Numerical methods for frictional contact problems and applications

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Résumé - Nous présentons ici une synthèse des différentes méthodes numériques développées au laboratoire ces dernières années pour résoudre les problèmes de contact unilatéral avec frottement. Les formulations variationnelles classiques associées au cas statique sont étendues au cas de chargements quasi-statiques complexes par l'introduction de termes en vitesse dans ces formulations. La généralisation aux problèmes de viscoélasticité et viscoplasticité est donnée. Les méthodes utilisées pour le contact sont de différents types : relaxation avec projection, programmation linéaire, point fixe sur conditions aux limites par pénalisation. Des techniques d'amélioration des performances des algorithmes sont présentées. L'efficacité des algorithmes est discutée sur des exemples test, et des applications à des exemples de type industriel sont présentées.

Abstract - *We present a synthesis of the different numerical methods developped these last few years in the laboratory to solve frictional unilateral contact problems. The classical variational formulations associated to the static case are extended to complex quasi-static loading cases by the use of velocity terms in the formulations. A generalization for viscoelasticity and viscoplasticity is given. Contact is dealt with by the use of different numerical methods : relaxation with projection, mathematical programming, fixed point on boundary conditions through penalization. Techniques for improving the algorithms are presented. Their efficiency is discussed on test examples and applications concerning industrial problems are presented.*

1. INTRODUCTION

Contact problems are caracterized by non linear relations between displacements, or rates of displacement and forces on a part of the boundary. The real contact area and the contact forces are unknown a priori and have to be determined by solving the whole problem. They are caracterised by a set of inequalities which lead to a complementarity problem [1]. In elasticity, the associated variational form is set as an implicit variational inequation. An extension is given for viscoelasticity and viscoplasticity by introducing a coupling with a linear or non linear differential equation.

The model is generalized to several body contact problems. Duvaut [2] gave a formulation of the Signorini problem with friction for the static case. We first give numerical methods based on punctual relaxation to solve this problem. Different

accelerating processes are used so as to obtain a powerfull algorithm even for large dimensional problem. A generalization for quasistatic evolutive problems is given. A velocity formulation leads to an incremental form solved by a step by step algorithm using the previous numerical techniques. For frictionless problems, mathematical programming methods are also used [1]. The efficiency is appreciated on a few test examples provided by a group of the GRECO 'Grandes Déformations et Endommagement' and applications to industrial examples are presented.

2. THE SIGNORINI PROBLEM WITH FRICTION

We first set a unilateral contact problem between an elastic solid and a rigid obstacle for the static case under small transformation hypothesis. It is a Signorini's problem with friction [2]. We denote by Ω the open set limiting the solid. Its boundary $\partial\Omega$ is separated as follows $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. On Γ_1, we impose homogeneous boundary conditions. On Γ_2, a force density Φ_2 is applied. The part Γ_3 is concerned by the contact with the rigid obstacle. We denote u the displacement field. On Γ_3, we partition the vector u (in fact in the sense of the trace of u) and the density contact force vector F into normal and tangential components (n is the exterior normal unit vector to Γ_3) :

$$u = u_N n + u_T , \qquad F = F_N n + F_T \tag{1}$$

In Ω, a force density Φ_1 , and a thermal strain are prescribed $e^0 = -\chi(\theta-\theta_0)$; χ is the dilatation tensor, θ the actual temperature field and θ_0 the initial one.

Unilateral conditions

The unilateral conditions, written on the part Γ_3 of the boundary, lead to the following complementarity conditions

$$u_N \leq 0, \qquad F_N \leq 0, \qquad u_N . F_N = 0 \tag{2}$$

The case $(u_N = 0, F_N \leq 0)$ characterizes the contact zone and the $(u_N < 0, F_N = 0)$ corresponds to the separate part.

Note 1 : for this formulation, Γ_3 is defined as the initial real contact area and only a loss of contact on Γ_3 will be allowed. The extension of the contact area cannot be modelized by these relations. More generally, we could write :

$$u_N - \alpha \leq 0, \qquad F_N \leq 0, \qquad (u_N - \alpha) F_N = 0 \tag{3}$$

where α is a given function defined on Γ_3 as the initial distance of each point of Γ_3 to the obstacle. For large displacements a satisfactory definition of α is a serious problem, but under a small transformation hypothesis there is no ambiguity and the distance may be defined according either to the normal to the solid or the normal to the obstacle. We have to note that under a small transformation hypothesis, relations (3) are incorrect because u_N is an infinitesimal function and α a finite one. Nevertheless, this will be used in the numerical model.

Friction conditions

We use here the classical Coulomb friction law. In spite of its elementary form, this friction law is capable of describing correctly numerous problems involving dry friction (especially for metal/metal contact) as long as the numerical treatement is carefully effected (see experimental comparison [29]). It is clear that lubrificated contact or problems involving very high temperatures will need more sophisticated laws. The Coulomb friction law is formulated :

$$|F_T| \leq \mu\,|F_N| \tag{4}$$

$$|F_T| < \mu\,|F_N| \Rightarrow \dot{u}_T = 0 \tag{5}$$

$$|F_T| = \mu\,|F_N| \Rightarrow \dot{u}_T = -\lambda\,F_T \qquad \lambda \geq 0 \tag{6}$$

The dot denotes the time relative derivation, and μ is the friction coefficient.

Note 2 : the no-contact situation ($F_N = F_T = 0$) is well described because the colinearity condition (6) between $\dot{u}_T$ and F_T (which is zero in this case) is then satisfied for any $\dot{u}_T$.

Note 3 : in the case of the unilateral conditions under consideration, the inequalities (3) enable us to replace $|F_N|$ by $-F_N$ in the relations (4)(5)(6).

