



HAL
open science

A Kalman filter-based strategy for space-frequency force reconstruction

Mathieu Aucejo

► **To cite this version:**

Mathieu Aucejo. A Kalman filter-based strategy for space-frequency force reconstruction. Internoise 2024, Aug 2024, Nantes, France. hal-04682354

HAL Id: hal-04682354

<https://hal.science/hal-04682354v1>

Submitted on 30 Aug 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A Kalman filter-based strategy for space-frequency force reconstruction

Mathieu AUCEJO¹

Laboratoire de Mécanique des Structures et des Systèmes Couplés, Cnam Paris
292 rue Saint-Martin - 75141 Paris Cedex 03

ABSTRACT

This contribution presents a Kalman filtering strategy for space-frequency reconstruction of mechanical sources. The proposed approach is based on the definition of an appropriate state-space model, where the state equation assumes that the force vector follows a random walk behavior, while the observation equation is the classical linear relationship between the force and the measured response through the transfer function matrix of the considered structure. One of the main challenges of the proposed approach is the fine-tuning of the covariance matrix associated with the process noise. In this work, it is estimated and adapted at each frequency during the filtering procedure. A numerical experiment is performed on a simply supported beam excited by a broadband point mechanical force to evaluate the reconstruction performance of the proposed approach. Further comparisons with other regularization strategies are also proposed to provide a fair overview of the results obtained.

1. INTRODUCTION

Force reconstruction problems are of great interest in many engineering applications, such as structural health monitoring, non-destructive testing, or noise and vibration control. The basic idea is to estimate the forces acting on a structure from the measured response. The main difficulty of this class of inverse problems is that they are generally ill-posed in Hadamard's sense, meaning that the solution is not unique and that small perturbations in the measured response can lead to large variations in the estimated solutions.

If we focus our scope on force reconstruction problems in the frequency domain, several strategies have been developed to deal with the intrinsic ill-posedness of the problem. Among the methods available in the literature, one can cite the Truncated Singular Value Decomposition [1] or methods deriving from the virtual work principle such as the Virtual Field Method [2] or the Force Analysis Technique [3]. However, despite the recent advent of machine learning techniques [4], the most widely used approach in the frequency domain is the regularization method, which consists in introducing a penalty (regularization) term to the minimization problem to stabilize it by constraining the space of admissible solutions. The most common regularization strategies are the Tikhonov regularization [5], the ℓ_q -regularization [6] or the mixed-norm regularization [7].

In this contribution, we propose a Kalman filter-based strategy for space-frequency force reconstruction. The proposed approach is based on the definition of an appropriate state-space

¹mathieu.aucejo@lecnam.net

model, where the state equation assumes that the force vector follows a random walk behavior, while the observation equation is the classical linear relationship between the force and the measured response through the transfer function matrix of the considered structure. One of the main challenges of the proposed approach is the fine-tuning of the covariance matrix associated with the process noise. In this work, it is estimated and adapted at each frequency during the filtering procedure. A numerical experiment is performed on a simply supported beam excited by a broadband point mechanical force to evaluate the reconstruction performance of the proposed approach. Further comparisons with other regularization strategies are also proposed to provide a fair overview of the results obtained.

2. KALMAN FILTER DERIVATION

This section describes the theoretical foundations of the proposed frequency-domain Kalman filtering, which rely on three main pillars: the definition of an appropriate state space model, the derivation of the filtering algorithm based on the Bayesian paradigm, and some practical considerations related to its implementation.

2.1. State space model

The state space model consists of a state equation, describing the evolution of the system state between two frequency steps, and an output equation relating the measured data to the system state. Here, the state of the system is the force vector to be reconstructed at a given frequency ω_k and noted \mathbf{u}_k , while the output is the measured response \mathbf{y}_k . From these definitions, the state space model can be defined as follows:

$$\begin{cases} \mathbf{u}_{k+1} = \mathbf{u}_k + \mathbf{w}_k \\ \mathbf{y}_k = \mathbf{H}_k \mathbf{u}_k + \mathbf{v}_k \end{cases}, \quad (1)$$

where \mathbf{H}_k is the transfer function matrix of the considered structure at frequency ω_k , while \mathbf{w}_k and \mathbf{v}_k are the process noise and measurement noise vectors, respectively. The noise vectors \mathbf{w}_k and \mathbf{v}_k are assumed to be two independent complex Gaussian variables with zero mean and covariance matrices denoted \mathbf{Q}_k and \mathbf{R}_k , respectively.

