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► To cite this version:

| Elie Aïdékon, Yueyun Hu, Zhan Shi. Boundedness of discounted tree sums. 2024. hal-04683431

HAL Id: hal-04683431

<https://hal.science/hal-04683431v1>

Preprint submitted on 2 Sep 2024

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BOUNDEDNESS OF DISCOUNTED TREE SUMS

ELIE AÏDÉKON, YUEYUN HU, AND ZHAN SHI

ABSTRACT. Let $(V(u), u \in \mathcal{T})$ be a (supercritical) branching random walk and $(\eta_u, u \in \mathcal{T})$ be marks on the vertices of the tree, distributed in an i.i.d. fashion. Following Aldous and Bandyopadhyay [2], for each infinite ray ξ of the tree, we associate the *discounted tree sum* $D(\xi)$ which is the sum of the $e^{-V(u)}\eta_u$ taken along the ray. The paper deals with the finiteness of $\sup_{\xi} D(\xi)$. To this end, we study the extreme behaviour of the local time processes of the paths $(V(u), u \in \xi)$. It answers a question of Nicolas Curien, and partially solves Open Problem 31 of Aldous and Bandyopadhyay [2]. We also present several open questions.

1. INTRODUCTION

Let $V := \{V(u), u \in \mathcal{T}\}$ be a discrete-time branching random walk on the real line \mathbb{R} , where \mathcal{T} is a Ulam-Harris tree which describes the genealogy of the particles and $V(u) \in \mathbb{R}$ is the position of the particle u . When a particle u is at n -th generation, we write $|u| = n$ for $n \geq 0$. The branching random walk V is constructed as follows: initially, there is a single particle \emptyset located at 0. The particle \emptyset is considered as the root of \mathcal{T} . At the first generation, the root dies and gives birth to a certain number of children, these children along with their spatial positions form a point process \aleph on \mathbb{R} and constitute the first generation of the branching random walk $\{V(u), |u| = 1\}$. We may identify $\aleph = \sum_{|u|=1} \delta_{\{V(u)\}}$. For the next generations, the process is constructed recursively: for each $|u| = n$ (if such u exists) with $n \geq 1$, the particle u dies in the $(n+1)$ -th generation and gives birth to an independent copy of \aleph shifted by $V(u)$. The collection of all children of all u together with their positions gives the $(n+1)$ -th generation. The whole system may survive forever or die out after some generations.

Define the measure $\mu(dx) := \mathbb{E}[\aleph(dx)]$. We assume that $\mu(\mathbb{R}) > 1$, so that \mathcal{T} is a supercritical Galton–Watson tree. We denote by \mathcal{S} the event of non-extinction of \mathcal{T} , and $\mathbb{P}^* := \mathbb{P}(\cdot \mid \mathcal{S})$. By an infinite ray ξ of \mathcal{T} , we mean a path $\xi = \{\xi_0 = \emptyset, \xi_1, \dots, \xi_n, \dots\}$ such that for all $n \geq 0$, ξ_n is the parent of ξ_{n+1} as vertices of \mathcal{T} . Denote by $\partial\mathcal{T}$ the set of all infinite rays. In particular $\partial\mathcal{T} \neq \emptyset$ on the event \mathcal{S} . Let

$$(1.1) \quad M := \sup_{\xi \in \partial\mathcal{T}} \sum_{n=0}^{\infty} e^{-V(\xi_n)}.$$

The starting point of this work is the following question: when do we have $M < \infty$, \mathbb{P}^* -a.s? This question was raised by Nicolas Curien (personal communication) in relation to the construction of self-similar Markov trees [6], see in particular the discussion in Section 3.4 there.

Let $m_n := \inf_{|u|=n} V(u)$ and $\varphi(t) := \log \int_{\mathbb{R}} e^{-tx} \mu(dx)$ for $t \in \mathbb{R}$. We have $M \geq \sup_{n \geq 0} e^{-m_n}$. Hence $M = \infty$ if $\liminf_{n \rightarrow \infty} m_n = -\infty$. This happens \mathbb{P}^* -a.s. when

Date: This version: September 2, 2024.

2010 Mathematics Subject Classification. 60F05, 60J45, 60J80.

Key words and phrases. Branching random walk, discounted tree sums, local times.

$\inf_{t>0} \frac{\varphi(t)}{t} \in (0, \infty)$. Actually, as soon as $\inf_{t>0} \frac{\varphi(t)}{t} < \infty$, we have the following law of large numbers ([15, 17, 8]): \mathbb{P}^* -a.s.,

$$(1.2) \quad \lim_{n \rightarrow \infty} \frac{m_n}{n} = - \inf_{t>0} \frac{\varphi(t)}{t} =: \gamma.$$

Notice that the upper bound

$$(1.3) \quad M \leq \sum_{n=0}^{\infty} e^{-m_n}$$

implies that $M < \infty$ a.s. when $\gamma > 0$. The critical case is therefore $\gamma = 0$, where the upper bound (1.3) is not good enough in general. From now on, we suppose that one of the following assumptions holds:

(H1) There exists $t > 0$ such that $\varphi(t) = 0$ and we let t^* be the minimal such t . The integral $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx)$ is well-defined and nonzero.

(H2) There exists $t^* > 0$ such that $\varphi(t^*) = 0$ and $\varphi(t) \in [0, \infty]$ for all $t > 0$. The integral $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx)$ is well-defined, is equal to 0 and $\int_{\mathbb{R}} x^2 e^{-t^*x} \mu(dx) \in (0, \infty)$.

In (H1) and (H2), we say that $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx)$ is well-defined if at least one of the quantities $\int_{(-\infty, 0)} |x| e^{-t^*x} \mu(dx)$ or $\int_{(0, \infty)} x e^{-t^*x} \mu(dx)$ is finite. Note that $\varphi(t^*) = 0$ yields that $m_n \rightarrow \infty$, \mathbb{P}^* -a.s. (see [20, Lemma 3.1]) and $\gamma \geq 0$.

Let us discuss the case (H1): if $\gamma > 0$, then $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx) > 0$. Otherwise, one necessarily has $\gamma = 0$. If furthermore $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx)$ is negative, resp. positive, then one necessarily has $\varphi(t) = \infty$ for all $t < t^*$, resp. for all $t > t^*$. The case (H1) with $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx) > 0$ and $\mu((-\infty, 0)) > 0$ will be denoted by **(H1')**.

In case (H2), one necessarily has $\gamma = 0$. The case (H2) is called “boundary case” in the literature, see [1] for the precise asymptotic of m_n and [20] for further references. There exist cases where $\gamma = 0$ but neither (H1) nor (H2) holds, for example when $\mu((-\infty, 0)) = 0$ and $\mu(\{0\}) > 1$. In that case $\gamma = 0$ while there is no $t > 0$ such that $\varphi(t) = 0$. We refer to the appendix of the arXiv version of [16] for such discussions.

Theorem 1.1. *Under (H1) or (H2), $M < \infty$ \mathbb{P}^* -a.s. Actually,*

$$\Xi_b := \sup_{\xi \in \partial\mathcal{T}} \sum_{n=0}^{\infty} (1 + |V(\xi_n)|)^{-b} < \infty$$

for all $b > 2$ under **(H1)**, and for all $b > 3$ under **(H2)**.

In the case $\gamma > 0$, $\Xi_b < \infty$ for $b > 1$ by (1.2) and (1.3). The fact that there exist rays such that $\limsup_{n \rightarrow \infty} \frac{V(\xi_n)}{n} < \infty$ implies that $\Xi_b = \infty$ for $b = 1$. On the other hand, according to Jaffuel [16], under (H2) with some further integrability conditions on μ , there exists an explicit constant $a_0 > 0$ such that \mathbb{P}^* -a.s.

$$\inf_{\xi \in \partial\mathcal{T}} \limsup_{n \rightarrow \infty} \frac{V(\xi_n)}{n^{1/3}} = a_0.$$

It implies that $\Xi_b = \infty$ for $b = 3$ in this case. Finally, the following theorem shows there are cases under **(H1')** where $\Xi_b = \infty$ for $b = 2$.

Theorem 1.2. *We suppose that (H1') holds and there exist $1 < \alpha_1 \leq \alpha_2$ such that*

$$\mathbb{E} \left[\sum_{|u|=1} e^{-t^*V(u)} |V(u)|^{\alpha_1} \right] < \infty$$

and for all $y > 0$ large enough,

$$\mathbb{P}(\exists |u| = 1 : V(u) \leq -y) \geq e^{-t^*y} y^{-\alpha_2}.$$

We further assume that

$$\mathbb{E} [\Sigma_V \log_+ \Sigma_V] < \infty,$$

where $\Sigma_V := \sum_{|u|=1} e^{-t^*V(u)}$ and $\log_+ x := \max(\log x, 0)$ for $x > 0$. Then there exists a constant $a_1 \in (0, \infty)$ such that \mathbb{P}^* -a.s.

$$\inf_{\xi \in \partial\mathcal{T}} \limsup_{n \rightarrow \infty} \frac{V(\xi_n)}{\sqrt{n \log n}} = a_1.$$

The issue of the finiteness of M is a particular case of the following problem stated in Aldous and Bandyopadhyay [2] as Open Problem 31, Section 4.4. It is related to the recursive distributional equation

$$(1.4) \quad X \stackrel{d}{=} \eta + \max_{|u|=1} e^{-V(u)} X_u$$

where on the right-hand side, conditionally on $(\eta, V(u), |u| = 1)$, the r.v. $(X_u, |u| = 1)$ are i.i.d. with the law of X . A possible endogenous solution of (1.4) can be constructed as follows. Consider a pair (η, \aleph) where $\eta > 0$ is a positive r.v. and \aleph is a point process as above. We do not suppose that η and \aleph are independent. We then construct recursively the branching random walk as before: for each vertex u at position $V(u)$, we take an independent copy of (η, \aleph) , call it (η_u, \aleph_u) , so that the children of u are at positions $V(u) + \aleph_u$. Following [2], we introduce the *discounted tree sum*

$$(1.5) \quad D(\xi) := \sum_{n=0}^{\infty} e^{-V(\xi_n)} \eta_{\xi_n}, \quad \forall \xi \in \partial\mathcal{T}.$$

Define

$$X := \sup_{\xi \in \partial\mathcal{T}} D(\xi).$$

Then X is a solution of (1.4). Observe that $X = M$ if $\eta \equiv 1$. Theorem 32 of [2] shows (among other results) that $X < \infty$ a.s. when $\mu((-\infty, 0]) = 0$, $E[\eta^p] < \infty$ for all $p \geq 1$ and $\varphi(t) < \infty$ for some $t \geq 1$. It was left open to study general conditions under which $X < \infty$. Again, it is clear from (1.2) that $X = \infty$ \mathbb{P}^* -a.s. if $\gamma < 0$. Notice that $\gamma > 0$ no longer ensures that $X < \infty$ a.s. because of the influence of the variable η . The property $\{X < \infty\}$ is inherited, namely, $\{X < \infty\} = \cap_{|u|=1} \{X^{(u)} < \infty\}$, where $X^{(u)}$ is defined as X but for the branching random walk indexed by the subtree of \mathcal{T} rooted at u . Therefore (see [19]) $\mathbb{P}^*(X = \infty) \in \{0, 1\}$.

Theorem 1.3. *Suppose that*

$$(1.6) \quad \zeta := \lim_{x \rightarrow \infty} \frac{-\log \mathbb{P}(\eta > x)}{\log x}$$

exists and $\zeta \in (0, \infty]$. Under (H1) or (H2),

- (i) *if $\zeta < t^*$, then $\mathbb{P}^*(X = \infty) = 1$.*
- (ii) *if $\zeta > t^*$, then $\mathbb{P}^*(X = \infty) = 0$.*

Remark 1.4. (1) *Theorem 1.3 (i) comes from the fact that $\sup_{u \in \mathcal{T}} e^{-V(u)} \eta_u = \infty$, \mathbb{P}^* -a.s., see Lemma 4.1. We refer to [4], [13] for much more precise results on this model, which is called last progeny modified branching random walk there.*

(2) *Theorem 1.3 (ii) was proved in [3] when $V(u) = c|u|$ for some constant $c > 0$.*

(3) If $\partial\mathcal{T}$ is equipped with the usual ultra metric, then in the case (ii), $\xi \mapsto D(\xi)$ is continuous on $\partial\mathcal{T}$. For more details, see Remark 4.2.

