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The harmonic oscillator on the Moyal-Groenewold plane: an approach via Lie groups and twisted Weyl tuples

Cédric Arhancet, Lukas Hagedorn, Christoph Kriegler and Pierre Portal

Abstract

This paper investigates the functional calculus of the harmonic oscillator on each Moyal-Groenewold plane, the noncommutative phase space which is a fundamental object in quantum mechanics. Specifically, we show that the harmonic oscillator admits a bounded $H^\infty(\Sigma_\omega)$ functional calculus for any angle $0 < \omega < \frac{\pi}{2}$ and even a bounded Hörmander functional calculus on the associated noncommutative L^p -spaces, where $\Sigma_\omega = \{z \in \mathbb{C}^* : |\arg z| < \omega\}$. To achieve these results, we develop a connection with the theory of 2-step nilpotent Lie groups by introducing a notion of twisted Weyl tuple and connecting it to some semigroups of operators previously investigated by Robinson via group representations. Along the way, we demonstrate that L^p -square-max decompositions lead to new insights between noncommutative ergodic theory and R -boundedness, and we prove a twisted transference principle, which is of independent interest. Our approach accommodates the presence of a constant magnetic field and they are indeed new even in the framework of magnetic Weyl calculus on classical L^p -spaces. Our results contribute to the understanding of functional calculi on noncommutative spaces and have implications for the maximal regularity of the most basic evolution equations associated to the harmonic oscillator.

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1 Introduction

Quantum Euclidean spaces \mathbb{R}_Θ^d , also recognized as Moyal-Groenewold planes [Gro46], [Moy49] or simply Moyal planes (although the name Moyal-Groenewold d -spaces is undeniably more suitable if $d > 2$), are central examples of *noncompact* noncommutative (spin) manifolds as observed initially in [GGISV04]. Their spectral action was computed in [Gal05]. These spaces are also used as leading examples of quantum locally compact metric spaces as first noticed in [Lat13]. They are locally compact counterparts of quantum tori and admit several equivalent definitions. Initially, in [Gro46], [Moy49], these entities were interpreted as noncommutative deformations of classical Euclidean spaces \mathbb{R}^d . From the more general point of view of Rieffel's deformation quantization theory, as described in the memoir [Rie93], one can describe \mathbb{R}_Θ^d using the algebra of classical smooth functions equipped with the Moyal star product

$$(1.1) \quad (f \star_\Theta g)(x) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x - \frac{1}{2}\Theta u) g(x + t) e^{-i\langle u, t \rangle} du dt, \quad f, g \in \mathcal{S}(\mathbb{R}^d)$$

which is a deformation of the classical product of functions. Here Θ denotes a $d \times d$ real skew-symmetric matrix. The relation between this deformation quantization and the operator-theoretic approach to quantum mechanics is given by the Weyl calculus.

Several analytic and geometric aspects of these spaces were investigated in [SuZ18] (Connes integration formula), [LSZ20] (Cwikel estimates), [MSX20] (quantum differentiability), pseudo-differential calculus [MSZ19], [GJP21] and [HWW22] (Fourier restriction estimates). We also refer to [NeS98] and [BrS11] for the construction of instantons in the case $d = 4$ and to [CaW10] and [CDMW11] for investigations of some properties of the spectral distance. Finally, sigma-Model Solitons on these noncommutative spaces were studied in [DLL15] using time-frequency analysis and Gabor analysis.

On a dual side, Quantum Euclidean spaces can be viewed as *twisted* group von Neumann algebras deformed by 2-cocycles. More precisely, each of these spaces can be identified with the von Neumann algebra $\text{VN}(\mathbb{R}^d, \sigma_\Theta)$ where $\sigma_\Theta: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{T}$ is a 2-cocycle defined by

$$(1.2) \quad \sigma_\Theta(s, t) \stackrel{\text{def}}{=} e^{\frac{1}{2}i\langle s, \Theta t \rangle} \quad s, t \in \mathbb{R}^d$$

for a $d \times d$ real skew-symmetric matrix Θ . It is a more algebraic way to define these structures, which allows one to introduce the associated noncommutative L^p -spaces $L^p(\mathbb{R}_\Theta^d)$. In particular, we can see the algebra \mathbb{R}_Θ^d as generated by d unbounded self-adjoint operators x_1, \dots, x_d («noncommutative spatial variables») satisfying the commutation relations

$$(1.3) \quad i[x_j, x_k] = \Theta_{jk}, \quad j, k \in \{1, \dots, d\}.$$

We show here that an advantage of this approach is that it connects with the famous work [Mac58] of Mackey, allowing us to exploit the «Mackey group» \mathbb{H}_Θ , which is a nilpotent Lie group defined by $\mathbb{H}_\Theta \stackrel{\text{def}}{=} \mathbb{R}^d \times \mathbb{T}$ and a *twisted* group law. To the best of our knowledge, it is the first time where this group is used in this context. This opens the door to more extensive use of the heavy machinery of analysis on Lie groups in the study of the Moyal-Groenewold planes.

Finally, it should be noted that exploring Moyal-Groenewold planes can be considered as a subset of the investigation of quantum phase spaces constructed from locally compact abelian groups equipped with a sufficiently regular 2-cocycle; see the recent [FulG23], [BCLPY23] and references therein for more information.

The interest for the analysis of these fuzzy structures is strengthened by their links with quantum field theory, quantum gravity, string theory and quantum Hall effect. Indeed, it is

widely accepted that when we reach the Planck length scale, the continuous nature of space-time collapses, leading to a somewhat indistinct or noncommutative space-time. We refer to the surveys [DoN01], [Sza03], [BaQ06], [ABJ08], [Riv07] and [Wul19] for a presentation of these topics.

In particular, Grosse and Wulkenhaar introduced in [GrW05] a remarkable quantum field model ϕ^4 on the four-dimensional Moyal plane \mathbb{R}_Θ^4 that is renormalizable to all orders. Here, we suppose that the real skew-symmetric matrix Θ is invertible. The associated action

$$(1.4) \quad S(\phi, m, \lambda, \Omega) \stackrel{\text{def}}{=} \int \left(\frac{1}{2} [\partial_i \phi \star \partial^i \phi] + \frac{\Omega^2}{2} (\tilde{x}_i \phi) \star (\tilde{x}^i \phi) + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) (x) \, d^4 x$$

is obtained by adding a «harmonic term» $\frac{\Omega^2}{2} (\tilde{x}_i \phi) \star (\tilde{x}^i \phi)$ to the naive action (this last one is obtained by replacing the usual commutative product by the Moyal star product (1.1) in the classical action), curing the UV/IR mixing curse of the naive model on \mathbb{R}_Θ^4 . Here the «frequency parameter» Ω is a positive dimensionless parameter, m is the mass, λ is the coupling constant and $\tilde{x}_i \stackrel{\text{def}}{=} 2 \sum_{j=1}^4 [\Theta^{-1}]_{ij} x_j$. A particularly striking aspect of this ϕ^4 model is that it exhibits more favourable behaviour in this noncommutative space compared to ϕ^4 models on classical spaces. Indeed, it is shown in [GrW04] that it has no Landau ghost contrarily to the case of the usual «commutative» ϕ^4 theory. Finally, note that the model is covariant under the Langmann-Szabo duality transformation [LaS02], i.e. we have $S(\phi, m, \lambda, \Omega) = \Omega^2 S(\phi, \frac{m}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega})$. Consequently, the frequency parameter can be restricted to $\Omega \in [0, 1]$. When $\Omega = 1$, the model is self-dual.

