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► **To cite this version:**

Bo Wang, Antonio Loria. Leader-Follower Formation and Tracking Control of Underactuated Surface Vessels. 2024. hal-04685614

HAL Id: hal-04685614

<https://hal.science/hal-04685614v1>

Preprint submitted on 3 Sep 2024

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Leader-Follower Formation and Tracking Control of Underactuated Surface Vessels

Bo Wang and Antonio Loría

Abstract—This paper presents a simple control approach for global trajectory tracking and formation control of underactuated surface vessels equipped with only two propellers. The control approach exploits the inherent cascaded structure of the vehicle dynamics and is divided into control designs at the kinematics level and the kinetics level. A controller with a low-gain feature is designed at the kinematics level by incorporating the cascaded system method, persistency of excitation, and the small-gain theorem. Furthermore, a PD+ controller is designed to achieve the velocity tracking at the kinetics level. The proposed control laws are partially linear and saturated linear and easy to implement. Based on a leader-follower scheme, our control approach applies to the formation tracking control problem of multi-vehicle systems under a directed spanning tree topology. Our main results guarantee uniform global asymptotic stability for the closed-loop system, which implies robustness with respect to perturbations.

Index Terms—Formation control, autonomous vehicles, underactuated systems, persistency of excitation.

I. INTRODUCTION

MOTION control of underactuated surface vessels has received much consideration over the last two decades due to its intrinsic nonlinear properties and practical applications in rescue, search, exploration, and reconnaissance missions. Due to the underactuation, a surface vessel equipped with only two propellers cannot be stabilized by continuous time-invariant feedback [1]. Consequently, trajectory tracking and set-point stabilization are usually studied as two separate problems in the context of underactuated surface vessels.

Trajectory tracking control of underactuated surface vessels has been thoroughly studied since the late 1990s—see, e.g., [2]–[7]. Early results primarily focus on local tracking [2] and semi-global tracking [3]. The first global tracking result utilizing the cascaded system approach for nonholonomic systems is reported in [8]. Later, using a similar cascaded system approach, the global tracking problem for underactuated surface vessels has been solved in [7]. Subsequently, several nonlinear control methods have been proposed in the literature based on the backstepping technique, for instance, in [4], [5], [9]. One characteristic of the controllers obtained through backstepping is that they are highly complex, making implementation and tuning challenging. To deal with modeling uncertainty and

disturbances, several robust control methods based on the sliding mode technique have been presented to solve the robust tracking control problem for underactuated surface vessels, for example, [10] and [11]. Using the cascaded system approach, the global path-following problem for underactuated surface vessels has been solved in [12].

Leader-follower formation tracking control is a natural extension of the classical trajectory tracking control problem to the multi-agent systems. The formation tracking control involves coordinating all the agents to achieve a predefined geometric configuration through local interactions and to follow a designated swarm leader. However, one essential difference between trajectory tracking and formation tracking lies in the fact that formation tracking is inherently *distributed*, necessitating that each agent relies on *relative* position measurements. In other words, agents lack GPS measurements, rendering control laws based on absolute positions unfeasible [13]. The leader-follower formation tracking control problem for underactuated surface vessels has been studied, e.g., in [14]–[17]. In [14], a barrier Lyapunov function-based formation control scheme is proposed to deal with line-of-sight range and bearing constraints for surface vessels. Using the adaptive backstepping method, formation tracking control with asymmetric range and bearing constraints is considered in [15]. Under the assumption that velocity measurements are unavailable, an output feedback formation controller has been proposed in [16] to solve the formation tracking problem, ensuring error convergence in a practical sense. In [17], a robust formation control and obstacle avoidance scheme for underactuated surface vessels is proposed for underactuated surface vessels with modeling uncertainty using the super-twisting control technique.

In this paper, we address the problem of leader-follower formation tracking control of underactuated surface vessels. That is, each surface vessel follows one leader, and only one swarm leader vessel has the information of the reference trajectory. The control design is based on the cascaded system approach, passivity-based control, small-gain theorem, and the construction of ISS Lyapunov functions. The main characteristics of our approach are:

- (i) Our controller benefits from the inherent cascaded structure of the vessel dynamics, resulting in a very natural and simple control structure—the control laws at the kinematics level are *linear* and *saturated linear*, and the control laws at the kinetics level are of the proportional-derivative plus dynamic compensation (PD+) type.
- (ii) In contrast to the classical high-gain approaches (such as

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backstepping and sliding mode control), our controllers exhibit the *low-gain* feature, which stabilizes the system without overly amplifying noise or disturbances and helps prevent input saturation.

- (iii) Our main results guarantee uniform global asymptotic stability (UGAS) of the origin for the closed-loop system. This is a property stronger than the (nonuniform) convergence typically encountered in the literature, and this ensures robustness with respect to perturbations.

Our control design is based on the methodologies described in [18], [19] for nonholonomic vehicles. To the best of the authors' knowledge, however, this marks the first application of a PD+ controller in formation control for *underactuated* surface vessels.

The structure of the remaining paper is as follows: Section II presents problem formulation and the main results. To enhance clarity, we initially present a result on leader-follower tracking control (limited to two surface vessels) and describe the control approach. Then, we present a result applicable to multiple leader-follower surface vessels operating over a directed spanning tree topology. Section III presents simulation results illustrating the practical applications of our theoretical findings. Finally, Section IV offers concluding remarks.

II. PROBLEM FORMULATION AND MAIN RESULTS

A. Cascades-Based Trajectory Tracking Control

Let us consider the Lagrangian model of a surface vessel with only two propellers that provide the surge force and yaw torque. The model is given by, e.g., in [4]–[6],

$$\dot{q} = J(q)v \quad (1a)$$

$$M\dot{v} + C(v)v + Dv = G\tau \quad (1b)$$

where $q = [x \ y \ \theta]^\top$ is the configuration of the surface vessel containing the Cartesian coordinates (x, y) and the orientation θ of the surface vessel in the fixed inertia frame; $v = [v_x \ v_y \ \omega]^\top$ is the generalized velocity vector consisting of the linear velocity (v_x, v_y) and the angular velocity ω in the body-fixed frame; and $\tau = [\tau_x \ \tau_\omega]^\top \in \mathbb{R}^2$ is the control input vector consisting of the surge force and the yaw torque. $J(\cdot)$ is the rotational matrix given by

$$J(q) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad (2)$$

$M = \text{diag}\{m_{11}, m_{22}, m_{33}\} > 0$ is the inertia matrix; $D = \text{diag}\{d_{11}, d_{22}, d_{33}\} > 0$ is the damping matrix. The Coriolis and centrifugal matrix $C(\cdot)$ is given by

$$C(v) = \begin{bmatrix} 0 & 0 & -m_{22}v_y \\ 0 & 0 & m_{11}v_x \\ m_{22}v_y & -m_{11}v_x & 0 \end{bmatrix}, \quad (3)$$

