Output-feedback consensus-formation control of nonholonomic vehicles with input constraints and time-varying delays.

Angel Paredes, Emmanuel Nuño, Antonio Loria

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***Abstract*— We provide a solution to the consensus-based formation control problem of multi-agent nonholonomic vehicles, under the assumption that the magnitudes of control inputs must satisfy certain pre-imposed bounds, to avoid saturation of the actuators. Also, we assume that the communications are affected by time-varying time-delays and the vehicles lack velocity measurements. The desired control objective is that all the vehicles converge to a given desired formation and that they reach consensus on the localization of the center of such formation and on their orientations. The control architecture exploits several features. From a control viewpoint, the systems' dynamics are split into their angular- and linear-motion parts, so each of these is driven by a separate distributed dynamic output-feedback controller, passively interconnected to the plant via a virtual spring. From a network-topology viewpoint the overall networked system contains two layers of interconnections with undirected graphs. Finally, from an analytical viewpoint, each vehicle-controller closed-loop-system may be regarded as a cascade with a δ-persistently exciting interconnection, which ensures the achievement of the control goal, in spite of the nonholonomic constraints.**

I. Introduction

The achievement of consensus for networked systems depends on the network's topology, the systems' dynamics, and the strength and nature of the interconnections [1], so our ability to analyze and control networked systems inextricably relies on more or less conservative assumptions related to one or another of these aspects. For instance, from a graph-theory point of view, the analysis of networks with directed graphs or time-varying topologies are particularly challenging [2], [3], so they may be considered on the assumption that the systems' dynamics is described by simple integrators [4], [5]. On the other hand, in many engineering applications networked systems are interconnected over channels with sufficient bandwidth to support bidirectional interconnections, but the use of technology may entail other complications. For instance, the communications may be intermittent [5], prone to loss of connectivity [6], or affected by delays [7].

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Yet, many control approaches that apply to networks with complex interconnections, but that are tailored for linear systems or simple integrators, do not extend to the case of muti-agent *nonlinear* systems, let alone of robots with nonholonomic constraints, as we study here. In their most general form, the dynamic models of nonholonomic systems are of order two: one equation models the vehicle's kinematics and another its velocity dynamics. The first one essentially captures the nonholonomic constraints which, as is well-known, prevent the use of static smooth controllers both in stabilization and tracking control tasks [8]. The second one is of the Euler-Lagrange type [9], [10], [11]. However, in a large number of works on control of multi-agent vehicles the dynamics equation is neglected—see *e.g.,* [12], in which the otherwise difficult problem of consensus over a directed ring is considered. In this paper, we adopt a full dynamic model, including the Lagrangian dynamics, but we consider differential-drive robots, whose inertia matrix is constant [13].

For such systems the problem of all systems' states acquiring a consensual value generally translates into making all the vehicles converge to positions defining a formation around a non-predefined rendezvous point [14] and, in some cases, also adopting a common orientation [15]. The former is referred to as partial consensus- and the latter as full consensus-formation. On the other hand, if, as in classical consensus problems, the rendezvous consensus posture depends on the systems' initial conditions, the systems' dynamics, and the network's topology, but is not pre-imposed as a reference, the type of consensus is commonly referred to as *leaderless*—see *e.g.,* [12]. In the case that the robots follow a predefined trajectory, described by a (or several leaders) the problem is referred to as leader-follower formation [16], [17], [18]. Both problems are fundamentally different, but equally significant as together they may form part of a more complex-maneuver mission [19].

Now, beyond the aspects that concern the network's interconnections and the systems' models, other aspects of practical nature must also be considered. These pertain to the fact that control engineering solutions to technological problems rely, by definition, on *sensors* that collect measures to provide *feedback*, as well as on *actuators*, which are physical systems too. This assertion is not a common place, but the recognition that some sensors are often faulty (*e.g.,* tachometers, which are often contaminated by noise), while actuators have limitations.

Therefore, it is also important to design controllers that, on one hand, rely only on position and orientation measurements [20], [21] and, on the other, that respect torque limitations imposed by the physics of the actuators [22], [23]. The latter, in particular, is an aspect that is disregarded when using kinematics-only-based dynamical models. To protect the actuators, saturation of the latter must be always avoided, but the extension of well-established controllers to comply with input constraints is very intricate [24], [25]. In general, simply introducing saturation functions in the control law, or replacing u with $\text{sat}(u)$ for that matter, is most commonly insufficient to achieve the control goal.

Thus, even though the consensus of nonholonomic mobile robots has been intensively studied via output-feedback control [20], [21], considering input saturation [22], [23], or in the presence of communication delays [9], [10], [23], to the best of our knowledge the problem has not been addressed considering all three aspects simultaneously. This is the case, *e.g.,* in [26], but not for nonholonomic vehicles. On the other hand, in [27] and [28] different output-feedback consensus-formation controllers are proposed for nonholonomic vehicles with delayed communications, but in the unrealistic case that the actuators do not saturate. Finally, consensus controllers satisfying pre-imposed bounds on the control inputs are proposed in [29], but these rely on full-state feedback and on the assumption that the interconnections are not affected by delays.

In this paper we design output-feedback dynamic controllers that are reminiscent of mass-spring-damper systems that are interconnected to the vehicles through virtual springs—cf. [27]. Through the spring interconnection, the controllers steer the second-order plant's positions and also injects damping into the plant. From a networked-systems viewpoint, it is important to stress that it is the controllers, and not the plants themselves, which are interconnected over a network. From a dynamical-systems viewpoint, the controllers are smooth and time-varying; they rely on persistency of excitation, which is a necessary condition to stabilize nonholonomic systems to an equilibrium point [8] and, more precisely, on a property, called δ-persistency of excitation, which is necessary for the attractivity of the origin in generic smooth nonlinear time-varying systems [30]. In addition, our controllers satisfy pre-imposed bounds, so actuator saturation is avoided. To the best of the authors' knowledge, this work provides the first solution to the full consensus-formation problem of nonholonomic vehicles without making use of velocity measurements, for which their interconnections are affected by time-varying delays, and avoiding actuator saturation.

The remainder of the paper is organized as follows. In Section II we describe the model and the problem statement; the control approach and the main statement are presented in Section III, while the proofs are given in Section IV. Some numerical simulations are provided in Section V and we wrap up the paper with concluding remarks in Section VI.

II. Problem formulation

We consider the consensus-formation control problem for a group of N robots with position and orientation coordinates

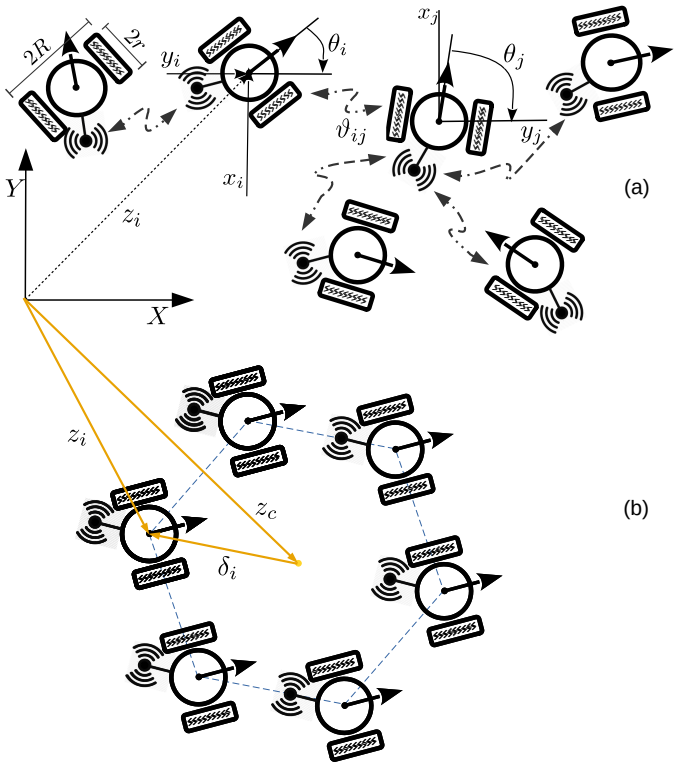

Fig. 1. Illustration of the control goal: for a group of scattered vehicles (a) to reach a consensus-based formation around an *a priori* unknown rendezvous set-point and acquire a common orientation (b).

$\mathbf{z}_i := [x_i, y_i]^\top \in \mathbb{R}^2$ and $\theta_i \in \mathbb{R}$, respectively, and $i \leqslant N$, with a dynamic model given by

$$\dot{\mathbf{x}}_i = \mathbf{G}(\mathbf{x}_i)\mathbf{v}_i, \qquad \mathbf{G}(\mathbf{x}_i) := \begin{bmatrix} \cos(\theta_i) & 0 \\ \sin(\theta_i) & 0 \\ 0 & 1 \end{bmatrix} \tag{1a}$$

$$\mathbf{M}_i\dot{\mathbf{v}}_i + \mathbf{F}_i\mathbf{v}_i = \mathbf{B}_i\boldsymbol{\tau}_i, \tag{1b}$$

where $\mathbf{M}_i := \text{diag}(m_i, I_i)$, $\mathbf{F}_i := \text{diag}(f_{vi}, f_{\omega i})$, I_i is the robot's inertia, m_i denotes its mass, and f_{vi} and $f_{\omega i}$ denote friction coefficients, so all these quantities are positive. Finally, the input matrix

$$\mathbf{B}_i = \frac{1}{r_i}\begin{bmatrix} 1 & 1 \\ 2R_i & -2R_i, \end{bmatrix}. \tag{2}$$

In the previous expression r_i is the wheel radius, and R_i is the wheel axle length—see top of Figure 1 above.

Remark 1: The system's dynamics has a simple Lagrangian form, with constant inertia matrix, because it is assumed that the center of mass is located on the axis connecting the centers of the wheels, which is characteristic of differential-wheel robots [13].

