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► **To cite this version:**

| Elie Aïdékon, Yueyun Hu, Zhan Shi. Boundedness of discounted tree sums. 2024. hal-04683431

**HAL Id: hal-04683431**

**<https://hal.science/hal-04683431>**

Preprint submitted on 2 Sep 2024

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# BOUNDEDNESS OF DISCOUNTED TREE SUMS

ELIE AÏDÉKON, YUEYUN HU, AND ZHAN SHI

ABSTRACT. Let  $(V(u), u \in \mathcal{T})$  be a (supercritical) branching random walk and  $(\eta_u, u \in \mathcal{T})$  be marks on the vertices of the tree, distributed in an i.i.d. fashion. Following Aldous and Bandyopadhyay [2], for each infinite ray  $\xi$  of the tree, we associate the *discounted tree sum*  $D(\xi)$  which is the sum of the  $e^{-V(u)}\eta_u$  taken along the ray. The paper deals with the finiteness of  $\sup_{\xi} D(\xi)$ . To this end, we study the extreme behaviour of the local time processes of the paths  $(V(u), u \in \xi)$ . It answers a question of Nicolas Curien, and partially solves Open Problem 31 of Aldous and Bandyopadhyay [2]. We also present several open questions.

## 1. INTRODUCTION

Let  $V := \{V(u), u \in \mathcal{T}\}$  be a discrete-time branching random walk on the real line  $\mathbb{R}$ , where  $\mathcal{T}$  is a Ulam-Harris tree which describes the genealogy of the particles and  $V(u) \in \mathbb{R}$  is the position of the particle  $u$ . When a particle  $u$  is at  $n$ -th generation, we write  $|u| = n$  for  $n \geq 0$ . The branching random walk  $V$  is constructed as follows: initially, there is a single particle  $\emptyset$  located at 0. The particle  $\emptyset$  is considered as the root of  $\mathcal{T}$ . At the first generation, the root dies and gives birth to a certain number of children, these children along with their spatial positions form a point process  $\aleph$  on  $\mathbb{R}$  and constitute the first generation of the branching random walk  $\{V(u), |u| = 1\}$ . We may identify  $\aleph = \sum_{|u|=1} \delta_{\{V(u)\}}$ . For the next generations, the process is constructed recursively: for each  $|u| = n$  (if such  $u$  exists) with  $n \geq 1$ , the particle  $u$  dies in the  $(n+1)$ -th generation and gives birth to an independent copy of  $\aleph$  shifted by  $V(u)$ . The collection of all children of all  $u$  together with their positions gives the  $(n+1)$ -th generation. The whole system may survive forever or die out after some generations.

Define the measure  $\mu(dx) := \mathbb{E}[\aleph(dx)]$ . We assume that  $\mu(\mathbb{R}) > 1$ , so that  $\mathcal{T}$  is a supercritical Galton–Watson tree. We denote by  $\mathcal{S}$  the event of non-extinction of  $\mathcal{T}$ , and  $\mathbb{P}^* := \mathbb{P}(\cdot \mid \mathcal{S})$ . By an infinite ray  $\xi$  of  $\mathcal{T}$ , we mean a path  $\xi = \{\xi_0 = \emptyset, \xi_1, \dots, \xi_n, \dots\}$  such that for all  $n \geq 0$ ,  $\xi_n$  is the parent of  $\xi_{n+1}$  as vertices of  $\mathcal{T}$ . Denote by  $\partial\mathcal{T}$  the set of all infinite rays. In particular  $\partial\mathcal{T} \neq \emptyset$  on the event  $\mathcal{S}$ . Let

$$(1.1) \quad M := \sup_{\xi \in \partial\mathcal{T}} \sum_{n=0}^{\infty} e^{-V(\xi_n)}.$$

The starting point of this work is the following question: when do we have  $M < \infty$ ,  $\mathbb{P}^*$ -a.s? This question was raised by Nicolas Curien (personal communication) in relation to the construction of self-similar Markov trees [6], see in particular the discussion in Section 3.4 there.

Let  $m_n := \inf_{|u|=n} V(u)$  and  $\varphi(t) := \log \int_{\mathbb{R}} e^{-tx} \mu(dx)$  for  $t \in \mathbb{R}$ . We have  $M \geq \sup_{n \geq 0} e^{-m_n}$ . Hence  $M = \infty$  if  $\liminf_{n \rightarrow \infty} m_n = -\infty$ . This happens  $\mathbb{P}^*$ -a.s. when

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*Date:* This version: September 2, 2024.

*2010 Mathematics Subject Classification.* 60F05, 60J45, 60J80.

*Key words and phrases.* Branching random walk, discounted tree sums, local times.

$\inf_{t>0} \frac{\varphi(t)}{t} \in (0, \infty)$ . Actually, as soon as  $\inf_{t>0} \frac{\varphi(t)}{t} < \infty$ , we have the following law of large numbers ([15, 17, 8]):  $\mathbb{P}^*$ -a.s.,

$$(1.2) \quad \lim_{n \rightarrow \infty} \frac{m_n}{n} = - \inf_{t>0} \frac{\varphi(t)}{t} =: \gamma.$$

Notice that the upper bound

$$(1.3) \quad M \leq \sum_{n=0}^{\infty} e^{-m_n}$$

implies that  $M < \infty$  a.s. when  $\gamma > 0$ . The critical case is therefore  $\gamma = 0$ , where the upper bound (1.3) is not good enough in general. From now on, we suppose that one of the following assumptions holds:

**(H1)** There exists  $t > 0$  such that  $\varphi(t) = 0$  and we let  $t^*$  be the minimal such  $t$ . The integral  $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx)$  is well-defined and nonzero.

**(H2)** There exists  $t^* > 0$  such that  $\varphi(t^*) = 0$  and  $\varphi(t) \in [0, \infty]$  for all  $t > 0$ . The integral  $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx)$  is well-defined, is equal to 0 and  $\int_{\mathbb{R}} x^2 e^{-t^*x} \mu(dx) \in (0, \infty)$ .

In (H1) and (H2), we say that  $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx)$  is well-defined if at least one of the quantities  $\int_{(-\infty, 0)} |x| e^{-t^*x} \mu(dx)$  or  $\int_{(0, \infty)} x e^{-t^*x} \mu(dx)$  is finite. Note that  $\varphi(t^*) = 0$  yields that  $m_n \rightarrow \infty$ ,  $\mathbb{P}^*$ -a.s. (see [20, Lemma 3.1]) and  $\gamma \geq 0$ .

Let us discuss the case (H1): if  $\gamma > 0$ , then  $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx) > 0$ . Otherwise, one necessarily has  $\gamma = 0$ . If furthermore  $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx)$  is negative, resp. positive, then one necessarily has  $\varphi(t) = \infty$  for all  $t < t^*$ , resp. for all  $t > t^*$ . The case (H1) with  $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx) > 0$  and  $\mu((-\infty, 0)) > 0$  will be denoted by **(H1')**.

In case (H2), one necessarily has  $\gamma = 0$ . The case (H2) is called ‘‘boundary case’’ in the literature, see [1] for the precise asymptotic of  $m_n$  and [20] for further references. There exist cases where  $\gamma = 0$  but neither (H1) nor (H2) holds, for example when  $\mu((-\infty, 0)) = 0$  and  $\mu(\{0\}) > 1$ . In that case  $\gamma = 0$  while there is no  $t > 0$  such that  $\varphi(t) = 0$ . We refer to the appendix of the arXiv version of [16] for such discussions.

**Theorem 1.1.** *Under (H1) or (H2),  $M < \infty$   $\mathbb{P}^*$ -a.s. Actually,*

$$\Xi_b := \sup_{\xi \in \partial \mathcal{T}} \sum_{n=0}^{\infty} (1 + |V(\xi_n)|)^{-b} < \infty$$

for all  $b > 2$  under **(H1)**, and for all  $b > 3$  under **(H2)**.

In the case  $\gamma > 0$ ,  $\Xi_b < \infty$  for  $b > 1$  by (1.2) and (1.3). The fact that there exist rays such that  $\limsup_{n \rightarrow \infty} \frac{V(\xi_n)}{n} < \infty$  implies that  $\Xi_b = \infty$  for  $b = 1$ . On the other hand, according to Jaffuel [16], under (H2) with some further integrability conditions on  $\mu$ , there exists an explicit constant  $a_0 > 0$  such that  $\mathbb{P}^*$ -a.s.

$$\inf_{\xi \in \partial \mathcal{T}} \limsup_{n \rightarrow \infty} \frac{V(\xi_n)}{n^{1/3}} = a_0.$$

It implies that  $\Xi_b = \infty$  for  $b = 3$  in this case. Finally, the following theorem shows there are cases under (H1') where  $\Xi_b = \infty$  for  $b = 2$ .

**Theorem 1.2.** *We suppose that (H1') holds and there exist  $1 < \alpha_1 \leq \alpha_2$  such that*

$$\mathbb{E} \left[ \sum_{|u|=1} e^{-t^*V(u)} |V(u)|^{\alpha_1} \right] < \infty$$

and for all  $y > 0$  large enough,

$$\mathbb{P}(\exists |u| = 1 : V(u) \leq -y) \geq e^{-t^*y} y^{-\alpha_2}.$$

We further assume that

$$\mathbb{E} [\Sigma_V \log_+ \Sigma_V] < \infty,$$

where  $\Sigma_V := \sum_{|u|=1} e^{-t^*V(u)}$  and  $\log_+ x := \max(\log x, 0)$  for  $x > 0$ . Then there exists a constant  $a_1 \in (0, \infty)$  such that  $\mathbb{P}^*$ -a.s.

$$\inf_{\xi \in \partial\mathcal{T}} \limsup_{n \rightarrow \infty} \frac{V(\xi_n)}{\sqrt{n \log n}} = a_1.$$

The issue of the finiteness of  $M$  is a particular case of the following problem stated in Aldous and Bandyopadhyay [2] as Open Problem 31, Section 4.4. It is related to the recursive distributional equation

$$(1.4) \quad X \stackrel{d}{=} \eta + \max_{|u|=1} e^{-V(u)} X_u$$

where on the right-hand side, conditionally on  $(\eta, V(u), |u| = 1)$ , the r.v.  $(X_u, |u| = 1)$  are i.i.d. with the law of  $X$ . A possible endogenous solution of (1.4) can be constructed as follows. Consider a pair  $(\eta, \aleph)$  where  $\eta > 0$  is a positive r.v. and  $\aleph$  is a point process as above. We do not suppose that  $\eta$  and  $\aleph$  are independent. We then construct recursively the branching random walk as before: for each vertex  $u$  at position  $V(u)$ , we take an independent copy of  $(\eta, \aleph)$ , call it  $(\eta_u, \aleph_u)$ , so that the children of  $u$  are at positions  $V(u) + \aleph_u$ . Following [2], we introduce the *discounted tree sum*

$$(1.5) \quad D(\xi) := \sum_{n=0}^{\infty} e^{-V(\xi_n)} \eta_{\xi_n}, \quad \forall \xi \in \partial\mathcal{T}.$$

Define

$$X := \sup_{\xi \in \partial\mathcal{T}} D(\xi).$$

Then  $X$  is a solution of (1.4). Observe that  $X = M$  if  $\eta \equiv 1$ . Theorem 32 of [2] shows (among other results) that  $X < \infty$  a.s. when  $\mu((-\infty, 0]) = 0$ ,  $E[\eta^p] < \infty$  for all  $p \geq 1$  and  $\varphi(t) < \infty$  for some  $t \geq 1$ . It was left open to study general conditions under which  $X < \infty$ . Again, it is clear from (1.2) that  $X = \infty$   $\mathbb{P}^*$ -a.s. if  $\gamma < 0$ . Notice that  $\gamma > 0$  no longer ensures that  $X < \infty$  a.s. because of the influence of the variable  $\eta$ . The property  $\{X < \infty\}$  is inherited, namely,  $\{X < \infty\} = \bigcap_{|u|=1} \{X^{(u)} < \infty\}$ , where  $X^{(u)}$  is defined as  $X$  but for the branching random walk indexed by the subtree of  $\mathcal{T}$  rooted at  $u$ . Therefore (see [19])  $\mathbb{P}^*(X = \infty) \in \{0, 1\}$ .

**Theorem 1.3.** *Suppose that*

$$(1.6) \quad \zeta := \lim_{x \rightarrow \infty} \frac{-\log \mathbb{P}(\eta > x)}{\log x}$$

*exists and  $\zeta \in (0, \infty]$ . Under (H1) or (H2),*

- (i) *if  $\zeta < t^*$ , then  $\mathbb{P}^*(X = \infty) = 1$ .*
- (ii) *if  $\zeta > t^*$ , then  $\mathbb{P}^*(X = \infty) = 0$ .*

**Remark 1.4.** (1) *Theorem 1.3 (i) comes from the fact that  $\sup_{u \in \mathcal{T}} e^{-V(u)} \eta_u = \infty$ ,  $\mathbb{P}^*$ -a.s., see Lemma 4.1. We refer to [4], [13] for much more precise results on this model, which is called last progeny modified branching random walk there.*

(2) *Theorem 1.3 (ii) was proved in [3] when  $V(u) = c|u|$  for some constant  $c > 0$ .*

(3) If  $\partial\mathcal{T}$  is equipped with the usual ultra metric, then in the case (ii),  $\xi \mapsto D(\xi)$  is continuous on  $\partial\mathcal{T}$ . For more details, see Remark 4.2.