Following Duvaut [2], this friction law can be written for static cases :

$$|F_T| \leq \mu\,|F_N| \tag{7}$$

$$|F_T| < \mu\,|F_N| \Rightarrow u_T = 0 \tag{8}$$

$$|F_T| = \mu\,|F_n| \Rightarrow u_T = -\lambda'\,F_T \qquad \lambda' \geq 0 \tag{9}$$

From a mechanical point of view, the validity of this static model is very limited because friction is clearly a path dependant phenomena which has to be set on displacement velocities. Nevertheless, the numerical methods for solving this problem will be used in the resolution of the incremental problem derived from the general relations (4)(5)(6). So we first focus on the following static formulation :

Problem P1 :
Be given the loads $(\varphi_1, \varphi_2) \in (L^2(\Omega))^3 \times (L^2(\Gamma_2))^3$, the prescribed strain $e^0 \in (L^2(\Omega))^6$. Find the displacement $u \in (H^1_0(\Omega))^3 = \left\{ v \in (H^1(\Omega))^3 ;\ v = 0 \text{ on } \Gamma_2 \right\}$, the strain $e \in (L^2(\Omega))^6$, the stress $\sigma \in (L^2(\Omega))^6$, the contact force density $F \in (H^{-\frac{1}{2}}(\Gamma_2))^3$ such that :

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) + e^0_{ij} \tag{10}$$

$$\sigma_{ij,j} = -\varphi_{1i} \text{ in } \Omega \tag{11}$$

$$\sigma_{ij} n_j = \varphi_{2i} \text{ on } \Gamma_2 \tag{12}$$

$$\sigma_{ij} n_j = F_i \text{ on } \Gamma_3 \tag{13}$$

$$\sigma_{ij} = K_{ijkl} e_{kl} \text{ in } \Omega \tag{14}$$

and verifying the unilateral conditions (2) and the friction relations (7)(8)(9). K_{ijkl} are the elasticity coefficients.

In other papers (Raous [4][5], Bouc and al. [6]), details can be found on the different choices of the functional scheme, the functional sets and the duality products which allow to set the problem under the following variational form :

Problème P2 : Let (φ_1, φ_2) and e^0 be given as in problem P1
Find $u \in \mathbb{K} = \left\{ v \in (H^1_0(\Omega))^3 ;\ v_N \leq 0 \text{ on } \Gamma_3 \right\}$ such that :

$$\forall\, v \in \mathbb{K} \qquad a(u, v-u) - (f, v-u) + j(v, u) - j(u, u) \geq 0 \tag{15}$$

with :
$$a(u,v) = \int_\Omega [Dv]^t K [Du]\,dx \tag{16}$$

$$(f,v) = \int_\Omega \varphi_1^t v\,dx + \int_{\Gamma_2} \varphi_2^t v\,d\gamma + \int_\Omega [Ke^0]^t [Dv]\,dx \tag{17}$$

$$j(u,v) = \int_{\Gamma_3} \mu\,|F_N(u)|\,|v_T|\,d\gamma \tag{18}$$

D is the symmetrical gradient mapping.

This problem is an implicit variational inequation because of the term $j(u,v)$ in which F_N depends on the solution . Though the convex cone $\mathbb{K}$ does not depend here on the solution, the nature of the problem is the same as that of a quasi-variational inequation. In fact, the dual formulation of problem P2 is a quasi-variational inequation : the unknown force density F_T is supposed to remain in the polar cone $\mathbb{K}^*$ depending on $\mu\,|F_N|$.
Through a fixed point method on the sliding limit $\mu\,|F_N(u)|$, we get a more classical variational inequation (see Panagiotopoulos [7]). We thus introduce a Tresca's friction problem, the sliding limit function g defined on Γ_3 being the new variable.

Problem P3 : Find $\bar{g}$ fixed point of the application $g \mapsto \mu|F_N(u)|$
where u is solution of the problem P4 depending on g

Problem P4 : Let $\{\varphi_1,\varphi_2\}$, e^0 be given as in problem P1. Let $g \in L^2(\Gamma_2)$,
find $u \in \mathbb{K}$ such that :

$$\forall\, v \in \mathbb{K} \qquad a(u,v-u) - (f,v-u) + j(v) - j(u) \geq 0 \tag{19}$$

$$\text{where } j(v) = \int_{\Gamma_2} g|v_T|\, d\gamma \tag{20}$$

We now have to solve the variational inequation problem P4 including the undifferentiable term (20). Thanks to the symmetry of the bilinear form a(.,.), we can write the problem P4 under the following equivalent minimization problem with constraints:

Problem P5 : Let $\{\varphi_1, \varphi_2\}$, e^0 and g be given as defined in problem P4,
Find $u \in \mathbb{K}$ such that :

$$\forall\, v \in \mathbb{K} \qquad \mathfrak{F}(u) \leq \mathfrak{F}(v) \tag{21}$$

$$\text{with : } \mathfrak{F}(v) = \frac{1}{2}\, a(v,v) - (f,v) + j(v) \tag{22}$$

Note 4 : The Coulomb friction problem is solved through the intermediate resolution of a sequence of Tresca's problems. We would like to underline that a Tresca's model, alone, would not be adequate for a problem involving unilateral conditions : g needs to be equal to zero on the non contact parts (u_N strictly negative) and these parts are unknown a priori. Tresca's friction is adequate only in very simple cases where the contact area and the normal contact forces are known a priori.

Mathematical remarks

The existence and uniqueness of the solution of the Signorini problem have been proved by Fichera [8], Stampacchia [9] thanks to an extension of the Lax-Milgram theorem. The static Tresca problem or the Coulomb problem with a given F_N (see Duvaut [2]) also have a unique solution : they are not unilateral problems, the contact area is given. The existence and uniqueness for the unilateral problem including static Coulomb's friction (i.e. problem P2) are not proved. The difficulties arise from the fact that F_N which is unknown in this problem, belongs to $H^{-1/2}(\Gamma)$ (this leads to compacity problems...). In the other problems, this given force is choosen only in $L^2(\Gamma)$.Nevertheless on the regularized problem introduced by Duvaut [2], Oden-Pires [11], Cocu [12], existence is proved and unicity is obtained for small friction coefficients. This concerns a non local friction law (Duvaut defines F_N through a convolution with a smooth function with compact support).

3. MULTIBODY CONTACT PROBLEM (SEE [13])

Let A and B be two deformable solids occupying the open sets Ω^A and Ω^B. Boundary conditions are set on Γ_1^A and Γ_1^B, volumic force densities φ_1^A and φ_1^B are applied in Ω^A and Ω^B, surfacic force densities φ_2^A and φ_2^B are given on Γ_2^A and Γ_2^B. The contact areas are Γ_3^A and Γ_3^B. P and Q denote an opposite pair of points of Γ_3^A and Γ_3^B as described on figure 1. The unit vector n^A and n^B are still the external vectors to Γ_3^A and Γ_3^B.

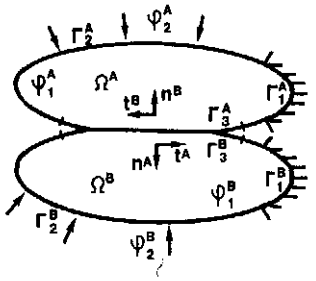


figure 1 : two body contact problem

Unilateral conditions

Under a small transformation hypothesis, we write the unilateral problem as:

$$u_N^P + u_N^Q \leq 0, \quad F_N^P = F_N^Q, \quad F_N^P \leq 0, \quad (u_N^P + u_N^Q).F_N^P = 0 \tag{23}$$

We have shown in [16] that this problem is still a complementarity problem thanks to a convenient choice of the duality product between the trace spaces.

