_{\alpha} V^1 + S_{12}^{\alpha}(U)\partial_{\alpha} V^2) = f^1(U, \nabla U) \\ S_{22}^0(U)\partial_t V^2 + \sum_{\alpha=1}^d (S_{21}^{\alpha}(U)\partial_{\alpha} V^1 + S_{22}^{\alpha}(U)\partial_{\alpha} V^2) = f^2(U, \nabla U) + \sum_{\alpha, \beta=1}^d \partial_{\alpha}(Z^{\alpha\beta}(U)\partial_{\beta} V^2) \end{cases} \quad (12)$$

with

$$S^0(U) = \begin{pmatrix} S_{11}^0(U) & 0 \\ 0 & S_{22}^0(U) \end{pmatrix} \quad \text{and} \quad S^{\alpha}(U) = \begin{pmatrix} S_{11}^{\alpha}(U) & S_{12}^{\alpha}(U) \\ S_{21}^{\alpha}(U) & S_{22}^{\alpha}(U) \end{pmatrix}. \quad (13)$$

The previous system can be regarded as a coupled system of a symmetric hyperbolic system for  $V^1$  and a parabolic system for  $V^2$ . Hence, for the initial value problem, the existence and the uniqueness of solutions local in time will be proved, based on this observation. This implies that, if the system (1) is entropy-dissipative and satisfies the assumption **A**, then it can be written in the normal form, and thus, the initial value problem for (1) is well posed.

**Remark 1.2.** *Contrary to the thesis [15] of S. Kawashima, we didn't make the assumption that the matrices  $S^0$  and the dissipation tensor in (11) are block-diagonal. It comes naturally from assumption **A** and the fact that the entropy  $\eta$  is dissipative.*

Furthermore, in [16] Kawashima and Shizuta proved that the symmetrizability of the system (1) can be characterized by the existence of an entropy function for (1). In addition, under a technical assumption (condition **N** in [16]) the authors obtained a normal form for (1) which is slightly different than what we obtained in (8). In fact, the difference lies on the fact that the component  $U^1$  remains unchanged when passing from (1) to (8). It is worth noting that D. Serre in [21] proved that condition **N** in [16] is equivalent to the assumption **A**. Let us underline that the notion of entropy for (1) (first introduced in [15]) used in [16] is completely different than what we defined in this paper.

As we will see below, unlike in [15] we don't need that the symbol  $B(\xi, u)$  is symmetric. Some local-in-time existence results will be achieved without this hypothesis.

### 1.3 Main results

The proofs of most of the results presented in this paper require a dyadic decomposition of Fourier variables, the so-called Littlewood-Paley decomposition that we recall in Appendix B, together with the definition of some functional spaces, such as Besov spaces. The reader is referred to [2] for more details.

According to System (12), it turns out that  $V^1$  verifies a hyperbolic system while  $V^2$  satisfies a parabolic system. One of the goals of this paper is to prove local in time existence of (12), based on the features of parabolic and hyperbolic systems. For that purpose, the hypothesis **B** on the matrices  $S^{\alpha}(U)$ , (for  $\alpha = 1, \dots, d$  and  $U \in \mathcal{U}$ ) can be weakened.

What we need is just the following:

**Assumptions BB:** Assume that for all  $\alpha, \beta = 1 \cdots, d$  and  $U \in \mathcal{U}$ ,

1. The matrix  $S^0(U)$  is symmetric, positive definite for every  $U \in \mathcal{U}$ , and is block diagonal,
2. The matrices  $S_{11}^\alpha(U)$  are symmetric,
3. the matrices  $Y^{\alpha\beta}(U)$  have the form (9) and inequality (4) holds true.
4.  $f$  has the form (10) .

Note that Condition **BB** is *weaker* than **B** since only the submatrix  $S_{11}^\alpha$  is required to be symmetric, not the whole matrix  $S^\alpha$ .

Before stating our first local existence result, let us motivate our functional framework. Since our general approach is based on energy estimates, we shall consider Besov spaces  $B_{p,r}^s$  built on  $L^2$ , that is  $p = 2$ . Furthermore, we shall restrict ourselves to the case  $r = 1$  since spaces of type  $B_{2,1}^s$  satisfy better properties than Sobolev spaces  $H^2 = B_{2,2}^s$  regarding embedding and parabolic maximal regularity. Typically, for the free heat equation, one can gain two full derivatives with respect to the regularity of the initial data after performing  $L^1$ -in-time integration (see Proposition 6), which is false for Sobolev spaces. Other good reasons for that choice will be explained throughout. In order to get  $L^\infty(0, T; \mathbb{R}^d)$  control on the functions  $S^0, S^\alpha, Y^{\alpha\beta}$  appearing in (11), it is suitable to work in spaces which will allow us to get  $L^\infty(0, T; \mathbb{R}^d)$  control of the unknown  $V$ . In the nonhomogeneous Besov spaces setting, this leads us to assume that the initial data  $V_0$  belongs to  $B_{2,1}^\theta$  with  $\theta \geq \frac{d}{2}$ . In order to guess what is the relevant solution space, we just use the fact that  $V^1$  is governed by a hyperbolic equation and that  $V^2$  may be seen as the solution to a parabolic equation with a source term. Then, starting from  $V_0^2 \in B_{2,1}^\theta$  (with  $\theta \geq \frac{d}{2}$ ) we expect for small time  $T$ , that  $V^2 \in C(0, T; B_{2,1}^\theta) \cap L^1(0, T; B_{2,1}^{\theta+2})$ , provided one can control the source term in  $L^1(0, T; B_{2,1}^\theta)$ , in particular  $S_{21}^\alpha(U) \partial_\alpha V^1$ . Hence, based on product laws, we will need that  $\nabla V^1 \in L^1(0, T; B_{2,1}^\theta)$ . Due to the fact that  $V^1$  is solution of a hyperbolic equation which ensures the conservation of the initial regularity but no gain of regularity, we thus have to assume, in addition, that  $V_0^1 \in B_{2,1}^{\theta+1}$ .

This motivates our first result, that can be stated as follows:

**Theorem 1.2.** *Let  $d \geq 1$  and  $s \geq \frac{d}{2}$ . Suppose that the partially diffusive hyperbolic system (11) (or equivalently (12)) satisfies assumption **BB**. If the initial data satisfies  $(V_0^1, V_0^2) \in B_{2,1}^{s+1} \times B_{2,1}^s$  and  $U_0 := V_0 + \bar{U}$  takes values in a bounded open subset  $\mathcal{O}_0$  of  $\mathcal{U}$  such that  $\bar{\mathcal{O}}_0 \subset \mathcal{U}$ , then there exists a time  $T_1 > 0$  depending only on the data and such that the following results hold true:*

**Existence:** *System (11) supplemented with the initial data  $V_0$  has a unique solution  $V = (V^1, V^2)$  in the class  $E_{T_1}$  defined by*

$$V^1 \in C([0, T_1]; B_{2,1}^{s+1}) \quad \text{and} \quad V^2 \in C([0, T_1]; B_{2,1}^s) \cap L_{T_1}^1(B_{2,1}^{s+2}).$$

*Moreover,  $U := V + \bar{U}$  belongs to a  $d_1$ -neighborhood of  $\mathcal{O}_0$  with  $d_1 < \text{dist}(\mathcal{O}_0, \partial\mathcal{U})$ .*

**Continuation criterion:** *If there exists  $T_2 > T_1$  such that  $V$  is defined on  $[0, T_2] \times \mathbb{R}^d$  and belongs to  $E_T$  for all  $T < T_2$ , and satisfies*

1.  $U$  belongs to an open bounded set  $\Omega$ , with  $\bar{\Omega} \in \mathcal{U}$ ,
2.  $\int_0^{T_2} \left( \|(\nabla V, \nabla^2 V^2)\|_{L^\infty} + \|\nabla V\|_{L^\infty}^2 \right) < \infty$ ,
3.  $\|\nabla V^1\|_{L^\infty([0, T_2] \times \mathbb{R}^d)} < \infty$ ,

then, there exists some  $T^* > T_2$  such that  $(V^1, V^2)$  may be continued on  $[0, T^*] \times \mathbb{R}^d$  to a solution of (11) which belongs to  $E_{T^*}$ .

Moreover, if the source term  $f^{22}(U, \nabla V^1)$  is quadratic in  $\nabla V^1$ , then Condition 3 is not needed.

**Remark 1.3.** Compared to the result of D. Serre in [20] and of that of Kawashima in [16], we here use different (and smaller) regularity indices for  $V^1$  and  $V^2$ . For instance, in the case  $d \geq 2$ , one may take data in  $B_{2,1}^{\frac{d}{2}+1} \times B_{2,1}^{\frac{d}{2}}$  instead of  $H^s$  for  $s > \frac{d}{2} + 1$  in Serre's work and  $s > 2 + \frac{d}{2}$  in Kawashima's work. Due to the embedding  $H^s \hookrightarrow B_{2,1}^{\frac{d}{2}+1} \hookrightarrow B_{2,1}^{\frac{d}{2}}$ , for  $s > \frac{d}{2} + 1$ , our regularity assumption is less than what D. Serre needed in [20]. In particular the component  $V_0^2$  can be taken in any space  $H^s$  with  $s > \frac{d}{2}$ . Finally, we recall that assumption **BB** does not require the symmetry of matrices  $Z^{\alpha\beta}$ .

**Remark 1.4.** If  $s > 1 + \frac{d}{2}$ , we have  $\partial_t V^1 \in C([0, T_1]; B_{2,1}^{s-1})$  while if  $s > 2 + \frac{d}{2}$ , we get  $\partial_t V^2 \in C([0, T_1]; B_{2,1}^{s-2})$ . Moreover as soon as  $s \geq 1 + \frac{d}{2}$  the Theorem 1.2 is still valid if we consider data  $(V_0^1, V_0^2)$  in  $B_{2,1}^s$ . We get then a unique solution  $(V^1, V^2)$  in the class

$$V^1 \in C([0, T_1]; B_{2,1}^s), \quad V^2 \in C([0, T_1]; B_{2,1}^s) \cap L_{T_1}^1(B_{2,1}^{s+2}) \quad \text{and} \quad \partial_t V \in L_{T_1}^1(B_{2,1}^s) \cap L_{T_1}^2(B_{2,1}^{s-1}).$$

**Remark 1.5.** Thanks to the explosion criterion, the lifetime is independent of  $s$ . We mean that if  $(V^1, V^2)$  is solution of (11) in  $C([0, T], B_{2,1}^{\frac{d}{2}+1} \times B_{2,1}^{\frac{d}{2}})$  with data  $(V_0^1, V_0^2) \in B_{2,1}^{s+1} \times B_{2,1}^s$ , for some  $s \geq \frac{d}{2}$ , then  $(V^1, V^2) \in C([0, T], B_{2,1}^{s'+1} \times B_{2,1}^{s'})$ , for all  $\frac{d}{2} \leq s' \leq s$ .

One may wonder whether the above statements extend to the so-called critical regularity setting as in the Navier-Stokes case. In fact, since the work of R. Danchin in [10], it is known that the barotropic compressible Navier-Stokes equations have a unique solution in the critical setting (which, here, amounts to assuming that the initial density and velocity respectively belong to  $B_{2,1}^{\frac{d}{2}}$  and  $B_{2,1}^{\frac{d}{2}-1}$ , see [10, 9]).

Our second local existence result consists in solving (11) in the critical regularity framework. For expository purpose, we here choose to work in the homogeneous framework. However, we have the same result in the nonhomogeneous framework.

To achieve the critical regularity framework, the following more restrictive structure conditions are needed:

**Assumption C.** For all  $\alpha, \beta = 1, \dots, d$  and  $U \in \mathcal{U}$ , we have

- (i)  $S_{22}^0$  is real symmetric positive definite and depends only on  $U^1$ ,
- (ii) the functions  $S_{21}^\alpha, S_{22}^\alpha$  are at most linear with respect to<sup>1</sup>  $U^2$ ,
- (iii) the functions  $(S_{11}^0)^{-1} S_{12}^\alpha$  depend only on  $U^1$  while  $(S_{11}^0)^{-1} S_{11}^\alpha$  are symmetric, depend only on  $U^2$  and are at most linear (i.e.  $D_{U^1}(S_{11}^0)^{-1} S_{11}^\alpha = 0$  and  $D_{U^2 U^2}^2 (S_{11}^0)^{-1} S_{11}^\alpha = 0$ ),
- (iv) the functions  $Z^{\alpha\beta}$  depend only on  $U^1$ .
- (v)  $f^1$  and  $f^2$  are functions of  $U$  only and satisfy  $f^1(\bar{U}) = 0$  and  $f^2(\bar{U}) = 0$ .

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<sup>1</sup>We say that a function  $k : (u, v) \mapsto k(u, v)$  is at most linear with respect to the variable  $v$  if  $D_{vv}^2 k = 0$ . In other words if  $k$  is a smooth function, there exist two functions  $k_1, k_2$  such that

$$k(u, v) = k_1(u)v + k_2(u). \tag{14}$$



Of course, according to Theorem 1.2, system (11) under the assumption **C**, supplemented with smooth initial data has a unique smooth solution. Our goal here is to obtain the same local result but with less regularity on the initial data.

Before stating our result let us introduce the following notation.

$$\mathcal{U}^1 = \{U^1 \in \mathbb{R}^p / \exists U^2 \in \mathbb{R}^{n-p}; U = (U^1, U^2) \in \mathcal{U}\}. \quad (15)$$

**Theorem 1.3.** *Let the structure assumptions **C** be in force and let  $O_0^1$  be a bounded open subset such that  $\overline{O_0^1} \subset \mathcal{U}^1$ . Let  $U_0 \in \mathcal{U}$  be an initial data such that  $U^1 \in O_0^1$ ,  $V_0^1 \in \dot{B}_{2,1}^{\frac{d}{2}}$  and  $V_0^2 \in \dot{B}_{2,1}^{\frac{d}{2}-1}$  with  $V_0 = U_0 - \overline{U}$ . Then, there exists a positive time  $T$  such that System (11) has a unique solution  $V$  with  $U = V + \overline{U}$  and  $U^1 \in O^1$ , where  $O^1$  is a  $d_1$ -neighbourhood of  $O_0^1$  with  $d_1 < \text{dist}(O_0^1, \partial\mathcal{U}^1)$ . Moreover  $V$  belongs to the class*

$$V^1 \in C([0, T]; \dot{B}_{2,1}^{\frac{d}{2}}), \quad V^2 \in C([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1}) \quad \text{and} \quad \partial_t V \in L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1}).$$

The rest of this paper unfolds as follows. In Sect. 2, we establish the local existence and continuation criterion for the system (11). Sect. 3 is devoted to the proof of our second local well-posedness result (Theorem 1.3). In appendix A, we set out some key results that will be of constant use in this article: maximal regularity of the linear parabolic equation, Gårding inequality, etc. In the last section (Sect. B), we briefly review some useful properties of Besov spaces.

Throughout this paper,  $(c_j)_{j \in \mathbb{Z}}$  stands for a positive sequence that verifies  $\|(c_j)\|_{l^1(\mathbb{Z})} = 1$ . Also,  $C$  designates a generic harmless constant, the value of which depends on the context.

## 2 Proof of Theorem 1.2

In this section, we prove the local existence of solutions to the symmetric partially diffusive system (11) subject to the following initial data

$$V|_{t=0} = V_0 \stackrel{\text{def}}{=} U_0 - \overline{U}, \quad (16)$$

in the general case where assumption **BB** is satisfied.

The proof of the local existence result will be based on an iterative scheme which consists in solving separately a (linear) hyperbolic equation and a (linear) fully parabolic equation. The two equations are coupled through lower order terms that will be ‘appropriate right-hand sides’. Taking advantage of product laws, composition properties (see Appendix) and the fact that  $s > \frac{d}{2}$  if  $d = 1$  and  $s \geq \frac{d}{2}$  if  $d \geq 2$ , it is easy to bound the sequence in the expected solution space on some fixed time interval  $[0, T]$  with  $T > 0$ . However, because the whole system is not fully parabolic, the strong convergence of the sequence is shown for a weaker norm corresponding to *a loss of one derivative*. The same restriction occurs as regards the uniqueness issue.

### 2.1 Existence for a suitable hyperbolic-parabolic linear system

Here we prove the local existence of solutions for a suitable linear system related to (11). We actually only keep the main order terms (the antisymmetric ones of order one for the first equation, and the second order ones for the second equation), and assume that the variable coefficients are ‘given’. More precisely, let  $Q_T \stackrel{\text{def}}{=} [0, T] \times \mathbb{R}^d$  ( $T$  is a positive constant). For  $\overline{U}$  in  $\mathcal{U}$  and  $U = \overline{U} + V$  with range in  $\mathcal{U}$  and  $\Theta_1, \Theta_2$ , some given functions on  $Q_T$ , we consider the following system:

$$\begin{cases} S_{11}^0(U) \partial_t \tilde{V}^1 + \sum_{\alpha=1}^d S_{11}^\alpha(U) \partial_\alpha \tilde{V}^1 = \Theta_1 \\ S_{22}^0(U) \partial_t \tilde{V}^2 - \sum_{\alpha, \beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U) \partial_\beta \tilde{V}^2) = \Theta_2 \end{cases} \quad (17)$$

supplemented with initial data  $\tilde{V}_0$ .

For the existence of solutions with high Sobolev regularity for the above linear system, the reader is referred to [15], [20] or [1].

We assume that for some  $s > -\frac{d}{2}$  and  $\sigma > -\frac{d}{2}$ , we have

$$V^1 \in \tilde{L}^\infty(0, T; B_{2,1}^\sigma) \quad \text{and} \quad \partial_t V^1 \in L^1(0, T; B_{2,1}^{\sigma-1}) \cap \tilde{L}^2(0, T; B_{2,1}^{\sigma-2}), \quad (18)$$

$$V^2 \in \tilde{L}^\infty(0, T; B_{2,1}^s) \cap L^1(0, T; B_{2,1}^{s+2}) \quad \text{and} \quad \partial_t V^2 \in L^1(0, T; B_{2,1}^s) \cap \tilde{L}^2(0, T; B_{2,1}^{s-1}), \quad (19)$$

$$\Theta_1 \in L^1(0, T; B_{2,1}^\sigma) \quad \text{and} \quad \Theta_2 \in L^1(0, T; B_{2,1}^s). \quad (20)$$

Assume also that, there exists  $\mathcal{O}$  a bounded open set in  $\mathbb{R}^d$  satisfying  $\overline{\mathcal{O}} \subset \mathcal{U}$  such that:

$$U(t, x) \in \mathcal{O} \quad \text{for all} \quad t \in [0, T], \quad x \in \mathbb{R}^d. \quad (21)$$

So, from (21), we have on the one hand: for any continuous function  $S : \mathbb{R}^n \mapsto \mathbb{R}$ , one can find a constant  $C = C(\overline{\mathcal{O}}, S)$ , such that

$$\|S(U)\|_{L^\infty(0, T; \mathbb{R}^d)} \leq C, \quad (22)$$

and on the other hand, as the matrix  $S^0(U)$  is symmetric positive definite for all  $U \in \mathcal{O}$ ,

$$C^{-1}I_n \leq S^0(U) \leq CI_n. \quad (23)$$

For  $\theta > -\frac{d}{2}$ , we set

$$\theta^{**} \stackrel{\text{def}}{=} \begin{cases} \frac{d}{2} & \text{if } \theta \leq 1 + \frac{d}{2} \\ \theta - 1 & \text{if } \theta > 1 + \frac{d}{2} \end{cases} \quad \text{and} \quad \theta^* = \begin{cases} \frac{d}{2} & \text{if } \theta < \frac{d}{2} \\ \theta & \text{if } \theta \geq \frac{d}{2}. \end{cases} \quad (24)$$

We first concentrate on the first equation of (17)<sub>1</sub>.

**Lemma 1** (Energy estimates for linearized hyperbolic equations). *Let us assume that  $S_{11}^0$  and  $S_{11}^\alpha$  (for  $\alpha = 1, \dots, d$ ) satisfy assumption **BB**. Suppose that  $V^1$  satisfies (18) with  $\sigma > -d/2$  and that  $U$  verifies (21). Assume also that  $\Theta_1$  verifies (20).*

*Then, there exists a constant  $C = C(\mathcal{O})$  such that for all  $t \in [0, T]$ , the following inequality holds*

$$\|\tilde{V}^1\|_{\tilde{L}_t^\infty(B_{2,1}^\sigma)} \leq e^{\Phi_1(t)} \left( \|\tilde{V}_0^1\|_{B_{2,1}^\sigma} + \int_0^t e^{-\Phi_1(\tau)} \|\Theta_1(\tau)\|_{B_{2,1}^\sigma} d\tau \right), \quad (25)$$

where  $\Phi_1(t) \stackrel{\text{def}}{=} C \int_0^t (\|(\partial_t V, \nabla V)\|_{L^\infty} + \|\nabla V\|_{B_{2,1}^{\sigma^{**}+1}})$ .

*Proof.* Applying the non-homogeneous dyadic block  $\Delta_j$  to the first equation of (17)<sub>1</sub> yields,

$$S_{11}^0(U) \partial_t \tilde{V}_j^1 + \sum_{\alpha=1}^d S_{11}^\alpha(U) \partial_\alpha \tilde{V}_j^1 = R_j^{11} + \Theta_{1,j},$$

where we define

$$\tilde{V}_j \stackrel{\text{def}}{=} \Delta_j \tilde{V}, \quad R_j^{11} \stackrel{\text{def}}{=} S_{11}^0(U) \sum_{\alpha=1}^d [(S_{11}^0(U))^{-1} S_{11}^\alpha(U), \Delta_j] (\partial_\alpha \tilde{V}^1),$$

and

$$\Theta_{1,j} \stackrel{\text{def}}{=} S_{11}^0(U) \sum_{\alpha=1}^d \Delta_j ((S_{11}^0(U))^{-1} \Theta_1).$$

Next, taking the scalar product in  $\mathbb{R}^{n_1}$  of this equation with  $V_j^1$ , integrating over the physical space along with integration by parts and using the symmetry properties of  $S_{11}^\alpha(U)$  gives:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} S_{11}^0(U) \tilde{V}_j^1 \cdot \tilde{V}_j^1 = \frac{1}{2} \int_{\mathbb{R}^d} (\partial_t(S_{11}^0(U)) + \partial_\alpha(S_{11}^\alpha(U))) \tilde{V}_j^1 \cdot \tilde{V}_j^1 + \int_{\mathbb{R}^d} (R_j^{11} + \Theta_{1,j}) \cdot \tilde{V}_j^1.$$

Cauchy-Schwarz inequality, inequalities (22) and (23) lead for some  $C = C(\mathcal{O})$ , to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} S_{11}^0(U) \tilde{V}_j^1 \cdot \tilde{V}_j^1 &\leq C \|(\partial_t V, \nabla V)\|_{L^\infty} \int_{\mathbb{R}^d} S_{11}^0(U) \tilde{V}_j^1 \cdot \tilde{V}_j^1 \\ &\quad + \|(R_j^{11}, \Theta_{1,j})\|_{L^2} \sqrt{\int_{\mathbb{R}^d} S_{11}^0(U) \tilde{V}_j^1 \cdot \tilde{V}_j^1}. \end{aligned}$$

Then, using Lemma 5 with  $X = \int_{\mathbb{R}^d} S_{11}^0(U) \tilde{V}_j^1 \cdot \tilde{V}_j^1$  and (23) again, one has

$$\|\tilde{V}_j^1\|_{L_T^\infty(L^2)} \leq \|\tilde{V}_{0,j}^1\|_{L^2} + C \int_0^T \|(\partial_t V, \nabla V)\|_{L^\infty} \|\tilde{V}_j^1\|_{L^2} + \|(R_j^{11}, \Theta_{1,j})\|_{L_T^1(L^2)} \quad (26)$$

whereafter we use the notations:

$$\tilde{V}_{0,j}^1 \stackrel{\text{def}}{=} \Delta_j \tilde{V}_0^1.$$

Let us now bound the terms  $R_j^{11}$  in  $L_T^1(L^2)$ . Using inequality (184) and Proposition 9, we obtain if  $\sigma \geq \frac{d}{2} + 1$ ,

$$\begin{aligned} \|R_j^{11}\|_{L^2} &\leq C c_j 2^{-j\sigma} \|\nabla ((S_{11}^0(U))^{-1} S_{11}^\alpha(U) - (S_{11}^0(\bar{U}))^{-1} S_{11}^\alpha(\bar{U}))\|_{L^\infty} \|\tilde{V}^1\|_{B_{2,1}^\sigma} \\ &\quad + C c_j 2^{-j\sigma} \|\nabla \tilde{V}^1\|_{L^\infty} \|\nabla ((S_{11}^0(U))^{-1} S_{11}^\alpha(U) - (S_{11}^0(\bar{U}))^{-1} S_{11}^\alpha(\bar{U}))\|_{B_{2,1}^{\sigma-1}}. \end{aligned} \quad (27)$$

Taking advantage of the embedding  $B_{2,1}^{\sigma-1} \hookrightarrow L^\infty$ , the previous inequality may be simplified as follows for all  $\sigma \geq \frac{d}{2} + 1$  for some  $C = C(\mathcal{O})$ :

$$\|R_j^{11}\|_{L^2} \leq C c_j 2^{-j\sigma} (\|\nabla V\|_{L^\infty} + \|V\|_{B_{2,1}^\sigma}) \|\tilde{V}^1\|_{B_{2,1}^\sigma}. \quad (28)$$

On the other side, combining (186), Proposition 10 and the embedding  $B_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty \cap B_{2,\infty}^{\frac{d}{2}}$ , we have for all  $-\frac{d}{2} < \sigma \leq \frac{d}{2} + 1$ ,

$$\|R_j^{11}\|_{L^2} \leq C c_j 2^{-j\sigma} \|V\|_{B_{2,1}^{\frac{d}{2}+1}} \|\tilde{V}^1\|_{B_{2,1}^\sigma}, \quad (29)$$

where the constant  $C$  depends on  $\mathcal{O}$ . Putting (28) (or (29)), into (26) gives us,

$$\begin{aligned} 2^{j\sigma} \|\tilde{V}_j^1\|_{L_T^\infty(L^2(\mathbb{R}^d))} &\leq 2^{j\sigma} \|\tilde{V}_{0,j}^1\|_{L^2} + C 2^{j\sigma} \int_0^T \|\tilde{V}_j^1\|_{L^2(\mathbb{R}^d)} \|(\partial_t V, \nabla V)\|_{L^\infty} \\ &\quad + C c_j \int_0^T \|V\|_{B_{2,1}^{\sigma^{**}+1}} \|\tilde{V}^1\|_{B_{2,1}^\sigma} + C 2^{j\sigma} \int_0^T \|\Theta_{1,j}\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

with  $\sigma^{**}$  defined in (24). Then, summing over  $j \geq -1$  and using Gronwall inequality gives us Inequality (25).  $\square$

Let us turn to the parabolic equation (17)<sub>2</sub>.