The way we address these problems is by looking at the local times of the random walk along a ray defined as for any $\xi \in \partial\mathcal{T}$ and $k \in \mathbb{Z}$,

$$(1.7) \quad N_\xi^k := \sum_{n=0}^{\infty} 1_{\{k \leq V(\xi_n) < k+1\}}.$$

Theorem 1.1 is then a consequence of the following result.

Theorem 1.5. *We have \mathbb{P}^* -a.s.*

$$\lim_{n \rightarrow \infty} \frac{1}{n^\kappa} \sup_{\xi \in \partial T} N_\xi^n = 0$$

for all $\kappa > 1$ under **(H1)** and all $\kappa > 2$ under **(H2)**.

We can readily prove Theorem 1.1.

Proof of Theorem 1.1. Let $b > 0$. As mentioned before, $\lim_{n \rightarrow \infty} m_n \rightarrow \infty$ \mathbb{P}^* -a.s under our assumptions. Since $\sum_{n=0}^{\infty} 1_{\{V(\xi_n) \geq 0\}} (1 + V(\xi_n))^{-b} \leq \sum_{n=0}^{\infty} (1 + n)^{-b} N_\xi^n$, we may conclude by Theorem 1.5. \square

To study the critical cases of Theorem 1.5, we will restrict our attention to some integer-valued right-continuous branching random walks. Recall that **(H1')** is the assumption **(H1)** together with the positivity of $\int_{\mathbb{R}} x e^{-t^*x} \mu(dx)$ and $\mu((-\infty, 0)) > 0$.

Theorem 1.6. *We suppose that μ is supported on $\mathbb{Z}_- \cup \{1\}$ and*

$$\mathbb{E} \left[\sum_{|u|, |v|=1, u \neq v} e^{-t^*V(u)} e^{-t^*V(v)} \right] < \infty.$$

Notice that N_ξ^k defined in (1.7) is now simply equal to $\sum_{n=0}^{\infty} 1_{\{V(\xi_n)=k\}}$ when $k \geq 0$.

(i) Under **(H1')**,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sup_{\xi \in \partial T} N_\xi^n = -\frac{t^*}{\log q}, \quad \mathbb{P}^*\text{-a.s.},$$

where $q \in (0, 1)$ is given by (5.3).

(ii) Under **(H2)**, then

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sup_{\xi \in \partial T} N_\xi^n = \frac{t^*}{2\theta}, \quad \mathbb{P}^*\text{-a.s.},$$

where θ is given by (5.4).

It implies that, for such branching random walks,

$$\sup_{\xi \in \partial\mathcal{T}} \limsup_{n \rightarrow \infty} \frac{1}{n} N_\xi^n = -\frac{t^*}{\log q}, \quad \text{under (H1')},$$

and

$$\sup_{\xi \in \partial\mathcal{T}} \limsup_{n \rightarrow \infty} \frac{1}{n^2} N_\xi^n = \frac{t^*}{2\theta}, \quad \text{under (H2)}.$$

One can ask the same question for \liminf instead of \limsup . Theorem 1.2 suggests that the renormalisation should be different under its conditions.

Open question 1.

- (i) Under the assumptions of Theorem 1.2, is $\sup_{\xi \in \partial \mathcal{T}} \liminf_{n \rightarrow \infty} \frac{\log n}{n} N_{\xi}^n \in (0, \infty)$? What is its value?
- (ii) Under the assumptions of Theorem 1.6 (ii), is $\sup_{\xi \in \partial \mathcal{T}} \liminf_{n \rightarrow \infty} \frac{1}{n^2} N_{\xi}^n \in (0, \infty)$? What is its value?

Open question 2. In Theorem 1.2, what is the value of a_1 ?

Open question 3. Let $a > 0$. Under the conditions of Theorem 1.6 (ii), what is the Hausdorff dimension of the rays $\xi \in \partial \mathcal{T}$ such that $\limsup_{n \rightarrow \infty} \frac{1}{n^2} N_{\xi}^n = a$? Same question with \liminf instead of \limsup .

The rest of the paper is organized as follows: Section 2 introduces the usual many-to-one formula. Theorem 1.5, Theorem 1.3, Theorem 1.6 are proved in Section 3, Section 4 and Section 5, respectively. A common tool in these proofs is the construction of some suitable optional lines in the sense of [10] by using the local times of the branching random walk. The proof of Theorem 1.2 is given in Section 6 and relies on a certain inhomogeneous Galton–Watson process, building on the fact that under (H1') and some additional assumptions, the minimum of the branching random walk is achieved by a large drop (see [5]). The Appendix A contains some estimates on random walks which are used in the course of the proof of Theorem 1.6.

Throughout this paper, c, c', c'' , eventually with some subscripts, denote positive constants whose values may vary from one line to another.

Acknowledgments. We thank Nicolas Curien for introducing us to this question and explaining the link with self-similar Markov trees.

Shortly before completion of this work, we learnt that Bastien Mallein (personal communication) proved that $M < \infty$ in the case of the branching Brownian motion by a similar argument, hence solving the question of Nicolas Curien in that case.

2. MANY-TO-ONE FORMULA

Since $\varphi(t^*) = 0$, $M_n := \sum_{|u|=n} e^{-t^* V(u)}$ defines a martingale. For $u \in \mathcal{T}$ and $0 \leq k \leq |u|$, we let u_k be the ancestor of u at generation k . In particular $u_0 = \emptyset$. We recall a very convenient tool in the study of branching random walks, namely the many-to-one formula, see [20, Theorem 1.1]:

Fact 2.1 (The many-to-one formula). *Under (H1) or (H2), there exists a one-dimensional random walk $(S_k)_{k \geq 0}$ such that $S_0 = 0$ and for any $k \geq 1$ and any measurable function $f : \mathbb{R}^k \rightarrow \mathbb{R}_+$, we have*

$$(2.1) \quad \mathbb{E} \left[\sum_{|u|=k} f(V(u_1), \dots, V(u_k)) \right] = \mathbb{E} \left[e^{t^* S_k} f(S_1, \dots, S_k) \right],$$

where the step distribution of S is given by

$$\mathbb{E}[h(S_1)] = \mathbb{E} \left[\sum_{|u|=1} e^{-t^* V(u)} h(V(u)) \right],$$

for any Borel bounded function h . In particular,

$$\mathbb{E}[S_1] = \mathbb{E} \left[\sum_{|u|=1} e^{-t^* V(u)} V(u) \right] = \int_{\mathbb{R}} x e^{-t^* x} \mu(dx).$$

For $x \in \mathbb{R}$, we write \mathbb{P}_x for a probability distribution under which the random walk $(S_n)_{n \geq 0}$ starts at x . Notice that $\mathbb{E}[S_1] \neq 0$ under (H1) whereas $\mathbb{E}[S_1] = 0$ under (H2). Under (H2), $(S_n)_{n \geq 0}$ is a centered random walk with finite variance so that by Lawler and Limic [18, Theorem 5.1.7]: there exists some positive constant c such that for any $a, b > 0$ and $x \in (-a, b)$,

$$(2.2) \quad \mathbb{P}_x(T_{(-\infty, -a]} < T_{[b, \infty)}) \geq c \frac{b - x + 1}{b + a + 1}$$

with the notation

$$(2.3) \quad T_A := \inf\{i \geq 1 : S_i \in A\}$$

for any Borel set A .

The many-to-one formula can be extended to optional lines in the sense of [10, Section 6]. In our applications, an optional line \mathcal{L} will be of the form

$$(2.4) \quad \mathcal{L} = \{\xi_k, (\xi, k) \in \partial\mathcal{T} \times \mathbb{N} \text{ such that } \tau(\xi) = k\}$$

where $\tau(\xi) := \inf\{k \geq 0 : (V(\xi_0), V(\xi_1), \dots, V(\xi_k)) \in A_k\}$ and for each $k \geq 0$, A_k is a Borel set of \mathbb{R}^{k+1} . A consequence of the many-to-one formula is that

$$(2.5) \quad \mathbb{E}[\#\mathcal{L}] = \mathbb{E}[e^{t^* S_\tau} \mathbf{1}_{\{\tau < \infty\}}]$$

where $\tau := \inf\{k \geq 0 : (S_0, S_1, \dots, S_k) \in A_k\}$. Here and before, we used the convention that $\inf \emptyset = \infty$. Finally, consider

$$(2.6) \quad M_{\mathcal{L}} := \sum_{u \in \mathcal{L}} e^{-t^* V(u)}.$$

Its first moment is

$$(2.7) \quad \mathbb{E}[M_{\mathcal{L}}] = \mathbb{P}(\tau < \infty).$$

It is a consequence of the many-to-one formula. We will need to compute its second moment. For real numbers s_0, \dots, s_k, s_{k+1} , let

$$(2.8) \quad p(s_0, \dots, s_k, s_{k+1}) := \mathbb{P}(k+1 \leq \tau < \infty \mid S_0 = s_0, \dots, S_k = s_k, S_{k+1} = s_{k+1})$$

and

$$(2.9) \quad \psi(s_0, \dots, s_k) := \mathbb{E} \left[\sum_{|u|, |v|=1, u \neq v} e^{-t^* V(u)} e^{-t^* V(v)} p_u p_v \right]$$

where $p_u = p(s_0, \dots, s_k, s_k + V(u))$, $p_v = p(s_0, \dots, s_k, s_k + V(v))$. With this notation,

$$(2.10) \quad \mathbb{E}[(M_{\mathcal{L}})^2] = \mathbb{E}[e^{-t^* S_\tau} \mathbf{1}_{\{\tau < \infty\}}] + \mathbb{E} \left[\sum_{k=0}^{\tau-1} e^{-t^* S_k} \psi(S_0, \dots, S_k) \right].$$

Let us prove it. For $w \in \mathcal{T}$, we write $w < \mathcal{L}$ if $(V(w_0), V(w_1), \dots, V(w_k)) \notin A_k$ for all $k \leq |w|$. By decomposing $(M_{\mathcal{L}})^2 = \sum_{x, y \in \mathcal{L}} e^{-t^* V(x)} e^{-t^* V(y)}$ with respect to the most recent common ancestor w of x and y , we get

$$(M_{\mathcal{L}})^2 = \sum_{w \in \mathcal{L}} e^{-2t^* V(w)} + \sum_{w < \mathcal{L}} \sum_{u \neq v \text{ children of } w} M_{\mathcal{L}}^u M_{\mathcal{L}}^v$$

with $M_{\mathcal{L}}^u := \sum_{x \in \mathcal{L}^u} e^{-t^* V(x)}$ for any $u \in \mathcal{T}$, where \mathcal{L}^u is the set of vertices x descendants of u which belong to \mathcal{L} . By the branching property and (2.7),

$$\mathbb{E} \left[\sum_{w < \mathcal{L}} \sum_{u \neq v \text{ children of } w} M_{\mathcal{L}}^u M_{\mathcal{L}}^v \right] = \mathbb{E} \left[\sum_{w < \mathcal{L}} \sum_{u \neq v \text{ children of } w} e^{-t^* V(u)} e^{-t^* V(v)} \tilde{p}_u \tilde{p}_v \right]$$

with $\tilde{p}_u := p(V(u_0), V(u_1), \dots, V(u))$ and the analog for \tilde{p}_v . Another use of the branching property shows that

$$\mathbb{E} \left[\sum_{w < \mathcal{L}} \sum_{u \neq v \text{ children of } w} e^{-t^* V(u)} e^{-t^* V(v)} \tilde{p}_u \tilde{p}_v \right] = \mathbb{E} \left[\sum_{w < \mathcal{L}} e^{-2t^* V(w)} \psi(V(w_0), V(w_1), \dots, V(w)) \right].$$

With another use of the many-to-one formula,

$$\begin{aligned} \mathbb{E} \left[\sum_{w \in \mathcal{L}} e^{-2t^* V(w)} \right] &= \mathbb{E} \left[e^{-t^* S_\tau} \mathbf{1}_{\{\tau < \infty\}} \right], \\ \mathbb{E} \left[\sum_{w < \mathcal{L}} e^{-2t^* V(w)} \psi(V(w_0), V(w_1), \dots, V(w)) \right] &= \mathbb{E} \left[\sum_{k=0}^{\tau-1} e^{-t^* S_k} \psi(S_0, \dots, S_k) \right]. \end{aligned}$$

It yields (2.10).