and the input matrix G is given by

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (4)$$

The trajectory tracking control problem consists in following a fictitious surface vessel with the same dynamical model

$$\dot{q}_d = J(q_d)v_d \quad (5a)$$

$$M\dot{v}_d + C(v_d)v_d + Dv_d = G\tau_d \quad (5b)$$

with bounded force reference $\tau_d = [\tau_{xd} \ \tau_{\omega d}]^\top$ and coordinates $q_d = [x_d \ y_d \ \theta_d]^\top$ and $v_d = [v_{xd} \ v_{yd} \ \omega_d]^\top$. The trajectory tracking control objective is to steer the error $q - q_d$ to zero. For the purposes of analysis, we transform the error from the global coordinate frame to the body-fixed frame on the surface vessel, *i.e.*,

$$\begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x - x_d \\ y - y_d \\ \theta - \theta_d \end{bmatrix}. \quad (6)$$

In the new coordinates, the error dynamics between the virtual reference vessel and the follower vessel become

$$\dot{e}_x = \omega e_y + v_x - v_{xd} \cos e_\theta - v_{yd} \sin e_\theta \quad (7a)$$

$$\dot{e}_y = -\omega e_x + v_y + v_{xd} \sin e_\theta - v_{yd} \cos e_\theta \quad (7b)$$

$$\dot{e}_\theta = \omega - \omega_d. \quad (7c)$$

The trajectory control problem is transformed into stabilizing the origin for the error dynamics (7).

As in [19], we address this problem by separating the stabilization tasks at the kinematics and the kinetics levels. That is, we design the virtual control laws $v^* = [v_x^* \ v_y^* \ \omega^*]^\top$ to stabilize the origin for the error kinematics (7). For the Lagrangian dynamics (1b) we design a PD+ controller to ensure that $v \rightarrow v^*$.

1) *Underactuation constraints*: Due to the underactuation, there is no external force that can directly control v_y . In other words, the virtual control v_y^* cannot be arbitrarily assigned. It can be seen from the Lagrangian dynamics (1b) that v_y depends on v_x and ω . Hence, given the virtual control laws v_x^* and ω^* , we define v_y^* by integrating the following differential equation:

$$\dot{v}_y^* = -\frac{m_{11}}{m_{22}}v_x^*\omega^* - \frac{d_{22}}{m_{22}}v_y^*, \quad v_y^*(0) = v_{yd}(0). \quad (8)$$

This acceleration constraint is a feasibility condition for the trajectory tracking control problem for underactuated surface vessels [17].

2) *Velocity tracking*: Let v_x^* and ω^* be given virtual control laws, and integrating (8) yields v_y^* . We want to design a controller such that the velocity error $\tilde{v} := [\tilde{v}_x \ \tilde{v}_y \ \tilde{\omega}]^\top := v - v^* \rightarrow 0$. For *fully-actuated* Lagrangian systems, this velocity tracking task can be easily achieved by the PD+ controller [20], [21]. Specifically, considering the fully-actuated Lagrangian system

$$M\dot{v} + C(v)v + Dv = u, \quad (9)$$

where $u = [u_x \ u_y \ u_\omega]^\top \in \mathbb{R}^3$, together with the controller

$$u = M\dot{v}^* + C(v)v^* + Dv^* - K_d\tilde{v}, \quad (10)$$

we obtain the closed-loop system in the error coordinates

$$M\dot{\tilde{v}} + C(v)\tilde{v} + [D + K_d]\tilde{v} = 0. \quad (11)$$

The total derivative of $V(\tilde{v}) := \frac{1}{2}\tilde{v}^\top M\tilde{v}$ yields

$$\dot{V}_{(11)} = -\tilde{v}^\top [D + K_d]\tilde{v} < 0 \quad (12)$$

for any $K_d = \text{diag}\{k_{dx}, k_{dy}, k_{d\omega}\} \geq 0$. The origin of the closed-loop system (11) is uniformly globally exponentially stable (UGES) if $K_d \geq 0$, and thus, we have $\tilde{v} \in \mathcal{L}_2$.

If we select $K_d = \text{diag}\{k_{dx}, 0, k_{d\omega}\}$, then the u_y -equation in (10) becomes

$$\begin{aligned} u_y &= m_{22}\dot{v}_y^* + m_{11}v_x\omega^* + d_{22}v_y^* \\ &= -m_{11}v_x^*\omega^* - d_{22}v_y^* + m_{11}v_x\omega^* + d_{22}v_y^* \\ &= m_{11}\tilde{v}_x\omega^*. \end{aligned}$$

This observation motivates us to design a controller for the *underactuated* system (1b). For underactuated surface vessels, we have $u_y \equiv 0$. Thus, if we use the control laws τ_x and τ_ω the same as u_x and u_ω in (10), respectively, *i.e.*,

$$\tau = G^\dagger(M\dot{v}^* + C(v)v^* + Dv^* - K_d\tilde{v}), \quad (13)$$

where G^\dagger represents the pseudoinverse of G , then the closed-loop system in the error coordinates is given by

$$M\dot{\tilde{v}} + C(v)\tilde{v} + [D + K_d]\tilde{v} = \begin{bmatrix} 0 \\ -m_{11}\tilde{v}_x\omega^* \\ 0 \end{bmatrix}. \quad (14)$$

The total derivative of $V(\tilde{v}) := \frac{1}{2}\tilde{v}^\top M\tilde{v}$ along trajectories of (14) yields

$$\dot{V}_{(14)} = -\tilde{v}^\top [D + K_d]\tilde{v} - m_{11}\omega^*\tilde{v}_x\tilde{v}_y. \quad (15)$$

Then, if ω^* is bounded, *i.e.*, if $|\omega^*| \leq \omega_M$, then using Young's inequality we obtain

$$\dot{V}_{(14)} \leq -\frac{1}{2}\tilde{v}^\top D\tilde{v} < 0, \quad (16)$$

provided that the control gain k_{dx} is selected as

$$k_{dx} \geq \frac{m_{11}^2\omega_M^2}{2d_{22}}. \quad (17)$$

We summarize the above analysis in the following proposition.

Proposition 1 (Velocity tracking): Consider the Lagrangian dynamics (1b) together with the control law (13), where $t \mapsto v^*(t)$ is defined on $[t_o, \infty)$ for any $t_o \geq 0$ and satisfies (8), and $K_d := \text{diag}\{k_{dx}, 0, k_{d\omega}\}$, with $k_{dx} \geq m_{11}^2\omega_M^2/(2d_{22})$ and $k_{d\omega} > 0$. Then, the origin of the closed-loop error system (14) is globally exponentially stable, uniformly in the initial conditions $\tilde{v}(0)$ and ω^* satisfying $|\omega^*|_\infty \leq \omega_M$.