The consensus-formation control problem that we address involves making the vehicles acquire a desired formation pattern relatively to a non pre-imposed rendezvous point that corresponds to the center of a formation pattern from a an initially scattered configuration—see Figure 1 for an illustration. In the latter, $\boldsymbol{\delta}_i \in \mathbb{R}^2$ denotes a given constant vector determining the position of the ith vehicle relatively to the non predefined center $\bar{\mathbf{z}}_c$ and $\bar{\mathbf{z}}_i := \mathbf{z}_i - \boldsymbol{\delta}_i$ denotes the vehicle's relative position error. That is, the vehicles are

said to achieve the formation consensus goal if $\mathbf{z}_i \to \mathbf{z}_c$ and $\theta_i \to \theta_c$ for all $i \leqslant N$ or, more precisely, if

$$\lim_{t\to\infty} v_i(t) = 0, \quad \lim_{t\to\infty} \bar{\mathbf{z}}_i(t) = \mathbf{z}_c, \tag{3}$$

$$\lim_{t\to\infty} \omega_i(t) = 0, \quad \lim_{t\to\infty} \theta_i(t) = \theta_c \qquad \forall\, i \leqslant N, \tag{4}$$

hold for all initial conditions. In the previous expressions, v_i and ω_i correspond to the linear-motion and angular velocities respectively and define $\mathbf{v}_i := [v_i \ \omega_i]^\top$ in (1b).

Then, for further development, we rewrite the kinematics model (1a) as

$$\dot{\bar{\mathbf{z}}}_i = \boldsymbol{\varphi}(\theta_i) v_i; \qquad \boldsymbol{\varphi}(\theta_i) := [\cos(\theta_i) \quad \sin(\theta_i)]^\top \tag{5}$$

$$\dot{\theta}_i = \omega_i. \tag{6}$$

On the other hand, for a differential-wheel drive vehicle the input torque in (1b) is given $\boldsymbol{\tau}_i = \mathbf{B}_i^{-1}\mathbf{u}_i$, where the input matrix $\mathbf{B}$ is defined in (2) and $\mathbf{u}_i := [u_{vi} \quad u_{\omega i}]^\top$ is now considered as the control input. Hence, for the sequel, we split the dynamics equation (1b) into

$$\dot{v}_i = \frac{1}{m_i}\left(u_{vi} - f_{vi} v_i\right) \tag{7a}$$

$$\dot{\omega}_i = \frac{1}{I_i}\left(u_{\omega i} - f_{\omega i}\omega_i\right). \tag{7b}$$

Now, as it is explained in the Introduction, the consensus-formation problem for nonholonomic vehicles has been extensively studied in the literature, in many forms, with different motivations, and under a variety of conditions, but to the best of the authors' knowledge, not under the following hypotheses, on the systems' individual dynamics and on the network, simultaneously.

Assumption 1 (on the agents): For each vehicle,

(a) the Cartesian positions (x_i, y_i) and the orientation θ_i are available from measurements, but not the velocities $\mathbf{v}_i$;

(b) the wheel applied torques $\boldsymbol{\tau}_i := [\tau_{li} \ \tau_{ri}]^\top$ must satisfy pre-defined given bounds $|\tau_{li}| \leqslant \bar{\tau}_{li}$ and $|\tau_{ri}| \leqslant \bar{\tau}_{ri}$ for the left and right wheel respectively;

Assumption 2 (on the network):

(a) Each agent, labeled $i \leqslant N$, receives information from a group of "neighbors" labeled $j \in \mathcal{N}_i$, where $\mathcal{N}_i \subset \mathbb{Z}$ is the set of indexes corresponding to robots transmitting information to the ith robot, but the information is received with a time-delay denoted $T_{ji}(t)$ that is assumed to be bounded by a known upper-bound $\overline{T}_{ji} \geqslant 0$ and has bounded time-derivatives.

(b) The interconnections are static and are modeled by an undirected, connected graph.

III. Physics-based Consensus-formation Controller

The output-feedback controller design is inspired by the vehicle's physics—cf. [27]. On one hand, we design, a controller for the angular-motion dynamics and, on the other, one for the linear-motion dynamics. In both cases, the controllers are distributed and dynamic; they are conceived as network-interconnected second-order mechanical systems designed to achieve consensus among themselves. Then, by virtue of their interconnection to the actual vehicles' dynamics, they steer the latter to consensus formation—see Figure 2.

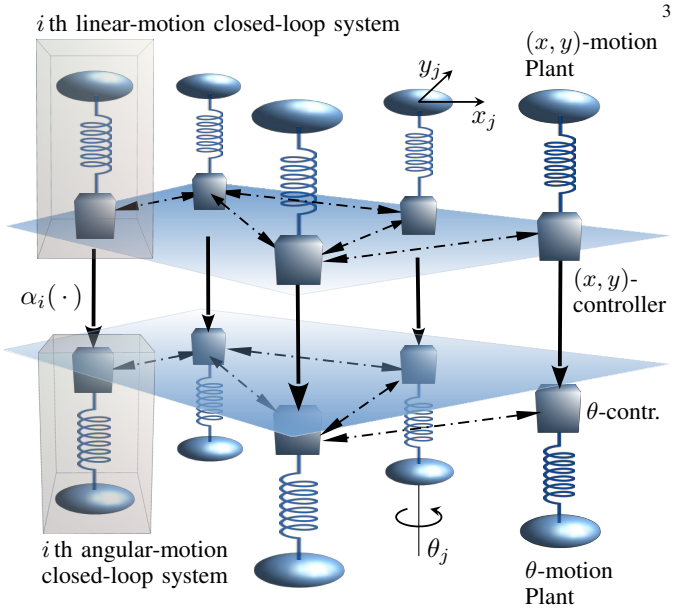

Fig. 2. Schematic representation of the controlled-systems architecture. From a control viewpoint, the systems' dynamics are split into angular- and linear-motion plants (represented by ellipses). To each of these, a distributed dynamic controller (represented by a cube) is hinged via a virtual spring. From a topological viewpoint the overall networked system contains two layers of interconnections: one containing a network of linear-motion *controllers* and another of angular-motion ones. Finally, from an analytical viewpoint, each system may be regarded as a cascade with a $\boldsymbol{\delta}$-PE interconnection $\boldsymbol{\alpha_i}$—look at the left of the image.

A. Angular-motion consensus-formation control

Consider the controlled second-order dynamical system

$$\ddot{\vartheta}_{\omega i} + d_{\omega i}\dot{\vartheta}_{\omega i} + p_{\omega i}\tilde{e}_{\omega i} = \nu_{\omega_i} \tag{8}$$

where $\dot{\vartheta}_{\omega i}, \vartheta_{\omega i} \in \mathbb{R}$ are the (virtual) angular velocity and orientation, respectively,

$$\tilde{e}_{\omega i} := \sum_{j\in\mathcal{N}_i} a_{ij}\Big[\vartheta_{\omega i} - \vartheta_{\omega j}(t - T_{ji}(t))\Big] \tag{9}$$

ν_{ω_i} is an external input, $T_{ji}(t)$ is a time-varying transmission delay from agent j to agent i, $a_{ij} = a_{ji} \in \{0, 1\}$, and $p_{\omega i}, d_{\omega i} > 0$ are the proportional and the damping injection gains, respectively. This equation represents the dynamics of a second-order system interconnected with $N_i := \mathrm{card}\{\mathcal{N}_i\}$ neighbors over a network with bidirectional interconnections and over which it received the delayed (virtual) position $\vartheta_{\omega j}(t - T_{ji}(t))$ from each qualifying neighbor. As the linear-motion dynamics (1b), the system (8) has simple Lagrangian dynamics (with unitary inertia), so, after [31, Proposition 1] the consensus manifold $\{(e_{\omega i}, \dot{\vartheta}_{\omega i}) = (0, 0); i \leqslant N\}$, where

$$e_{\omega i} := \sum_{j\in\mathcal{N}_i} a_{ij}\big[\vartheta_{\omega i} - \vartheta_{\omega j}\big], \tag{10}$$

is uniformly globally asymptotically stable if the damping injection gain is such that

$$d_{\omega i} > \frac{1}{2} p_{\omega i} \sum_{i\in\mathcal{N}_i} a_{ij}\left(\varepsilon_i + \frac{\overline{T}_{ji}^2}{\varepsilon_j}\right), \tag{11}$$

for some $\varepsilon_i > 0$ and all $i \in N$. Furthermore, also after [31, Proposition 1], the system is input-output stable with input[1] $\boldsymbol{\nu}_\omega$ and output $(\boldsymbol{e}_\omega, \dot{\boldsymbol{\vartheta}}_\omega)$. This is significant because it implies, directly, that $e_{\omega i}(t)$ and $\dot{\vartheta}_{\omega i}(t)$ are bounded for all t if so is $\nu_{\omega_i}(t)$. We use this property to establish the main result.

Now let the input $\nu_{\omega i} := -u_{\omega i}$ and let $u_{\omega i}$, which was introduced in (7b), be set to

$$u_{\omega i} = -k_{\omega i}\tanh(\theta_i - \vartheta_{\omega i}), \quad k_{\omega i} > 0. \tag{12}$$

Then, the closed-loop system corresponding to the angular-motion dynamics results in

$$\dot{\theta}_i = \omega_i \tag{13a}$$
$$\dot{\omega}_i = -\frac{1}{I_i}\big[k_{\omega i}\tanh(\theta_i - \vartheta_{\omega i}) + f_{\omega i}\omega_i\big] \tag{13b}$$
$$\ddot{\vartheta}_{\omega i} = -d_{\omega i}\dot{\vartheta}_{\omega i} - p_{\omega i}\tilde{e}_{\omega i} + k_{\omega i}\tanh(\theta_i - \vartheta_{\omega i}). \tag{13c}$$

After [26, Proposition 1], the limits in (4) hold for the trajectories of the networked systems (13).