**Lemma 2** (Energy estimates for linearized parabolic equations). *Let us assume that  $S_{22}^0$  and  $Z^{\alpha\beta}$  (for  $\alpha, \beta = 1, \dots, d$ ) satisfy assumption **BB** and that  $\Theta_2$  verifies (20). Suppose also that  $V^2$  satisfies (19) with  $s > -d/2$  and that  $U$  verifies (21).*

*Then, there exist two constants  $c, C = C(\mathcal{O})$  such that for all  $t \in [0, T]$ , the following inequality holds*

$$\|\tilde{V}^2\|_{\tilde{L}_t^\infty(B_{2,1}^s)} + \frac{c}{2}\|\tilde{V}^2\|_{L_t^1(B_{2,1}^{s+2})} \leq e^{\Phi_2(t)} \left( \|\tilde{V}_0^2\|_{B_{2,1}^s} + \int_0^t e^{-\Phi_2(\tau)} \|\Theta_2(\tau)\|_{B_{2,1}^s} \right), \quad (30)$$

where

$$\begin{aligned} \Phi_2(t) \stackrel{\text{def}}{=} C \int_0^t \left( 1 + \|\partial_t V\|_{L^\infty} + (1 + \|V\|_{B_{2,1}^{s*}})^2 \left( \|V^1\|_{B_{2,1}^{s*} \cap B_{2,1}^{s**}}^2 \right. \right. \\ \left. \left. + \|V^2\|_{B_{2,1}^{s*}} \|V^2\|_{B_{2,1}^{s*+2}} + \|V^2\|_{B_{2,1}^{s**}} \|V^2\|_{B_{2,1}^{s**+2}} \right) \right). \end{aligned}$$

*Proof.* Applying  $\Delta_j$  to the second equation of (17) gives

$$S_{22}^0(U) \partial_t \tilde{V}_j^2 - \sum_{\alpha, \beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U) \partial_\beta \tilde{V}_j^2) = \Theta_{2,j} + R_j^2,$$

where we denote:

$$\begin{aligned} \Theta_{2,j} \stackrel{\text{def}}{=} S_{22}^0(U) \Delta_j ((S_{22}^0)^{-1}(U) \Theta_2), \quad \tilde{V}_j^2 = \Delta_j \tilde{V}^2, \\ R_j^2 \stackrel{\text{def}}{=} S_{22}^0(U) \left( \sum_{\alpha, \beta=1}^d [\Delta_j, (S_{22}^0)^{-1}(U) Z^{\alpha\beta}] (\partial_\alpha \partial_\beta \tilde{V}^2) \right) \\ + S_{22}^0(U) \sum_{\alpha, \beta=1}^d \Delta_j \left( (S_{22}^0)^{-1}(U) \partial_\alpha (Z^{\alpha\beta}(U)) \partial_\beta \tilde{V}^2 \right). \end{aligned}$$

Taking the  $L^2(\mathbb{R}^d)$  inner product of the above equation with  $\tilde{V}_j^2$ , we easily get, for  $j \geq -1$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} S_{22}^0(U) \tilde{V}_j^2 \cdot \tilde{V}_j^2 - \sum_{\alpha, \beta=1}^d \int_{\mathbb{R}^d} Z^{\alpha\beta}(U) \partial_\alpha \partial_\beta \tilde{V}_j^2 \cdot \tilde{V}_j^2 = \frac{1}{2} \int_{\mathbb{R}^d} (\partial_t (S_{22}^0(U))) \tilde{V}_j^2 \cdot \tilde{V}_j^2 \\ + \int_{\mathbb{R}^d} (R_j^2 + \Theta_{2,j}) \cdot \tilde{V}_j^2. \quad (31) \end{aligned}$$

Note that, under the condition (3) we have by making use of Lemma 6, for any  $j \geq 0$ ,

$$- \sum_{\alpha, \beta=1}^d \int_{\mathbb{R}^d} Z^{\alpha\beta}(U) \partial_\alpha \partial_\beta \tilde{V}_j^2 \cdot \tilde{V}_j^2 \geq c \left\| \nabla \tilde{V}_j^2 \right\|_{L^2}^2 - \varepsilon_c \left\| \nabla^2 \tilde{V}_j^2 \right\|_{L^2} \left\| \tilde{V}_j^2 \right\|_{L^2} - C(\varepsilon_c, \mathcal{O}) \left\| \tilde{V}_j^2 \right\|_{L^2(\mathbb{R}^d)}^2$$

where  $c$  is positive constant depending on  $\mathcal{O}$  and  $\varepsilon_c > 0$  may be chosen as small as we want. Owing to Bernstein inequality and choosing  $\varepsilon_c$  small enough, we deduce on the one hand

$$- \sum_{\alpha, \beta=1}^d \int_{\mathbb{R}^d} Z^{\alpha\beta}(U) \partial_\alpha \partial_\beta \tilde{V}_j^2 \cdot \tilde{V}_j^2 \geq 2^{2j} \frac{c}{2} \left\| \tilde{V}_j^2 \right\|_{L^2(\mathbb{R}^d)}^2 - C \left\| \tilde{V}_j^2 \right\|_{L^2(\mathbb{R}^d)}^2, \quad \text{for all } j \geq 0. \quad (32)$$

On the other hand, Bernstein inequality ensures that

$$\sum_{\alpha, \beta=1}^d \int_{\mathbb{R}^d} Z^{\alpha\beta}(U) \partial_\alpha \partial_\beta \tilde{V}_{-1}^2 \cdot \tilde{V}_{-1}^2 \leq C \left\| \nabla^2 \tilde{V}_{-1}^2 \right\|_{L^2} \left\| \tilde{V}_{-1}^2 \right\|_{L^2} \leq C \left\| \tilde{V}_{-1}^2 \right\|_{L^2}^2, \quad (33)$$

with  $C > 0$  depending on  $Z$ .

Then, from (22) (23) and using (32), (33), inequality (31) becomes for all  $j \geq -1$ :

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} S_{22}^0(U) \tilde{V}_j^2 \cdot \tilde{V}_j^2 + c2^{2j} S_{22}^0(U) \tilde{V}_j^2 \cdot \tilde{V}_j^2 \leq C(1 + \|\partial_t V\|_{L^\infty}) \int_{\mathbb{R}^d} S_{22}^0(U) \tilde{V}_j^2 \cdot \tilde{V}_j^2 \\ + C \|(R_j^2, \Theta_{2,j})\|_{L^2} \sqrt{\int_{\mathbb{R}^d} S_{22}^0(U) \tilde{V}_j^2 \cdot \tilde{V}_j^2}. \end{aligned}$$

So taking  $X = \int S_{22}^0(U) \tilde{V}_j^2 \cdot \tilde{V}_j^2$  in Lemma 5 and using again (23), one has for all  $t \in [0, T]$  and  $j \geq -1$ ,

$$\begin{aligned} 2^{js} \left\| \tilde{V}_j^2(t) \right\|_{L^2} + 2^{(s+2)j} c \int_0^t \left\| \tilde{V}_j^2 \right\|_{L^2} \leq \left\| \tilde{V}_{0,j}^2 \right\|_{L^2} + C2^{js} \|\tilde{V}_j^2\|_{L_t^\infty(L^2(\mathbb{R}^d))} \int_0^t (1 + \|\partial_t V\|_{L^\infty}) \\ + 2^{js} \int_0^t \|(R_j^2, \Theta_{2,j})\|_{L^2}, \quad (34) \end{aligned}$$

where we used the notation  $\tilde{V}_{0,j}^2 \stackrel{\text{def}}{=} \Delta_j \tilde{V}_0^2$ . Owing to (22), we have for some  $C = C(\mathcal{O})$ ,

$$\begin{aligned} \|R_j^2\|_{L_T^1(L^2(\mathbb{R}^d))} \leq C \sum_{\alpha, \beta=1}^d \left\| [\Delta_j, (S_{22}^0)^{-1}(U) Z^{\alpha\beta}] (\partial_\alpha \partial_\beta \tilde{V}^2) \right\|_{L_T^1(L^2(\mathbb{R}^d))} \\ + C \sum_{\alpha, \beta=1}^d \left\| \Delta_j (S_{22}^0)^{-1}(U) \partial_\alpha (Z^{\alpha\beta}(U)) \partial_\beta \tilde{V}^2 \right\|_{L_T^1(L^2(\mathbb{R}^d))}. \end{aligned}$$

According to Proposition 9 we know that for all  $\theta > -\frac{d}{2}$  (see (24) for the definition of  $\theta^*$ )

$$\|ab\|_{B_{2,1}^\theta} \leq C \|a\|_{B_{2,1}^{\theta^*}} \|b\|_{B_{2,1}^\theta}. \quad (35)$$

Taking  $\theta = s > -\frac{d}{2}$  and combining with Proposition 10, non homogeneous Besov embeddings give us: for all  $\alpha, \beta$

$$\left\| (S_{22}^0(U))^{-1} \partial_\alpha (Z^{\alpha\beta}(U)) \partial_\beta \tilde{V}^2 \right\|_{L_T^1(B_{2,1}^{s+1})} \leq C(1 + \|V\|_{B_{2,1}^{s+1}}) \int_0^T \|V\|_{B_{2,1}^{s+1}} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s+1}}.$$

Next, with the aid of inequality (184) and Proposition 10 one obtains: for  $s \geq \frac{d}{2} + 1$

$$\begin{aligned} \sum_{j \geq -1} \sum_{\alpha, \beta=1}^d 2^{js} \left\| [\Delta_j, (S_{22}^0)^{-1}(U) Z^{\alpha\beta}(U)] \partial_\alpha \partial_\beta \tilde{V}^2 \right\|_{L_T^1(L^2)} \leq C \int_0^T \left( \|\nabla V\|_{L^\infty} \left\| \nabla \tilde{V}^2 \right\|_{B_{2,1}^s} \right. \\ \left. + \left\| \nabla^2 \tilde{V}^2 \right\|_{L^\infty} \|V\|_{B_{2,1}^s} \right). \quad (36) \end{aligned}$$

The previous inequality may be simplified by using Besov embedding. We have

$$\sum_{j \geq -1} \sum_{\alpha, \beta=1}^d 2^{js} \left\| [\Delta_j, (S_{22}^0)^{-1}(U) Z^{\alpha\beta}(U)] \partial_\alpha \partial_\beta \tilde{V}^2 \right\|_{L_T^1(L^2)} \leq C \int_0^T \|\nabla V\|_{B_{2,1}^{s-1}} \left\| \nabla \tilde{V}^2 \right\|_{B_{2,1}^s}. \quad (37)$$

On the other side, for  $-\frac{d}{2} < s \leq \frac{d}{2} + 1$ , the inequality (186) combined with the embedding  $B_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty \cap B_{2,\infty}^{\frac{d}{2}}$  provides, for some  $C = C(\mathcal{O})$ ,

$$\sum_{j \geq -1} \sum_{\alpha, \beta=1}^d 2^{js} \left\| [\Delta_j, (S_{22}^0)^{-1}(U) Z^{\alpha\beta}(U)] \partial_\alpha \partial_\beta \tilde{V}^2 \right\|_{L_T^1(L^2)} \leq C \int_0^T \|V\|_{B_{2,1}^{\frac{d}{2}+1}} \left\| \nabla \tilde{V}^2 \right\|_{B_{2,1}^s}. \quad (38)$$

Putting all this information into (34), summing the obtained inequality over  $j \geq -1$  implies:

$$\begin{aligned} & \left\| \tilde{V}^2 \right\|_{\tilde{L}_T^\infty(B_{2,1}^{s_1})} + c \left\| \tilde{V}^2 \right\|_{L_T^1(B_{2,1}^{s_1+2})} \leq \left\| \tilde{V}_0^2 \right\|_{B_{2,1}^{s_1}} + C \int_0^T (1 + \|\partial_t V\|_{L^\infty}) \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1}} \\ & + C \int_0^T (1 + \|V\|_{B_{2,1}^{s_1}^*}) \|V\|_{B_{2,1}^{s_1+1}^*} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1+1}} + C \int_0^T \|V\|_{B_{2,1}^{s_1+1}^*} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1+1}} + \int_0^T \|\Theta_2\|_{B_{2,1}^{s_1}}. \end{aligned} \quad (39)$$

Using interpolation inequality combined with Young's inequality yields

$$\begin{aligned} (1 + \|V\|_{B_{2,1}^{s_1}^*}) \|V\|_{B_{2,1}^{s_1+1}^*} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1+1}} & \leq (1 + \|V\|_{B_{2,1}^{s_1}^*}) \left( \|V^1\|_{B_{2,1}^{s_1+1}^*} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1+1}} + \|V^2\|_{B_{2,1}^{s_1+1}^*} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1+1}} \right) \\ & \leq (1 + \|V\|_{B_{2,1}^{s_1}^*}) \left( \|V^1\|_{B_{2,1}^{s_1+1}^*} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1}}^{\frac{1}{2}} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1+2}}^{\frac{1}{2}} + \|V^2\|_{B_{2,1}^{s_1+1}^*}^{\frac{1}{2}} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1}}^{\frac{1}{2}} \|V^2\|_{B_{2,1}^{s_1+2}^*}^{\frac{1}{2}} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1+2}}^{\frac{1}{2}} \right) \\ & \leq \frac{c}{2} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1+2}} + C(1 + \|V\|_{B_{2,1}^{s_1}^*})^2 \left( \|V^1\|_{B_{2,1}^{s_1+1}^*}^2 + \|V^2\|_{B_{2,1}^{s_1}^*} \|V^2\|_{B_{2,1}^{s_1+2}^*} \right) \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1}}. \end{aligned} \quad (40)$$

Similarly, we have

$$\|V\|_{B_{2,1}^{s_1+1}^*} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1+1}} \leq \frac{c}{2} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1+2}} + C \left( \|V^1\|_{B_{2,1}^{s_1+1}^*}^2 + \|V^2\|_{B_{2,1}^{s_1}^*} \|V^2\|_{B_{2,1}^{s_1+2}^*} \right) \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1}}. \quad (41)$$

Plugging now, the inequality (40) into (39), we arrive at

$$\begin{aligned} & \left\| \tilde{V}^2 \right\|_{\tilde{L}_T^\infty(B_{2,1}^{s_1})} + \frac{c}{2} \left\| \tilde{V}^2 \right\|_{L_T^1(B_{2,1}^{s_1+2})} \leq \left\| \tilde{V}_0^2 \right\|_{B_{2,1}^{s_1}} + C \int_0^T \left( 1 + \|\partial_t V\|_{L^\infty} + (1 + \|V\|_{B_{2,1}^{s_1}^*})^2 \left( \|V^1\|_{B_{2,1}^{s_1}^*}^2 \right. \right. \\ & \left. \left. + \|V^1\|_{B_{2,1}^{s_1+1}^*}^2 + \|V^2\|_{B_{2,1}^{s_1}^*} \|V^2\|_{B_{2,1}^{s_1+2}^*} + \|V^2\|_{B_{2,1}^{s_1}^*} \|V^2\|_{B_{2,1}^{s_1+2}^*} \right) \right) \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s_1}} + \int_0^T \|\Theta_2\|_{B_{2,1}^{s_1}}, \end{aligned}$$

and the desired estimate follows from Gronwall inequality.  $\square$

To bound  $\partial_t \tilde{V}^1$  and  $\partial_t \tilde{V}^2$ , we need the following lemma.

**Lemma 3** (Estimates of  $\partial_t V$  for linearized equations). *Assume that the functions  $\Theta_1$  and  $\Theta_2$  are in  $L^1(0; T, B_{2,1}^\gamma)$ , for  $\gamma > -\frac{d}{2}$ . Then, the following inequality holds true*

$$\begin{aligned} & \int_0^T \left\| (\partial_t \tilde{V}^1, \partial_t \tilde{V}^2) \right\|_{B_{2,1}^\gamma} \leq C(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{s_1}^*)}) \left( T \left\| \tilde{V}^1 \right\|_{\tilde{L}_T^\infty(B_{2,1}^{\gamma+1})} \right. \\ & \left. + \int_0^T \|V\|_{B_{2,1}^{\gamma+1}^*} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{\gamma+1}} + \int_0^T \left\| \tilde{V}^2 \right\|_{B_{2,1}^{\gamma+2}} + \int_0^T \left( \|\Theta_1\|_{B_{2,1}^\gamma} + \|\Theta_2\|_{B_{2,1}^\gamma} \right) \right). \end{aligned} \quad (42)$$

Moreover, if  $\Theta_1, \Theta_2 \in \tilde{L}^2(0; T, B_{2,1}^\gamma)$  for  $\gamma > -\frac{d}{2}$ , then

$$\begin{aligned} & \left\| (\partial_t \tilde{V}^1, \partial_t \tilde{V}^2) \right\|_{\tilde{L}_T^2(B_{2,1}^\gamma)} \leq C(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{s_1}^*)}) \left( \sqrt{T} \left\| \tilde{V}^1 \right\|_{\tilde{L}_T^\infty(B_{2,1}^{\gamma+1})} \right. \\ & \left. + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{\gamma+1}^*)} \left\| \tilde{V}^2 \right\|_{\tilde{L}_T^2(B_{2,1}^{\gamma+1})} + \left\| \tilde{V}^2 \right\|_{\tilde{L}_T^2(B_{2,1}^{\gamma+2})} + \|(\Theta_1, \Theta_2)\|_{\tilde{L}_T^2(B_{2,1}^\gamma)} \right). \end{aligned} \quad (43)$$

*Proof.* We recall that :

$$\partial_t \tilde{V}^1 = -(S_{11}^0(U))^{-1} \sum_{\alpha=1}^d S_{11}^\alpha(U) \partial_\alpha \tilde{V}^1 + (S_{11}^0(U))^{-1} \Theta_1. \quad (44)$$

So, we have for all  $\gamma > -d/2$ , using (35)

$$\left\| \partial_t \tilde{V}^1 \right\|_{B_{2,1}^\gamma} \leq (1 + \|V\|_{B_{2,1}^{\gamma^*}}) \left( \left\| \tilde{V}^1 \right\|_{B_{2,1}^{\gamma+1}} + \|\Theta_1\|_{B_{2,1}^\gamma} \right). \quad (45)$$

In the same spirit, we want to express  $\partial_t \tilde{V}^2$  in  $L^1(B_{2,1}^s)$  that verifies the following equation:

$$\partial_t \tilde{V}^2 = (S_{22}^0(U))^{-1} \sum_{\alpha, \beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U) \partial_\beta \tilde{V}^2) + (S_{22}^0(U))^{-1} \Theta_2. \quad (46)$$

The only thing that changes is the diffusive term which can be treated using the product inequality (35) of Proposition 9 and the decomposition

$$(S_{22}^0(U))^{-1} \partial_\alpha (Z^{\alpha\beta}(U) \partial_\beta \tilde{V}^2) = (S_{22}^0(U))^{-1} \partial_\alpha (Z^{\alpha\beta}(U)) \partial_\beta \tilde{V}^2 + (S_{22}^0(U))^{-1} Z^{\alpha\beta}(U) \partial_\alpha \partial_\beta \tilde{V}^2. \quad (47)$$

We have for all  $\gamma > -\frac{d}{2}$

$$\begin{aligned} \left\| (S_{22}^0(U))^{-1} \partial_\alpha (Z^{\alpha\beta}(U) \partial_\beta \tilde{V}^2) \right\|_{B_{2,1}^\gamma} &\leq C(1 + \|V\|_{B_{2,1}^{\gamma^*}}) \|V\|_{B_{2,1}^{\gamma^*+1}} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{\gamma+1}} \\ &\quad + C(1 + \|V\|_{B_{2,1}^{\gamma^*}}) \left\| \tilde{V}^2 \right\|_{B_{2,1}^{\gamma+2}}, \end{aligned}$$

which, combined with (45), concludes the proof of (42).

Finally, to prove (43), we use again (35), Proposition 10 and inequalities (44), (46) and (47). We then obtain, for all  $\alpha, \beta$

- $\left\| (S_{11}^0(U))^{-1} S_{11}^\alpha(U) \partial_\alpha \tilde{V}^1 \right\|_{\tilde{L}_T^2(B_{2,1}^\gamma)} \leq C(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{\gamma^*})}) \sqrt{T} \left\| \nabla \tilde{V}^1 \right\|_{\tilde{L}_T^\infty(B_{2,1}^\gamma)},$
- $\left\| (S_{22}^0(U))^{-1} \partial_\alpha (Z^{\alpha\beta}(U)) \partial_\beta \tilde{V}^2 \right\|_{\tilde{L}_T^2(B_{2,1}^\gamma)} \leq C(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{\gamma^*})}) \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{\gamma^*+1})} \left\| \tilde{V}^2 \right\|_{\tilde{L}_T^2(B_{2,1}^{\gamma+1})},$
- $\left\| (S_{22}^0(U))^{-1} Z^{\alpha\beta}(U) \partial_\alpha \partial_\beta \tilde{V}^2 \right\|_{\tilde{L}_T^2(B_{2,1}^\gamma)} \leq C(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{\gamma^*})}) \left\| \tilde{V}^2 \right\|_{\tilde{L}_T^2(B_{2,1}^{\gamma+2})},$
- $\left\| (S_{11}^0(U))^{-1} \Theta_1 \right\|_{\tilde{L}_T^2(B_{2,1}^\gamma)} \leq C(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{\gamma^*})}) \|\Theta_1\|_{\tilde{L}_T^2(B_{2,1}^\gamma)},$
- $\left\| (S_{22}^0(U))^{-1} \Theta_2 \right\|_{\tilde{L}_T^2(B_{2,1}^\gamma)} \leq C(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{\gamma^*})}) \|\Theta_2\|_{\tilde{L}_T^2(B_{2,1}^\gamma)}.$

Putting together these inequalities yields (43), which completes the proof of Lemma 3.  $\square$

## 2.2 Local Existence

First, for given smooth functions  $U$  with range in  $\mathcal{U}$  and  $V_0$ , we shall consider the following linear equations with variable coefficients from (11):

$$\begin{cases} S_{11}^0(U) \partial_t \tilde{V}^1 + \sum_{\alpha=1}^d S_{11}^\alpha(U) \partial_\alpha \tilde{V}^1 = \Theta_1 \\ S_{22}^0(U) \partial_t \tilde{V}^2 - \sum_{\alpha, \beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U) \partial_\beta \tilde{V}^2) = \Theta_2 \end{cases} \quad (48)$$

with initial data

$$\tilde{V}|_{t=0} = \tilde{V}_0, \quad (49)$$

and (see Assumption **BB** for the definitions of  $f^1$  and  $f^2$ )

$$\begin{aligned}\Theta_1 &\stackrel{\text{def}}{=} f^1(U, \nabla U) - \sum_{\alpha=1}^d S_{12}^\alpha(U) \partial_\alpha V^2, \\ \Theta_2 &\stackrel{\text{def}}{=} f^2(U, \nabla U) - \sum_{\alpha=1}^d (S_{21}^\alpha(U) \partial_\alpha V^1 + S_{22}^\alpha(U) \partial_\alpha V^2).\end{aligned}\tag{50}$$

Let us set

$$\tilde{U} = \tilde{V} + \bar{U}.\tag{51}$$

Let  $s \geq \frac{d}{2}$  and  $\sigma = s + 1$ . We assume that

$$U_0(x) = V_0 + \bar{V}(x) \in \mathcal{O}_0 \quad \text{for any } x \in \mathbb{R}^d,\tag{52}$$

where  $\mathcal{O}_0$  is a bounded open set in  $\mathbb{R}^n$  satisfying  $\bar{\mathcal{O}}_0 \subset \mathcal{U}$ .

Furthermore, we assume that  $V$  satisfies

$$(\mathcal{H}1): \quad V^1 \in \tilde{L}^\infty(0, T; B_{2,1}^\sigma) \quad \text{and} \quad \partial_t V^1 \in L^1(0, T; B_{2,1}^{\sigma-1}) \cap \tilde{L}^2(0, T; B_{2,1}^{\sigma-2}),$$

$$(\mathcal{H}2): \quad V^2 \in \tilde{L}^\infty(0, T; B_{2,1}^s) \cap L^1(0, T; B_{2,1}^{s+2}) \quad \text{and} \quad \partial_t V^2 \in L^1(0, T; B_{2,1}^s) \cap \tilde{L}^2(0, T; B_{2,1}^{s-1}),$$

( $\mathcal{H}3$ ): there exists a bounded open set in  $\mathbb{R}^d$  satisfying  $U \subset \bar{\mathcal{O}}$  such that:

$$U(t, x) \in \bar{\mathcal{O}} \quad \text{for } t \in [0, T], \quad x \in \mathbb{R}^d.\tag{53}$$

We set:

$$M_1 \stackrel{\text{def}}{=} \|V^1\|_{\tilde{L}_T^\infty(B_{2,1}^\sigma)} + \|V^2\|_{\tilde{L}_T^\infty(B_{2,1}^s)} \quad \text{and} \quad M_2 \stackrel{\text{def}}{=} \|V^2\|_{L_T^1(B_{2,1}^{s+2})}.\tag{54}$$

From (53), one can deduce that for any continuous function  $S : \mathbb{R}^n \mapsto \mathbb{R}$ , there exists a constant  $C = C(\bar{\mathcal{O}}, S)$  such that:

$$\|S(U)\|_{L^\infty(0, T; \mathbb{R}^d)} \leq C.\tag{55}$$

We aim at proving local in time a priori estimates of (48) with (49) in the space  $E_T^\sigma$  defined by

$$E_T^\sigma = \left\{ V \in \tilde{C}(0, T; B_{2,1}^\sigma \times B_{2,1}^s) : \text{conditions } (\mathcal{H}1) - (\mathcal{H}3) \text{ are satisfied} \right\}.\tag{56}$$

In other words, we are going to prove that the set  $E_T^\sigma$  is invariant under the mapping defined by  $V \mapsto \tilde{V}$  with  $\tilde{V}$  satisfying (48), (49). Lemmas 1, 2 and 3 will come into play. To state it more precisely, we need to control  $\Theta_1$  and  $\Theta_2$ , defined in (50), in  $L_T^1(B_{2,1}^\sigma) \cap \tilde{L}_T^2(B_{2,1}^{s-1})$  and  $L_T^1(B_{2,1}^s) \cap \tilde{L}_T^2(B_{2,1}^{s-1})$ , respectively.

Thanks to inequality (188) of Proposition 9 and Proposition 10, we have for all  $\alpha = 1, \dots, d$  and all  $\theta \geq \frac{d}{2}$ ,

$$\begin{aligned}\|S_{12}^\alpha(U) \partial_\alpha V^2\|_{L_T^1(B_{2,1}^\sigma)} &\leq C \int_0^T \left( (1 + \|V\|_{B_{2,1}^\theta}) \|V^2\|_{B_{2,1}^{\frac{d}{2}+1}} + (1 + \|V\|_{B_{2,1}^{\frac{d}{2}}}) \|V^2\|_{B_{2,1}^{\theta+1}} \right) \\ &\leq C(1 + M_1) \|V^2\|_{L_T^1(B_{2,1}^{\theta+1})}.\end{aligned}$$

We deduce that (remember the form of  $f^1(U)$ , see (10))

$$\begin{aligned}\|\Theta_1\|_{L_T^1(B_{2,1}^\sigma)} &\leq C \|V\|_{L_T^1(B_{2,1}^\sigma)} + C(1 + M_1) \|V^2\|_{L_T^1(B_{2,1}^{s+2})} \\ &\leq CT \|V^1\|_{L_T^\infty(B_{2,1}^\sigma)} + \sqrt{T} \|V^2\|_{L_T^2(B_{2,1}^\sigma)} + C(1 + M_1) \|V^2\|_{L_T^1(B_{2,1}^{s+2})} \\ &\leq C(1 + M_1)(T + M_2) + \sqrt{TM_1 M_2},\end{aligned}\tag{57}$$

where in the last inequality we used an interpolation inequality.