In the proposition, we make an abuse of notation in writing ' $v^*(t)$ '. Indeed, $v^* = [v_x^* \ v_y^* \ \omega^*]^\top$, where v_x^* and ω^* are virtual control laws to be defined in function of t and e , and v_y^* is defined dynamically via (8). Therefore, in the statement of Proposition 1 $v^*(t)$ must be considered as a continuous function of closed-loop states, evaluated along the system's trajectories. With this under consideration, the remaining task is to design the control laws v_x^* and ω^* to stabilize the origin for the error kinematics (7), under the constraint that v_y^* must satisfy (8). As we show below, v_x^* and ω^* may be defined as functions of (t, e_x) and (t, e_θ) , respectively.

3) Control of the error kinematics: Using $v = v^* + \tilde{v}$ and the error coordinates $e := [e_x \ e_y \ e_\theta]^\top$, the error kinematics (7) becomes

$$\dot{e} = f(t, e, v^*) + g(t, e, \tilde{v})\tilde{v}, \quad (18)$$

where the drift of the nominal system is

$$f(t, e, v^*) := \begin{bmatrix} \omega^*e_y + v_x^* - v_{xd}\cos e_\theta - v_{yd}\sin e_\theta \\ -\omega^*e_x + v_y^* + v_{xd}\sin e_\theta - v_{yd}\cos e_\theta \\ \omega^* - \omega_d \end{bmatrix}, \quad (19)$$

and the input-gain matrix g is given by

$$g(t, e, \tilde{v}) := \begin{bmatrix} 1 & 0 & e_y \\ 0 & 1 & -e_x \\ 0 & 0 & 1 \end{bmatrix}. \quad (20)$$

The control goal for (8) and (18) is to asymptotically stabilize, globally and uniformly, an equilibrium containing $\{e = 0\}$. Replacing the latter in the right-hand side of (19) we see that $f(t, 0, v^*) = 0$ if and only if $v^* = v_d$, where $v_d := [v_{xd} \ v_{yd} \ \omega_d]^\top$. Therefore, it results sensible that for the purpose of analysis, we rewrite (8) in terms of the velocity error $\bar{v}_y := v_y^* - v_{yd}$, which is obtained by subtracting the second equation in (5b) from (8), to obtain

$$\dot{\bar{v}}_y = -\frac{d_{22}}{m_{22}}\bar{v}_y - \frac{m_{11}}{m_{22}}[v_x^*(t, e_x)\omega^*(t, e_\theta) - v_{xd}(t)\omega_d(t)], \quad (21)$$

The right-hand side of (21) equals to zero if $\bar{v}_y = 0$, $v_x^* = v_{xd}$, and $\omega^* = \omega_d$. Therefore, $v_x^*(t, e_x)$ and $\omega^*(t, e_\theta)$ must be designed to satisfy $v_x^*(t, 0) = v_{xd}$ and $\omega^*(t, 0) = \omega_d$.

Now, for the purpose of analysis, based on the statement of Proposition 1, we regard the system (14), (18), and (21) as a cascaded system, *i.e.*,

$$\Sigma_1 : \begin{bmatrix} \dot{e} \\ \dot{\bar{v}}_y \end{bmatrix} = \begin{bmatrix} f(t, e, v^*(t, e, \bar{v}_y)) \\ -\frac{d_{22}}{m_{22}}\bar{v}_y - \frac{m_{11}}{m_{22}}[v_x^*\omega^* - v_{xd}\omega_d] \end{bmatrix} + \begin{bmatrix} g(\cdot) \\ 0 \end{bmatrix} \tilde{v} \quad (22a)$$

$$\Sigma_2 : M\dot{\tilde{v}} = -C(v(t))\tilde{v} - [D + K_d]\tilde{v} + \Phi(t, \tilde{v}) \quad (22b)$$

where $\Phi(t, \tilde{v}) := [0 \ -m_{11}\tilde{v}_x\omega^*(t, e_\theta(t)) \ 0]^\top$.

In Equations (22) above the system Σ_1 is non-autonomous, its state variables are e and \bar{v}_y , and it is perturbed by \tilde{v} , which is the state of Σ_2 . Indeed, note that f in (19) is a function of t through the reference trajectories $v_{xd}(t)$ and $v_{yd}(t)$ and depends on e , also, through the functions $v_x^*(t, e_x)$, $\omega^*(t, e_\theta)$. In addition, f depends on the state \bar{v}_y —see the term $v_y^* = \bar{v}_y + v_{yd}(t)$ in the second element in (19). On the other hand, Equation (22b) is that of another non-autonomous system with state \tilde{v} and depends on time through the trajectories $v(t)$ and $e_\theta(t)$. Strictly speaking, these equations are well-posed provided that the solutions are forward complete [22]. This is shown further below.

Then, [23, Theorem 2] may be invoked to assess uniform global asymptotic stability of the origin for the overall closed-loop system (14), (18), and (21), provided that:

(A1) Under control laws $v_x^*(t, e)$ and $\omega^*(t, e_\theta)$ the origin $\{(e, \bar{v}_y) = (0, 0)\}$ is UGAS for the nominal system

$$\Sigma_{1o} : \begin{bmatrix} \dot{e} \\ \dot{\bar{v}}_y \end{bmatrix} = \begin{bmatrix} f(t, e, v^*(t, e, \bar{v}_y)) \\ -\frac{d_{22}}{m_{22}}\bar{v}_y - \frac{m_{11}}{m_{22}}[v_x^*\omega^* - v_{xd}\omega_d] \end{bmatrix} \quad (23)$$

and we dispose of a polynomial non-strict Lyapunov function [24, Proposition 4.8].

(A2) The function g in (20) satisfies the condition of linear growth in $|e|$:

$$|g(t, e, \tilde{v})| \leq \alpha_1(|\tilde{v}|)|e| + \alpha_2(|\tilde{v}|), \quad (24)$$

where $\alpha_1, \alpha_2 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ are continuous.

(A3) The origin $\{\tilde{v} = 0\}$ is UGES for (14), uniformly in $e(t)$ and $v(t)$.

Condition (A2) holds trivially while, as explained above, Condition (A3) holds after Proposition 1 provided that $t \mapsto \omega^*(t, e_\theta(t))$ is uniformly bounded. For the latter and Condition (A1) to hold we introduce the use of the control laws

$$v_x^*(t, e_x) := -k_x e_x + v_{xd}(t) \quad (25a)$$

$$\omega^*(t, e_\theta) := -k_\theta \tanh(e_\theta) + \omega_d(t), \quad (25b)$$

where k_x and k_θ are positive design parameters to be determined later. Note that since $\omega_d \in \mathcal{L}_\infty$ by assumption, we have $|\omega^*|_\infty \leq k_\theta + \bar{\omega}_d =: \omega_M$, as required to invoke Proposition 1 in the analysis of (14) and (18).