Friction condition

The static Coulomb friction law can be written :

$$F_T^P = F_T^Q \tag{24}$$

$$|F_T^P| \leq \mu\, |F_N^P| \tag{25}$$

$$|F_T^P| < \mu |F_N^P| \Rightarrow u_T^P + u_T^Q = 0 \tag{26}$$

$$|F_T^P| = \mu |F_N^P| \Rightarrow \exists\, \lambda \geq 0 \text{ such that } u_T^P + u_T^Q = -\lambda\, F_T^P \tag{27}$$

This problem will be treated after discretization by a change of variable (see [13]). We will see in paragraph 7 the application to a multibody contact problem.

4. VARIATIONAL FORMULATION FOR QUASI-STATIC PROBLEMS

We first set the quasi-static problem using the general Coulomb friction law (4)(5)(6) (written in terms of rates of displacement) in the case of a given contact boundary (no unilateral conditions). For an evolutive problem, it is clear that friction is an irreversible phenomena which has to be written in terms of rates of displacement. It is a path dependant phenomena and the final solution does not depend only on the final force state but on all the intermediate states.

Problem P6 :
Be given the forces $\{\varphi_1, \varphi_2\} : [0,T] \longrightarrow (L^2(\Omega))^3 \times (L^2(\Gamma_2))^3$, the thermal strain $e^0 : [0,T] \longrightarrow (L^2(\Omega))^6$, find $u : [0,T] \longrightarrow (H_0^1(\Omega))^3$ such that :
for $t = 0$ $u(0) = u_0$ be given in Ω
for $t \in]0,T]$ (10)(11)(12)(13)(14) and (4)(5)(6) be verified.

Restricting the dynamical formulation of Duvaut-Lions [2] to the quasi-static case, we remove the inertial terms and we get the following variational formulation.

Problem P7 :
(φ_1, φ_2) and e^0 be given as in problem P6, find $u(t) \in U = (H_0^1(\Omega))^3$ such that :
for $t = 0$ $u(0) = u_0$ be given in Ω
for $t \in]0,T]$ $\forall v \in U$ $\quad a(u,v-\dot{u})-f(v-\dot{u})+j(v)-j(\dot{u}) \geq 0$ (28)

In view of the numerical resolution, we give now an incremental formulation by approximating the derivatives by finite differences. Let $(t_0,\ldots,t_k,\ldots,t_M)$ be a partition of the interval $[0,T]$, we get a forward finite difference approximation of $\dot{u}$ by :

$$\dot{u}(t_{k+1}) = (u(t_{k+1})-u(t_k))/(t_{k+1}-t_k) \tag{29}$$

Substituting $u(t_k)$ by u^k and $f(t^k)$ by f^k and setting
$\Delta u^k = u^{k+1} - u^k$ and $\Delta t^k = t^{k+1} - t^k$, we write (28) under the following form :

$$a(u^k+\Delta u^k, v-\Delta u^k) - (f^k+\Delta f^k, v-\Delta u^k) + j(v) - j(\Delta u^k) \geq 0 \tag{30}$$
$$a(\Delta u^k, v-\Delta u^k) - (\Delta f^k, v-\Delta u^k) + j(v) - j(\Delta u^k) \geq - a(u^k, v-\Delta u^k) + (f^k, v-\Delta u_k) \tag{31}$$

The left part of the inequality (31) is connected to the equilibrium deflect R^k, i.e. the friction force at the previous time t_k.

$$-a(u^k, v-\Delta u^k) + (f^k, v-\Delta u^k) = (R^k, v-\Delta u^k) \tag{32}$$

So, we get a problem formally analogous to the static case one :

Problem P8 :
find $\Delta u^k \in U$ such that
$$\forall v \in U \quad a(\Delta u^k, v-\Delta u^k) - (R^k+\Delta f^k, v-\Delta u^k) + j(v) - j(\Delta u^k) \geq 0 \tag{33}$$

The non penetration condition corresponding to the initial unilateral problem is introduced by projecting Δu^k such as to assume :

$$u^k + \Delta u^k \leq 0 \tag{34}$$

Consequently, we have to solve problem P8 which is of the same form as the static problem P4. We note that we have for each step an extra load coming from the friction force of the previous step and that the projection is now different as shown in (34). We have to underline that the solution depends on the time increment size and that the equivalence with the initial problem occurs only for $\Delta t \longrightarrow 0$. So, the time discretization has to be small, except if the loads are piecewise linear (which is often the case). In this case, we take one increment on each piece.

5. EXTENSION TO THE VISCOELASTICITY AND VISCOPLASTICITY CASES

We have shown in [14] how to set a viscoelasticity problem as a variational equation coupled to a differential equation. This form has been extended to contact problems both in viscoelasticity and viscoplasticity in [15] and [16]. These cases are often concerned with thermal effects as it has been set in Problem P1. We set :

Problem P9 :
Let the loads $f = (\varphi_1, \varphi_2)$,the prescribed strain e^0 be given, find u and Y such that:

$$u(0) = u_0$$
$$\forall v \in U \quad a(u,v-\dot{u}) - (f,v-\dot{u}) + \int_\Omega [KY]^t D(v-\dot{u})dx + j(v) - j(\dot{u}) \geq 0 \tag{35}$$
$$\forall t \in]0,T] \quad \dot{Y} = F(Y,u,t) \tag{36}$$

For the viscoelasticity case, a Maxwell's model leads to the following form of the general form (36) where $Y = \xi$ is the viscous strain :

$$\dot{\xi} = \eta^{-1} K (Du + e^0 - \xi) \tag{37}$$

η and K are respectively the viscosity and the elasticity matrices caracterised by the Young modulus E, the Poisson coefficient ν, and the relaxation times τ_c (cubic part) and τ_s (deviatoric part). Usually τ_c is taken infinite [14].

In viscoplasticity, we consider a Norton-Hoff law including isotropic and cinematic hardening introduced by Lemaitre-Chaboche (p.309 of [17]). In this case Y is a vector $(\xi^p, X)^t$ where ξ^p is the viscoplastic strain and X (which is a deviatoric part) caracterises the hardening. (36) becomes :

$$\dot{\xi}^p = 3/2 \ \dot{p}(\sigma' - X)/J \tag{38}$$

$$\dot{X} = 2/3 \ c(a\dot{\xi}^p - X\dot{p}) \tag{39}$$

where $\dot{p} = \langle (J-k)/\eta \rangle^n$ ($\langle \ . \ \rangle$ means the positive part), σ' is the deviatoric part of $\left[K(Du + e^0 - \xi_p)\right]$, J is the second invariant of the deviatoric part of $(\sigma - X)$ and η, a, c, k, n are coefficients.