3. PROOF OF THEOREM 1.5

It is enough to prove that \mathbb{P}^* -a.s. for n large enough,

$$(3.1) \quad \sup_{\xi \in \partial \mathcal{T}} N_\xi^n \leq n^\kappa$$

for any $\kappa > 1$ under (H1) and $\kappa > 2$ under (H2). Fix such a κ . Let $a > 0$ be a small constant whose value will be determined later. Without loss of generality, we will prove (3.1) where N_ξ^n is rather defined as

$$N_\xi^n = \sum_{\ell=0}^{\infty} \mathbf{1}_{\{V(\xi_\ell) \in [an, a(n+1)]\}}.$$

For any $u \in \mathcal{T}$ and $k \in \mathbb{Z}$, we let

$$N_u^n := \sum_{\ell=0}^{|u|} \mathbf{1}_{\{V(u_\ell) \in [an, a(n+1)]\}}.$$

We first prove (3.1) under (H1). We introduce the following set

$$\mathcal{L}_n := \{u \in \mathcal{T} : N_u^n \geq \lfloor n^\kappa \rfloor, \max_{0 \leq k < |u|} N_{u_k}^n < \lfloor n^\kappa \rfloor\}$$

which is the optional line of the particles in the branching random walk stopped when they first visited $\lfloor n^\kappa \rfloor$ times the set $[an, a(n+1)]$. It corresponds to (2.4) with $\tau(\xi)$ given by

$$(3.2) \quad \tau_n(\xi) := \inf\{k \geq 0 : N_{\xi_k}^n \geq \lfloor n^\kappa \rfloor\}.$$

Note that

$$(3.3) \quad \left\{ \sup_{\xi \in \partial \mathcal{T}} N_\xi^n \geq n^\kappa \right\} \subset \{\mathcal{L}_n \neq \emptyset\}.$$

Equation (2.5) yields that

$$(3.4) \quad \mathbb{E}(\#\mathcal{L}_n) = \mathbb{E}(e^{t^* S_{\tau_n}} \mathbf{1}_{\{\tau_n < \infty\}}),$$

where

$$(3.5) \quad \tau_n := \inf \left\{ i \geq 0 : \sum_{\ell=0}^i \mathbf{1}_{\{S_\ell \in [an, a(n+1)]\}} \geq \lfloor n^\kappa \rfloor \right\}.$$

We have $S_{\tau_n} \in [an, a(n+1)]$ on $\{\tau_n < \infty\}$. If we set $\Sigma_n := \sum_{\ell=0}^{\infty} 1_{\{S_\ell \in [an, a(n+1)]\}}$, then

$$(3.6) \quad \mathbb{P}(\mathcal{L}_n \neq \emptyset) \leq \mathbb{E}[\#\mathcal{L}_n] \leq e^{at^*(n+1)} \mathbb{P}(\Sigma_n \geq \lfloor n^\kappa \rfloor).$$

Under (H1), the random walk $(S_n)_{n \geq 0}$ either drifts to $+\infty$ a.s. or to $-\infty$ a.s. according to $\mathbb{E}[S_1] > 0$ or $\mathbb{E}[S_1] < 0$. Suppose for example that $S_n \rightarrow \infty$ a.s. In particular $\mathbb{P}(S_n \geq 0, \forall n \geq 0) > 0$. Choose $a > 0$ such that $\mathbb{P}(S_1 > a) > 0$. For any $n \geq 0$ and $x \in [an, a(n+1)]$,

$$\mathbb{P}_x(T_{[an, a(n+1)]} = \infty) \geq \mathbb{P}(S_n > a, \forall n \geq 1) =: c > 0$$

with the notation (2.3) for $T_{[an, a(n+1)]}$. By the Markov property, for any $\ell \geq 1$, $\mathbb{P}(\Sigma_n \geq \ell) \leq (1-c)^{\ell-1}$. In view of (3.6), using that $\kappa > 1$, we get that $\sum_n \mathbb{P}(\mathcal{L}_n \neq \emptyset)$ is summable. The Borel–Cantelli lemma implies that $\mathcal{L}_n = \emptyset$ for n large enough, and we deduce from (3.3) that (3.1) holds. The same reasoning deals with the case $S_n \rightarrow -\infty$.

It remains to prove (3.1) under (H2). We apply the same arguments using the standard strategy of introducing a barrier. Recall that under (H2), $m_n \rightarrow +\infty$ \mathbb{P}^* -a.s. Let $c > 0$. Now $\mathcal{L}_n = \mathcal{L}_n(c)$ is defined as

$$\mathcal{L}_n := \{u \in \mathcal{T} : N_u^n \geq \lfloor n^\kappa \rfloor, \max_{0 \leq k < |u|} N_{u_k}^n < \lfloor n^\kappa \rfloor, \min_{0 \leq k \leq |u|} V(u_k) > -c\}.$$

It is of the form (2.4) with $\tau(\xi) = \tau_n(\xi)$ of (3.2) if $\min_{0 \leq k \leq \tau_n(\xi)} V(\xi_k) > -c$ and $\tau(\xi) = \infty$ otherwise. Using (3.5) for the definition of τ_n , the many-to-one formula (2.5) tells us that

$$\mathbb{P}(\mathcal{L}_n \neq \emptyset) \leq \mathbb{E}[\#\mathcal{L}_n] = \mathbb{E}[e^{t^* S_{\tau_n}} 1_{\{\min_{0 \leq i \leq \tau_n} S_i > -c\}}] \leq e^{at^*(n+1)} \mathbb{P}\left(\min_{0 \leq i \leq \tau_n} S_i > -c\right).$$

With the notation (2.3) and $\Sigma_n^{(c)} := \sum_{\ell=0}^{T_{(-\infty, -c]}} 1_{\{S_\ell \in [an, a(n+1)]\}}$, we get

$$\mathbb{P}(\mathcal{L}_n \neq \emptyset) \leq e^{at^*(n+1)} \mathbb{P}(\Sigma_n^{(c)} \geq \lfloor n^\kappa \rfloor).$$

We now choose $a > 0$ small enough such that $\mathbb{P}(S_1 < -a) > 0$. By (2.2), there exists $c_1 > 0$ such that for any $n \geq 1$ and $x \in [an, a(n+1)]$,

$$\mathbb{P}_x(T_{(-\infty, -c]} < T_{[an, a(n+1)]}) \geq \frac{c_1}{n}.$$

This, in view of the Markov property, implies that for any $\ell \geq 1$,

$$\mathbb{P}(\Sigma_n^{(c)} \geq \ell) \leq \left(1 - \frac{c_1}{n}\right)^{\ell-1}.$$

Hence we get that

$$\mathbb{P}(\mathcal{L}_n \neq \emptyset) \leq e^{at^*(n+1)} e^{-c_1(\lfloor n^\kappa \rfloor - 1)n^{-1}}.$$

For any $\kappa > 2$, the sum over n converges. The Borel–Cantelli lemma yields that almost surely, for all large n , $\mathcal{L}_n = \emptyset$, hence $\sup_{\xi \in \partial \mathcal{T}} N_\xi^n \leq n^\kappa$ on the event $\{\inf_{u \in \mathcal{T}} V(u) > -c\}$. Since \mathbb{P}^* -a.s. $\inf_{u \in \mathcal{T}} V(u) > -\infty$, we conclude by letting $c \rightarrow \infty$. \square

4. PROOF OF THEOREM 1.3

4.1. Proof of Part (i). It is a consequence of the following lemma.

Lemma 4.1. *Under the conditions of Theorem 1.3 (i), $\sup_{u \in \mathcal{T}} e^{-V(u)} \eta_u = +\infty$ \mathbb{P}^* -a.s.*

Proof of the lemma. Let $b > 0$ be small enough such that $t^* - 2b > \zeta(1+b)$. For a point process $\widehat{\mathfrak{N}} \leq \mathfrak{N}$, let $\widehat{\mu} := \mathbb{E}[\widehat{\mathfrak{N}}]$, $\widehat{\varphi}(t) := \log \int_{\mathbb{R}} e^{-tx} \widehat{\mu}(dx)$. Since $\varphi(t^*) = 0$ and $\varphi(t) > 0$ for all $t < t^*$, monotone convergence implies that we can choose $\widehat{\mathfrak{N}} \leq \mathfrak{N}$ such that:

- (i) there exists $a > 0$ such that $\widehat{\mathfrak{N}}(\mathbb{R}) \leq a$ a.s.,

- (ii) the support of $\widehat{\aleph}$ is in $(-a, a)$ a.s.,
- (iii) $\widehat{\varphi}(0) > 0$,
- (iv) $\widehat{\varphi}(s) = 0$ for some $s \in (0, t^*)$ which satisfies $s - 2b > \zeta(1 + b)$.

We let $(V(u), u \in \widehat{\mathcal{T}})$ be the branching random walk associated to the point process $\widehat{\aleph}$, which is contained in the original branching random walk $(V(u), u \in \mathcal{T})$, i.e. $\widehat{\mathcal{T}} \subset \mathcal{T}$. Necessarily $\widehat{\varphi}(t^*) < 0$. Since $s\widehat{\varphi}'(s) < 0 = \widehat{\varphi}(s)$, Biggins' theorem [8] implies that $\sum_{|u|=n, u \in \widehat{\mathcal{T}}} e^{-sV(u)}$ converges a.s. and in L^1 to some random variable \widehat{W} which is positive on $\{\partial\widehat{\mathcal{T}} \neq \emptyset\}$. Let

$$\widehat{\mathcal{L}}_k := \{u \in \widehat{\mathcal{T}} : V(u) \geq k, \max_{0 \leq j < |u|} V(u_j) < k\}$$

be the set of particles which first cross level k . Notice that for any $u \in \widehat{\mathcal{L}}_k$, $V(u) \in [k, k+a]$. Theorem 6.1 in [10] shows that $\sum_{u \in \widehat{\mathcal{L}}_k} e^{-sV(u)}$ also converges a.s. to \widehat{W} . Since

$$\sum_{u \in \widehat{\mathcal{L}}_k} e^{-sV(u)} \leq e^{-sk} \#\widehat{\mathcal{L}}_k,$$

we deduce that a.s. on $\{\widehat{W} > 0\}$, for k large enough,

$$\#\widehat{\mathcal{L}}_k \geq e^{(s-b)k}.$$

On the other hand, (1.6) implies the existence of a constant $c > 0$ such that for any $x \geq 1$, $\mathbb{P}(\eta \leq x) \leq 1 - cx^{-\zeta-\varepsilon}$ where $\varepsilon > 0$ is such that $(1+b)(\zeta+\varepsilon) = s-2b$. By the branching property applied to the optional line $\widehat{\mathcal{L}}_k$,

$$\mathbb{P}\left(\forall u \in \widehat{\mathcal{L}}_k, \eta_u \leq e^{(1+b)k}, \#\widehat{\mathcal{L}}_k \geq e^{(s-b)k}\right) \leq (1 - ce^{-(s-2b)k})^{e^{(s-b)k}} \leq e^{-ce^{bk}}.$$

The right-hand side is summable in k , hence the Borel–Cantelli lemma shows that a.s. on $\{\widehat{W} > 0\}$, for all k large enough one can find $u \in \widehat{\mathcal{L}}_k$ such that $\eta_u > e^{(1+b)k}$. For such a u , $e^{-V(u)}\eta_u \geq e^{-k-a}e^{(1+b)k} = e^{bk-a}$. Therefore $\mathbb{P}(\sup_{u \in \mathcal{T}} e^{-V(u)}\eta_u = \infty) \geq \mathbb{P}(\widehat{W} > 0) > 0$. Since $\{\sup_{u \in \mathcal{T}} e^{-V(u)}\eta_u = \infty\}$ is an inherited property, $\mathbb{P}^*(\sup_{u \in \mathcal{T}} e^{-V(u)}\eta_u = \infty) = 1$. \square