The rationale behind the control strategy is that, on one hand, the virtual mechanical systems (13b) are interconnected among themselves in a manner that they reach consensus and, on the other, each system is interconnected to the angular motion dynamics via the nonlinear virtual-spring mechanical-force term $\tanh(\theta_i - \vartheta_{\omega i})$, so ultimately, all the systems reach angular-position consensus. The same rationale is applied next to the linear-motion dynamics, with certain modifications to accommodate for the nonholonomic constraints.

B. Linear-motion consensus-formation control

Consider now the linear-motion dynamics (5)-(7a). As for the angular-motion dynamics, the controllers for (5)-(7a) are designed as networked dynamical systems conceived to achieve consensus among themselves and are individually interconnected to the linear-motion dynamics of each vehicle. More precisely, the controllers are given by the set of equations

$$\ddot{\boldsymbol{\vartheta}}_{vi} = -d_{vi}\dot{\boldsymbol{\vartheta}}_{vi} - p_{vi}\tilde{\mathbf{e}}_{vi} + k_{vi}\boldsymbol{\varphi}(\theta_i)^\top \mathbf{tanh}(\bar{\mathbf{z}}_i - \boldsymbol{\vartheta}_{vi}) \tag{14}$$
$$u_{vi} = -k_{vi}\boldsymbol{\varphi}(\theta_i)^\top \mathbf{tanh}(\bar{\mathbf{z}}_i - \boldsymbol{\vartheta}_{vi}), \tag{15}$$

where

$$\mathbf{tanh}(\boldsymbol{\vartheta}_{vi} - \bar{\mathbf{z}}_i) := \begin{bmatrix}\tanh(\vartheta_{xi} - \bar{x}_i) & \tanh(\vartheta_{yi} - \bar{y}_i)\end{bmatrix}^\top$$

and

$$\tilde{\mathbf{e}}_{vi} := \sum_{j\in\mathcal{N}_i} a_{ij}\Big[\boldsymbol{\vartheta}_{vi} - \boldsymbol{\vartheta}_{vj}(t - T_{ji}(t))\Big]. \tag{16}$$

In closed loop with (5)-(7a), we obtain the closed-loop system's equations

$$\Sigma_{vi} := \begin{cases} \dot{\bar{\mathbf{z}}}_i = \boldsymbol{\varphi}(\theta_i)v_i \\ \dot{v}_i = -\dfrac{1}{m_i}\left[f_{vi}v_i + k_{vi}\boldsymbol{\varphi}(\theta_i)^\top \mathbf{tanh}(\bar{\mathbf{z}}_i - \boldsymbol{\vartheta}_{vi})\right] \\ \ddot{\boldsymbol{\vartheta}}_{vi} = -d_{vi}\dot{\boldsymbol{\vartheta}}_{vi} - k_{vi}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi} - \bar{\mathbf{z}}_i) - p_{vi}\tilde{\mathbf{e}}_{vi}. \end{cases} \tag{17}$$

[1]Notation: $\boldsymbol{\nu}_\omega := [\nu_{\omega 1}\cdots\nu_{\omega N}]^\top$, similarly for $\boldsymbol{e}_\omega$ and $\dot{\boldsymbol{\vartheta}}_\omega$. In [31, Proposition 1] it is stated that the system is input-to-*state* stable, with input $\boldsymbol{\nu}_\omega$ and state $(\boldsymbol{e}_\omega, \dot{\boldsymbol{\vartheta}}_\omega)$, which results in an abuse of terminology.

Now, as we explained earlier, the trajectories of the networked systems (13) are guaranteed to achieve consensus if the systems are interconnected over an undirected graph containing a spanning tree (Assumption 2b). Therefore, it is guaranteed that $\theta_i(t) \to \theta_c$ as $t \to \infty$. On the other hand, in view of the nonholonomic constraints, we see that, for any fixed $\theta_i = \theta_c$ the set of equations (17) admit many equilibria, other than the desired one $\{(\bar{\mathbf{z}}_i, v_i, \boldsymbol{\vartheta}_{vi}, \dot{\boldsymbol{\vartheta}}_{vi}) = (\mathbf{z}_c, 0, \mathbf{z}_c, 0)\}$ for all i, $j \leqslant N$. To see this, we first observe that

$$\boldsymbol{\vartheta}_{vi} - \boldsymbol{\vartheta}_{vj}(t - T_{ji}(t)) = \boldsymbol{\vartheta}_{vi} - \boldsymbol{\vartheta}_{vj} + \int_{t-T_{ji}(t)}^{t} \dot{\boldsymbol{\vartheta}}_{vj}(\sigma)d\sigma, \tag{18}$$

and then we set $v_i = 0, \dot{\boldsymbol{\vartheta}}_{vi} = 0$ for all $i \leqslant N$. It follows that, for any fixed θ_c, on $\{\theta_i = \theta_c\}$, the equilibria of Σ_{vi} are the solutions of

$$\boldsymbol{\varphi}(\theta_c)^\top \mathbf{tanh}(\bar{\mathbf{z}}_i - \boldsymbol{\vartheta}_{vi}) = 0, \tag{19a}$$
$$k_{vi}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi} - \bar{\mathbf{z}}_i) + p_{vi}\boldsymbol{e}_{vi} = \mathbf{0}, \tag{19b}$$

where

$$\boldsymbol{e}_{vi} := \sum_{j\in\mathcal{N}_i} a_{ij}\big[\boldsymbol{\vartheta}_{vi} - \boldsymbol{\vartheta}_{vj}\big]. \tag{20}$$

Now, introducing the annihilator of $\boldsymbol{\varphi}$, *i.e.,* $\boldsymbol{\varphi}(\theta)^\perp := [-\sin(\theta) \;\; \cos(\theta)]^\top$, which satisfies $\boldsymbol{\varphi}(\theta)^{\perp\top}\boldsymbol{\varphi}(\theta) = 0$, for any $\theta \in \mathbb{R}$, we see that the set of Equations in (19) admit solutions $(\bar{\mathbf{z}}_i, \boldsymbol{\vartheta}_{vi}) \equiv (\bar{\mathbf{z}}_i^*, \boldsymbol{\vartheta}_{vi}^*)$ such that $\mathbf{tanh}(\bar{\mathbf{z}}_i^* - \boldsymbol{\vartheta}_{vi}^*) = c_i\boldsymbol{\varphi}(\theta_c)^\perp = (k_{vi}/p_{vi})\boldsymbol{e}_{vi}^*$, with $c_i \in \mathbb{R}\backslash\{0\}$, $\boldsymbol{e}_{vi}^* \neq 0$, and $\bar{\mathbf{z}}_i^* \neq \boldsymbol{\vartheta}_{vi}^*$. To ensure that the equilibria are bound to the points in the consensus manifold, the equation

$$\boldsymbol{\varphi}(\theta_i)^{\perp\top}\mathbf{tanh}(\bar{\mathbf{z}}_i - \boldsymbol{\vartheta}_{vi}) = 0 \tag{21}$$

must be satisfied in addition to (19), regardless of the value of θ_i. Indeed, because $\boldsymbol{\varphi}(\theta_i)$ is orthogonal to $\boldsymbol{\varphi}(\theta_i)^\perp$ and $\tanh(\cdot)$ is odd, strictly increasing and $\tanh(s) = 0$ if and only if $s = 0$, the only set of points that satisfy simultaneously (19a) and (21), independently of θ_i, is bound to points for which necessarily $\bar{\mathbf{z}}_i = \boldsymbol{\vartheta}_{vi}$. Thus, to remove the unwanted equilibria, we redesign the angular-motion controller's dynamics (13b) by adding a term that vanishes only if (21) holds. Let

$$\alpha_i(t, \theta_i, \boldsymbol{\vartheta}_{vi}, \bar{\mathbf{z}}) := k_{\alpha i}\psi_i(t)\boldsymbol{\varphi}(\theta_i)^{\perp\top}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi} - \bar{\mathbf{z}}_i), \tag{22}$$

where $k_{\alpha i} > 0$, and let the following hold.

Assumption 3: For each $i \leqslant N$, ψ_i in (22) is bounded, differentiable, there exists $\bar{\psi}_i > 0$ such that

$$\max\{\sup_{t\geqslant 0}|\psi_i(t)|, \; \sup_{t\geqslant 0}|\dot{\psi}_i(t)|\} \leqslant \bar{\psi}_i,$$

as well as μ_i and $T_i > 0$ such that

$$\int_t^{t+T} \dot{\psi}_i(s)^2 ds \geqslant \mu, \qquad \forall\, t \geqslant 0. \tag{23}$$

Then, reconsider the controller's dynamics and let it be redefined as

$$\begin{aligned}\ddot{\vartheta}_{\omega i} = &- d_{\omega i}\dot{\vartheta}_{\omega i} - p_{\omega i}\tilde{e}_{\omega i} + k_{\omega i}\tanh(\theta_i - \vartheta_{\omega i}) \\ &+ k_{\alpha i}\psi_i(t)\boldsymbol{\varphi}(\theta_i)^{\perp\top}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi} - \bar{\mathbf{z}}_i).\end{aligned} \tag{24}$$

The last term on the right-hand side of (24), *i.e.,*, α_i, is uniformly δ-Persistently exciting with respect to the function

$h(\theta_i, \bar{\mathbf{z}}_i, \boldsymbol{\vartheta}_{vi}) := \boldsymbol{\varphi}(\theta_i)^{\perp\top}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi} - \bar{\mathbf{z}}_i)$—cf. [30]. That is, roughly speaking, α_i has the property that it is persistently exciting as long as $\boldsymbol{\varphi}(\theta_i)^{\perp\top}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi} - \bar{\mathbf{z}}_i) \neq 0$. Therefore, it prevents the systems' trajectories to converge and remain close to the set of solutions to (19), unless they converge and remain close to the consensus manifold. To better see this, let us assume that, at some instant t, the systems' trajectories came to satisfy $(\bar{\mathbf{z}}_i(t), \theta_i(t), \boldsymbol{\vartheta}_{vi}(t)) = (\bar{\mathbf{z}}_i^*, \theta_c, \boldsymbol{\vartheta}_{vi}^*)$, which are solution to (19a). Therefore, from (21), we would have $\boldsymbol{\varphi}(\theta_i(t))^{\perp\top}\mathbf{tanh}(\bar{\mathbf{z}}_i(t) - \boldsymbol{\vartheta}_{vi}(t)) \neq 0$, which in turn implies, via the systems' equations (13a), (13b), and (24), that $\omega_i(t)$, $\vartheta_{\omega i}(t)$, and $\dot{\vartheta}_{\omega i}(t) \neq 0$. Therefore $\theta_i(t) \neq$ const., which is a contradiction. This stabilization mechanism is at the basis of the so-called δ-persistently-exciting controllers for nonholonomic systems—see [29] and references therein—and leads to our main result, which is stated next.