So we get a variational inequation problem coupled with a differential equation which is linear in viscoelasticity and strongly non linear in viscoplasticity. The first problem is a global relation whose resolution is presented in paragraph 6 and the second one will be solved locally element by element after the finite element discretization. Numerical methods for solving this differential equation problem are presented in [16][18][19][20]. Among the explicit and implicit methods which have been tested, the θ-method is our favorite.

6. NUMERICAL METHODS USED IN THE FINITE ELEMENT CODE PROTIS [21]

6.1. Relaxation method with projection (SORP)

We extend to the friction case the Successive Over Relaxation Method with Projection (SORP) introduced in [22][23] for the Signorini problem. A finite element discretization is done with linear elements P1 under the classical plane hypothesis (plane strain, plane stress or cylindrical symmetry). We have to underline that for the case of finite elements with a higher degree of approximation, the projection on the constraint convex is much more difficult and that this point is often neglected in the litterature. The contact variables are treated in local normal tangential coordinates. The convenient rotations are done on the finite element matrix. The discrete problem associated to problem P5 is written :

Problem P10 :

Find $\bar{u} \in K^h \subset \mathbb{R}^{2N}$ such that :

$$J(u_1, u_2, \ldots, u_{2N}) \leq J(v_1, v_2, \ldots, v_{2N}) \ \forall \ \bar{v} \in K^h \tag{40}$$

$\bar{u}$ is the displacement vector of components u_i ; i=1,2N N is the total node number

K^h is the approximate convex defined by $K^h = \prod_{i=1}^{2N} K_i$ with : $K_{2\alpha} = K_{2\alpha-1} = \mathbb{R}$ if $\alpha \notin I^h$

$K_{2\alpha-1} = \mathbb{R}$ if $\alpha \in I^h$

$K_{2\alpha} = \mathbb{R}^-$ if $\alpha \in I^h$

I^h is the set of the indices of the contact nodes.

$$J(u_1,u_2,\ldots,u_{2N}) = \frac{1}{2}\sum_{i=1}^{2N}\sum_{i=1}^{2N} a_{ij}u_iu_j - \sum_{i=1}^{2N} f_iu_i + \sum_{\alpha\in I_h} G_{2\alpha-1}\,|u_{2\alpha-1}| \qquad (41)$$

$a_{ij} = a(\omega_i,\omega_j)$ is the general term of the elasticity finite element matrix
$f_i = (f,\omega_i)$ where ω_i are the shape functions.

$$G_{2\alpha-1} = \int_{\Gamma_3} g\,\omega_{2\alpha-1}\,d\gamma$$

A punctual relaxation process defines the solution of problem P10 as the limit of the solutions u_i^{n+1} of the successive problems P11 set on one component after the other.

Problem P11 :
Find $u_i^{n+1} \in K_i$ such that for i = 1, ... 2N :

$$J(u_1^{n+1},u_2^{n+1},\ldots,u_i^{n+1},u_{i+1}^{n},\ldots,u_{2N}^{n}) \le J(u_1^{n+1},u_2^{n+1},\ldots,v,u_{i+1}^{n},\ldots,u_{2N}^{n}) \quad \forall\, v \in K_i \qquad (42)$$

The convergence of the algorithm is given in [22] for this special form of the convex K^h for the Signorini problem.

The problem P11 is a pretty simple one dimensional minimization problem. According to the value of i, it will be one of the three following problems :
- if $i = 2\alpha$ or $i = 2\alpha-1$ with $\alpha \notin I^h$: minimization on $\mathbb{R}$ of a quadratic functional (the last term of J(u) in (41) is not present). An overrelaxation method is used.

$$u_i^{n+\frac{1}{2}} = \frac{1}{a_{ii}}\left(f_i - \sum_{j=1}^{i-1} a_{ij}\,u_j^{n+1} - \sum_{j=i+1}^{2N} a_{ij}\,u_j^{n}\right) \qquad (43)$$

$$u_i^{n+1} = (1-\omega)u_i^{n} + \omega\,u_i^{n+\frac{1}{2}} \qquad (44)$$

ω is the relaxation coefficient. $\omega \in]0,2[$
- if $i = 2\alpha$ with $\alpha \in I^h$ minimization on $\mathbb{R}^-$ of the same quadratic functional. The form (43) is still available and we introduce a projection on $\mathbb{R}^-$ in (44) which is replaced by:

$$u_i^{n+1} = \mathrm{Proj}_{\mathbb{R}^-}\left\{(1-\omega)u_i^{n} + \omega\,u_i^{n+\frac{1}{2}}\right\} \qquad (45)$$

- if $i = 2\alpha-1$ with $\alpha \in I^h$ minimization on $\mathbb{R}$ of the non differentiable functional (41) containing an absolute valor. As the problem is set on $\mathbb{R}$ it is easy to solve it by considering the two alternatives, the solution u_i^{n+1} to be positive or negative, and adding in (43) $\pm G_i$. The relaxation process (45) remains without projection.

Turning back to the complete problem P3, a fixed point method is used on the sliding limit g, i.e. on its discrete form $\bar{G}$, to insure the Coulomb relations to be true. Starting with $\bar{G}^0$ equal to zero (friction less case), the resolution of problem P10 gives the solution $\bar{u}^k$ and the normal contact forces $\bar{F}_N^k$ for the step k. The next $\bar{G}^{k+1}$ is constructed by :

$$\bar{G}^{k+1} = \mu\left|\bar{F}_N^k\right| \qquad (46)$$

The process involving a resolution of problem P10 for each step is stopped when the variation of $\bar{G}$ in the sense of an euclidian norm is smaller than a given ϵ_1 (we usually take $\epsilon_1 = 0.1\%$). It is very important to note that $\bar{F}_N^k$ is directly obtained as the residu of the minimization of (41) and that it is not computed from the stress solution which would give a very bad estimation. And so is $\bar{G}^k$.

Acceleration with a diagonal process.

In reality, the first updatings of $\bar{G}^k$ are done after only a partial resolution of problem P10. The relaxation algorithm is stopped by a relative variation test on the current solution $\bar{u}^k$ (Euclidian norm) which is asked to be lower than a coefficient ϵ_2. This coefficient is large for the first iterations on $\bar{G}^k$ ($\epsilon_2 \cong 10^{-1}$) and decreases, progressively when k increases, down to a final value of $.5\ 10^{-5}$. Usually the number of iterations k on $\bar{G}^k$ is lower than 10. This diagonal process is very efficient and reduces the resolution time in a proportion which may be of 3 or 4.