To control  $\Theta_1$  in  $\tilde{L}_T^2(B_{2,1}^{s-1})$ , we use inequality (197) for  $f^1$  and (35) (with  $\theta = s-1$ ) combined with (192) for the others terms. One has

$$\begin{aligned} \|\Theta_1\|_{\tilde{L}_T^2(B_{2,1}^{s-1})} &\leq C\sqrt{T}(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{\frac{d}{2}})}) \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{s-1})} + C(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{(s-1)^*})})\sqrt{T} \|V^2\|_{L_T^2(B_{2,1}^s)} \\ &\leq C\sqrt{T}(1 + M_1)(M_1 + \sqrt{M_1M_2}). \end{aligned} \quad (58)$$

Next, the inequality (188) together with interpolation inequality ensures that

$$\begin{aligned} \|f^{21}(U)\|_{L_T^1(B_{2,1}^s)} &\leq C \|V\|_{L_T^1(B_{2,1}^s)} \leq CTM_1 \\ \|f^{22}(U)\|_{L_T^1(B_{2,1}^s)} &\leq C \|(V, \nabla V^1)\|_{L_T^1(B_{2,1}^s)} \leq CTM_1, \end{aligned}$$

and for all  $\alpha = 1, \dots, d$

$$\begin{aligned} \|S_{21}^\alpha(U)\partial_\alpha V^1\|_{L_T^1(B_{2,1}^s)} &\leq CT(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^s)}) \|V^1\|_{\tilde{L}_T^\infty(B_{2,1}^{s+1})} \leq CT(1 + M_1)M_1, \\ \|S_{22}^\alpha(U)\partial_\alpha V^2\|_{L_T^1(B_{2,1}^s)} &\leq C\sqrt{T}(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^s)}) \|V^2\|_{\tilde{L}_T^2(B_{2,1}^{s+1})} \\ &\leq C\sqrt{T}(1 + M_1)\sqrt{M_1M_2} \end{aligned}$$

so that

$$\|\Theta_2\|_{L_T^1(B_{2,1}^s)} \leq C(1 + M_1)^2(T + \sqrt{TM_2}) + \|f^{23}\|_{L_T^1(B_{2,1}^s)}. \quad (59)$$

Finally, using (again) (197) and the fact that  $(s-1)^* \leq s$  yields

- $\|f^{21}\|_{\tilde{L}_T^2(B_{2,1}^{s-1})} \leq \sqrt{T}C(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{(s-1)^*})}) \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{s-1})} \leq \sqrt{T}C(1 + M_1)M_1$
- $\|f^{22}\|_{\tilde{L}_T^2(B_{2,1}^{s-1})} \leq \sqrt{T}C(1 + \|(V, \nabla V^1)\|_{\tilde{L}_T^\infty(B_{2,1}^{(s-1)^*})}) \|(V, \nabla V^1)\|_{\tilde{L}_T^\infty(B_{2,1}^{s-1})} \leq \sqrt{T}C(1 + M_1)M_1,$

whereas inequalities (35) and (192) provide for all  $\alpha = 1, \dots, d$

- $\|f^{23}\|_{\tilde{L}_T^2(B_{2,1}^{s-1})} \leq (1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{(s-1)^*})}) \|\nabla V\|_{\tilde{L}_T^2(B_{2,1}^{(s-1)^*})} \|\nabla V\|_{\tilde{L}_T^\infty(B_{2,1}^{s-1})} \\ \leq C(1 + M_1)(\sqrt{T}M_1 + \sqrt{M_1M_2})M_1,$
- $\|S_{21}^\alpha(U)\partial_\alpha V^1\|_{\tilde{L}_T^2(B_{2,1}^{s-1})} \leq C\sqrt{T}(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{(s-1)^*})}) \|\nabla V^1\|_{\tilde{L}_T^\infty(B_{2,1}^{s-1})} \leq C\sqrt{T}(1 + M_1)M_1$
- $\|S_{22}^\alpha(U)\partial_\alpha V^2\|_{\tilde{L}_T^2(B_{2,1}^{s-1})} \leq C\sqrt{T}(1 + \|V\|_{\tilde{L}_T^\infty(B_{2,1}^{(s-1)^*})}) \|\nabla V^2\|_{\tilde{L}_T^\infty(B_{2,1}^{s-1})} \leq C\sqrt{T}(1 + M_1)M_1.$

We thus obtain

$$\|\Theta_2\|_{L_T^2(B_{2,1}^{s-1})} \leq C(1 + M_1)M_1\sqrt{M_1M_2} + C\sqrt{T}(1 + M_1)M_1. \quad (60)$$

**Lemma 4** (Invariant set under iterations). *Let  $d \geq 1$ ,  $s \geq d/2$  and  $\sigma = s+1$ . Suppose that the initial data satisfy  $(V_0^1, V_0^2) \in B_{2,1}^\sigma \times B_{2,1}^s$  and (52). Then, there exists a time  $T_0 > 0$  depending only on  $d, \mathcal{O}_0$  and on the data, such that if  $V \in E_{T_0}^\sigma$ , the unique solution  $\tilde{V}$  of the Cauchy problem (48)-(49) belongs to the same space  $E_{T_0}^\sigma$ .*

*Proof.* Here, for  $U = (U^1, U^2)$  a given smooth function on  $Q_T$ , we assume that, setting  $V = U - \bar{U}$ ,

$$\text{(H1)} \quad \|V^1\|_{\tilde{L}_T^\infty(B_{2,1}^\sigma)} \leq 4 \|V_0^1\|_{B_{2,1}^\sigma},$$

$$\text{(H2)} \quad \|V^2\|_{\tilde{L}_T^\infty(B_{2,1}^s)} \leq 2 \|V_0^1\|_{B_{2,1}^s},$$

$$\text{(H3)} \quad \int_0^T \|V^2\|_{B_{2,1}^{s+2}} \leq \eta,$$

$$\text{(H4)} \quad \int_0^T \|(\partial_t V^1, \partial_t V^2)\|_{B_{2,1}^s} + \|(\partial_t V^1, \partial_t V^2)\|_{\tilde{L}_T^2(B_{2,1}^{s-1})} \leq \eta_1$$

$$\text{(H5)} \quad |U(t, x) - U_0(x)| \leq d_1 \quad \text{for any } (t, x) \in Q_T,$$

where  $\eta, \eta_1 > 0$  are constants, and  $d_1$  satisfies  $d_1 < \text{dist}(\mathcal{O}_0, \partial\mathcal{U})$ ,  $\tilde{V} \in \mathcal{O}$  with  $\mathcal{O} \stackrel{\text{def}}{=} d_1$ -neighborhood of  $\mathcal{O}_0$ .

We are going to prove that under suitable assumptions on  $T$  and  $\eta, \eta_1$  (to be specified below) if conditions **(H1)** to **(H5)** are satisfied for  $V$ , then they are actually satisfied for  $\tilde{V}$ .

Let us define  $M_0$  by

$$M_0 \stackrel{\text{def}}{=} 4 \|V_0^1\|_{B_{2,1}^\sigma} + 2 \|V_0^2\|_{B_{2,1}^s} \quad (61)$$

so that if **(H1)** to **(H3)** are satisfied for  $V$ , then (see (54) for the definition of  $M_1$  and  $M_2$ )

$$M_2 \leq \eta \quad \text{and} \quad M_1 \leq M_0. \quad (62)$$

Let us assume that  $\varepsilon, \eta_1$  and  $T$  satisfy

$$\eta < \min \left( 1, \frac{\|V_0^1\|_{B_{2,1}^\sigma}}{2(1+M_0)} \right), \quad \eta_1 < \frac{\ln(2)}{8C} \quad \text{and} \quad T < \min \left( \frac{\|V_0^1\|_{B_{2,1}^\sigma}}{2(1+M_0)}, \frac{(\ln(2))^2}{64C^2 M_0} \right). \quad (63)$$

According to inequalities (25) (with  $\sigma \geq \frac{d}{2} + 1$ ) and (57), we have

$$\begin{aligned} \|\tilde{V}^1\|_{\tilde{L}_T^\infty(B_{2,1}^\sigma)} &\leq e^{\Phi_1(T)} \left( \|V_0^1\|_{B_{2,1}^\sigma} + C(1+M_1)(T+M_2) \right) \\ &\leq e^{\Phi_1(T)} \left( \|V_0^1\|_{B_{2,1}^\sigma} + C(1+M_0)(T+\eta) \right) \end{aligned} \quad (64)$$

where in the last inequality we used inequality (62). We recall (see (24) for the definition of  $\sigma^{**}$ ) that

$$\Phi_1(t) = C \int_0^t (\|V\|_{B_{2,1}^{\sigma^{**}+1}} + \|(\partial_t V, \nabla V)\|_{L^\infty}).$$

As  $\sigma^{**} \leq \sigma - 1$ , then we deduce from interpolation inequality, Besov embedding, **(H3)** and (62) that

$$\begin{aligned} \Phi_1(T) &\leq \int_0^T \|\partial_t V\|_{B_{2,1}^{\sigma-1}} + T \|V^1\|_{\tilde{L}_T^\infty(B_{2,1}^\sigma)} + \sqrt{T} \|V^2\|_{\tilde{L}_T^2(B_{2,1}^\sigma)} \\ &\leq C(\eta_1 + TM_1 + \sqrt{TM_1 M_2}) \leq C(\eta_1 + TM_0 + \sqrt{TM_0 \eta}). \end{aligned} \quad (65)$$

Using Conditions (63) to (65), we deduce that **(H1)** is satisfied for  $\tilde{V}$ .

In order to prove that **(H2)** and **(H3)** are satisfied for  $\tilde{V}$ , we have to use (30). For that purpose we need to introduce a new element. Indeed as we can see in the estimate (57) and (64), to complete the proof of bounds of  $\tilde{V}^1$  in  $\tilde{L}_T^\infty(B_{2,1}^\sigma)$  we have to get  $\|V^2\|_{L_T^1(B_{2,1}^{s+2})}$  as small as possible. In fact it is clear that  $\|(\nabla V^2, \nabla \tilde{V}^2)\|_{L_T^1(B_{2,1}^{\frac{d}{2}})}$  tends to 0 for  $T$  going to 0; but we do not know how fast. It's worth pointing out that we don't face this problem if  $V_0^1$  and  $V_0^2$  have the same regularity, that is,  $V_0^1, V_0^2 \in B_{2,1}^\sigma$ , for  $\sigma \geq \frac{d}{2} + 1$ . To overcome the difficulty, we

decompose  $\tilde{V}^2$  into the sum of the solution  $V^L$  to the following linear parabolic system with constant coefficients:

$$\begin{cases} \overline{S_{22}^0} \partial_t V^L - \overline{Z}^{\alpha\beta} \partial_\alpha \partial_\beta V^L = 0 \\ V^L(0) = V^2(0), \end{cases} \quad (66)$$

where we recall that  $\overline{S} = S(\overline{U})$  for all function  $S$ , and the discrepancy  $\tilde{V}^S$  to  $V^L$  ( $\tilde{V}^S \stackrel{\text{def}}{=} \tilde{V}^2 - V^L$ ). By definition,  $\tilde{V}^S|_{t=0} \equiv 0$  and  $\tilde{V}^S$  satisfies the parabolic system

$$S_{22}^0(U) \partial_t \tilde{V}^S - \sum_{\alpha, \beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U) \partial_\beta \tilde{V}^S) = (\overline{S_{22}^0} - S_{22}^0(U)) \partial_t V^L + \sum_{\alpha, \beta=1}^d \partial_\alpha (r^{\alpha\beta}(U) \partial_\beta V^L) + \Theta_2, \quad (67)$$

where  $r^{\alpha\beta}(U) = Z^{\alpha\beta}(U) - \overline{Z}^{\alpha\beta}$ .

Since  $\tilde{V}^S(0) = 0$ , we expect  $\left\| \nabla \tilde{V}^S \right\|_{L_T^1(B_{2,1}^{s+2})}$  to be small enough, for some time  $T$  depending only on initial data, through  $V^L$ . More precisely, since, according to Proposition 6, we have  $V^L \in L_T^1(B_{2,1}^{s+2})$  for all  $T > 0$ , for all  $\varepsilon > 0$ , there exists  $T_\varepsilon > 0$  such that

$$\int_0^{T_\varepsilon} \left( \|\partial_t V^L\|_{B_{2,1}^s} + \|V^L\|_{B_{2,1}^{s+2}} \right) < \varepsilon. \quad (68)$$

Moreover, for all  $T > 0$

$$\|V^L\|_{\tilde{L}_T^\infty(B_{2,1}^s)} \leq \|V_0^2\|_{B_{2,1}^s}. \quad (69)$$

In the sequel, we assume that

$$T < T_\varepsilon. \quad (70)$$

In order to establish **(H2)** and **(H3)** for  $\tilde{V}^2$ , it suffices to prove that: if  $V^S \stackrel{\text{def}}{=} V^2 - V^L$  satisfies the following condition

$$\textbf{(HS)} \quad \|V^S\|_{\tilde{L}_T^\infty(B_{2,1}^s)} + \int_0^T \|V^S(\tau)\|_{B_{2,1}^{s+2}} d\tau \leq \frac{\eta}{2},$$

then **(HS)** is also satisfied for  $\tilde{V}^S$ . Indeed if **(HS)** is satisfied for  $\tilde{V}^S$ , then, from the decomposition  $\tilde{V}^2 = V^L + \tilde{V}^S$  and inequalities (68), we have

$$\begin{aligned} \|\tilde{V}^2\|_{\tilde{L}_T^\infty(B_{2,1}^s)} &\leq \|(\tilde{V}^S, V^L)\|_{\tilde{L}_T^\infty(B_{2,1}^s)} \leq \frac{\eta}{2} + \|V_0^2\|_{B_{2,1}^s}, \\ \int_0^T \|\tilde{V}^2(\tau)\|_{B_{2,1}^{s+2}} d\tau &\leq \int_0^T \|(\tilde{V}^S, V^L)(\tau)\|_{B_{2,1}^{s+2}} d\tau \leq \varepsilon + \frac{\eta}{2} \end{aligned}$$

from which, we deduce **(H2)** and **(H3)** for  $\tilde{V}^2$ , provided that

$$\eta < 2 \|V_0^2\|_{B_{2,1}^s} \quad \text{and} \quad \varepsilon < \frac{\eta}{2}. \quad (71)$$

Applying inequality (30) with  $s \geq \frac{d}{2}$  to  $\tilde{V}^S$  we have

$$\begin{aligned} \|\tilde{V}^S\|_{\tilde{L}_T^\infty(B_{2,1}^s)} + \frac{c}{2}\|\tilde{V}^S\|_{L_T^1(B_{2,1}^s)} &\leq e^{\Phi_2(T)} \left( \int_0^T \|(\Theta_2(\tau))\|_{B_{2,1}^s} \right. \\ &\quad \left. + \int_0^T \|(\overline{S_{22}^0} - S_{22}^0(U))\partial_t V^L\|_{B_{2,1}^s} + \sum_{\alpha,\beta=1}^d \int_0^T \|\partial_\alpha(r^{\alpha\beta}(U))\partial_\beta V^L\|_{B_{2,1}^s} \right), \end{aligned} \quad (72)$$

where we recall that

$$\begin{aligned} \Phi_2(T) = C \int_0^T &\left( 1 + \|\partial_t V\|_{L^\infty} + (1 + \|V\|_{B_{2,1}^{s^*}})^2 \left( \|V^1\|_{B_{2,1}^{s^*} \cap B_{2,1}^{s^{**}}}^2 \right. \right. \\ &\quad \left. \left. + \|V^2\|_{B_{2,1}^{s^*}} \|V^2\|_{B_{2,1}^{s^{**}+2}} + \|V^2\|_{B_{2,1}^{s^{**}}} \|V^2\|_{B_{2,1}^{s^{**}+2}} \right) \right), \end{aligned}$$

and  $s^*$  and  $s^{**}$  have been defined in (24). Remarking that  $s^* \leq s$  and  $s^{**} \leq s$  for  $s \geq \frac{d}{2}$ , we infer that

$$\begin{aligned} \Phi_2(T) &\leq C \left( T + \eta_1 + (1 + M_1)^2 (TM_1^2 + M_1 M_2) \right) \\ &\leq C \left( T + \eta_1 + (1 + M_0)^2 TM_0^2 + M_0 \eta \right). \end{aligned} \quad (73)$$

Next, using the fact that the numerical product maps  $B_{2,1}^s \times B_{2,1}^s$  in  $B_{2,1}^s$  for all  $s \geq \frac{d}{2}$ , and the composition estimates in proposition 10, we deduce that: for all  $\alpha, \beta = 1, \dots, d$

$$\begin{aligned} \bullet &\|(\overline{S^0} - S_{22}^0(U))\partial_t V^L\|_{L_T^1(\dot{B}_{2,1}^s)} \leq C \|V\|_{\tilde{L}_T^\infty(B_{2,1}^s)} \|\partial_t V^L\|_{L_T^1(B_{2,1}^s)} \leq CM_1 \varepsilon \leq CM_0 \varepsilon, \\ \bullet &\|\partial_\alpha(r^{\alpha\beta}(U))\partial_\beta V^L\|_{L_T^1(B_{2,1}^s)} \leq C \left( \|V\|_{L_T^\infty(B_{2,1}^s)} \|V^L\|_{L_T^1(B_{2,1}^{s+2})} \right. \\ &\quad \left. + \sqrt{T} \|V^1\|_{L_T^\infty(B_{2,1}^s)} \|V^L\|_{L_T^2(B_{2,1}^{s+1})} + \|V^2\|_{L_T^2(B_{2,1}^{s+1})} \|V^L\|_{\tilde{L}_T^2(B_{2,1}^{s+1})} \right) \\ &\leq CM_1 \varepsilon + C\sqrt{T}M_1 \leq M_0 \varepsilon + C\sqrt{T}M_0 \\ \bullet &\|f^{23}\|_{L_T^1(B_{2,1}^s)} \leq C(1 + \|V\|_{L_T^\infty(B_{2,1}^s)}) \|\nabla V\|_{\tilde{L}_T^2(B_{2,1}^s)}^2 \\ &\leq C(1 + \|V\|_{L_T^\infty(B_{2,1}^s)}) (\|\nabla V^1\|_{\tilde{L}_T^2(B_{2,1}^s)}^2 + \|\nabla V^L\|_{\tilde{L}_T^2(B_{2,1}^s)}^2 + \|\nabla V^S\|_{\tilde{L}_T^2(B_{2,1}^s)}^2) \\ &\leq C(1 + M_0)(TM_0 + M_0 \varepsilon + \frac{\eta^2}{4}). \end{aligned} \quad (74)$$

Plugging (74) (73) and (59) (remember (62)) into (72) leads to

$$\begin{aligned} \|\tilde{V}^S\|_{\tilde{L}_T^\infty(B_{2,1}^s)} + \frac{c}{2}\|\tilde{V}^S\|_{L_T^1(B_{2,1}^s)} &\leq e^{C \left( T + \eta_1 + (1 + M_0)^2 (TM_0^2 + M_0 \eta) \right)} \left( CM_0 \varepsilon + C\sqrt{T}M_0 \right. \\ &\quad \left. + C(1 + M_0)^2 (T + \sqrt{T}\eta) + C(1 + M_0)(TM_0 + M_0 \varepsilon + \frac{\eta^2}{4}) \right). \end{aligned}$$

We deduce that  $\tilde{V}^S$  satisfies (HS) if (for instance)

- ▶  $C \left( T + \eta_1 + (1 + M_0)^2 (TM_0^2 + M_0 \eta) \right) < \ln(2),$
- ▶  $CM_0 \varepsilon + C\sqrt{T}M_0 + C(1 + M_0)^2 (T + \sqrt{T}\eta) + CM_0 \varepsilon < \frac{\eta}{8},$
- ▶  $C\frac{\eta^2}{4}(1 + M_0) < \frac{\eta}{8}.$

We may choose  $\eta_1$  as in (63) and  $T, \eta, \varepsilon$  satisfying,

$$T < \min \left( 1, \frac{\eta^2}{16C^2(1+M_0)^4}, \frac{\ln(2)}{8C(1+M_0)^4} \right); \quad \eta < \min \left( 1, \frac{1}{2(1+M_0)C}, \frac{\ln(2)}{8C(1+M_0)^3} \right);$$

$$\varepsilon < \frac{\eta}{16CM_0(1+M_0)}. \quad (75)$$

In order to prove that (H4) is satisfied for  $\tilde{V}$  we have to use (42), (43) and interpolation inequality in Besov Spaces. It holds

$$\begin{aligned} \int_0^T \left\| (\partial_t \tilde{V}^1, \partial_t \tilde{V}^2) \right\|_{B_{2,1}^s} &\leq C(1+M_0) \left( T \left\| \tilde{V}^1 \right\|_{\tilde{L}_T^\infty(B_{2,1}^\sigma)} \right. \\ &+ \int_0^T \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s+2}} + \int_0^T \|V\|_{B_{2,1}^{s+1}} \left\| \tilde{V}^2 \right\|_{B_{2,1}^{s+1}} + \int_0^T \left( \|\Theta_2\|_{B_{2,1}^s} + \|\Theta_1\|_{B_{2,1}^\sigma} \right), \\ &\left\| (\partial_t \tilde{V}^1, \partial_t \tilde{V}^2) \right\|_{\tilde{L}_T^2(B_{2,1}^{s-1})} \leq C(1+M_1) \left( \sqrt{T} \left\| \tilde{V}^1 \right\|_{\tilde{L}_T^\infty(B_{2,1}^s)} \right. \\ &\left. + M_1 \left\| \tilde{V}^2 \right\|_{\tilde{L}_T^2(B_{2,1}^s)} + \left\| \tilde{V}^2 \right\|_{\tilde{L}_T^2(B_{2,1}^{s+1})} + \|(\Theta_1, \Theta_2)\|_{\tilde{L}_T^2(B_{2,1}^{s-1})} \right). \end{aligned}$$

As  $\tilde{V}$  satisfies (H1)-(H3) (hence  $\tilde{V}$  also satisfies (62)), using the assumption that  $\eta, T < 1$  (see (75)) and inserting (57), (58), (60) and (59) (combined with the last inequality of (74)) in the previous two inequalities, we deduce that

$$\begin{aligned} \left\| (\partial_t \tilde{V}^1, \partial_t \tilde{V}^2) \right\|_{L_T^1(B_{2,1}^s)} + \left\| (\partial_t \tilde{V}^1, \partial_t \tilde{V}^2) \right\|_{\tilde{L}_T^2(B_{2,1}^{s-1})} \\ \leq C(1+M_0)^2(1+\sqrt{M_0})\sqrt{T} + C(M_0\varepsilon + \frac{\eta^2}{4})(1+M_0) + C(1+M_0)\sqrt{M_0\eta} \\ \leq C(1+M_0)^{\frac{5}{2}}(\sqrt{T} + \varepsilon + \sqrt{\eta}), \end{aligned}$$

from which we deduce that  $\tilde{V}$  satisfies (H4) provided that

$$\varepsilon < \frac{\eta_1}{8C(1+M_0)^{\frac{5}{2}}}; \quad T < \frac{\eta_1^2}{8^2C^2(1+M_0)^5}; \quad \eta < \frac{\eta_1^2}{8^2C^2(1+M_0)^5}. \quad (76)$$

Finally combining (H4) and the embedding  $L^\infty \hookrightarrow B_{2,1}^a$  with  $a \geq \frac{d}{2}$  leads to:

$$|\tilde{U}(t, x) - \tilde{U}_0(t, x)| \leq C \int_0^T \left\| (\partial_t V^1, \partial_t V^2)(\tau) \right\|_{B_{2,1}^s} d\tau \leq C\eta_1 \quad \text{for all } (t, x) \in Q_T,$$

which yields (H5) with  $\tilde{U}$  instead of  $U$ , if

$$C\eta_1 < d_1. \quad (77)$$

We take  $T_0 \leq T$  with  $T > 0$  satisfying the above conditions. Let us notice that the parameters  $\eta, \eta_1, \varepsilon$  and the time  $T$  depend only on  $d, d_1$  and on the data. Moreover, we can find some constant  $\varepsilon = \varepsilon(d, d_1, M_0)$  and  $c_0, C_0$  depending only on  $d, d_1$  such that (see (68) for the definition of  $T_\varepsilon$ )

$$T_0 < \min \left( 1, C_0, T_\varepsilon, \frac{c_0}{(1+M_0)^{10}} \right). \quad (78)$$

□

Based on Lemma 4, we further establish the local existence of (11) with (16).

**Proposition 1.** *Let  $d \geq 1$  and  $s \geq \frac{d}{2}$  and  $\sigma = s + 1$ . Assume that the initial data  $V_0$  satisfies  $(V_0^1, V_0^2) \in B_{2,1}^\sigma \times B_{2,1}^\sigma$  and (52).*

*Then, there exists a positive time  $T_1 (\leq T_0)$  depending only on  $d$ , on  $\mathcal{O}_0$  and on the initial data such that the Cauchy problem (11) with (16) has a unique solution  $V \in E_{T_1}^\sigma$ .*

*Proof.* The idea is to construct iteratively a sequence of smooth approximate local solutions.

We introduce the successive approximation sequence<sup>2</sup>  $\{V_n\}_{n=3}^\infty$  for the Cauchy problem (48)-(49) as follows:

$$V_3^1 = 0_{\mathbb{R}^{n_1}} \quad \text{and} \quad V_3^2 = V^L, \quad (79)$$

where  $V^L$  is solution of the parabolic system (66).

For  $n \geq 3$ , we set  $U_n = V_n + \bar{U}$  and  $V_{n+1}$  solution of:

$$\begin{aligned} S_{11}^0(U_n) \partial_t V_{n+1}^1 + \sum_{\alpha=1}^d S_{11}^\alpha(U_n) \partial_\alpha V_{n+1}^1 &= f^1(U_n, \nabla U_n) - \sum_{\alpha=1}^d S_{12}^\alpha(U_n) \partial_\alpha V_n^2, \\ S_{22}^0(U_n) \partial_t V_{n+1}^2 - \sum_{\alpha,\beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U_n) \partial_\beta V_{n+1}^2) &= f^2(U_n, \nabla U_n) \\ &\quad - \sum_{\alpha=1}^d (S_{21}^\alpha(U_n) \partial_\alpha V_n^1 + S_{22}^\alpha(U_n) \partial_\alpha V_n^2), \end{aligned} \quad (80)$$

with the initial data<sup>3</sup>

$$V_{n+1}|_{t=0} \stackrel{\text{def}}{=} S_{n+1} V_0, \quad \text{where } S_n \text{ is the cut-off operator defined in (179)}. \quad (81)$$

Moreover, it holds that for all  $\theta \in \mathbb{R}$  and  $n \geq 3$ , the operator  $S_n$  maps  $B_{2,1}^\theta$  into itself and  $\|S_n V_0\|_{B_{2,1}^\theta} \leq \|V_0\|_{B_{2,1}^\theta}$ .

According to [20, 1, 15] for all  $\theta \geq 2$ , there exists a time  $\tilde{T}_\theta > 0$  such that the Cauchy problem (80)-(81) has a unique solution in  $C([0, \tilde{T}_\theta], H^\theta)$ . By Lemma 4, there exists  $0 < T_0 \leq \tilde{T}_\sigma$  depending only on  $d$ ,  $\mathcal{O}_0$  and the initial data  $V_0$  such that the sequence  $\{V_n\}_{n=3}^\infty$  is well defined on  $Q_{T_0}$  for all  $n \geq 0$ , and is uniformly bounded with respect to  $n$ , that is,  $V_n \in E_{T_0}^\sigma$ . Next, in the case<sup>4</sup>  $d \geq 2$  or  $d = 1$  and  $s > \frac{1}{2}$ , it will be shown that  $\{V_n\}_{n=3}^\infty$  is a Cauchy sequence in the space

$$F_T = \{V \stackrel{\text{def}}{=} (V^1, V^2) : V \in \tilde{C}([0, T]; B_{2,1}^{\sigma-1} \times B_{2,1}^{s-1}), V^2 \in L_T^1 B_{2,1}^{s+1}\}$$

for  $T \leq T_0$ . The reason for lowering regularity is the usual loss of one derivative when proving stability estimates for quasilinear hyperbolic (diffusive) systems.

Define  $\tilde{V}_n = V_{n+1} - V_n$  for any  $n \geq 1$ . Take the difference between the equation (80) for the  $n + 1$ -th step and the  $n$ -th step to get

$$\begin{aligned} S_{11}^0(U_n) \partial_t \tilde{V}_n^1 + \sum_{\alpha=1}^d S_{11}^\alpha(U_n) \partial_\alpha \tilde{V}_n^1 &= h, \\ S_{22}^0(U_n) \partial_t \tilde{V}_n^2 - \sum_{\alpha,\beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U_n) \partial_\beta \tilde{V}_n^2) &= g, \end{aligned} \quad (82)$$

<sup>2</sup>For better readability,  $n$  starts at 3.

<sup>3</sup>Note that we smoothed out the data  $V_0$  in order to get a smooth solution of the Cauchy problem (80)-(81).