Hörmander functional calculus Recall the following famous theorem of Mihlin [Mik56] which is a cornerstone of harmonic analysis. This theorem establishes sufficient conditions for a function to act as a bounded Fourier multiplier on the space $L^p(\mathbb{R}^d)$. Notably, the function may possess a singularity at 0. We refer also [Zim89, Proposition 3 p. 204] for a generalization for UMD vector-valued L^p -spaces and to [HvNVW23, Corollary 13.2.15 p. 273] for an operator-valued version. In the next statement describing this result, $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions on \mathbb{R}^d and \hat{g} is the Fourier transform of a function g of $\mathcal{S}(\mathbb{R}^d)$.

Theorem 1.1 (Mihlin) *Suppose that $f \in L^\infty(\mathbb{R}^d)$. Assume also that for every distributional derivative ∂_α where $\alpha = (\alpha_1, \dots, \alpha_d) \in \{0, 1\}^d$, we have*

$$(1.5) \quad \sup_{\xi \neq 0} |\xi|^{|\alpha|} |\partial_\alpha f(\xi)| < \infty, \quad \text{where } |\alpha| \stackrel{\text{def}}{=} \alpha_1 + \dots + \alpha_d.$$

If $1 < p < \infty$ then the operator $M_f: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$, $g \mapsto \mathcal{F}^{-1}(f\hat{g})$ extends to a bounded operator on the Banach space $L^p(\mathbb{R}^d)$, where \mathcal{F}^{-1} is the inverse Fourier transform.

In 1960, Hörmander significantly contributed in [Hor60, Theorem 2.5 p. 120] to the field of Fourier multiplier theorems by establishing what is currently recognized as the Mihlin-Hörmander multiplier theorem. Hörmander showed that the conclusion of Mihlin's theorem remains valid when we replace the Mihlin condition (1.5) by an estimate on an L^2 -average. We refer to [Ste70, p. 96], [Gra14a, Theorem 6.2.7 p. 446], [Hor03, Theorem 7.9.5 p. 243] and [Duo01, Corollary 8.11 p. 165] for different proofs of this classical result.

Theorem 1.2 (Hörmander) *Let $f: \mathbb{R}^d \rightarrow \mathbb{C}$ be a bounded function of class $C^{\lfloor \frac{d}{2} \rfloor + 1}$ on $\mathbb{R}^d - \{0\}$. Suppose that for any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ of \mathbb{N}^d*

$$(1.6) \quad \sup_{R>0} R^{|\alpha|} \left(\frac{1}{R^d} \int_{R<|\xi|<2R} |\partial_\alpha f(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty, \quad 0 \leq |\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1.$$

If $1 < p < \infty$ then the operator $M_f: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$, $g \mapsto \mathcal{F}^{-1}(f\hat{g})$ extends to a bounded operator on the Banach space $L^p(\mathbb{R}^d)$, where \mathcal{F}^{-1} is the inverse Fourier transform.

We refer to [Hyt04] for a nice discussion on the difference between these theorems. It is worth noting that the assumption (1.6) is more general than the condition $|\xi|^{|\alpha|} |\partial_\alpha f(\xi)| \leq C$ for all $\alpha \in \mathbb{N}^d$ with $0 \leq |\alpha| \leq \lfloor \frac{d}{2} \rfloor + 1$, which is often unfortunately referred to as the Mihlin condition in the literature, as in the short historical account [Gra21, Section 1]. A recurring theme in harmonic analysis involves providing proofs for various adaptations of these two theorems, as discussed in the following papers [AnL86], [Ank90], [Ion02] (symmetric spaces), [JMP14], [JMP18], [GJP21], [CGPT23b] (group von Neumann algebras) and [CGPT23] (Schur multipliers).

Theorem 1.2 implies the following result for the Laplacian Δ on \mathbb{R}^d .

Corollary 1.3 (Hörmander) *Let $f: \mathbb{R}_+ \rightarrow \mathbb{C}$ be a bounded function of class $C^{\lfloor \frac{d}{2} \rfloor + 1}$ on \mathbb{R}_+^* . Suppose that*

$$(1.7) \quad \sup_{R>0} R^{2k-1} \int_R^{2R} |f^{(k)}(s)|^2 ds < \infty, \quad 0 \leq k \leq \lfloor \frac{d}{2} \rfloor + 1.$$

If $1 < p < \infty$ then the operator $f(-\Delta): \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$, $g \mapsto \mathcal{F}^{-1}(f(|x|^2)\hat{g})$ extends to a bounded operator on the Banach space $L^p(\mathbb{R}^d)$.

If $s > 0$, using the usual Sobolev norm

$$(1.8) \quad \|f\|_{W^{s,2}(\mathbb{R})} \stackrel{\text{def}}{=} \left\| [\hat{f}(t)(1+t^2)^{\frac{s}{2}}] \right\|_{L^2(\mathbb{R})}$$

and the inhomogeneous Sobolev space $W^{s,2}(\mathbb{R}) \stackrel{\text{def}}{=} \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{W^{s,2}(\mathbb{R})} < \infty\}$ it was later observed (though it is historically difficult to determine exactly who first noted this), that this theorem could be generalized to encompass all measurable functions f that meet the criterion

$$(1.9) \quad \|f\|_{\text{Hör}_2^s(\mathbb{R}_+^*)} \stackrel{\text{def}}{=} \sup_{t>0} \|x \mapsto \eta(x) \cdot f(tx)\|_{W^{s,2}(\mathbb{R})} < \infty,$$

where $\eta:]0, \infty[\rightarrow \mathbb{R}$ is any non-zero function of class C^∞ with compact support and $s > \frac{d}{2}$. We refer to [Pel18, Theorem 6.24 p. 79] and to [Duo01, Theorem 8.10 p. 164], [Gra14b, Theorem 7.5.5 p. 565] for a proof of slight variants and to [Kri09, p. 63], [Lit65, Lemma 1], [CT77, Theorem 4.7] for related information.

Finally, note that the norm of radial Fourier multipliers which are bounded on the space $L^p(\mathbb{R}^d)$ was described more recently in the paper [HNS11] if $d \geq 4$ for $1 < p < \frac{2d-2}{d+1}$. When the symbol f of these radial Fourier multipliers M_f is compactly supported in the fixed dyadic annulus $\{\xi : \frac{1}{2} < |\xi| < 2\}$, their result takes a particularly satisfying form, namely f induces a bounded operator $M_f: L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ if and only if $\hat{f} \in L^p(\mathbb{R}^d)$. For $d = 4$, this was extended in the paper [Cla18] to the range $1 < p < \frac{36}{29}$. Note that proving such a result for $1 < p < \frac{2d}{d+1}$ would have far reaching consequences (implying the Bochner-Riesz conjecture). These results on radial Fourier multipliers are much stronger than Theorem 1.2.

Hörmander's theorem gives results about the functional calculus for the operator $-\Delta$, i.e. for the generator of the heat semigroup on \mathbb{R}^d . A traditional broadening of this context involves replacing this semigroup by more general symmetric contraction semigroups of operators acting on classical or noncommutative L^p -spaces. If $(T_t)_{t \geq 0}$ is such a semigroup with generator $-A$ acting on a noncommutative L^p -space $L^p(\mathcal{M})$, then the operator A is positive and self-adjoint

on $L^2(\mathcal{M})$. The spectral theorem thus allows one to obtain a functional calculus $\mathcal{L}^\infty(\mathbb{R}_+) \rightarrow B(L^2(\mathcal{M}))$, $f \mapsto f(A)$ where the operator $f(A): L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ is well-defined and bounded on the Hilbert space $L^2(\mathcal{M})$. If $1 < p < \infty$, the function f is said to be an L^p -spectral multiplier if $f(A)$ extends to a bounded operator on $L^p(\mathcal{M})$. Giving a version of Hörmander's theorem in this context entails demonstrating that f is an L^p -spectral multiplier if it fulfils (1.9) for some $s > 0$. This often requires additional or different assumptions.