Condensation

In a lot of cases, the number of contact node M is small in front of the total number of nodes N. Then, it is interesting to reduce the size of the problem to the only contact variables which are concerned by the contact non linearities. Let us formally part the finite element matrix A (general term a_{ij}) as shown on fig. 2 :

$$\begin{matrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{matrix}$$

figure 2 : the FEM matrix

The parts containing the indice c concern the contact variables. The initial problem can be set on $J^*(u)$ defined by :

$$J^*(\bar{u}_c) = \frac{1}{2}\, \bar{u}_c^t\, A^*_{cc}\, \bar{u}_c - \bar{f}^{*t}\, \bar{u}_c - \bar{G}^t \left|\bar{u}_c\right| \tag{47}$$

where $\bar{u}_c$ is the vector of the contact displacements

$$A^*_{cc} = A_{cc} - A_{cf}\, A_{ff}^{-1}\, A_{fc} \tag{48}$$

$$f^* = - A_{ff}^{-1}\, A_{fc}\, f \tag{49}$$

the inversion of A_{ff} is not necessary, the term $A_{ff}^{-1}\, A_{fc}$ is computed directly by 2M resolutions involving the matrix A_{ff} and the columns of A_{fc} as different second members. This is not expensive because the factorization of A_{ff} is effected only once. We have to underline that A^*_{cc} is now a full symetrical matrix but its dimension is only 2M.

Matrix storage

For solving the complete problem without condensation a sparse matrix storage (or Morse storage) is used. This is very efficient for the Gauss-Seidel type method because no products by zero are done. (see table 4 in paragraph 8).

For solving the problem after condensation, a skyline storage is used because the computation of A^*_{cc} is done with a Cholesky variante method which is most efficient with this storage. In any case, the matrix A^*_{cc} is full.

Choice of the relaxation coefficient ω

There are no theoretical results on the optimum relaxation coefficient for the constrained problem. It depends on the number of constraints which are saturated on the final solution. We have experimentally observed that for the frictionless problem, this optimal coefficient remains close to the one of bilateral contact . For the unilateral problem with friction the sensitivity according to this coefficient is smaller (see [5]) and it is easier to get a convenient relaxation coefficient. The optimal values of ω are usually between 1.5 and 1.9.

To avoid the determination of the optimum coefficient ω, a Gauss-Seidel method accelerated by a Aitken procedure is actually developped [33].

6.2. Mathematical programming method (LEMKE)

This kind of method is applied on the complementarity problem form (2) together with the relation between u_N and F_N given through the equilibriums and the constitutive equations. The method has been introduced by Lemke (see in reference [3]) for the Signorini problem and has been extended to the friction case by Klarbring [25]. It is used by Barbosa-Feijoo [26] both on contact problems and plasticity problems. It is a direct method which is an extension of the Simplex technics to a problem including constraints. An extension to the two body contact problem has been presented in Pinto-Raous [27]. This method is always used on the problem after condensation. Details of the method can be found in the previous references and the comparative efficiency is presented in the paragraph 8.

6.3. Fixed Point method on Boundary Conditions -FPBC- (see [28])

This method could be compared to a strong penalization method because of the use of the Irons coefficient to impose the boundary conditions.

It is an iterative method using the direct solution of a classical elasticity problem with a given state of the contact boundary. On this part, boundary conditions and given forces are introduced or removed for the next iteration according to a sequence of tests on the previous solution such as to avoid penetration, traction forces and to assume the relations (7)(8)(9).

The method works pretty well for frictionless problems but a few instabilities are observed for cases including friction. This method is very efficient for specific cases. We use it a lot for the study of cracked viscoplastic solids with large scale temperature variations. The (frictionless) unilateral problem is set by the conditions of non penetration of one edge of the crack on the other. Because of the non homogeneous temperature effects, the finite element matrix has to be assembled again from time to time. We do this updating of the matrix when we have to change the boundary conditions. So, in this case there is no need for extra computation to deal with unilateral contact because a new factorization has to be effected in any case.

7. APPLICATIONS

7.1. Contact of a long bar on a plane surface

This example and the next one have been treated by 6 different laboratories of the groupe 'Validation of computer code' of the GRECO 'Grandes Déformations et Endommagement". This example has the advantage of being very elementary and that of giving different contact states according to the loadings and the friction coefficient values.

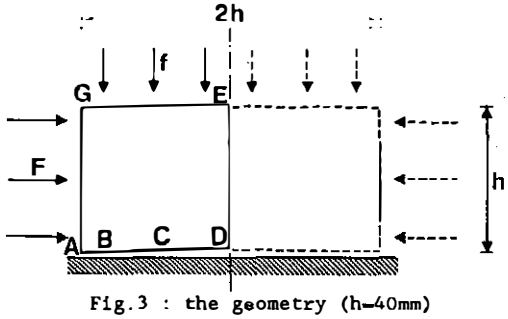


Fig.3 : the geometry (h=40mm) and the loadings N=230 M=32

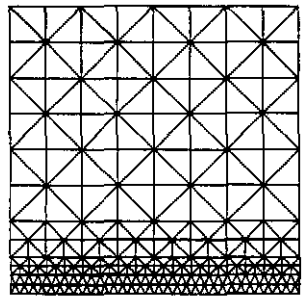


Fig.4 : the mesh

Constitutive law : plane strain elasticity E=13000 daN/mm^2, ν=0.2
Boundary conditions : u_x=0 on DE, u_x=u_y=0 at D.
Unilateral conditions with friction on the side AD : μ=0.2 or μ=1 (academic)
Loadings : force densities f and F are applied respectively on EG and GA.
Mesh : we use a 230 node mesh with 32 contact nodes N=230 M=32
Results : we give on the table 1, the length (and the number of nodes) of the separate part AB, the sliding part BC and the stick part CD for different choices of the loads f and F, and of the friction coefficient μ.