2.2. Filtering algorithm

The Kalman filter is a recursive algorithm that estimates the state of a linear dynamic system from a series of noisy measurements [8]. It is composed of two main steps, which consists in updating the state estimate by combining the prior estimate with the information brought by a new measurement. From a Bayesian perspective, a Kalman filter can be divided into four parts [9]:

0. Model definition – It consists mainly in rewriting the state space model given by Equation 1 in Bayesian terms, namely:

$$\begin{aligned} \mathbf{u}_{k+1} &\sim p(\mathbf{u}_{k+1} | \mathbf{u}_k) = \mathcal{N}_c(\mathbf{u}_{k+1} | \mathbf{u}_k, \mathbf{Q}_k) \\ \mathbf{y}_k &\sim p(\mathbf{y}_k | \mathbf{u}_k) = \mathcal{N}_c(\mathbf{y}_k | \mathbf{H}_k \mathbf{u}_k, \mathbf{R}_k), \end{aligned} \quad (2)$$

where $\mathcal{N}_c(\mathbf{x} | \mathbf{m}, \mathbf{P})$ is the complex multivariate Gaussian distribution over the random vector \mathbf{x} with mean vector \mathbf{m} and covariance matrix \mathbf{P} .

1. Initialization – This step requires the prior knowledge of the initial state \mathbf{u}_0 , as well as the prediction of the state $\tilde{\mathbf{u}}_1$ at the next frequency step. These requirements are summarized as follows:

$$\mathbf{u}_0 \sim p(\mathbf{u}_0) = \mathcal{N}_c(\mathbf{u}_0 | \hat{\mathbf{u}}_0, \mathbf{P}_0), \quad (3)$$

where $\hat{\mathbf{u}}_0$ and \mathbf{P}_0 are the mean vector and covariance matrix of the prior estimate, respectively.

$$\mathbf{u}_1 \sim p(\mathbf{u}_1 | \mathbf{u}_0) = \int_{\mathbf{u}_0} p(\mathbf{u}_1 | \mathbf{u}_0) p(\mathbf{u}_0) d\mathbf{u}_0 = \mathcal{N}_c(\mathbf{u}_1 | \tilde{\mathbf{u}}_1, \tilde{\mathbf{P}}_1), \quad (4)$$

where $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{P}}_1$ are the mean vector and covariance matrix of the predicted estimate, respectively. These quantities are expressed as:

$$\tilde{\mathbf{u}}_1 = \hat{\mathbf{u}}_0 \text{ and } \tilde{\mathbf{P}}_1 = \mathbf{P}_0 + \mathbf{Q}_0. \quad (5)$$

2. State update – This step consists of applying the Bayes' rule to update our prediction of the input vector, given the information provided by the measurement at the current frequency step k , that is:

$$\mathbf{u}_k \sim p(\mathbf{u}_k | \mathbf{y}_{1:k}) \propto p(\mathbf{y}_k | \mathbf{u}_k) p(\mathbf{u}_k | \mathbf{y}_{1:k-1}) = \mathcal{N}_c(\mathbf{u}_k | \hat{\mathbf{u}}_k, \mathbf{P}_k), \quad (6)$$

where $\mathbf{y}_{1:k} = \{\mathbf{y}_1, \dots, \mathbf{y}_k\}$ is the set of all the measurements up to the current frequency step k , while $\hat{\mathbf{u}}_k$ and \mathbf{P}_k are the mean vector and covariance matrix of the updated estimate, respectively. The latter quantities are expressed as:

$$\hat{\mathbf{u}}_k = \tilde{\mathbf{u}}_k + \mathbf{K}_k (\mathbf{y}_k - \mathbf{H}_k \tilde{\mathbf{u}}_k) \text{ and } \mathbf{P}_k = (\mathbf{I}_{n_u} - \mathbf{K}_k \mathbf{H}_k) \tilde{\mathbf{P}}_k, \quad (7)$$

where $\mathbf{K}_k = \tilde{\mathbf{P}}_k \mathbf{H}_k^H (\mathbf{H}_k \tilde{\mathbf{P}}_k \mathbf{H}_k^H + \mathbf{R}_k)^{-1}$ is the Kalman gain, \mathbf{I}_{n_u} is the identity matrix of size n_u (n_u : number of reconstruction points) and \mathbf{x}^H is the Hermitian adjoint of \mathbf{x} .

3. State prediction – This step consists of predicting the state of the system at the next frequency step $k+1$, that is:

$$\mathbf{u}_{k+1} \sim p(\mathbf{u}_{k+1} | \mathbf{y}_{1:k}) = \int_{\mathbf{u}_k} p(\mathbf{u}_{k+1} | \mathbf{u}_k) p(\mathbf{u}_k | \mathbf{y}_{1:k}) d\mathbf{u}_k = \mathcal{N}_c(\mathbf{u}_{k+1} | \tilde{\mathbf{u}}_{k+1}, \tilde{\mathbf{P}}_{k+1}), \quad (8)$$

where $\tilde{\mathbf{u}}_{k+1}$ and $\tilde{\mathbf{P}}_{k+1}$ are the mean vector and covariance matrix of the predicted estimate, respectively. These quantities are expressed as:

$$\tilde{\mathbf{u}}_{k+1} = \hat{\mathbf{u}}_k \text{ and } \tilde{\mathbf{P}}_{k+1} = \mathbf{P}_k + \mathbf{Q}_k. \quad (9)$$

The practical implementation of the standard Kalman filter deriving from the previous steps is summarized in Alg. 1. Note that the algorithm can be run in both forward or backward mode, depending on the availability of an accurate starting solution at the first frequency step.