Remark 1: The control laws in (25) are reminiscent of those proposed (without saturation) in [8], where cascades-and-persistency-of-excitation-based tracking control of non-holonomic vehicles was originally proposed for the first time. See also the subsequent work [7] and some of the references therein.

It is left to analyze the stability of the nominal system Σ_{1o} in (23), *i.e.*, to verify Condition (A1) above. To that end, we employ another cascades argument. Indeed, the system Σ_{1o} may be regarded as an inner-loop cascade in which the e_θ dynamics is decoupled from the rest, *i.e.*, substituting (25) into (19), the system in (23) may be written in the form:

$$\Sigma'_1 : \begin{cases} \dot{e}_p = A(t)e_p + \bar{v}_y + B_1(t, e_p, e_\theta) \\ \dot{\tilde{v}}_y = -\frac{d_{22}}{m_{22}}\tilde{v}_y + \frac{m_{11}}{m_{22}}k_x\omega_d(t)e_x + B_2(t, e_p, e_\theta) \end{cases} \quad (26a)$$

$$(26b)$$

$$\Sigma'_2 : \quad \dot{e}_\theta = -k_\theta \tanh(e_\theta) \quad (27)$$

where $e_p := [e_x \ e_y]^\top$, $\bar{v}_y := [0 \ \tilde{v}_y]^\top$,

$$A(t) := \begin{bmatrix} -k_x & \omega_d(t) \\ -\omega_d(t) & 0 \end{bmatrix},$$

$$B_1(\cdot) := \begin{bmatrix} v_{xd}(1 - \cos(e_\theta)) - v_{yd} \sin(e_\theta) - k_\theta e_y \tanh(e_\theta) \\ v_{xd} \sin(e_\theta) + v_{yd}(1 - \cos(e_\theta)) + k_\theta e_x \tanh(e_\theta) \end{bmatrix},$$

and

$$B_2(t, e_p, e_\theta) := [v_{xd} - k_x e_x] \frac{m_{11}}{m_{22}} k_\theta \tanh(e_\theta).$$

Following [23, Theorem 2], we conclude that for the system (26)-(27), the origin is UGAS if:

(B1) The origin of the system Σ'_1 with $\{e_\theta = 0\}$, that is,

$$\Sigma''_1 : \dot{e}_p = A(t)e_p + \bar{v}_y \quad (28a)$$

$$\Sigma''_2 : \dot{\tilde{v}}_y = -\frac{d_{22}}{m_{22}}\tilde{v}_y + \frac{m_{11}}{m_{22}}k_x\omega_d(t)e_x, \quad (28b)$$

is UGAS with a polynomial Lyapunov function.

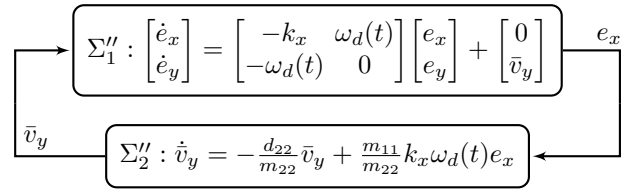


Fig. 1. Small-gain feedback representation of the system (28).

(B2) The functions B_1 and B_2 satisfy the linear-growth condition

$$|B_i(t, e_p, e_\theta)| \leq \alpha_{1i}(|e_\theta|)|e_p| + \alpha_{2i}(|e_\theta|), \quad (29)$$

where $i \in \{1, 2\}$, and α_{1i} and α_{2i} are class \mathcal{K} functions.

(B3) The origin $\{e_\theta = 0\}$ is GAS and LES for (27).

Conditions (B2) and (B3) hold trivially. For Condition (B1), the system (28) may be regarded as the feedback interconnection of two subsystems, as shown in Fig. 1. The stability of the origin of the feedback interconnection may be concluded by invoking the small-gain theorem (refer to the appendix) [25, Theorem 8.2.1].

Proposition 2 (Nominal system): Consider the feedback-interconnected system (28). Suppose that there exists a positive constant $\bar{\omega}_d$ such that $\max\{|\omega_d|_\infty, |\dot{\omega}_d|_\infty\} \leq \bar{\omega}_d$. Suppose that ω_d is persistently exciting, *i.e.*, there exist $\mu, T > 0$ such that

$$\int_t^{t+T} \omega_d(s)^2 ds \geq \mu, \quad \forall t \geq 0. \quad (30)$$

Moreover, we assume that $\bar{\omega}_d$ satisfies

$$\bar{\omega}_d \leq \left(\frac{d_{22}}{m_{11}} \cdot \frac{\mu^2}{4\mu T + T^2} \right)^{\frac{1}{3}}. \quad (31)$$

Then there exists a constant $\bar{k}_x > 0$ such that for all $k_x \in (0, \bar{k}_x]$, the origin of the feedback interconnection (28) is UGAS.

Proof: We will complete the proof by showing the following:

- 1) Σ'_1 has \mathcal{L}_2 -gain $\gamma(\Sigma'_1) \leq \gamma_1$;
- 2) Σ'_2 has \mathcal{L}_2 -gain $\gamma(\Sigma'_2) \leq \gamma_2$;
- 3) The small-gain condition $\gamma_1 \cdot \gamma_2 < 1$ holds.

Item 1) We will prove item 1) by constructing an ISS Lyapunov function for Σ'_1 .

First, for the linear system $\Sigma''_{1o} : \dot{e}_p = A(t)e_p$, we show that

$$W_1(t, e_p) := \frac{1}{2}[\Upsilon_{\omega_d^2}(t) + \alpha](e_x^2 + e_y^2) - \omega_d(t)e_x e_y \quad (32)$$

is a strict Lyapunov function, where α is a constant satisfying

$$\alpha \geq \max \left\{ \bar{\omega}_d, \frac{2\bar{\omega}_d^2}{k_x} \left(1 + \frac{T}{4\mu}(1 + k_x)^2 \right) \right\}. \quad (33)$$

The functional $\Upsilon : (\mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}) \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is defined as

$$\Upsilon_{\omega_d^2}(t) := 1 + 2\bar{\omega}_d^2 T - \frac{2}{T} \int_t^{t+T} \int_t^m \omega_d^2(s) ds dm, \quad (34)$$

and it satisfies, for all $t \geq 0$,

$$1 + \bar{\omega}_d^2 T \leq \Upsilon_{\omega_d^2}(t) \leq 1 + 2\bar{\omega}_d^2 T. \quad (35)$$