μ	F daN/mm^2	f daN/mm^2	Separate part AB mm	Sliding part BC mm	Stick part CD mm
1	10	-5	3.75 (3 nodes)	18.75 (15 nodes)	17.5 (15 nodes)
1	15	-5	3.75 (3 nodes)	26.25 (21 nodes)	10 (9 nodes)
0.2	10	-5	0	40 (32 nodes)	0
0.2	10	-15	0	23.75 (19 nodes)	16.25 (14 nodes)
0.2	10	-25	0	3.75 (3 nodes)	36.25 (30 nodes)

Table 1 : contact states for different loading cases

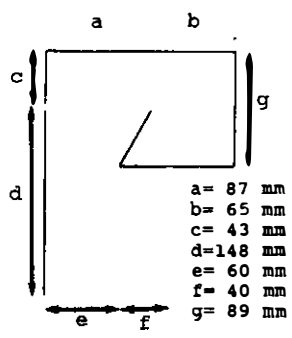
NODE	U_N	U_T	F_N	F_T
1	-0.5698E-3	0.1466E-1	0.	0.
3	-0.2709E-3	0.1375E-1	0.	0.
5	-0.4861E-4	0.1282E-1	0.	0.
6	0.	0.1180E-1	-1.157	-1.158
7	0.	0.1073E-1	-2.157	-2.156
13	0.	0.9679E-2	-2.834	-2.832
15	0.	0.8647E-2	-3.401	-3.399
16	0.	0.7647E-2	-3.871	-3.870
17	0.	0.6683E-2	-4.291	-4.290
27	0.	0.5757E-2	-4.657	-4.657
30	0.	0.4872E-2	-5.008	-5.008
31	0.	0.4031E-2	-5.329	-5.329
32	0.	0.3239E-2	-5.644	-5.645
45	0.	0.2497E-2	-5.948	-5.950
48	0.	0.1815E-2	-6.269	-6.270
49	0.	0.1199E-2	-6.608	-6.607

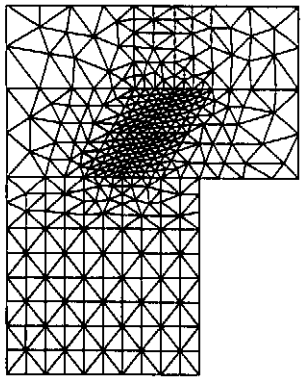
NODE	U_N	U_T	F_N	F_T
50	0.	0.6662E-3	-6.994	-6.993
65	0.	0.2432E-3	-7.503	-7.503
69	0.	0.	-8.162	-7.602
83	0.	0.	-8.595	-5.073
87	0.	0.	-8.676	-4.332
100	0.	0.	-8.750	-3.728
115	0.	0.	-8.799	-3.226
132	0.	0.	-8.865	-2.787
150	0.	0.	-8.903	-2.410
170	0.	0.	-8.943	-2.060
191	0.	0.	-8.961	-1.727
206	0.	0.	-8.995	-1.409
214	0.	0.	-9.011	-1.114
221	0.	0.	-9.035	-0.8269
226	0.	0.	-9.044	-0.5424
229	0.	0.	-9.074	-0.2599

Table 2 : displacements and forces for the contact nodes case μ=1 F=10 f=-5

It can be observed on the table 2 that the relations (2) and (4)(5)(6) are verified with a very good accuracy.

7.2. A dovetail assembling (proposed by the CETIM-Paris).
This example is a two body contact problem with an oblique contact zone.





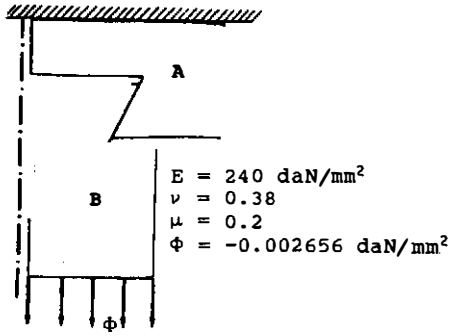


Fig.5. : geometry
(plane stress hypothesis)
thickness = 6.4 mm

Fig.6 : the mesh
N = 384
M = 38

Fig.7. : displacements
(amplified twenty times)

caracteristics E = 240 daN/mm² ; ν = 0.38 ; μ = 0.2 ; φ = -.002656 daN/mm²

The nodal forces and the nodal displacements on the contact area for the corresponding nodes are given in the table 3.

NODE	U_N	U_T	F_N	F_T
130	0.1539	-0.2609	0.	0.
140	0.1502	-0.2570	0.	0.
156	0.1461	-0.2533	0.	0.
184	0.1412	-0.2499	-0.1046	0.0209
205	0.1361	-0.2466	-0.1656	0.0331
210	0.1308	-0.2435	-0.2070	0.0414
197	0.1255	-0.2406	-0.2361	0.0472
180	0.1200	-0.2379	-0.2619	0.0523
167	0.1145	-0.2353	-0.2786	0.0557
164	0.1087	-0.2329	-0.2960	0.0591
165	0.1030	-0.2305	-0.3072	0.0614
163	0.0971	-0.2282	-0.3171	0.0634
155	0.0906	-0.2260	-0.3168	0.0633
152	0.0838	-0.2239	-0.3120	0.0624
147	0.0763	-0.2219	-0.3059	0.0611
141	0.0684	-0.2200	-0.2915	0.0583
137	0.0595	-0.2185	-0.2630	0.0526
136	0.0501	-0.2173	-0.1985	0.0397
133	0.0394	-0.2169	-0.0828	0.0166

NODE	U_N	U_T	F_N	F_T
216	-0.1576	0.02889	0.	0.
217	-0.1518	0.02894	0.	0.
218	-0.1464	0.02882	0.	0.
220	-0.1412	0.02868	-0.1046	0.0209
221	-0.1361	0.02840	-0.1657	0.0331
223	-0.1308	0.02811	-0.2070	0.0414
226	-0.1255	0.02763	-0.2361	0.0471
229	-0.1200	0.02711	-0.2619	0.0523
231	-0.1145	0.02642	-0.2787	0.0558
230	-0.1087	0.02572	-0.2961	0.0589
232	-0.1030	0.02498	-0.3072	0.0614
239	-0.0971	0.02402	-0.3171	0.0634
244	-0.0906	0.02293	-0.3169	0.0633
250	-0.0838	0.02148	-0.3120	0.0624
256	-0.0763	0.01974	-0.3059	0.0611
264	-0.0684	0.01750	-0.2915	0.0583
276	-0.0595	0.01478	-0.2630	0.0526
288	-0.0501	0.01173	-0.1985	0.0397
303	-0.0394	0.00881	-0.0828	0.0166

Table 3 : solution on the contact area

The vertical displacements of the lower edge is -0.33mm (variation between -0.327 and -0.329).

7.3. The punch problem with loading and unloading

The specifications given by Curnier [32] are used. This example has the advantage of having some analytical results and uses loading history including unloading.