The way we address these problems is by looking at the local times of the random walk along a ray defined as for any  $\xi \in \partial\mathcal{T}$  and  $k \in \mathbb{Z}$ ,

$$(1.7) \quad N_\xi^k := \sum_{n=0}^{\infty} \mathbf{1}_{\{k \leq V(\xi_n) < k+1\}}.$$

Theorem 1.1 is then a consequence of the following result.

**Theorem 1.5.** *We have  $\mathbb{P}^*$ -a.s.*

$$\lim_{n \rightarrow \infty} \frac{1}{n^\kappa} \sup_{\xi \in \partial\mathcal{T}} N_\xi^n = 0$$

for all  $\kappa > 1$  under **(H1)** and all  $\kappa > 2$  under **(H2)**.

We can readily prove Theorem 1.1.

*Proof of Theorem 1.1.* Let  $b > 0$ . As mentioned before,  $\lim_{n \rightarrow \infty} m_n \rightarrow \infty$   $\mathbb{P}^*$ -a.s under our assumptions. Since  $\sum_{n=0}^{\infty} \mathbf{1}_{\{V(\xi_n) \geq 0\}} (1 + V(\xi_n))^{-b} \leq \sum_{n=0}^{\infty} (1 + n)^{-b} N_\xi^n$ , we may conclude by Theorem 1.5.  $\square$

To study the critical cases of Theorem 1.5, we will restrict our attention to some integer-valued right-continuous branching random walks. Recall that **(H1')** is the assumption **(H1)** together with the positivity of  $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx)$  and  $\mu((-\infty, 0)) > 0$ .

**Theorem 1.6.** *We suppose that  $\mu$  is supported on  $\mathbb{Z}_- \cup \{1\}$  and*

$$\mathbb{E} \left[ \sum_{|u|, |v|=1, u \neq v} e^{-t^*V(u)} e^{-t^*V(v)} \right] < \infty.$$

Notice that  $N_\xi^k$  defined in (1.7) is now simply equal to  $\sum_{n=0}^{\infty} \mathbf{1}_{\{V(\xi_n)=k\}}$  when  $k \geq 0$ .

(i) Under **(H1')**,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sup_{\xi \in \partial\mathcal{T}} N_\xi^n = -\frac{t^*}{\log q}, \quad \mathbb{P}^*\text{-a.s.},$$

where  $q \in (0, 1)$  is given by (5.3).

(ii) Under **(H2)**, then

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sup_{\xi \in \partial\mathcal{T}} N_\xi^n = \frac{t^*}{2\theta}, \quad \mathbb{P}^*\text{-a.s.},$$

where  $\theta$  is given by (5.4).

It implies that, for such branching random walks,

$$\sup_{\xi \in \partial\mathcal{T}} \limsup_{n \rightarrow \infty} \frac{1}{n} N_\xi^n = -\frac{t^*}{\log q}, \quad \text{under (H1')},$$

and

$$\sup_{\xi \in \partial\mathcal{T}} \limsup_{n \rightarrow \infty} \frac{1}{n^2} N_\xi^n = \frac{t^*}{2\theta}, \quad \text{under (H2)}.$$

One can ask the same question for  $\liminf$  instead of  $\limsup$ . Theorem 1.2 suggests that the renormalisation should be different under its conditions.

**Open question 1.**

(i) Under the assumptions of Theorem 1.2, is  $\sup_{\xi \in \partial \mathcal{T}} \liminf_{n \rightarrow \infty} \frac{\log n}{n} N_{\xi}^n \in (0, \infty)$ ? What is its value?

(ii) Under the assumptions of Theorem 1.6 (ii), is  $\sup_{\xi \in \partial \mathcal{T}} \liminf_{n \rightarrow \infty} \frac{1}{n^2} N_{\xi}^n \in (0, \infty)$ ? What is its value?

**Open question 2.** In Theorem 1.2, what is the value of  $a_1$ ?

**Open question 3.** Let  $a > 0$ . Under the conditions of Theorem 1.6 (ii), what is the Hausdorff dimension of the rays  $\xi \in \partial \mathcal{T}$  such that  $\limsup_{n \rightarrow \infty} \frac{1}{n^2} N_{\xi}^n = a$ ? Same question with  $\liminf$  instead of  $\limsup$ .

The rest of the paper is organized as follows: Section 2 introduces the usual many-to-one formula. Theorem 1.5, Theorem 1.3, Theorem 1.6 are proved in Section 3, Section 4 and Section 5, respectively. A common tool in these proofs is the construction of some suitable optional lines in the sense of [10] by using the local times of the branching random walk. The proof of Theorem 1.2 is given in Section 6 and relies on a certain inhomogeneous Galton–Watson process, building on the fact that under (H1') and some additional assumptions, the minimum of the branching random walk is achieved by a large drop (see [5]). The Appendix A contains some estimates on random walks which are used in the course of the proof of Theorem 1.6.

Throughout this paper,  $c, c', c''$ , eventually with some subscripts, denote positive constants whose values may vary from one line to another.

**Acknowledgments.** We thank Nicolas Curien for introducing us to this question and explaining the link with self-similar Markov trees.

*Shortly before completion of this work, we learnt that Bastien Mallein (personal communication) proved that  $M < \infty$  in the case of the branching Brownian motion by a similar argument, hence solving the question of Nicolas Curien in that case.*

## 2. MANY-TO-ONE FORMULA

Since  $\varphi(t^*) = 0$ ,  $M_n := \sum_{|u|=n} e^{-t^*V(u)}$  defines a martingale. For  $u \in \mathcal{T}$  and  $0 \leq k \leq |u|$ , we let  $u_k$  be the ancestor of  $u$  at generation  $k$ . In particular  $u_0 = \emptyset$ . We recall a very convenient tool in the study of branching random walks, namely the many-to-one formula, see [20, Theorem 1.1]:

**Fact 2.1** (The many-to-one formula). *Under (H1) or (H2), there exists a one-dimensional random walk  $(S_k)_{k \geq 0}$  such that  $S_0 = 0$  and for any  $k \geq 1$  and any measurable function  $f : \mathbb{R}^k \rightarrow \mathbb{R}_+$ , we have*

$$(2.1) \quad \mathbb{E} \left[ \sum_{|u|=k} f(V(u_1), \dots, V(u_k)) \right] = \mathbb{E} \left[ e^{t^*S_k} f(S_1, \dots, S_k) \right],$$

where the step distribution of  $S$  is given by

$$\mathbb{E}[h(S_1)] = \mathbb{E} \left[ \sum_{|u|=1} e^{-t^*V(u)} h(V(u)) \right],$$

for any Borel bounded function  $h$ . In particular,

$$\mathbb{E}[S_1] = \mathbb{E} \left[ \sum_{|u|=1} e^{-t^*V(u)} V(u) \right] = \int_{\mathbb{R}} x e^{-t^*x} \mu(dx).$$

For  $x \in \mathbb{R}$ , we write  $\mathbb{P}_x$  for a probability distribution under which the random walk  $(S_n)_{n \geq 0}$  starts at  $x$ . Notice that  $\mathbb{E}[S_1] \neq 0$  under (H1) whereas  $\mathbb{E}[S_1] = 0$  under (H2). Under (H2),  $(S_n)_{n \geq 0}$  is a centered random walk with finite variance so that by Lawler and Limic [18, Theorem 5.1.7]: there exists some positive constant  $c$  such that for any  $a, b > 0$  and  $x \in (-a, b)$ ,

$$(2.2) \quad \mathbb{P}_x(T_{(-\infty, -a]} < T_{[b, \infty)}) \geq c \frac{b - x + 1}{b + a + 1}$$

with the notation

$$(2.3) \quad T_A := \inf\{i \geq 1 : S_i \in A\}$$

for any Borel set  $A$ .

The many-to-one formula can be extended to optional lines in the sense of [10, Section 6]. In our applications, an optional line  $\mathcal{L}$  will be of the form

$$(2.4) \quad \mathcal{L} = \{\xi_k, (\xi, k) \in \partial\mathcal{T} \times \mathbb{N} \text{ such that } \tau(\xi) = k\}$$

where  $\tau(\xi) := \inf\{k \geq 0 : (V(\xi_0), V(\xi_1), \dots, V(\xi_k)) \in A_k\}$  and for each  $k \geq 0$ ,  $A_k$  is a Borel set of  $\mathbb{R}^{k+1}$ . A consequence of the many-to-one formula is that

$$(2.5) \quad \mathbb{E}[\#\mathcal{L}] = \mathbb{E}[e^{t^* S_\tau} \mathbf{1}_{\{\tau < \infty\}}]$$

where  $\tau := \inf\{k \geq 0 : (S_0, S_1, \dots, S_k) \in A_k\}$ . Here and before, we used the convention that  $\inf \emptyset = \infty$ . Finally, consider

$$(2.6) \quad M_{\mathcal{L}} := \sum_{u \in \mathcal{L}} e^{-t^* V(u)}.$$

Its first moment is

$$(2.7) \quad \mathbb{E}[M_{\mathcal{L}}] = \mathbb{P}(\tau < \infty).$$

It is a consequence of the many-to-one formula. We will need to compute its second moment. For real numbers  $s_0, \dots, s_k, s_{k+1}$ , let

$$(2.8) \quad p(s_0, \dots, s_k, s_{k+1}) := \mathbb{P}(k+1 \leq \tau < \infty \mid S_0 = s_0, \dots, S_k = s_k, S_{k+1} = s_{k+1})$$

and

$$(2.9) \quad \psi(s_0, \dots, s_k) := \mathbb{E} \left[ \sum_{|u|, |v|=1, u \neq v} e^{-t^* V(u)} e^{-t^* V(v)} p_u p_v \right]$$

where  $p_u = p(s_0, \dots, s_k, s_k + V(u))$ ,  $p_v = p(s_0, \dots, s_k, s_k + V(v))$ . With this notation,

$$(2.10) \quad \mathbb{E}[(M_{\mathcal{L}})^2] = \mathbb{E} [e^{-t^* S_\tau} \mathbf{1}_{\{\tau < \infty\}}] + \mathbb{E} \left[ \sum_{k=0}^{\tau-1} e^{-t^* S_k} \psi(S_0, \dots, S_k) \right].$$

Let us prove it. For  $w \in \mathcal{T}$ , we write  $w < \mathcal{L}$  if  $(V(w_0), V(w_1), \dots, V(w_k)) \notin A_k$  for all  $k \leq |w|$ . By decomposing  $(M_{\mathcal{L}})^2 = \sum_{x, y \in \mathcal{L}} e^{-t^* V(x)} e^{-t^* V(y)}$  with respect to the most recent common ancestor  $w$  of  $x$  and  $y$ , we get

$$(M_{\mathcal{L}})^2 = \sum_{w \in \mathcal{L}} e^{-2t^* V(w)} + \sum_{w < \mathcal{L}} \sum_{u \neq v \text{ children of } w} M_{\mathcal{L}}^u M_{\mathcal{L}}^v$$

with  $M_{\mathcal{L}}^u := \sum_{x \in \mathcal{L}^u} e^{-t^* V(x)}$  for any  $u \in \mathcal{T}$ , where  $\mathcal{L}^u$  is the set of vertices  $x$  descendants of  $u$  which belong to  $\mathcal{L}$ . By the branching property and (2.7),

$$\mathbb{E} \left[ \sum_{w < \mathcal{L}} \sum_{u \neq v \text{ children of } w} M_{\mathcal{L}}^u M_{\mathcal{L}}^v \right] = \mathbb{E} \left[ \sum_{w < \mathcal{L}} \sum_{u \neq v \text{ children of } w} e^{-t^* V(u)} e^{-t^* V(v)} \tilde{p}_u \tilde{p}_v \right]$$

with  $\tilde{p}_u := p(V(u_0), V(u_1), \dots, V(u))$  and the analog for  $\tilde{p}_v$ . Another use of the branching property shows that

$$\mathbb{E} \left[ \sum_{w < \mathcal{L}} \sum_{u \neq v \text{ children of } w} e^{-t^*V(u)} e^{-t^*V(v)} \tilde{p}_u \tilde{p}_v \right] = \mathbb{E} \left[ \sum_{w < \mathcal{L}} e^{-2t^*V(w)} \psi(V(w_0), V(w_1), \dots, V(w)) \right].$$

With another use of the many-to-one formula,

$$\begin{aligned} \mathbb{E} \left[ \sum_{w \in \mathcal{L}} e^{-2t^*V(w)} \right] &= \mathbb{E} \left[ e^{-t^*S_\tau} \mathbf{1}_{\{\tau < \infty\}} \right], \\ \mathbb{E} \left[ \sum_{w < \mathcal{L}} e^{-2t^*V(w)} \psi(V(w_0), V(w_1), \dots, V(w)) \right] &= \mathbb{E} \left[ \sum_{k=0}^{\tau-1} e^{-t^*S_k} \psi(S_0, \dots, S_k) \right]. \end{aligned}$$

It yields (2.10).