For instance, Christ, Mauceri and Meda proved in [Chr91, Theorem 1 p. 74] and [MaM90] that the sub-Laplacian of a stratified Lie group G satisfies a Hörmander theorem for any $s > \frac{Q}{2}$, where Q is the homogeneous dimension of G . An alternative proof of this result has been given by Sikora in [Sik92]. Although the dimensional parameter $\frac{Q}{2}$ is very natural in this context, it is not always the most effective choice. Indeed, Hebisch significantly advanced this area in his work [Heb93, Theorem 1.1 p. 233], where he demonstrated an improved condition for sub-Laplacians on products of Heisenberg-type groups. This development builds on and enhances the earlier work of Müller and Stein [MuS94], who studied direct products of Heisenberg groups with Euclidean groups. Specifically, Hebisch proved that the sufficient condition $s > \frac{Q}{2}$ can be more effectively replaced by $s > \frac{d}{2}$, where d represents the topological dimension of the corresponding Lie group. Analogous results were proven by Cowling and Sikora in [CoS01, Theorem 1.1 p. 2] for the compact Lie group $SU(2)$ and by Martini and Müller in [MaM13] for the 6-dimensional free 2-step nilpotent Lie group $N_{3,2}$ on three generators. See also [MaM14b], [Mar15] and [Nie23] for similar results for other classes of groups. Actually, as proved in the paper [MaM16, Theorem 3], it is known that one cannot do better than $\frac{d}{2}$ for any 2-step stratified Lie group of topological dimension d . Finally, it is worth noting that the approach of Duong in [Duo96, Theorem 3] for sub-Laplacians on stratified Lie groups relates the L^1 -norm of the complex time heat kernels of a sub-Laplacian on a stratified Lie group with the order of regularity s needed in the Hörmander multiplier theorem.

On a *compact* smooth Riemannian manifold M of dimension $d \geq 2$ without boundary, a second order self-adjoint positive elliptic differential operator A also satisfies a Hörmander theorem for any $s > \frac{d}{2}$, by [SeS89, Theorem 3.1. p. 723] (see also [Sog17, Theorem 5.3.1 p. 155]). See also [DOS02, p. 469] and [Xu07] for other proofs. It is worth noting that, in the recent paper [MMN23], it is proved that, for a general class of smooth second-order differential operators associated with a sub-Riemannian structure on a smooth d -dimensional manifold, regularity of order $s > \frac{d}{2}$ is necessary for a Hörmander theorem to hold.

The literature on spectral multiplier theorems is even broader, and includes, in particular, [Ale94] (sub-Laplacians on Lie groups of polynomial growth), [Blu03], [DOS02], [KuU15] (homogeneous metric measure spaces), [CCMS17] (Kohn Laplacian on some spheres), [GCM⁺01, Har19] (Ornstein-Uhlenbeck operator), [Heb90] (Schrödinger operators), [MaM14a] (Grushin operators), [Med90] (negative generators of symmetric semigroups), [MPR07], [MPR15] (Hodge Laplacian on the Heisenberg group), [CGHM94] (Laplacians on groups of exponential growth), [SWX23] (partial harmonic oscillator), and [Sha21, Sha22] (abstract group generators with finite speed of propagation and Sobolev embedding properties).

The present paper relies on two abstract approaches to spectral multiplier theorems: a characterisation using R -bounds in the spirit of H^∞ functional calculus (and the aforementioned approach of Duong) proven in [KrW18] (and generalized in [HaP23]), and a transference result in the style of [CoW76] and [NPS23].

The harmonic oscillator on the Moyal-Groenewold plane \mathbb{R}_Θ^d Our spectral multiplier theorem is primarily designed for the harmonic oscillator on Quantum Euclidean spaces (introduced in [Gro46]), which can be seen as a natural generalization of the classical harmonic oscillator when the space is quantized. This operator is formally defined by $\sum_{k=1}^d (\partial_k^2 + X_k^2)$

where X_k is the left multiplication by the «spatial variable» x_k and where ∂_k is some suitable «quantum partial derivation» acting on the Banach space $L^p(\mathbb{R}_\Theta^d)$ and satisfying

$$(1.10) \quad i[X_j, X_k] = \Theta_{jk}, \quad [\partial_j, \partial_k] = 0 \quad \text{and} \quad i[\partial_j, X_k] = \delta_{j=k}, \quad 1 \leq j, k \leq d.$$

We obtain in Theorem 12.4 a bounded Hörmander functional calculus of order $s > d + \frac{1}{2}$ on the space $L^p(\mathbb{R}_\Theta^d)$ for a shifted version of the harmonic oscillator. Note that our approach can be used for a variant of this operator when «magnetic fields» are present. We refer to [NeP20] for the case of the classical (euclidean) harmonic oscillator. Operators with the same algebraic structure (i.e. sums of squares of generators of a Heisenberg algebra), acting on UMD Banach lattices, have been studied in the paper [NPS23]. The case of the operator $\sum_{k=1}^d X_k^2$ acting on Quantum Euclidean spaces was considered in the PhD thesis of Sharma [Sha22], and motivated our work on this topic. Going from $\sum_{k=1}^d X_k^2$ to the true harmonic oscillator $\sum_{k=1}^d (\partial_k^2 + X_k^2)$ was the motivation to generalise the Weyl calculus studied in [NeP20] to the Θ -Weyl calculus studied here.

Note also that it is shown in [GRV06] that the propagator of the Grosse-Wulkenhaar model defined by (1.4) is the kernel of the inverse of a shifted «classical» harmonic oscillator $-\Delta + \Omega^2 \tilde{x} + m^2$, allowing a second proof in [GMRV06] of the renormalizability of this model to all orders. Recall that the «frequency parameter» Ω is a positive dimensionless parameter. This operator is also connected to noncommutative geometry, see [GaW13], where a related spectral triple is introduced. Our result on the harmonic oscillator $\sum_{k=1}^d (\partial_k^2 + X_k^2)$ (which is different from the operator $-\Delta + \Omega^2 \tilde{x} + m^2$) combined with the result [JMX06, Lemma 3.5 p. 28] allows us to obtain the (uniform) boundedness of the H^∞ functional calculus of the operator $\sum_{k=1}^d (\partial_k^2 + X_k^2) + m^2$ for any «mass» $m > 0$.

Application to maximal regularity of evolution equations Recall that a Hörmander type spectral multiplier theorem implies the boundedness of the $H^\infty(\Sigma_\omega)$ functional calculus [HvNVW18, Chapter 10], [Haa06] for any $\omega > 0$, where $\Sigma_\omega \stackrel{\text{def}}{=} \{z \in \mathbb{C}^* : |\arg z| < \omega\}$ is the sector of angle 2ω . Our result thus implies that the harmonic oscillator on \mathbb{R}_Θ^d has maximal regularity. Let us recall classical concepts of maximal regularity, and refer to [Den21], [HvNVW23], [KuW04], [Mon09] (see also [Are04]) and the references therein for a much more complete account. If A is a closed densely defined linear operator on a Banach space X , we can consider the following inhomogeneous autonomous Cauchy problem

$$(1.11) \quad \begin{cases} y'(t) + A(y(t)) = f(t), & t \in]0, T[\\ y(0) = x_0 \end{cases}$$

where $f:]0, T[\rightarrow X$ is a function of $L^1(]0, T[, X)$ with $0 < T < \infty$.

If $-A$ is the generator of a strongly continuous semigroup $(T_t)_{t \geq 0}$ on a Banach space X then by [ABHN11, Proposition 3.1.16 p. 118] the Cauchy problem (1.11) admits a unique *mild* solution given by

$$(1.12) \quad y(t) = T_t(x_0) + \int_0^t T_{t-s}(f(s)) ds, \quad 0 \leq t < T.$$

Suppose that $1 < p < \infty$. If the right-hand side f of (1.11) belongs to the Banach space $L^p(]0, T[, X)$, «optimal regularity» would imply that both y' and Ay should also belong to the space $L^p(]0, T[, X)$. We are led to say, following [HvNVW23, Definition 17.2.4 p. 577] and [HvNVW23, Proposition 17.2.11 p. 581], that the operator A has maximal L^p -regularity

on $]0, T[$ if for any vector-valued function $f \in L^p(]0, T[, X)$ the mild solution of the Cauchy problem (1.11) is a «strong L^p -solution», i.e. satisfies the following two conditions:

1. y takes values in $\text{dom } A$ almost everywhere,
2. $Ay \in L^p(]0, T[, X)$.