Proposition 1: Consider the system (1) in closed loop with the controller defined by (14), (15), (12), and (24). Then, the following hold.

(i) Let k_{vi} and $k_{\omega i}$, be positive constants such that, for any given saturation levels $\bar{\tau}_{li}$ and $\bar{\tau}_{ri}$ and each wheel axle length R_i,

$$\frac{4R_i}{r_i}\min\{\bar{\tau}_{li},\ \bar{\tau}_{ri}\} > 2\sqrt{2}R_i k_{vi} + k_{\omega i}. \tag{25}$$

Then, the left and right torques satisfy $|\tau_{li}| < \bar{\tau}_{li}$ and $|\tau_{ri}| < \bar{\tau}_{ri}$, so the actuators do not saturate.

(ii) If in addition, Assumptions 1–2 hold and, for any $i \leqslant N$,

$$d_{vi} > \frac{1}{2}p_{vi}\sum_{i\in\mathcal{N}_i} a_{ij}\left(\beta_i + \frac{\bar{T}_{ji}^2}{\beta_j}\right), \tag{26}$$

$$d_{\omega i} > \frac{1}{2}p_{\omega i}\sum_{i\in\mathcal{N}_i} a_{ij}\left(\varepsilon_i + \frac{\bar{T}_{ji}^2}{\varepsilon_j}\right), \tag{27}$$

for all $j \in \mathcal{N}_i$, $\varepsilon_i > 0$, and $\beta_i > 0$ arbitrarily chosen, the desired control objectives (3) and (4) hold for any initial conditions.

Remark 2: Note that the choice of the controller gains k_{vi} and $k_{\omega i}$ is arbitrary; as any positive values are fit to guarantee the convergence of the trajectories to the consensus manifold. However, they may lead to actuator saturation, which is why condition (25) relies on the knowledge of the actuators maximum torque. On the other hand, if the delays are large and the proportional gains are increased, so must the required damping gains. This is observed in conditions (26) and (27). This is required to compensate for the undesired destabilizing effects induced by the delays in the communications.

Remark 3: We focus on the so-called leaderless consensus-based formation problem, under which the consensus rendezvous point is not specified. However, the solution to the leader-follower consensus-formation problem is straightforward. In the leader-follower scenario, all the robots are required to achieve a desired formation at a desired center $\mathbf{z}_d$ with a desired orientation θ_d, with these reference values being known only by a nonempty set of followers. In such a case, the result follows by simply modifying the error equations $\tilde{\mathbf{e}}_{vi}$ and $\tilde{e}_{\omega i}$ as

$$\tilde{\mathbf{e}}_{vi} := \sum_{j\in\mathcal{N}_i} a_{ij}\Big(\boldsymbol{\vartheta}_{vi} - \boldsymbol{\vartheta}_{vj}(t - T_{ji}(t))\Big) + b_i(\boldsymbol{\vartheta}_{vi} - \mathbf{z}_d),$$

and

$$\tilde{e}_{\omega i} := \sum_{j\in\mathcal{N}} a_{ij}\left(\vartheta_{\omega i} - \vartheta_{\omega j}(t - T_{ji}(t))\right) + b_i(\vartheta_{\omega i} - \theta_d),$$

respectively, where $b_i > 0$ if the ith-vehicle knows the desired center and the desired orientation, and it is $b_i = 0$, otherwise.

IV. Proof of Proposition 1

The closed-loop system is given by the equations (17), (13a), (13b), and (24), which constitute a dynamical system in feedback form. However, loosely speaking, the proof relies on a cascades argument. In order to consider these systems in cascaded form, *i.e.,* for the purpose of analysis, we replace the state variable θ_i in Eqs. (17) with an arbitrary trajectory $t \mapsto \theta_i(t)$, so these equations take the form

$$\dot{\bar{\mathbf{z}}}_i = \boldsymbol{\varphi}(\theta_i(t))v_i \tag{28a}$$

$$\dot{v}_i = -\frac{1}{m_i}\left[f_{vi}v_i + k_{vi}\boldsymbol{\varphi}(\theta_i(t))^\top\mathbf{tanh}(\bar{\mathbf{z}}_i - \boldsymbol{\vartheta}_{vi})\right] \tag{28b}$$

$$\ddot{\boldsymbol{\vartheta}}_{vi} = -d_{vi}\dot{\boldsymbol{\vartheta}}_{vi} - k_{vi}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi} - \bar{\mathbf{z}}_i) - p_{vi}\tilde{\mathbf{e}}_{vi}. \tag{28c}$$

In this form, Σ_{vi} in (17) is a nonlinear *time-varying* system with state $(\bar{\mathbf{z}}_i, v_i, \boldsymbol{\vartheta}_{vi}, \dot{\boldsymbol{\vartheta}}_{vi})$. The solutions to (28), however, are defined only on the interval of definition of $t \mapsto \theta_i(t)$. If (and only if) this interval extends to infinity we may consider the system (28) in cascade with (13a), (13b), and (24), with α_i playing the role of the cascade interconnection. Hence, the behavior of the closed-loop solutions may be inferred by that of the *separate* nonlinear time-varying systems (28) on one hand, and (13a), (13b), and (24) on the other. This, provided that the trajectories are forward complete (*i.e.,* that the interval of existence of solutions extends to infinity) and that they remain bounded under the cascade interconnection. The proof of Proposition 1 starts with the establishment of these two properties.

A. Proof of forward completeness

Consider the linear-motion closed-loop dynamics Σ_{vi} in (17) and the Lyapunov-Krasovskii functional

$$V := \sum_{i=1}^{N}\left[\frac{1}{p_{vi}}\mathcal{E}_{vi} + \Upsilon_{vi} + \frac{1}{4}\sum_{j\in\mathcal{N}_i} a_{ij}|\boldsymbol{\vartheta}_{vi} - \boldsymbol{\vartheta}_{vj}|^2\right] \tag{29}$$

with

$$\mathcal{E}_{vi} := \frac{1}{2}\Big[m_i v_i^2 + |\dot{\boldsymbol{\vartheta}}_{vi}|^2 + 2k_{vi}\big[\ln(\cosh(\vartheta_{xi} - \bar{x}_i)) + \ln(\cosh(\vartheta_{yi} - \bar{y}_i))\big]\Big],$$

$$\Upsilon_{vi} := \frac{1}{2\beta_i}\sum_{j\in\mathcal{N}_i} a_{ij}\bar{T}_{ji}\int_{-\bar{T}_{ji}}^{0}\int_{t+\eta}^{t}|\dot{\boldsymbol{\vartheta}}_{vj}(\sigma)|^2 d\sigma d\eta, \quad \beta_i > 0.$$

The functional $V(t, v, \boldsymbol{e}_{\vartheta_v}, \dot{\boldsymbol{\vartheta}}_v, (\boldsymbol{\vartheta}_v - \bar{\mathbf{z}}))$ is positive definite and radially unbounded, since so is $\ln(\cosh(\cdot)) = 0$, $\Upsilon_{vi}(t) \geqslant 0$ for all $t \geqslant 0$, and all the other terms are quadratic.

Now, evaluating the total derivative along the trajectories of (17) we obtain

$$\dot{V} = -\sum_{i=1}^{N}\left[\frac{f_{vi}}{p_{vi}}v_i^2 + \frac{d_{vi}}{p_{vi}}|\dot{\boldsymbol{\vartheta}}_{vi}|^2 + \sum_{j\in\mathcal{N}_i} a_{ij}\dot{\boldsymbol{\vartheta}}_{vj}^\top \int_{t-\bar{T}_{ji}}^{t} \dot{\boldsymbol{\vartheta}}_{vj}(\sigma)\, d\sigma \right.$$
$$\left. + \frac{1}{2\beta_i}\sum_{j\in\mathcal{N}} a_{ij}\bar{T}_{ji}\left(\int_{t-\bar{T}_{ji}}^{t} |\dot{\boldsymbol{\vartheta}}_{vj}(\sigma)|\, d\sigma - \bar{T}_{ji}|\dot{\boldsymbol{\vartheta}}_{vj}(t)|^2\right)\right],$$

which, after a successive application of Young and Cauchy-Schwartz inequalities leads to—cf. [27]

$$\dot{V} \leqslant -\sum_{i=1}^{N}\left(\frac{f_{vi}}{p_{vi}}v_i^2 + \left[\frac{d_{vi}}{p_{vi}} - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\frac{\beta_i}{2} + \frac{\bar{T}_{ji}^2}{2\beta_j}\right)\right]|\dot{\boldsymbol{\vartheta}}_{vi}|^2\right).$$

Hence, after (26), for each $i \leqslant N$, there exist constants c_{1i} and $c_{2i} > 0$ such that

$$\dot{V} \leqslant -\sum_{i=1}^{N} c_{1i}v_i^2 - c_{2i}|\dot{\boldsymbol{\vartheta}}_{vi}|^2 \leqslant 0. \tag{30}$$

From the above and the positivity of V it also follows that $\dot{V} \leqslant V$. Strictly speaking, the above computations are valid only on the interval of existence of the solutions.