### 3. PROOF OF THEOREM 1.5

It is enough to prove that  $\mathbb{P}^*$ -a.s. for  $n$  large enough,

$$(3.1) \quad \sup_{\xi \in \partial \mathcal{T}} N_\xi^n \leq n^\kappa$$

for any  $\kappa > 1$  under (H1) and  $\kappa > 2$  under (H2). Fix such a  $\kappa$ . Let  $a > 0$  be a small constant whose value will be determined later. Without loss of generality, we will prove (3.1) where  $N_\xi^n$  is rather defined as

$$N_\xi^n = \sum_{\ell=0}^{\infty} \mathbf{1}_{\{V(\xi_\ell) \in [an, a(n+1)]\}}.$$

For any  $u \in \mathcal{T}$  and  $k \in \mathbb{Z}$ , we let

$$N_u^n := \sum_{\ell=0}^{|u|} \mathbf{1}_{\{V(u_\ell) \in [an, a(n+1)]\}}.$$

We first prove (3.1) under (H1). We introduce the following set

$$\mathcal{L}_n := \{u \in \mathcal{T} : N_u^n \geq \lfloor n^\kappa \rfloor, \max_{0 \leq k < |u|} N_{u_k}^n < \lfloor n^\kappa \rfloor\}$$

which is the optional line of the particles in the branching random walk stopped when they first visited  $\lfloor n^\kappa \rfloor$  times the set  $[an, a(n+1)]$ . It corresponds to (2.4) with  $\tau(\xi)$  given by

$$(3.2) \quad \tau_n(\xi) := \inf\{k \geq 0 : N_{\xi_k}^n \geq \lfloor n^\kappa \rfloor\}.$$

Note that

$$(3.3) \quad \left\{ \sup_{\xi \in \partial \mathcal{T}} N_\xi^n \geq n^\kappa \right\} \subset \{\mathcal{L}_n \neq \emptyset\}.$$

Equation (2.5) yields that

$$(3.4) \quad \mathbb{E}(\#\mathcal{L}_n) = \mathbb{E}(e^{t^*S_{\tau_n}} \mathbf{1}_{\{\tau_n < \infty\}}),$$

where

$$(3.5) \quad \tau_n := \inf \left\{ i \geq 0 : \sum_{\ell=0}^i \mathbf{1}_{\{S_\ell \in [an, a(n+1)]\}} \geq \lfloor n^\kappa \rfloor \right\}.$$



We have  $S_{\tau_n} \in [an, a(n+1)]$  on  $\{\tau_n < \infty\}$ . If we set  $\Sigma_n := \sum_{\ell=0}^{\infty} 1_{\{S_\ell \in [an, a(n+1)]\}}$ , then

$$(3.6) \quad \mathbb{P}(\mathcal{L}_n \neq \emptyset) \leq \mathbb{E} [\#\mathcal{L}_n] \leq e^{at^*(n+1)} \mathbb{P}(\Sigma_n \geq \lfloor n^\kappa \rfloor).$$

Under (H1), the random walk  $(S_n)_{n \geq 0}$  either drifts to  $+\infty$  a.s. or to  $-\infty$  a.s. according to  $\mathbb{E}[S_1] > 0$  or  $\mathbb{E}[S_1] < 0$ . Suppose for example that  $S_n \rightarrow \infty$  a.s. In particular  $\mathbb{P}(S_n \geq 0, \forall n \geq 0) > 0$ . Choose  $a > 0$  such that  $\mathbb{P}(S_1 > a) > 0$ . For any  $n \geq 0$  and  $x \in [an, a(n+1)]$ ,

$$\mathbb{P}_x(T_{[an, a(n+1)]} = \infty) \geq \mathbb{P}(S_n > a, \forall n \geq 1) =: c > 0$$

with the notation (2.3) for  $T_{[an, a(n+1)]}$ . By the Markov property, for any  $\ell \geq 1$ ,  $\mathbb{P}(\Sigma_n \geq \ell) \leq (1-c)^{\ell-1}$ . In view of (3.6), using that  $\kappa > 1$ , we get that  $\sum_n \mathbb{P}(\mathcal{L}_n \neq \emptyset)$  is summable. The Borel–Cantelli lemma implies that  $\mathcal{L}_n = \emptyset$  for  $n$  large enough, and we deduce from (3.3) that (3.1) holds. The same reasoning deals with the case  $S_n \rightarrow -\infty$ .

It remains to prove (3.1) under (H2). We apply the same arguments using the standard strategy of introducing a barrier. Recall that under (H2),  $m_n \rightarrow +\infty$   $\mathbb{P}^*$ -a.s. Let  $c > 0$ . Now  $\mathcal{L}_n = \mathcal{L}_n(c)$  is defined as

$$\mathcal{L}_n := \{u \in \mathcal{T} : N_u^n \geq \lfloor n^\kappa \rfloor, \max_{0 \leq k < |u|} N_{u_k}^n < \lfloor n^\kappa \rfloor, \min_{0 \leq k \leq |u|} V(u_k) > -c\}.$$

It is of the form (2.4) with  $\tau(\xi) = \tau_n(\xi)$  of (3.2) if  $\min_{0 \leq k \leq \tau_n(\xi)} V(\xi_k) > -c$  and  $\tau(\xi) = \infty$  otherwise. Using (3.5) for the definition of  $\tau_n$ , the many-to-one formula (2.5) tells us that

$$\mathbb{P}(\mathcal{L}_n \neq \emptyset) \leq \mathbb{E} [\#\mathcal{L}_n] = \mathbb{E} [e^{t^* S_{\tau_n}} 1_{\{\min_{0 \leq i \leq \tau_n} S_i > -c\}}] \leq e^{at^*(n+1)} \mathbb{P}\left(\min_{0 \leq i \leq \tau_n} S_i > -c\right).$$

With the notation (2.3) and  $\Sigma_n^{(c)} := \sum_{\ell=0}^{T_{(-\infty, -c]}} 1_{\{S_\ell \in [an, a(n+1)]\}}$ , we get

$$\mathbb{P}(\mathcal{L}_n \neq \emptyset) \leq e^{at^*(n+1)} \mathbb{P}(\Sigma_n^{(c)} \geq \lfloor n^\kappa \rfloor).$$

We now choose  $a > 0$  small enough such that  $\mathbb{P}(S_1 < -a) > 0$ . By (2.2), there exists  $c_1 > 0$  such that for any  $n \geq 1$  and  $x \in [an, a(n+1)]$ ,

$$\mathbb{P}_x(T_{(-\infty, -c]} < T_{[an, a(n+1)]}) \geq \frac{c_1}{n}.$$

This, in view of the Markov property, implies that for any  $\ell \geq 1$ ,

$$\mathbb{P}(\Sigma_n^{(c)} \geq \ell) \leq \left(1 - \frac{c_1}{n}\right)^{\ell-1}.$$

Hence we get that

$$\mathbb{P}(\mathcal{L}_n \neq \emptyset) \leq e^{at^*(n+1)} e^{-c_1(\lfloor n^\kappa \rfloor - 1)n^{-1}}.$$

For any  $\kappa > 2$ , the sum over  $n$  converges. The Borel–Cantelli lemma yields that almost surely, for all large  $n$ ,  $\mathcal{L}_n = \emptyset$ , hence  $\sup_{\xi \in \partial \mathcal{T}} N_\xi^n \leq n^\kappa$  on the event  $\{\inf_{u \in \mathcal{T}} V(u) > -c\}$ . Since  $\mathbb{P}^*$ -a.s.  $\inf_{u \in \mathcal{T}} V(u) > -\infty$ , we conclude by letting  $c \rightarrow \infty$ .  $\square$

## 4. PROOF OF THEOREM 1.3

4.1. **Proof of Part (i).** It is a consequence of the following lemma.

**Lemma 4.1.** *Under the conditions of Theorem 1.3 (i),  $\sup_{u \in \mathcal{T}} e^{-V(u)} \eta_u = +\infty$   $\mathbb{P}^*$ -a.s.*

*Proof of the lemma.* Let  $b > 0$  be small enough such that  $t^* - 2b > \zeta(1+b)$ . For a point process  $\widehat{\mathfrak{N}} \leq \mathfrak{N}$ , let  $\widehat{\mu} := \mathbb{E}[\widehat{\mathfrak{N}}]$ ,  $\widehat{\varphi}(t) := \log \int_{\mathbb{R}} e^{-tx} \widehat{\mu}(dx)$ . Since  $\varphi(t^*) = 0$  and  $\varphi(t) > 0$  for all  $t < t^*$ , monotone convergence implies that we can choose  $\widehat{\mathfrak{N}} \leq \mathfrak{N}$  such that:

- (i) there exists  $a > 0$  such that  $\widehat{\mathfrak{N}}(\mathbb{R}) \leq a$  a.s.,

- (ii) the support of  $\widehat{\aleph}$  is in  $(-a, a)$  a.s.,
- (iii)  $\widehat{\varphi}(0) > 0$ ,
- (iv)  $\widehat{\varphi}(s) = 0$  for some  $s \in (0, t^*)$  which satisfies  $s - 2b > \zeta(1 + b)$ .

We let  $(V(u), u \in \widehat{\mathcal{T}})$  be the branching random walk associated to the point process  $\widehat{\aleph}$ , which is contained in the original branching random walk  $(V(u), u \in \mathcal{T})$ , i.e.  $\widehat{\mathcal{T}} \subset \mathcal{T}$ . Necessarily  $\widehat{\varphi}(t^*) < 0$ . Since  $s\widehat{\varphi}'(s) < 0 = \widehat{\varphi}(s)$ , Biggins' theorem [8] implies that  $\sum_{|u|=n, u \in \widehat{\mathcal{T}}} e^{-sV(u)}$  converges a.s. and in  $L^1$  to some random variable  $\widehat{W}$  which is positive on  $\{\partial\widehat{\mathcal{T}} \neq \emptyset\}$ . Let

$$\widehat{\mathcal{L}}_k := \{u \in \widehat{\mathcal{T}} : V(u) \geq k, \max_{0 \leq j < |u|} V(u_j) < k\}$$

be the set of particles which first cross level  $k$ . Notice that for any  $u \in \widehat{\mathcal{L}}_k$ ,  $V(u) \in [k, k+a]$ . Theorem 6.1 in [10] shows that  $\sum_{u \in \widehat{\mathcal{L}}_k} e^{-sV(u)}$  also converges a.s. to  $\widehat{W}$ . Since

$$\sum_{u \in \widehat{\mathcal{L}}_k} e^{-sV(u)} \leq e^{-sk} \#\widehat{\mathcal{L}}_k,$$

we deduce that a.s. on  $\{\widehat{W} > 0\}$ , for  $k$  large enough,

$$\#\widehat{\mathcal{L}}_k \geq e^{(s-b)k}.$$

On the other hand, (1.6) implies the existence of a constant  $c > 0$  such that for any  $x \geq 1$ ,  $\mathbb{P}(\eta \leq x) \leq 1 - cx^{-\zeta-\varepsilon}$  where  $\varepsilon > 0$  is such that  $(1+b)(\zeta+\varepsilon) = s-2b$ . By the branching property applied to the optional line  $\widehat{\mathcal{L}}_k$ ,

$$\mathbb{P}\left(\forall u \in \widehat{\mathcal{L}}_k, \eta_u \leq e^{(1+b)k}, \#\widehat{\mathcal{L}}_k \geq e^{(s-b)k}\right) \leq (1 - ce^{-(s-2b)k})^{e^{(s-b)k}} \leq e^{-ce^{bk}}.$$

The right-hand side is summable in  $k$ , hence the Borel–Cantelli lemma shows that a.s. on  $\{\widehat{W} > 0\}$ , for all  $k$  large enough one can find  $u \in \widehat{\mathcal{L}}_k$  such that  $\eta_u > e^{(1+b)k}$ . For such a  $u$ ,  $e^{-V(u)}\eta_u \geq e^{-k-a}e^{(1+b)k} = e^{bk-a}$ . Therefore  $\mathbb{P}\left(\sup_{u \in \mathcal{T}} e^{-V(u)}\eta_u = \infty\right) \geq \mathbb{P}\left(\widehat{W} > 0\right) > 0$ . Since  $\{\sup_{u \in \mathcal{T}} e^{-V(u)}\eta_u = \infty\}$  is an inherited property,  $\mathbb{P}^*(\sup_{u \in \mathcal{T}} e^{-V(u)}\eta_u = \infty) = 1$ .  $\square$