It is a commonly known fact [HvNVW23, Theorem 17.2.15 p. 586] that this property implies that the operator $-A$ generates an analytic semigroup on X . Note also that by [Mon09, Theorem 17.2.15 p. 586] this property is independent of p . A classical result [HvNVW23, Corollary 17.3.6 p. 631] (combined with [HvNVW23, Lemma 17.2.16 p. 590]) says that an operator admitting a bounded $H^\infty(\Sigma_\theta)$ functional calculus of angle $0 < \theta < \frac{\pi}{2}$ on a UMD Banach space X has maximal L^p -regularity on $]0, T[$. Consequently, our main result implies that the harmonic oscillator on the Moyal-Groenewold plane has L^p -maximal regularity on $]0, T[$ on the Banach space $L^q(\mathbb{R}_\Theta^d)$ for any $1 < p, q < \infty$. This result is in the continuity of [McD24] where a preliminary theory of partial differential equations on this structure is presented, and of [GJP21], where singular integral theory and pseudodifferential calculus are thoroughly investigated. We also refer to [RST23] for more harmonic analysis results in $L^q(\mathbb{R}_\Theta^d)$.

A consequence of maximal L^p -regularity is the estimate [HvNVW23, p. 577]

$$(1.13) \quad \|y'\|_{L^p(]0, T[, X)} + \|Ay\|_{L^p(]0, T[, X)} \leq C \|f\|_{L^p(]0, T[, X)},$$

for a positive constant $C \geq 0$ which is independent of f . By means of linearization techniques, this estimate allows an effective approach to quasilinear problems via the contraction mapping principle. We refer to [HvNVW23, Chapter 18] for an extensive description of these applications.

Approach and structure of the paper In Section 2, we collect background information in operator theory, functional calculus theory, and noncommutative analysis.

To describe the set up of our approach in Section 3, let us first recall that the Stone-von Neumann theorem [Hal13, Theorem 14.8 p. 286], [Mor17, Theorem 11.24 p. 501] states that, if (A_1, \dots, A_n) and (B_1, \dots, B_n) are self-adjoint operators acting irreducibly on a complex Hilbert space H , and satisfying the exponentiated commutation relations

$$(1.14) \quad e^{isA_j} e^{itA_k} = e^{itA_k} e^{isA_j}$$

$$(1.15) \quad e^{isB_j} e^{itB_k} = e^{itB_k} e^{isB_j}, \quad s, t \in \mathbb{R}, \quad j, k \in \{1, \dots, n\}$$

$$(1.16) \quad e^{itA_j} e^{isB_k} = e^{-its\delta_{j=k}} e^{isB_k} e^{itA_j},$$

then the operators are unitarily equivalent to the «position operators» (X_1, \dots, X_n) and «momentum operators» (Q_1, \dots, Q_n) defined on $L^2(\mathbb{R}^d)$ by

$$(Q_j f)(x) = x_j f(x), \quad (P_j f)(x) = -i\partial_j f(x), \quad x \in \mathbb{R}^d.$$

In [NeP20, Definition 3.1], the authors investigated Weyl pairs (A_1, \dots, A_n) and (B_1, \dots, B_n) of closed densely defined operators acting on a complex Banach space X satisfying the previous relations. In Section 3, we generalize this theory by introducing (see Definition 3.1) the notion of Θ -Weyl tuple $A = (A_1, \dots, A_n)$ of operators acting on a Banach space X . Loosely speaking, if $\Theta \in M_n(\mathbb{R})$ denotes a fixed skew-symmetric matrix, we say that (A_1, \dots, A_n) is a Θ -Weyl tuple if we have some associated groups of operators which satisfy the following commutation relations

$$(1.17) \quad e^{itA_j} e^{isA_k} = e^{i\Theta_{jk}ts} e^{isA_k} e^{itA_j}, \quad j, k \in \{1, \dots, n\}, \quad s, t \in \mathbb{R}.$$

We show in Example 3.3 how to associate to a d -dimensional Weyl pair as in (1.14) a Θ -Weyl tuple for some suitable matrix Θ . In Lemma 3.4, we construct a projective representation $\mathbb{R}^n \rightarrow \mathbf{B}(X)$, $t \mapsto e^{it \cdot A}$ of \mathbb{R}^n on X associated with each Θ -Weyl tuple (A_1, \dots, A_n) . This projective representation allows us to define the associated notion of Θ -Weyl functional calculus $a \mapsto a(A)$ by the formula (3.8).

In Section 4, we prove a twisted transference result (Proposition 4.4) that is a generalization of the transference principle popularized in [CoW76]. Note that this result can also be used in the more general setting described in [FulG23] which encompasses the context of [Lei11] and [LeL22]. It estimates the norms of operators $a(A)$ defined through the Θ -Weyl calculus by the norm of some twisted convolution operator. Finally, we introduce a specific concrete example of a Θ -Weyl tuple $A = (A_1, \dots, A_n)$ acting on the space $L^p(\mathbb{R}^n)$, which we call the universal Θ -Weyl tuple A_{univ}^Θ . This object plays a fundamental role in the sequel as we transfer properties of this universal Θ -Weyl tuple to *arbitrary* Θ -Weyl tuples.

We focus on operators $a(A)$ that arise as spectral multipliers associated with a strongly continuous semigroup $(T_t)_{t \geq 0}$ generated by the abstract «sum of squares» $\sum_{k=1}^n A_k^2$. We give some preliminary information about this semigroup in Section 5 by connecting it to the theory of 2-step nilpotent Lie groups, using the «Mackey group» \mathbb{H}_Θ (as well as to some semigroups of operators previously investigated by Robinson). We focus on the case where A_{univ}^Θ is the universal Θ -Weyl tuple defined in (4.12). We are able to show in Proposition 5.8, that the operator $\sum_{k=1}^n A_k^2$ has a bounded $H^\infty(\Sigma_\omega)$ functional calculus on the Bochner space $L^p(\mathbb{R}^n, S^p)$ for any $\omega > \frac{\pi}{2}$. An interesting technical aspect of our paper at this point is that we use contractively regular semigroups that are not necessarily positive. Prior to our work, the application of these semigroups in analysis seemed to be quite limited. In this paper, they naturally emerge in the context of twisted convolutions.

In Section 6, and in subsequent sections of the paper, we need to examine in depth the semigroup and its integral description. We show that there exists a tensor decomposition of this semigroup which essentially reduces the difficulties to the case of the matrix $\Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

It turns out that the spectrum of $\sum_{k=1}^n A_k^2$ is contained in $[\alpha, \infty)$ for some constant $\alpha \geq 0$ depending on the eigenvalues of the skew-symmetric matrix Θ . We show even more, namely that the operator $\sum_{k=1}^n A_k^2 - \alpha \text{Id}_{L^p(\mathbb{R}^n, S^p)}$ has a bounded $H^\infty(\Sigma_\omega)$ functional calculus, for any angle $\omega \in (\frac{\pi}{2}, \pi)$, see Corollary 6.7.