Next, let us consider the angular-motion closed-loop equations (13a), (13b), and (24) and the Lyapunov-Krasovskii functional

$$W := \sum_{i=1}^{N}\left[\frac{1}{p_{\omega i}}\mathcal{E}_{\omega i} + \Upsilon_{\omega i} + \frac{1}{4}\sum_{j\in\mathcal{N}_i} a_{ij}\left(\vartheta_{\omega i} - \vartheta_{\omega j}\right)^2\right]$$

where, for each $i \leqslant N$,

$$\mathcal{E}_{\omega i} := \frac{1}{2}\left[I_i\omega_i^2 + \dot{\vartheta}_{\omega i}^2 + 2k_{\omega i}\ln(\cosh(\vartheta_{\omega i} - \theta_i))\right],$$

$$\Upsilon_{\omega i} := \frac{1}{2\varepsilon_i}\sum_{j\in\mathcal{N}_i} a_{ij}\bar{T}_{ji}\int_{-\bar{T}_{ji}}^{0}\int_{t+\eta}^{t} |\dot{\vartheta}_{\omega j}(\sigma)|^2 d\sigma d\eta,$$

and $\varepsilon_i > 0$. The functional W is positive definite and radially unbounded in ω_i, $(\vartheta_{\omega i} - \theta_i)$, $\dot{\vartheta}_{\omega i}$, and $e_{\omega i}$. Also, its total time derivative along the trajectories of (13a), (13b), and (24) yields, upon applying the Young and Cauchy Schwartz inequalities,

$$\dot{W} \leqslant -\sum_{i=1}^{N}\left(\frac{f_{\omega i}}{p_{\omega i}}\omega_i^2 + \left[\frac{d_{\omega i}}{p_{\omega i}} - \sum_{j\in\mathcal{N}_i} a_{ij}\left(\frac{\varepsilon_i}{2} + \frac{\bar{T}_{ji}^2}{2\varepsilon_j}\right)\right]|\dot{\vartheta}_{\omega i}|^2 \right.$$
$$\left. - \frac{1}{p_{\omega i}}\alpha_i\dot{\vartheta}_{\omega i}\right).$$

So, after (27), for each $i \leqslant N$, there exists $\lambda_{\omega i} > 0$ such that

$$\dot{W} \leqslant -\sum_{i=1}^{N}\left[\frac{f_{\omega i}}{p_{\omega i}}\omega_i^2 + \lambda_{\omega i}|\dot{\vartheta}_{\omega i}|^2 - \frac{1}{p_{\omega i}}\alpha_i\dot{\vartheta}_{\omega i}\right]. \tag{31}$$

Furthermore, note that α_i, which is defined in (22), is bounded; more precisely, $|\alpha_i| \leqslant \sqrt{2}\bar{\psi}_i k_{\alpha i}$. Hence, there exists $a > 0$ such that $\dot{W} \leqslant W + a$.

Thus, we conclude that the total derivative of $\mathcal{V} := V + W$ along the overall closed-loop system (17), (13a), (13b), and (24) satisfies $\dot{\mathcal{V}} \leqslant \mathcal{V} + a$. Integrating on both sides of the latter along the systems' trajectories we conclude that the latter exist over $[t_\circ, \infty)$ for any $t_\circ$.

B. Proof of Boundedness

Because the interval of existence of solutions extends to infinity, the closed-loop system defined by (17), (13a), (13b), and (24) may be regarded as a cascaded system constituted by (28), (13a), (13b), and (24). In this setting, (28) is seen as decoupled from the rest of the dynamics and α_i as the interconnection term. Then, the two systems may be analyzed separately.

Consider, first, the linear-motion closed-loop dynamics (28) and the Lyapunov-Krasovskii functional V defined in (29). The total derivative of V along the trajectories of (28) satisfies (30). Integrating along the trajectories on both sides of the second inequality in (30) we obtain that V is uniformly bounded along the trajectories and so are the latter. More precisely, v_i, $\boldsymbol{e}_{\vartheta_{vi}}$, $\dot{\boldsymbol{\vartheta}}_{vi}$, and $|\boldsymbol{\vartheta}_{vi} - \bar{\mathbf{z}}_i| \in \mathcal{L}_\infty$ for all $i \leqslant N$.

Next, consider the angular-motion closed-loop dynamics, given by the equations (13a), (13b), and (24). The latter equation is that of a Lagrangian system with the Identity as inertia matrix and input $\nu_{\omega i} = k_{\omega i}\tanh(\theta_i - \vartheta_{\omega i}) + k_{\alpha i}\psi_i(t)\boldsymbol{\varphi}(\theta_i)^{\perp\top}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi} - \bar{\mathbf{z}}_i)$, which is bounded. So, as explained in Section III-A, after [31, Proposition 1], (24) is input-output stable and because $\nu_{\omega i}$ is bounded, so are $e_{\omega i}$ and $\dot{\vartheta}_{\omega_i}$. Then, after Equation (13b), which corresponds to an exponentially stable first-order system perturbed by the bounded input $\tanh(\theta_i - \vartheta_{\omega i})$ and with output ω_i, we also conclude that $\omega_i \in \mathcal{L}_\infty$ and, in turn, from the same equation we conclude that $\dot{\omega}_i \in \mathcal{L}_\infty$.

The boundedness of other functions follows after further computations and are established below as needed, in the proof of convergence of the trajectories.

C. Proof of convergence

Again, we start by studying the linear-motion dynamics (28), so we turn back to the Lyapunov-Krasovskii functional V in (29), which satisfies (30). Then, by integrating along the trajectories on both sides of the first inequality, and since V is bounded along the trajectories, we have that v_i and $\dot{\boldsymbol{\vartheta}}_{vi} \in \mathcal{L}_2$. From this, it follows that both converge to zero: To see that $v_i \to 0$ follows from the fact that $v_i \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{v}_i \in \mathcal{L}_\infty$—see [32, Lemma 3.2.5]; that $\dot{v}_i$ is bounded follows from the fact that both terms on the right-hand side of (28b) are bounded. That $\dot{\boldsymbol{\vartheta}}_{vi} \to 0$ follows also from [32, Lemma 3.2.5] and the fact that $\dot{\boldsymbol{\vartheta}}_{vi} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\ddot{\boldsymbol{\vartheta}}_{vi} \in \mathcal{L}_\infty$. The former was established above; the boundedness of $\ddot{\boldsymbol{\vartheta}}_{vi}$ follows by observing that all the terms on the right-hand side of (28c) are bounded. Indeed, by adding all terms on both sides of the equation (18), over $i \leqslant N$, we obtain

$$\tilde{\boldsymbol{e}}_{vi} = \boldsymbol{e}_{vi} + \sum_{i\leqslant N}\int_{t-T_{ji}(t)}^{t} \dot{\boldsymbol{\vartheta}}_{vj}(\sigma)d\sigma. \tag{32}$$

So, since $\boldsymbol{e}_{vi}$ and $\dot{\boldsymbol{\vartheta}}_{vj} \in \mathcal{L}_\infty$ for all $j \leqslant N$, and $|T_{ji}(t)| \leqslant \bar{T}_{ji}$ for all $t \geqslant 0$ and all i, $j \leqslant N$, it follows that $\tilde{\boldsymbol{e}}_{vi} \in \mathcal{L}_\infty$.

Next, we establish the convergence of $\dot{v}_i$, using Barbălat's Lemma. To that end, we verify that $\dot{v}_i$ is uniformly continuous, which holds because all the terms on the right-hand side of

$$\ddot{v}_i = -\frac{1}{m_i}\left(f_{vi}\dot{v}_i + k_{vi}\omega_i(t)\boldsymbol{\varphi}(\theta_i(t))^{\perp\top}\mathbf{tanh}(\bar{\mathbf{z}}_i - \boldsymbol{\vartheta}_{vi})\right.$$

$$+k_{vi}\boldsymbol{\varphi}(t)(\theta_i(t))^\top \begin{bmatrix}\text{sech}^2(\bar{x}_i-\vartheta_{xi})(\dot{\bar{x}}_i-\dot{\vartheta}_{xi})\\ \text{sech}^2(\bar{y}_i-\vartheta_{yi})(\dot{\bar{y}}_i-\dot{\vartheta}_{yi})\end{bmatrix}\Bigg), \tag{33}$$

which results from differentiating on both sides of (28b), are bounded. Therefore, noting that

$$\lim_{t\to\infty}\int_0^t \dot{v}_i(\sigma)d\sigma = \lim_{t\to\infty} v_i(t) - v_i(0) = -v_i(0),$$

we conclude after Barbălat's Lemma that $\dot{v}_i \to 0$. Thus, from the latter, the fact that $v_i \to 0$, and (28b), we obtain that

$$\lim_{t\to\infty} \boldsymbol{\varphi}(\theta_i(t))^\top \tanh(\bar{\mathbf{z}}_i(t)-\boldsymbol{\vartheta}_{vi}(t)) = 0. \tag{34}$$

On the other hand, one can also establish that $\ddot{v}_i \to 0$—see farther below, which in turn, together with $\dot{\boldsymbol{\vartheta}}_{vi} \to 0$ and $\dot{v}_i \to 0$, imply from (33) that

$$\lim_{t\to\infty} \omega_i(t)\boldsymbol{\varphi}(\theta_i(t))^{\perp\top}\mathbf{tanh}(\bar{\mathbf{z}}_i(t)-\boldsymbol{\vartheta}_{vi}(t)) = 0. \tag{35}$$

After (34) and (35) we obtain that the trajectories converge to the manifold where both $\boldsymbol{\varphi}(\theta_i)^\top \tanh(\bar{\mathbf{z}}_i-\boldsymbol{\vartheta}_{vi}) = 0$ and $\omega_i\boldsymbol{\varphi}(\theta_i)^{\perp\top}\mathbf{tanh}(\bar{\mathbf{z}}_i-\boldsymbol{\vartheta}_{vi}) = 0$ hold.