**4.2. Proof of Part (ii).** Suppose now that  $\zeta > t^*$ . Let  $r \in (t^*, \zeta)$  and  $b \in (0, 1)$  be such that  $t^* - br < 0$ . By (1.6), there exists a constant  $c > 0$  such that for all  $x \geq 1$ ,

$$(4.1) \quad \mathbb{P}(\eta \geq x) \leq cx^{-r}.$$

By Theorem 1.5 (already proved in Section 3), there exists  $\kappa > 0$  such that  $\sup_{\xi} N_{\xi}^n \leq \lfloor n^{\kappa} \rfloor$  for all  $n$  large enough. Since

$$D(\xi) = \sum_{n=0}^{\infty} e^{-V(\xi_n)} \eta_{\xi_n} \leq \sum_{n=0}^{\infty} e^{-n} \sum_{u \in \xi : n \leq V(u) < n+1} \eta_u \leq \sum_{n=0}^{\infty} e^{-n} N_{\xi}^n \sup_{u \in \mathcal{T}} \eta_u \mathbf{1}_{\{n \leq V(u) < n+1\}},$$

it is enough to show that for  $n$  large enough, for all  $u \in \mathcal{T}$ ,

$$(4.2) \quad \eta_u \mathbf{1}_{\{n \leq V(u) < n+1\}} \leq e^{bn}.$$

For  $u \in \mathcal{T}$  and  $n \geq 0$ , define  $N_u^n := \sum_{\ell=0}^{|u|} \mathbf{1}_{\{n \leq V(u_{\ell}) < n+1\}}$ . For  $n \geq 0$  and  $k \geq 1$ , we consider the optional line

$$\mathcal{L}_n(k) := \{u \in \mathcal{T} : \max_{0 \leq j < |u|} N_{u_j}^n < k, N_u^n = k\}.$$

It is the optional line (2.4) associated to  $\tau(\xi) = \inf\{\ell \geq 0 : N_{\xi_\ell}^n = k\}$ . The many-to-one formula (2.5) implies that  $\mathbb{E}[\#\mathcal{L}_n(k)] \leq e^{t^*(n+1)}$ . By the branching property and the union bound, we get that

$$\mathbb{P}(\exists u \in \mathcal{L}_n(k) : \eta_u \geq e^{bn}) \leq e^{t^*(n+1)} \mathbb{P}(\eta \geq e^{bn}) \leq ce^{t^*(n+1)-brn}$$

by (4.1). The sum on the right-hand side over  $k \leq \lfloor n^\kappa \rfloor$  and  $n \geq 0$  is finite. The Borel–Cantelli lemma implies that for  $n$  large enough,  $\eta_u \mathbf{1}_{\{n \leq V(u) < n+1, N_u^n \leq \lfloor n^\kappa \rfloor\}} \leq e^{bn}$  for all  $u \in \mathcal{T}$  a.s. Since  $\sup_{u \in \mathcal{T}} N_u^n \leq \lfloor n^\kappa \rfloor$  a.s. for  $n$  large, we proved (4.2).  $\square$

**Remark 4.2.** *Suppose that  $\zeta > t^*$ . For  $\xi, \xi' \in \partial\mathcal{T}$ , let  $d_{\partial\mathcal{T}}(\xi, \xi') := 2^{-|\xi \wedge \xi'|}$  be the usual ultra metric, where  $|\xi \wedge \xi'| := \sup\{n \geq 0 : \xi_n = \xi'_n\}$ . Recall from (1.5) that  $D(\xi) = \sum_{n=0}^{\infty} e^{-V(\xi_n)} \eta_{\xi_n}$ . Since  $|D(\xi) - D(\xi')| \leq \sum_{n=|\xi \wedge \xi'|+1}^{\infty} (e^{-V(\xi_n)} \eta_{\xi_n} + e^{-V(\xi'_n)} \eta_{\xi'_n})$ , an application of (4.2) yields that  $\mathbb{P}^*$ -a.s.,  $D(\xi^{(k)}) \rightarrow D(\xi)$  for  $k \rightarrow \infty$ , if  $d_{\partial\mathcal{T}}(\xi, \xi^{(k)}) \rightarrow 0$ . In other words,  $\mathbb{P}^*$ -a.s.,  $\xi \mapsto D(\xi)$  is continuous on  $\partial\mathcal{T}$ .*

## 5. PROOF OF THEOREM 1.6

Recall that  $(S_n, n \geq 0)$  is associated to the branching random walk through (2.1). From the assumptions on  $\mu$ ,  $(S_n, n \geq 0)$  is integer-valued and  $\mathbb{P}(S_1 \geq 2) = 0$ . We introduce the local times

$$(5.1) \quad \ell_n := \sum_{j=0}^n \mathbf{1}_{\{S_j=0\}}$$

and the inverse local times defined by  $\sigma_0 := 0$  and for  $j \geq 1$

$$(5.2) \quad \sigma_j := \inf\{k > \sigma_{j-1} : S_k = 0\}$$

with the convention that  $\inf \emptyset = \infty$ . In the case (i) (namely under (H1')),  $\mathbb{E}[S_1] > 0$  and  $\mathbb{P}(S_1 < 0) > 0$ . We let

$$(5.3) \quad q := \mathbb{P}(\sigma_1 < \infty) \in (0, 1).$$

In the case (ii) (namely under (H2)), the random walk  $(S_n, n \geq 0)$  is centered with finite variance and by Lemma A.2 there exists some  $\theta > 0$  such that

$$(5.4) \quad q(n) := \mathbb{P}\left(\min_{0 \leq i \leq \sigma_1} S_i \leq -n\right) \sim \frac{\theta}{n}, \quad n \rightarrow \infty.$$

Recall that the minimum of the branching random walk goes to  $\infty$ ,  $\mathbb{P}^*$ -a.s. in both cases. Notice that the sets  $\{u \in \mathcal{T} : V(u) = k, \max_{0 \leq j < |u|} V(u_j) < k\}$  indexed by  $k \geq 0$  form a Galton–Watson process. According to (2.5), its mean offspring is  $e^{t^*}$ . In particular, the line  $\{u \in \mathcal{T} : V(u) = n, \max_{0 \leq j < |u|} V(u_j) < n\}$  has size  $e^{t^*n+o(n)}$  as  $n \rightarrow \infty$   $\mathbb{P}^*$ -a.s. By considering the descendants of these particles, we see that in order to prove that  $\sup_{\xi \in \partial\mathcal{T}} N_\xi^n \leq dn$  as  $n \rightarrow \infty$ , it is enough to show that there exists  $\varepsilon_d > 0$  such that for  $n$  large enough

$$(5.5) \quad \mathbb{P}(N_* > dn) \leq e^{-t^*n - \varepsilon_d n}$$

where  $N_* := \sup_{\xi \in \partial\mathcal{T}} N_\xi^0$ . Indeed, it would be a consequence of the Borel–Cantelli lemma. Similarly, to prove that  $\sup_{\xi \in \partial\mathcal{T}} N_\xi^n \geq dn$  as  $n \rightarrow \infty$ , it is enough to show that there exists  $\varepsilon'_d > 0$  such that for  $n$  large enough

$$(5.6) \quad \mathbb{P}(N_* > dn) \geq e^{-t^*n + \varepsilon'_d n}.$$

For  $u \in \mathcal{T}$  and  $k \geq 0$ , we let  $N_u^k := \sum_{i=0}^{|u|} \mathbf{1}_{\{V(u_i)=k\}}$ .

5.1. **Proof of Theorem 1.6 (i).** Let  $d > -\frac{t^*}{\log q}$ . We want to prove (5.5). For notational brevity, we treat  $dn$  as an integer. We consider the optional line

$$\mathcal{L}_n := \{u \in \mathcal{T} : N_u^0 > dn, \max_{0 \leq j < |u|} N_{u_j}^0 \leq dn\}$$

which is the set of particles stopped when their local time at level 0 exceeds  $dn$ . It is of the form (2.4) with  $\tau(\xi)$  given by

$$(5.7) \quad \sigma_{dn}(\xi) := \inf\{k \geq 0 : N_{\xi_k}^0 > dn\}.$$

The many-to-one formula (2.5) implies that

$$\mathbb{E}[\#\mathcal{L}_n] = \mathbb{P}(\sigma_{dn} < \infty) = q^{dn}$$

with the notation (5.2). Then (5.5) follows from  $\mathbb{P}(N_* > dn) \leq \mathbb{P}(\mathcal{L}_n \neq \emptyset) \leq \mathbb{E}[\#\mathcal{L}_n]$ .

We turn to the lower bound. Let  $d \in (0, -\frac{t^*}{\log q})$ . We want to prove (5.6). Take  $K \geq 1$  large enough such that  $e^{t^*} q_K^d > 1$  where

$$q_K := \mathbb{P}(\sigma_1 < \infty, \min_{0 \leq n \leq \sigma_1} S_n \geq -K).$$

We let

$$\mathcal{L}_n(K) := \{u \in \mathcal{T} : N_u^0 > dn, \max_{0 \leq j < |u|} N_{u_j}^0 \leq dn, \min_{0 \leq j \leq |u|} V(u_j) \geq -K\}$$

so that if  $\mathcal{L}_n(K)$  is nonempty then  $N_* > dn$ . Using the notation (5.7),  $\mathcal{L}_n(K)$  is of the form (2.4) with  $\tau(\xi) = \sigma_{dn}(\xi)$  if  $\min_{0 \leq j \leq \sigma_{dn}(\xi)} V(\xi_j) \geq -K$ , and  $\tau(\xi) = \infty$  otherwise. We introduce

$$M_{\mathcal{L}_n(K)} := \sum_{u \in \mathcal{L}_n(K)} e^{-t^*V(u)}.$$

We have by (2.7)

$$(5.8) \quad \mathbb{E}[M_{\mathcal{L}_n(K)}] = \mathbb{P}(\tau < \infty) = \mathbb{P}(\sigma_{dn} < \infty, \min_{0 \leq j \leq \sigma_{dn}} S_j \geq -K) = q_K^{dn}$$

where now  $\tau = \sigma_{dn}$  if  $\min_{0 \leq j \leq \sigma_{dn}} S_j \geq -K$ , and  $\tau = \infty$  otherwise. Recall the notation  $p(s_0, \dots, s_{k+1})$  and  $\psi(s_0, \dots, s_k)$  in (2.8) and (2.9), which read in our setting

$$\begin{aligned} p(s_0, \dots, s_k, s_{k+1}) &= \mathbb{P}(\tau < \infty \mid S_0 = s_0, \dots, S_k = s_k, S_{k+1} = s_{k+1}) \\ &= \mathbb{P}(\sigma_{dn} < \infty, \min_{0 \leq j \leq \sigma_{dn}} S_j \geq -K \mid S_0 = s_0, \dots, S_k = s_k, S_{k+1} = s_{k+1}) \end{aligned}$$

and

$$\psi(s_0, \dots, s_k) = \mathbb{E} \left[ \sum_{|u|, |v|=1, u \neq v} e^{-t^*V(u)} e^{-t^*V(v)} p_u p_v \right]$$

where  $p_u = p(s_0, \dots, s_k, s_k + V(u))$ ,  $p_v = p(s_0, \dots, s_k, s_k + V(v))$ . The second moment of  $M_{\mathcal{L}_n(K)}$  is given by (2.10), i.e.

$$(5.9) \quad \mathbb{E}[(M_{\mathcal{L}_n(K)})^2] = \mathbb{E} [e^{-t^*S_\tau} \mathbf{1}_{\{\tau < \infty\}}] + \mathbb{E} \left[ \sum_{k=0}^{\tau-1} e^{-t^*S_k} \psi(S_0, \dots, S_k) \right].$$

Notice that

$$\psi(S_0, \dots, S_k) \mathbf{1}_{\{k < \tau\}} = \psi(S_0, \dots, S_k) \mathbf{1}_{\{k < \sigma_{dn}, \min_{0 \leq j \leq k} S_j \geq -K\}}.$$

Indeed,  $p(s_0, \dots, s_{k+1}) = \mathbb{P}(k+1 \leq \tau < \infty \mid S_0 = s_0, \dots, S_{k+1} = s_{k+1}) = \mathbf{1}_{\{\sigma_{dn} \geq k+1\}} \mathbb{P}(\sigma_{dn} < \infty, \min_{0 \leq j \leq \sigma_{dn}} S_j \geq -K \mid S_0 = s_0, \dots, S_{k+1} = s_{k+1})$ . Therefore  $\psi(S_0, \dots, S_k) = 0$  on  $\{\sigma_{dn} \leq k\} \cup \{\min_{0 \leq j \leq k} S_j < -K\}$ , and the above equality follows.