Algorithm 1: Standard Frequency domain Kalman filter

Input: $\mathbf{y}_k, \mathbf{H}_k, \hat{\mathbf{u}}_0, \mathbf{P}_0, \mathbf{Q}_k, \mathbf{R}_k$

Output: $\hat{\mathbf{u}}_k, \mathbf{P}_k$

1. Initialization – $\tilde{\mathbf{u}}_1 = \hat{\mathbf{u}}_0, \tilde{\mathbf{P}}_1 = \mathbf{P}_0 + \mathbf{Q}_0$

for each time step $k > 0$ **do**

 2. State update

$$\mathbf{K}_k = \tilde{\mathbf{P}}_k \mathbf{H}_k^H (\mathbf{H}_k \tilde{\mathbf{P}}_k \mathbf{H}_k^H + \mathbf{R}_k)^{-1}$$

$$\hat{\mathbf{u}}_k = \tilde{\mathbf{u}}_k + \mathbf{K}_k (\mathbf{y}_k - \mathbf{H}_k \tilde{\mathbf{u}}_k)$$

$$\mathbf{P}_k = (\mathbf{I}_{n_u} - \mathbf{K}_k \mathbf{H}_k) \tilde{\mathbf{P}}_k$$

 3. State prediction – $\tilde{\mathbf{u}}_{k+1} = \hat{\mathbf{u}}_k, \tilde{\mathbf{P}}_{k+1} = \mathbf{P}_k + \mathbf{Q}_k$

end

2.3. Practical considerations

The main challenges of the proposed approach concern the computation of an accurate starting solution and the proper tuning of the covariance matrices associated with the process and measurement noise vectors.

3.1 Computation of the starting solution

Due to the recursive nature of the Kalman filter, it is crucial for a proper estimation of the excitation field at the very first frequencies to have an accurate starting solution. In this work, the starting solution is computed by a Bayesian regularization, which consists in estimating \mathbf{u}_0 as the solution of the following optimization problem:

$$(\hat{\mathbf{u}}_0, \hat{\tau}_i) = \underset{(\mathbf{u}_0, \tau_i)}{\operatorname{argmax}} p(\mathbf{y}_0 | \mathbf{u}_0) \prod_{i=1}^{n_u} p(u_{0i} | \tau_i) p(\tau_i), \quad (10)$$

where the likelihood function $p(\mathbf{y}_0 | \mathbf{u}_0)$ and the prior distribution $p(u_{0i} | \tau_i)$ and $p(\tau_i)$ are chosen as follows:

$$\begin{aligned} p(\mathbf{y}_0 | \mathbf{u}_0) &= \mathcal{N}_c(\mathbf{y}_0 | \mathbf{H}_0 \mathbf{u}_0, \mathbf{R}_0) \\ p(u_{0i} | \tau_i) &= \mathcal{N}_c(u_{0i} | 0, \tau_i^{-1}) \quad . \\ p(\tau_i) &= \mathcal{G}(\tau_i | \alpha_i, \beta_i), \end{aligned} \quad (11)$$

Here, $\mathcal{G}(\tau_i | \alpha, \beta)$ is the gamma distribution over the precision parameter τ_i with shape parameter $\alpha = 1$ and rate parameter $\beta = 10^{-18}$.

The optimization problem Equation 10 is solved by an iterative procedure, not described here for the sake of brevity, which is initialized with the solution of the standard Tikhonov regularization problem.

3.2 Noise covariance matrices tuning

The tuning of the covariance matrices associated with the process noise and measurement noise vectors requires special attention as it can greatly affect the quality of the filtering process. In this work, the measurement noise covariance matrix \mathbf{R}_k is assumed to be constant and diagonal. Its entries, i.e. the noise variances of each measurement channel, are estimated from the data using the described proposed in Ref. [10].