<sup>4</sup>The limitation to that case arises from technical difficulties in establishing product laws, as apparent in (35) for instance. An alternative approach to deal with the one-dimensional case and  $s = \frac{1}{2}$  consist in replacing  $F_T$  by the space  $L_T^\infty(\dot{B}_{2,\infty}^{\frac{1}{2}}) \times \left( L_T^\infty(\dot{B}_{2,\infty}^{-\frac{1}{2}}) \cap \tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{3}{2}}) \right)$  and adapt the proof in section 3.3

with  $h = h_1 + h_2$ ,  $g = g_1 + g_2 + g_3 + g_4 + g_5 + g_6 + g_7 + g_8$  and

$$\begin{aligned}
h_1 &\stackrel{\text{def}}{=} -S_{11}^0(U_n) \sum_{\alpha=1}^d ((S_{11}^0(U_n))^{-1} S_{11}^\alpha(U_n) - (S_{11}^0(U_{n-1}))^{-1} S_{11}^\alpha(U_{n-1})) \partial_\alpha V_n^1, \\
h_2 &\stackrel{\text{def}}{=} -S_{11}^0(U_n) \sum_{\alpha=1}^d ((S_{11}^0(U_n))^{-1} S_{11}^\alpha(U_n) - (S_{11}^0(U_{n-1}))^{-1} S_{12}^\alpha(U_{n-1})) \partial_\alpha V_{n-1}^2 \\
&\quad - \sum_{\alpha=1}^d S_{12}^\alpha(U_n) \partial_\alpha \tilde{V}_{n-1}^2 + f^1(U_n, \nabla U_n) - S_{11}^0(U_n) S_{11}^0(U_{n-1})^{-1} f^1(U_{n-1}, \nabla U_{n-1}), \\
g_1 &\stackrel{\text{def}}{=} -(S_{22}^0(U_n) - S_{22}^0(U_{n-1})) \partial_t V_n^2, \quad g_2 \stackrel{\text{def}}{=} - \sum_{\alpha=1}^d (S_{22}^\alpha(U_n) - S_{22}^\alpha(U_{n-1})) \partial_\alpha V_{n-1}^2, \\
g_3 &\stackrel{\text{def}}{=} \sum_{\alpha, \beta=1}^d \partial_\alpha ((Z^{\alpha\beta}(U_{n-1}) - Z^{\alpha\beta}(U_n)) \partial_\beta V_n^2), \\
g_4 &\stackrel{\text{def}}{=} - \sum_{\alpha=1}^d ((S_{21}^\alpha(U_n) - S_{21}^\alpha(U_{n-1})) \partial_\alpha V_{n-1}^1), \\
g_5 &\stackrel{\text{def}}{=} - \sum_{\alpha=1}^d S_{21}^\alpha(U_n) \partial_\alpha \tilde{V}_{n-1}^1, \quad g_6 \stackrel{\text{def}}{=} - \sum_{\alpha=1}^d S_{22}^\alpha(U_n) \partial_\alpha \tilde{V}_{n-1}^2, \\
g_7 &\stackrel{\text{def}}{=} f^2(U_n, \nabla U_n) - f^2(U_{n-1}, \nabla U_{n-1}), \quad g_8 \stackrel{\text{def}}{=} \sum_{\alpha, \beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U_n)) \partial_\beta \tilde{V}_n^2.
\end{aligned}$$

Let us notice that all estimates established in the proof of Lemma 4 are also valid for  $U_n$ , for  $n \geq 3$ . In particular, the right-hand sides of (73) and (65) are smaller than  $\ln 2$ . Apply thus (25) (with  $\sigma \rightarrow \sigma - 1$ ) to (82)<sub>1</sub> and (30) (with  $s \rightarrow s - 1$ ) to (82)<sub>2</sub> to get<sup>5</sup>

$$\left\| \tilde{V}_n^1 \right\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{\sigma-1})} \leq 2 \left( \left\| \tilde{V}_{0,n}^1 \right\|_{B_{2,1}^{\sigma-1}} + \int_0^{T_0} \|h(\tau)\|_{B_{2,1}^\sigma} d\tau \right), \quad (83)$$

and

$$\left\| \tilde{V}_n^2 \right\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1})} + \frac{c}{2} \left\| \tilde{V}_n^2 \right\|_{L_{T_0}^1(B_{2,1}^{s+1})} \leq 2 \left( \left\| \tilde{V}_{0,n}^2 \right\|_{B_{2,1}^{s-1}} + \int_0^{T_0} \|g(\tau)\|_{B_{2,1}^s} d\tau \right). \quad (84)$$

It remains only to estimate the terms on the right-hand side of (83) and (84) by using the same type of estimates as in the previous section and the property satisfied by  $T_0, \eta, \varepsilon, \eta_1$  in the proof of Lemma 4.

On the one hand, using the fact that the space  $B_{2,1}^\theta$  (for  $\theta \geq \frac{d}{2}$ ) is an algebra for numerical product of functions and composition result (see Proposition 10), there exists a constant  $C = C(\mathcal{O}_0, d)$  such that:

$$\begin{aligned}
\|h_1\|_{L_{T_0}^1(B_{2,1}^{\sigma-1})} &\leq C(1 + \|V_n\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{\sigma-1})}) \left( T_0 \left\| \tilde{V}_{n-1}^1 \right\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{\sigma-1})} \|V_n^1\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^\sigma)} \right. \\
&\quad \left. + \sqrt{T_0} \left\| \tilde{V}_{n-1}^2 \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^s)} \|V_n^1\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^\sigma)} \right),
\end{aligned}$$

<sup>5</sup>Here we need  $s - 1$  and  $\sigma - 1$  to be larger than  $-d/2$ , whence the restriction on the regularity exponent if  $d = 1$ .

$$\|h_2\|_{L_{T_0}^1(B_{2,1}^{\sigma_{-1}})} \leq C(1 + \|V_n\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{\sigma_{-1}})}) \left( \sqrt{T_0} \left\| \tilde{V}_{n-1}^1 \right\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{\sigma_{-1}})} \|V_n^2\|_{\tilde{L}_{T_0}^2(B_{2,1}^{\sigma_{-1}})} \right. \\ \left. + T_0 \left\| \tilde{V}_{n-1} \right\|_{L_{T_0}^\infty(B_{2,1}^{\sigma_{-1}})} + \left\| \tilde{V}_{n-1}^2 \right\|_{L_{T_0}^1(B_{2,1}^{\sigma_{-1}})} + \left\| \tilde{V}_{n-1}^2 \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^{\sigma_{-1}})} \|V_n^2\|_{\tilde{L}_{T_0}^2(B_{2,1}^{\sigma_{-1}})} \right),$$

$$\|g_3\|_{L_{T_0}^1(B_{2,1}^{s-1})} \leq C\sqrt{T_0} \left\| \tilde{V}_{n-1}^1 \right\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^s)} \|V_n^2\|_{\tilde{L}_{T_0}^2(B_{2,1}^{s+1})} + \left\| \tilde{V}_{n-1}^2 \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^s)} \|V_n^2\|_{\tilde{L}_{T_0}^2(B_{2,1}^{s+1})}.$$

On the other hand, inequality (35) (with  $\theta = s - 1$ ) combined with proposition 10 ensures the existence of  $C = C(\mathcal{O}, d)$  such that

$$\|g_1\|_{L_{T_0}^1(B_{2,1}^{s-1})} \leq C \left( \sqrt{T_0} \left\| \tilde{V}_{n-1}^1 \right\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{(s-1)^*})} \left\| \partial_t V_n^2 \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^{s-1})} \right. \\ \left. + \left\| \tilde{V}_{n-1}^2 \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^{(s-1)^*})} \left\| \partial_t V_n^2 \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^{s-1})} \right), \\ \|(g_2, g_4)\|_{L_{T_0}^1(B_{2,1}^{s-1})} \leq C \left( T_0 \left\| \tilde{V}_{n-1}^1 \right\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{(s-1)^*})} \|V_n\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1})} \right. \\ \left. + \sqrt{T_0} \left\| \tilde{V}_{n-1}^2 \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^{(s-1)^*})} \|V_n\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1})} \right), \\ \|g_5\|_{L_{T_0}^1(B_{2,1}^{s-1})} \leq C c_j 2^{-j(s-1)} \left( T_0 + T_0 \|V_n\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{(s-1)^*})} \right) \left\| \tilde{V}_{n-1}^1 \right\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^s)}, \\ \|g_6\|_{L_{T_0}^1(B_{2,1}^{s-1})} \leq C c_j 2^{-j(s-1)} \left( \sqrt{T_0} + \sqrt{T_0} \|V_n\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{(s-1)^*})} \right) \left\| \tilde{V}_{n-1}^2 \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^s)}, \\ \|g_8\|_{L_{T_0}^1(L^2(\mathbb{R}^d))} \leq C c_j 2^{-j(s-1)} (1 + \|V_n\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{(s-1)^*})}) \|V_n\|_{\tilde{L}_{T_0}^2(B_{2,1}^{(s-1)^*+1})} \left\| \tilde{V}_{n-1}^2 \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^s)}$$

with  $(s-1)^* = \frac{d}{2}$  if  $-\frac{d}{2} < s-1 < \frac{d}{2}$  and  $(s-1)^* = s-1$  if  $s-1 \geq \frac{d}{2}$ . Similarly using (195) (when  $-\frac{d}{2} < s-1 < \frac{d}{2}$ ) and (193) (when  $s-1 \geq \frac{d}{2}$ ) and remembering the form of  $f^{21}, f^{22}, f^{23}$  (see Assumption **BB**) yields

$$g_7 = f_n^{21} - f_{n-1}^{21} + f_n^{22} - f_{n-1}^{22} + f_n^{23} - f_{n-1}^{23}$$

with

$$\|f_n^{21} - f_{n-1}^{21}\|_{L_{T_0}^1(B_{2,1}^{s-1})} \leq C T_0 (1 + \|(V_{n-1}, V_n)\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{(s-1)^*})}) \left\| \tilde{V}_{n-1} \right\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1})} \\ \|f_n^{22} - f_{n-1}^{22}\|_{L_{T_0}^1(B_{2,1}^{s-1})} \leq C T_0 (1 + \|(V_{n-1}, V_n, \nabla V_n^1, \nabla V_{n-1}^1)\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{(s-1)^*})}) \left\| (\tilde{V}, \nabla \tilde{V}^1) \right\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1})} \\ \|f_n^{23} - f_{n-1}^{23}\|_{L_{T_0}^1(B_{2,1}^{s-1})} \leq C \left\| \tilde{V}_{n-1} \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^{(s-1)^*})} \sqrt{T} \|V_n\|_{\tilde{L}_{T_0}^2(B_{2,1}^s)} \|\nabla V_n\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{(s-1)^*})} \\ + (1 + M_0) \left\| \nabla \tilde{V}_{n-1} \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^{(s-1)^*})} \left\| (\nabla V_{n-1}, \nabla V_n) \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^s)}.$$

Remember that  $V_n \in E_{T_0}^\sigma$ , which implies that (see the proof of Lemma 4 for the definition  $\eta, \eta_1$ ):

$$\|V_n^1\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^\sigma)} + \|V_n^2\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^s)} \leq M_0; \quad \|V_n^2\|_{\tilde{L}_{T_0}^1(B_{2,1}^{s+1})} \leq \eta; \\ \left\| \partial_t V_n^2 \right\|_{\tilde{L}_{T_0}^2(B_{2,1}^{s-1})} + \left\| \partial_t V_n \right\|_{\tilde{L}_{T_0}^1(B_{2,1}^s)} \leq \eta_1.$$



Summing (83) and (84) over  $j \geq -1$ , using the following inequality coming from interpolation inequality in Besov spaces

$$\begin{aligned} \|V_n^2\|_{\tilde{L}_{T_0}^2(B_{2,1}^{s+1})} &\leq C \sqrt{\|V_n^2\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^s)} \|V_n^2\|_{L_{T_0}^1(B_{2,1}^{s+2})}} \leq C \sqrt{M_0 \eta} \\ \|\tilde{V}_{n-1}^2\|_{\tilde{L}_{T_0}^2(B_{2,1}^s)} &\leq C \sqrt{\|\tilde{V}_{n-1}^2\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1})} \|\tilde{V}_{n-1}^2\|_{L_{T_0}^1(B_{2,1}^{s+1})}} \leq C \|\tilde{V}_{n-1}\|_{F_T^\sigma} \end{aligned}$$

we end up with

$$\|\tilde{V}_n^1\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1})} \leq 2 \|\tilde{V}_{0,n}^1\|_{B_{2,1}^{s-1}} + C \left( \sqrt{T_0} M_0 + \sqrt{M_0 \eta} \right) \|\tilde{V}_{n-1}\|_{F_T^\sigma} + (1 + M_0) \|\tilde{V}_n^2\|_{L_{T_0}^1(B_{2,1}^s)},$$

and

$$\begin{aligned} \|\tilde{V}_n^2\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1})} + c \|\tilde{V}_n^2\|_{L_{T_0}^1(B_{2,1}^{s+1})} &\leq 2 \left( \|\tilde{V}_{0,n}^2\|_{B_{2,1}^{s-1}} + C \left( T_0 + \eta_1 + T_0 M_0 + \sqrt{T_0} \sqrt{M_0 \eta} \right. \right. \\ &\quad \left. \left. + \sqrt{T_0} M_0 + \sqrt{M_0 \eta} + (1 + M_0) \sqrt{M_0 \eta} \right) \|\tilde{V}_n^2\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1}) \cap L_{T_0}^1(B_{2,1}^{s+1})} \right. \\ &\quad \left. + C \left( \sqrt{T_0} \sqrt{M_0 \eta} + \sqrt{M_0 \eta} + \eta_1 + M_0 \sqrt{T_0} \right) \|\tilde{V}_{n-1}\|_{F_T^\sigma} \right). \end{aligned}$$

Next, the conditions satisfied by  $T_0, \eta, \eta_1, \varepsilon$  in Step 5 of the proof of Lemma 6 allow us to simplify the previous inequalities as follows,

$$\|\tilde{V}_n^1\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1})} \leq 2 \|\tilde{V}_{0,n}^1\|_{B_{2,1}^{s-1}} + C \left( \sqrt{T_0} M_0 + \sqrt{M_0 \eta} \right) \|\tilde{V}_{n-1}\|_{F_T^\sigma} + (1 + M_0) \|\tilde{V}_n^2\|_{L_{T_0}^1(B_{2,1}^s)} \quad (85)$$

and

$$\|\tilde{V}_n^2\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1}) \cap L_{T_0}^1(B_{2,1}^{s+1})} \leq 2 \|\tilde{V}_{0,n}^2\|_{B_{2,1}^{s-1}} + C \left( \sqrt{M_0 \eta} + \eta_1 + M_0 \sqrt{T_0} \right) \|\tilde{V}_{n-1}\|_{F_T^\sigma}. \quad (86)$$

Finally, putting together (85) and (86) (let us notice that we multiply (85) by  $1/2(1 + M_0)$  so that its last term is absorbed by the left-hand side of (86) ) we get, for large enough  $C$ ,

$$\begin{aligned} \frac{1}{2(1 + M_0)} \|\tilde{V}_n^1\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1})} + \|\tilde{V}_n^2\|_{\tilde{L}_{T_0}^\infty(B_{2,1}^{s-1}) \cap L_{T_0}^1(B_{2,1}^{s+1})} &\leq \frac{1}{1 + M_0} \|\tilde{V}_{0,n}^1\|_{B_{2,1}^{s-1}} + \|\tilde{V}_{0,n}^2\|_{B_{2,1}^{s-1}} \\ &\quad + C \left( \sqrt{T_0} M_0 + \sqrt{M_0(\eta + \varepsilon)} \right) \|\tilde{V}_{n-1}\|_{F_T^\sigma}. \end{aligned}$$

It implies that

$$\|\tilde{V}_n\|_{F_T^\sigma} \leq C \left( \|\tilde{V}_{0,n}^1\|_{B_{2,1}^{s-1}} + \|\tilde{V}_{0,n}^2\|_{B_{2,1}^{s-1}} \right) + C \left( \sqrt{T_0} M_0 + \sqrt{M_0(\eta + \varepsilon)} \right) \|\tilde{V}_{n-1}\|_{F_T^\sigma}. \quad (87)$$

Take  $T_1$  so small that

$$T_1 \leq T_0 \quad \text{and} \quad C \left( \sqrt{T_1} M_0 + \sqrt{M_0(\eta + \varepsilon)} \right) \leq \frac{1}{2}. \quad (88)$$

Then it follows from (87) and the fact that  $\|(S_{n+1} - S_n)a\|_{B_{2,1}^{\theta-1}} \simeq n^{-1} \|a\|_{B_{2,1}^\theta}$ , for all  $\theta \in \mathbb{R}$ , that  $(\tilde{V}_n)_{n=3}^\infty$  is a Cauchy sequence in  $F_{T_1}^\sigma$ . There exists a distribution  $V \in F_{T_1}^\sigma$  such that  $(V_n - V) \rightarrow 0$  strongly in  $F_{T_1}^\sigma$  as  $n \rightarrow \infty$ . On the other hand, doing an argument totally similar to what has been done in [2, Chap. 10], one can prove that  $V$  satisfy  $(\mathcal{H}1) - (\mathcal{H}3)$ . The

property of strong convergence enables us to pass to the limit in the system (80)-(81) and to conclude that  $V$  is a solution to (11) and (16) satisfying (H1) – (H3).

What remains is to check that  $(V^1, V^2) \in \tilde{C}([0, T_1]; B_{2,1}^\sigma \times B_{2,1}^s)$  and then  $(V^1, V^2) \in C([0, T_1]; B_{2,1}^\sigma \times B_{2,1}^s)$ . This can be achieved by following the ideas in [2, Chap. 10]. For the sake of completeness, we provide the proof for  $V^1$ . Combining the fact that  $V^1 \in L^\infty(0, T; B_{2,1}^\sigma)$  and  $V^2 \in L^\infty(0, T; B_{2,1}^s)$  with the information that  $V \in C([0, T], L^2)$ , which comes from  $\partial_t V \in L_{T_1}^1(B_{2,1}^s) \subset L_{T_1}^1(L^2)$ , this ensures that  $V^1 \in \tilde{C}([0, T]; B_{2,1}^\sigma)$  and  $V^2 \in \tilde{C}([0, T]; B_{2,1}^s)$ . Indeed, for any  $J \in \mathbb{N}$  and  $(t, t_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ , one may write

$$\begin{aligned} \|V^1(t) - V^1(t_0)\|_{B_{2,1}^\sigma} &\leq \sum_{-1 \leq j \leq J} 2^{j\sigma} \|\Delta_j V(t) - \Delta_j V(t_0)\|_{L^2} + 2 \sum_{j > J} 2^{j\sigma} \|\Delta_j V^1\|_{L_T^\infty(L^2)} \\ &\leq 2^{J\sigma} \sum_{-1 \leq j \leq J} \|V(t) - V(t_0)\|_{L^2} + 2 \sum_{j > J} 2^{j\sigma} \|\Delta_j V^1\|_{L_T^\infty(L^2)}. \end{aligned}$$

Now, for any given  $J$  the first term goes to 0 if  $t \rightarrow t_0$  while the second term tends to 0 if  $J \rightarrow \infty$ . The same argument implies that  $\tilde{V}^2 \in \tilde{C}([0, T]; B_{2,1}^s)$ . This completes the proof of  $V \in E_{T_1}^\sigma$ . Hence, the local existence part of the solutions is complete eventually.

Concerning the uniqueness, we set  $\tilde{V} \stackrel{\text{def}}{=} V_2 - V_1$ , where  $V_1$  and  $V_2$  are two solutions to the system (11) and are subject to the same initial data, respectively. Then the error solution  $\tilde{V}$  satisfies the equation (82) where instead of  $U_n, U_{n-1}, V_n, \tilde{V}_{n-1}$  we have  $U_2, U_1, V_1, \tilde{V}$  respectively. Then taking advantage on (87) (recall that here  $\tilde{V}_n = \tilde{V}_{n-1} = \tilde{V}$ ) and (88) we conclude that  $\tilde{V} = 0$  in  $F_{T_1}^\sigma$ . This finishes the proof of Proposition 1.  $\square$

## 2.3 Proof of the continuation criterion

This section is devoted to the proof of the following continuation criterion.

**Proposition 2.** *Let  $0 < T_0 < \infty$ . Under the hypotheses of Theorem 1.2, assume that the system (11) has a solution  $(V^1, V^2)$  on  $[0, T_0[ \times \mathbb{R}^d$  which belongs to  $C([0, T]; B_{2,1}^\sigma) \times \left( C([0, T]; B_{2,1}^s) \cap L^1(0, T; B_{2,1}^{s+2}) \right)$ , for all  $T < T_0$  and satisfies*

1.  $U$  belongs to an open bounded set  $\Omega$ , with  $\bar{\Omega} \in \mathcal{U}$ ,

2.  $\int_0^{T_0} \left( \|(\nabla V, \nabla^2 V^2)\|_{L^\infty} + \|\nabla V\|_{L^\infty}^2 \right) < \infty$ ,

3.  $\sup_{(t,x) \in [0, T_0[ \times \mathbb{R}^d} \nabla V^1(t, x) < \infty$ .

There exists some  $T^* > T_0$  such that  $(V^1, V^2)$  may be continued on  $[0, T^*] \times \mathbb{R}^d$  to a solution of (11) which belongs to  $C([0, T^*]; B_{2,1}^\sigma) \times \left( C([0, T^*]; B_{2,1}^s) \cap L^1(0, T^*; B_{2,1}^{s+2}) \right)$ .

Moreover, if the source term  $f^{22}(U, \nabla V^1)$  is quadratic in  $\nabla V^2$ , then the last assumption is not needed.

*Proof.* It is not difficult to prove that

$$\|\partial_t V\|_{L^\infty} \leq C(1 + \|\nabla V\|_{L^\infty} + \|\nabla V\|_{L^\infty} \|\nabla V\|_{L^\infty} + \|\nabla^2 V^2\|_{L^\infty}) \text{ on } [0, T_0[. \quad (89)$$

Throughout this proof, we denote by  $C$  a positive constant depending only on  $d, \|V\|_{L^\infty([0, T_0] \times \mathbb{R}^d)}$  and the matrices involved in the system (11). Since  $V$  satisfies (17) with  $V = \tilde{V}$ , and

$$\begin{aligned}\Theta_1 &= f^1 - \sum_{\alpha=1}^d S_{12}^\alpha(U) \partial_\alpha V^2 \\ \Theta_2 &= f^{21} + f^{22} + f^{23} - \sum_{\alpha=1}^d (S_{21}^\alpha(U) \partial_\alpha V^1 + S_{22}^\alpha(U) \partial_\alpha V^2)\end{aligned}\tag{90}$$

then, for all  $T < T_0$ , we have from (26) and (34)

$$\begin{aligned}\blacktriangleright \|V^1\|_{\tilde{L}_T^\infty(B_{2,1}^\sigma)} &\leq \|V_0^1\|_{B_{2,1}^\sigma} + C \int_0^T \|(\partial_t V, \nabla V)\|_{L^\infty} \|V^1\|_{B_{2,1}^\sigma} + \|\Theta_1\|_{L_T^1(B_{2,1}^\sigma)} \\ &\quad + \int_0^T \sum_{j \geq -1} 2^{j\sigma} \|R_j^{11}\|_{L^2}, \\ \blacktriangleright \|V^2\|_{\tilde{L}_T^\infty(B_{2,1}^\sigma)} + c \|V^2\|_{\tilde{L}_T^1(B_{2,1}^{s+2})} &\leq \|V_0^2\|_{B_{2,1}^s} \\ &\quad + \int_0^T (1 + \|\partial_t V\|_{L^\infty}) \|V^2\|_{B_{2,1}^s} + \int_0^T \sum_{j \geq -1} 2^{js} \|R_j^2\|_{L^2} + \|\Theta_2\|_{L_T^1(B_{2,1}^\sigma)}.\end{aligned}\tag{91}$$

According to (27) and interpolation inequality, we have

$$\begin{aligned}\int_0^T \sum_{j \geq -1} 2^{j\sigma} \|R_j^{11}\|_{L^2} &\leq C \int_0^T \|\nabla V\|_{L^\infty} \|V^1\|_{B_{2,1}^\sigma} + C \int_0^T \|\nabla V^1\|_{L^\infty} \|(V^1, V^2)\|_{B_{2,1}^\sigma} \\ &\leq C \int_0^T \|\nabla V\|_{L^\infty} \|V^1\|_{B_{2,1}^\sigma} + C \int_0^T \|\nabla V^1\|_{L^\infty}^2 \|V^2\|_{B_{2,1}^{\sigma-1}} + \|V^2\|_{L_T^1(B_{2,1}^{\sigma+1})}.\end{aligned}$$

Next, the product estimate (188) combined with interpolation inequality ensures that

$$\begin{aligned}\bullet \|f^1(U)\|_{L_T^1(B_{2,1}^\sigma)} &\leq C \|V\|_{L_T^1(B_{2,1}^\sigma)} \leq C \int_0^T \|V^1\|_{B_{2,1}^\sigma} + C \int_0^T \|V^2\|_{B_{2,1}^{\sigma-1}} + \|V^2\|_{L_T^1(B_{2,1}^{\sigma+1})} \\ \bullet \|S_{12}^\alpha(U) \partial_\alpha V^2\|_{L_T^1(B_{2,1}^\sigma)} &\leq C \|\nabla V^2\|_{L_T^1(B_{2,1}^\sigma)} + C \int_0^T \|\nabla V^2\|_{L^\infty} (1 + \|V\|_{B_{2,1}^\sigma}) \\ &\leq C \|V^2\|_{L_T^1(B_{2,1}^{\sigma+1})} + C \int_0^T \|\nabla V^2\|_{L^\infty} \|V^1\|_{B_{2,1}^\sigma} + C \int_0^T \|\nabla V^2\|_{L^\infty}^2 \|V^2\|_{B_{2,1}^{\sigma-1}}.\end{aligned}$$

Furthermore, combining (36), the product estimate (188), interpolation inequality on  $V^2$  along with Young inequality yields for all  $\varepsilon > 0$

$$\begin{aligned}\star \int_0^T \sum_{j \geq -1} 2^{js} \|R_j^2\|_{L^2} &\leq C \int_0^T \|\nabla V\|_{L^\infty} \|(V^1, V^2)\|_{B_{2,1}^{s+1}} + C \int_0^T \|\nabla^2 V^2\|_{L^\infty} \|V\|_{B_{2,1}^s} \\ &\leq C \int_0^T \|\nabla V\|_{L^\infty} \|V^1\|_{B_{2,1}^{s+1}} + C(\varepsilon) \int_0^T \|\nabla V\|_{L^\infty}^2 \|V^2\|_{B_{2,1}^s} \\ &\quad + \varepsilon^2 \|V^2\|_{L_T^1(B_{2,1}^{s+2})} + C \int_0^T \|\nabla^2 V^2\|_{L^\infty} \|V\|_{B_{2,1}^s}\end{aligned}$$

and

$$\begin{aligned}
\star & \|f^{21}(U)\|_{L_T^1(B_{2,1}^s)} \leq C \int_0^T \|V\|_{B_{2,1}^s}, \\
\star & \|f^{23}(U)\|_{L_T^1(B_{2,1}^s)} \leq C \int_0^T \|\nabla V\|_{L^\infty} \|\nabla V\|_{B_{2,1}^s} + \int_0^T (1 + \|V\|_{B_{2,1}^s}) \|\nabla V\|_{L^\infty}^2 \\
& \leq C \int_0^T \|\nabla V\|_{L^\infty} \|V^1\|_{B_{2,1}^{s+1}} + C(\varepsilon) \int_0^T \|\nabla V\|_{L^\infty}^2 \|V^2\|_{B_{2,1}^s} + \varepsilon^2 \|V^2\|_{L_T^1(B_{2,1}^{s+2})} \\
& + \int_0^T (1 + \|V\|_{B_{2,1}^s}) \|\nabla V\|_{L^\infty}^2, \\
\star & \|(S_{21}^\alpha(U)\partial_\alpha V^1, S_{22}^\alpha(U)\partial_\alpha V^2)\|_{L_T^1(B_{2,1}^s)} \leq C \int_0^T \|(\nabla V^1, \nabla V^2)\|_{B_{2,1}^s} \\
& + C \int_0^T (1 + \|V\|_{B_{2,1}^s}) \|(\nabla V^1, \nabla V^2)\|_{L^\infty} \\
& \leq C \int_0^T \|V^1\|_{B_{2,1}^{s+1}} + C(\varepsilon) \int_0^T \|V\|_{B_{2,1}^s} + \varepsilon^2 \|V^2\|_{L_T^1(B_{2,1}^{s+2})} \\
& + C \int_0^T (1 + \|V\|_{B_{2,1}^s}) \|(\nabla V^1, \nabla V^2)\|_{L^\infty}.
\end{aligned}$$