In (5.9),

$$\mathbb{E} [e^{-t^*S_\tau} \mathbf{1}_{\{\tau < \infty\}}] = \mathbb{P}(\tau < \infty) = q_K^{dn}$$

by (5.8). The last term of (5.9) is

$$(5.10) \quad \mathbb{E} \left[ \sum_{k=0}^{\sigma_{dn}-1} e^{-t^* S_k} \psi(S_0, \dots, S_k) \mathbf{1}_{\{\min_{0 \leq j \leq k} S_j \geq -K\}} \right].$$

We bound  $p(s_0, \dots, s_k, s_{k+1})$  for  $s_0, \dots, s_k$  such that  $\ell := \sum_{j=0}^k \mathbf{1}_{\{s_j=0\}} \leq dn$  and  $\min_{0 \leq j \leq k} s_j \geq -K$ . By the strong Markov property at the first hitting time of 0 after time  $k+1$ , we see that  $p(s_0, \dots, s_k, s_{k+1}) \leq q_K^{dn-\ell}$ . Hence  $\psi(s_0, \dots, s_k) \leq c' q_K^{2(dn-\ell)}$  with

$$c' = \mathbb{E} \left[ \sum_{|u|, |v|=1, u \neq v} e^{-t^* V(u)} e^{-t^* V(v)} \right].$$

Therefore the expectation in (5.10) is less than

$$c' q_K^{2dn} \mathbb{E} \left[ \sum_{k=0}^{\sigma_{dn}-1} e^{-t^* S_k} q_K^{-2\ell_k} \mathbf{1}_{\{\min_{0 \leq j \leq k} S_j \geq -K\}} \right]$$

with the notation (5.1). Discussing over the value of  $\ell_k$  and using the strong Markov property at  $\sigma_{\ell-1}$  below, we obtain that

$$\begin{aligned} \mathbb{E} \left[ \sum_{k=0}^{\sigma_{dn}-1} e^{-t^* S_k} q_K^{-2\ell_k} \mathbf{1}_{\{\min_{0 \leq j \leq k} S_j \geq -K\}} \right] &= \sum_{\ell=1}^{dn} q_K^{-2\ell} \mathbb{E} \left[ \sum_{k=\sigma_{\ell-1}}^{\sigma_{\ell}-1} e^{-t^* S_k} \mathbf{1}_{\{\min_{0 \leq j \leq k} S_j \geq -K, \sigma_{\ell} < \infty\}} \right] \\ &= c'' \sum_{\ell=1}^{dn} q_K^{-2\ell} \mathbb{P} \left( \sigma_{\ell-1} < \infty, \min_{0 \leq j \leq \sigma_{\ell-1}} S_j \geq -K \right) \\ &= c'' \sum_{\ell=1}^{dn} q_K^{-(\ell+1)}, \end{aligned}$$

with

$$c'' := \mathbb{E} \left[ \sum_{k=0}^{\sigma_1-1} e^{-t^* S_k} \mathbf{1}_{\{\min_{0 \leq j \leq k} S_j \geq -K, \sigma_1 < \infty\}} \right].$$

By Lemma A.1,  $c'' < \infty$ . We get that the second moment of  $M_{\mathcal{L}_n(K)}$  is bounded by  $c''' q_K^{dn}$ . Recall (5.8). Since

$$\mathbb{P}(N^* > dn) \geq \mathbb{P}(\#\mathcal{L}_n(K) > 0) = \mathbb{P}(M_{\mathcal{L}_n(K)} > 0) \geq \frac{\mathbb{E} [M_{\mathcal{L}_n(K)}]^2}{\mathbb{E} [(M_{\mathcal{L}_n(K)})^2]} \geq \frac{1}{c'''} q_K^{dn},$$

and  $q_K^d > e^{-t^*}$ , we proved (5.6) and then the lower bound in Theorem 1.6 (i).

**5.2. Proof of Theorem 1.6 (ii). Upper bound.** Let  $d > \frac{t^*}{2\theta}$ . We show (5.5). We consider a family of continuous non-increasing functions  $f_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $f_n(0) = n$ ,  $f_n(t) > 0$  for  $0 \leq t < dn^2$  and  $f_n(t) = 0$  for all  $t \geq dn^2$ . We introduce the optional line

$$(5.11) \quad \mathcal{L}_n := \{u \in \mathcal{T} : V(u) \leq -f_n(N_u^0), \forall 0 \leq j < |u|, V(u_j) > -f_n(N_{u_j}^0)\}$$

where we have stopped the particles  $u$  which go below  $-f_n(N_u^0)$  for the first time. It is the optional line (2.4) associated to  $\tau(\xi)$  which is

$$(5.12) \quad \tau_n(\xi) := \inf\{k \geq 0 : V(\xi_k) \leq -f_n(N_{\xi_k}^0)\}.$$

We notice that  $\{N_* > dn^2\} \subset \{\mathcal{L}_n \neq \emptyset\}$  as  $f_n(dn^2) = 0$  so that  $\mathbb{P}(N_* > dn^2) \leq \mathbb{E}[\#\mathcal{L}_n]$ . It follows from (2.5) that

$$(5.13) \quad \mathbb{P}(N_* > dn^2) \leq \mathbb{E}[e^{t^* S_{\tau_n}}] =: b_n,$$

where  $\tau_n := \inf\{i \geq 0 : S_i \leq -f_n(\ell_i)\}$  with the notation (5.1) for the local time process  $(\ell_i)$ . Observe that  $\tau_n \leq \sigma_{\lceil dn^2 \rceil}$ , with  $\sigma_j$  defined in (5.2). In the case (ii),  $(S_n)$  is a centered random walk with finite variance, so  $\tau_n$  is finite almost surely.

To estimate  $b_n$ , we discuss on the value of  $\ell_{\tau_n}$ . Note that

$$(5.14) \quad b_n \leq \mathbb{E} \left[ e^{-t^* f_n(\ell_{\tau_n})} \right] = \sum_{k=1}^{\lceil dn^2 \rceil} e^{-t^* f_n(k)} \mathbb{P}(\ell_{\tau_n} = k)$$

where we used the fact that  $\ell_{\tau_n} \leq \lceil dn^2 \rceil$ . Recall the notation  $q(n)$  in (5.4). We observe that, for any  $k \geq 1$ ,

$$(5.15) \quad \begin{aligned} \mathbb{P}(\ell_{\tau_n} = k) &= \mathbb{P}\left(\forall 1 \leq j < k, \min_{\sigma_{j-1} \leq i \leq \sigma_j} S_i > -f_n(j), \min_{\sigma_{k-1} \leq i \leq \sigma_k} S_i \leq -f_n(k)\right) \\ &= \prod_{j=1}^{k-1} (1 - q(f_n(j))) q(f_n(k)) \end{aligned}$$

with the convention  $\prod_{\emptyset} = 1$ . Hence, for all  $k \geq 1$ ,

$$(5.16) \quad e^{-t^* f_n(k)} \mathbb{P}(\ell_{\tau_n} = k) \leq e^{-t^* f_n(k) - \sum_{j=1}^{k-1} q(f_n(j))}.$$

As  $d > \frac{t^*}{2\theta}$ , we may choose and fix some  $\theta' \in (0, \theta)$  such that  $\alpha := \sqrt{2d\frac{\theta'}{t^*}} > 1$ . By (5.4), there exists  $k_0 \geq 1$  such that for all  $k \geq k_0$ ,  $q(k) \geq \frac{\theta'}{k}$ . Let  $f(t) := \alpha \sqrt{(1 - \frac{t}{d})_+}$  and  $f_n(t) := nf(t/n^2)$  for  $t \geq 0$ . We check that for any  $t \leq dn^2$ ,

$$(5.17) \quad t^* f_n(t) + \theta' \int_0^t \frac{ds}{f_n(s)} = \alpha t^* n.$$

It implies that  $t^* f_n(k) + \theta' \sum_{j=1}^{k-1} \frac{1}{f_n(j)} \geq \alpha t^* n$  for any  $k \geq 0$ . Let  $m_n$  be the minimal integer such that  $f_n(m_n + 2) < k_0$ . We can check that  $f_n(m_n)$  is bounded in  $n$ . For  $k \leq m_n$ , we get

$$t^* f_n(k) + \sum_{j=1}^{k-1} q(f_n(j)) \geq t^* f_n(k) + \theta' \sum_{j=1}^{k-1} \frac{1}{f_n(j)} \geq \alpha t^* n - \frac{2\theta'}{k_0}$$

while for  $k > m_n$ , we write

$$t^* f_n(k) + \sum_{j=1}^{k-1} q(f_n(j)) \geq \sum_{j=1}^{m_n-1} q(f_n(j)) \geq \alpha t^* n - \frac{2\theta'}{k_0} - t^* f_n(m_n).$$

In view of (5.16), equation (5.14) becomes  $b_n \leq e^{-\alpha t^* n \lceil dn^2 \rceil} e^{\frac{2\theta'}{k_0} + t^* f_n(m_n)}$ , proving (5.5).

**5.3. Proof of Theorem 1.6 (ii). Lower bound.** Let  $0 < d < \frac{t^*}{2\theta}$ . For  $\varrho \in (d, \frac{t^*}{2\theta})$ , let  $\alpha := \sqrt{2\varrho\frac{\theta}{t^*}}$  and take  $f_n(t) = nf(t/n^2)$  with  $f(t) = \alpha \sqrt{(1 - \frac{t}{\varrho})_+}$  for  $t \geq 0$ . Note that  $\alpha < 1$ . We set  $d_n := \lceil dn^2 \rceil + 1$  for concision. Notice from (5.4) and (5.17) that uniformly in  $k \leq d_n$ ,

$$(5.18) \quad e^{-t^* f_n(k)} \prod_{j=0}^{k-1} (1 - q(f_n(j))) = e^{-\alpha t^* n + o(n)}.$$

We consider a slightly modified version of the optional line  $\mathcal{L}_n$  defined in (5.11): Let  $(\lambda_n)_{n \geq 0}$  be a sequence of positive real numbers such that  $\lambda_n = o(n)$  as  $n \rightarrow \infty$ . Consider

$$\begin{aligned} \mathcal{L}_n^{(d)} &:= \{u \in \mathcal{T} : V(u) \leq -f_n(N_u^0), |V(u) + f_n(N_u^0)| \leq \lambda_n, \\ &\quad \forall 0 \leq j < |u|, V(u_j) > -f_n(N_{u_j}^0), N_u^0 = d_n\}. \end{aligned}$$

In other words, we only select those  $u \in \mathcal{L}_n$  such that  $N_u^0 = d_n$  and the overshoot  $|V(u) + f_n(N_u^0)|$  is less than  $\lambda_n$ . If  $\mathcal{L}_n^{(d)}$  is not empty, then  $N_* \geq d_n > dn^2$ . We use the definition of  $\tau_n(\xi)$  in (5.12). The optional line  $\mathcal{L}_n^{(d)}$  is of the form (2.4) with  $\tau(\xi)$  given by  $\tau(\xi) := \tau_n(\xi)$  if  $N_{\xi_{\tau_n(\xi)}}^0 = d_n$  and  $|V(\xi_{\tau_n(\xi)}) + f_n(d_n)| \leq \lambda_n$ , and  $\tau(\xi) := \infty$  otherwise. Let

$$M_{\mathcal{L}_n^{(d)}} := \sum_{u \in \mathcal{L}_n^{(d)}} e^{-t^*V(u)}.$$

We will use (2.7), with the notation  $\tau = \tau_n$  on the event  $\{\ell_{\tau_n} = d_n, |S_{\tau_n} + f_n(d_n)| \leq \lambda_n\}$ , and  $\tau = \infty$  otherwise, where as before,  $\tau_n := \inf\{i \geq 0 : S_i \leq -f_n(\ell_i)\}$ . We then get that  $\mathbb{E} \left[ M_{\mathcal{L}_n^{(d)}} \right] = \mathbb{P}(\ell_{\tau_n} = d_n, |S_{\tau_n} + f_n(d_n)| \leq \lambda_n)$ , and by the strong Markov property at  $\sigma_{d_n}$ ,

$$\mathbb{P}(\ell_{\tau_n} = d_n, |S_{\tau_n} + f_n(d_n)| \leq \lambda_n) = \prod_{j=0}^{d_n-1} (1 - q(f_n(j))) x_n,$$

where

$$x_n := \mathbb{P} \left( T'_{f_n(d_n)} < \sigma_1, |S_{T'_{f_n(d_n)}} + f_n(d_n)| \leq \lambda_n \right)$$

with  $T'_{f_n(d_n)} := \inf\{i \geq 0 : S_i \leq -f_n(d_n)\}$ . By (A.2), if we choose  $\lambda_n := K_{\lfloor f_n(d_n) \rfloor}$  (which we will), then  $\lambda_n = o(n)$  and  $x_n \sim \frac{\theta}{f_n(d_n)} \sim q(f_n(d_n))$  as  $n \rightarrow \infty$ . Then by (5.15) and (5.18), we get that

$$(5.19) \quad \mathbb{E} \left[ M_{\mathcal{L}_n^{(d)}} \right] = e^{-\alpha t^* n + t^* f_n(d_n) + o(n)}.$$