The time derivative of $\Upsilon_{\omega_d^2}(t)$ is given by

$$\begin{aligned}\dot{\Upsilon}_{\omega_d^2}(t) &= -\frac{2}{T} \int_t^{t+T} \omega_d^2(s) ds + 2\omega_d^2(t) \\ &\leq -\frac{2\mu}{T} + 2\omega_d^2(t).\end{aligned}\quad (36)$$

It follows from (33) and (35), along with Young's inequality, that

$$\frac{1}{2}|e_p|^2 \leq W_1(t, e_p) \leq \frac{1}{2}[1 + 2\bar{\omega}_d^2 T + \alpha + \bar{\omega}_d]|e_p|^2. \quad (37)$$

The total derivative of W_1 along trajectories of Σ''_1 yields

$$\begin{aligned}\dot{W}_1|_{\Sigma''_1} &= -k_x[\Upsilon_{\omega_d^2}(t) + \alpha]e_x^2 - \left(\frac{1}{T} \int_t^{t+T} \omega_d^2(s) ds\right)|e_p|^2 \\ &\quad + 2\omega_d^2(t)e_x^2 + [k_x\omega_d(t) - \dot{\omega}_d(t)]e_x e_y.\end{aligned}\quad (38)$$

It follows from (33) and Young's inequality that

$$\dot{W}_1|_{\Sigma''_1} \leq -\frac{\mu}{2T}|e_p|^2. \quad (39)$$

Hence, for the linear system $\dot{e}_p = A(t)e_p$, W_1 is a strict Lyapunov function, and the origin of the system is UGES.

Next, we show that W_1 is an ISS Lyapunov function for Σ''_1 . The total derivative of W_1 along trajectories of Σ''_1 yields

$$\begin{aligned}\dot{W}_1 &\leq -\frac{\mu}{2T}|e_p|^2 + \frac{\partial W_1}{\partial e_y} \bar{v}_y \\ &\leq -\frac{\mu}{2T}|e_p|^2 + \left| \frac{\partial W_1}{\partial e_y} \right| |\bar{v}_y| \\ &\leq -\frac{\mu}{2T}|e_p|^2 + \beta|e_p| |\bar{v}_y|\end{aligned}$$

where $\beta := 1 + 2\bar{\omega}_d^2 T + \alpha + \bar{\omega}_d$. It follows from Young's inequality that

$$\dot{W}_1 \leq -\frac{\mu}{4T}|e_p|^2 + \frac{T\beta^2}{\mu} \bar{v}_y^2. \quad (40)$$

Let $V_1 := \frac{2T}{\mu} W_1$ and we have

$$\dot{V}_1 \leq -\frac{1}{2}|e_p|^2 + \frac{\gamma_1^2}{2} \bar{v}_y^2, \quad (41)$$

where

$$\gamma_1 := \frac{2T\beta}{\mu}. \quad (42)$$

It follows that Σ''_1 has \mathcal{L}_2 -gain $\gamma(\Sigma''_1) \leq \gamma_1$.

Item 2) The total derivative of the storage function $V_2 := \frac{m_{22}}{2d_{22}} \bar{v}_y^2$ along trajectories of Σ''_2 yields

$$\begin{aligned}\dot{V}_2 &= -\bar{v}_y^2 + \frac{m_{11}}{d_{22}} k_x \omega_d(t) e_x \bar{v}_y \\ &\leq -\bar{v}_y^2 + \frac{m_{11}}{d_{22}} k_x \bar{\omega}_d |e_x| |\bar{v}_y| \\ &\leq -\frac{1}{2} \bar{v}_y^2 + \frac{m_{11}^2}{2d_{22}^2} k_x^2 \bar{\omega}_d^2 e_x^2 \\ &= -\frac{1}{2} \bar{v}_y^2 + \frac{\gamma_2^2}{2} e_x^2,\end{aligned}\quad (43)$$

where

$$\gamma_2 := \frac{m_{11} k_x \bar{\omega}_d}{d_{22}}. \quad (44)$$

It follows that Σ''_2 has \mathcal{L}_2 -gain $\gamma(\Sigma''_2) \leq \gamma_2$. Note that γ_2 can be made arbitrarily small by selecting the parameter k_x .

Item 3) From (42) and (44), we have

$$\begin{aligned}\gamma_1 \cdot \gamma_2 &= \frac{2T\beta}{\mu} \cdot \frac{m_{11}}{d_{22}} k_x \bar{\omega}_d \\ &< \frac{2Tm_{11}}{\mu d_{22}} k_x \bar{\omega}_d \left(1 + 2\bar{\omega}_d^2 T + 2\bar{\omega}_d\right. \\ &\quad \left. + \frac{2\bar{\omega}_d^2}{k_x} \left(1 + \frac{T}{4\mu}(1 + k_x^2)\right)\right).\end{aligned}\quad (45)$$

We see that the right-hand side of the inequality (45) is a continuous function of k_x , and it monotonically decreases as k_x decreases. Therefore, taking the limit as k_x tends to zero yields

$$\gamma_1 \cdot \gamma_2 < \frac{4Tm_{11}\bar{\omega}_d^3}{\mu d_{22}} \left(1 + \frac{T}{4\mu}\right). \quad (46)$$

Finally, $\bar{\omega}_d$ satisfies (31), which implies that the small-gain condition $\gamma_1 \cdot \gamma_2 < 1$ holds. The zero-state detectability conditions for Σ''_1 and Σ''_2 can be easily verified. It follows from Theorem 3 that the origin of the feedback interconnection (28a)-(28b) is UGAS, which completes the proof. ■

The proof of Proposition 2 not only demonstrates the UGAS of the feedback interconnection (28) but also provides a strict Lyapunov function. Because $\gamma_1 \cdot \gamma_2 < 1$, there exists $\lambda > 0$ such that $\gamma_1 < \lambda < \frac{1}{\gamma_2}$. Then, $\mathcal{V} := V_1 + \lambda^2 V_2$ is a strict Lyapunov function for (28). Due to the fact that \mathcal{V} is quadratic, we verify Condition (B1), and thus, the origin of the system (26)-(27) is UGAS. Consequently, we verify Condition (A1) and conclude that the origin of the closed-loop system (18) with control laws (8) and (25) is UGAS [23, Theorem 2]. Thus, we have the following.

Theorem 1 (Trajectory tracking): Consider the system (1) together with the control laws (8), (13), and (25). Suppose that there exists a positive constant $\bar{\omega}_d$ such that $\max\{|\omega_d|_\infty, |\dot{\omega}_d|_\infty\} \leq \bar{\omega}_d$. Furthermore, suppose that ω_d is persistently exciting in the sense of (30), and that $\bar{\omega}_d$ satisfies (31). Then there exists a constant $\bar{k}_x > 0$ such that for all $k_x \in (0, \bar{k}_x]$ and all $k_\theta \in (0, \infty)$, there exists a constant $\bar{k}_{dx} > 0$, such that for all $k_{dx} \in [\bar{k}_{dx}, \infty)$ and all $k_{d\omega} \in (0, \infty)$, the trajectory tracking control problem is solved, and the origin of the closed-loop system, $(e, \bar{v}_y, \tilde{v}) = (0, 0, 0)$, is UGAS.