The next goal is then to enhance the boundedness of H^∞ functional calculus of the operator $\mathcal{A} - \alpha \text{Id}_{L^p(\mathbb{R}^n, S^p)}$ to a bounded Hörmander functional calculus with respect to (non-holomorphic) spectral multipliers defined on the positive half-axis. For this task, we proceed by showing R -boundedness of the semigroup generated by this operator to the maximal angle $\frac{\pi}{2}$. In doing so, we control the dependency on the angle as required in [KrW18]. Such R -boundedness is usually proved with estimates of the radially decreasing kernel of the semigroup together with boundedness of the maximal Hardy-Littlewood operator. In the short Section 7, we explain how to use the boundedness of this maximal operator on the Bochner space $L^p(\mathbb{R}^n, S^p)$ in our context, preparing the ground for the Section 10.

In Section 8, we prove a stability result for R -boundedness under the operation of tensor product which will be useful in Section 10. In Section 9, we connect L^p -square-max decompositions of [GJP17] to R -boundedness by showing, in Proposition 9.2, that the existence of an L^p -square-max decomposition leads to an R -bound.

Such an L^p -square-max decomposition is critical to our approach, and gives, in Section 10, the R -sectoriality (indeed a more general result) of the shifted «sum of squares» on $L^p(\mathbb{R}^n)$ for the universal Θ -Weyl tuple A_{univ}^Θ . A more elaborate argument relying on similar ideas gives R -sectoriality on the Bochner space $L^p(\mathbb{R}^n, S^p)$. In Theorem 10.8, we deduce the desired

boundedness of the Hörmander calculus on the Bochner space $L^p(\mathbb{R}^n, S^p)$ by using the sufficient abstract condition of [KrW18].

In Section 11, we return to general Θ -Weyl tuples of operators acting on noncommutative L^p -spaces. We are able to use the transference result from Corollary 4.6 to transfer the boundedness of the Hörmander functional calculus of the «sum of squares» $\sum_{j=1}^n A_j^2$ of the universal Θ -Weyl tuple A_{univ}^Θ from Theorem 10.8 to the case of the Hörmander functional calculus of the sum of squares of a *general* Θ -Weyl tuple (see Corollary 11.5). In a final Proposition, we briefly discuss the case of Θ -Weyl tuples acting on UMD Banach spaces, generalizing the case of noncommutative L^p -spaces, in which many results of the whole section still hold, but for which, for now, we are only able to show the boundedness of the $H^\infty(\Sigma_\omega)$ functional calculus of the associated (shifted) harmonic oscillator for an angle $\omega > \frac{\pi}{2}$. We finish the section by giving applications to Schur multipliers in Example 11.10.

We are finally able to apply our theory to the Moyal-Groenewold plane \mathbb{R}_Θ^d (for any skew-symmetric matrix $\Theta \in M_d(\mathbb{R})$) in Section 12. Noncommutative spatial variables X_k and quantum derivations ∂_k are rigorously introduced by considering the associated groups of operators. In Proposition 12.2, we observe that these semigroups form a Θ -Weyl tuple where $\Theta \stackrel{\text{def}}{=} \begin{pmatrix} \theta & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}$. This allows us to define the harmonic oscillator $\sum_{k=1}^d (X_k^2 + \partial_k^2)$ and to apply the theory developed in the previous sections to conclude that a suitable shifted version $\sum_{k=1}^d (X_k^2 + \partial_k^2) - \alpha \text{Id}$ admits a Hörmander functional calculus.

2 Preliminaries

Gaussian integrals If $a > 0$, we have

$$(2.1) \quad \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{\sqrt{\pi}}{\sqrt{a}} \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{2a^{3/2}}.$$

Bounded holomorphic semigroups We refer to [Paz83] for more information on semigroups of operators. Suppose $0 < \theta \leq \frac{\pi}{2}$. A strongly continuous semigroup $(T_t)_{t \geq 0}$ of operators acting on a Banach space X is called a bounded holomorphic semigroup of angle θ if it has a bounded holomorphic extension to $\Sigma_{\theta'} \stackrel{\text{def}}{=} \{z \in \mathbb{C}^* : |\arg z| < \theta'\}$ for each $0 < \theta' < \theta$. The following characterization is [HvNVW18, Theorem G.5.3 p. 541] and [ABHN11, Theorem 3.7.19 p. 157].

Proposition 2.1 *A strongly continuous bounded semigroup $(T_t)_{t \geq 0}$ of operators acting on a Banach space X with generator $-A$ is a bounded holomorphic semigroup if and only if $T_t(X) \subset \text{dom } A$ for any $t > 0$ and $\sup_{t > 0} t \|AT_t\|_{X \rightarrow X} < \infty$.*

Functional calculus of sectorial operators For any angle $\sigma \in (0, \pi)$, we will use the open sector symmetric around the positive real half-axis with opening angle 2σ

$$\Sigma_\sigma^+ = \Sigma_\sigma \stackrel{\text{def}}{=} \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \sigma\}.$$

It will be convenient to set $\Sigma_0^+ \stackrel{\text{def}}{=} (0, \infty)$. We refer to [Haa06], [HvNVW18], [JMX06] for background on sectorial operators and their H^∞ functional calculus, as well as R -boundedness. A sectorial operator $A: \text{dom } A \subset X \rightarrow X$ of type σ on a Banach space X is a closed densely

defined operator such that $\sigma(A) \subset \overline{\Sigma_\sigma^+}$ for some $\sigma \in [0, \pi)$ and such that $\sup\{\|zR(z, A)\| : z \notin \overline{\Sigma_{\sigma'}}\} < \infty$ for any $\sigma' \in (\sigma, \pi)$. We introduce the angle of sectoriality

$$(2.2) \quad \omega(A) = \inf \sigma.$$

An operator A is called R -sectorial of type σ if it is sectorial of type σ and if for any $\theta \in (\sigma, \pi)$ the set

$$(2.3) \quad \{zR(z, A) : z \notin \overline{\Sigma_\theta}\}$$

is R -bounded. Recall the following result [HvNVW18, Example 10.1.2 p. 362].

Proposition 2.2 *If $-A$ is the generator of a bounded strongly continuous semigroup $(T_t)_{t \geq 0}$ on a Banach space X then the operator A is sectorial of type $\frac{\pi}{2}$.*

Moreover, by [HvNVW18, Example 10.1.3 p. 362], A is sectorial of type $< \frac{\pi}{2}$ if and only if $-A$ generates a bounded holomorphic strongly continuous semigroup. If A is a sectorial operator on a reflexive Banach space X , we have by [Haa06, Proposition 2.1.1 (h) p. 21] or [HvNVW18, Proposition 10.1.9 p. 367] a decomposition

$$(2.4) \quad X = \text{Ker } A \oplus \overline{\text{Ran } A}.$$

For any $0 < \theta < \pi$, let $H^\infty(\Sigma_\theta)$ be the algebra of all bounded analytic functions $f : \Sigma_\theta \rightarrow \mathbb{C}$, equipped with the supremum norm $\|f\|_{H^\infty(\Sigma_\theta)} \stackrel{\text{def}}{=} \sup\{|f(z)| : z \in \Sigma_\theta\}$. Let $H_0^\infty(\Sigma_\theta) \subset H^\infty(\Sigma_\theta)$ be the subalgebra of bounded analytic functions $f : \Sigma_\theta \rightarrow \mathbb{C}$ for which there exist $s, c > 0$ such that $|f(z)| \leq \frac{c|z|^s}{(1+|z|)^{2s}}$ for any $z \in \Sigma_\theta$.

Given a sectorial operator A of type $0 < \omega < \pi$, a bigger angle $\theta \in (\omega, \pi)$, and a function $f \in H_0^\infty(\Sigma_\theta)$, one may define a bounded operator $f(A)$ by means of a Cauchy integral on Σ_θ (see e.g. [Haa06, Section 2.3] or [KuW04, Section 9]). The resulting mapping $H_0^\infty(\Sigma_\theta) \rightarrow B(X)$, $f \mapsto f(A)$ is an algebra homomorphism. By definition, A has a bounded $H_0^\infty(\Sigma_\theta)$ functional calculus provided that this homomorphism is bounded, that is if there exists a positive constant C such that $\|f(A)\|_{X \rightarrow X} \leq C \|f\|_{H_0^\infty(\Sigma_\theta)}$ for any $f \in H_0^\infty(\Sigma_\theta)$. In the case where A has a dense range, the latter boundedness condition allows a natural extension of $f \mapsto f(A)$ to the full algebra $H^\infty(\Sigma_\theta)$.