4.2. Proof of Part (ii). Suppose now that $\zeta > t^*$. Let $r \in (t^*, \zeta)$ and $b \in (0, 1)$ be such that $t^* - br < 0$. By (1.6), there exists a constant $c > 0$ such that for all $x \geq 1$,

$$(4.1) \quad \mathbb{P}(\eta \geq x) \leq cx^{-r}.$$

By Theorem 1.5 (already proved in Section 3), there exists $\kappa > 0$ such that $\sup_{\xi} N_{\xi}^n \leq \lfloor n^{\kappa} \rfloor$ for all n large enough. Since

$$D(\xi) = \sum_{n=0}^{\infty} e^{-V(\xi_n)} \eta_{\xi_n} \leq \sum_{n=0}^{\infty} e^{-n} \sum_{u \in \xi : n \leq V(u) < n+1} \eta_u \leq \sum_{n=0}^{\infty} e^{-n} N_{\xi}^n \sup_{u \in \mathcal{T}} \eta_u \mathbf{1}_{\{n \leq V(u) < n+1\}},$$

it is enough to show that for n large enough, for all $u \in \mathcal{T}$,

$$(4.2) \quad \eta_u \mathbf{1}_{\{n \leq V(u) < n+1\}} \leq e^{bn}.$$

For $u \in \mathcal{T}$ and $n \geq 0$, define $N_u^n := \sum_{\ell=0}^{|u|} \mathbf{1}_{\{n \leq V(u_{\ell}) < n+1\}}$. For $n \geq 0$ and $k \geq 1$, we consider the optional line

$$\mathcal{L}_n(k) := \{u \in \mathcal{T} : \max_{0 \leq j < |u|} N_{u_j}^n < k, N_u^n = k\}.$$

It is the optional line (2.4) associated to $\tau(\xi) = \inf\{\ell \geq 0 : N_{\xi_\ell}^n = k\}$. The many-to-one formula (2.5) implies that $\mathbb{E}[\#\mathcal{L}_n(k)] \leq e^{t^*(n+1)}$. By the branching property and the union bound, we get that

$$\mathbb{P}(\exists u \in \mathcal{L}_n(k) : \eta_u \geq e^{bn}) \leq e^{t^*(n+1)} \mathbb{P}(\eta \geq e^{bn}) \leq ce^{t^*(n+1)-bn}$$

by (4.1). The sum on the right-hand side over $k \leq \lfloor n^\kappa \rfloor$ and $n \geq 0$ is finite. The Borel–Cantelli lemma implies that for n large enough, $\eta_u \mathbf{1}_{\{n \leq V(u) < n+1, N_u^n \leq \lfloor n^\kappa \rfloor\}} \leq e^{bn}$ for all $u \in \mathcal{T}$ a.s. Since $\sup_{u \in \mathcal{T}} N_u^n \leq \lfloor n^\kappa \rfloor$ a.s. for n large, we proved (4.2). \square

Remark 4.2. Suppose that $\zeta > t^*$. For $\xi, \xi' \in \partial\mathcal{T}$, let $d_{\partial\mathcal{T}}(\xi, \xi') := 2^{-|\xi \wedge \xi'|}$ be the usual ultra metric, where $|\xi \wedge \xi'| := \sup\{n \geq 0 : \xi_n = \xi'_n\}$. Recall from (1.5) that $D(\xi) = \sum_{n=0}^{\infty} e^{-V(\xi_n)} \eta_{\xi_n}$. Since $|D(\xi) - D(\xi')| \leq \sum_{n=|\xi \wedge \xi'|+1}^{\infty} (e^{-V(\xi_n)} \eta_{\xi_n} + e^{-V(\xi'_n)} \eta_{\xi'_n})$, an application of (4.2) yields that \mathbb{P}^* -a.s., $D(\xi^{(k)}) \rightarrow D(\xi)$ for $k \rightarrow \infty$, if $d_{\partial\mathcal{T}}(\xi, \xi^{(k)}) \rightarrow 0$. In other words, \mathbb{P}^* -a.s., $\xi \mapsto D(\xi)$ is continuous on $\partial\mathcal{T}$.

5. PROOF OF THEOREM 1.6

Recall that $(S_n, n \geq 0)$ is associated to the branching random walk through (2.1). From the assumptions on μ , $(S_n, n \geq 0)$ is integer-valued and $\mathbb{P}(S_1 \geq 2) = 0$. We introduce the local times

$$(5.1) \quad \ell_n := \sum_{j=0}^n \mathbf{1}_{\{S_j=0\}}$$

and the inverse local times defined by $\sigma_0 := 0$ and for $j \geq 1$

$$(5.2) \quad \sigma_j := \inf\{k > \sigma_{j-1} : S_k = 0\}$$

with the convention that $\inf \emptyset = \infty$. In the case (i) (namely under (H1')), $\mathbb{E}[S_1] > 0$ and $\mathbb{P}(S_1 < 0) > 0$. We let

$$(5.3) \quad q := \mathbb{P}(\sigma_1 < \infty) \in (0, 1).$$

In the case (ii) (namely under (H2)), the random walk $(S_n, n \geq 0)$ is centered with finite variance and by Lemma A.2 there exists some $\theta > 0$ such that

$$(5.4) \quad q(n) := \mathbb{P}\left(\min_{0 \leq i \leq \sigma_1} S_i \leq -n\right) \sim \frac{\theta}{n}, \quad n \rightarrow \infty.$$

Recall that the minimum of the branching random walk goes to ∞ , \mathbb{P}^* -a.s. in both cases. Notice that the sets $\{u \in \mathcal{T} : V(u) = k, \max_{0 \leq j < |u|} V(u_j) < k\}$ indexed by $k \geq 0$ form a Galton–Watson process. According to (2.5), its mean offspring is e^{t^*} . In particular, the line $\{u \in \mathcal{T} : V(u) = n, \max_{0 \leq j < |u|} V(u_j) < n\}$ has size $e^{t^*n+o(n)}$ as $n \rightarrow \infty$ \mathbb{P}^* -a.s. By considering the descendants of these particles, we see that in order to prove that $\sup_{\xi \in \partial\mathcal{T}} N_\xi^n \leq dn$ as $n \rightarrow \infty$, it is enough to show that there exists $\varepsilon_d > 0$ such that for n large enough

$$(5.5) \quad \mathbb{P}(N_* > dn) \leq e^{-t^*n - \varepsilon_d n}$$

where $N_* := \sup_{\xi \in \partial\mathcal{T}} N_\xi^0$. Indeed, it would be a consequence of the Borel–Cantelli lemma. Similarly, to prove that $\sup_{\xi \in \partial\mathcal{T}} N_\xi^n \geq dn$ as $n \rightarrow \infty$, it is enough to show that there exists $\varepsilon'_d > 0$ such that for n large enough

$$(5.6) \quad \mathbb{P}(N_* > dn) \geq e^{-t^*n + \varepsilon'_d n}.$$

For $u \in \mathcal{T}$ and $k \geq 0$, we let $N_u^k := \sum_{i=0}^{|u|} \mathbf{1}_{\{V(u_i)=k\}}$.

5.1. Proof of Theorem 1.6 (i). Let $d > -\frac{t^*}{\log q}$. We want to prove (5.5). For notational brevity, we treat dn as an integer. We consider the optional line

$$\mathcal{L}_n := \{u \in \mathcal{T} : N_u^0 > dn, \max_{0 \leq j < |u|} N_{u_j}^0 \leq dn\}$$

which is the set of particles stopped when their local time at level 0 exceeds dn . It is of the form (2.4) with $\tau(\xi)$ given by

$$(5.7) \quad \sigma_{dn}(\xi) := \inf\{k \geq 0 : N_{\xi_k}^0 > dn\}.$$

The many-to-one formula (2.5) implies that

$$\mathbb{E}[\#\mathcal{L}_n] = \mathbb{P}(\sigma_{dn} < \infty) = q^{dn}$$

with the notation (5.2). Then (5.5) follows from $\mathbb{P}(N_* > dn) \leq \mathbb{P}(\mathcal{L}_n \neq \emptyset) \leq \mathbb{E}[\#\mathcal{L}_n]$.

We turn to the lower bound. Let $d \in (0, -\frac{t^*}{\log q})$. We want to prove (5.6). Take $K \geq 1$ large enough such that $e^{t^*} q_K^d > 1$ where

$$q_K := \mathbb{P}(\sigma_1 < \infty, \min_{0 \leq n \leq \sigma_1} S_n \geq -K).$$

We let

$$\mathcal{L}_n(K) := \{u \in \mathcal{T} : N_u^0 > dn, \max_{0 \leq j < |u|} N_{u_j}^0 \leq dn, \min_{0 \leq j \leq |u|} V(u_j) \geq -K\}$$

so that if $\mathcal{L}_n(K)$ is nonempty then $N_* > dn$. Using the notation (5.7), $\mathcal{L}_n(K)$ is of the form (2.4) with $\tau(\xi) = \sigma_{dn}(\xi)$ if $\min_{0 \leq j \leq \sigma_{dn}(\xi)} V(\xi_j) \geq -K$, and $\tau(\xi) = \infty$ otherwise. We introduce

$$M_{\mathcal{L}_n(K)} := \sum_{u \in \mathcal{L}_n(K)} e^{-t^* V(u)}.$$

We have by (2.7)

$$(5.8) \quad \mathbb{E}[M_{\mathcal{L}_n(K)}] = \mathbb{P}(\tau < \infty) = \mathbb{P}(\sigma_{dn} < \infty, \min_{0 \leq j \leq \sigma_{dn}} S_j \geq -K) = q_K^{dn}$$

where now $\tau = \sigma_{dn}$ if $\min_{0 \leq j \leq \sigma_{dn}} S_j \geq -K$, and $\tau = \infty$ otherwise. Recall the notation $p(s_0, \dots, s_{k+1})$ and $\psi(s_0, \dots, s_k)$ in (2.8) and (2.9), which read in our setting

$$\begin{aligned} p(s_0, \dots, s_k, s_{k+1}) &= \mathbb{P}(\tau < \infty \mid S_0 = s_0, \dots, S_k = s_k, S_{k+1} = s_{k+1}) \\ &= \mathbb{P}(\sigma_{dn} < \infty, \min_{0 \leq j \leq \sigma_{dn}} S_j \geq -K \mid S_0 = s_0, \dots, S_k = s_k, S_{k+1} = s_{k+1}) \end{aligned}$$

and

$$\psi(s_0, \dots, s_k) = \mathbb{E} \left[\sum_{|u|, |v|=1, u \neq v} e^{-t^* V(u)} e^{-t^* V(v)} p_u p_v \right]$$

where $p_u = p(s_0, \dots, s_k, s_k + V(u))$, $p_v = p(s_0, \dots, s_k, s_k + V(v))$. The second moment of $M_{\mathcal{L}_n(K)}$ is given by (2.10), i.e.

$$(5.9) \quad \mathbb{E}[(M_{\mathcal{L}_n(K)})^2] = \mathbb{E} [e^{-t^* S_\tau} \mathbf{1}_{\{\tau < \infty\}}] + \mathbb{E} \left[\sum_{k=0}^{\tau-1} e^{-t^* S_k} \psi(S_0, \dots, S_k) \right].$$

Notice that

$$\psi(S_0, \dots, S_k) \mathbf{1}_{\{k < \tau\}} = \psi(S_0, \dots, S_k) \mathbf{1}_{\{k < \sigma_{dn}, \min_{0 \leq j \leq k} S_j \geq -K\}}.$$

Indeed, $p(s_0, \dots, s_{k+1}) = \mathbb{P}(k+1 \leq \tau < \infty \mid S_0 = s_0, \dots, S_{k+1} = s_{k+1}) = \mathbf{1}_{\{\sigma_{dn} \geq k+1\}} \mathbb{P}(\sigma_{dn} < \infty, \min_{0 \leq j \leq \sigma_{dn}} S_j \geq -K \mid S_0 = s_0, \dots, S_{k+1} = s_{k+1})$. Therefore $\psi(S_0, \dots, S_k) = 0$ on $\{\sigma_{dn} \leq k\} \cup \{\min_{0 \leq j \leq k} S_j < -K\}$, and the above equality follows.