The process noise covariance matrix \mathbf{Q}_k is more difficult to tune because it reflects the degree of confidence in the state transition model (state equation). A common practice is to assume that this matrix is isotropic and to set the value of the variance parameter to an arbitrarily large value. However, this approach is not satisfactory, since it does not take into account the specific characteristics of the system under consideration. In fact, setting the variance parameter to too low a value can lead to the state being fixed at a constant value given by the starting solution, while setting it to too high a value can lead to a poor convergence of the filtering process. Several approaches have been proposed in the literature to estimate the variance parameter, such as the L-curve criterion [11]. Other strategies aim at estimating the covariance matrix as a whole, e.g. using an expectation-maximization algorithm [12]. In this work, the process noise covariance matrix is updated at each frequency by revisiting and adapting the time-domain Bayesian approach proposed by Sedehi et al. in Ref [13]. The estimation procedure can be summarized as follows:

1. Prediction step at $k-1$ – The predictive prior distribution of the covariance matrix \mathbf{Q}_{k-1} is given by:

$$p(\mathbf{Q}_{k-1} | \hat{\mathbf{u}}_{k-1}) = \mathcal{F}\mathcal{W}_c(\mathbf{Q}_{k-1} | \Psi_{k-1}, \nu_{k-1}), \quad (12)$$

where $\mathcal{IW}_c(\mathbf{X}|\Psi, \nu)$ is the complex inverse Wishart distribution over the complex random matrix \mathbf{X} with scale matrix Ψ and degrees of freedom ν .

2. Update at step k – After some calculation not detailed here, the posterior distribution of the covariance matrix \mathbf{Q}_k is given by:

$$p(\mathbf{Q}_{k-1}|\hat{\mathbf{u}}_k) = \mathcal{IW}_c(\mathbf{Q}_{k-1}|\Psi_k, \nu_k), \quad (13)$$

where Ψ_k and ν_k are the updated scale matrix and degrees of freedom, respectively. They are expressed as:

$$\Psi_k = \Psi_{k-1} + (\hat{\mathbf{u}}_k - \hat{\mathbf{u}}_{k-1})(\hat{\mathbf{u}}_k - \hat{\mathbf{u}}_{k-1})^H \text{ and } \nu_k = \nu_{k-1} + 1. \quad (14)$$

By assuming that the process noise varies slowly, it is possible to assume that $p(\mathbf{Q}_k|\hat{\mathbf{u}}_k) \approx p(\mathbf{Q}_{k-1}|\hat{\mathbf{u}}_k)$. Finally, the updated covariance matrix \mathbf{Q}_k is chosen as the mean of the updated predictive prior distribution, that is²:

$$\mathbf{Q}_k = \frac{\Psi_k}{\nu_k - n_u} \text{ with } \nu_k > n_u. \quad (15)$$

Finally, to initialize the procedure, the scale matrix Ψ_0 and the degrees of freedom ν_0 must be defined by the user. In the rest of this paper, these parameters are chosen so that $\Psi_0 = 10^{-10} \mathbf{I}_{n_u}$ and $\nu_0 = n_u + 1$.

3. NUMERICAL EXPERIMENT

The structure under consideration is a simply supported stainless steel beam. The beam is 1 m long, 3 cm wide and 1 cm thick. The material properties are $E = 210$ GPa for the Young's modulus and $\rho = 7850$ kg.m⁻³ for the density. A structural damping factor is assumed and set to 1%.

Along the beam, a set of 20 accelerometers are mounted on the structure and one of these sensors is collocated with the excitation at $x_0 = 63$ cm as presented in Fig. 1.

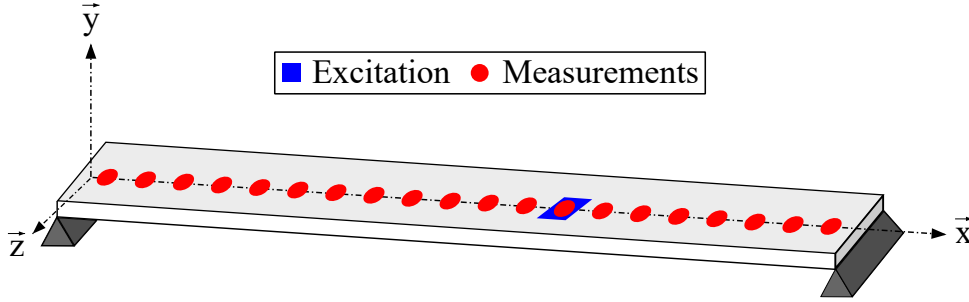


Figure 1: Definition of the numerical experiment

Regarding the excitation spectrum to identify, it is supposed that the structure is excited by a white noise point force with mean 2 N and variance 0.1 N² between 100 Hz and 1 kHz. The frequency resolution is set to 0.5 Hz. In this frequency range, the beam has 4 modes (211 Hz, 375 Hz, 586 Hz and 844 Hz).

In this numerical experiment, the noiseless vibration data have been generated using a finite element model. Then, they have corrupted by an additive Gaussian white noise with a controlled signal-to-noise ratio set to 25 dB. On the other hand, the transfer functions matrices, required to construct the state-space representation, have been computed analytically from a classical mode expansion using the first 10 modes of the structure (i.e. up to 2.3 kHz).