We compute now the second moment of  $M_{\mathcal{L}_n^{(d)}}$ . Recall (2.10) and the notation  $p(s_0, \dots, s_{k+1})$  and  $\psi(s_0, \dots, s_k)$  in (2.8) and (2.9). Note that  $p(s_0, \dots, s_{k+1}) = \mathbb{P}(k+1 \leq \tau < \infty | S_0 = s_0, \dots, S_{k+1} = s_{k+1}) = \mathbf{1}_{\{\tau_n \geq k+1\}} \mathbb{P}(\ell_{\tau_n} = d_n, |S_{\tau_n} + f_n(d_n)| \leq \lambda_n | S_0 = s_0, \dots, S_{k+1} = s_{k+1})$ , one has  $\psi(s_0, \dots, s_k) = 0$  on  $\{\tau_n \leq k\} \cup \{\ell_k > d_n\}$ . Therefore

$$\mathbb{E} \left[ \sum_{k=0}^{\tau-1} e^{-t^* S_k} \psi(s_0, \dots, s_k) \right] = \mathbb{E} \left[ \sum_{k=0}^{\tau_n-1} \mathbf{1}_{\{\ell_k \leq d_n\}} e^{-t^* S_k} \psi(s_0, \dots, s_k) \right].$$

Let us compute  $p(s_0, \dots, s_{k+1})$ . Let  $\ell = \sum_{j=0}^k \mathbf{1}_{\{s_j=0\}}$ . Suppose that  $\ell \leq d_n - 1$ . By the strong Markov property of  $S$  at the first hitting time of 0 after time  $k+1$ ,  $p(s_0, \dots, s_{k+1})$  is less than

$$\mathbb{P} \left( \forall 1 \leq j < d_n(\ell), \min_{\sigma_{j-1} \leq i \leq \sigma_j} S_i > -f_n(j+\ell), \min_{\sigma_{d_n(\ell)-1} \leq i \leq \sigma_{d_n(\ell)}} S_i \leq -f_n(d_n) \right)$$

with  $d_n(\ell) := d_n - \ell$ . It is  $\prod_{j=\ell+1}^{d_n-1} (1 - q(f_n(j))) q(f_n(d_n))$ . We deduce that

$$\psi(s_0, \dots, s_k) \leq c' \left( \prod_{j=\ell+1}^{d_n-1} (1 - q(f_n(j))) q(f_n(d_n)) \right)^2$$

with  $c' := \mathbb{E} \left[ \sum_{|u|, |v|=1, u \neq v} e^{-t^*V(u)} e^{-t^*V(v)} \right]$ . Therefore, in view of (5.18), uniformly over  $s_0, \dots, s_k$ ,

$$\psi(s_0, \dots, s_k) \leq e^{2t^*(f_n(d_n) - f_n(\ell)) + o(n)}.$$



When  $\ell = d_n$ , we use the bound  $\psi(s_0, \dots, s_k) \leq c'$  since  $p(s_0, \dots, s_{k+1})$  is always smaller than 1. We get

$$\mathbb{E} \left[ \sum_{k=0}^{\tau-1} e^{-t^* S_k} \psi(S_0, \dots, S_k) \right] \leq e^{o(n)} \mathbb{E} \left[ \sum_{k=0}^{\tau_n-1} \mathbf{1}_{\{\ell_k \leq d_n\}} e^{-t^* S_k} e^{2t^*(f_n(d_n) - f_n(\ell_k))} \right].$$

We discuss on the local times  $\ell_k$  to obtain

$$\mathbb{E} \left[ \sum_{k=0}^{\tau_n-1} \mathbf{1}_{\{\ell_k \leq d_n\}} e^{-t^* S_k} e^{2t^*(f_n(d_n) - f_n(\ell_k))} \right] = \sum_{\ell=1}^{d_n} \mathbb{E} \left[ \sum_{k=\sigma_{\ell-1}}^{\sigma_{\ell}} \mathbf{1}_{\{k < \tau_n\}} e^{-t^* S_k} \right] e^{2t^*(f_n(d_n) - f_n(\ell))}.$$

By the strong Markov property,

$$\begin{aligned} \mathbb{E} \left[ \sum_{k=\sigma_{\ell-1}}^{\sigma_{\ell}} \mathbf{1}_{\{k < \tau_n\}} e^{-t^* S_k} \right] &\leq \mathbb{P}(\tau_n \geq \sigma_{\ell-1}) \mathbb{E} \left[ \sum_{k=0}^{\infty} \mathbf{1}_{\{\min_{0 \leq i \leq k} S_i \geq -f_n(\ell)\}} e^{-t^* S_k} \right] \\ &\leq c'' \mathbb{P}(\tau_n \geq \sigma_{\ell-1}) e^{t^* f_n(\ell)} \end{aligned}$$

with  $c'' := \sup_{x \geq 0} \mathbb{E} \left[ \sum_{k=0}^{\infty} \mathbf{1}_{\{\min_{0 \leq i \leq k} S_i + x \geq 0\}} e^{-t^* S_k - t^* x} \right]$  which is finite by Lemma A.1. Since  $\mathbb{P}(\tau_n \geq \sigma_{\ell-1}) = \prod_{j=1}^{\ell-1} (1 - q(f_n(j)))$ , we obtain by (5.18)  $\mathbb{P}(\tau_n \geq \sigma_{\ell-1}) e^{t^* f_n(\ell)} = e^{-\alpha t^* n + 2t^* f_n(\ell) + o(n)}$  hence

$$\mathbb{E} \left[ \sum_{k=0}^{\tau-1} e^{-t^* S_k} \psi(S_0, \dots, S_k) \right] \leq e^{-\alpha t^* n + 2t^* f_n(d_n) + o(n)}.$$

Moreover  $\mathbb{E} \left[ e^{-t^* S_{\tau}} \mathbf{1}_{\{\tau < \infty\}} \right] = \mathbb{E} \left[ e^{-t^* S_{\tau_n}} \mathbf{1}_{\{\ell_{\tau_n} = d_n, |S_{\tau_n} + f_n(d_n)| \leq \lambda_n\}} \right] \leq e^{t^* f_n(d_n) + \lambda_n} \mathbb{P}(\ell_{\tau_n} = d_n)$ . By (5.15) and (5.18),  $\mathbb{P}(\ell_{\tau_n} = d_n) = e^{-\alpha t^* n + t^* f_n(d_n) + o(n)}$ . Recall that  $\lambda_n = o(n)$ . Therefore (2.10) implies  $\mathbb{E}[(M_{\mathcal{L}_n^{(d)}})^2] \leq e^{-\alpha t^* n + 2t^* f_n(d_n) + o(n)}$ . From

$$\mathbb{P}(N_* > dn^2) \geq \mathbb{P}(M_{\mathcal{L}_n^{(d)}} > 0) \geq \frac{(\mathbb{E}[M_{\mathcal{L}_n^{(d)}}])^2}{\mathbb{E}[M_{\mathcal{L}_n^{(d)}}^2]},$$

we get  $\mathbb{P}(N_* > dn^2) \geq e^{-\alpha t^* n + o(n)}$  which yields (5.6) since  $\alpha < 1$ .

## 6. PROOF OF THEOREM 1.2

Recall that  $\mathbb{P}^*$ -a.s.,  $\min_{|u|=n} V(u) \rightarrow \infty$ . Let  $\Theta := \inf_{\xi \in \partial \mathcal{T}} \limsup_{n \rightarrow \infty} \frac{V(\xi_n)}{\sqrt{n \log n}}$ . For any  $a > 0$ , the event  $\{\Theta \geq a\}$  is inherited in the sense that  $\{\Theta \geq a\} = \bigcap_{|u|=1} \{\Theta^{(u)} \geq a\}$ , where  $\Theta^{(u)}$  is defined exactly as  $\Theta$  but for the branching random walk indexed by the subtree  $\mathcal{T}$  rooted at  $u$ . Hence (see [19])  $\mathbb{P}^*(\Theta \geq a) \in \{0, 1\}$ . It follows that  $\mathbb{P}^*$ -a.s.,  $\Theta$  is a constant. We only need to check that  $\Theta \in (0, \infty)$ , which will be done separately in the next two subsections.

**6.1. Lower bound: Proof of  $\Theta > 0$ .** Let  $a > 0$  whose value will be determined later. Fix  $K > 0$ . We will show that for any  $n_0 \geq 1$ ,

$$(6.1) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left( \exists |u| = n : -K \leq V(u_{\ell}) \leq a \sqrt{\ell \log \ell}, \forall n_0 \leq \ell \leq n \right) = 0.$$

In fact, on the event  $\{\inf_{u \in \mathcal{T}} V(u) \geq -K\}$ , (6.1) yields that a.s. for all  $n_0$ , there is no  $\xi \in \partial \mathcal{T}$  such that  $V(\xi_{\ell}) \leq a \sqrt{\ell \log \ell}, \forall n_0 \leq \ell$ , which a fortiori yields that  $\Theta \geq a$ . Since  $\mathbb{P}^*$ -a.s.  $\inf_{u \in \mathcal{T}} V(u) > -\infty$ , we get  $\Theta \geq a$ .

It remains to show (6.1). Let  $C > 0$ ,  $L > 1$  and  $r_i := n_0 + \lfloor C i^2 \log i \rfloor$  for  $1 \leq i \leq L$ , and consider the set

$$\mathcal{L}_{L, (r_i)} := \{|u| = r_L : -K \leq V(u_{r_i}) \leq a \sqrt{r_i \log r_i}, \forall 1 \leq i \leq L\}.$$



Note that the probability term in (6.1) is less than  $\mathbb{P}(\#\mathcal{L}_{L,(r_i)} > 0)$  for any  $L$  (indeed we will let  $L \rightarrow \infty$ ). By the Markov inequality and the many-to-one formula (2.1),

$$\begin{aligned} \mathbb{P}(\#\mathcal{L}_{L,(r_i)} > 0) &\leq \mathbb{E}[\#\mathcal{L}_{L,(r_i)}] \\ &= \mathbb{E}\left[e^{t^* S_{r_L}} \mathbf{1}_{\{-K \leq S_{r_i} \leq a\sqrt{r_i \log r_i}, \forall 1 \leq i \leq L\}}\right] \\ &\leq e^{at^* \sqrt{r_L \log r_L}} \mathbb{P}\left(\cap_{1 \leq i \leq L} \{-K \leq S_{r_i} \leq a\sqrt{r_i \log r_i}\}\right). \end{aligned}$$

By the Markov property for the random walk  $S$ ,

$$(6.2) \quad \mathbb{P}\left(\cap_{1 \leq i \leq L} \{-K \leq S_{r_i} \leq a\sqrt{r_i \log r_i}\}\right) \leq \prod_{i=1}^L \mathbb{P}\left(S_{r_{i+1}-r_i} \leq a\sqrt{r_i \log r_i} + K\right).$$

Let us recall some useful estimates on the random walk  $S$ . Under (H1'),  $\mathbb{E}(S_1) = \int_{\mathbb{R}} xe^{-t^*x} \mu(dx) =: \mathfrak{m} > 0$ . By assumption,  $c_{\alpha_1} := \mathbb{E}(|S_1|^{\alpha_1}) = \mathbb{E}\left[\sum_{|u|=1} e^{-t^*V(u)} |V(u)|^{\alpha_1}\right] < \infty$  for some  $\alpha_1 > 1$ . Applying [12, Lemma 2.1], we get some constant  $c' > 0$  such that for all  $n \geq 1$ ,  $y \geq n^{\max(1/\alpha_1, 1/2)}$  and  $x > 0$ ,

$$\mathbb{P}\left(S_n \leq \mathfrak{m}n - x, \min_{1 \leq i \leq n} (S_i - S_{i-1}) \geq -y\right) \leq c'e^{-x/y}.$$

Using  $\mathbb{P}(S_1 \leq -y) \leq c_{\alpha_1} y^{-\alpha_1}$ , for any  $y > 0$ , we get that

$$(6.3) \quad \mathbb{P}\left(S_n \leq \mathfrak{m}n - x\right) \leq c'e^{-x/y} + c_{\alpha_1} n y^{-\alpha_1}.$$

We apply (6.3) to the probability terms on the right-hand side of (6.2). Since  $r_{i+1} - r_i \sim 2Ci \log i$  and  $a\sqrt{r_i \log r_i} \sim a\sqrt{C}i \log i$ , we will choose  $C > 0$  such that

$$C\mathfrak{m} > a\sqrt{C}.$$

For all large  $i$ ,  $a\sqrt{r_i \log r_i} + K \leq \frac{\mathfrak{m}}{2}(r_{i+1} - r_i)$ . Applying (6.3) to  $n = r_{i+1} - r_i$ ,  $x = \frac{\mathfrak{m}}{2}(r_{i+1} - r_i)$  and  $y = (r_{i+1} - r_i)^{1-\delta}$  with  $0 < \delta < \min(\frac{1}{2}, 1 - \frac{1}{\alpha_1})$ , we have for all large  $i$ , say  $i \geq i_1$ ,

$$\mathbb{P}\left(S_{r_{i+1}-r_i} \leq a\sqrt{r_i \log r_i} + K\right) \leq \mathbb{P}\left(S_{r_{i+1}-r_i} \leq \frac{\mathfrak{m}}{2}(r_{i+1} - r_i)\right) \leq (r_{i+1} - r_i)^{-\delta'},$$

where  $\delta' \in (0, \alpha_1(1 - \delta) - 1)$  is a positive constant. It follows that

$$\mathbb{P}(\#\mathcal{L}_{L,(r_i)} > 0) \leq e^{at^* \sqrt{r_L \log r_L}} e^{-\delta' \sum_{i=i_1}^L \log(r_{i+1}-r_i)}.$$

Note that  $\sum_{i=i_1}^L \log(r_{i+1} - r_i) \sim L \log L$  and  $at^* \sqrt{r_L \log r_L} \sim at^* \sqrt{C}L \log L$ . Now we choose an arbitrary constant  $a > 0$  such that  $a^2 t^* < \delta' \mathfrak{m}$ , and we can find  $C > 0$  such that  $C\mathfrak{m} > a\sqrt{C}$  and  $at^* \sqrt{C} < \delta'$ , so that  $e^{at^* \sqrt{r_L \log r_L}} e^{-\delta' \sum_{i=i_1}^L \log(r_{i+1}-r_i)} \rightarrow 0$  as  $L \rightarrow \infty$ . This proves (6.1).