We will make use of the following elementary general result about $H^\infty(\Sigma_\omega)$ calculus for sectorial operators shifted to the right.

Lemma 2.3 *Let A be a sectorial operator of angle ω_0 on a Banach space X and $\omega > \omega_0$. The following statements are equivalent:*

1. $A + \varepsilon$ has a bounded $H^\infty(\Sigma_\omega)$ functional calculus on X for some $\varepsilon > 0$,
2. $A + \varepsilon$ has a bounded $H^\infty(\Sigma_\omega)$ functional calculus on X for all $\varepsilon > 0$.

Proof : Let $\varepsilon, \varepsilon' > 0$ be such that $\varepsilon' + A$ has a bounded $H^\infty(\Sigma_\omega)$ functional calculus on X . Both $\varepsilon + A$ and $\varepsilon' + A$ are invertible and we can write $\varepsilon + A = (\varepsilon' + A) + B$ with $B \stackrel{\text{def}}{=} (\varepsilon' - \varepsilon)\text{Id}_X$. By [Haa06, Corollary 5.5.5 p. 119], the operator $\varepsilon + A$ admits a bounded $H^\infty(\Sigma_\omega)$ functional calculus. ■

The following is extracted from [Kat62].

Proposition 2.4 *Let $(e^{tA})_{t \geq 0}$ and $(e^{tB})_{t \geq 0}$ be two strongly continuous semigroups on a Banach space X such that*

$$e^{sA}e^{tB} = e^{cst}e^{tB}e^{sA}, \quad 0 \leq s, t < \infty$$

for some constant $c \in \mathbb{C}$ then the subspace $\text{dom}(BA) \cap \text{dom}(AB)$ is dense in X , and $(AB - BA)f = cf$ if $f \in \text{dom}(BA) \cap \text{dom}(AB)$.

Convolution on locally compact groups If G is a locally compact group equipped with a left Haar measure μ_G , recall that the convolution product of two functions f and g is given, when it exists, by

$$(2.5) \quad (f * g)(t) \stackrel{\text{def}}{=} \int_G f(s)g(s^{-1}t) d\mu_G(s), \quad t \in G.$$

2-cocycles Let G be a locally compact group equipped with a left Haar measure μ_G . We first recall that a continuous 2-cocycle on G with values in \mathbb{T} is a continuous map $\sigma: G \times G \rightarrow \mathbb{T}$ such that

$$(2.6) \quad \sigma(s, t)\sigma(st, r) = \sigma(s, tr)\sigma(t, r), \quad s, t, r \in G$$

such that $\sigma(s, e) = \sigma(e, s) = 1$ for any $s \in G$. The cocycle is said to be normalized if we have $\sigma(s, s^{-1}) = \sigma(s^{-1}, s)$ for any $s \in G$. Observe that the cocycle defined in (1.2) is normalized. The set of 2-cocycles is denoted by $Z^2(G, \mathbb{T})$, which is a group under pointwise multiplication. As observed in [Kle74, p. 216], we have $\overline{\sigma(s, t)} = \sigma(t^{-1}, s^{-1})$ for any $s, t \in G$. It is easy to check that

$$(2.7) \quad \sigma(s, s^{-1}t) = \overline{\sigma(t^{-1}, s)}.$$

Twisted group von Neumann algebra Recall that a map $\pi: G \rightarrow B(H)$ for some Hilbert space H is called a (strongly) continuous σ -projective unitary representation on G if each π_s is unitary, the map $G \rightarrow H, s \mapsto \pi_s \xi$ is continuous for any $\xi \in H$ and if we have

$$(2.8) \quad \pi_s \pi_t = \sigma(s, t) \pi_{st}, \quad s, t \in G.$$

We refer to the nice survey [Pac08] for a description of some applications of projective representations of locally compact groups, and to [Kle74] and to [FeD88, Chapter VIII, Section 10] for more information. The left regular σ -representation λ_σ is given by $\lambda_\sigma: G \rightarrow B(L^2(G))$ and

$$(2.9) \quad (\lambda_{\sigma, s} f)(t) \stackrel{\text{def}}{=} \sigma(s, s^{-1}t) f(s^{-1}t), \quad s, t \in G, f \in L^2(G).$$

We have

$$(2.10) \quad \lambda_{\sigma, s} \lambda_{\sigma, t} = \sigma(s, t) \lambda_{\sigma, st}, \quad (\lambda_{\sigma, s})^* = \overline{\sigma(s, s^{-1})} \lambda_{\sigma, s^{-1}}.$$

For any $f \in L^1(G)$, we let $\lambda_\sigma(f) \stackrel{\text{def}}{=} \int_G f(s) \lambda_{\sigma, s} d\mu_G(s)$. We define the twisted group von Neumann algebra $VN(G, \sigma)$ of the group G to be the von Neumann algebra generated by $\{\lambda_\sigma(f) : f \in L^1(G)\}$ (or equivalently by $\{\lambda_{\sigma, s} : s \in G\}$) in the algebra $B(L^2(G))$.

A central extension Suppose that G and K are two locally compact groups. Each neutral element is denoted by e . An extension of G by K is an exact sequence $\{e\} \rightarrow K \xrightarrow{i} H \xrightarrow{j} G \rightarrow \{e\}$ where H is a locally compact group, i is a topological isomorphism onto its image and where j is continuous and open. It is said to be a central extension if $i(K)$ is contained in the center of the group H .

Assume that G is second countable. Let $\sigma: G \times G \rightarrow \mathbb{T}$ be a continuous 2-cocycle on G . Consider the set

$$(2.11) \quad G_\sigma \stackrel{\text{def}}{=} G \times \mathbb{T}$$

endowed with the group law $(s, z) \cdot (t, w) \stackrel{\text{def}}{=} (st, zw\sigma(s, t))$ where $s, t \in G$ and $z, w \in \mathbb{T}$. As in [Mac58, Section 2], [Par69, Section 3] and [Var85, Theorem 7.8], we equip G_σ with the product topology. Then G_σ becomes a second countable locally compact group with left Haar measure $\mu_G \otimes \mu_{\mathbb{T}}$ by [Var85, Remark p. 255] and [Par69, p. 13]. We warn the reader that in the theory of Mackey, we can use cocycles that are only Borel measurable instead of continuous. In this case, the used topology on the «Mackey group» G_σ is the «Weil topology» and this is not identical in general to the product topology on $G \times \mathbb{T}$.

Introducing the continuous maps $i: \mathbb{T} \rightarrow G_\sigma, z \mapsto (e, z)$ and $j: G_\sigma \rightarrow G, (s, z) \mapsto s$ then by [Var85, Theorem 7.8] the triple (G_σ, i, j) defines a central extension of the group G by \mathbb{T} . This means that we have a short exact sequence

$$(2.12) \quad \{e\} \rightarrow \mathbb{T} \xrightarrow{i} G_\sigma \xrightarrow{j} G \rightarrow \{e\}$$

such that the image of \mathbb{T} is a subgroup of the center of the group G_σ .

We will use the following result [Var85, Theorem 7.21 p. 262] or [Rag94, Corollary 5.2 p. 220].