In the general case where  $f^{22}$  depends on  $U$  and  $\nabla V^1$ , Inequality (192) ensures that

$$\star \|f^{22}(U, \nabla V^1)\|_{L_T^1(B_{2,1}^s)} \leq C(\|(V, \nabla V^1)\|_{L^\infty}) \int_0^T \|(V, \nabla V^1)\|_{B_{2,1}^s},$$

whereas, if  $f^{22}(U, \nabla V^1)$  is quadratic in terms of  $\nabla V^1$ , that is,  $f^{22}(U, \nabla V^1)$  is a finite combination of  $v_3(U)\nabla V^1 \otimes \nabla V^1$ , where  $v_3$  is a smooth function, then one has

$$\star \|f^{22}(U, \nabla V^1)\|_{L_T^1(B_{2,1}^s)} \leq C \int_0^T \|\nabla V^1\|_{L^\infty} \|\nabla V^1\|_{B_{2,1}^s} + C \int_0^T \|\nabla V^1\|_{L^\infty}^2 (1 + \|V\|_{B_{2,1}^s}).$$

Putting this information into (91) and setting

$$X_\varepsilon(t) = \|V^1\|_{\tilde{L}_t^\infty(B_{2,1}^s)} + \frac{1}{\varepsilon} \|V^2\|_{\tilde{L}_t^\infty(B_{2,1}^s)} + \frac{c}{\varepsilon} \|V^2\|_{L_t^1(B_{2,1}^{s+2})},$$

we end up with

$$\begin{aligned}
X_\varepsilon(T) \leq & \|V_0^2\|_{B_{2,1}^s} + \|V_0^1\|_{B_{2,1}^s} + \int_0^T X_\varepsilon(\tau) \left( \|\partial_t V\|_{L^\infty} + \mathcal{Y}_\varepsilon(\tau) \right) d\tau \\
& + C \|V^2\|_{L_T^1(B_{2,1}^{s+2})} + \varepsilon \|V^2\|_{L_T^1(B_{2,1}^{s+2})}
\end{aligned}$$

with, for all  $\varepsilon > 0$ ,  $\mathcal{Y}_\varepsilon \in L^1(0, T_0)$  according to the assumption of Proposition 2. Choosing  $\varepsilon$  small enough the last two terms of the right-hand side may be absorbed by the left-hand side. Applying then Gronwall inequality and taking advantage of (89), we conclude that  $(V^1, V^2) \in \tilde{L}^\infty(0, T_0; B_{2,1}^s) \times \left( \tilde{L}^\infty(0, T_0; B_{2,1}^s) \cap L^1(0, T_0; B_{2,1}^{s+2}) \right)$  and thus  $(V^1, V^2) \in L^\infty(0, T_0; B_{2,1}^s) \times \left( L^\infty(0, T_0; B_{2,1}^s) \cap L^1(0, T_0; B_{2,1}^{s+2}) \right)$ . By replacing  $M_0$  in the lower bound (78) that we have obtained for the existence time, we obtain an  $\varepsilon > 0$  such that (11) with data  $(V^1(T_0 - \varepsilon), V^2(T_0 - \varepsilon))$  has a solution on  $[0, 2\varepsilon]$ . Since the solution  $(V^1, V^2)$  is unique on  $[0, T_0)$  this provides a continuation of  $(V^1, V^2)$  beyond  $T_0$ .  $\square$

### 3 Local existence in critical spaces

We seek to establish as in the previous section the local well-posedness but, with lower regularity assumptions on the initial data. In other words, we want to prove Theorem 1.3. In fact it's well known since the works of R. Danchin in [10], that the barotropic compressible Navier-Stokes equation has a unique solution in the critical setting (here critical spaces means that the initial density and velocity belong to  $\dot{B}_{2,1}^{\frac{d}{2}}$  and  $\dot{B}_{2,1}^{\frac{d}{2}-1}$ , respectively). This motivates us to prove the local existence in spaces less regular than in the previous section for a class of systems of type (1). The price to pay is that some restrictions on the structure of matrices (assumption **C**) are needed.

We recall that the initial data  $(U_0^1, U_0^2)$  belongs to  $O_0^1 \times \mathbb{R}^{n_1}$  with  $O_0^1$  a bounded open set such that  $\overline{O_0^1} \subset \mathcal{U}^1$  (see (15)). Let us point out that the component  $U_0^2$  of the initial data isn't necessary bounded. Hence we aim at proving local in time existence with initial large data and unbounded second component. This is one of the motivation of this section.

Let us first state some a priori estimates for the system (11) under the assumptions **C**. Here we omit the lower order source term  $f$  in (11) for simplicity, since it is only responsible for the large-time behavior of solutions.

#### 3.1 A priori estimates.

Let  $U$  be a smooth solution of the Cauchy problem (11) and (16) on  $Q_T = [0, T] \times \mathbb{R}^d$  satisfying  $V^1 \in \tilde{L}^\infty(0, T; \dot{B}_{2,1}^{\frac{d}{2}})$  and  $V^2 \in \tilde{L}^\infty(0, T; \dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L^1(0, T; \dot{B}_{2,1}^{\frac{d}{2}+1})$  with  $\partial_t V^2 \in L^1(0, T; \dot{B}_{2,1}^{\frac{d}{2}-1})$ . We set:

$$M_1 \stackrel{\text{def}}{=} \|V^1\|_{\tilde{L}^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|V^2\|_{\tilde{L}^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \quad (92)$$

and

$$M_2 \stackrel{\text{def}}{=} \int_0^T \|V^2(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} dt \quad M_3 \stackrel{\text{def}}{=} \int_0^T \|\partial_t V^2(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} dt. \quad (93)$$

We assume also that there exists a bounded open subset  $O^1$  of  $\mathcal{U}^1$  satisfying  $\overline{O^1} \subset \mathcal{U}^1$  such that:

$$U^1(t, x) \in O^1 \quad \text{for } t \in [0, T], x \in \mathbb{R}^d. \quad (94)$$

Due to (94) and the continuity of the function  $U^1 \mapsto S_{22}^0(U^1)$  we have the following inequalities that are similar to (23):

$$C^{-1}I_{n_1} \leq S_{22}^0(U^1) \leq CI_{n_1}, \quad (95)$$

where the positive constant  $C$  depends only on  $O^1$ . We have the following results.

**Proposition 3.** *There exists a constant  $C$  depending only on  $d, O^1$  and data such that*

$$\begin{aligned} \|V^1\|_{\tilde{L}^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} &\leq 4 \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \quad \text{and} \quad \|V^2\|_{\tilde{L}^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq 2 \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}, \\ \int_0^T \|V^2(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} d\tau + \int_0^T \|(\partial_t V^1(\tau), \partial_t V^2(\tau))\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} d\tau &< C, \\ V^1(t, x) &\in O^1 \quad \text{for } (t, x) \in Q_T. \end{aligned}$$

We divide the proof of the proposition into several steps.

**Step 1: Estimates for  $V^1$ .**

**Proposition 4.** *Let the structure assumptions  $\mathbf{C}$  be in force and  $d \geq 2$ . Then there exists a constant  $C = C(O^1, M_1)$  such that for all  $m \in \mathbb{Z}$ , the following inequalities hold true.*

$$\sum_{j \geq m} 2^{j\sigma} \|V_j^1\|_{L_T^\infty(L^2)} \leq \sum_{j \geq m} 2^{j\sigma} \|V_{0,j}^1\|_{L^2} + CM_2 \left( \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + 1 \right), \quad (96)$$

where we define  $V_j^1 \stackrel{\text{def}}{=} \dot{\Delta}_j V^1$ ,  $V_{0,j}^1 \stackrel{\text{def}}{=} \dot{\Delta}_j(V_0^1)$ .

*Proof.* Throughout the proof,  $C$  stands for a positive constant that depends only on  $O^1$  and on other parameters which are independent of our system. From (11) and assumption  $\mathbf{C}$ ,  $V^1$  is solution of the following equation

$$\partial_t V^1 + \sum_{\alpha=1}^d (A_{11}^\alpha(U^2) \partial_\alpha V^1 + A_{12}^\alpha(U^1) \partial_\alpha V^2) = 0,$$

with  $A_{11}^\alpha \stackrel{\text{def}}{=} (S_{11}^0)^{-1} S_{11}^\alpha$  and  $A_{12}^\alpha \stackrel{\text{def}}{=} (S_{11}^0)^{-1} S_{12}^\alpha$  for all  $\alpha = 1, \dots, d$ .

Applying the non-homogeneous dyadic block to the above equation yields,

$$\partial_t V_j^1 + \sum_{\alpha=1}^d A_{11}^\alpha(U^2) \partial_\alpha V_j^1 = - \sum_{\alpha=1}^d \dot{\Delta}_j (A_{12}^\alpha(U^1) \partial_\alpha V^2) + R_j^{11}, \text{ with } R_j^{11} \stackrel{\text{def}}{=} [A_{11}^\alpha(U^2), \dot{\Delta}_j](\partial_\alpha V^1).$$

Next, following the classical procedure for hyperbolic system (see the Step 1 of the proof of Lemma 4) we arrive at

$$\begin{aligned} 2^{j\sigma} \|V_j^1\|_{L_T^\infty(L^2)} &\leq 2^{j\sigma} \|V_j^1(0)\|_{L^2} + C 2^{j\sigma} \sum_{\alpha=1}^d \int_0^t \|\nabla A_{11}^\alpha(U^2)\|_{L^\infty} \|V_j\|_{L^2} \\ &\quad + C 2^{j\sigma} \sum_{\alpha=1}^d \int_0^t \left( \|\dot{\Delta}_j (A_{12}^\alpha(U^1) \partial_\alpha V^2)\|_{L^2} + \|R_j^{11}\|_{L^2} \right). \end{aligned} \quad (97)$$

It is clear that for closing our estimate, we need that for all  $t \in [0, T]$ ,  $\nabla A_{11}^\alpha(U(t)) \in L^\infty(\mathbb{R}^d)$ . However, our critical functional framework does not ensure that  $\nabla V^1(t)$  belongs to  $L^\infty(\mathbb{R}^d)$ . Hence, to be under control, the terms  $A_{11}^\alpha$ , for  $\alpha = 1, \dots, d$  have to depend only on  $U^2$ . Moreover, as  $U^2$  does not need to be bounded on  $Q_T$ , one has to assume that  $A_{11}^\alpha$  is at most linear with respect to  $U^2$ .

The terms  $\|(\dot{\Delta}_j (A_{12}^\alpha(U^1) \partial_\alpha V^2))\|_{L_T^1(L^2)}$  for  $\alpha = 1, \dots, d$  may be bounded according to the stability of the space  $\dot{B}_{2,1}^{\frac{d}{2}}$  by numerical product and Proposition 10. One has

$$\begin{aligned} \|\dot{\Delta}_j (A_{12}^\alpha(U^1) \partial_\alpha V^2)\|_{L_T^1(L^2)} &\leq C 2^{-\frac{d}{2}j} c_j \|A_{12}^\alpha(U^1) \partial_\alpha V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}})} \\ &\leq C c_j (1 + \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \|V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}. \end{aligned} \quad (98)$$

As we can see, in the above estimate, to bound the term  $\|\dot{\Delta}_j (A_{12}^\alpha(U^1) \partial_\alpha V^2)\|_{L_T^1(L^2)}$ , we need that  $A_{12}^\alpha(U) - A_{12}^\alpha(\bar{U})$  is in  $L^\infty(0, T; \dot{B}_{2,1}^{\frac{d}{2}})$ . Hence we are stuck to the framework where all the matrices  $A_{12}^\alpha(U)$  depend only on  $U^1$ .

Thanks to Proposition 8, we have the following bound for  $R_j^{11}$ :

$$\|R_j^{11}\|_{L_T^1(L^2)} \leq C 2^{-j\frac{d}{2}} c_j \|\nabla A_{11}^\alpha(U^2)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}})} \|V^1\|_{L_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}.$$

Using the fact that  $A_{11}^\alpha(U^2)$  is at most linear, we finally get

$$\|R_j^{11}\|_{L_T^1(L^2)} \leq C2^{-j\frac{d}{2}}c_j \|V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}. \quad (99)$$

Plugging (99) and (98) into (97), using  $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$  and summing over  $j \geq m$  yields (96).  $\square$

The second step is devoted to bounding  $V^2$  and  $\partial_t V$ .

**Step 2: Estimates for  $V^2$  and  $\partial_t V$ .** We turn to the second equation of (11), under assumptions **C**, namely

$$S_{22}^0(U^1)\partial_t V^2 + \sum_{\alpha=1}^d (S_{21}^\alpha(U)\partial_\alpha V^1 + S_{22}^\alpha(U)\partial_\alpha V^2) - \sum_{\alpha,\beta=1}^d \partial_\alpha(Z^{\alpha\beta}(U^1)\partial_\beta V^2) = 0,$$

which can be rewritten as follows:

$$\begin{aligned} S_{22}^0(U_m^1)\partial_t V^2 - \sum_{\alpha,\beta=1}^d \partial_\alpha(Z^{\alpha\beta}(U_m^1)\partial_\beta V^2) &= - \sum_{\alpha=1}^d (S_{21}^\alpha(U)\partial_\alpha V^1 + S_{22}^\alpha(U)\partial_\alpha V^2) \\ &+ (S_{22}^0(U_m^1) - S_{22}^0(U^1))\partial_t V^2 + \sum_{\alpha,\beta=1}^d \partial_\alpha((Z^{\alpha\beta}(U^1) - Z^{\alpha\beta}(U_m^1))\partial_\beta V^2), \end{aligned} \quad (100)$$

where we denote

$$U_m \stackrel{\text{def}}{=} \bar{U} + V_m \stackrel{\text{def}}{=} \bar{U} + \dot{S}_m V = \bar{U} + \sum_{j \leq m-1} \dot{\Delta}_j V. \quad (101)$$

Note that the localisation of  $Z^{\alpha\beta}$  and  $S_{22}^0$  in (100) allows us to consider a parabolic equation with smooth (and decaying) coefficients. The perturbation terms induced by these localizations (that is the last two terms of r.h.s of (100)) may be treated as harmless source terms, since the prefactors  $S_{22}^0(U_m^1) - S_{22}^0(U^1)$  and  $Z^{\alpha\beta}(U^1) - Z^{\alpha\beta}(U_m^1)$  are sufficiently small, provided that we choose  $m$  large enough.

If  $V^1$  belongs to  $\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})$ , then  $U_m^1$  tends to  $U^1$  uniformly on  $[0, T] \times \mathbb{R}^d$ . Hence we have  $U_m^1 \in O^1$  for  $m$  large enough. Furthermore, taking advantage of Bernstein's inequality, there exists a constant  $C > 0$  independent of  $m$  so that for all real numbers  $\gamma \geq \frac{d}{2}$ :

$$\|V_m^1\|_{\dot{B}_{2,1}^\gamma} \leq C2^{m(\gamma-\frac{d}{2})} \|V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \quad (102)$$

We aim at getting uniform estimates on  $V^2$  in suitable Besov spaces. For that, as in the previous section, we consider the unknown  $V^S \stackrel{\text{def}}{=} V^2 - V^L$  where  $V^L$  stands for the solution of (66). This function satisfies the following parabolic system:

$$\begin{cases} S_{22}^0(U_m^1)\partial_t V^S - \sum_{\alpha,\beta=1}^d \partial_\alpha(Z^{\alpha\beta}(U_m^1)\partial_\beta V^S) \\ = \sum_{\alpha,\beta=1}^d \partial_\alpha((Z^{\alpha\beta}(U^1) - Z^{\alpha\beta}(U_m^1))\partial_\beta V^S) + R^t + R^{21} + R^{22} + R^L, \\ V^S(0) = 0, \end{cases} \quad (103)$$

where  $r^{\alpha\beta}(U^1) = Z^{\alpha\beta}(U^1) - \bar{Z}^{\alpha\beta}$  and

$$\begin{aligned} R^t &\stackrel{\text{def}}{=} (S_{22}^0(U_m^1) - S_{22}^0(U^1))\partial_t V^2, \quad R^{22} \stackrel{\text{def}}{=} - \sum_{\alpha=1}^d S_{22}^\alpha(U)\partial_\alpha V^2, \\ R^L &\stackrel{\text{def}}{=} (\bar{S}_{22}^0 - S_{22}^0(U_m^1))\partial_t V^L + \sum_{\alpha,\beta=1}^d \partial_\alpha(r^{\alpha\beta}(U^1)\partial_\beta V^L), \quad R^{21} \stackrel{\text{def}}{=} - \sum_{\alpha=1}^d S_{21}^\alpha(U)\partial_\alpha V^1. \end{aligned} \quad (104)$$

We have the following result.

**Proposition 5.** *Under the hypotheses of proposition 4 there exists a constant  $C$  depending on  $O^1$  and  $\|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$ ,  $\|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}$  such that setting*

$$\begin{aligned} \mathfrak{A}^S(T) &\stackrel{\text{def}}{=} \|V^S\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|V^S\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}; \quad \mathfrak{A}^L(T) \stackrel{\text{def}}{=} \|\partial_t V^L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|V^L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ \mathfrak{A}_m^S(T) &\stackrel{\text{def}}{=} \sum_{j=-\infty}^m \left( 2^{(\frac{d}{2}-1)j} \|\dot{\Delta}_j V^2\|_{L_T^\infty(L^2)} + 2^{(\frac{d}{2}+1)j} \|\dot{\Delta}_j V^2\|_{L_T^1(L^2)} \right) \end{aligned} \quad (105)$$

and assuming in addition  $T \leq 1$ , we have, for all  $m \in \mathbb{Z}$

$$\begin{aligned} &\left( 1 - CT - 2^m C \|\partial_t V^1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \right) \mathfrak{A}_m^S(T) \\ &\leq C \left( 2^m \sqrt{T} M_1 + \|V - V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + (1 + M_1)^2 \sqrt{T} \right) \mathfrak{A}^S(T) + C(1 + M_1) (\mathfrak{A}^S(T))^2 \\ &+ C(1 + M_1)^2 \left( \sqrt{T \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}} \mathfrak{A}^L(T) + \mathfrak{A}^L(T) \right) + C(1 + M_1) \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \mathfrak{A}^L(T) \\ &+ C \left( \|V^1 - V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + 2^m \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \right) M_3. \end{aligned} \quad (106)$$

Moreover,

$$\|(\partial_t V^1, \partial_t V^2)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq C(1 + M_1)^2 \left( \sqrt{T} M_1 + \sqrt{T} \sqrt{M_1 M_2} + M_2 \right). \quad (107)$$

*Proof.* Let us stress that  $V_m^1$  is bounded since,  $\|V_m^1\|_{L^\infty} \leq C\|V^1\|_{L^\infty}$  and  $V^1$  is bounded. Apply  $\dot{\Delta}_j$  to (103) to get :

$$S_{22}^0(U_m^1) \partial_t V_j^S - Z^{\alpha\beta}(U_m^1) \partial_\alpha \partial_\beta V_j^S = R_j^L + R_j^{21} + R_j^{22} + R_j^S + R_j^t + E_j^S + E_j^t,$$

where we denote:

$$\begin{aligned} R_j^L &\stackrel{\text{def}}{=} \dot{\Delta}_j R^L; \quad R_j^t \stackrel{\text{def}}{=} \dot{\Delta}_j R^t; \quad R_j^{21} \stackrel{\text{def}}{=} \dot{\Delta}_j R^{21}; \quad R_j^{22} \stackrel{\text{def}}{=} \dot{\Delta}_j R^{22}; \\ R_j^S &\stackrel{\text{def}}{=} \sum_{\alpha, \beta=1}^d \left[ \dot{\Delta}_j, Z^{\alpha\beta}(U_m^1) \right] \partial_\alpha \partial_\beta V^S + \sum_{\alpha, \beta=1}^d \dot{\Delta}_j (\partial_\alpha (Z^{\alpha\beta}(U_m^1))) \partial_\beta V^S \\ E_j^m &\stackrel{\text{def}}{=} \sum_{\alpha, \beta=1}^d \dot{\Delta}_j \partial_\alpha (Z^{\alpha\beta}(U^1) - Z^{\alpha\beta}(U_m^1)) \partial_\beta V^S, \quad E_j^t \stackrel{\text{def}}{=} -[\dot{\Delta}_j, S_{22}^0(U_m^1)](\partial_t V^S). \end{aligned}$$

Perform the energy method for parabolic system (see the Step 2 of the proof of Lemma 4) to get:

$$\begin{aligned} \|V_j^S\|_{L_T^\infty(L^2)} + 2^{2j} c \int_0^t \|V_j^S\|_{L^2} &\leq C \int_0^t \|V_j^S\|_{L^2} (1 + \|\partial_t (S_{22}^0(U_m))\|_{L^\infty}) \\ &+ \int_0^t \|(R_j^L, R_j^{21}, R_j^{22}, R_j^S, E_j^S, R_j^t, E_j^t)\|_{L^2}, \end{aligned} \quad (108)$$

where  $c > 0$  and  $C > 0$  depending on  $O^1$ .

Note that the embedding  $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$  and the fact that  $U^1$  is bounded combined with (102) lead to :

$$\|\partial_t (S_{22}^0(U_m^1))\|_{L^\infty} \leq C \|\partial_t V_m^1\|_{L^\infty} \leq C \|\partial_t V_m^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq C 2^m \|\partial_t V^1\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}. \quad (109)$$



Owing to Proposition 9 and 10, we have for all  $\alpha, \beta = 1 \cdots, d$

$$\begin{aligned} \|(\bar{S}_{22}^0 - S_{22}^0(U_m^1))\partial_t V^L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} &\leq C \|V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\partial_t V^L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})}, \\ \|\partial_\alpha(r^{\alpha\beta}(U^1)\partial_\beta V^L)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} &\leq C \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|V^L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}, \end{aligned}$$

which combined with (102) yields the following bound on  $R_j^L$ ,

$$\|R_j^L\|_{L_T^1(L^2)} \leq C 2^{-j(\frac{d}{2}-1)} c_j \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \left( \|\partial_t V^L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|V^L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \right). \quad (110)$$

The next step is to bound  $R_j^{22}$  and  $R_j^{21}$  in  $L_T^1(L^2)$ . The term  $R_j^{21}$  can be decomposed as

$$R_j^{21} = - \sum_{\alpha=1}^d S_{21}^\alpha(\bar{U}) \partial_\alpha V^1 + \sum_{\alpha=1}^d \left( S_{21}^\alpha(\bar{U}) - S_{21}^\alpha(U) \right) \partial_\alpha V^1.$$

Now bearing in mind the structure of the matrices  $S_{21}^\alpha$  (that is, the second derivative of  $S_{21}^\alpha$  w.r.t.  $V^2$  vanishes) and using directly Propositions 11 (especially (196) with  $s = \frac{d}{2}$ ) and 9 to the second term in the previous identity yields

$$\begin{aligned} \|R_j^{21}\|_{L_T^1(L^2)} &\leq C 2^{-j(\frac{d}{2}-1)} c_j \left( (T + T \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \right. \\ &\quad \left. + \sqrt{T}(1 + \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \right) \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}. \quad (111) \end{aligned}$$

Since, for all  $\alpha = 1, \dots, d$ ,  $S_{22}^\alpha$  has the same structure as  $S_{21}^\alpha$ , the term  $R_j^{22}$  may be bounded by employing the same techniques allowing to get (111). There holds

$$\begin{aligned} \|R_j^{22}\|_{L_T^1(L^2)} &\leq C 2^{-j(\frac{d}{2}-1)} c_j \left( \sqrt{T} + \sqrt{T} \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \right. \\ &\quad \left. + (1 + \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \right) \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})}. \quad (112) \end{aligned}$$

To bound the term  $R_j^t$  we take advantage of Propositions 9 and 10 that give

$$\|R_j^t\|_{L_T^1(L^2)} \leq C 2^{-j(\frac{d}{2}-1)} c_j \|V^1 - V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\partial_t V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})}. \quad (113)$$

Bounding the last term  $R_j^S$  and  $E_j^m$  involves Propositions 8 (with  $\sigma = \frac{d}{2} - 1$ ), 10 (with  $s = \frac{d}{2}$ ) and 9 combined with the fact that  $U_m^1$  is bounded (let us notice that Propositions 8, 9 require that  $d \geq 2$ ). We have for all  $\alpha, \beta = 1, \dots, d$ ,

$$\begin{aligned} \blacktriangleright &\left\| \left[ \dot{\Delta}_j, Z^{\alpha\beta}(U_m^1) \right] (\partial_\alpha \partial_\beta V^S) \right\|_{L_T^1(L^2)} \leq C 2^{-j(\frac{d}{2}-1)} c_j \sqrt{T} \|\nabla V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla V^S\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1})}, \\ \blacktriangleright &\left\| \dot{\Delta}_j (\partial_\alpha (Z^{\alpha\beta}(U_m^1)) \partial_\beta V^S) \right\|_{L_T^1(L^2)} \leq C 2^{-j(\frac{d}{2}-1)} c_j \sqrt{T} \|\nabla V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla V^S\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1})}, \\ \blacktriangleright &\left\| \dot{\Delta}_j \partial_\alpha ((Z^{\alpha\beta}(U^1) - Z^{\alpha\beta}(U_m^1)) \partial_\beta V^S) \right\|_{L_T^1(L^2)} \leq C 2^{-j(\frac{d}{2}-1)} c_j \|V - V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla V^S\|_{\tilde{L}_T^1(\dot{B}_{2,1}^{\frac{d}{2}})}, \\ \blacktriangleright &\left\| [\dot{\Delta}_j, S_{22}^0(U_m^1)] (\partial_t V^2) \right\|_{L_T^1(L^2)} \leq C 2^{-j(\frac{d}{2}-1)} c_j \|\nabla V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\partial_t V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-2})}. \end{aligned} \quad (114)$$

Then, owing to (102), we deduce then

$$\begin{aligned} \|E_j^m\|_{L_T^1(L^2)} &\leq C2^{-j(\frac{d}{2}-1)}c_j \|V - V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|V^S\|_{\tilde{L}_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \\ &\leq C2^{-j(\frac{d}{2}-1)}c_j C \|V - V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \mathfrak{A}^S(T), \end{aligned} \quad (115)$$

and

$$\|R_j^S\|_{L_T^1(L^2)} \leq C2^{-j(\frac{d}{2}-1)}2^m c_j \sqrt{T} \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|V^S\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})},$$

which, combined with interpolation inequality yields

$$\|R_j^S\|_{L_T^1(L^2)} \leq C2^{-j(\frac{d}{2}-1)}2^m c_j \sqrt{T} \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \mathfrak{A}^S(T). \quad (116)$$

Using Bernstein inequality and the injection  $\dot{B}_{2,1}^{\frac{d}{2}-1} \hookrightarrow \dot{B}_{2,1}^{\frac{d}{2}-2}$  in the last inequality of (114) insures that

$$\|E_j^t\|_{L_T^1(L^2)} \leq C2^{-j(\frac{d}{2}-1)}2^m c_j \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\partial_t V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})}. \quad (117)$$

Inserting (117), (116), (115), (113), (112), (111), (110) and (109) into (108), then summing over  $j \leq m$ , and using the definition of  $M_1, M_2, M_3$  (we use also  $T \leq 1$ ), we end up with

$$\begin{aligned} &(1 - CT - 2^m C \|\partial_t V^1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})}) \mathfrak{A}_m^S(T) \\ &\leq C \left( 2^m \sqrt{T} \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|V - V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \right) \mathfrak{A}^S(T) + CM_1 \mathfrak{A}^L(T) \\ &\quad + c_j C \left( \|V^1 - V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + 2^m \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \right) \|\partial_t V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \\ &\quad + C(1 + M_1) \left( \sqrt{T}(1 + \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})}) M_1 + \sqrt{T} \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} + \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})}^2 \right). \end{aligned}$$

Finally, using again interpolation inequality to  $V^2$  gives

$$\begin{aligned} \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} &\leq \|(V^L, V^S)\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \\ &\leq \mathfrak{A}^S + \sqrt{\|V^L\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \|V^L\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}} \\ &\leq \mathfrak{A}^S + \sqrt{\|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \mathfrak{A}^L}, \end{aligned} \quad (118)$$

and simplifying redundant terms yields (106).