**6.2. Upper bound: Proof of  $\Theta < \infty$ .** We want to construct an infinite ray  $\xi \in \partial\mathcal{T}$  such that  $\limsup_{n \rightarrow \infty} \frac{V(\xi_n)}{\sqrt{n \log n}} < \infty$ . The idea is to construct a (inhomogeneous) Galton-Watson process  $(\#\mathcal{Z}_i)_{i \geq 0}$  which will survive with positive probability. Let  $r_1 > 1$  be a large integer and consider  $r_i := r_1 + \lfloor i^2 \log i \rfloor$  for  $i \geq 2$ . Let  $\mathcal{Z}_1 = \{u \in \mathcal{T} : |u| = r_1\}$  be the set of particles alive in the  $r_1$ -th generation of  $\mathcal{T}$ . Let  $C > 0$  whose value will be determined later. For  $i \geq 2$ , we construct the set  $\mathcal{Z}_i$  recursively by letting

$$\begin{aligned} \mathcal{Z}_i := \cup_{v \in \mathcal{Z}_{i-1}} \left\{ u \in \mathcal{T}_v, |u| = r_i : V(u_k) - V(v) \leq C(k - r_{i-1}), \forall r_{i-1} \leq k \leq r_i, \right. \\ \left. V(u_{r_{i-1}}) - V(v) \geq 2C \log i, V(u) - V(v) \leq C \log i \right\}, \end{aligned}$$

where  $\mathcal{T}_v$  denotes the set of descendants of  $v$ . We note that for each  $i$ ,  $\mathcal{Z}_i$  is a subset of  $\{u \in \mathcal{T} : |u| = r_i\}$ . By the branching property,  $(\mathcal{Z}_i)$  constitutes an inhomogeneous Galton-Watson tree. If  $(\mathcal{Z}_i)$  survives with positive probability, on the survival set of  $(\mathcal{Z}_i)$ , we may find an infinite ray for the system  $(\mathcal{Z}_i)$ , say  $\xi$ . Then  $\xi$  can be also considered as an infinite ray of  $\partial\mathcal{T}$ . Moreover by definition of  $(\mathcal{Z}_i)$ ,  $V(\xi_{r_i}) - V(\xi_{r_{i-1}}) \leq C \log i$  for any  $i \geq 2$ , hence  $V(\xi_i) \leq \sum_{j=1}^i C \log j \leq 2Ci \log i$  for all large  $i$ .

Therefore for any large  $n$ , there is a unique  $j$  such that  $r_j \leq n < r_{j+1}$ , and  $V(\xi_n) \leq V(\xi_{r_j}) + C(n - r_j) \leq 2Cj \log j + C(r_{j+1} - r_j) \sim 4Cj \log j$ . Since  $\sqrt{n \log n} \geq \sqrt{r_j \log r_j} \sim \sqrt{2}j \log j$ , we see that for all large  $n$ ,  $V(\xi_n) \leq 3C\sqrt{n \log n}$ . This yields  $\Theta \leq 3C$  with positive probability. Since  $\Theta$  is deterministic, we get  $\Theta \leq 3C$ , the desired upper bound.

It remains to show that  $(\mathcal{Z}_i)$  can survive with positive probability. We will make a coupling between  $(\#\mathcal{Z}_i)$  and a homogenous Galton-Watson process. Let  $b_i := r_i - r_{i-1} - 1$  and

$$\nu_i := \sum_{|w|=b_i} \mathbf{1}_{\{V(w_k) \leq Ck, \forall k \leq b_i, V(w) \geq 2C \log i\}} \mathbf{1}_{\{\exists u \in \mathcal{T}_w : |u|=b_i+1, V(u) \leq C \log i\}}.$$

Observe that  $(\#\mathcal{Z}_i)$  is stochastically larger than an inhomogeneous Galton-Watson process, say  $(Z'_i)$ , with  $\nu_i$  as the reproduction law of the particles in the  $(i-1)$ th generation of  $Z'$ . It is enough to show that  $(Z'_i)$  can survive with positive probability (if we choose  $C$  sufficiently large).

We claim that for any  $K_0 \geq 1$ ,

$$(6.4) \quad \limsup_{C \rightarrow \infty} \limsup_{i \rightarrow \infty} \mathbb{P}(\nu_i \leq K_0) \leq q_{\mathcal{T}},$$

with  $q_{\mathcal{T}} := \mathbb{P}(\mathcal{T} \text{ is finite}) \in [0, 1)$  is the extinction probability of the Galton-Watson tree  $\mathcal{T}$ . In fact, (6.4) is an equality, as  $\nu_i = 0$  for all large  $i$  on the set of extinction of  $\mathcal{T}$ .

Assume for the moment (6.4). Let  $q' \in (q_{\mathcal{T}}, 1)$ . Let  $K_0$  be an integer such that  $K_0(1 - q') > 1$ . We may choose (and then fix)  $C > 0$  large enough such that for all large  $i$ ,  $\mathbb{P}(\nu_i \leq K_0) \leq q'$ . Then for all large  $i$ ,  $\nu_i$  is stochastically larger than a Bernoulli variable which equals  $K_0$  with probability  $1 - q'$  and 0 with probability  $q'$ . Since the (homogenous) Galton-Watson process with such a Bernoulli variable as reproduction law is supercritical, we see that the inhomogeneous Galton-Watson process  $(Z'_i)$  will survive with positive probability.

The rest of this subsection is devoted to the proof of (6.4).

Recall that  $\mathbb{E}[S_1] = \mathfrak{m} > 0$ . Let  $C > \mathfrak{m}$  be large enough such that

$$(6.5) \quad i^{Ct^*} b_i^{-\alpha_2} \rightarrow \infty, \quad \text{as } i \rightarrow \infty.$$

Let for  $\ell \geq 0$ ,  $\mathcal{F}_\ell := \sigma\{u, V(u) : |u| \leq \ell\}$  be the  $\sigma$ -field generated by the BRW up to generation  $\ell$ . Conditionally on  $\mathcal{F}_{b_i}$ ,  $\nu_i$  is a sum of independent Bernoulli random variables:

$$\mathbb{E}[\nu_i | \mathcal{F}_{b_i}] = \sum_{|w|=b_i} \mathbf{1}_{\{V(w_k) \leq Ck, \forall k \leq b_i, V(w) \geq 2C \log i\}} \vartheta(V(w) - C \log i),$$

with  $\vartheta(y) := \mathbb{P}(\exists |u| = 1 : V(u) \leq -y)$ . By assumption for all large  $y$ ,  $\vartheta(y) \geq e^{-t^*y} y^{-\alpha_2}$ . We also have

$$\vartheta(y) \leq \mathbb{E}\left[\sum_{|u|=1} \mathbf{1}_{\{V(u) \leq -y\}}\right] \leq e^{-t^*y} \mathbb{E}\left[\sum_{|u|=1} e^{-t^*V(u)} \mathbf{1}_{\{V(u) \leq -y\}}\right] \leq c_{\alpha_1} e^{-t^*y} y^{-\alpha_1},$$

with  $c_{\alpha_1} := \mathbb{E}[\sum_{|u|=1} e^{-t^*V(u)} |V(u)|^{\alpha_1}] < \infty$ . Therefore

$$(6.6) \quad \mathbb{E}[\nu_i | \mathcal{F}_{b_i}] \geq i^{Ct^*} (Cb_i)^{-\alpha_2} W'_i,$$

with

$$W'_i := \sum_{|w|=b_i} 1_{\{V(w_k) \leq Ck, \forall k \leq b_i, V(w) \geq 2C \log i\}} e^{-t^* V(w)}.$$

Observe that

$$\begin{aligned} \text{Var}(\nu_i | \mathcal{F}_{b_i}) &= \sum_{|w|=b_i} 1_{\{V(w_k) \leq Ck, \forall k \leq b_i, V(w) \geq 2C \log i\}} \vartheta(V(w) - C \log i) (1 - \vartheta(V(w) - C \log i)) \\ &\leq \mathbb{E}[\nu_i | \mathcal{F}_{b_i}]. \end{aligned}$$

This implies that for any  $\varepsilon \in (0, 1)$ ,

$$\begin{aligned} \mathbb{P}\left(\nu_i - \mathbb{E}[\nu_i | \mathcal{F}_{b_i}] \leq -\varepsilon \mathbb{E}[\nu_i | \mathcal{F}_{b_i}], W'_i \geq \varepsilon\right) &\leq \mathbb{E}\left[\frac{\text{Var}(\nu_i | \mathcal{F}_{b_i})}{\varepsilon^2 (\mathbb{E}[\nu_i | \mathcal{F}_{b_i}])^2} 1_{\{W'_i \geq \varepsilon\}}\right] \\ &\leq \mathbb{E}\left[\frac{1}{\varepsilon^2 \mathbb{E}[\nu_i | \mathcal{F}_{b_i}]} 1_{\{W'_i \geq \varepsilon\}}\right] \\ &\leq \frac{1}{\varepsilon^3 i^{Ct^*} (Cb_i)^{-\alpha_2}} \\ &\rightarrow 0, \end{aligned}$$

as  $i \rightarrow \infty$ , where the last convergence follows from (6.5).

On the set  $\{W'_i \geq \varepsilon\}$ ,  $\mathbb{E}[\nu_i | \mathcal{F}_{b_i}] \geq \varepsilon i^{Ct^*} (Cb_i)^{-\alpha_2} \geq K_0/(1 - \varepsilon)$  for all large  $i$ . Therefore,

$$\mathbb{P}(\nu_i \leq K_0) \leq \mathbb{P}\left(\nu_i - \mathbb{E}[\nu_i | \mathcal{F}_{b_i}] \leq -\varepsilon \mathbb{E}[\nu_i | \mathcal{F}_{b_i}], W'_i \geq \varepsilon\right) + \mathbb{P}(W'_i < \varepsilon),$$

yielding that

$$(6.7) \quad \limsup_{i \rightarrow \infty} \mathbb{P}(\nu_i \leq K_0) \leq \limsup_{i \rightarrow \infty} \mathbb{P}(W'_i < \varepsilon).$$

To estimate  $\mathbb{P}(W'_i < \varepsilon)$ , we compare  $W'_i$  with  $W_i$ , where

$$W_i := \sum_{|w|=b_i} e^{-t^* V(w)}.$$

Since by assumption  $\mathbb{E}\left[\sum_{|u|=1} e^{-t^* V(u)} \log_+(\sum_{|u|=1} e^{-t^* V(u)})\right] < \infty$ , we may apply Biggin's theorem ([8]) to see that  $W_i \rightarrow W_\infty$  which is positive  $\mathbb{P}^*$ -a.s. By the many-to-one formula (2.1),

$$\begin{aligned} \mathbb{E}[W_i - W'_i] &= 1 - \mathbb{P}\left(S_k \leq Ck, \forall k \leq b_i, S_{b_i} \geq 2C \log i\right) \\ &\leq \mathbb{P}\left(\max_{k \geq 1} \frac{S_k}{k} > C\right) + \mathbb{P}\left(S_{b_i} < 2C \log i\right). \end{aligned}$$

Recall that  $\mathbb{E}[S_1] = m > 0$  and  $b_i \sim 2i \log i$  as  $i \rightarrow \infty$ . The law of large numbers yields that  $\mathbb{P}(S_{b_i} < 2C \log i) \rightarrow 0$ . Using  $\mathbb{P}(W'_i < \varepsilon) \leq \mathbb{P}(W_i \leq 2\varepsilon) + \mathbb{P}(W_i - W'_i > \varepsilon) \leq \mathbb{P}(W_i \leq 2\varepsilon) + \frac{1}{\varepsilon} \mathbb{E}[W_i - W'_i]$ , we get that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \mathbb{P}(\nu_i \leq K_0) &\leq \limsup_{i \rightarrow \infty} \mathbb{P}(W'_i < \varepsilon) \\ &\leq \limsup_{i \rightarrow \infty} \mathbb{P}(W_i \leq 2\varepsilon) + \frac{1}{\varepsilon} \mathbb{P}\left(\max_{k \geq 1} \frac{S_k}{k} > C\right). \end{aligned}$$

Since  $\mathbb{P}(\max_{k \geq 1} \frac{S_k}{k} > C) \rightarrow 0$  as  $C \rightarrow \infty$  and  $\mathbb{P}(W_\infty = 0) = q_T$ , we complete the proof of (6.4) by letting  $C \rightarrow \infty$  then  $\varepsilon \rightarrow 0$ .