Theorem 2.5 *Let G be a connected Lie group. Let $\sigma: G \times G \rightarrow \mathbb{T}$ be a continuous 2-cocycle. Then there exists a unique analytic structure on G_σ which converts G_σ into a Lie group.*

Let \mathfrak{a} and \mathfrak{g} be Lie algebras. An extension of \mathfrak{g} by \mathfrak{a} is a short exact sequence $0 \rightarrow \mathfrak{a} \xrightarrow{\lambda} \mathfrak{h} \xrightarrow{\mu} \mathfrak{g} \rightarrow 0$ where \mathfrak{h} is a Lie algebra, μ is a surjective homomorphism from \mathfrak{h} onto \mathfrak{g} and where λ is an injective homomorphism from \mathfrak{a} onto the kernel of μ . Two extensions $0 \rightarrow \mathfrak{a} \xrightarrow{\lambda} \mathfrak{h} \xrightarrow{\mu} \mathfrak{g} \rightarrow 0$ and $0 \rightarrow \mathfrak{a} \xrightarrow{\lambda'} \mathfrak{h}' \xrightarrow{\mu'} \mathfrak{g} \rightarrow 0$ are said to be equivalent if there exists a homomorphism $\nu: \mathfrak{h} \rightarrow \mathfrak{h}'$ such that $\lambda' = \nu \circ \lambda$ and $\mu = \mu' \circ \nu$. In this case, ν is an isomorphism. An extension is said to be central if \mathfrak{a} is contained in the center of the Lie algebra \mathfrak{h} , that is if $[\mathfrak{a}, \mathfrak{h}] = 0$.

A 2-cocycle on a Lie algebra \mathfrak{g} with values in an abelian Lie algebra \mathfrak{a} , is a bilinear map $c: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{a}$, which is antisymmetric (i.e. satisfies $c(X, Y) = -c(Y, X)$ for any $X, Y \in \mathfrak{g}$) and satisfying the cocycle identity

$$c([X, Y], Z) + c([Z, X], Y) + c([Y, Z], X) = 0, \quad X, Y, Z \in \mathfrak{g}.$$

For any linear form $f: \mathfrak{g} \rightarrow \mathbb{C}$, one can define a 2-cocycle c_f by the formula $c_f(X, Y) \stackrel{\text{def}}{=} f([X, Y])$ where $X, Y \in \mathfrak{g}$. Such a 2-cocycle is called a 2-coboundary or a trivial 2-cocycle on \mathfrak{g} . A 2-cocycle c_1 is said to be equivalent to a 2-cocycle c_2 if the difference $c_1 - c_2$ is trivial.

Recall that there is a one-to-one correspondence between the equivalence classes of central extensions of \mathfrak{g} by an abelian Lie algebra \mathfrak{a} and the equivalence classes of cocycles. In particular, for any central extension \mathfrak{h} of \mathfrak{g} by \mathfrak{a} there exists by [Sch08, p. 68] a 2-cocycle $c: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{a}$ such that the Lie algebra \mathfrak{h} is isomorphic to the Lie algebra defined by the vector space $\mathfrak{g} \oplus \mathfrak{a}$ and by the bracket

$$(2.13) \quad [(X, Z), (Y, Z')] = [X, Y]_{\mathfrak{g}} + c(X, Y), \quad X, Y \in \mathfrak{g}, Z, Z' \in \mathfrak{a}.$$

In the situation of Theorem 2.5, we obtain by [Sch08, p. 64] that the Lie algebra \mathfrak{g}_σ of the Lie group G_σ is a central extension of the Lie algebra \mathfrak{t} of the Lie group \mathbb{T} by the Lie algebra \mathfrak{g} of the Lie group G . This means that we have a short exact sequence

$$(2.14) \quad 0 \rightarrow \mathfrak{t} \rightarrow \mathfrak{g}_\sigma \rightarrow \mathfrak{g} \rightarrow 0.$$

with $[\mathfrak{t}, \mathfrak{g}_\sigma] = 0$. Assume that the 2-cocycle $\sigma: G \times G \rightarrow \mathbb{T}$ is smooth. If $\gamma_t: \mathbb{R} \rightarrow G$ and $\beta_s: \mathbb{R} \rightarrow G$ are smooth curves in G such that $\frac{d}{dt}|_{t=0}\gamma_t = X$ and $\frac{d}{ds}|_{s=0}\beta_s = Y$ then the formula

$$(2.15) \quad c(X, Y) = \frac{d^2}{dt ds}|_{t=0, s=0}\sigma(\gamma_t, \beta_s) - \frac{d^2}{dt ds}|_{t=0, s=0}\sigma(\beta_s, \gamma_t),$$

defines by a particular case of [KhW09, Proposition 3.14 p. 26] a 2-cocycle $c: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{t}$ defining the corresponding central extension (2.14). See also [Sun91, Corollary 1.8] for a different but similar formula.

It is well-established that the representations of the group G and the σ -projective representations of the Mackey group G_σ are closely related, as detailed in [Mac58, Section 2], [FeD88, Chapter VIII, Section 10] and [Kle74, Corollary p. 223]. Roughly speaking, the theory of σ -projective representations of G can be reduced to the theory of unitary representations of the slightly larger group G_σ . We need a part of this correspondence for σ -projective representations of locally compact groups on *Banach spaces*, which we now define. If X is a Banach space, a map $\pi: G \rightarrow \mathbb{B}(X)$ is called a (strongly) continuous σ -projective representation on G if the map $G \rightarrow X$, $s \mapsto \pi_s(x)$ is continuous for any $x \in X$ and if we have the equality (2.8).

Proposition 2.6 *Let X be a Banach space. Consider a second countable locally compact group G equipped with a continuous 2-cocycle $\sigma: G \times G \rightarrow \mathbb{T}$. For each continuous σ -projective representation $\pi: G \rightarrow \mathbb{B}(X)$, the map $\tilde{\pi}: G_\sigma \rightarrow \mathbb{B}(X)$, $(s, z) \mapsto z\pi_s$ defines a continuous representation of the group G_σ .*

Proof : For any elements (s, z) and (t, w) of the group G_σ , we have the equalities $\tilde{\pi}(s, z)\tilde{\pi}(t, w) = zw\pi_s\pi_t$ and $\tilde{\pi}((s, z) \cdot (t, w)) = \tilde{\pi}(st, zw\sigma(s, t)) = zw\sigma(s, t)\pi_{st} \stackrel{(2.8)}{=} zw\pi_s\pi_t$. Consequently, $\tilde{\pi}$ is a representation of the group G_σ . The continuity is obvious. \blacksquare

3 Weyl calculus for tuples of C_0 -groups with Θ -commutation relations

Let $n \in \mathbb{N}$. In all the paper, $\Theta \in M_n(\mathbb{R})$ denotes a fixed skew-symmetric matrix, i.e. satisfying $\Theta^T = -\Theta$, where Θ^T is the transpose of the matrix Θ . In the sequel, it is convenient to introduce the upper and lower triangle parts of the matrix Θ by $(\Theta^\uparrow)_{jk} \stackrel{\text{def}}{=} \delta_{j \leq k} \Theta_{jk}$ and $(\Theta^\downarrow)_{jk} \stackrel{\text{def}}{=} \delta_{j \geq k} \Theta_{jk}$ where $1 \leq j, k \leq n$. We clearly have the equalities $\Theta = \Theta^\uparrow + \Theta^\downarrow$ (recall that Θ vanishes on the diagonal) and $(\Theta^\uparrow)^T = -\Theta^\downarrow$.

We start by introducing the following fundamental definition. Independently, Gerhold and Shalit introduced in [GeS23, Definition 1.1] a similar notion in the Hilbert space setting.

Definition 3.1 *Let X be a Banach space. For each $k \in \{1, \dots, n\}$, consider some strongly continuous bounded group $(e^{itA_k})_{t \in \mathbb{R}}$ of operators acting on X with generator iA_k . We say that (A_1, \dots, A_n) is a Θ -Weyl tuple if the groups satisfy the following commutation relations:*

$$(3.1) \quad e^{itA_j} e^{isA_k} = e^{i\Theta_{jk}ts} e^{isA_k} e^{itA_j}, \quad j, k \in \{1, \dots, n\}, s, t \in \mathbb{R}.$$

We will use the following useful notation that we will see as a generalization of [Hal13, notation 14.3 p. 284] and [NeP20, (4.1)].