In (5.9),

$$\mathbb{E} [e^{-t^* S_\tau} \mathbf{1}_{\{\tau < \infty\}}] = \mathbb{P}(\tau < \infty) = q_K^{dn}$$

by (5.8). The last term of (5.9) is

$$(5.10) \quad \mathbb{E} \left[\sum_{k=0}^{\sigma_{dn}-1} e^{-t^* S_k} \psi(S_0, \dots, S_k) \mathbf{1}_{\{\min_{0 \leq j \leq k} S_j \geq -K\}} \right].$$

We bound $p(s_0, \dots, s_k, s_{k+1})$ for s_0, \dots, s_k such that $\ell := \sum_{j=0}^k \mathbf{1}_{\{s_j=0\}} \leq dn$ and $\min_{0 \leq j \leq k} s_j \geq -K$. By the strong Markov property at the first hitting time of 0 after time $k+1$, we see that $p(s_0, \dots, s_k, s_{k+1}) \leq q_K^{dn-\ell}$. Hence $\psi(s_0, \dots, s_k) \leq c' q_K^{2(dn-\ell)}$ with

$$c' = \mathbb{E} \left[\sum_{|u|, |v|=1, u \neq v} e^{-t^* V(u)} e^{-t^* V(v)} \right].$$

Therefore the expectation in (5.10) is less than

$$c' q_K^{2dn} \mathbb{E} \left[\sum_{k=0}^{\sigma_{dn}-1} e^{-t^* S_k} q_K^{-2\ell_k} \mathbf{1}_{\{\min_{0 \leq j \leq k} S_j \geq -K\}} \right]$$

with the notation (5.1). Discussing over the value of ℓ_k and using the strong Markov property at $\sigma_{\ell-1}$ below, we obtain that

$$\begin{aligned} \mathbb{E} \left[\sum_{k=0}^{\sigma_{dn}-1} e^{-t^* S_k} q_K^{-2\ell_k} \mathbf{1}_{\{\min_{0 \leq j \leq k} S_j \geq -K\}} \right] &= \sum_{\ell=1}^{dn} q_K^{-2\ell} \mathbb{E} \left[\sum_{k=\sigma_{\ell-1}}^{\sigma_{\ell}-1} e^{-t^* S_k} \mathbf{1}_{\{\min_{0 \leq j \leq k} S_j \geq -K, \sigma_{\ell} < \infty\}} \right] \\ &= c'' \sum_{\ell=1}^{dn} q_K^{-2\ell} \mathbb{P} \left(\sigma_{\ell-1} < \infty, \min_{0 \leq j \leq \sigma_{\ell-1}} S_j \geq -K \right) \\ &= c'' \sum_{\ell=1}^{dn} q_K^{-(\ell+1)}, \end{aligned}$$

with

$$c'' := \mathbb{E} \left[\sum_{k=0}^{\sigma_1-1} e^{-t^* S_k} \mathbf{1}_{\{\min_{0 \leq j \leq k} S_j \geq -K, \sigma_1 < \infty\}} \right].$$

By Lemma A.1, $c'' < \infty$. We get that the second moment of $M_{\mathcal{L}_n(K)}$ is bounded by $c''' q_K^{dn}$. Recall (5.8). Since

$$\mathbb{P}(N^* > dn) \geq \mathbb{P}(\#\mathcal{L}_n(K) > 0) = \mathbb{P}(M_{\mathcal{L}_n(K)} > 0) \geq \frac{\mathbb{E} [M_{\mathcal{L}_n(K)}]^2}{\mathbb{E} [(M_{\mathcal{L}_n(K)})^2]} \geq \frac{1}{c'''} q_K^{dn},$$

and $q_K^d > e^{-t^*}$, we proved (5.6) and then the lower bound in Theorem 1.6 (i).

5.2. Proof of Theorem 1.6 (ii). Upper bound. Let $d > \frac{t^*}{2\theta}$. We show (5.5). We consider a family of continuous non-increasing functions $f_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $f_n(0) = n$, $f_n(t) > 0$ for $0 \leq t < dn^2$ and $f_n(t) = 0$ for all $t \geq dn^2$. We introduce the optional line

$$(5.11) \quad \mathcal{L}_n := \{u \in \mathcal{T} : V(u) \leq -f_n(N_u^0), \forall 0 \leq j < |u|, V(u_j) > -f_n(N_{u_j}^0)\}$$

where we have stopped the particles u which go below $-f_n(N_u^0)$ for the first time. It is the optional line (2.4) associated to $\tau(\xi)$ which is

$$(5.12) \quad \tau_n(\xi) := \inf\{k \geq 0 : V(\xi_k) \leq -f_n(N_{\xi_k}^0)\}.$$

We notice that $\{N_* > dn^2\} \subset \{\mathcal{L}_n \neq \emptyset\}$ as $f_n(dn^2) = 0$ so that $\mathbb{P}(N_* > dn^2) \leq \mathbb{E}[\#\mathcal{L}_n]$. It follows from (2.5) that

$$(5.13) \quad \mathbb{P}(N_* > dn^2) \leq \mathbb{E}[e^{t^* S_{\tau_n}}] =: b_n,$$

where $\tau_n := \inf\{i \geq 0 : S_i \leq -f_n(\ell_i)\}$ with the notation (5.1) for the local time process (ℓ_i) . Observe that $\tau_n \leq \sigma_{\lceil dn^2 \rceil}$, with σ_j defined in (5.2). In the case (ii), (S_n) is a centered random walk with finite variance, so τ_n is finite almost surely.

To estimate b_n , we discuss on the value of ℓ_{τ_n} . Note that

$$(5.14) \quad b_n \leq \mathbb{E} [e^{-t^* f_n(\ell_{\tau_n})}] = \sum_{k=1}^{\lceil dn^2 \rceil} e^{-t^* f_n(k)} \mathbb{P}(\ell_{\tau_n} = k)$$

where we used the fact that $\ell_{\tau_n} \leq \lceil dn^2 \rceil$. Recall the notation $q(n)$ in (5.4). We observe that, for any $k \geq 1$,

$$(5.15) \quad \begin{aligned} \mathbb{P}(\ell_{\tau_n} = k) &= \mathbb{P}\left(\forall 1 \leq j < k, \min_{\sigma_{j-1} \leq i \leq \sigma_j} S_i > -f_n(j), \min_{\sigma_{k-1} \leq i \leq \sigma_k} S_i \leq -f_n(k)\right) \\ &= \prod_{j=1}^{k-1} (1 - q(f_n(j))) q(f_n(k)) \end{aligned}$$

with the convention $\prod_{\emptyset} = 1$. Hence, for all $k \geq 1$,

$$(5.16) \quad e^{-t^* f_n(k)} \mathbb{P}(\ell_{\tau_n} = k) \leq e^{-t^* f_n(k) - \sum_{j=1}^{k-1} q(f_n(j))}.$$

As $d > \frac{t^*}{2\theta}$, we may choose and fix some $\theta' \in (0, \theta)$ such that $\alpha := \sqrt{2d\frac{\theta'}{t^*}} > 1$. By (5.4), there exists $k_0 \geq 1$ such that for all $k \geq k_0$, $q(k) \geq \frac{\theta'}{k}$. Let $f(t) := \alpha \sqrt{(1 - \frac{t}{d})_+}$ and $f_n(t) := nf(t/n^2)$ for $t \geq 0$. We check that for any $t \leq dn^2$,

$$(5.17) \quad t^* f_n(t) + \theta' \int_0^t \frac{ds}{f_n(s)} = \alpha t^* n.$$

It implies that $t^* f_n(k) + \theta' \sum_{j=1}^{k-1} \frac{1}{f_n(j)} \geq \alpha t^* n$ for any $k \geq 0$. Let m_n be the minimal integer such that $f_n(m_n + 2) < k_0$. We can check that $f_n(m_n)$ is bounded in n . For $k \leq m_n$, we get

$$t^* f_n(k) + \sum_{j=1}^{k-1} q(f_n(j)) \geq t^* f_n(k) + \theta' \sum_{j=1}^{k-1} \frac{1}{f_n(j)} \geq \alpha t^* n - \frac{2\theta'}{k_0}$$

while for $k > m_n$, we write

$$t^* f_n(k) + \sum_{j=1}^{k-1} q(f_n(j)) \geq \sum_{j=1}^{m_n-1} q(f_n(j)) \geq \alpha t^* n - \frac{2\theta'}{k_0} - t^* f_n(m_n).$$

In view of (5.16), equation (5.14) becomes $b_n \leq e^{-\alpha t^* n} \lceil dn^2 \rceil e^{\frac{2\theta'}{k_0} + t^* f_n(m_n)}$, proving (5.5).

5.3. Proof of Theorem 1.6 (ii). Lower bound. Let $0 < d < \frac{t^*}{2\theta}$. For $\varrho \in (d, \frac{t^*}{2\theta})$, let $\alpha := \sqrt{2\varrho\frac{\theta}{t^*}}$ and take $f_n(t) = nf(t/n^2)$ with $f(t) = \alpha \sqrt{(1 - \frac{t}{\varrho})_+}$ for $t \geq 0$. Note that $\alpha < 1$. We set $d_n := \lceil dn^2 \rceil + 1$ for concision. Notice from (5.4) and (5.17) that uniformly in $k \leq d_n$,

$$(5.18) \quad e^{-t^* f_n(k)} \prod_{j=0}^{k-1} (1 - q(f_n(j))) = e^{-\alpha t^* n + o(n)}.$$

We consider a slightly modified version of the optional line \mathcal{L}_n defined in (5.11): Let $(\lambda_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lambda_n = o(n)$ as $n \rightarrow \infty$. Consider

$$\begin{aligned} \mathcal{L}_n^{(d)} &:= \{u \in \mathcal{T} : V(u) \leq -f_n(N_u^0), |V(u) + f_n(N_u^0)| \leq \lambda_n, \\ &\quad \forall 0 \leq j < |u|, V(u_j) > -f_n(N_{u_j}^0), N_u^0 = d_n\}. \end{aligned}$$

In other words, we only select those $u \in \mathcal{L}_n$ such that $N_u^0 = d_n$ and the overshoot $|V(u) + f_n(N_u^0)|$ is less than λ_n . If $\mathcal{L}_n^{(d)}$ is not empty, then $N_* \geq d_n > dn^2$. We use the definition of $\tau_n(\xi)$ in (5.12). The optional line $\mathcal{L}_n^{(d)}$ is of the form (2.4) with $\tau(\xi)$ given by $\tau(\xi) := \tau_n(\xi)$ if $N_{\xi_{\tau_n(\xi)}}^0 = d_n$ and $|V(\xi_{\tau_n(\xi)}) + f_n(d_n)| \leq \lambda_n$, and $\tau(\xi) := \infty$ otherwise. Let

$$M_{\mathcal{L}_n^{(d)}} := \sum_{u \in \mathcal{L}_n^{(d)}} e^{-t^* V(u)}.$$

We will use (2.7), with the notation $\tau = \tau_n$ on the event $\{\ell_{\tau_n} = d_n, |S_{\tau_n} + f_n(d_n)| \leq \lambda_n\}$, and $\tau = \infty$ otherwise, where as before, $\tau_n := \inf\{i \geq 0 : S_i \leq -f_n(\ell_i)\}$. We then get that $\mathbb{E}[M_{\mathcal{L}_n^{(d)}}] = \mathbb{P}(\ell_{\tau_n} = d_n, |S_{\tau_n} + f_n(d_n)| \leq \lambda_n)$, and by the strong Markov property at σ_{d_n} ,

$$\mathbb{P}(\ell_{\tau_n} = d_n, |S_{\tau_n} + f_n(d_n)| \leq \lambda_n) = \prod_{j=0}^{d_n-1} (1 - q(f_n(j))) x_n,$$

where

$$x_n := \mathbb{P}(T'_{f_n(d_n)} < \sigma_1, |S_{T'_{f_n(d_n)}} + f_n(d_n)| \leq \lambda_n)$$

with $T'_{f_n(d_n)} := \inf\{i \geq 0 : S_i \leq -f_n(d_n)\}$. By (A.2), if we choose $\lambda_n := K_{\lfloor f_n(d_n) \rfloor}$ (which we will), then $\lambda_n = o(n)$ and $x_n \sim \frac{\theta}{f_n(d_n)} \sim q(f_n(d_n))$ as $n \rightarrow \infty$. Then by (5.15) and (5.18), we get that

$$(5.19) \quad \mathbb{E}[M_{\mathcal{L}_n^{(d)}}] = e^{-\alpha t^* n + t^* f_n(d_n) + o(n)}.$$