²This choice differs from that made in Ref. [13], where the covariance matrix is chosen as the mode of the updated predictive prior distribution.

3.1. Application of the proposed strategy

Figures 2 and 3 present the results of the proposed Kalman filtering approach when running in the forward mode (i.e. from low to high frequencies) and in the backward mode (i.e. from high to low frequencies), respectively. The analysis of these figures shows that the force location is properly identified, and the force amplitude is correctly estimated for both modes. A closer look at the identified force spectrum reveals a slower convergence of the filter in the forward mode than in the backward mode. This behavior is related to the fact that the filter is initialized with a more accurate solution in the backward mode than in the forward mode (see Fig. 4).

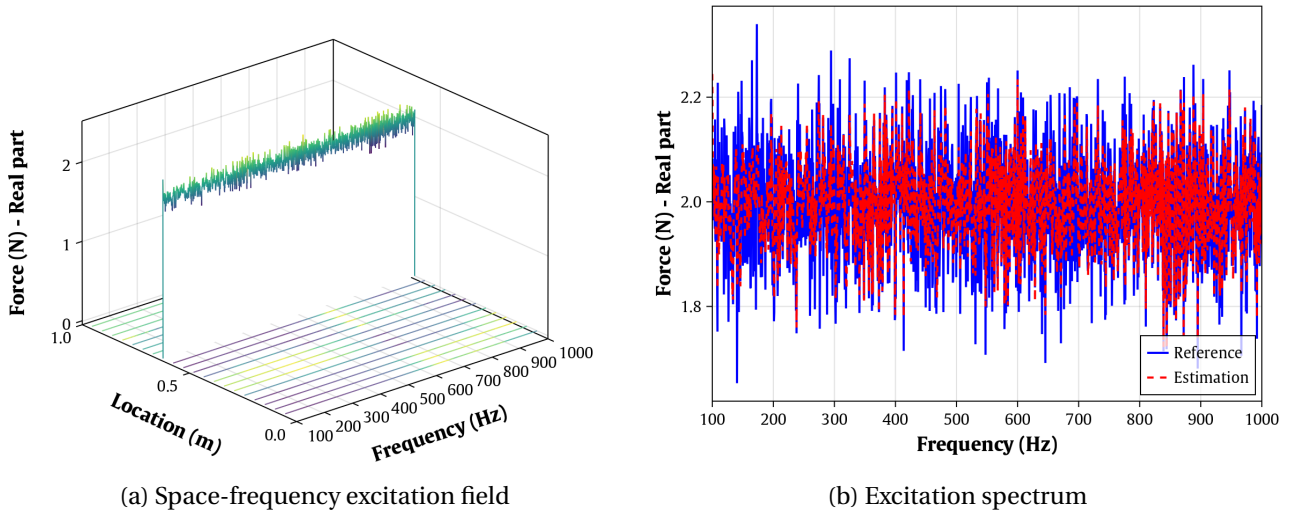


Figure 2: Input force estimated by the proposed Kalman filter run in the forward mode

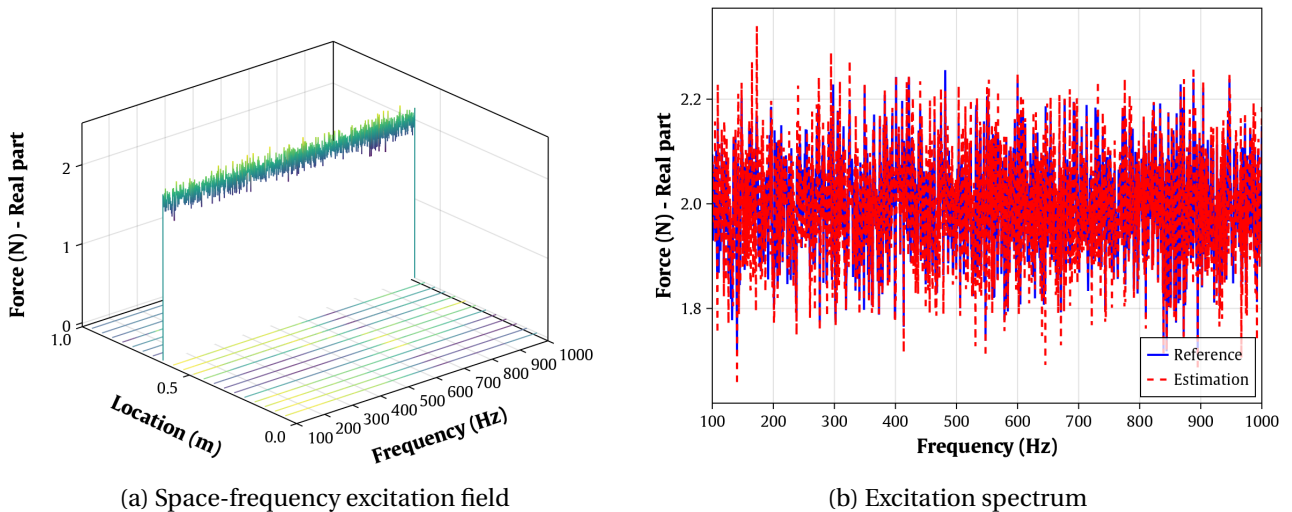


Figure 3: Input force estimated by the proposed Kalman filter run in the backward mode