To prove (107) we recall that

$$\partial_t V^1 + \sum_{\alpha=1}^d (A_{11}^\alpha(U) \partial_\alpha V^1 + A_{12}^\alpha(U) \partial_\alpha V^2) = 0,$$

$$\partial_t V^2 + (S_{22}^0(U^1))^{-1} \sum_{\alpha=1}^d (S_{21}^\alpha(U) \partial_\alpha V^1 + S_{22}^\alpha(U) \partial_\alpha V^2) - (S_{22}^0(U^1))^{-1} \sum_{\alpha,\beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U^1) \partial_\beta V^2) = 0.$$

Then, thanks to Propositions 9 and 11 and remembering the structure assumption **C**, we have for all  $\alpha, \beta = 1, \dots, d$ ,

$$\begin{aligned}
\blacklozenge & \|A_{11}^\alpha(U)\partial_\alpha V^1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq C(T + \sqrt{T} \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})}) \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}, \\
\blacklozenge & \|A_{12}^\alpha(U)\partial_\alpha V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq C\sqrt{T}(1 + \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})}, \\
\blacklozenge & \|(S_{22}^0(U^1))^{-1}S_{21}^\alpha(U)\partial_\alpha V^1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq C(1 + \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})})(T \\
& \quad + T \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \sqrt{T} \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})}) \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}, \\
\blacklozenge & \|(S_{22}^0(U^1))^{-1}S_{22}^\alpha(U)\partial_\alpha V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq C(1 + \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})})(\sqrt{T} \\
& \quad + \sqrt{T} \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})} \|V^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})}, \\
\blacklozenge & \|(S_{22}^0(U^1))^{-1}\partial_\alpha(Z^{\alpha\beta}(U^1)\partial_\beta V^2)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq C(1 + \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})})^2 \|V^2\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}.
\end{aligned}$$

We deduce by means of interpolation inequality (118) and of the fact  $T \leq 1$ , that

$$\|(\partial_t V^1, \partial_t V^2)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq C(1 + M_1)^2 \left( \sqrt{T}M_1 + \sqrt{T}\sqrt{M_1M_2} + M_2 \right).$$

This completes the proof of the Proposition.  $\square$

Let us now state the last ingredient needed to prove Proposition 3.

**Step 3: Closing the estimates** In this part we are going to prove the priori estimate of the solution  $V$  in the set determined by the following constraints:

$$\begin{aligned}
\text{(C1)} & \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \leq 4 \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \\
\text{(C2)} & \|V^2\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq 2 \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}, \\
\text{(C3)} & \|V^1 - V_m^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \leq \eta_m, \\
\text{(C4)} & \|V^S\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \int_0^T \|V^S\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \leq \eta, \\
\text{(C5)} & \int_0^T \|\partial_t V\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \leq \eta_T, \\
\text{(C6)} & |U^1(t, x) - U_0^1(x)| \leq d_1 \quad \text{for any } (t, x) \in Q_T, \\
\text{(C7)} & \|V^2 - V_m^2\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq \eta_m,
\end{aligned}$$

where  $\eta_T, \eta_m > 0$  and  $0 < \eta < 1$  are constants, that will be determined in the sequel;  $d_1 < \text{dist}(O_0, \partial\mathcal{U}^1)$ . Let  $O^1$  be a bounded open convex set in  $\mathbb{R}^n$  satisfying

$$O^1 = d_1 - \text{neighborhood of } \mathcal{O}_0^1. \quad (119)$$

Then, from (C6), we have  $U^1(t, x) \in O^1$  for any  $(t, x) \in Q_T$ .

We are going to prove that under suitable assumption on  $T(\leq 1)$  and  $\eta$  (to be specified below) if Conditions (C1) to (C7) are satisfied, then they are actually satisfied with *strict* inequalities. Since all those conditions depend continuously on the time and are strictly satisfied initially, a basic bootstrap argument insures that (C1) to (C7) are indeed satisfied for  $T$ .

Let  $\eta^L > 0$ . According to Proposition 6, there exists  $T^L > 0$  such that  $V^L$  verifies (see Proposition 5 for the definition of  $\mathfrak{A}^L(T)$ )

$$\mathfrak{A}^L(T^L) \leq \eta^L. \quad (120)$$

From (C1) to (C5) and the definition of  $M_1, M_2, M_3$  in (92),(93) we have on the one hand

$$M_1 \leq 4 \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + 2 \|V_0^2\|_{\dot{B}_{2,1}^s} \quad \text{and} \quad M_3 \leq \eta_T. \quad (121)$$

Bearing in mind that  $V^2 = V^L + V^S$  and taking advantage of (163), we have on the other hand

$$\|V^2\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq \eta + \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \quad \text{and} \quad M_2 \leq \|V^L, V^S\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \leq \eta + \eta^L. \quad (122)$$

So, up to changing  $M_1$  (resp.  $M_2$ ) in the right-hand side of the first inequality of (121) (resp. second inequality of (122)), we can suppose that  $M_1$  (resp.  $M_2$ ) depends only on initial data (resp.  $\eta, \eta^L$ ).

We shall assume that  $M_2$  (so  $\eta, \eta^L$ ) satisfies

$$CM_2 < \frac{1}{2}. \quad (123)$$

Recall that, from Proposition 4,  $V^1$  satisfies for all  $m \in \mathbb{Z}$ ,

$$\sum_{j \geq m} 2^{j\frac{d}{2}} \|V_j^1\|_{L_T^\infty(L^2)} \leq \sum_{j \geq m} 2^{j\frac{d}{2}} \|V_j^1(0)\|_{L^2} + CM_2 \left( \|V^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + 1 \right). \quad (124)$$

Taking the limit when  $m$  goes to  $-\infty$  in the last inequality and using (123) we deduce that (C1) is satisfied with strict inequality. Then, the inequality (124) becomes

$$\sum_{j \geq m} 2^{j\frac{d}{2}} \|V_j^1\|_{L_T^\infty(L^2)} \leq \sum_{j \geq m} 2^{j\frac{d}{2}} \|V_j^1(0)\|_{L^2} + CM_2 \left( 4 \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + 1 \right).$$

Hence, (C3) is satisfied with strict inequality provided that  $M_2$  and  $m$  satisfy

$$\sum_{j \geq m} 2^{j\frac{d}{2}} \|V_j^1(0)\|_{L^2} < \frac{\eta_m}{2} \quad \text{and} \quad M_2 < \frac{\eta_m}{2C(4 \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + 1)}. \quad (125)$$

Next, as  $T \leq 1$ , Proposition 5 can be applied. Under assumptions (C4), (C3), (C4) the inequality (106) can be reduced to:

$$\begin{aligned} & \left( 1 - CT - 2^m \sqrt{T} CM_1 - C\eta_m - 2^m \eta_T (1 + M_1)^2 \sqrt{T} - C(1 + M_1)\eta \right) \mathfrak{A}^S(T) \\ & \leq C(\eta_m + 2^m M_2)\eta_T + C(1 + M_1)^2 \left( \sqrt{T} \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \eta^L + \eta^L \right) + C(1 + M_1) \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \eta^L. \end{aligned} \quad (126)$$

Assuming that

$$T < \frac{1}{16C}; \quad \sqrt{T} \leq \frac{1}{16C2^m(1 + M_1)^2}; \quad \eta_m \leq \frac{1}{16C}; \quad \eta_T < \frac{1}{32C}; \quad \eta < \frac{1}{32C(1 + M_1)} \quad (127)$$

so that the prefactor of the left-hand side of (126) is less than  $1/2$ , and remembering Inequality (120) and the fact  $M_2 \leq \eta_m$  (according to (125)), we can simplify (126) as follows:

$$\mathfrak{A}^S(T) \leq C(1 + 2^m)\eta_m\eta_T + C(1 + M_1)^2 \left( (\|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + 1)^{1/2} + \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \right) (\eta^L + \sqrt{\eta^L})$$

from which we deduce that (C4) is satisfied with strict inequality if (for instance)  $\eta, \eta^L, T$  and  $M_2$  verify

$$(\eta^L + \sqrt{\eta^L}) < \frac{\eta}{16C(1 + M_1)^2 \left( (\|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + 1)^{1/2} + \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \right)}; \quad \eta_T\eta_m < \frac{\eta}{16C(1 + 2^m)}. \quad (128)$$

Let us notice that under conditions (128) and (127), the inequality (106) of Proposition 5 becomes

$$\mathfrak{A}_m^S(T) < \frac{1}{2}\mathfrak{A}^S(T) + \eta < 2\eta.$$

Then combining this inequality and inequality (164) which provides

$$\sum_{j \geq m} 2^{j(\frac{d}{2}-1)} \|\dot{\Delta}_j V^L\|_{L^\infty(L^2)} \leq \sum_{j \geq m} 2^{j(\frac{d}{2}-1)} \|\dot{\Delta}_j V_0^2\|_{L^2}$$

we deduce that assumption (C7) holds with strict inequality if

$$2\eta < \frac{\eta_m}{2} \quad \text{and} \quad \sum_{j \geq m} 2^{j(\frac{d}{2}-1)} \|\dot{\Delta}_j V_0^2\|_{L^2} < \frac{\eta_m}{2}. \quad (129)$$

Under assumption (C3) and from (122), we deduce that (C2) is satisfied with strict inequality provided that

$$\eta < \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}. \quad (130)$$

Let us apply (107). We get, under  $M_2 \leq 1$

$$\|(\partial_t V^1, \partial_t V^2)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} \leq C(1 + M_1)^2 \left( \sqrt{T}M_1 + \sqrt{T}\sqrt{M_1} + M_2 \right).$$

Hence (C5) is satisfied with a strict inequality provided that  $T, M_2$  (so  $\eta, \eta^L$ ) satisfy

$$\sqrt{T} < \frac{\eta_T}{2C(1 + M_1)^2(M_1 + \sqrt{M_1})} \quad \text{and} \quad M_2 < \frac{\eta_T}{2C(1 + M_1)^2}. \quad (131)$$

In order to check whether (C6) is satisfied, we use the fact that:

$$V^1 - V_0^1 = \dot{S}_{m+1}(V^1 - V_0^1) + (\text{Id} - \dot{S}_{m+1})(V^1 - V_0^1).$$

Then, the embedding  $\dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty$  and the fact that  $\dot{\Delta}_j(\text{Id} - \dot{S}_{m+1}) = 0$  if  $j < m$  ensure that for all  $(t, x) \in Q_T$

$$|V^1(t, x) - V_0^1(x)| \leq C \left( \int_0^t \left\| \dot{S}_{m+1} \partial_t V^1(\tau) \right\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau + \sum_{j \geq m-1} 2^{j\frac{d}{2}} \left\| \dot{\Delta}_j (V^1(t) - V_0^1) \right\|_{L^2} \right).$$

Hence using (C3), (C5) and Bernstein inequality lead to:

$$|U^1(t, x) - U_0^1(t, x)| \leq C(2^{m+1}\eta_T + \eta_m)$$

which yields (C6) with a strict inequality if

$$C\eta_T < \frac{d_1}{2^{m+2}} \quad \text{and} \quad C\eta_m < \frac{d_1}{2}. \quad (132)$$

The continuity of the operator  $\dot{S}_m$  and (C6) guarantee that: for all  $(t, x) \in Q_T$ ,

$$|U_m^1(t, x) - U_0^1(x)| \leq \sup_{0 \leq t \leq T} \|U^1(t, \cdot) - U_0^1(x)\|_{L^\infty} < d_1. \quad (133)$$

Then, (C6) and (132) imply that  $U_m^1(t, x) \in O^1$ , for all,  $(t, x) \in Q_T$ .

**Remark 3.1.** *The existence time  $T$  may be bounded from below by*

$$T \geq \min \left\{ \mathbf{P}; T^L \right\}. \quad (134)$$

where  $\mathbf{P}$  is a non-increasing positive function of  $\|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$  and  $\|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}$  which depends also on the matrices involved in system (11) and  $d$ .

### 3.2 The Proof of Existence

We use a standard scheme:

1. We smooth out the data and get a sequence of smooth solutions  $(V_n^1, V_n^2)$  to (11) under the assumption **C** on a bounded interval  $[0, T^n]$  which may depend on  $n$ .
2. We exhibit a positive lower bound  $T \leq 1$  for  $T^n$  and prove uniform estimates in the space

$$\mathcal{E}_T = \tilde{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}}) \times \left( \tilde{C}([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1}) \right)$$

for the smooth solution  $(V_n^1, V_n^2)$ .

3. We use compactness to prove that the sequence converges, up to extraction, to a solution of (11).

**First step:** We smooth out the data as follows:

$$V_{0,n} = (\dot{S}_n - \dot{S}_{-n})V_0.$$

Note that  $V_{0,n}$  belongs to all nonhomogeneous Besov spaces  $B_{2,1}^s$ , and that

$$\forall n \in \mathbb{N}; \quad \|V_{0,n}^1\|_{B_{2,1}^{\frac{d}{2}}} \leq C_n \|V_0^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}; \quad \|V_{0,n}^2\|_{B_{2,1}^{\frac{d}{2}-1}} \leq C_n \|V_0^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}.$$

$V_{0,n}^1$  (resp.  $V_{0,n}^2$ ) tends to  $V_0^1$  (resp.  $V_0^2$ ) in  $\dot{B}_{2,1}^{\frac{d}{2}}$  (resp.  $\dot{B}_{2,1}^{\frac{d}{2}-1}$ ).

Now, according to Theorem 1.2 (note that assumption **(C)** implies assumption **(BB)**), one can solve (11) under assumption **(C)** with the smooth data  $(V_{0,n}^1, V_{0,n}^2)$ . For all  $s \geq \frac{d}{2}$ , we get a solution  $(V_n^1, V_n^2)$  on a non trivial time interval  $[0, T]$  such that

$$(V_n^1, V_n^2) \in \tilde{C}([0, T]; B_{2,1}^{s+1}) \times \left( \tilde{C}([0, T]; B_{2,1}^s) \cap L_T^1(B_{2,1}^{s+2}) \right). \quad (135)$$

**Second step** : Let  $T^{n,s}$  be the lifespan of  $V_n$ , that is, the supremum of all  $T$  such that the system (11) under assumption (C) with data  $V_{0,n}$  has a solution which satisfies (135). Let  $T \in [0, \min(1, T^{n,s})]$ ,  $m, \eta, \eta_m$  and  $\eta_T$  be chosen such that conditions (123), (125), (127), (128), (129), (130), (131) and (132) are satisfied (this can be ensured independently of  $n$ ). Then  $V_n$  satisfies (C1)- (C7) thus is bounded independently of  $n$  on  $[0, T]$ .

We still have to showcase that  $T^{n,s}$  may be bounded by below by the supremum  $\bar{T}$  of all times  $T$  such that the conditions obtained in the stability step of Subsection 3.1. This is actually a consequence of the uniform bounds we have just obtained, and of Remark 3.1. We thus have  $T^{n,s} \geq \bar{T}$ .

**Last step:** We now have to prove that  $(V_n)_n$  tends (up to a subsequence) to some function  $V$  which belongs to  $\mathcal{E}_T$  and satisfies (11) with assumption (C). The proof is based on Ascoli's theorem, Fatou's Lemma and compact embeddings in Besov spaces. As similar arguments have been employed in [2, chapter 10], or [8], the details are left to the reader.

### 3.3 Uniqueness

Assume that we are given  $(V_1^1, V_1^2)$  and  $(V_2^1, V_2^2)$ , two solutions of (11) under assumption C (with the same data), satisfying the regularity assumptions of Theorem 1.3. Let us put  $U_i = V_i + \bar{U}$ . In order to show that these two solutions coincide, we shall give estimates for  $(\tilde{V}^1, \tilde{V}^2) \stackrel{\text{def}}{=} (V_2^1 - V_1^1, V_2^2 - V_1^2)$ . The proof will consist in obtaining suitable a priori estimates for the following system:

$$\begin{cases} \partial_t \tilde{V}^1 + \sum_{\alpha=1}^d A_{11}^\alpha(U_2^2) \tilde{V}^1 = h \\ S_{22}^0(U_{2,m}^1) \partial_t \tilde{V}^2 - \sum_{\alpha,\beta=1}^d Z^{\alpha\beta}(U_{2,m}^1) \partial_\alpha \partial_\beta \tilde{V}^2 = g \end{cases} \quad (136)$$

with  $h = h_1 + h_2 + h_3$  and  $g = \sum_{k=1}^7 g_k$ , where

$$h_1 = \sum_{\alpha=1}^d (A_{11}^\alpha(U_2^2) - A_{11}^\alpha(U_1^2)) \partial_\alpha V_1^1, \quad h_2 = - \sum_{\alpha=1}^d A_{12}^\alpha(U_2^1) \partial_\alpha \tilde{V}^2,$$

$$h_3 = - \sum_{\alpha=1}^d (A_{12}^\alpha(U_2^1) - A_{12}^\alpha(U_1^1)) \partial_\alpha V_1^2,$$

$$g_1 = g_{1,1} + g_{1,2} = -(S_{22}^0(U_{2,m}^1) - S_{22}^0(U_2^1)) \partial_t \tilde{V}^2 - ((S_{22}^0(U_1^1) - S_{22}^0(U_2^1))) \partial_t V_1^2,$$

$$g_2 = - \sum_{\alpha=1}^d (S_{22}^\alpha(U_2) - S_{22}^\alpha(U_1)) \partial_\alpha V_1^2, \quad g_3 = - \sum_{\alpha=1}^d S_{22}^\alpha(U_2) \partial_\alpha \tilde{V}^2$$

$$g_4 = g_{4,1} + g_{4,2} = - \sum_{\alpha=1}^d S_{21}^\alpha(U_2) \partial_\alpha \tilde{V}^1 - \sum_{\alpha=1}^d (S_{21}^\alpha(U_2) - S_{21}^\alpha(U_1)) \partial_\alpha V_1^1,$$

$$g_5 = - \sum_{\alpha,\beta=1}^d \partial_\alpha (Z^{\alpha\beta}(U_{2,m}^1)) \partial_\beta \tilde{V}^2,$$

$$g_6 = \sum_{\alpha,\beta=1}^d \partial_\alpha \left( (Z^{\alpha\beta}(U_2^1) - Z^{\alpha\beta}(U_{2,m}^1)) \partial_\beta \tilde{V}^2 \right), \quad g_7 = \sum_{\alpha,\beta=1}^d \partial_\alpha \left( (Z^{\alpha\beta}(U_2^1) - Z^{\alpha\beta}(U_1^1)) \partial_\beta V_1^2 \right).$$

Like in the previous section, one cannot avoid a loss of one derivative in the stability estimates, which leads us to proving the uniqueness in the function space

$$\mathfrak{F}_T = \tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1}) \times \left( \tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-2}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1}) \right).$$

In the two-dimensional case, the regularity index of the second component  $\tilde{V}^2$  is only 0, so that some product laws fail when estimating some terms on the right-hand side of the equation for  $\tilde{V}^2$  (e.g.  $g_2$  and  $g_4$ ). The reason why is that the product of functions only maps  $\dot{B}_{2,1}^0 \times \dot{B}_{2,1}^0$  in the larger space  $\dot{B}_{2,\infty}^{-1}$ , rather than in  $\dot{B}_{2,1}^{-1}$ . Consequently, we look for estimates of  $\tilde{V}^1$  in  $L_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}-1})$  and  $\tilde{V}^2$  in  $L_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}-2})$ . Yet, another problem arises due to  $\dot{B}_{2,\infty}^1 \not\hookrightarrow L^\infty$ , the term  $f_1$  in the right-hand side of the first equation of (136) cannot be estimated properly. This difficulty may be bypassed by making use of the following logarithmic interpolation (see [8] or Lemma 7 for nonhomogeneous case)

$$\|a\|_{\tilde{L}_T^\rho(\dot{B}_{p,1}^s)} \leq C \frac{\|a\|_{\tilde{L}_T^\rho(\dot{B}_{p,1}^s)}}{\eta} \log \left( e + \frac{\|a\|_{\tilde{L}_T^\rho(\dot{B}_{p,\infty}^{s-1})} + \|a\|_{\tilde{L}_T^\rho(\dot{B}_{p,\infty}^{s+1})}}{\|a\|_{\tilde{L}_T^\rho(\dot{B}_{p,\infty}^s)}} \right), \text{ for all } s \in \mathbb{R}. \quad (137)$$

This being said, we are going to estimate  $(\tilde{V}^1, \tilde{V}^2)$  in<sup>6</sup>

$$L_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}-1}) \times L_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}-2}) \cap \tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}}).$$

Apply operator  $\dot{\Delta}_j$  to (136)<sub>1</sub> to get

$$\partial_t \tilde{V}_j^1 + \sum_{\alpha=1}^d A_{11}^\alpha(U_2^2) \tilde{V}_j^1 = h_j + \tilde{R}_j \quad \text{with} \quad h_j = \dot{\Delta}_j h, \quad \tilde{R}_j = - \sum_{\alpha=1}^d [\dot{\Delta}_j, A_{11}^\alpha(U_2^2)] (\partial_\alpha \tilde{V}^1).$$

As  $\tilde{V}^1(0) = 0$ , following the energy method and remembering that  $A_{11}^\alpha(U_2^2)$  is at most linear yields

$$\|\tilde{V}_j^1\|_{L_T^\infty(L^2)} \leq \|\tilde{V}_j^1\|_{L_T^\infty(L^2)} \|\nabla V_2^2\|_{L_T^1(L^\infty)} + \|(\tilde{R}_j, h_j)\|_{L_T^1(L^2)}.$$

Next bounding  $\|(R_j, h_j)\|_{L_T^\infty(L^2)}$  may be achieved by combining Propositions 8 and 9. We have

$$\begin{aligned} 2^{j(\frac{d}{2}-1)} \|\tilde{R}_j\|_{L_T^\infty(L^2)} &\leq C \left\| \tilde{V}^1 \right\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}-1})} \|\nabla V_2^2\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}}) \cap L_T^1(L^\infty)}, \\ 2^{j(\frac{d}{2}-1)} \|h_j\|_{L_T^1(L^2)} &\leq C \int_0^T \left\| \tilde{V}^2 \right\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla V_1^1\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}} + \int_0^T (1 + \|V_2^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \left\| \nabla \tilde{V}^2 \right\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}} \\ &\quad + \left\| \tilde{V}^1 \right\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}-1})} \|\nabla V_1^2\|_{\tilde{L}_T^1(\dot{B}_{2,1}^{\frac{d}{2}})}. \end{aligned}$$

Finally using the embedding  $L_T^1(\dot{B}_{2,1}^{\frac{d}{2}}) \hookrightarrow \tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}}) \cap L_T^1(L^\infty)$ , we arrive at

$$\begin{aligned} \left\| \tilde{V}^1 \right\|_{L_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}-1})} &\leq C \|(V_2^2, V_1^2)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \left\| \tilde{V}^1 \right\|_{L_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}-1})} \\ &\quad + C \int_0^T \left( 1 + \|V_1^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|V_2^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \left\| \tilde{V}^2 \right\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

We note that by virtue of the Lebesgue dominated convergence theorem,  $\|(V_2^2, V_1^2)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}$  tends to 0 when  $T$  goes to 0, and hence there exists a positive time (which we still denote by  $T$ )

<sup>6</sup>Obviously, this problem does not occur in dimension  $d \geq 3$ , and one can provide a simpler proof of uniqueness in the space  $\mathfrak{F}_T$ , with no need of logarithmic estimates.



such that the first term on the right-hand side may be absorbed by the left-hand side. Then, making use of inequality (137), and setting

$$\begin{aligned} M_1(T) &= \|(V_1^1, V_2^1)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} + \|(V_1^2, V_2^2)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \quad \text{and} \\ M_2(T) &= \|(\partial_t V_1, \partial_t V_2)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|(V_1^2, V_2^2)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})} \end{aligned} \quad (138)$$

we end up with the following inequality:

$$\begin{aligned} \|\tilde{V}^1\|_{L_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}-1})} &\leq C(1+M_1) \|\tilde{V}^2\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}})} \log \left( e + \frac{\|(V_1^2, V_2^2)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} + \|(V_1^2, V_2^2)\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1})}}{\|\tilde{V}^2\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}})}} \right) \\ &\leq C(1+M_1) \|\tilde{V}^2\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}})} \log \left( e + \frac{TM_1 + M_2}{\|\tilde{V}^2\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}})}} \right). \end{aligned} \quad (139)$$

We now bound  $\tilde{V}^2$ . Apply the operator  $\dot{\Delta}_j$  to (136)<sub>2</sub> to get

$$S_{22}^0(U_{2,m}^1) \partial_t \tilde{V}_j^2 - \sum_{\alpha,\beta=1}^d Z^{\alpha\beta}(U_{2,m}^1) \partial_\alpha \partial_\beta \tilde{V}_j^2 = g_j + \mathfrak{G}_j,$$

with

$$\begin{aligned} \mathfrak{G}_j &= S_{22}^0(U_{2,m}^1) \sum_{\alpha,\beta=1}^d \left[ \dot{\Delta}_j, (S_{22}^0(U_{2,m}^1))^{-1} Z^{\alpha\beta}(U_{2,m}^1) \right] (\partial_\alpha \partial_\beta \tilde{V}_j^2) \\ g_j &= S_{22}^0(U_{2,m}^1) \dot{\Delta}_j \left( (S_{22}^0(U_{2,m}^1))^{-1} g \right). \end{aligned}$$

Following the energy method for parabolic system we get

$$\|\tilde{V}_j^2\|_{L_T^\infty(L^2)} + 2^{2j} \|\tilde{V}_j^2\|_{L_T^1(L^2)} \leq \|\tilde{V}_j^2\|_{L_T^\infty(L^2)} \|\partial_t V_{2,m}^1\|_{L_T^1(L^\infty)} + \|(g_j, \mathfrak{G}_j)\|_{L_T^\infty(L^2)}.$$

Since  $d \geq 2$ , taking advantage of commutator estimates (see Proposition 8), one may write

$$\begin{aligned} \|\mathfrak{G}_j\|_{L_T^1(L^2)} &\leq C 2^{-j(\frac{d}{2}-2)} \int_0^T \|\nabla U_{2,m}^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla \tilde{V}^2\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-2}} \\ &\leq C 2^{-j(\frac{d}{2}-2)} \sqrt{T} \|\nabla U_{2,m}^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla \tilde{V}^2\|_{\tilde{L}_T^2(\dot{B}_{2,\infty}^{\frac{d}{2}-2})}, \end{aligned} \quad (140)$$

from which and the fact that

$$\|\tilde{V}_j^2\|_{L_T^\infty(L^2)} \|\partial_t V_{2,m}^1\|_{L_T^1(L^\infty)} \leq C \|\tilde{V}_j^2\|_{L_T^\infty(L^2)} \|\partial_t V_{2,m}^1\|_{L_T^1(\dot{B}_{2,1}^{\frac{d}{2}})} \leq C 2^m M_2,$$

we deduce that

$$\begin{aligned} \|\tilde{V}_j^2\|_{L_T^\infty(L^2)} + 2^{2j} \|\tilde{V}_j^2\|_{L_T^1(L^2)} &\leq C 2^m M_2(T) \|\tilde{V}_j^2\|_{L_T^\infty(L^2)} \\ &\quad + C 2^{-j(\frac{d}{2}-2)} (2^m \sqrt{T} M_1(T)) \tilde{\mathfrak{U}}(T) + \|g_j\|_{L_T^\infty(L^2)}, \end{aligned} \quad (141)$$

where hereafter we define

$$\tilde{\mathfrak{U}}(T) = \|\tilde{V}^2\|_{L_T^\infty(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} + \|\tilde{V}^2\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}})}.$$

In order to bound all the  $g_j$ 's, we may exploit that the product of two functions maps  $\dot{B}_{2,\infty}^{\frac{d}{2}-1} \times \dot{B}_{2,1}^{\frac{d}{2}-1}$  or  $\dot{B}_{2,\infty}^{\frac{d}{2}-2} \times \dot{B}_{2,1}^{\frac{d}{2}}$  to  $\dot{B}_{2,\infty}^{\frac{d}{2}-2}$  (recall that  $d \geq 2$ ), and Propositions 8, 9 and 10, adapted to the spaces  $\tilde{L}_T^\rho(\dot{B}_{2,r}^s)$ .