We compute now the second moment of $M_{\mathcal{L}_n^{(d)}}$. Recall (2.10) and the notation $p(s_0, \dots, s_{k+1})$ and $\psi(s_0, \dots, s_k)$ in (2.8) and (2.9). Note that $p(s_0, \dots, s_{k+1}) = \mathbb{P}(k+1 \leq \tau < \infty | S_0 = s_0, \dots, S_{k+1} = s_{k+1}) = \mathbf{1}_{\{\tau_n \geq k+1\}} \mathbb{P}(\ell_{\tau_n} = d_n, |S_{\tau_n} + f_n(d_n)| \leq \lambda_n | S_0 = s_0, \dots, S_{k+1} = s_{k+1})$, one has $\psi(s_0, \dots, s_k) = 0$ on $\{\tau_n \leq k\} \cup \{\ell_k > d_n\}$. Therefore

$$\mathbb{E} \left[\sum_{k=0}^{\tau-1} e^{-t^* S_k} \psi(s_0, \dots, s_k) \right] = \mathbb{E} \left[\sum_{k=0}^{\tau_n-1} \mathbf{1}_{\{\ell_k \leq d_n\}} e^{-t^* S_k} \psi(s_0, \dots, s_k) \right].$$

Let us compute $p(s_0, \dots, s_{k+1})$. Let $\ell = \sum_{j=0}^k \mathbf{1}_{\{s_j=0\}}$. Suppose that $\ell \leq d_n - 1$. By the strong Markov property of S at the first hitting time of 0 after time $k+1$, $p(s_0, \dots, s_{k+1})$ is less than

$$\mathbb{P}(\forall 1 \leq j < d_n(\ell), \min_{\sigma_{j-1} \leq i \leq \sigma_j} S_i > -f_n(j+\ell), \min_{\sigma_{d_n(\ell)-1} \leq i \leq \sigma_{d_n(\ell)}} S_i \leq -f_n(d_n))$$

with $d_n(\ell) := d_n - \ell$. It is $\prod_{j=\ell+1}^{d_n-1} (1 - q(f_n(j))) q(f_n(d_n))$. We deduce that

$$\psi(s_0, \dots, s_k) \leq c' \left(\prod_{j=\ell+1}^{d_n-1} (1 - q(f_n(j))) q(f_n(d_n)) \right)^2$$

with $c' := \mathbb{E}[\sum_{|u|, |v|=1, u \neq v} e^{-t^* V(u)} e^{-t^* V(v)}]$. Therefore, in view of (5.18), uniformly over s_0, \dots, s_k ,

$$\psi(s_0, \dots, s_k) \leq e^{2t^*(f_n(d_n) - f_n(\ell)) + o(n)}.$$

When $\ell = d_n$, we use the bound $\psi(s_0, \dots, s_k) \leq c'$ since $p(s_0, \dots, s_{k+1})$ is always smaller than 1. We get

$$\mathbb{E} \left[\sum_{k=0}^{\tau-1} e^{-t^* S_k} \psi(S_0, \dots, S_k) \right] \leq e^{o(n)} \mathbb{E} \left[\sum_{k=0}^{\tau_n-1} \mathbf{1}_{\{\ell_k \leq d_n\}} e^{-t^* S_k} e^{2t^*(f_n(d_n) - f_n(\ell_k))} \right].$$

We discuss on the local times ℓ_k to obtain

$$\mathbb{E} \left[\sum_{k=0}^{\tau_n-1} \mathbf{1}_{\{\ell_k \leq d_n\}} e^{-t^* S_k} e^{2t^*(f_n(d_n) - f_n(\ell_k))} \right] = \sum_{\ell=1}^{d_n} \mathbb{E} \left[\sum_{k=\sigma_{\ell-1}}^{\sigma_{\ell}} \mathbf{1}_{\{k < \tau_n\}} e^{-t^* S_k} \right] e^{2t^*(f_n(d_n) - f_n(\ell))}.$$

By the strong Markov property,

$$\begin{aligned} \mathbb{E} \left[\sum_{k=\sigma_{\ell-1}}^{\sigma_{\ell}} \mathbf{1}_{\{k < \tau_n\}} e^{-t^* S_k} \right] &\leq \mathbb{P}(\tau_n \geq \sigma_{\ell-1}) \mathbb{E} \left[\sum_{k=0}^{\infty} \mathbf{1}_{\{\min_{0 \leq i \leq k} S_i \geq -f_n(\ell)\}} e^{-t^* S_k} \right] \\ &\leq c'' \mathbb{P}(\tau_n \geq \sigma_{\ell-1}) e^{t^* f_n(\ell)} \end{aligned}$$

with $c'' := \sup_{x \geq 0} \mathbb{E} \left[\sum_{k=0}^{\infty} \mathbf{1}_{\{\min_{0 \leq i \leq k} S_i + x \geq 0\}} e^{-t^* S_k - t^* x} \right]$ which is finite by Lemma A.1. Since $\mathbb{P}(\tau_n \geq \sigma_{\ell-1}) = \prod_{j=1}^{\ell-1} (1 - q(f_n(j)))$, we obtain by (5.18) $\mathbb{P}(\tau_n \geq \sigma_{\ell-1}) e^{t^* f_n(\ell)} = e^{-\alpha t^* n + 2t^* f_n(\ell) + o(n)}$ hence

$$\mathbb{E} \left[\sum_{k=0}^{\tau-1} e^{-t^* S_k} \psi(S_0, \dots, S_k) \right] \leq e^{-\alpha t^* n + 2t^* f_n(d_n) + o(n)}.$$

Moreover $\mathbb{E} [e^{-t^* S_{\tau}} \mathbf{1}_{\{\tau < \infty\}}] = \mathbb{E} [e^{-t^* S_{\tau_n}} \mathbf{1}_{\{\ell_{\tau_n} = d_n, |S_{\tau_n} + f_n(d_n)| \leq \lambda_n\}}] \leq e^{t^* f_n(d_n) + \lambda_n} \mathbb{P}(\ell_{\tau_n} = d_n)$. By (5.15) and (5.18), $\mathbb{P}(\ell_{\tau_n} = d_n) = e^{-\alpha t^* n + t^* f_n(d_n) + o(n)}$. Recall that $\lambda_n = o(n)$. Therefore (2.10) implies $\mathbb{E}[(M_{\mathcal{L}_n^{(d)}})^2] \leq e^{-\alpha t^* n + 2t^* f_n(d_n) + o(n)}$. From

$$\mathbb{P}(N_* > dn^2) \geq \mathbb{P}(M_{\mathcal{L}_n^{(d)}} > 0) \geq \frac{(\mathbb{E}[M_{\mathcal{L}_n^{(d)}}])^2}{\mathbb{E}[M_{\mathcal{L}_n^{(d)}}^2]},$$

we get $\mathbb{P}(N_* > dn^2) \geq e^{-\alpha t^* n + o(n)}$ which yields (5.6) since $\alpha < 1$.

6. PROOF OF THEOREM 1.2

Recall that \mathbb{P}^* -a.s., $\min_{|u|=n} V(u) \rightarrow \infty$. Let $\Theta := \inf_{\xi \in \partial \mathcal{T}} \limsup_{n \rightarrow \infty} \frac{V(\xi_n)}{\sqrt{n \log n}}$. For any $a > 0$, the event $\{\Theta \geq a\}$ is inherited in the sense that $\{\Theta \geq a\} = \cap_{|u|=1} \{\Theta^{(u)} \geq a\}$, where $\Theta^{(u)}$ is defined exactly as Θ but for the branching random walk indexed by the subtree \mathcal{T} rooted at u . Hence (see [19]) $\mathbb{P}^*(\Theta \geq a) \in \{0, 1\}$. It follows that \mathbb{P}^* -a.s., Θ is a constant. We only need to check that $\Theta \in (0, \infty)$, which will be done separately in the next two subsections.

6.1. Lower bound: Proof of $\Theta > 0$. Let $a > 0$ whose value will be determined later. Fix $K > 0$. We will show that for any $n_0 \geq 1$,

$$(6.1) \quad \lim_{n \rightarrow \infty} \mathbb{P} \left(\exists |u| = n : -K \leq V(u_{\ell}) \leq a\sqrt{\ell \log \ell}, \forall n_0 \leq \ell \leq n \right) = 0.$$

In fact, on the event $\{\inf_{u \in \mathcal{T}} V(u) \geq -K\}$, (6.1) yields that a.s. for all n_0 , there is no $\xi \in \partial \mathcal{T}$ such that $V(\xi_{\ell}) \leq a\sqrt{\ell \log \ell}, \forall n_0 \leq \ell$, which a fortiori yields that $\Theta \geq a$. Since \mathbb{P}^* -a.s. $\inf_{u \in \mathcal{T}} V(u) > -\infty$, we get $\Theta \geq a$.

It remains to show (6.1). Let $C > 0$, $L > 1$ and $r_i := n_0 + \lfloor C i^2 \log i \rfloor$ for $1 \leq i \leq L$, and consider the set

$$\mathcal{L}_{L, (r_i)} := \{|u| = r_L : -K \leq V(u_{r_i}) \leq a\sqrt{r_i \log r_i}, \forall 1 \leq i \leq L\}.$$

Note that the probability term in (6.1) is less than $\mathbb{P}(\#\mathcal{L}_{L,(r_i)} > 0)$ for any L (indeed we will let $L \rightarrow \infty$). By the Markov inequality and the many-to-one formula (2.1),

$$\begin{aligned} \mathbb{P}(\#\mathcal{L}_{L,(r_i)} > 0) &\leq \mathbb{E}[\#\mathcal{L}_{L,(r_i)}] \\ &= \mathbb{E}\left[e^{t^* S_{r_L}} \mathbf{1}_{\{-K \leq S_{r_i} \leq a\sqrt{r_i \log r_i}, \forall 1 \leq i \leq L\}}\right] \\ &\leq e^{at^* \sqrt{r_L \log r_L}} \mathbb{P}\left(\cap_{1 \leq i \leq L} \{-K \leq S_{r_i} \leq a\sqrt{r_i \log r_i}\}\right). \end{aligned}$$

By the Markov property for the random walk S ,

$$(6.2) \quad \mathbb{P}\left(\cap_{1 \leq i \leq L} \{-K \leq S_{r_i} \leq a\sqrt{r_i \log r_i}\}\right) \leq \prod_{i=1}^L \mathbb{P}\left(S_{r_{i+1}-r_i} \leq a\sqrt{r_i \log r_i} + K\right).$$

Let us recall some useful estimates on the random walk S . Under (H1'), $\mathbb{E}(S_1) = \int_{\mathbb{R}} x e^{-t^* x} \mu(dx) =: \mathfrak{m} > 0$. By assumption, $c_{\alpha_1} := \mathbb{E}(|S_1|^{\alpha_1}) = \mathbb{E}\left[\sum_{|u|=1} e^{-t^* V(u)} |V(u)|^{\alpha_1}\right] < \infty$ for some $\alpha_1 > 1$. Applying [12, Lemma 2.1], we get some constant $c' > 0$ such that for all $n \geq 1$, $y \geq n^{\max(1/\alpha_1, 1/2)}$ and $x > 0$,

$$\mathbb{P}\left(S_n \leq \mathfrak{m}n - x, \min_{1 \leq i \leq n} (S_i - S_{i-1}) \geq -y\right) \leq c' e^{-x/y}.$$

Using $\mathbb{P}(S_1 \leq -y) \leq c_{\alpha_1} y^{-\alpha_1}$, for any $y > 0$, we get that

$$(6.3) \quad \mathbb{P}\left(S_n \leq \mathfrak{m}n - x\right) \leq c' e^{-x/y} + c_{\alpha_1} n y^{-\alpha_1}.$$

We apply (6.3) to the probability terms on the right-hand side of (6.2). Since $r_{i+1} - r_i \sim 2Ci \log i$ and $a\sqrt{r_i \log r_i} \sim a\sqrt{C}i \log i$, we will choose $C > 0$ such that

$$C\mathfrak{m} > a\sqrt{C}.$$

For all large i , $a\sqrt{r_i \log r_i} + K \leq \frac{\mathfrak{m}}{2}(r_{i+1} - r_i)$. Applying (6.3) to $n = r_{i+1} - r_i$, $x = \frac{\mathfrak{m}}{2}(r_{i+1} - r_i)$ and $y = (r_{i+1} - r_i)^{1-\delta}$ with $0 < \delta < \min(\frac{1}{2}, 1 - \frac{1}{\alpha_1})$, we have for all large i , say $i \geq i_1$,

$$\mathbb{P}\left(S_{r_{i+1}-r_i} \leq a\sqrt{r_i \log r_i} + K\right) \leq \mathbb{P}\left(S_{r_{i+1}-r_i} \leq \frac{\mathfrak{m}}{2}(r_{i+1} - r_i)\right) \leq (r_{i+1} - r_i)^{-\delta'},$$

where $\delta' \in (0, \alpha_1(1 - \delta) - 1)$ is a positive constant. It follows that

$$\mathbb{P}(\#\mathcal{L}_{L,(r_i)} > 0) \leq e^{at^* \sqrt{r_L \log r_L}} e^{-\delta' \sum_{i=i_1}^L \log(r_{i+1} - r_i)}.$$

Note that $\sum_{i=i_1}^L \log(r_{i+1} - r_i) \sim L \log L$ and $at^* \sqrt{r_L \log r_L} \sim at^* \sqrt{C}L \log L$. Now we choose an arbitrary constant $a > 0$ such that $a^2 t^* < \delta' \mathfrak{m}$, and we can find $C > 0$ such that $C\mathfrak{m} > a\sqrt{C}$ and $at^* \sqrt{C} < \delta'$, so that $e^{at^* \sqrt{r_L \log r_L}} e^{-\delta' \sum_{i=i_1}^L \log(r_{i+1} - r_i)} \rightarrow 0$ as $L \rightarrow \infty$. This proves (6.1).