Proof: Consider the function

$$U(e, \bar{v}_y, \tilde{v}) = |e|^2 + |\tilde{v}|^2 + |\bar{v}_y|^2 \quad (47)$$

After direct computation of \dot{U} along the trajectories of (14), (18), and (21), we see that there exist a and $b > 0$ such that $\dot{v}(t) \leq av(t) + b$, where $v(t) := U(e(t), \bar{v}_y(t), \tilde{v}(t))$. Therefore, the closed-loop system is forward complete. After Propositions 1 and 2, and the argumentation on (22) above, the statement follows. ■

B. Formation Tracking control

In this subsection, we extend the previous result to the case of formation tracking control. Let us consider a group of n

surface vessels that are required to advance in formation with the Lagrangian models

$$\dot{q}_i = J(q_i)v_i \quad (48a)$$

$$M\dot{v}_i + C(v_i)v_i + Dv_i = G\tau_i, \quad (48b)$$

where for the i th vessel, $q_i = [x_i \ y_i \ \theta_i]^\top$ is the configuration containing the Cartesian coordinates (x_i, y_i) and the orientation θ_i in the fixed inertia frame; $v_i = [v_{xi} \ v_{yi} \ \omega_i]^\top$ is the generalized velocity vector consisting of the linear velocity (v_{xi}, v_{yi}) and the angular velocity ω_i in the body-fixed frame; and $\tau_i = [\tau_{xi} \ \tau_{\omega_i}]^\top$ is the control input vector consisting of the surge force and the yaw torque, respectively.

The formation tracking control problem consists in making the n surface vessels take specific postures assigned by the topology designer while guiding the swarm along a path determined by a virtual reference surface vessel labeled as agent-0. It is assumed that the i th surface vessel follows a leader, indexed $i - 1$, thus forming a directed spanning tree graph communication topology. In other words, each surface vessel has only one leader, but it may have several followers. Additionally, the swarm has only one swarm leader vessel labeled as agent-1, which is the sole entity with access to the reference trajectory generated by the virtual leader agent-0.

Similar to (6), we define formation error $e_i = [e_{xi} \ e_{yi} \ e_{\theta_i}]^\top$ for the i th surface vessel in its local coordinates as

$$\begin{bmatrix} e_{xi} \\ e_{yi} \\ e_{\theta_i} \end{bmatrix} = \begin{bmatrix} \cos \theta_i & \sin \theta_i & 0 \\ -\sin \theta_i & \cos \theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i - x_{i-1} - d_{xi} \\ y_i - y_{i-1} - d_{yi} \\ \theta_i - \theta_{i-1} \end{bmatrix}, \quad (49)$$

where (d_{xi}, d_{yi}) is the desired relative displacement between the i th vessel and its leader, and defines the geometry of the formation. Thus, the error dynamics between the i th surface vessel and its leader become

$$\dot{e}_{xi} = \omega_i e_{yi} + v_{xi} - v_{x(i-1)} \cos e_{\theta_i} - v_{y(i-1)} \sin e_{\theta_i} \quad (50a)$$

$$\dot{e}_{yi} = -\omega_i e_{xi} + v_{yi} + v_{x(i-1)} \sin e_{\theta_i} - v_{y(i-1)} \cos e_{\theta_i} \quad (50b)$$

$$\dot{e}_{\theta_i} = \omega_i - \omega_{i-1}. \quad (50c)$$

For $i = 1$, we recover the error dynamics for the case of trajectory tracking, *i.e.*, $v_0 := v_d$. Then, for $i \leq n$, similar to (8) and (25), we introduce the virtual controls

$$v_{xi}^* := -k_{xi}e_{xi} + v_{x(i-1)} \quad (51a)$$

$$\omega_i^* := -k_{\theta_i} \tanh(e_{\theta_i}) + \omega_{i-1}, \quad (51b)$$

and v_{yi}^* given by the integration of the differential equation

$$\dot{v}_{yi}^* = -\frac{m_{11}}{m_{22}}v_{xi}^*\omega_i^* - \frac{d_{22}}{m_{22}}v_{yi}^*, \quad v_{yi}^*(0) = v_{y(i-1)}(0). \quad (52)$$

With $v_i^* = [v_{xi}^* \ v_{yi}^* \ \omega_i^*]^\top$, the actual control τ_i is given by

$$\tau_i = G^\dagger(M\dot{v}_i^* + C(v_i)v_i^* + Dv_i^* - K_{di}\tilde{v}_i), \quad (53)$$

where the velocity error $\tilde{v}_i := [\tilde{v}_{xi} \ \tilde{v}_{yi} \ \tilde{\omega}_i]^\top := v_i - v_i^*$ and the gain matrix $K_{di} = \text{diag}\{k_{dxi}, 0, k_{d\omega_i}\} \geq 0$.

Next, for the purpose of analysis, let us define $\Delta v_i := [\Delta v_{xi} \ \Delta v_{yi} \ \Delta \omega_i]^\top := v_i - v_d$ and $\bar{v}_{yi} := v_{yi}^* - v_{y(i-1)}$ for all $i \leq n$. Then, we replace v_i with $v_i^* + \tilde{v}_i$ in (50), substitute

(51) into (50) and (52), and substitute (53) into (48b), which yields the closed-loop error system

$$\dot{\xi}_i = F(t, \xi_i) + H(e_i)\Delta v_{i-1} + G(e_i)\tilde{v}_i \quad (54a)$$

$$M\dot{\tilde{v}}_i = -C(v_i(t))\tilde{v}_i - [D + K_{di}]\tilde{v}_i + \Phi_i(t, \tilde{v}_i), \quad (54b)$$

where—*cf.* (22), $\xi_i = [e_{pi}^\top \ \bar{v}_{yi} \ e_{\theta_i}]^\top$, $e_{pi} = [e_{xi} \ e_{yi}]^\top$,

$$F(t, \xi_i) := \begin{bmatrix} A(t)e_{pi} + \bar{v}_{yi} + B_1(t, e_{pi}, e_{\theta_i}) \\ -\frac{d_{22}}{m_{22}}\bar{v}_{yi} + \frac{m_{11}}{m_{22}}k_{xi}\omega_d(t)e_{xi} + B_2(t, e_{pi}, e_{\theta_i}) \\ -k_{\theta_i} \tanh e_{\theta_i} \end{bmatrix},$$