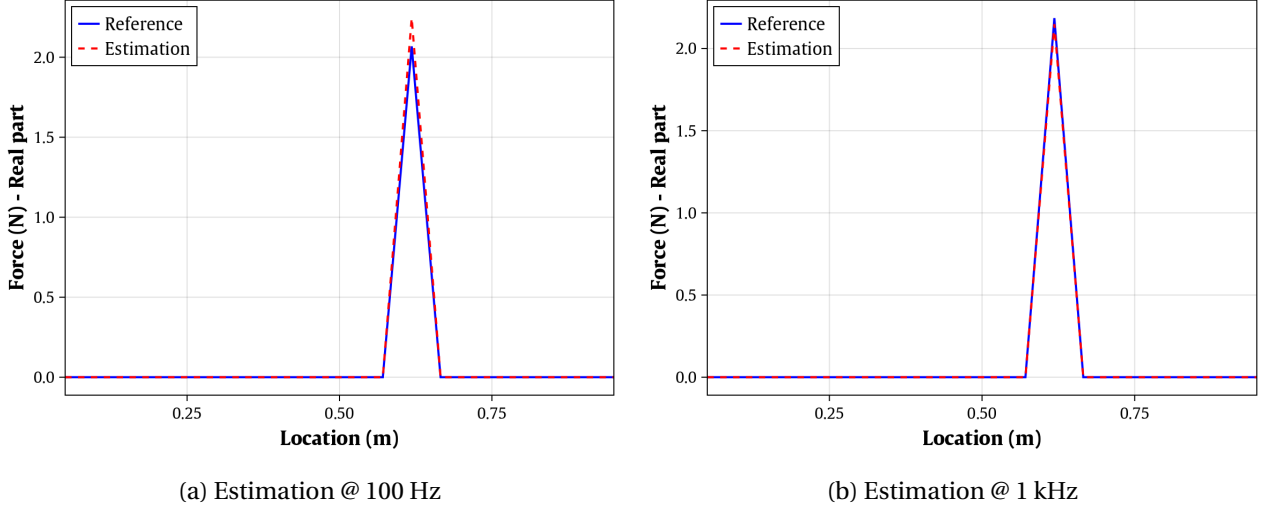


Figure 4: Initial solution used to initialize the Kalman filter - (a) Forward mode and (b) Backward mode

3.2. Comparison with other strategies

The performance of the proposed Kalman filtering approach is compared with two other regularization strategies, namely the multiplicative ℓ_q -regularization [14] and the multiplicative $\ell_{p,q}$ -regularization [15]. The multiplicative ℓ_q -regularization consists in solving the following minimization problem at each frequency:

$$\hat{\mathbf{u}}_k = \underset{\mathbf{u}_k \setminus \{0\}}{\operatorname{argmin}} \left\| \mathbf{y}_k - \mathbf{H}_k \mathbf{u}_k \right\|_{\mathbf{R}}^2 \cdot \|\mathbf{u}_k\|_q^q, \quad (16)$$

while the multiplicative $\ell_{p,q}$ -regularization considers all the frequencies at ones by solving the following minimization problem:

$$\hat{\mathbf{U}} = \underset{\mathbf{U} \setminus \{0\}}{\operatorname{argmin}} \left\| \mathbf{Y} - \bar{\mathbf{H}} \mathbf{U} \right\|_{\mathbf{R}}^2 \cdot \|\mathbf{U}\|_{p,q}^q. \quad (17)$$

In the previous equation, $\mathbf{Y} = \operatorname{vec}(\mathbf{y})$, $\mathbf{U} = \operatorname{vec}(\mathbf{u})$ (the "vec" function stands for vectorization), $\bar{\mathbf{H}} = \operatorname{diag}(\mathbf{H}_1, \dots, \mathbf{H}_{n_f})$ and $\bar{\mathbf{R}} = \mathbf{I}_{n_f} \otimes \mathbf{R}$ (n_f : number of frequencies).

Figure 5 shows that the ℓ_q -regularization identifies the excitation field reasonably well, except around some of the resonance frequencies of the structure (211 Hz and 586 Hz). This behavior is related to the fact that the ℓ_q -regularization solves the problem at each frequency independently. On the other hand, the $\ell_{p,q}$ -regularization, as shown in Fig. 6, provides a consistent reconstruction of the excitation field due to its ability to properly reflect the space-frequency characteristics of the source.