6.2. Upper bound: Proof of $\Theta < \infty$. We want to construct an infinite ray $\xi \in \partial\mathcal{T}$ such that $\limsup_{n \rightarrow \infty} \frac{V(\xi_n)}{\sqrt{n \log n}} < \infty$. The idea is to construct a (inhomogeneous) Galton-Watson process $(\#\mathcal{Z}_i)_{i \geq 0}$ which will survive with positive probability. Let $r_1 > 1$ be a large integer and consider $r_i := r_1 + \lfloor i^2 \log i \rfloor$ for $i \geq 2$. Let $\mathcal{Z}_1 = \{u \in \mathcal{T} : |u| = r_1\}$ be the set of particles alive in the r_1 -th generation of \mathcal{T} . Let $C > 0$ whose value will be determined later. For $i \geq 2$, we construct the set \mathcal{Z}_i recursively by letting

$$\begin{aligned} \mathcal{Z}_i := \cup_{v \in \mathcal{Z}_{i-1}} \Big\{ & u \in \mathcal{T}_v, |u| = r_i : V(u_k) - V(v) \leq C(k - r_{i-1}), \forall r_{i-1} \leq k \leq r_i, \\ & V(u_{r_{i-1}}) - V(v) \geq 2C \log i, V(u) - V(v) \leq C \log i \Big\}, \end{aligned}$$

where \mathcal{T}_v denotes the set of descendants of v . We note that for each i , \mathcal{Z}_i is a subset of $\{u \in \mathcal{T} : |u| = r_i\}$. By the branching property, (\mathcal{Z}_i) constitutes an inhomogeneous Galton-Watson tree. If (\mathcal{Z}_i) survives with positive probability, on the survival set of (\mathcal{Z}_i) , we may find an infinite ray for the system (\mathcal{Z}_i) , say ξ . Then ξ can be also considered as an infinite ray of $\partial\mathcal{T}$. Moreover by definition of (\mathcal{Z}_i) , $V(\xi_{r_i}) - V(\xi_{r_{i-1}}) \leq C \log i$ for any $i \geq 2$, hence $V(\xi_i) \leq \sum_{j=1}^i C \log j \leq 2Ci \log i$ for all large i .

Therefore for any large n , there is a unique j such that $r_j \leq n < r_{j+1}$, and $V(\xi_n) \leq V(\xi_{r_j}) + C(n - r_j) \leq 2Cj \log j + C(r_{j+1} - r_j) \sim 4Cj \log j$. Since $\sqrt{n \log n} \geq \sqrt{r_j \log r_j} \sim \sqrt{2j \log j}$, we see that for all large n , $V(\xi_n) \leq 3C\sqrt{n \log n}$. This yields $\Theta \leq 3C$ with positive probability. Since Θ is deterministic, we get $\Theta \leq 3C$, the desired upper bound.

It remains to show that (\mathcal{Z}_i) can survive with positive probability. We will make a coupling between $(\#\mathcal{Z}_i)$ and a homogenous Galton-Watson process. Let $b_i := r_i - r_{i-1} - 1$ and

$$\nu_i := \sum_{|w|=b_i} 1_{\{V(w_k) \leq Ck, \forall k \leq b_i, V(w) \geq 2C \log i\}} 1_{\{\exists u \in \mathcal{T}_w : |u|=b_i+1, V(u) \leq C \log i\}}.$$

Observe that $(\#\mathcal{Z}_i)$ is stochastically larger than an inhomogeneous Galton-Watson process, say (Z'_i) , with ν_i as the reproduction law of the particles in the $(i-1)$ th generation of Z' . It is enough to show that (Z'_i) can survive with positive probability (if we choose C sufficiently large).

We claim that for any $K_0 \geq 1$,

$$(6.4) \quad \limsup_{C \rightarrow \infty} \limsup_{i \rightarrow \infty} \mathbb{P}(\nu_i \leq K_0) \leq q_{\mathcal{T}},$$

with $q_{\mathcal{T}} := \mathbb{P}(\mathcal{T} \text{ is finite}) \in [0, 1)$ is the extinction probability of the Galton-Watson tree \mathcal{T} . In fact, (6.4) is an equality, as $\nu_i = 0$ for all large i on the set of extinction of \mathcal{T} .

Assume for the moment (6.4). Let $q' \in (q_{\mathcal{T}}, 1)$. Let K_0 be an integer such that $K_0(1 - q') > 1$. We may choose (and then fix) $C > 0$ large enough such that for all large i , $\mathbb{P}(\nu_i \leq K_0) \leq q'$. Then for all large i , ν_i is stochastically larger than a Bernoulli variable which equals K_0 with probability $1 - q'$ and 0 with probability q' . Since the (homogenous) Galton-Watson process with such a Bernoulli variable as reproduction law is supercritical, we see that the inhomogeneous Galton-Watson process (Z'_i) will survive with positive probability.

The rest of this subsection is devoted to the proof of (6.4).

Recall that $\mathbb{E}[S_1] = \mathfrak{m} > 0$. Let $C > \mathfrak{m}$ be large enough such that

$$(6.5) \quad i^{Ct^*} b_i^{-\alpha_2} \rightarrow \infty, \quad \text{as } i \rightarrow \infty.$$

Let for $\ell \geq 0$, $\mathcal{F}_\ell := \sigma\{u, V(u) : |u| \leq \ell\}$ be the σ -field generated by the BRW up to generation ℓ . Conditionally on \mathcal{F}_{b_i} , ν_i is a sum of independent Bernoulli random variables:

$$\mathbb{E}[\nu_i | \mathcal{F}_{b_i}] = \sum_{|w|=b_i} 1_{\{V(w_k) \leq Ck, \forall k \leq b_i, V(w) \geq 2C \log i\}} \vartheta(V(w) - C \log i),$$

with $\vartheta(y) := \mathbb{P}(\exists |u| = 1 : V(u) \leq -y)$. By assumption for all large y , $\vartheta(y) \geq e^{-t^*y} y^{-\alpha_2}$. We also have

$$\vartheta(y) \leq \mathbb{E}\left[\sum_{|u|=1} 1_{\{V(u) \leq -y\}}\right] \leq e^{-t^*y} \mathbb{E}\left[\sum_{|u|=1} e^{-t^*V(u)} 1_{\{V(u) \leq -y\}}\right] \leq c_{\alpha_1} e^{-t^*y} y^{-\alpha_1},$$

with $c_{\alpha_1} := \mathbb{E}[\sum_{|u|=1} e^{-t^*V(u)} |V(u)|^{\alpha_1}] < \infty$. Therefore

$$(6.6) \quad \mathbb{E}[\nu_i | \mathcal{F}_{b_i}] \geq i^{Ct^*} (Cb_i)^{-\alpha_2} W'_i,$$

with

$$W'_i := \sum_{|w|=b_i} 1_{\{V(w_k) \leq Ck, \forall k \leq b_i, V(w) \geq 2C \log i\}} e^{-t^* V(w)}.$$

Observe that

$$\begin{aligned} \text{Var}(\nu_i | \mathcal{F}_{b_i}) &= \sum_{|w|=b_i} 1_{\{V(w_k) \leq Ck, \forall k \leq b_i, V(w) \geq 2C \log i\}} \vartheta(V(w) - C \log i) (1 - \vartheta(V(w) - C \log i)) \\ &\leq \mathbb{E}[\nu_i | \mathcal{F}_{b_i}]. \end{aligned}$$

This implies that for any $\varepsilon \in (0, 1)$,

$$\begin{aligned} \mathbb{P}\left(\nu_i - \mathbb{E}[\nu_i | \mathcal{F}_{b_i}] \leq -\varepsilon \mathbb{E}[\nu_i | \mathcal{F}_{b_i}], W'_i \geq \varepsilon\right) &\leq \mathbb{E}\left[\frac{\text{Var}(\nu_i | \mathcal{F}_{b_i})}{\varepsilon^2 (\mathbb{E}[\nu_i | \mathcal{F}_{b_i}])^2} 1_{\{W'_i \geq \varepsilon\}}\right] \\ &\leq \mathbb{E}\left[\frac{1}{\varepsilon^2 \mathbb{E}[\nu_i | \mathcal{F}_{b_i}]} 1_{\{W'_i \geq \varepsilon\}}\right] \\ &\leq \frac{1}{\varepsilon^3 i^{Ct^*} (Cb_i)^{-\alpha_2}} \\ &\rightarrow 0, \end{aligned}$$

as $i \rightarrow \infty$, where the last convergence follows from (6.5).

On the set $\{W'_i \geq \varepsilon\}$, $\mathbb{E}[\nu_i | \mathcal{F}_{b_i}] \geq \varepsilon i^{Ct^*} (Cb_i)^{-\alpha_2} \geq K_0/(1 - \varepsilon)$ for all large i . Therefore,

$$\mathbb{P}(\nu_i \leq K_0) \leq \mathbb{P}\left(\nu_i - \mathbb{E}[\nu_i | \mathcal{F}_{b_i}] \leq -\varepsilon \mathbb{E}[\nu_i | \mathcal{F}_{b_i}], W'_i \geq \varepsilon\right) + \mathbb{P}(W'_i < \varepsilon),$$

yielding that

$$(6.7) \quad \limsup_{i \rightarrow \infty} \mathbb{P}(\nu_i \leq K_0) \leq \limsup_{i \rightarrow \infty} \mathbb{P}(W'_i < \varepsilon).$$

To estimate $\mathbb{P}(W'_i < \varepsilon)$, we compare W'_i with W_i , where

$$W_i := \sum_{|w|=b_i} e^{-t^* V(w)}.$$

Since by assumption $\mathbb{E}\left[\sum_{|u|=1} e^{-t^* V(u)} \log_+(\sum_{|u|=1} e^{-t^* V(u)})\right] < \infty$, we may apply Biggin's theorem ([8]) to see that $W_i \rightarrow W_\infty$ which is positive \mathbb{P}^* -a.s. By the many-to-one formula (2.1),

$$\begin{aligned} \mathbb{E}[W_i - W'_i] &= 1 - \mathbb{P}(S_k \leq Ck, \forall k \leq b_i, S_{b_i} \geq 2C \log i) \\ &\leq \mathbb{P}\left(\max_{k \geq 1} \frac{S_k}{k} > C\right) + \mathbb{P}(S_{b_i} < 2C \log i). \end{aligned}$$

Recall that $\mathbb{E}[S_1] = \mathfrak{m} > 0$ and $b_i \sim 2i \log i$ as $i \rightarrow \infty$. The law of large numbers yields that $\mathbb{P}(S_{b_i} < 2C \log i) \rightarrow 0$. Using $\mathbb{P}(W'_i < \varepsilon) \leq \mathbb{P}(W_i \leq 2\varepsilon) + \mathbb{P}(W_i - W'_i > \varepsilon) \leq \mathbb{P}(W_i \leq 2\varepsilon) + \frac{1}{\varepsilon} \mathbb{E}[W_i - W'_i]$, we get that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \mathbb{P}(\nu_i \leq K_0) &\leq \limsup_{i \rightarrow \infty} \mathbb{P}(W'_i < \varepsilon) \\ &\leq \limsup_{i \rightarrow \infty} \mathbb{P}(W_i \leq 2\varepsilon) + \frac{1}{\varepsilon} \mathbb{P}\left(\max_{k \geq 1} \frac{S_k}{k} > C\right). \end{aligned}$$

Since $\mathbb{P}(\max_{k \geq 1} \frac{S_k}{k} > C) \rightarrow 0$ as $C \rightarrow \infty$ and $\mathbb{P}(W_\infty = 0) = q_T$, we complete the proof of (6.4) by letting $C \rightarrow \infty$ then $\varepsilon \rightarrow 0$.