Now, that $\ddot{v}_i \to 0$ follows from Barbălat's Lemma, provided that $\ddot{v}_i$ is uniformly continuous. Indeed, since $\dot{v}_i \to 0$, we have

$$\lim_{t\to\infty}\int_0^t \ddot{v}_i(\sigma)d\sigma = \lim_{t\to\infty}\dot{v}_i(t) - \dot{v}_i(0) = -\dot{v}_i(0).$$

Uniform continuity of $\ddot{v}_i$ follows from the fact that $v_i^{(3)} \in \mathcal{L}_\infty$. To see that the latter holds true, note that $v_i^{(3)}$ is a continuous function of the bounded functions $\ddot{v}_i$, $\dot{\omega}_i$, $\dot{\boldsymbol{\vartheta}}_{vi}$, $\ddot{\boldsymbol{\vartheta}}_{vi}$, $\boldsymbol{\varphi}(\cdot)^{\perp\top}\mathbf{tanh}(\cdot)$, $\boldsymbol{\varphi}(\cdot)^\top\text{sech}^2(\cdot)$, as well as of the partial derivatives of the latter evaluated along bounded trajectories.

Now, under (34), which we established to hold, we consider two cases in which (35) holds too: either because $\lim_{t\to\infty}\boldsymbol{\varphi}(\theta_i(t))^{\perp\top}\mathbf{tanh}(\bar{\mathbf{z}}_i(t)-\boldsymbol{\vartheta}_{vi}(t)) = 0$, irrespectively of the behavior of $\omega_i(t)$, which is bounded, or because $\omega_i \to 0$ irrespectively of the behavior of $\boldsymbol{\varphi}(\theta_i(t))^{\perp\top}\mathbf{tanh}(\bar{\mathbf{z}}_i(t)-\boldsymbol{\vartheta}_{vi}(t))$, which is also bounded.

In the first case, since $\boldsymbol{\varphi}(\theta_i)^\perp$ and $\boldsymbol{\varphi}(\theta_i)$ belong to orthogonal spaces and (34) holds, necessarily, $\lim_{t\to\infty}\mathbf{tanh}(\bar{\mathbf{z}}_i(t)-\boldsymbol{\vartheta}_{vi}(t)) = \mathbf{0}$. Now, considering the term $\mathbf{tanh}(\bar{\mathbf{z}}_i-\boldsymbol{\vartheta}_{vi}) =: \nu_{vi}$ as an input, the equation (28c) has the form of a network of Lagrangian systems as in (8), which we know from [31, Proposition 1] to be input-output stable and asymptotically stable with $\nu_{vi} \equiv 0$. Hence, since $\nu_{vi}(t) \to 0$ for all $i \leqslant N$, it follows that $\boldsymbol{e}_{vi} \to 0$. That is, $\lim_{t\to\infty}[\boldsymbol{\vartheta}_{vi}(t)-\boldsymbol{\vartheta}_{vj}(t)] = 0$, for all $i,\, j \leqslant N$, so we conclude that

$$\lim_{t\to\infty}\bar{\mathbf{z}}_i(t) = \lim_{t\to\infty}\boldsymbol{\vartheta}_{vi}(t) = \mathbf{z}_c, \quad \forall\, i \leqslant N.$$

That is, consensus formation is achieved in the linear motion coordinates. On the other hand, $\mathbf{tanh}(\bar{\mathbf{z}}_i(t)-\boldsymbol{\vartheta}_{vi}(t)) \to 0$ also implies that $\alpha_i \to 0$ along the systems' trajectories. That is, α_i in (24) constitutes an additive vanishing input to the system (13), which is input-output stable—cf. [26, Proposition 1]. Therefore, the same vanishing-input argument as above leads to the conclusion that $\lim_{t\to\infty}\theta_i = \theta_c$, $\lim_{t\to\infty}\vartheta_{\omega i}(t) = \vartheta_c$, and $\lim_{t\to\infty}\omega_i(t) = 0$. The statement of Proposition 1 follows in this case.

Alternatively, if (35) holds because $\omega_i \to 0$ irrespectively of the behavior of $\boldsymbol{\varphi}(\theta_i(t))^{\perp\top}\mathbf{tanh}(\bar{\mathbf{z}}_i(t)-\boldsymbol{\vartheta}_{vi}(t))$, then we observe that the following holds.

Claim 1: If $\dot{\vartheta}_{\omega i} \in \mathcal{L}_\infty$ and ω_i converges to zero, so do the functions $\dot{\omega}_i$, $\ddot{\omega}_i$, $\dot{\vartheta}_{\omega i}$, $\ddot{\vartheta}_{\omega i}$, and $\vartheta_{\omega i}^{(3)}$.

Under the statement of Claim 1, which is proved farther below, and the fact that v_i and $\dot{\boldsymbol{\vartheta}}_{vi} \to 0$, all the terms on the right-hand side of

$$\begin{aligned}k_{\alpha i}\dot{\psi}_i(t)\boldsymbol{\varphi}(\theta_i(t))^{\perp\top}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi}-\bar{\mathbf{z}}_i) &= \vartheta_{\omega i}^{(3)} + d_{\omega i}\ddot{\vartheta}_{\omega i}\\ &+ k_{\omega i}\text{sech}^2(\vartheta_{\omega i}-\theta_i)(\dot{\vartheta}_{\omega i}-\omega_i)\\ &+ p_{\omega i}\dot{\tilde{e}}_{\omega i} + k_{\alpha i}\psi_i(t)\omega_i\boldsymbol{\varphi}(\theta_i(t))^\top\mathbf{tanh}(\boldsymbol{\vartheta}_{vi}-\bar{\mathbf{z}}_i)\\ &- k_{\alpha i}\psi_i(t)\boldsymbol{\varphi}(\theta_i(t))^{\perp\top}\begin{bmatrix}\text{sech}^2(\vartheta_{vix}-\bar{x}_i)(\dot{\vartheta}_{vix}-\dot{\bar{x}}_i)\\ \text{sech}^2(\vartheta_{viy}-\bar{y}_i)(\dot{\vartheta}_{viy}-\dot{\bar{y}}_i)\end{bmatrix},\end{aligned}$$

which results from differentiating on both sides of (24), individually converge to zero. That is,

$$\lim_{t\to\infty}\dot{\psi}_i(t)\boldsymbol{\varphi}(\theta_i)^{\perp\top}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi}(t)-\bar{\mathbf{z}}_i(t)) = 0$$

and, since $\dot{\psi}_i(t)$ is persistently exciting (by assumption), it necessarily holds that

$$\lim_{t\to\infty}\boldsymbol{\varphi}(\theta_i)^{\perp\top}\mathbf{tanh}(\boldsymbol{\vartheta}_{vi}(t)-\bar{\mathbf{z}}_i(t)) = 0.$$

Again, in view of (34) and the orthogonality of $\boldsymbol{\varphi}(\theta_i)$ and $\boldsymbol{\varphi}(\theta_i)^\perp$, we have $\lim_{t\to\infty}\mathbf{tanh}(\bar{\mathbf{z}}_i(t)-\boldsymbol{\vartheta}_{vi}(t)) = 0$. The statement follows as in the previous case.

Proof of Claim 1: First, we differentiate on both sides of (13b); we obtain

$$\ddot{\omega}_i = -\frac{1}{I_i}\Big[f_{\omega i}\dot{\omega}_i + k_{\omega i}\text{sech}^2(\theta_i-\vartheta_{\omega i})(\omega_i-\dot{\vartheta}_{\omega i})\Big]. \tag{36}$$

Since $\omega_i \to 0$, it is bounded. Then, since $\dot{\vartheta}_{\omega i}$ and $\text{sech}^2(\cdot)$ are also bounded $\dot{\omega}_i \in \mathcal{L}_\infty$. The latter follows from the fact that (36) constitutes a first-order stable filter with bounded input and output $\dot{\omega}_i$. We also conclude from (36) that $\ddot{\omega}_i \in \mathcal{L}_\infty$, which implies that $\dot{\omega}_i$ is uniformly continuous. Furthermore, observing that the derivatives of all the terms on the right-hand side of (36) are bounded, we deduce that $\omega_i^{(3)} \in \mathcal{L}_\infty$, so $\ddot{\omega}_i$ is also uniformly continuous. Then, we apply Barbălat's Lemma successively to conclude that $\dot{\omega}_i \to 0$ and, then, $\ddot{\omega}_i \to 0$. Indeed, note that $\omega_i \to 0$ implies that $\lim_{t\to\infty}\int_0^t\dot{\omega}_i(\sigma)d\sigma = -\omega_i(0)$ so, in turn, $\lim_{t\to\infty}\int_0^t\ddot{\omega}_i(\sigma)d\sigma = -\dot{\omega}_i(0)$. It follows that all the terms on the right-hand side of

$$\begin{aligned}\frac{k_{\omega i}}{I_i}\text{sech}^2(\theta_i-\vartheta_{\omega i})\dot{\vartheta}_{\omega i} =\\ -\ddot{\omega}_i - \frac{1}{I_i}\Big[f_{\omega i}\dot{\omega}_i + k_{\omega i}\text{sech}^2(\theta_i-\vartheta_{\omega i})\omega_i\Big],\end{aligned} \tag{37}$$

which is equivalent to (36), converge to zero. Now, since ω_i, $\dot{\omega}_i \to 0$, it follows from (13b) and the nature of $\tanh$, that $|\theta_i-\vartheta_{\omega i}| \to 0$ so $|\theta_i-\vartheta_{\omega i}|$ is bounded. In turn, since $\text{sech}^2(s)$ is bounded and separated from zero for all bounded $|s|$, we conclude from (37) that $\dot{\vartheta}_{\omega i} \to 0$.

The statement that $\ddot{\vartheta}_{\omega i} \to 0$ and $\vartheta_{\omega i}^{(3)} \to 0$ follows along similar lines as for the proof of convergence of $\dot{\omega}_i$ and $\ddot{\omega}_i$ above, *i.e.,* using (24) and applying successively Barbălat's Lemma. ■

V. Numerical simulations

We realized two simulation experiments: one pertaining to the pre-bounded control inputs (12) and (14) and another in which the saturating function $\tanh(s)$ is replaced by its argument s, resulting in a non pre-bounded control input. The simulation involves six differential-drive vehicles whose respective controllers communicate according to the topology depicted in Fig. 3. Each interconnection weight is set to one.