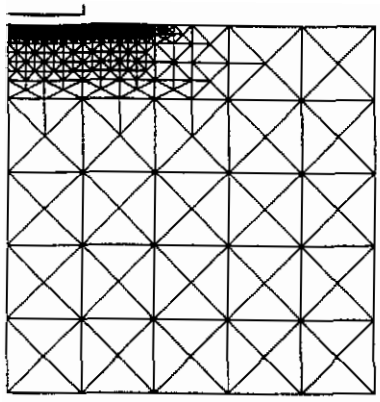


Fig.8. The mesh
N=484 M=32

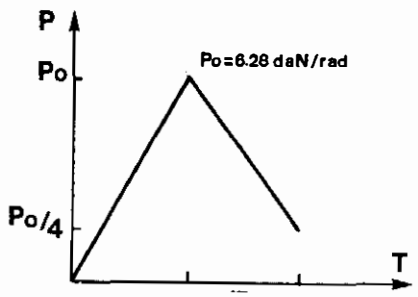


Fig.9. Loading evolution

Elasticity under revolution symmetry hypothesis E = 314 daN/mm^2, ν=0, μ = 0.4.

On the figures 10 and 11, the variations of $F_T/\mu * F_N$ along the radius of the punch are plotted. The sliding zones are caracterized by the value 1 of this quotient. The other parts are sticked. On the figure 10, the comparison with a semi analytical solution gives an error of 7% : this is due to the weakness of the use of P1 elements for axisymmetrical problems. The figure 11 shows the solution after the unloading step. This complex solution showing three different zones (stick, inward-slip, outward-slip) has been reached with quite the same number of iterations than the first one (see table 4).

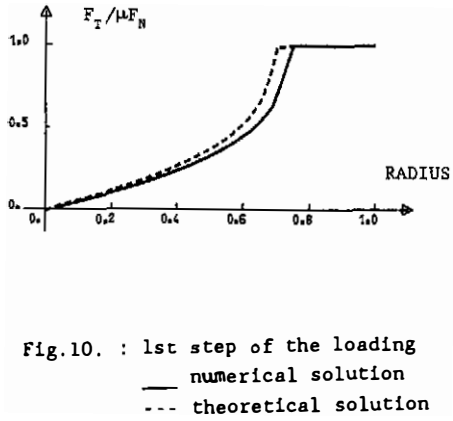


Fig.10. : 1st step of the loading
___ numerical solution
--- theoretical solution (semi analytical)

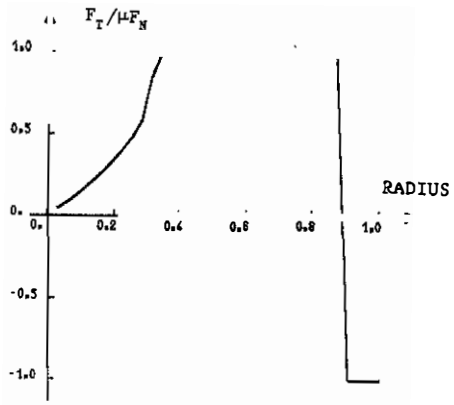


Fig.11. : 2nd step of the loading

7.4. Structure assembling : bolted junction under internal pressure loading

This example deals with a real scale model on which experimental measurements have been done. This work has been supported by Technicatome (Les Milles - France). It is a four body contact problem including three contact areas with different friction coefficients. It is a three step loading : first the closure pressure is applied, then the pressure on the lower part and finally the pressure on the upper part. The displacements with an amplicator coefficient of 100 are given on the following figures (elasticity under revolution symmetry hypothesis). N=674, M=54.

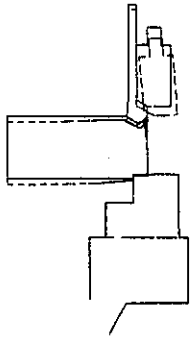


Fig.12.closure pressure

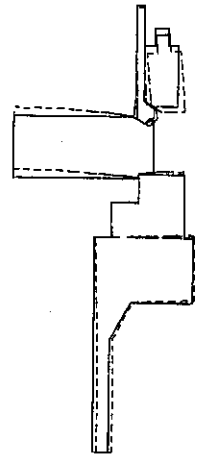


Fig.13. pressure on the lower part

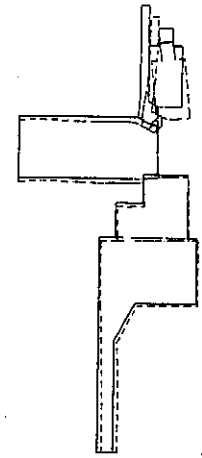
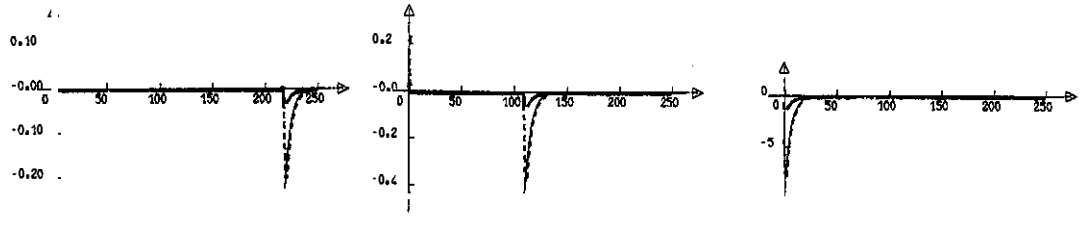


Fig.14. pressure on the upper part

An identification of the different friction coefficients has been done on the first loading step and the comparison between the numerical results and experimental datas for all the loading steps can be found in [29] and [30]. The errors are less than 15% on the most significant quantities.

7.5. Friction modelling for metal forming processes

We focus on the behaviour of the sheet metal under the blankholder during the stamping process. This study has been supported by Regienov (Paris-France). It is a very thin structure with a large number of contact nodes (N=784, M=251). Details about the results can be found in [31]. The following figure shows the evolution of the contact forces according to the increase of the tangential displacement Δ applied to the righthand extremity. The extension of the sliding area can be observed along with the good verification of Coulomb's law (F_T/μ is also plotted).



$\Delta = 5.10^{-4}$ $\Delta = 5.10^{-3}$ $\Delta = 0.25$

Fig.15. contact forces evolution

___ normal force F_N __ tangential force F_T --- quotient F_T/μ

7.6. Cracked viscoplastic solid under alternative loading

This example deals with the study of the progression of a crack in a structure submitted to high variations of temperature. The contact between the edges of the crack is frictionless. The viscoplastic law has been given in paragraph 6 (relations (38)(39)).