To compare all the previous results quantitatively, three indicators are used: the global relative error (GRE), the execution time t_e and the memory footprint m_f . The GRE is defined as:

$$\text{GRE} = \frac{\|\mathbf{u} - \hat{\mathbf{u}}\|_1}{\|\mathbf{u}\|_1}, \quad (18)$$

where \mathbf{u} and $\hat{\mathbf{u}}$ are the true and estimated excitation fields, respectively.

Table 1 summarizes the performance of the different strategies considered in the paper. The results show that the proposed Kalman filtering approach provides a consistent reconstruction

of the excitation field, with a GRE of 2.51% and 2.39% for the forward and backward modes, respectively. The execution time and memory footprint are also reasonable and similar for the forward and backward modes, with values around 0.29 s and 156 MiB. On the other hand, the results provided by the $\ell_{p,q}$ -regularization are in line with those of the proposed Kalman filtering approach, with a GRE of 2.73%. However, the execution time and memory footprint are respectively about 3 times and 4 times higher than those of the proposed Kalman filtering approach. Finally, the multiplicative ℓ_q -regularization has the worst performance, mainly due to the large discrepancies observed around the resonance frequencies of the structure.

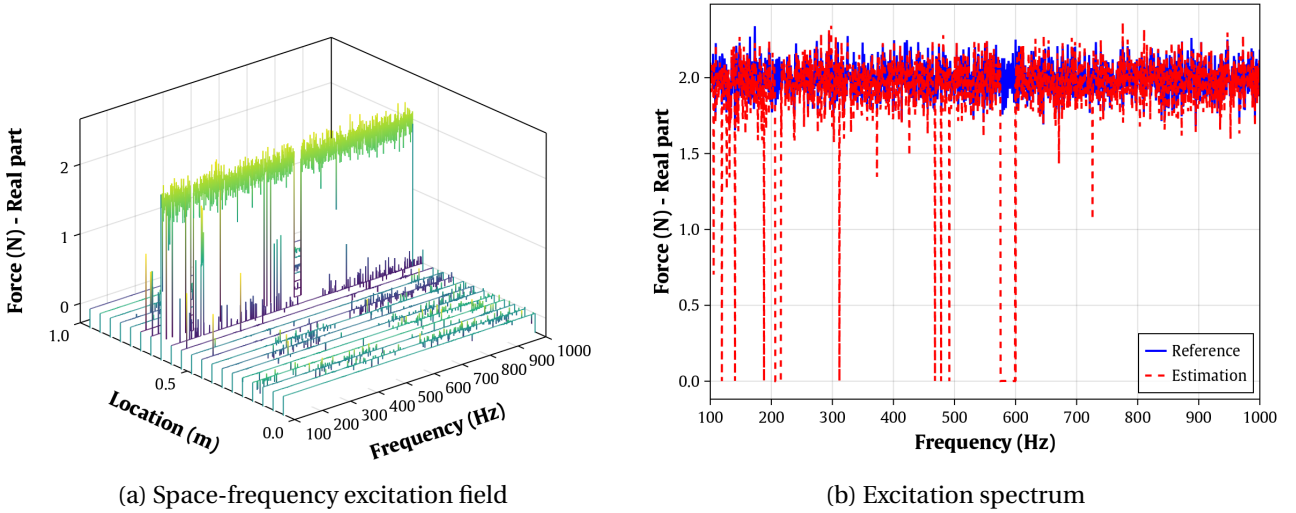


Figure 5: Input force estimated by the multiplicative ℓ_q regularization for $q = 0.5$