## APPENDIX A. ESTIMATES ON RANDOM WALKS

**Lemma A.1.** *Let  $(S_n, n \geq 0)$  be a real-valued random walk such that  $\mathbb{P}(S_1 \neq 0) > 0$ . Then*

$$\sup_{x \geq 0} \mathbb{E} \left[ \sum_{k=0}^{\infty} \mathbf{1}_{\{\min_{0 \leq i \leq k} S_i + x \geq 0\}} e^{-S_k - x} \right] < \infty.$$

*Proof.* Let  $v_0 := 0$  and  $v_j := \inf\{n > v_{j-1} : S_n \leq S_{v_{j-1}}\}$  for  $j \geq 1$ , be the weak descending ladder times of  $S$ , with the convention that  $\inf \emptyset = \infty$ . We have

$$\mathbb{E} \left[ \sum_{k=0}^{\infty} \mathbf{1}_{\{\min_{0 \leq i \leq k} S_i + x \geq 0\}} e^{-S_k} \right] = \sum_{j=0}^{\infty} \mathbb{E} \left[ \mathbf{1}_{\{S_{v_j} \geq -x, v_j < \infty\}} e^{-S_{v_j}} \right] \mathbb{E} \left[ \sum_{k=0}^{\infty} e^{-S_k} \mathbf{1}_{\{k < v_1\}} \right].$$

The term  $\mathbb{E} \left[ \sum_{k=0}^{\infty} e^{-S_k} \mathbf{1}_{\{k < v_1\}} \right]$  can be evaluated through the ladder heights. Let  $v_0^+ := 0, v_j^+ := \inf\{i > v_{j-1}^+ : S_i > S_{v_{j-1}^+}\}$  for  $j \geq 1$ . By the time-reversal for the random walk,

$$\mathbb{E} \left[ \sum_{k=0}^{\infty} e^{-S_k} \mathbf{1}_{\{k < v_1\}} \right] = \mathbb{E} \left[ \sum_{j=0}^{\infty} e^{-S_{v_j^+}} \mathbf{1}_{\{v_j^+ < \infty\}} \right] = \sum_{j=0}^{\infty} \left( \mathbb{E} \left[ e^{-S_{v_1^+}} \mathbf{1}_{\{v_1^+ < \infty\}} \right] \right)^j < \infty$$

as  $\mathbb{E} \left[ e^{-S_{v_1^+}} \mathbf{1}_{\{v_1^+ < \infty\}} \right] < 1$ . Finally,  $\sum_{j=0}^{\infty} \mathbb{E} \left[ \mathbf{1}_{\{S_{v_j} \geq -x, v_j < \infty\}} e^{-S_{v_j}} \right] \leq \sum_{z=0}^{\lceil x \rceil} e^z U^-(z)$  where

$$U^-(z) := \sum_{j=0}^{\infty} \mathbb{P}(S_{v_j} \in [-z, -z+1), v_j < \infty).$$

By the classical renewal theorem (see for example [14, Theorem II.4.2]),  $\sup_{z \geq 0} U^-(z) = c < \infty$ . Therefore

$$\sum_{j=0}^{\infty} \mathbb{E} \left[ \mathbf{1}_{\{S_{v_j} \geq -x, v_j < \infty\}} e^{-S_{v_j}} \right] \leq \frac{ce}{e-1} e^{\lceil x \rceil}.$$

It completes the proof.  $\square$

**Lemma A.2.** *Let  $(S_n, n \geq 0)$  be a random walk with values in  $\mathbb{Z}$  such that  $\mathbb{P}(S_1 \geq 2) = 0$ . In agreement with (5.2), let  $\sigma_1 := \inf\{k \geq 1 : S_k = 0\}$ . Suppose that  $\mathbb{E}[S_1] = 0$  and  $\mathbb{E}[S_1^2] < \infty$ . Then there exists  $\theta > 0$  such that*

$$(A.1) \quad \mathbb{P}(T'_{-n} < \sigma_1) \sim \frac{\theta}{n}, \quad n \rightarrow \infty,$$

where  $T'_{-n} := \inf\{i \geq 0 : S_i \leq -n\}$ . Furthermore, there exists some sequence  $(K_n)_{n \geq 0}$  such that  $K_n \geq 1, K_n = o(n)$  as  $n \rightarrow \infty$  and

$$(A.2) \quad \mathbb{P}(T'_{-n} < \sigma_1, |S_{T'_{-n}} + n| \leq K_n) \sim \frac{\theta}{n}, \quad n \rightarrow \infty.$$

*Proof.* We will prove the lemma with  $\theta := \mathbb{E}[|S_{v_1}| \mathbf{1}_{\{v_1 < \sigma_1\}}]$  where  $v_1 := \inf\{i \geq 1 : S_i < 0\}$ . The optional stopping theorem applied to  $T'_{-n} \wedge \sigma_1$  implies that for  $-n < x < 0$ ,

$$x = \mathbb{E}_x[S_{\sigma_1 \wedge T'_{-n}}] = \mathbb{E}_x[S_{T'_{-n}} \mathbf{1}_{\{T'_{-n} < \sigma_1\}}] \leq -n \mathbb{P}_x(T'_{-n} < \sigma_1).$$

Hence  $\mathbb{P}_x(T'_{-n} < \sigma_1) \leq \frac{|x|}{n}$  for any  $-n < x < 0$ , and also for any  $x \leq -n$ . By the strong Markov property applied to  $v_1$ ,

$$(A.3) \quad \mathbb{P}(T'_{-n} < \sigma_1) = \mathbb{E} \left[ \mathbf{1}_{\{v_1 < \sigma_1\}} \mathbb{P}_{S_{v_1}}(T'_{-n} < \sigma_1) \right] \leq \frac{\theta}{n}.$$

Discussing the last value taken by  $S$  before leaving  $[-(n-1), -1]$ , we have for any  $-n < x < 0$ ,

$$\mathbb{E}_x[|S_{T'_n} + n| \mathbf{1}_{\{T'_n < \sigma_1\}}] = \sum_{k=1}^{n-1} G_n(x, -k) \mathbb{E}[|S_1 + n - k| \mathbf{1}_{\{S_1 \leq k-n\}}]$$

where  $G_n(x, -k) := \sum_{\ell=0}^{\infty} \mathbb{P}_x(S_\ell = -k, \ell < T'_n \wedge \sigma_1)$ . Hence the inequality

$$(A.4) \quad \mathbb{E}_x[|S_{T'_n} + n| \mathbf{1}_{\{T'_n < \sigma_1\}}] \leq \sum_{k=1}^{n-1} G_n(x, -k) \mathbb{E}[|S_1| \mathbf{1}_{\{S_1 \leq k-n\}}].$$

As in (2.3), let  $T_{\{-k\}} = \inf\{i \geq 1 : S_i = -k\}$ . In the sequel, the constant  $c$  does not depend on  $n, k, x$  and can change its value from line to line. By (2.2), for all  $-k < x < 0$

$$\mathbb{P}_x(T_{\{-k\}} < \sigma_1) \leq c \frac{|x|}{k}.$$

We can suppose that the inequality stays true when  $x \leq -k$  by taking a bigger  $c$  if necessary. Moreover, by (2.2), for all  $0 < k < n$ ,

$$\mathbb{P}_{-k}(T_{\{-k\}} > T'_n \wedge \sigma_1) \geq c \max\left(\frac{1}{k}, \frac{1}{n-k}\right).$$

The last two inequalities imply that  $G_n(x, -k) \leq c \min(k, n-k) \frac{|x|}{k}$  for all  $-n < x < 0$  and  $0 < k < n$ . Splitting the sum on the right-hand side of (A.4) according to whether  $k \leq \lfloor \frac{n}{2} \rfloor$  or  $k > \lfloor \frac{n}{2} \rfloor$  gives

$$(A.5) \quad \mathbb{E}_x[|S_{T'_n} + n| \mathbf{1}_{\{T'_n < \sigma_1\}}] \leq c|x|(I_1 + I_2)$$

where

$$I_1 := \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \mathbb{E}[|S_1| \mathbf{1}_{\{S_1 \leq k-n\}}], \quad I_2 := \frac{1}{n} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} k \mathbb{E}[|S_1| \mathbf{1}_{\{S_1 \leq -k\}}].$$

We observe that

$$I_1 \leq \lfloor \frac{n}{2} \rfloor \mathbb{E}[|S_1| \mathbf{1}_{\{2S_1 \leq -n\}}] \leq \mathbb{E}[S_1^2 \mathbf{1}_{\{2S_1 \leq -n\}}] \rightarrow 0$$

as  $n \rightarrow \infty$  by dominated convergence. Similarly, since  $\sum_{k=1}^{\min(p,q)} k \leq cp \min(p, q)$  for  $p = |S_1|$  and  $q = \lfloor \frac{n}{2} \rfloor$ ,

$$I_2 \leq c \mathbb{E}\left[S_1^2 \frac{\min(|S_1|, n)}{n}\right] \rightarrow 0.$$

We deduce that  $\lim_{n \rightarrow \infty} \mathbb{E}_x[|S_{T'_n} + n| \mathbf{1}_{\{T'_n < \sigma_1\}}] = 0$ . Writing

$$x = \mathbb{E}_x[S_{\sigma_1 \wedge T'_n}] = -n \mathbb{P}_x(T'_n < \sigma_1) + \mathbb{E}_x[(S_{T'_n} + n) \mathbf{1}_{\{T'_n < \sigma_1\}}],$$

it implies that for all fixed  $x < 0$ ,  $\mathbb{P}_x(T'_n < \sigma_1) \sim \frac{|x|}{n}$ . By Fatou's lemma,

$$\liminf_{n \rightarrow \infty} n \mathbb{P}(T'_n < \sigma_1) = \liminf_{n \rightarrow \infty} \mathbb{E}[\mathbf{1}_{\{v_1 < \sigma_1\}} n \mathbb{P}_{S_{v_1}}(T'_n < \sigma_1)] \geq \theta.$$

Together with (A.3), it completes the proof of (A.1).

To prove (A.2), it is sufficient to show that we can find a sequence  $K_n = o(n)$  with  $K_n \geq 1$ , such that

$$(A.6) \quad \mathbb{P}(T'_n < \sigma_1, |S_{T'_n} + n| > K_n) = o\left(\frac{1}{n}\right), \quad n \rightarrow \infty.$$

In fact, the probability term in (A.6) is less than

$$\mathbb{P}(S_{v_1} \leq -n) + \mathbb{E}\left[\mathbf{1}_{\{v_1 < \sigma_1, S_{v_1} > -n\}} \mathbb{P}_{S_{v_1}}(T'_n < \sigma_1, |S_{T'_n} + n| > K_n)\right] =: I_3 + I_4.$$

By the Markov inequality,

$$I_3 \leq \frac{1}{n} \mathbb{E} [ |S_{v_1}| \mathbf{1}_{\{S_{v_1} \leq -n\}} ].$$

According to Doney [11, Corollary 1],  $\mathbb{E} [|S_{v_1}|] < \infty$ , hence

$$I_3 = o\left(\frac{1}{n}\right), \quad n \rightarrow \infty.$$

For  $I_4$ , using (A.5), we have

$$\begin{aligned} I_4 &\leq \frac{1}{K_n} \mathbb{E} \left[ \mathbf{1}_{\{v_1 < \sigma_1, S_{v_1} > -n\}} \mathbb{E}_{S_{v_1}} ( |S_{T'_{-n}} + n | \mathbf{1}_{\{T'_{-n} < \sigma_1\}} ) \right] \\ &\leq \frac{c(I_1 + I_2)}{K_n} \mathbb{E} [|S_{v_1}|] \leq \frac{c'\varepsilon_n}{K_n}, \end{aligned}$$

with some positive constant  $c'$  and  $\varepsilon_n := \mathbb{E}[S_1^2 \mathbf{1}_{\{2S_1 \leq -n\}}] + \mathbb{E}\left[S_1^2 \frac{\min(|S_1|, n)}{n}\right] \rightarrow 0$ .

If we choose  $K_n := \max(1, n\sqrt{\varepsilon_n})$ , then  $K_n = o(n)$  and  $I_4 \leq \frac{c'}{n}\varepsilon_n^{1/2} = o\left(\frac{1}{n}\right)$ . Thus, we obtain (A.6) and complete the proof of Lemma A.2.  $\square$

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