From the definition of  $g_j$  and the fact that  $U_{i,m}^1$  is bounded, one may write

$$2^{j(\frac{d}{2}-2)} \|g_j\|_{L_T^1(L^2)} \leq C \left\| (S_{22}^0(U_{2,m}^1))^{-1} g \right\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} \leq C(1 + M_1) \|g\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})}. \quad (142)$$

To bound  $g_2$  and  $g_{4,2}$  we use the inequality (198) of Proposition 11 and that the product of two functions maps  $\dot{B}_{2,\infty}^{\frac{d}{2}-1} \times \dot{B}_{2,1}^{\frac{d}{2}-1}$  in  $\dot{B}_{2,\infty}^{\frac{d}{2}-2}$ . It holds

$$\begin{aligned} \|g_2\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} &\leq C\sqrt{T} \left\| \tilde{V}^2 \right\|_{\tilde{L}_T^2(\dot{B}_{2,\infty}^{\frac{d}{2}-1})} (1 + \|V_2^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \left\| \nabla V_1^2 \right\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}-1})} \\ &\quad + \int_0^T \left( \left\| \tilde{V}^1 \right\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}} \|V_1^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \left\| \tilde{V}^1 \right\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}} \right) \left\| \nabla V_1^2 \right\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}, \\ \|g_{4,2}\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} &\leq C\sqrt{T} \left\| \tilde{V}^2 \right\|_{\tilde{L}_T^2(\dot{B}_{2,\infty}^{\frac{d}{2}-1})} (1 + \|V_2^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \left\| \nabla V_1^1 \right\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}-1})} \\ &\quad + \int_0^T \left( \left\| \tilde{V}^1 \right\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}} \|V_1^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \left\| \tilde{V}^1 \right\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}} \right) \left\| \nabla V_1^1 \right\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}. \end{aligned}$$

Besides, using (196) and the product law  $\dot{B}_{2,\infty}^{\frac{d}{2}-2} \times \dot{B}_{2,1}^{\frac{d}{2}}$  in  $\dot{B}_{2,\infty}^{\frac{d}{2}-2}$  yields

$$\begin{aligned} \|g_{4,1}\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} &\leq C \int_0^T (1 + \|V_2^2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) (1 + \|V_2^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) + \|V_2^1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left\| \nabla \tilde{V}^1 \right\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-2}} \\ \|g_3\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} &\leq C(\sqrt{T} + \|V_2^2\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{\frac{d}{2}})}) (1 + \|V_2^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) + \sqrt{T} \|V_2^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \left\| \nabla \tilde{V}^2 \right\|_{\tilde{L}_T^2(\dot{B}_{2,\infty}^{\frac{d}{2}-2})}. \end{aligned}$$

Next, thanks to (193) and to the fact that the numerical product maps  $\dot{B}_{2,\infty}^{\frac{d}{2}-2} \times \dot{B}_{2,1}^{\frac{d}{2}}$  to  $\dot{B}_{2,\infty}^{\frac{d}{2}-2}$ , we get

$$\begin{aligned} \|g_{1,1}\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} &\leq C \|V_2^1 - V_{2,m}^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} (1 + \|(V_2^1, V_{m,2}^1)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \left\| \partial_t \tilde{V}^2 \right\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})}, \\ \|g_5\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} &\leq C\sqrt{T} \left\| \nabla V_{2,m}^1 \right\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \left\| \nabla \tilde{V}^2 \right\|_{\tilde{L}_T^2(\dot{B}_{2,\infty}^{\frac{d}{2}-2})}. \end{aligned} \quad (143)$$

Combining inequality (194) and the product law  $\dot{B}_{2,\infty}^{\frac{d}{2}-1} \times \dot{B}_{2,1}^{\frac{d}{2}-1} \rightarrow \dot{B}_{2,\infty}^{\frac{d}{2}-2}$  we have the following bound for  $g_{1,2}$ :

$$\|g_{1,2}\|_{L_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} \leq C(1 + \|(V_2^1, V_1^1)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \int_0^T \left\| \tilde{V}^1 \right\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}} \left\| \partial_t V_1^2 \right\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}}. \quad (144)$$

Finally, thanks to the product law  $\dot{B}_{2,\infty}^{\frac{d}{2}-1} \times \dot{B}_{2,1}^{\frac{d}{2}} \rightarrow \dot{B}_{2,\infty}^{\frac{d}{2}-1}$ , Proposition 10 (especially (193)) and Bernstein inequality, we get

$$\begin{aligned} \|g_6\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} &\leq C \|V_2^1 - V_{2,m}^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} (1 + \|(V_2^1, V_{m,2}^1)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \left\| \nabla \tilde{V}^2 \right\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-1})} \\ \|g_7\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} &\leq C(1 + \|(V_1^1, V_2^1)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}) \int_0^T \left\| \tilde{V}^1 \right\|_{\dot{B}_{2,\infty}^{\frac{d}{2}}} \left\| \nabla V_1^2 \right\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned} \quad (145)$$

Multiplying (141) by  $2^{j(\frac{d}{2}-2)}$ , taking into account the above estimates, using many times Bernstein inequality and interpolation inequality, one concludes that (for simplicity we assume that

$T \leq 1$  and  $M_2(T) \leq 1$ ):

$$\begin{aligned} \tilde{\mathfrak{U}}(T) &\leq C_{M_1} \left( 2^m M_2(T)(1 + \sqrt{T}) + M_2(T) + \sqrt{M_2(T)} + \|V_2^1 - V_{2,m}^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \right) \tilde{\mathfrak{U}}(T) \\ &\quad + C_{M_1} \int_0^T \left( 1 + \|(V_1^2, V_2^2)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\partial_t V_1^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \right) \|\tilde{V}^1\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}} \\ &\quad \quad \quad + C_{M_1} \|V_2^1 - V_{2,m}^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \|\partial_t \tilde{V}^2\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})}, \end{aligned}$$

where  $C_{M_1}$  depends on  $M_1$ . In order to pursue our proof for uniqueness, we need to estimate  $\|\partial_t \tilde{V}^2\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-2}}$ . More precisely we are going to bound  $\|\partial_t \tilde{V}^2\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})}$  in terms of  $\tilde{\mathfrak{U}}(T)$ . Starting from (136)<sub>2</sub>, we have

$$\partial_t \tilde{V}^2 = (S_{22}^0(U_{2,m}^1))^{-1} \sum_{\alpha, \beta=1}^d Z^{\alpha\beta}(U_{2,m}^1) \partial_\alpha \partial_\beta \tilde{V}^2 + (S_{22}^0(U_{2,m}^1))^{-1} g. \quad (146)$$

Then, bounding  $\partial_t \tilde{V}^2$  in  $L^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})$  is similar to what we did to bound  $\|g_j\|_{L_T^1(L^2)}$ . In fact, combining the product law  $\dot{B}_{2,\infty}^{\frac{d}{2}-2} \times \dot{B}_{2,1}^{\frac{d}{2}} \rightarrow \dot{B}_{2,\infty}^{\frac{d}{2}-2}$  (taking into account the structure of  $(S_{22}^0(U_{2,m}^1))^{-1} S_{22}^\alpha(U_{2,m})$ ), and Propositions 10 and 11 we have

$$\|\partial_t \tilde{V}^2\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} \leq C_{M_1} \left( \|\nabla^2 \tilde{V}^2\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} + \|g\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})} \right).$$

As  $\|g\|_{\tilde{L}_T^1(\dot{B}_{2,\infty}^{\frac{d}{2}-2})}$  has already been bounded above we deduce that

$$\begin{aligned} \tilde{\mathfrak{U}}(T) &\leq C_{M_1} \left( 2^m (M_2(T) + \sqrt{T}) + M_2(T) + \sqrt{M_2(T)} + \|V_2^1 - V_{2,m}^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})} \right) \tilde{\mathfrak{U}}(T) \\ &\quad + C_{M_1} \int_0^T \left( 1 + \|(V_1^2, V_2^2)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\partial_t V_1^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \right) \|\tilde{V}^1\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}}. \quad (147) \end{aligned}$$

Now, the first term on the right-hand side may be absorbed by the left-hand side, provided that, first,  $m$  is taken large enough, and then  $T$  is sufficiently small. In fact, the term  $\|V_2^1 - V_{2,m}^1\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})}$  tends to zero when  $m$  goes to  $\infty$  (recall that  $V_2^1 \in \tilde{L}_T^\infty(\dot{B}_{2,1}^{\frac{d}{2}})$ ). Next, we note that by virtue of the Lebesgue dominated convergence theorem,  $M_2(T)$  tends to 0 when  $T$  goes to 0. Note that  $M_1$  can be seen as independent on time since, it is bounded by the initial data. Hence, so does  $C_{M_1}$ . We end up with the following inequality:

$$\tilde{\mathfrak{U}}(T) \leq C_{M_1} \int_0^T \left( 1 + \|(V_1^2, V_2^2)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\partial_t V_1^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \right) \|\tilde{V}^1\|_{\dot{B}_{2,\infty}^{\frac{d}{2}-1}}.$$

We plug (139) into this inequality and we use the fact that the function  $r \mapsto r \log(e + \frac{1}{r})$  is increasing, to eventually get

$$\tilde{\mathfrak{U}}(T) \leq C_{M_1} \int_0^T \left( 1 + \|(V_1^2(\tau), V_2^2(\tau))\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\partial_t V_1^2(\tau)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \right) \tilde{\mathfrak{U}}(\tau) \log \left( e + \frac{M_2(T)}{\tilde{\mathfrak{U}}(\tau)} \right) d\tau.$$

As

$$1 + \|(V_1^2, V_2^2)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\partial_t V_1^2\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \in L_T^1 \quad \text{and} \quad \int_0^T \frac{1}{r \log(e + \frac{1}{r})} dr = \infty,$$

Osgood's lemma entails that  $\tilde{\mathcal{U}}(t) = 0$  for all  $0 \leq t \leq T$ . This means that  $(V_1^1, V_1^2)$  and  $(V_2^1, V_2^2)$  coincide on  $[0, T]$  for small  $T > 0$ . Appealing to a connectivity argument yields uniqueness on the whole interval existence, which completes the proof.

## 4 Application to the compressible Navier-Stokes system

In the Eulerian description, a general compressible fluid evolving in some open set  $\Omega$  of  $\mathbb{R}^d$  is characterized at every material point  $x$  in  $\Omega$  and time  $t \in \mathbb{R}$  by its velocity field  $u = u(t, x) \in \mathbb{R}^d$ , density  $\rho \in \mathbb{R}_+$ , pressure  $p = p(t, x) \in \mathbb{R}$ , internal energy by unit mass  $e = e(t, x) \in \mathbb{R}$ , entropy by unit mass  $s = s(t, x) \in \mathbb{R}$  and absolute temperature  $\theta = \theta(t, x) \in \mathbb{R}_+$ . In the absence of external forces, those quantities are governed by:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho \partial_t u + \rho u \cdot \nabla u - \operatorname{div}(2\mu D u + \lambda I_d \operatorname{div}(u)) + \nabla p = 0, \\ \rho e_\theta (\partial_t \theta + u \cdot \nabla \theta) + \theta p_\theta \operatorname{div}(u) - \operatorname{div}(k \nabla \theta) = \Psi, \end{cases} \quad (148)$$

with

$$\Psi \stackrel{\text{def}}{=} \frac{\mu}{2} \sum_{i,j=1}^d (\partial_{x_j} u^i + \partial_{x_i} u^j)^2 + \lambda (\operatorname{div}(u))^2. \quad (149)$$

In order to get a closed system of  $d + 2$  equations for the  $d + 2$  unknowns  $(\rho, u, \theta)$ , we have to supplement System (148) with closure relations interrelating  $\rho, p, e$  and  $\theta$ . Here we assume:

### Assumption D

1. The thermodynamic quantities  $p$  and  $e$  are smooth functions of  $\rho, \theta > 0$  such that the first law of thermodynamics

$$\theta ds = de + pd\left(\frac{1}{\rho}\right), \quad (150)$$

holds, and we assume that for all  $\rho > 0$  and  $\theta > 0$ ,

$$p_\rho(\rho, \theta) := \frac{\partial p}{\partial \rho} > 0 \quad \text{and} \quad e_\theta(\rho, \theta) := \frac{\partial e}{\partial \theta} > 0. \quad (151)$$

2. The viscosity coefficients  $\lambda, \mu$  and the heat conductivity  $k$  are smooth functions of  $\rho, \theta > 0$  and satisfy the following condition

$$\mu > 0, \quad \nu \stackrel{\text{def}}{=} 2\mu + \lambda > 0 \quad \text{and} \quad k > 0. \quad (152)$$

Let us next write (148) as a symmetric hyperbolic partially diffusive system. Set

$$\mathcal{U} \stackrel{\text{def}}{=} \{(\rho, u, \theta) \in \mathbb{R}^{d+2} / \rho > 0, \theta > 0\}.$$

For some positive constants  $\bar{\rho} > 0$  and  $\bar{\theta} > 0$ , define

$$U \stackrel{\text{def}}{=} (\rho, u, \theta), \quad \bar{U} \stackrel{\text{def}}{=} (\bar{\rho}, 0_{\mathbb{R}^d}, \bar{\theta}) \quad \text{and} \quad V \stackrel{\text{def}}{=} U - \bar{U}.$$

Then, (148) is reduced to

$$S^0(U) \frac{d}{dt} U + \sum_{\alpha=1}^d S^\alpha(U) \partial_\alpha U - \sum_{\alpha, \beta=1}^d \partial_\alpha (Y^{\alpha\beta} \partial_\beta V) = H(U) \quad (153)$$

where

$$S^0(U) \stackrel{\text{def}}{=} \begin{pmatrix} \frac{p_\rho}{\rho} & 0 & 0 \\ 0 & \rho \text{Id} & 0 \\ 0 & 0 & \frac{\rho e_\theta}{\theta} \end{pmatrix}, \quad H(U) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ \Psi - k \nabla \theta \cdot \nabla \left(\frac{1}{\theta}\right) \end{pmatrix}, \quad (154)$$

$$\sum_{\alpha=1}^d S^\alpha(U) \xi_\alpha \stackrel{\text{def}}{=} \begin{pmatrix} \frac{p_\rho}{\rho} u \cdot \xi & p_\rho \xi & 0 \\ p_\rho^T \xi & \rho(u \cdot \xi) \text{Id} & p_\theta^T \xi \\ 0 & p_\theta \xi & \frac{\rho e_\theta}{\theta} u \cdot \xi \end{pmatrix}, \quad (155)$$

and

$$\sum_{\alpha, \beta=1}^d Y^{\alpha\beta} \xi_\alpha \xi_\beta \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ 0 & \sum_{\alpha, \beta=1}^d Z^{\alpha\beta} \xi_\alpha \xi_\beta \end{pmatrix} \quad (156)$$

with

$$Z^{\alpha\beta} \xi_\alpha \xi_\beta \stackrel{\text{def}}{=} \begin{pmatrix} \mu |\xi|^2 + (\mu + \lambda) \xi \otimes \xi \text{Id} & 0 \\ 0 & \frac{k}{\theta} |\xi|^2 \end{pmatrix}. \quad (157)$$

We note that  $S^0(U)$  is a diagonal positive definite matrix for all  $U \in \mathcal{U}$ , and that the matrices  $S^\alpha(U)$ , for  $\alpha = 1, \dots, d$ , are real symmetric. Furthermore a simple calculation shows that: for all  $X, \xi \in \mathbb{R}^d$ ,  $Y \in \mathbb{R}$ ,

$$\sum_{\alpha, \beta=1}^d \langle Z^{\alpha\beta} \xi_\alpha \xi_\beta A, A \rangle \geq \min(\mu, \nu) |X|^2 + \frac{k}{\theta} Y^2 \quad (158)$$

where  $A \stackrel{\text{def}}{=} (X, Y)$  and  $\langle \cdot, \cdot \rangle$  denotes the standard inner product in  $\mathbb{R}^{d+1}$ . As for the right-hand side  $H$ , it is regarded as a lower order (quadratic) term, that satisfies the 4th condition of **BB**.

These considerations lead to the following result which is a direct application of Theorem 1.2:

**Theorem 4.1** (Local existence). *Let Assumption **D** be in force and let  $s > \frac{d}{2}$  if  $d = 1$  and  $s \geq \frac{d}{2}$  if  $d \geq 2$ . Let  $\bar{\rho} > 0$ ,  $\bar{\theta} > 0$  be arbitrary fixed real numbers. Suppose that the initial data  $(\rho_0, u_0, \theta_0) \in \mathcal{U}$  satisfies  $\rho_0 - \bar{\rho} \in B_{2,1}^{s+1}$ ,  $u_0 \in B_{2,1}^s$  and  $\theta_0 - \bar{\theta} \in B_{2,1}^s$ . Then, there exists some  $T > 0$  such that the problem (148) supplemented with the initial data  $(\rho_0, u_0, \theta_0)$  has a unique solution  $(\rho, u, \theta) \in \mathcal{U}$  on  $Q_T \stackrel{\text{def}}{=} [0, T] \times \mathbb{R}^d$  such that  $\rho$  and  $\theta$  are bounded away from zero on  $Q_T$ . More precisely,  $\inf_{Q_T} \{\rho(t, x)\} > 0$  and  $\inf_{Q_T} \{\theta(t, x)\} > 0$ . Moreover we have*

$$\rho - \bar{\rho} \in C([0, T_1]; B_{2,1}^{s+1}) \quad \text{and} \quad (u, \theta - \bar{\theta}) \in C([0, T_1]; B_{2,1}^s) \cap L_{T_1}^1(B_{2,1}^{s+2}).$$

The system constituted by the first two equations in (148) is the so-called barotropic compressible Navier-Stokes system (this corresponds to the case where the temperature is constant). It reads:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho + \rho \operatorname{div}(u) = 0 \\ \rho \partial_t u + \rho u \cdot \nabla u - \operatorname{div}(2\mu(\rho)Du + \lambda(\rho) \operatorname{div}(u)I_d) + \nabla p(\rho) = 0. \end{cases} \quad (159)$$

Without difficulties, we can see that, if

$$\mu(\rho) > 0 \quad \text{and} \quad \nu(\rho) > 0, \quad \text{for all } \rho > 0, \quad (160)$$

then the system (159) satisfies the assumption **C**. Then, Theorem 1.3 can be applied and we recover the following result of Danchin in [10]:

**Theorem 4.2** (Local existence). *Let Condition (160) be satisfied. Let  $\bar{\rho} > 0$  be an arbitrary fixed constant. Suppose that the initial data  $(\rho_0, u_0)$  satisfy  $\rho_0 - \bar{\rho} \in \dot{B}_{2,1}^{\frac{d}{2}}$ ,  $u_0 \in \dot{B}_{2,1}^{\frac{d}{2}-1}$  and  $\rho_0 > 0$ . Then, System (159) supplemented with the initial data  $(\rho_0, u_0)$  has a unique solution  $(\rho, u)$  on  $Q_T$  with  $T > 0$  which satisfies  $\inf_{Q_T} \{\rho(t, x)\} > 0$ . Moreover we have*

$$\rho - \bar{\rho} \in C([0, T]; \dot{B}_{2,1}^{\frac{d}{2}}) \quad \text{and} \quad u \in C([0, T]; \dot{B}_{2,1}^{\frac{d}{2}-1}) \cap L_T^1(\dot{B}_{2,1}^{\frac{d}{2}+1}).$$

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## Appendix

### A Some inequalities

Here we gather a few technical results that have been used repeatedly in the paper. The first one is the following well known result which consists in integrating suitably a differential inequality.

**Lemma 5.** *Let  $X : [0, T] \rightarrow \mathbb{R}_+$  be a continuous function such that  $X$  is differentiable.*

*Assume that there exists a constant  $B \geq 0$  and a measurable function  $A : \mathbb{R}_+ \rightarrow [0, T]$  such that*

$$\frac{1}{2} \frac{d}{dt} X + BX \leq AX^{\frac{1}{2}} \quad \text{a.e on } [0, T].$$

*Then, for all  $t \in [0, T]$ , we have*

$$X^{\frac{1}{2}}(t) + B \int_0^t X^{\frac{1}{2}} \leq X^{\frac{1}{2}}(0) + \int_0^t A.$$

*Proof.* For all  $\varepsilon > 0$ , the following inequality is verified by  $X$ :

$$\frac{1}{2} \frac{d}{dt} (X + \varepsilon^2) + B(X + \varepsilon^2) \leq A(X + \varepsilon^2)^{1-\frac{1}{2}} + B\varepsilon(X + \varepsilon^2)^{\frac{1}{2}}.$$

As  $X(t) + \varepsilon^2 > 0$  for all  $t \in [0, T]$ , we have

$$\frac{d}{dt} (X + \varepsilon^2)^{\frac{1}{2}} + B(X + \varepsilon^2)^{\frac{1}{2}} \leq A + B\varepsilon.$$

Then, integrating the previous inequality over  $[0, T]$  yields

$$(X(t) + \varepsilon^2)^{\frac{1}{2}} + B \int_0^t (X(\tau) + \varepsilon^2)^{\frac{1}{2}} d\tau \leq \int_0^t A(\tau) d\tau + t\varepsilon B.$$

To obtain the desired inequality, it suffices to let  $\varepsilon$  go to zero.  $\square$

Let  $V$  be a solution of the following linear parabolic equation:

$$\begin{cases} S\partial_t V - Z(D)V = 0 \\ V(0) = V_0 \end{cases} \quad (161)$$

where  $S$  is a symmetric positive definite matrix and  $Z \in C^\infty(\mathbb{R}^d; \mathcal{M}_n(\mathbb{C}))$  is homogeneous of degree  $\gamma \in \mathbb{R}$  and such that the matrix  $Z(\xi)$  satisfies for some constant  $c > 0$

$$(Z(\xi)z \cdot z) \geq c|\xi|^\gamma |z|^2, \quad \xi \in \mathbb{R}^d \setminus \{0\}, \quad z \in \mathbb{C}^n. \quad (162)$$

We have the following statement that turns out to be the key to proving our local existence result.

**Proposition 6.** *There exist universal positive constants  $c, C$  such that for all  $s \in \mathbb{R}$ ,  $T \in \mathbb{R}^+$  and  $m \in \mathbb{N}$ , the following estimates hold:*

$$\|V\|_{\tilde{L}_T^\infty(B_{2,1}^s)} \leq C \|V_0\|_{B_{2,1}^s}, \quad (163)$$

$$\sum_{j \geq m} 2^{js} \|\Delta_j V\|_{L_T^\infty(L^2)} \leq C \sum_{j \geq m} 2^{js} \|\Delta_j V_0\|_{L^2}, \quad (164)$$

$$\sum_{j \geq m} 2^{js} \|(\Delta_j(D^\gamma V), \Delta_j(\partial_t V))\|_{L_T^1(L^2)} \leq C \sum_{j \geq m} (1 - e^{-c2^{j\gamma}T}) 2^{js} \|\Delta_j V_0\|_{L^2}, \quad (165)$$

$$\|(\Delta_{-1}V, \partial_t \Delta_{-1}(V))\|_{L_T^1(L^2)} \leq CT \|V^2(0)\|_{B_{2,1}^s}. \quad (166)$$

Furthermore, in the case  $Z \in C^\infty(\mathbb{R}^d \setminus \{0\}; \mathcal{M}_n(\mathbb{C}))$ , the inequalities (163) and (165) are also valid in the homogeneous framework (i.e., with  $\Delta_j$ ,  $m \in \mathbb{Z}$  and homogeneous Besov norms instead of  $\Delta_j$ ,  $m \in \mathbb{N}$  and nonhomogeneous Besov norms respectively).

*Proof.* We provide only the proof in the nonhomogeneous case, which is an easy adaptation of the similar result for the "ordinary" heat equation stated in [6]. First apply  $\Delta_j$  to (66), take the  $L^2$  inner product with  $V_j \stackrel{\text{def}}{=} \Delta_j V$ , then use Plancherel's theorem. One gets

$$\frac{d}{dt} \|\widehat{V}_j\|_{L_S^2(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d} Z(\xi) \widehat{V}_j(\xi) \cdot \widehat{V}_j(\xi) d\xi = 0, \quad \text{with} \quad \|V\|_{L_S^2(\mathbb{R}^d)}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} SV \cdot V.$$

Next using the strong ellipticity condition (162), we get for all  $j \geq -1$ ,

$$\frac{d}{dt} \|\widehat{V}_j\|_{L_S^2(\mathbb{R}^d)}^2 + c \int_{\mathbb{R}^d} |\xi|^\gamma |\widehat{V}_j(\xi)|^2 d\xi \leq 0, \quad (167)$$

with  $c$  being a universal constant. Since  $|\xi|^\gamma |\widehat{V}_j(\xi)|^2 \geq \kappa 2^{j\gamma} |\widehat{V}_j(\xi)|^2$  for  $j \geq 0$  where  $\kappa > 0$  is a universal constant, combining with the fact  $S$  is a constant, symmetric, positive definite matrix, we get

$$\frac{d}{dt} \|\widehat{V}_j\|_{L^2_S(\mathbb{R}^d)}^2 + c\kappa 2^{j\gamma} \|\widehat{V}_j\|_{L^2_S(\mathbb{R}^d)}^2 \leq 0, \text{ for all } j \geq 0.$$

This leads to (up to a slight modification of  $c$ ):

$$\|V_j(t)\|_{L^2_S(\mathbb{R}^d)} \leq e^{-c2^{j\gamma}t} \|V_{0,j}\|_{L^2_S(\mathbb{R}^d)}, \text{ for all } j \geq 0, t \in [0, T] \quad (168)$$

which provides (164). Another consequence of the previous inequality is

$$\|V_j\|_{L^q_T(L^2(\mathbb{R}^d))} \leq \left( \frac{1 - e^{-c2^{j\gamma}Tq}}{c2^{j\gamma}Tq} \right)^{\frac{1}{q}} \|V_{0,j}\|_{L^2}, \text{ for all } j \geq 0, q \in [1, \infty) \quad (169)$$

which combined with the fact that

$$\partial_t V = S^{-1}Z(D)V \quad (170)$$

and Bernstein inequality provides for all  $j \geq 0, q \in [1, \infty)$

$$\|\Delta_j(\partial_t V^L)\|_{L^q_T(L^2(\mathbb{R}^d))} \leq C \left( \frac{1 - e^{-c2^{j\gamma}Tq}}{c2^{j\gamma}Tq} \right)^{\frac{1}{q}} 2^{j\gamma} \|V_{0,j}\|_{L^2}, \quad (171)$$

from which we deduce inequality (165).