The physical parameters and the actuator bounds are organized in Table I. These parameters include the mass m_i, the moment of inertia I_i, the wheels' axis distance R_i, the wheel radius r_i, and the maximal torque $\bar{\tau}_i$ for each vehicle. The initial conditions and offsets defining a triangular formation are presented in Table II. These initial conditions include the initial positions $(x_i(0), y_i(0))$ and the initial orientations $\theta_i(0)$ of the vehicles, along with the offsets $(\delta_{xi}, \delta_{yi})$, which define the formation.

TABLE I
Physical parameters and actuator bounds

Index	m_i [kg]	I_i [kg m^2]	R_i [m]	r_i [m]	$\bar{\tau}_i$ [N]
$1, 2$	1.52	0.4	0.12	0.0266	0.8
$3, 4$	1.9	0.5	0.15	0.0333	0.9
$5, 6$	0.95	0.25	0.075	0.0166	0.7

TABLE II
Initial conditions

Index	$x_i(0)$	$y_i(0)$	$\theta_i(0)$	δ_{xi}	δ_{yi}
1	4	10	$-3\pi/4$	-5	-2.75
2	10	10	$-\pi/2$	-2.5	1.75
3	19	10	$3\pi/4$	0	6.75
4	7	10	$\pi/2$	2.5	1.75
5	16	10	$-\pi/4$	5	-2.75
6	13	10	$\pi/4$	0	-2.75

The control gains for each robot are set to satisfy the conditions (25), (26), and (27). They are defined as follows: $k_{vi} = 0.6$, $k_{\omega i} = 1$, $p_{vi} = 5.5$, $p_{\omega i} = 6$, $d_{vi} = 52$, $d_{\omega i} = 30$, and $k_{\alpha i} = 30$ for all $i \in N$. The persistently exciting function $t \mapsto \psi_i(t)$ is defined as $\psi_i(t) := 1.25 + \sin(0.05t)$. For a fair comparison, we used identical initial conditions, gains, and persistently exciting functions for both control schemes.

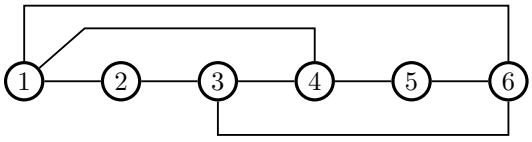

Fig. 3. Undirected-graph topology used in the numerical simulations

In order to emulate the time-varying UDP/IP internet delays, we use a uniform delay signal for all the agents, based on a normal Gaussian distribution with mean, variance, and seed set to 0.45, 0.005, and 45, respectively, as depicted in Fig. 4. For this example, the upper bound of delays is set as $\bar{T}_{ji} = 0.65$s. It is important to mention that, to model the system's behavior under more challenging conditions, the emulated delays are intentionally larger compared to real-world Internet delays.

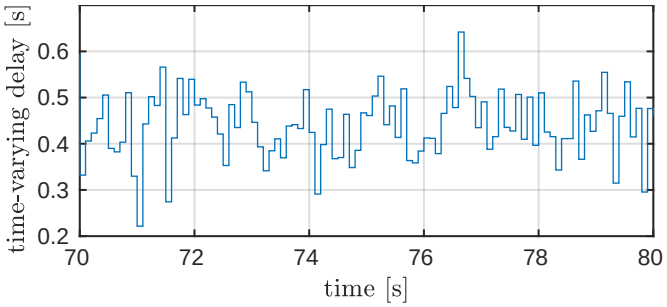

Fig. 4. Emulated UDP/IP Internet delay, as a piece-wise constant function taking random values in the interval XXX. We show a snapshot of a 10s-short window of simulation.

The vehicles' paths on the plane are illustrated in Figures 5 and 6, for the non pre-bounded and the pre-bounded control laws, respectively. In both cases, the formation goal is achieved, and the final orientations are depicted by pointing arrows. In the same order, in Figures 7 and 9 are depicted the systems' trajectories $\bar{z}(t)$ and $\bar{\theta}(t)$; it is appreciated that under the action of the non pre-bounded unbounded control law the systems take a longer time to reach consensus, and with a somewhat undesired transient behavior. Finally, in Figures 8 and 10 we show the control torques in both cases. Note that there is a factor of ten to twenty between the torques applied in one case and the other; moreover, the proposed scheme maintains the required torques within the saturation levels.

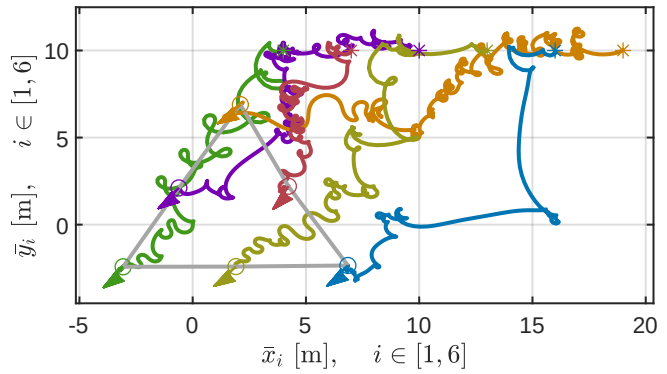

Fig. 5. Vehicles' paths on the plane, sitrred by the non pre-bounded controller (the final orientation of each agent is represented by an arrow.

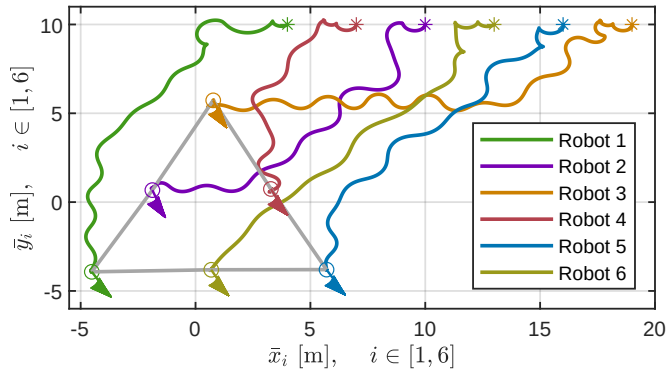

Fig. 6. Vehicles' paths on the plane, sitrred by the pre-bounded controller (the final orientation of each agent is represented by an arrow.

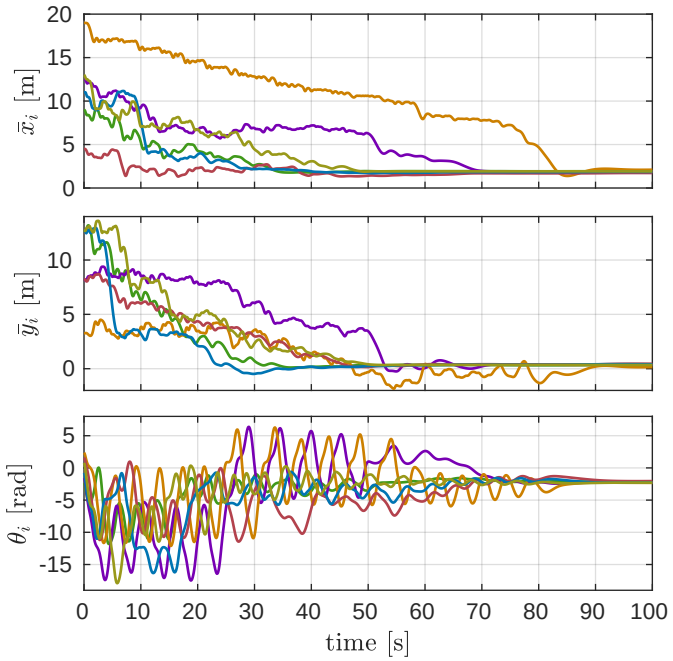

Fig. 7. Consensus under the action of the non pre-bounded control law.

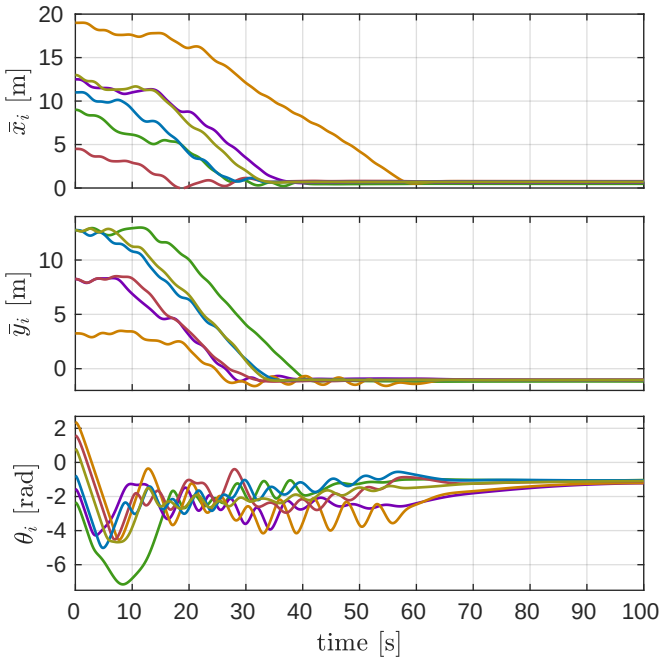

Fig. 9. Consensus under the action of the pre-bounded control law.