APPENDIX A. ESTIMATES ON RANDOM WALKS

Lemma A.1. *Let $(S_n, n \geq 0)$ be a real-valued random walk such that $\mathbb{P}(S_1 \neq 0) > 0$. Then*

$$\sup_{x \geq 0} \mathbb{E} \left[\sum_{k=0}^{\infty} \mathbf{1}_{\{\min_{0 \leq i \leq k} S_i + x \geq 0\}} e^{-S_k - x} \right] < \infty.$$

Proof. Let $v_0 := 0$ and $v_j := \inf\{n > v_{j-1} : S_n \leq S_{v_{j-1}}\}$ for $j \geq 1$, be the weak descending ladder times of S , with the convention that $\inf \emptyset = \infty$. We have

$$\mathbb{E} \left[\sum_{k=0}^{\infty} \mathbf{1}_{\{\min_{0 \leq i \leq k} S_i + x \geq 0\}} e^{-S_k} \right] = \sum_{j=0}^{\infty} \mathbb{E} \left[\mathbf{1}_{\{S_{v_j} \geq -x, v_j < \infty\}} e^{-S_{v_j}} \right] \mathbb{E} \left[\sum_{k=0}^{\infty} e^{-S_k} \mathbf{1}_{\{k < v_1\}} \right].$$

The term $\mathbb{E} \left[\sum_{k=0}^{\infty} e^{-S_k} \mathbf{1}_{\{k < v_1\}} \right]$ can be evaluated through the ladder heights. Let $v_0^+ := 0$, $v_j^+ := \inf\{i > v_{j-1}^+ : S_i > S_{v_{j-1}^+}\}$ for $j \geq 1$. By the time-reversal for the random walk,

$$\mathbb{E} \left[\sum_{k=0}^{\infty} e^{-S_k} \mathbf{1}_{\{k < v_1\}} \right] = \mathbb{E} \left[\sum_{j=0}^{\infty} e^{-S_{v_j^+}} \mathbf{1}_{\{v_j^+ < \infty\}} \right] = \sum_{j=0}^{\infty} \left(\mathbb{E} \left[e^{-S_{v_1^+}} \mathbf{1}_{\{v_1^+ < \infty\}} \right] \right)^j < \infty$$

as $\mathbb{E} \left[e^{-S_{v_1^+}} \mathbf{1}_{\{v_1^+ < \infty\}} \right] < 1$. Finally, $\sum_{j=0}^{\infty} \mathbb{E} \left[\mathbf{1}_{\{S_{v_j} \geq -x, v_j < \infty\}} e^{-S_{v_j}} \right] \leq \sum_{z=0}^{\lceil x \rceil} e^z U^-(z)$ where

$$U^-(z) := \sum_{j=0}^{\infty} \mathbb{P}(S_{v_j} \in [-z, -z+1), v_j < \infty).$$

By the classical renewal theorem (see for example [14, Theorem II.4.2]), $\sup_{z \geq 0} U^-(z) = c < \infty$. Therefore

$$\sum_{j=0}^{\infty} \mathbb{E} \left[\mathbf{1}_{\{S_{v_j} \geq -x, v_j < \infty\}} e^{-S_{v_j}} \right] \leq \frac{ce}{e-1} e^{\lceil x \rceil}.$$

It completes the proof. \square

Lemma A.2. *Let $(S_n, n \geq 0)$ be a random walk with values in \mathbb{Z} such that $\mathbb{P}(S_1 \geq 2) = 0$. In agreement with (5.2), let $\sigma_1 := \inf\{k \geq 1 : S_k = 0\}$. Suppose that $\mathbb{E}[S_1] = 0$ and $\mathbb{E}[S_1^2] < \infty$. Then there exists $\theta > 0$ such that*

$$(A.1) \quad \mathbb{P}(T'_{-n} < \sigma_1) \sim \frac{\theta}{n}, \quad n \rightarrow \infty,$$

where $T'_{-n} := \inf\{i \geq 0 : S_i \leq -n\}$. Furthermore, there exists some sequence $(K_n)_{n \geq 0}$ such that $K_n \geq 1$, $K_n = o(n)$ as $n \rightarrow \infty$ and

$$(A.2) \quad \mathbb{P}(T'_{-n} < \sigma_1, |S_{T'_{-n}} + n| \leq K_n) \sim \frac{\theta}{n}, \quad n \rightarrow \infty.$$

Proof. We will prove the lemma with $\theta := \mathbb{E}[|S_{v_1}| \mathbf{1}_{\{v_1 < \sigma_1\}}]$ where $v_1 := \inf\{i \geq 1 : S_i < 0\}$. The optional stopping theorem applied to $T'_{-n} \wedge \sigma_1$ implies that for $-n < x < 0$,

$$x = \mathbb{E}_x[S_{\sigma_1 \wedge T'_{-n}}] = \mathbb{E}_x[S_{T'_{-n}} \mathbf{1}_{\{T'_{-n} < \sigma_1\}}] \leq -n \mathbb{P}_x(T'_{-n} < \sigma_1).$$

Hence $\mathbb{P}_x(T'_{-n} < \sigma_1) \leq \frac{|x|}{n}$ for any $-n < x < 0$, and also for any $x \leq -n$. By the strong Markov property applied to v_1 ,

$$(A.3) \quad \mathbb{P}(T'_{-n} < \sigma_1) = \mathbb{E}[\mathbf{1}_{\{v_1 < \sigma_1\}} \mathbb{P}_{S_{v_1}}(T'_{-n} < \sigma_1)] \leq \frac{\theta}{n}.$$

Discussing the last value taken by S before leaving $[-(n-1), -1]$, we have for any $-n < x < 0$,

$$\mathbb{E}_x[|S_{T'_{-n}} + n| \mathbf{1}_{\{T'_{-n} < \sigma_1\}}] = \sum_{k=1}^{n-1} G_n(x, -k) \mathbb{E}[|S_1 + n - k| \mathbf{1}_{\{S_1 \leq k-n\}}]$$

where $G_n(x, -k) := \sum_{\ell=0}^{\infty} \mathbb{P}_x(S_\ell = -k, \ell < T'_{-n} \wedge \sigma_1)$. Hence the inequality

$$(A.4) \quad \mathbb{E}_x[|S_{T'_{-n}} + n| \mathbf{1}_{\{T'_{-n} < \sigma_1\}}] \leq \sum_{k=1}^{n-1} G_n(x, -k) \mathbb{E}[|S_1| \mathbf{1}_{\{S_1 \leq k-n\}}].$$

As in (2.3), let $T_{\{-k\}} = \inf\{i \geq 1 : S_i = -k\}$. In the sequel, the constant c does not depend on n, k, x and can change its value from line to line. By (2.2), for all $-k < x < 0$

$$\mathbb{P}_x(T_{\{-k\}} < \sigma_1) \leq c \frac{|x|}{k}.$$

We can suppose that the inequality stays true when $x \leq -k$ by taking a bigger c if necessary. Moreover, by (2.2), for all $0 < k < n$,

$$\mathbb{P}_{-k}(T_{\{-k\}} > T'_{-n} \wedge \sigma_1) \geq c \max\left(\frac{1}{k}, \frac{1}{n-k}\right).$$

The last two inequalities imply that $G_n(x, -k) \leq c \min(k, n-k) \frac{|x|}{k}$ for all $-n < x < 0$ and $0 < k < n$. Splitting the sum on the right-hand side of (A.4) according to whether $k \leq \lfloor \frac{n}{2} \rfloor$ or $k > \lfloor \frac{n}{2} \rfloor$ gives

$$(A.5) \quad \mathbb{E}_x[|S_{T'_{-n}} + n| \mathbf{1}_{\{T'_{-n} < \sigma_1\}}] \leq c|x|(I_1 + I_2)$$

where

$$I_1 := \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \mathbb{E}[|S_1| \mathbf{1}_{\{S_1 \leq k-n\}}], \quad I_2 := \frac{1}{n} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} k \mathbb{E}[|S_1| \mathbf{1}_{\{S_1 \leq -k\}}].$$

We observe that

$$I_1 \leq \lfloor \frac{n}{2} \rfloor \mathbb{E}[|S_1| \mathbf{1}_{\{2S_1 \leq -n\}}] \leq \mathbb{E}[S_1^2 \mathbf{1}_{\{2S_1 \leq -n\}}] \rightarrow 0$$

as $n \rightarrow \infty$ by dominated convergence. Similarly, since $\sum_{k=1}^{\min(p,q)} k \leq cp \min(p, q)$ for $p = |S_1|$ and $q = \lfloor \frac{n}{2} \rfloor$,

$$I_2 \leq c \mathbb{E}\left[S_1^2 \frac{\min(|S_1|, n)}{n}\right] \rightarrow 0.$$

We deduce that $\lim_{n \rightarrow \infty} \mathbb{E}_x[|S_{T'_{-n}} + n| \mathbf{1}_{\{T'_{-n} < \sigma_1\}}] = 0$. Writing

$$x = \mathbb{E}_x[S_{\sigma_1 \wedge T'_{-n}}] = -n \mathbb{P}_x(T'_{-n} < \sigma_1) + \mathbb{E}_x[(S_{T'_{-n}} + n) \mathbf{1}_{\{T'_{-n} < \sigma_1\}}],$$

it implies that for all fixed $x < 0$, $\mathbb{P}_x(T'_{-n} < \sigma_1) \sim \frac{|x|}{n}$. By Fatou's lemma,

$$\liminf_{n \rightarrow \infty} n \mathbb{P}(T'_{-n} < \sigma_1) = \liminf_{n \rightarrow \infty} \mathbb{E}[\mathbf{1}_{\{v_1 < \sigma_1\}} n \mathbb{P}_{S_{v_1}}(T'_{-n} < \sigma_1)] \geq \theta.$$

Together with (A.3), it completes the proof of (A.1).

To prove (A.2), it is sufficient to show that we can find a sequence $K_n = o(n)$ with $K_n \geq 1$, such that

$$(A.6) \quad \mathbb{P}(T'_{-n} < \sigma_1, |S_{T'_{-n}} + n| > K_n) = o\left(\frac{1}{n}\right), \quad n \rightarrow \infty.$$

In fact, the probability term in (A.6) is less than

$$\mathbb{P}(S_{v_1} \leq -n) + \mathbb{E}\left[\mathbf{1}_{\{v_1 < \sigma_1, S_{v_1} > -n\}} \mathbb{P}_{S_{v_1}}(T'_{-n} < \sigma_1, |S_{T'_{-n}} + n| > K_n)\right] =: I_3 + I_4.$$

By the Markov inequality,

$$I_3 \leq \frac{1}{n} \mathbb{E} [|S_{v_1}| \mathbf{1}_{\{S_{v_1} \leq -n\}}].$$

According to Doney [11, Corollary 1], $\mathbb{E} [|S_{v_1}|] < \infty$, hence

$$I_3 = o\left(\frac{1}{n}\right), \quad n \rightarrow \infty.$$

For I_4 , using (A.5), we have

$$\begin{aligned} I_4 &\leq \frac{1}{K_n} \mathbb{E} \left[\mathbf{1}_{\{v_1 < \sigma_1, S_{v_1} > -n\}} \mathbb{E}_{S_{v_1}} (|S_{T'_{-n}} + n| \mathbf{1}_{\{T'_{-n} < \sigma_1\}}) \right] \\ &\leq \frac{c(I_1 + I_2)}{K_n} \mathbb{E} [|S_{v_1}|] \leq \frac{c' \varepsilon_n}{K_n}, \end{aligned}$$

with some positive constant c' and $\varepsilon_n := \mathbb{E}[S_1^2 \mathbf{1}_{\{2S_1 \leq -n\}}] + \mathbb{E} \left[S_1^2 \frac{\min(|S_1|, n)}{n} \right] \rightarrow 0$.

If we choose $K_n := \max(1, n\sqrt{\varepsilon_n})$, then $K_n = o(n)$ and $I_4 \leq \frac{c'}{n} \varepsilon_n^{1/2} = o\left(\frac{1}{n}\right)$. Thus, we obtain (A.6) and complete the proof of Lemma A.2. \square

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