It remains to deal with the case  $j = -1$ . From (167), we deduce that:

$$\frac{d}{dt} \|\widehat{V}_{-1}\|_{L^2_S(\mathbb{R}^d)}^2 \leq 0.$$

This implies that

$$\|V_{-1}\|_{L^\infty_T(L^2(\mathbb{R}^d))} \leq \|V_{0,-1}\|_{L^2} \quad (172)$$

which combined with (170) and Bernstein inequality yields (166). At the same time we get (163) from (168) and (172).  $\square$

So far, the operators considered were independent of  $x$ . In the following result, we explain how to handle a second order operator with variable coefficients.

**Lemma 6** (Gårding inequality). *Let  $U : \mathbb{R}^d \mapsto \mathbb{R}^n$  be a bounded function. Assume that the operator  $Z(U)\nabla_x$  is strongly elliptic in the sense of (3). Then, there exists a positive constant  $c$  (depending on the ellipticity constant) and a constant  $\varepsilon = \varepsilon(c, U) > 0$ , chosen as small as we want, such that the following inequality holds true for all smooth function  $f : \mathbb{R}^d \rightarrow \mathbb{R}^{n-n_1}$ :*

$$\begin{aligned} & - \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta}(U(x)) \partial_\alpha \partial_\beta f^i(x) f^j(x) dx \\ & \geq c \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - \varepsilon \|\nabla^2 f\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} - C \|f\|_{L^2(\mathbb{R}^d)}^2, \end{aligned} \quad (173)$$

where  $C = C(c, \varepsilon, U) > 0$  depends only on  $\varepsilon$ , the range of  $U$  and the ellipticity constant.



**Remark A.1.** The ‘standard’ Gårding inequality has a better form than what we stated in inequality (173), but, provided that we keep the divergence form of the operator  $Z(U)\nabla_x$ . In fact, under the assumptions of Lemma 6, there exist positive constants  $c, C$  such that for any function  $f : \mathbb{R}^d \rightarrow \mathbb{R}^{n-p}$ , the following inequality holds true:

$$- \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} \partial_\beta \left( Z_{ij}^{\alpha\beta}(U) \partial_\alpha f^i \right) (x) \partial_\beta f^j(x) dx \geq c \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - C \|f\|_{L^2(\mathbb{R}^d)}^2. \quad (174)$$

Although Inequality (173) seems not so good because of the second order derivatives in the right-hand side, it will be useful for us once combined with Bernstein inequality, since it will be applied to spectrally localized functions  $f$ .

*Proof of Lemma 6.* As a first, suppose that the functions  $Z_{ij}^{\alpha\beta}$  are constant. Then, in light of Fourier-Plancherel theorem and of (3), we have

$$\begin{aligned} - \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta} \partial_\alpha \partial_\beta f^i(x) f^j(x) dx &= \mathcal{R}e \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta} \xi_\alpha \xi_\beta \widehat{f^i} \overline{\widehat{f^j}} d\xi \\ &= \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta} \xi_\alpha \xi_\beta \left( \mathcal{R}e(\widehat{f^i}) \mathcal{R}e(\widehat{f^j}) + \mathcal{I}m(\widehat{f^i}) \mathcal{I}m(\widehat{f^j}) \right) d\xi \\ &\geq c_1 \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 \end{aligned}$$

where  $c_1$  in the constant appearing in (3). Hence, (173) is true in this special case.

We next consider the case of variable coefficients. Suppose first that the function  $U$  has range in a small ball about  $\bar{U}$ . Taking  $\bar{U} = 0$  for notational simplicity and using the preceding case, we have

$$\begin{aligned} - \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta}(U) \partial_\alpha \partial_\beta f^i(x) f^j(x) dx &= - \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta}(0) \partial_\alpha \partial_\beta f^i(x) f^j(x) dx \\ &\quad - \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} (Z_{ij}^{\alpha\beta}(U) - Z_{ij}^{\alpha\beta}(0)) \partial_\alpha \partial_\beta f^i(x) f^j(x) dx \\ &\geq c_1 \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} (Z_{ij}^{\alpha\beta}(U) - Z_{ij}^{\alpha\beta}(0)) \partial_\alpha \partial_\beta f^i(x) f^j(x) dx. \end{aligned}$$

If the image of  $U$  is so small that  $Z_{ij}^{\alpha\beta}$  have a very small oscillation then we see that the second term of the last inequality may be bounded by  $\varepsilon \|\nabla^2 f\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}$ , for  $\varepsilon$  small.

Finally, we consider the general case. Construct a partition of unit in  $\mathbb{R}^d$  as follows: We can write the range  $G$  of  $U$  which is bounded, as

$$G \subset \bigcup_{k=1}^N B_k$$

where  $B_k \stackrel{\text{def}}{=} B(a_k, \varepsilon)$  is a ball of center  $a_k \in G$  and radius  $\varepsilon > 0$ , which can be taken as small as we want. We denote

$$\Omega_k \stackrel{\text{def}}{=} U^{-1}(B_k),$$

and consider a partition of unity such that

$$1 = \sum_{k=1}^N \omega_k^2(x) \quad \forall x \in \mathbb{R}^d \quad \omega_k \geq 0, \quad \text{Supp } \omega_k \subset \Omega_k. \quad (175)$$

Then, by the Leibniz' rule of differentiation of the product of functions, Cauchy-Schwarz inequality and the estimate of the case obtained above, we have,

$$\begin{aligned}
& - \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta}(U) \partial_\alpha \partial_\beta f^i(x) f^j(x) dx = - \sum_{\alpha, \beta, i, j} \sum_{k=1}^N \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta}(U) \omega_k^2(x) \partial_\alpha \partial_\beta f^i(x) f^j(x) dx \\
& = - \sum_{\alpha, \beta, i, j} \sum_{k=1}^N \int_{\Omega_k} Z_{ij}^{\alpha\beta}(U) \partial_\alpha \partial_\beta (\omega_k f^i)(x) \omega_k f^j(x) dx \\
& + 2 \sum_{\alpha, \beta, i, j} \sum_{k=1}^N \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta}(U) f^i(x) \partial_\alpha \omega_k(x) \partial_\beta f^j(x) \omega_k(x) dx \\
& + \sum_{\alpha, \beta, i, j} \sum_{k=1}^N \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta}(U) \partial_\alpha \partial_\beta (\omega_k)(x) f^i f^j(x) \omega_k(x) dx \\
& \geq \sum_{k=1}^N \left( c_1 \|\nabla(\omega_k f)\|_{L^2(\mathbb{R}^d)}^2 - \varepsilon C(\omega_k) \|\nabla^2 f\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} \right. \\
& \quad \left. - C(\omega_k) \|\nabla f\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} - C(\omega_k) \|f\|_{L^2(\mathbb{R}^d)}^2 \right). \quad (176)
\end{aligned}$$

Relation (175) will be used for the first term of (176). In fact using Leibniz formula we can observe that

$$2 \|\nabla(\omega_k f)\|_{L^2}^2 \geq \|\omega_k \nabla f\|_{L^2}^2 - 2 \|f \nabla \omega_k\|_{L^2}^2 \geq \|\omega_k \nabla f\|_{L^2}^2 - 2C(\omega_k) \|f\|_{L^2}^2.$$

Using Young's inequality for the term  $C(\omega_k) \|\nabla f\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)}$ , that is,

$$C(\omega_k) \|\nabla f\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} \leq \frac{c_1}{4} \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 + C(\omega_k) \|f\|_{L^2(\mathbb{R}^d)}^2$$

we deduce the following inequality for  $C = C(\omega_k)$ :

$$- \sum_{\alpha, \beta, i, j} \int_{\mathbb{R}^d} Z_{ij}^{\alpha\beta}(U) \partial_\alpha \partial_\beta f^i(x) f^j(x) dx \geq \frac{c_1}{4} \|\nabla f\|_{L^2}^2 - \varepsilon C \|\nabla^2 f\|_{L^2(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} - C \|f\|_{L^2(\mathbb{R}^d)}^2.$$

This gives the desired result.  $\square$

The following logarithmic interpolation inequality in nonhomogeneous Besov spaces is very useful to prove the uniqueness of the solution of system (11) under the assumption **C** in nonhomogeneous Besov space. We haven't used this inequality in this paper, but instead we have used its original (inequality (137)) that has been proved by R. Danchin in [8, p. 1319] for a class of homogeneous Besov spaces. We adapt the result to the nonhomogeneous case.

**Lemma 7.** *For any  $(p, \rho) \in [1, \infty]^2$ ,  $T > 0$ ,  $s \in \mathbb{R}$  and  $\eta \in (0, 1]$  we have*

$$\|a\|_{\tilde{L}_T^\rho(B_{p,1}^s)} \leq C \frac{\|a\|_{\tilde{L}_T^\rho(B_{p,1}^s)}}{\eta} \log \left( e + \frac{\|a\|_{\tilde{L}_T^\rho(B_{p,\infty}^{s+\eta})}}{\|a\|_{\tilde{L}_T^\rho(B_{p,\infty}^s)}} \right). \quad (177)$$

*Proof.* We split the norm  $\|a\|_{\tilde{L}_T^\rho(B_{p,r}^s)}$  into low and high frequencies. We have

$$\begin{aligned}
\|a\|_{\tilde{L}_T^\rho(B_{p,1}^s)} &= \sum_{q=-1}^m 2^{qs} \|\Delta_q a\|_{L_T^\rho(L^p)} + \sum_{q=m+1}^{\infty} 2^{q(s+\eta)} \|\Delta_q a\|_{L_T^\rho(L^p)} 2^{-q\eta} \\
&\leq (m+2) \sup_{q \geq -1} 2^{qs} \|\Delta_q a\|_{L_T^\rho(L^p)} + \frac{2^{-\eta(m+1)}}{1-2^{-\eta}} \sup_{q \geq -1} 2^{q(s+\eta)} \|\Delta_q a\|_{L_T^\rho(L^p)}.
\end{aligned}$$

As  $\eta > 0$  we deduce that

$$\|a\|_{\tilde{L}_T^\rho(B_{p,1}^s)} \lesssim (m+2) \|a\|_{\tilde{L}_T^\rho(B_{p,\infty}^s)} + \eta^{-1} 2^{-\eta m} \|a\|_{\tilde{L}_T^\rho(B_{p,\infty}^{s+\eta})}.$$

Choosing  $m$  to be the integer part of

$$\frac{1}{\eta} \log_2 \left( \frac{\|a\|_{\tilde{L}_T^\rho(B_{p,\infty}^{s+\eta})}}{\|a\|_{\tilde{L}_T^\rho(B_{p,\infty}^s)}} \right)$$

yields the desired inequality.  $\square$

## B Littlewood-Paley decomposition and Besov spaces

In this section we present some results on the Littlewood-Paley decomposition and Besov spaces, the source of which can be found in [2, Chap. 2].

To define the Littlewood-Paley decomposition, we fix some smooth radial non increasing function  $\chi$  with  $\text{Supp}\chi \subset B(0, \frac{4}{3})$  and  $\chi \equiv 1$  on  $B(0, \frac{3}{4})$ , then set  $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$  so that

$$\chi + \sum_{j \geq 0} \varphi(2^{-j}\cdot) = 1 \text{ on } \mathbb{R}^d \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\cdot) = 1 \text{ on } \mathbb{R}^d \setminus \{0\}.$$

In order to pursue our definition, we agree that for  $S$  a function,  $z$  a distribution and  $a$  a real constant, the operator  $S(aD)$  is defined by

$$S(aD)z \stackrel{\text{def}}{=} \mathcal{F}^{-1} S(a\cdot) \mathcal{F} z, \quad (178)$$

when the right-hand side makes sense. Let us now state

$$\dot{S}_j = \chi(2^{-j}D) \text{ for all } j \in \mathbb{Z} \quad \text{and} \quad S_j \stackrel{\text{def}}{=} \dot{S}_j \text{ for all } j \geq 0, \quad S_j = 0 \text{ for all } j \leq -1. \quad (179)$$

We define the homogeneous dyadic block  $\dot{\Delta}_j$  and nonhomogeneous dyadic block  $\Delta_j$  as

$$\begin{aligned} \dot{\Delta}_j &\stackrel{\text{def}}{=} \varphi(2^{-j}D) \text{ for all } j \in \mathbb{Z} \\ \Delta_j &= \dot{\Delta}_j \text{ for all } j \geq 0, \quad \Delta_{-1} = \dot{S}_0 \quad \text{and} \quad \Delta_j = 0 \text{ for all } j < -1 \end{aligned} \quad (180)$$

and we define  $\mathcal{S}'_h$  to be the set of tempered distributions  $z$  such that

$$\lim_{j \rightarrow -\infty} \dot{S}_j z = 0. \quad (181)$$

Following [2], we introduce the homogeneous Besov semi-norms (resp. nonhomogeneous Besov norms):

$$\|z\|_{\dot{B}_{p,r}^s} \stackrel{\text{def}}{=} \|2^{js} \|\dot{\Delta}_j z\|_{L^p}\|_{l^r} \quad (\text{resp. } \|z\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \|2^{js} \|\Delta_j z\|_{L^p}\|_{l^r}). \quad (182)$$

Then, for any  $s \in \mathbb{R}$  and  $(p, r) \in [1, \infty]$  we define the homogeneous Besov spaces  $\dot{B}_{p,r}^s$  (resp. nonhomogeneous Besov spaces  $B_{p,r}^s$ ) to be the subset of those  $z$  in  $\mathcal{S}'_h$  (resp. the subset of those  $z$  in the tempered distribution space  $\mathcal{S}'$ ) such that  $\|z\|_{\dot{B}_{p,r}^s}$  (resp.  $\|z\|_{B_{p,r}^s}$ ) is finite.

The study of non-stationary PDEs requires spaces of type  $L^\rho(0, T; X)$  (endowed with the norm  $\|z\|_{L_T^\rho(X)} = \| \|z(t, \cdot)\|_X \|_{L_T^\rho}$ ) for appropriate Banach spaces  $X$ . Here we expect  $X$  to be a Besov space. However, for technical reasons, it is sometimes more suitable to use the Chemin-Lerner spaces that are defined below:

**Definition B.1.** Let  $\rho$  in  $[1, \infty]$  and time  $T \in [0, \infty]$ . We set

$$\|z\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \stackrel{\text{def}}{=} \|2^{js} \|\Delta_j z\|_{L_T^\rho(L^p)}\|_{l^r} \quad \text{with} \quad \|z\|_{L_T^\rho(L^p)} \stackrel{\text{def}}{=} \|z\|_{L^\rho(0,T;L^p)}.$$

We then define the space  $\tilde{L}_T^\rho(B_{p,r}^s)$  to be the set of tempered distributions  $z$  over  $[0, T] \times \mathbb{R}^d$  such that  $\|z\|_{\tilde{L}_T^\rho(B_{p,r}^s)} < \infty$ .

We have a similar definition in the homogeneous framework.

We set  $\tilde{C}([0, T]; B_{p,r}^s) = \tilde{L}_T^\rho(B_{p,r}^s) \cap C([0, T]; B_{p,r}^s)$ . Let us emphasize that, according to the Minkowski inequality, we have:

$$\|z\|_{\tilde{L}_T^\rho(B_{p,r}^s)} \leq \|z\|_{L_T^\rho(B_{p,r}^s)}, \text{ if } r \geq \rho \quad \text{and} \quad \|z\|_{L_T^\rho(B_{p,r}^s)} \leq \|z\|_{\tilde{L}_T^\rho(B_{p,r}^s)}, \text{ if } r \leq \rho. \quad (183)$$

Even though most of the functions we shall consider here will have range in the set of vectors or matrices, we shall keep the same notation for Besov spaces pertaining to this case.

One of the main motivations for using Littlewood-Paley decomposition when dealing with PDEs is that the derivatives act almost as dilations on distributions with Fourier transform supported in a ball or an annulus, as regards  $L^p$  norms. This is exactly what is stated in the following proposition:

**Proposition 7** (Bernstein inequality). Let  $0 < r < R$ .

- There exists a constant  $C$  so that, for any  $k \in \mathbb{N}$ , pair  $(q_1, q_2)$  in  $[1, \infty]^2$  with  $q_2 \geq q_1 \geq 1$  and function  $u$  of  $L^{q_1}$  with  $\hat{u}$  supported in the ball  $B(0, \lambda R)$  of  $\mathbb{R}^d$  for some  $\lambda > 0$ , we have  $D^k u \in L^{q_2}$  and

$$\|D^k u\|_{L^{q_2}(\mathbb{R}^d)} \leq C^{k+1} \lambda^{k+d(\frac{1}{q_1}-\frac{1}{q_2})} \|u\|_{L^{q_1}(\mathbb{R}^d)}.$$

- For any smooth function  $M$  on  $\mathbb{R}^d \setminus \{0\}$  with homogeneity  $\gamma$ , there exists a constant  $C$  such that for any  $\lambda > 0$  and any function  $u$  with Fourier transform  $\hat{u}$  supported in annulus  $\{\xi \in \mathbb{R}^d \mid r\lambda \leq |\xi| \leq R\lambda\}$  of  $\mathbb{R}^d$ , we have

$$\|M(D)u\|_{L^{q_2}(\mathbb{R}^d)} \leq C^{k+1} \lambda^{\gamma+d(\frac{1}{q_1}-\frac{1}{q_2})} \|u\|_{L^{q_1}(\mathbb{R}^d)}.$$

- There exists a constant  $C$  so that for any  $k \in \mathbb{N}$ ,  $q \in [1, \infty]$  and function  $u$  of  $L^q$  with  $\text{Supp } \hat{u} \subset \{\xi \in \mathbb{R}^d \mid r\lambda \leq |\xi| \leq R\lambda\}$  for some  $\lambda > 0$ , we have

$$\lambda^k \|u\|_{L^q(\mathbb{R}^d)} \leq C^{k+1} \|D^k u\|_{L^q(\mathbb{R}^d)}.$$

In order to bound the commutator terms, we use the following results:

**Proposition 8.** We designate by  $\mathbb{B}_{2,r}^s$  both  $B_{2,r}^s$  and  $\dot{B}_{2,r}^s$ . The following inequalities hold true: If  $\sigma > 0$

$$\|[a, \Delta_j]b\|_{L^2} \leq 2^{-q\sigma} C c_j (\|\nabla a\|_{L^\infty} \|b\|_{\mathbb{B}_{2,1}^{\sigma-1}} + \|b\|_{L^\infty} \|\nabla a\|_{\mathbb{B}_{2,1}^{\sigma-1}}) \quad \text{with} \quad \sum_j c_j = 1. \quad (184)$$

In particular, if  $\sigma \geq \frac{d}{2} + 1$ , then it holds

$$\|[a, \Delta_j]b\|_{L^2} \leq 2^{-q\sigma} C c_j \|\nabla a\|_{\mathbb{B}_{2,1}^{\sigma-1}} \|b\|_{\mathbb{B}_{2,1}^{\sigma-1}} \quad \text{with} \quad \sum_{j \geq -1} c_j = 1. \quad (185)$$

Moreover if  $-\frac{d}{2} < \sigma \leq \frac{d}{2} + 1$ , then

$$\| [a, \Delta_j] b \|_{L^2} \leq 2^{-q\sigma} C c_j \| \nabla a \|_{\mathbb{B}_{2,\infty}^{\frac{d}{2}} \cap L^\infty} \| b \|_{\mathbb{B}_{2,1}^{\sigma-1}} \quad \text{with} \quad \sum_j c_j = 1. \quad (186)$$

We have also

$$\| [a, \Delta_j] b \|_{L^2} \leq 2^{-q\sigma} C \| \nabla a \|_{\mathbb{B}_{2,\infty}^{\frac{d}{2}} \cap L^\infty} \| b \|_{\mathbb{B}_{2,\infty}^{\sigma-1}} \quad \text{for all} \quad -d/2 \leq \sigma < d/2 + 1. \quad (187)$$

Similar results hold true if we replace  $L^2$  by  $L_T^\rho(L^2)$  in the l.h.s and the spaces  $L_T^\rho(\mathbb{B}_{2,r}^s)$  (or  $\tilde{L}_T^\rho(\mathbb{B}_{2,r}^s)$ ) in the r.h.s.

The following product laws in Besov spaces have been used repeatedly.

**Proposition 9.** *Let  $(s, r) \in ]0, \infty[ \times [1, \infty]$ . Then  $B_{p,r}^s \cap L^\infty$  is an algebra and we have*

$$\| ab \|_{B_{2,r}^s} \leq C (\| a \|_{L^\infty} \| b \|_{B_{2,r}^s} + \| b \|_{L^\infty} \| a \|_{B_{2,r}^s}). \quad (188)$$

Moreover, if  $-d/2 < s \leq d/2$ , then the following inequality holds:

$$\| ab \|_{B_{2,1}^s} \leq C \| a \|_{B_{2,1}^{\frac{d}{2}}} \| b \|_{B_{2,1}^s} \quad (189)$$

and if  $-d/2 \leq s < d/2$ ,

$$\| ab \|_{B_{2,\infty}^s} \leq C \| a \|_{B_{2,1}^{\frac{d}{2}}} \| b \|_{B_{2,\infty}^s}. \quad (190)$$

Finally, if  $s > d/2$  (or  $s = d/2$  and  $r = 1$ ),

$$\| ab \|_{B_{2,r}^s} \leq C \| a \|_{B_{2,r}^s} \| b \|_{B_{2,r}^s}. \quad (191)$$

The above estimates hold if we replace the nonhomogeneous Besov spaces with the corresponding homogeneous Besov spaces. We have similar results for the spaces  $L_T^\rho(B_{2,r}^s)$ ,  $\tilde{L}_T^\rho(B_{2,r}^s)$ ,  $L_T^\rho(\dot{B}_{2,r}^s)$  and  $\tilde{L}_T^\rho(\dot{B}_{2,r}^s)$  see [7, 14].

Among the results necessary to prove our Theorems, 1.3 and 1.2, we have the following one.

**Proposition 10.** *Let  $f$  be a function in  $C^\infty(\mathbb{R})$ . Let  $r \in [1, \infty]$  and  $s \in ]0, \infty[$ . Then, if  $f(0) = 0$ , for every real-valued function  $u$  in  $B_{2,r}^s$ , the function  $f \circ u$  belongs to  $B_{2,r}^s$  and we have*

$$\| f \circ u \|_{B_{2,r}^s} \leq C(f', \| u \|_{L^\infty}) \| u \|_{B_{2,r}^s}. \quad (192)$$

More generally,

$$\| f \circ u - f \circ v \|_{B_{2,r}^s} \leq C(f', \| u, v \|_{L^\infty}) (1 + \| u \|_{B_{2,1}^{\frac{d}{2}}} + \| v \|_{B_{2,1}^{\frac{d}{2}}}) \| u - v \|_{B_{2,r}^s}. \quad (193)$$

Furthermore, if  $-\frac{d}{2} \leq s < \frac{d}{2}$  then the last inequality remains valid for  $r = \infty$ , that is,

$$\| f \circ u - f \circ v \|_{B_{2,\infty}^s} \leq C(f', \| u, v \|_{L^\infty}) (1 + \| u \|_{B_{2,1}^{\frac{d}{2}}} + \| v \|_{B_{2,1}^{\frac{d}{2}}}) \| u - v \|_{B_{2,\infty}^s}. \quad (194)$$

Finally if  $-\frac{d}{2} < s \leq \frac{d}{2}$ , then we still an inequality similar to (193)

$$\| f \circ u - f \circ v \|_{B_{2,1}^s} \leq C(f', \| u, v \|_{L^\infty}) (1 + \| u \|_{B_{2,1}^{\frac{d}{2}}} + \| v \|_{B_{2,1}^{\frac{d}{2}}}) \| u - v \|_{B_{2,1}^s}. \quad (195)$$

Similar results holds for homogeneous Besov spaces and Chemin-Lerner spaces.

*Proof.* The proof of (192) can be found in [2, pages 94 and 104] while (193), (194) and (195) can be obtained by adapting the proof of first inequality of [2, page 449].  $\square$

Finally, the following composition estimates enable us to handle the 0-th order terms in System (11):

**Proposition 11.** *Let  $0 \leq n_1 \leq n$  and  $m$  be three integers. Let  $f : (X, Y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n-n_1} \mapsto f(X, Y) \in \mathbb{R}^m$  be a smooth function on  $\mathbb{R}^n$ . Assume that  $f$  is at most linear with respect to  $Y$  (that is, the second derivative  $D_Y^2 f$  of  $f$  with respect to the component  $Y$  vanishes).*

*If  $f$  vanishes at  $0_{\mathbb{R}^n}$ , then for any  $0 < s \leq \frac{d}{2}$  the following inequality holds true*

$$\|f(u, v)\|_{B_{2,1}^s} \leq C(f', \|u\|_{L^\infty})(\|v\|_{B_{2,1}^s} (1 + \|u\|_{B_{2,1}^{\frac{d}{2}}}) + \|u\|_{B_{2,1}^s}). \quad (196)$$

Furthermore if  $-\frac{d}{2} < s \leq \frac{d}{2}$ , then we have for some  $C = C(f', \|u_1, u_2\|_{L^\infty})$ :

$$\begin{aligned} \|f(u_1, v_1) - f(u_2, v_2)\|_{B_{2,1}^s} &\leq C \|v_2 - v_1\|_{B_{2,1}^s} (1 + \|u_2\|_{B_{2,1}^{\frac{d}{2}}}) \\ &+ C(1 + \|u_1\|_{B_{2,1}^{\frac{d}{2}}} + \|u_2\|_{B_{2,1}^{\frac{d}{2}}}) \left( \|u_2 - u_1\|_{B_{2,1}^{\frac{d}{2}}} \|v_1\|_{B_{2,1}^s} + \|u_1 - u_2\|_{B_{2,1}^s} \right). \end{aligned} \quad (197)$$

Finally if  $-\frac{d}{2} \leq s < \frac{d}{2}$  then we have

$$\begin{aligned} \|f(u_1, v_1) - f(u_2, v_2)\|_{B_{2,\infty}^s} &\leq C \|v_2 - v_1\|_{B_{2,\infty}^s} (1 + \|u_2\|_{B_{2,1}^{\frac{d}{2}}}) \\ &+ C(1 + \|u_1\|_{B_{2,1}^{\frac{d}{2}}} + \|u_2\|_{B_{2,1}^{\frac{d}{2}}}) \left( \|u_2 - u_1\|_{B_{2,\infty}^s} \|v_1\|_{B_{2,1}^{\frac{d}{2}}} + \|u_1 - u_2\|_{B_{2,\infty}^s} \right), \end{aligned} \quad (198)$$

where  $C = C(f', \|u_1, u_2\|_{L^\infty})$ .

*Proof.* The Taylor formula and the fact  $D_Y^2 f = 0$  guarantee the existence of two smooth functions  $\Lambda$  and  $\Gamma$  defined on  $\mathbb{R}^{n_1}$  such that

$$f(u, v) = \Lambda(u)v + \Gamma(u) \quad (\text{with } \Gamma(0_{\mathbb{R}^{n_1}}) = 0_{\mathbb{R}^m} \text{ if } f(0_{\mathbb{R}^{n_1}}, 0_{\mathbb{R}^{n-n_1}}) = 0_{\mathbb{R}^m}.$$

If  $0 < s \leq \frac{d}{2}$  then, applying the inequalities (189) and (192) to the term  $\Lambda(u)v$  yields the first term of the right-hand side of inequality (196). Next, using Proposition 10 (recall that  $s > 0$ ) for the term  $\Lambda(u)$  gives the second term of (196).

To prove (197) (resp. (198)), we use the above decomposition to get

$$f(u_2, v_2) - f(u_1, v_1) = \Lambda(u_2)(v_2 - v_1) + (\Lambda(u_2) - \Lambda(u_1))v_1 + (\Gamma(u_2) - \Gamma(u_1)).$$

Having this decomposition in hand, the first two terms of the last equality may be handled by the inequalities (189) (resp. (190)) and (192) (resp. (194)). Concerning the last one, we use Inequality (195) (resp. (194)).  $\square$

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