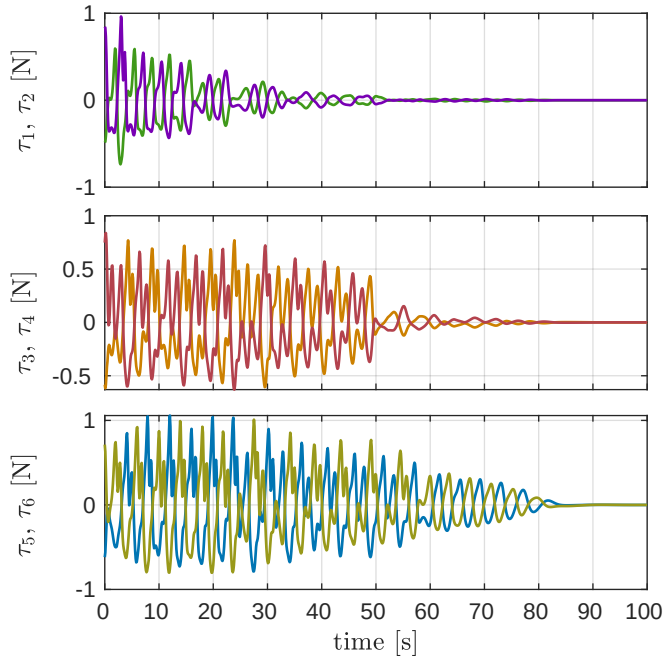

Fig. 8. Torques under the action of the non pre-bounded control law.

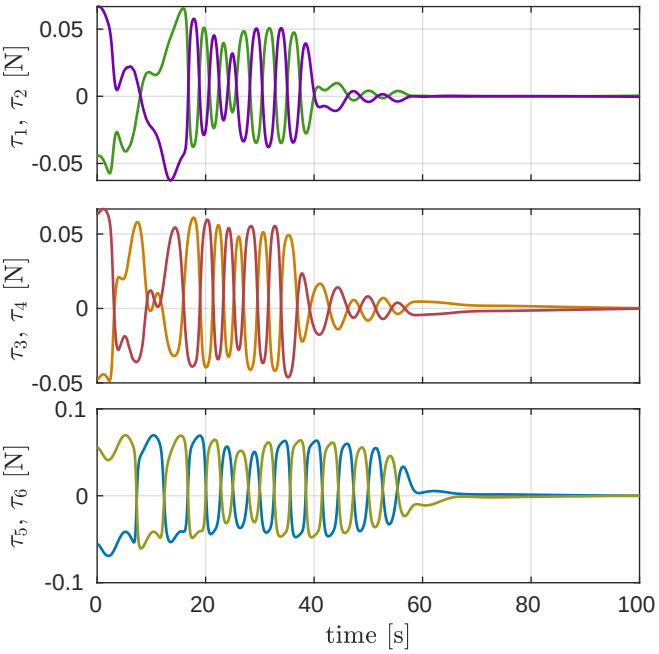

Fig. 10. Torques under the action of the pre-bounded control law.

VI. Conclusions

We solved the consensus-based formation problem for multi-agent nonholonomic vehicles in the scenario that the communications induce time-varying delays and actuator saturation must be avoided. The proposed controller is smooth, time-varying, dynamic, and relies on output feedback. The controller injects damping through its second-order dynamics and this damping back-propagates to the plant. We assume that the interconnection topology of the vehicles is static and undirected.

Current research is devoted to extending these results to consider time-varying topologies and/or directed interconnections. Another possible future research avenue is to eliminate the assumption that the time-delays are differentiable. For this, one possible avenue is to design a strict Lyapunov function, *e.g.,* along the lines of [31], but in the latter reference the controllers rely on full-state feedback and the extension to the higher relative-degree case is far from evident.

REFERENCES

[1] E. Panteley, "A stability-theory perspective to synchronisation of heterogeneous networks," Habilitation à diriger des recherches (DrSc dissertation). Université Paris Sud, Orsay, France, 2015, available online: https://hal.archives-ouvertes.fr/tel-01262772/.

[2] Y. Cao and W. Ren, *Distributed Coordination of Multi-agent Networks: Emergent Problems, Models, and Issues*. Springer-Verlag, 2011.

[3] M. Mesbahi and M. Egerstedt, *Graph theoretic methods in multiagent networks*. Princeton University Press, 2010.

[4] S. Martin, I.-C. Morărescu, and D. Nešić, "Time scale modeling for consensus in sparse directed networks with time-varying topologies," in *2016 IEEE 55th Conference on Decision and Control (CDC)*. IEEE, 2016, pp. 7–12.

[5] S. Martin and A. Girard, "Continuous-time consensus under persistent connectivity and slow divergence of reciprocal interaction weights," *SIAM Journal on Control and Optimization*, 2013.

[6] H. A. Poonawala and M. W. Spong, "Preserving strong connectivity in directed proximity graphs," *IEEE Trans. Automatic Control*, vol. 62, no. 9, pp. 4392–4404, Sep. 2017.

[7] L. Moreau, "Stability of continuous-time distributed consensus algorithms," in *2004 43rd IEEE Conference on Decision and Control (CDC) (IEEE Cat. No.04CH37601)*, vol. 4, Dec. 2004, pp. 3998–4003 Vol.4.

[8] D. A. Lizárraga., "Obstructions to the existence of universal stabilizers for smooth control systems," *Math. Control, Sign. Syst.*, vol. 16, pp. 255–277, 2004.

[9] H. Wang, "Consensus of networked mechanical systems with communication delays: A unified framework," *IEEE Transactions on Automatic Control*, vol. 59, no. 6, pp. 1571–1576, 2014.

[10] G. Wang, "Consensus algorithm for multiagent systems with nonuniform communication delays and its application to nonholonomic robot rendezvous," *IEEE Transactions on Control of Network Systems*, vol. 10, no. 3, pp. 1496–1507, 2023.

[11] K. D. Do, Z.-P. Jiang, and J. Pan, "A global output-feedback controller for simultaneous tracking and stabilization of unicycle-type mobile robots," *IEEE Trans. on Robotics and Automation*, vol. 20, no. 3, pp. 589–594, 2004.

[12] P. Lu, W. Yu, G. Chen, and X. Yu, "Leaderless consensus of ring-networked mobile robots via distributed saturated control," *IEEE Transactions on Industrial Electronics*, vol. 67, no. 12, pp. 10 723–10 731, 2020.

[13] S. G. Tzafestas, *Introduction to mobile robot control*. Elsevier Inc, First ed., 2013.

[14] Z. Lin, B. Francis, and M. Maggiore, "Necessary and sufficient graphical conditions for formation control of unicycles," *IEEE Trans. Autom. Control*, vol. 50, no. 1, pp. 121–127, 2005.

[15] D. Dimarogonas and K. Kyriakopoulos, "On the rendezvous problem for multiple nonholonomic agents," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 916–922, May 2007.

[16] Z. Peng, G. Wen, A. Rahmani, and Y. Yu, "Distributed consensus-based formation control for multiple nonholonomic mobile robots with a specified reference trajectory," *International Journal of Systems Science*, vol. 46, no. 8, pp. 1447–1457, 2015.

[17] X. Li and M. Wang, "Consensus control for wheeled mobile robots under input saturation constraint," *IEEE Access*, vol. 8, pp. 177 125–177 130, 2020.

[18] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, 2015.

[19] S. Zhao, "Affine formation maneuver control of multiagent systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 12, pp. 4140–4155, 2018.

[20] Y. Cheng, R. Jia, H. Du, G. Wen, and W. Zhu, "Robust finite-time consensus formation control for multiple nonholonomic wheeled mobile robots via output feedback," *Int. J. Robust Nonlinear Control*, vol. 28, no. 6, pp. 2082–2096, 2018.

[21] X. Liang, H. Wang, Y. Liu, W. Chen, and T. Liu, "Formation control of nonholonomic mobile robots without position and velocity measurements," *IEEE Trans. on Robotics*, vol. 34, no. 2, pp. 434–446, 2018.

[22] A. Ajorlou, M. Asadi, M. Aghdam, and S. Blouin, "Distributed consensus control of unicycle agents in the presence of external disturbances," *Systems & Control Letters*, vol. 82, pp. 86–90, 2015.

[23] G. Wang, "Consensus algorithm for multi-agent systems with nonuniform communication delays and its application to nonholonomic robot rendezvous," *IEEE Transactions on Control of Network Systems*, 2022.

[24] A. Zavala-Río, E. Aguiñaga-Ruiz, and V. Santibáñez, "Global trajectory tracking through output feedback for robot manipulators with bounded inputs," *Asian Journal of Control*, vol. 13, no. 3, pp. 430–438, 2011.

[25] A. Zavala-Río and V. Santibáñez, "A natural saturating extension of the pd–with–desired–gravity compensation control law for robot manipulators with bounded inputs," *IEEE Transactions on Robotics*, vol. 23, no. 2, pp. 386–391, 2007.

[26] A. I. Paredes-Lopez, E. Nuño, E. Cruz-Zavala, and C. I. Aldana, "Output-feedback consensus of delayed networks of euler-lagrange agents with bounded controllers," *IEEE Control Systems Letters*, vol. 6, pp. 1903–1908, 2022.

[27] E. Nuño, A. Loría, E. Panteley, and E. Restrepo, "Rendezvous of nonholonomic robots via output-feedback control under time-varying delays," *IEEE Transactions on Control Systems Technology*, vol. 30, no. 6, pp. 2707–2716, 2022.

[28] J. Romero, E. Nuño, E. Restrepo, and I. Sarras, "Global consensus-based formation control of nonholonomic mobile robots with time-varying delays and without velocity measurements," *IEEE Transactions on Automatic Control*, vol. 69, no. 1, pp. 355–362, 2024.

[29] E. Nuño, A. Loría, A. I. Paredes, and T. Hernández, "Consensus-based formation control of multiple nonholonomic vehicles under input constraints," *IEEE Control Systems Letters*, vol. 6, no. 3, pp. 2767–2772, 2022.

[30] A. Loría, E. Panteley, D. Popović, and A. Teel, "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable non-autonomous systems," *IEEE Trans. on Automat. Contr.*, vol. 50, no. 2, pp. 183–198, 2005.

[31] E. Nuño, I. Sarras, A. L. anbd M. Maghenem, E. Cruz-Zavala, and E. Panteley, "Strict lyapunov-krasovskiĭ functionals for undirected networks of Euler-Lagrange systems with time-varying delays," *Systems and Control Letters*, vol. 135, p. 104579, 2020.

[32] P. Ioannou and J. Sun, *Robust adaptive control*. New Jersey, USA: Prentice Hall, 1996.