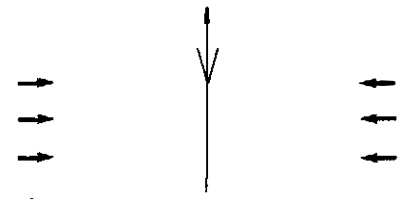


Fig.16. the structure (the loading is harmonic)

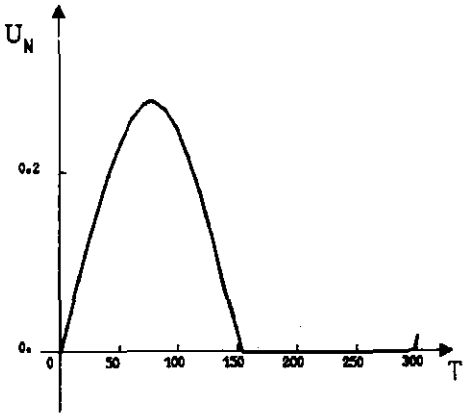


Fig.17. displacement of a contact node

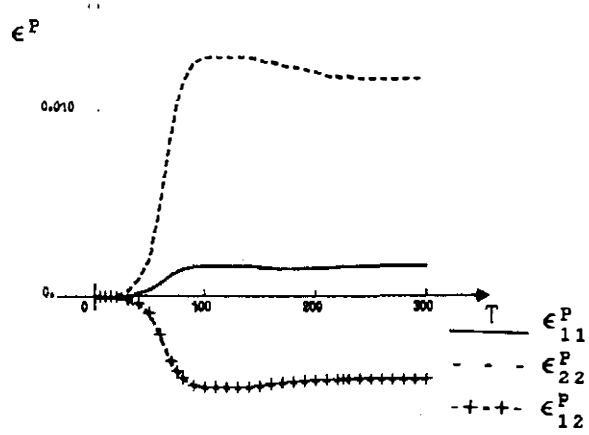


Fig.18.viscoplastic strain near the crack tip

The unilateral contact has been solved by the FPBC method described in paragraph 6.3 and the viscoplastic equations have been integrated by a Runge-Kutta method. Details can be found in [20].

8. COMPARATIVE COMPUTATIONAL TIMES

Computational times (on a Microvax II) corresponding to the examples referenced in paragraph 7 are presented in table 4. The methods are the ones presented in paragraph 6 either with a skyline storage or a Morse storage (sparse matrix storage), with or without condensation. To give an idea of the cost of a contact problem, the computational time for a close classical elasticity problem on the same mesh is given as a 'reference' : the unilateral and friction conditions are replaced in this case by classical boundary conditions. The global number of iterations N_{SORP} is given together with the number N_G of updating of $\bar{G}^k$.

Example		Method	N_{SORP}/N_G	Total time	Solver time
	Reference	Cholesky		20"	4"
		SORP+Skyline (ω=1.88)	90	1'49"	1'35"
§ 7.1		SORP+Morse (ω=1.88)	90	1'34"	33"
	Frictionless	SORP+Skyline+Cond.(ω=1.74)	30	1'21"	1.5"
with	μ=0	Lemke+Skyline+Cond.		1'10"	3"
F=10		FPBC+Skyline	2	28"	8"
f=-5	Friction	SORP+Skyline (ω=1.86)	99/8	1'58"	1'44"
	μ=0.2	SORP+Morse (ω=1.86)	99/8	1'34"	36"
		SORP+Skyline+Cond. (ω=1.72)	41/8	1'26"	2"
N=230	Friction	SORP+Skyline (ω=1.72)	200/8	3'42"	3'28"
M=32	μ=1	SORP+Morse (ω=1.72)	200/8	2'13"	1'12"
		SORP+Skyline+Cond. (ω=1.67)	101/10	1'30"	5"
	Reference	Cholesky		47"	12"
§ 7.2	Frictionless	SORP+Skyline+Cond. (ω=1.87)	212	5'7"	15"
N=384	μ=0	Lemke+Skyline+Cond.		2'34"	7"
M=38	Friction μ=0.2	SORP+Skyline+Cond. (ω=1.58)	505/9	8'59"	36"
	Reference	Cholesky		1'38"	13"
§ 7.3		SORP+Skyline (ω=1.89)	166/10	8'1"	6'33"
N=484	1st.load.step	SORP+Morse (ω=1.89)	166/10	5'48"	56"
M=32		SORP+Skyline+Cond. (ω=1.58)	30/10	4'56"	1.5"
	2nd.load.step	SORP+Skyline+Cond. (ω=1.58)	30/10	4'60"	1.5"
	Reference	Cholesky		1'26"	28"
§ 7.4	μ_i=0 1stload.step	Lemke+Skyline+Cond.		7'49"	30"
N=674	1stload.step	SORP+Skyline+Cond. (ω=1.82)	686/9		28'23"
M=54	$\mu_i \neq 0$ 2ndload.step	SORP+Skyline+Cond. (ω=1.82)	669/9		27'27"
	3rdload.step	SORP+Skyline+cond. (ω=1.82)	595/9	1h39'12"	24'27"
§ 7.5	Step of fig.12a	SORP+Morse (ω=1.90)	742/151		8'32"
N=754	Step of fig.12b	SORP+Morse (ω=1.90)	5472/1232		50'06"
M=251	Step of fig.12c	SORP+Morse (ω=1.90)	5931/1464	21h	55'20"

Table 4. Computational times

In conclusion, we can note that it is important to have different solvers to treat unilateral problems with friction because the efficiency of each method depends on the nature of the problem. We can summarize this by a few remarks.

For frictionless problems, the Lemke algorithm with condensation and the FPBC seem to be the best ones.

Condensation is always suitable when the number M of contact nodes is small compared to the total number of nodes N. The only exception is when the finite element matrix changes (in case of thermal effects for example) because in such a case the computation of A^*_{cc} (see(48)) has to be done again and it will be costly.

When the number of contact nodes is large a Successive OverRelaxation method with Projection using a sparse matrix storage (Morse) is more adequate. The example 7.5 is pretty stiff because the solution includes a part of the boundary where the contact is associated to a very small compression force. Before convergence is reached the numerical

solution oscillates between two states ($u_N = \epsilon'$, $F_N = 0$ - no contact) and ($u_N = 0$, $F_N = \epsilon''$ - contact with small compression) where ϵ' and ϵ'' are two little negative quantities.

Condensation is very important for the multibody contact problems because the initial matrix is ill conditioned due to the connection relations between the different solids which have been meshed separatively.

Acknowledgements.

Partial support for the developments described here has been provided by Technicatome (Les Milles-France) and Regienov (Rueil-Malmaison-France) which are gratefully acknowledged.

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