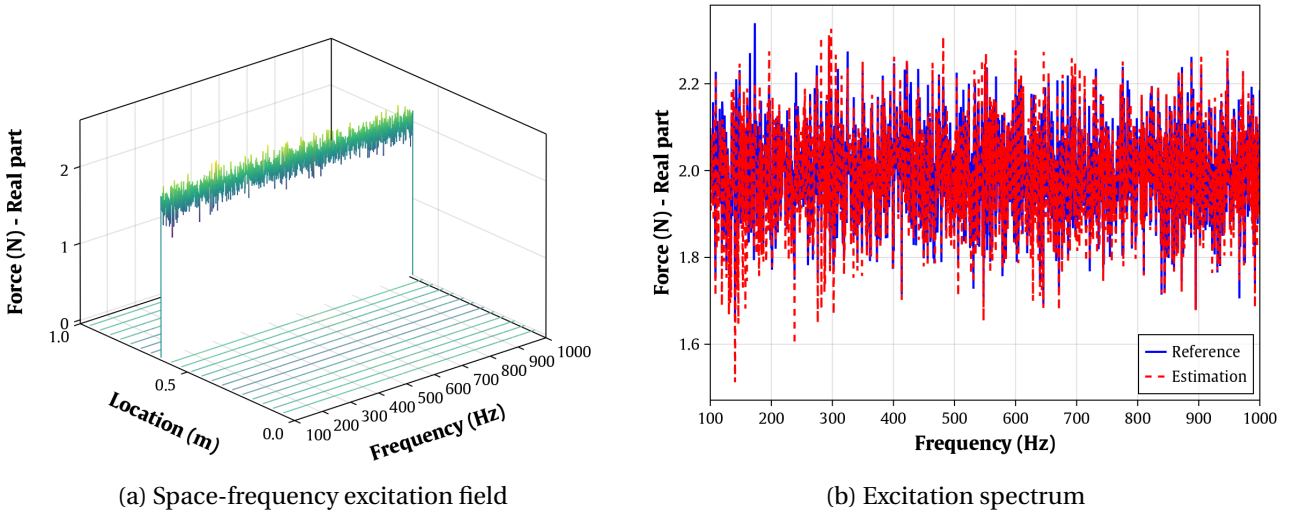


Figure 6: Input force estimated by the multiplicative $\ell_{p,q}$ regularization for $(p, q) = (2, 0.5)$

4. CONCLUSION

This contribution introduced a frequency-domain Kalman filter for broadband sparse source reconstruction. The present Kalman filter is similar to that classically used for time-domain applications. To be successful, the filter must be properly tuned and initialized. For this purpose, dedicated strategies have been implemented. The numerical experiment has

demonstrated the potential benefits of the proposed strategy, as it allows to obtain consistent reconstructions even at resonance frequencies, where sparse regularization generally fails.

Table 1: Comparison of the different strategies

Method	GRE (%)	t_e (s)	m_f (MiB)
Kalman Filter (forward/backward)	2.51/2.39	0.292	156
ℓ_q -regularization	17.06	1.573	1035
$\ell_{p,q}$ -regularization	2.73	0.967	676

REFERENCES

1. A. N. Thite and D. J. Thompson. The quantification of structure-borne transmission paths by inverse methods. Part 1: Improved singular value rejection methods. *Journal of Sound and Vibration*, 264:411–431, 2003.
2. A. Berry, O. Robin, and F. Pierron. Identification of dynamic loading on a bending plate using the Virtual Fields Method. *Journal of Sound and Vibration*, 333:7151–7164, 2014.
3. C. Pézerat and J.-L. Guyader. Force analysis technique. *Acta Acustica*, 86:322–332, 2000.
4. C. Wang, D. Chen, J. Chen, X. Lai, and T. He. Deep regression adaptation networks with model-based transfer learning for dynamic load identification in the frequency domain. *Engineering Applications of Artificial Intelligence*, 102:104244, 2021.
5. A. N. Thite and D. J. Thompson. The quantification of structure-borne transmission paths by inverse methods. Part 2: Use of regularization techniques. *Journal of Sound and Vibration*, 264:433–451, 2003.
6. M. Aucejo. Structural source identification using a generalized tikhonov regularization. *Journal of Sound and Vibration*, 333:5693–5707, 2014.
7. A. Rezayat, V. Nassiri, B. De Pauw, J. Ertveldt, and S. Vanlanduit. Identification of dynamic forces using group-sparsity in frequency domain. *Mechanical Systems and Signal Processing*, 70–71:756–768, 2016.
8. R. E. Kalman. A new approach to linear filtering and prediction problems. *Journal of Basic Engineering*, 82:35–45, 1960.
9. S. Särkkä. *Bayesian Filtering and Smoothing*. Cambridge University Press, First edition, 2013.
10. D. Garcia. Robust smoothing of gridded data in one and higher dimensions with missing values. *Computational Statistics and Data Analysis*, 54:1167–1178, 2010.
11. E. Lourens, E. Reynders, G. De Roeck, G. Degrande, and G. Lombaert. An augmented kalman filter for force identification in structural dynamics. *Mechanical Systems and Signal Processing*, 27:446–460, 2012.
12. D. Teymouri, O. Sedehi, L. S. Katafygiotis, and C. Papadimitriou. A bayesian expectation-maximization (bem) methodology for joint input-state estimation and virtual sensing of structures. *Mechanical Systems and Signal Processing*, 169:108602, 2022.
13. O. Sedehi, C. Papadimitriou, D. Teymouri, and L. S. Katafygiotis. Sequential bayesian estimation of state and input in dynamical systems using output-only measurements. *Mechanical Systems and Signal Processing*, 131:659–688, 2019.
14. M. Aucejo and O. De Smet. A multiplicative regularization for force reconstruction. *Mechanical Systems and Signal Processing*, 85:730–745, 2017.
15. M. Aucejo and O. De Smet. A space-frequency multiplicative regularization for force reconstruction problems. *Mechanical Systems and Signal Processing*, 104:1–18, 2018.