$$H(e_i) := \begin{bmatrix} 1 - \cos e_{\theta_i} & -\sin e_{\theta_i} & e_{yi} \\ \sin e_{\theta_i} & 1 - \cos e_{\theta_i} & -e_{xi} \\ \frac{m_{11}}{m_{22}}k_{\theta_i} \tanh e_{\theta_i} & 0 & \frac{m_{11}}{m_{22}}k_{xi}e_{xi} \\ 0 & 0 & 0 \end{bmatrix},$$

$$G(e_i) := \begin{bmatrix} 1 & 0 & e_{yi} \\ 0 & 1 & -e_{xi} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$\bar{v}_{yi} := [0 \ \bar{v}_{yi}]^\top$, $\Phi_i(t, \tilde{v}) := [0 \ -m_{11}\tilde{v}_{xi}\omega_i^*(t, e_{\theta_i}(t)) \ 0]^\top$, and matrices $A(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$ are defined in (26)-(27). The overall closed-loop multi-agent kinematics system has the cascaded form:

$$\dot{\xi}_n = F(t, \xi_n) + H(e_n)\Delta v_{n-1} + G(e_n)\tilde{v}_n \quad (55a)$$

⋮

$$\dot{\xi}_2 = F(t, \xi_2) + H(e_2)\Delta v_1 + G(e_2)\tilde{v}_2 \quad (55b)$$

$$\dot{\xi}_1 = F(t, \xi_1) + G(e_1)\tilde{v}_1. \quad (55c)$$

We propose our second main result as follows.

Theorem 2 (Formation tracking): Consider the Lagrangian systems (48) together with the control laws (51)-(53) for $i \leq n$. Define $v_0 := v_d$ and suppose that there exists a positive constant $\bar{\omega}_d$ such that $\max\{|\omega_d|_\infty, |\dot{\omega}_d|_\infty\} \leq \bar{\omega}_d$. Furthermore, suppose that ω_d is persistently exciting in the sense of (30), and that $\bar{\omega}_d$ satisfies (31). Then there exists a constant $\bar{k}_x > 0$ such that for all $i \leq n$, all $k_{xi} \in (0, \bar{k}_x]$, and all $k_{\theta_i} \in (0, \infty)$, there exists a constant $\bar{k}_{dx} > 0$ such that for all $k_{dxi} \geq \bar{k}_{dx}$ and all $k_{d\omega_i} \geq 0$, the formation tracking control problem is solved, and the origin of the closed-loop multi-agent system, $(\xi, \tilde{v}) = (0, 0)$, where $\xi := [\xi_1^\top \ \dots \ \xi_n^\top]^\top$ and $\tilde{v} = [\tilde{v}_1^\top \ \dots \ \tilde{v}_n^\top]^\top$, is UGAS.

Proof: We will complete the proof by induction. For $i = 1$, we recover the trajectory tracking control problem, and the closed-loop system, composed of (55c) and (54b) with $i = 1$, has a nested cascaded form. It follows from Proposition 1 that the origin of (54b) with $i = 1$ is UGES, and thus, $\tilde{v}_1 \rightarrow 0$ exponentially. Then, on the set $\{\tilde{v}_1 = 0\}$, the nominal system $\dot{e}_1 = F(t, e_1)$ also has a cascaded form, as shown in (26)-(27). Following Proposition 2 and recursively using the cascades argument, we conclude that the origin $(\xi_1, \tilde{v}_1) = (0, 0)$ is UGAS. Furthermore, it follows from the proof of Proposition 2 that $\xi_1 \rightarrow 0$ exponentially near the origin due to the quadratic Lyapunov function \mathcal{V} . By virtue of the control design, $\tilde{v}_1 \rightarrow 0$ and $\xi_1 \rightarrow 0$ exponentially near the origin implies that $\Delta v_1 \rightarrow 0$ exponentially near the origin. Denote

the closed-loop system (55c) and (54b) with $i = 1$ as $\dot{\zeta}_1 = F_{cl,1}(t, \zeta_1)$, where $\zeta_1 := [\xi_1^\top \tilde{v}_1^\top]^\top$.

Next, let $i = 2$ and consider the closed-loop system

$$\dot{\xi}_2 = F(t, \xi_2) + H(e_2)\Delta v_1 + G(e_2)\tilde{v}_2 \quad (56a)$$

$$\dot{\zeta}_1 = F_{cl,1}(t, \zeta_1) \quad (56b)$$

$$M\dot{\tilde{v}}_2 = -C(v_2)\tilde{v}_2 - [D + K_{d2}]\tilde{v}_2 + \Phi_2(t, \tilde{v}_2). \quad (56c)$$

Note that the system (56) also has a nested cascaded form. The origin of (56c) is UGES following Proposition 1, and thus, $\tilde{v}_2 \rightarrow 0$ exponentially. The origin of (56b) is UGAS, and $\zeta_1 \rightarrow 0$ exponentially near the origin. Therefore, to analyze the stability of the origin for (56), we only need to establish UGAS for the system $\dot{\xi}_2 = F(t, \xi_2)$ and invoking again [23, Theorem 2], because the matrices $H(e_2)$ and $G(e_2)$ satisfy the condition of linear growth in $|e_2|$. The system $\dot{\xi}_2 = F(t, \xi_2)$ once again exhibits a cascaded form, as illustrated in equations (26)-(27). Therefore, following Proposition 2 again, we conclude that $(\xi_2, \zeta_1, \tilde{v}_2) = (0, 0, 0)$ is a UGAS equilibrium of (56). Denote the closed-loop system (56) as $\dot{\zeta}_2 = F_{cl,2}(t, \zeta_2)$, where $\zeta_2 := [\xi_2^\top \zeta_1^\top \tilde{v}_2^\top]^\top$.

For $i = 3$, the closed-loop system is

$$\dot{\xi}_3 = F(t, \xi_3) + H(e_3)\Delta v_2 + G(e_3)\tilde{v}_3 \quad (57a)$$

$$\dot{\zeta}_2 = F_{cl,2}(t, \zeta_2) \quad (57b)$$

$$M\dot{\tilde{v}}_3 = -C(v_3)\tilde{v}_3 - [D + K_{d3}]\tilde{v}_3 + \Phi_3(t, \tilde{v}_3). \quad (57c)$$

The previous arguments, as in the case of $i = 2$, now apply to equations (57). Therefore, the result follows by induction. ■

III. SIMULATION RESULTS

We consider a group of four surface vessels in the linear communication topology following a virtual leader. That is, each surface vessel has only one follower except for agent-4. Each surface vessel is considered to be modeled by (1a)-(1b) with nominal parameters

$$m_{11} = 1.412, \quad m_{22} = 1.982, \quad m_{33} = 0.354,$$

$$d_{11} = 3.436, \quad d_{22} = 12.99, \quad d_{33} = 0.864,$$

where the parameters are from a laboratory-size surface vessel used in [26]. All parameters are given in SI units. The desired geometric formation shape of the four surface vessels is a diamond configuration, *i.e.*, $(d_{x1}, d_{y1}) = (0, 0)$, $(d_{x2}, d_{y2}) = (-2, -2)$, $(d_{x3}, d_{y3}) = (4, 0)$, and $(d_{x4}, d_{y4}) = (-2, -2)$. The reference trajectory is generated by the virtual leader with input $(\tau_{x0}, \tau_{\omega0}) = (2, 0.3 \cdot \sin(0.3t))$ and with zero initial conditions. The initial conditions of the follower agents are randomly generated as $q_1(0) = [1.46, 0.45, 1.33]^\top$, $q_2(0) = [-3.45, 2.25, 1.02]^\top$, $q_3(0) = [-5.63, -4.94, -0.18]^\top$, and $q_4(0) = [-1.17, -5.23, -0.78]^\top$. The control gains are set to $k_{\theta i} = k_{\dot{\theta} i} = 0.2$, and $K_{di} = \text{diag}\{10, 0, 10\}$. The simulation results are shown in Figs. 2 and 3 (dash-dotted lines), which illustrate the physical paths described by the swarm and the formation tracking error trajectories for each leader-follower pair. As shown in Fig. 3 (dash-dotted lines), the formation tracking errors converge to zero asymptotically.

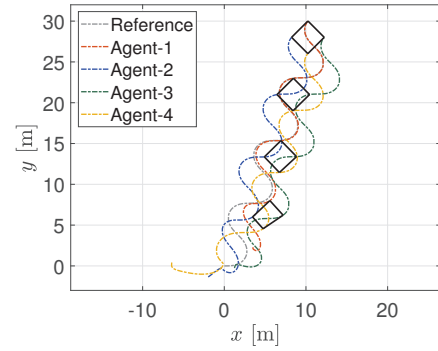


Fig. 2. Illustration of the paths in formation tracking.

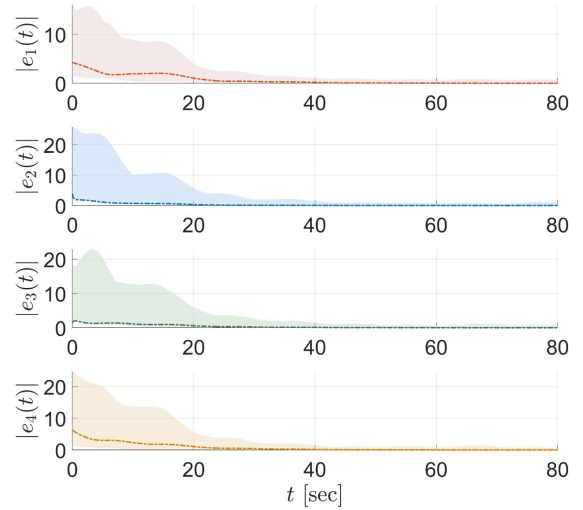


Fig. 3. Convergence of the relative errors (in norm) for each leader-follower pair. The shadowed colored regions represent the envelopes containing the trajectories of 100 simulation tests with randomly chosen initial conditions and under the influence of perturbations.

To validate the robustness of our approach statistically, Fig. 3 provides the evolution of the formation tracking errors with *parameter uncertainties* and *input disturbances* over 100 runs with random initial states, *i.e.*, $(x_i, y_i, v_i) \sim \mathcal{N}(0, 5^2)$ and $\theta_i \sim \mathcal{U}(-\pi, \pi)$. Each model parameter of the four follower vessels is randomly generated in each run by multiplying a random variable factor $\eta \sim \mathcal{U}(0.5, 1.5)$ to the nominal parameter. In other words, up to $\pm 50\%$ parameter uncertainties are considered in the simulations. Moreover, white noise disturbances $W(t) := [W_x(t) \ W_y(t) \ W_\omega(t)]^\top$ are added to the surface vessel dynamics with noise power 0.1 and sample time 0.01. The simulation results of the 100 runs are shown in Fig. 3. The results indicate that all formation tracking error trajectories converge to and remain within a small neighborhood around zero.

IV. CONCLUSION

We present a simple formation tracking controller for underactuated surface vessels equipped with only two propellers. The control laws at the kinematics level are *linear* and *saturated linear* and exhibit a low-gain feature. At the kinetics

level, our approach extends the *classical PD+ controller* to underactuated vehicle systems, ensuring uniform global asymptotic stability for the closed-loop system. Future research will focus on extending the presented approach to underactuated surface vessels with more complex models and under relaxed assumptions, such as switching topologies, communication delays, and heterogeneous networks.

APPENDIX: \mathcal{L}_2 -GAIN AND THE SMALL-GAIN THEOREM

We say that the \mathcal{L}_2 -gain of the dynamical system

$$\Sigma : \begin{cases} \dot{x} = f(x, u), & x \in \mathbb{R}^n, u \in \mathbb{R}^m \\ y = h(x, u), & y \in \mathbb{R}^p \end{cases}$$

is less than or equal to γ if it is dissipative with respect to the supply rate $s(u, y) := \frac{1}{2}\gamma^2|u|^2 - \frac{1}{2}|y|^2$. That is, there exists a continuously differentiable storage function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\dot{V} := \frac{\partial V}{\partial x}(x)f(x, u) \leq \frac{\gamma^2}{2}|u|^2 - \frac{1}{2}|h(x, u)|^2, \quad \forall x, u.$$

The \mathcal{L}_2 -gain of Σ is defined as

$$\gamma(\Sigma) := \inf\{\gamma : \Sigma \text{ has } \mathcal{L}_2\text{-gain} \leq \gamma\}.$$

Consider the feedback interconnection of two systems

$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i, u_i) \end{cases}, \quad i = 1, 2,$$

where $u_1 = -y_2$ and $u_2 = y_1$. Denote the storage function of Σ_1 and Σ_2 by V_1 and V_2 , respectively.

Theorem 3 (Small-gain theorem [25]): Suppose that the systems Σ_1 and Σ_2 have \mathcal{L}_2 -gains such that $\gamma(\Sigma_1) \leq \gamma_1$ and $\gamma(\Sigma_2) \leq \gamma_2$, with $\gamma_1 \cdot \gamma_2 < 1$. Suppose that V_1 and V_2 satisfying that

$$\dot{V}_i := \frac{\partial V_i}{\partial x_i}(x_i)f_i(x_i, u_i) \leq \frac{\gamma_i^2}{2}|u_i|^2 - \frac{1}{2}|y_i|^2, \quad i = 1, 2,$$

are proper and have global minima at $(x_1, x_2) = (0, 0)$, and suppose that Σ_1 and Σ_2 are zero-state detectable. Then the origin of the feedback interconnection $(x_1, x_2) = (0, 0)$ is globally asymptotically stable.

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