Definition 3.2 *Let (A_1, \dots, A_n) be a Θ -Weyl tuple. Let $(t_1, \dots, t_n) \in \mathbb{R}^n$. We define*

$$(3.2) \quad e^{it \cdot A} = e^{-\frac{1}{2}i\langle t, \Theta^\uparrow t \rangle} \prod_{k=1}^n e^{it_k A_k} = e^{-\frac{1}{2}i\langle t, \Theta^\uparrow t \rangle} e^{it_1 A_1} e^{it_2 A_2} \dots e^{it_n A_n}.$$

Example 3.3 Suppose that $n = 2d$ for some integer $d \geq 1$. Consider a d -dimensional Weyl pair $(A, B) = (A_1, \dots, A_d, B_1, \dots, B_d)$ as in [NeP20, Definition 3.1] i.e. satisfying the relations

(1.14), (1.15) and (1.16). We will use the matrix $\Theta = \begin{bmatrix} 0 & -I_d \\ I_d & 0 \end{bmatrix}$. We have $\Theta^\dagger = \begin{bmatrix} 0 & -I_d \\ 0 & 0 \end{bmatrix}$.

By introducing the notation $(A_{d+1}, \dots, A_{2d}) \stackrel{\text{def}}{=} (B_1, \dots, B_d)$, we can write $(A_1, \dots, A_{2d}) = (A_1, \dots, A_d, B_1, \dots, B_d)$ and this notation defines a Θ -Weyl tuple. Furthermore, if $t = (u, v) = (u_1, \dots, u_d, v_1, \dots, v_d)$ is an element of \mathbb{R}^{2d} , we obtain $\langle t, \Theta^\dagger t \rangle = -u \cdot v$. Consequently, we recover the notation

$$(3.3) \quad e^{i(uA+vB)} \stackrel{\text{def}}{=} e^{\frac{1}{2}iu \cdot v} e^{iu_1 A_1} \dots e^{iu_d A_d} e^{iv_1 B_1} \dots e^{iv_d B_d}$$

of [Hal13, Notation 14.3 p. 284] with $\hbar = 1$ and d instead of n and the one of [NeP20, (4.1)]. In particular, the case $d = 1$ gives the equality

$$(3.4) \quad e^{i(uA_1+vB_2)} \stackrel{\text{def}}{=} e^{\frac{1}{2}iuv} e^{iuA_1} e^{ivB_2}, \quad u, v \in \mathbb{R}.$$

Now, we prove a commutation rule for the operators defined by the equation (3.2). This result says that we have a projective representation of \mathbb{R}^n with respect to the 2-cocycle defined in (1.2).

Lemma 3.4 *Let (A_1, \dots, A_n) be a Θ -Weyl tuple. Let $(t_1, \dots, t_n), (s_1, \dots, s_n) \in \mathbb{R}^n$. Then*

$$(3.5) \quad e^{it \cdot A} \circ e^{is \cdot A} = e^{\frac{1}{2}i \langle t, \Theta s \rangle} e^{i(t+s) \cdot A}.$$

Moreover, the map $\mathbb{R}^n \rightarrow \mathcal{B}(X)$, $t \mapsto e^{it \cdot A}$ is continuous when the space $\mathcal{B}(X)$ is equipped with the strong operator topology.

Proof : According to the commutation relations from Definition 3.1, we have for any $j \in \{1, \dots, n-1\}$,

$$(3.6) \quad \begin{aligned} \left(\prod_{k=j}^n e^{it_k A_k} \right) e^{is_j A_j} &= e^{it_j A_j} e^{it_{j+1} A_{j+1}} \dots e^{it_n A_n} e^{is_j A_j} \\ &\stackrel{(3.1)}{=} e^{i \sum_{k=j+1}^n \Theta_{kj} t_k s_j} e^{it_j A_j} e^{is_j A_j} \prod_{k=j+1}^n e^{it_k A_k} \\ &= e^{-is_j \langle \Theta^\dagger t \rangle_j} e^{i(t_j+s_j) A_j} \prod_{k=j+1}^n e^{it_k A_k} \end{aligned}$$

since $\sum_{k=j+1}^n \Theta_{kj} t_k s_j = -\sum_{k=j+1}^n \Theta_{jk} t_k s_j = -s_j \langle \Theta^\dagger t \rangle_j$. Using the equality (3.6) n times with $j = 1, \dots, j = n$ to gradually eliminate the factor $\prod_{j=1}^n e^{is_j A_j}$, we deduce that

$$\begin{aligned} e^{it \cdot A} \circ e^{is \cdot A} &\stackrel{(3.2)}{=} e^{-\frac{1}{2}i \langle t, \Theta^\dagger t \rangle} e^{-\frac{1}{2}i \langle s, \Theta^\dagger s \rangle} \left(\prod_{k=1}^n e^{it_k A_k} \right) \left(\prod_{j=1}^n e^{is_j A_j} \right) \\ &\stackrel{(3.6)}{=} e^{-\frac{1}{2}i \langle (t, \Theta^\dagger t) + (s, \Theta^\dagger s) \rangle} e^{-is_1 \langle \Theta^\dagger t \rangle_1} e^{i(t_1+s_1) A_1} \left(\prod_{k=2}^n e^{it_k A_k} \right) \left(\prod_{j=2}^n e^{is_j A_j} \right) \\ &\stackrel{(3.6)}{=} \dots = e^{-\frac{1}{2}i \langle (t, \Theta^\dagger t) + (s, \Theta^\dagger s) \rangle} e^{-i \sum_{k=1}^n s_k \langle \Theta^\dagger t \rangle_k} \prod_{k=1}^n e^{i(t_k+s_k) A_k}. \end{aligned}$$

We conclude since the exponents in the exponential function simplify to

$$\begin{aligned}
-\frac{1}{2}i[\langle t, \Theta^\dagger t \rangle + \langle s, \Theta^\dagger s \rangle] - i\langle s, \Theta^\dagger t \rangle &= -\frac{1}{2}i[\langle t, \Theta^\dagger t \rangle + \langle s, \Theta^\dagger s \rangle + 2\langle s, \Theta^\dagger t \rangle] \\
&= -\frac{1}{2}i[\langle t, \Theta^\dagger t \rangle + \langle s, \Theta^\dagger s \rangle + \langle s, \Theta^\dagger t \rangle + \langle t, \Theta^\dagger s \rangle] + \frac{1}{2}i\langle t, \Theta^\dagger s \rangle - \frac{1}{2}i\langle s, \Theta^\dagger t \rangle \\
&= -\frac{1}{2}i\langle (t+s), \Theta^\dagger (t+s) \rangle + \frac{1}{2}i\langle t, \Theta^\dagger s \rangle - \frac{1}{2}i\langle s, \Theta^\dagger t \rangle \\
&= -\frac{1}{2}i\langle (t+s), \Theta^\dagger (t+s) \rangle + \frac{1}{2}i\langle t, \Theta^\dagger s \rangle + \frac{1}{2}i\langle \Theta^\dagger s, t \rangle \\
&= -\frac{1}{2}i\langle (t+s), \Theta^\dagger (t+s) \rangle + \frac{1}{2}i\langle t, \Theta s \rangle.
\end{aligned}$$

The strong continuity is a consequence of the joint continuity of the composition of operators on bounded sets for the strong operator topology, see [EFHN15, Proposition C.19 p. 517]. ■

Example 3.5 Using the notations of Example 3.3, for any elements $t = (u_1, \dots, u_d, v_1, \dots, v_d)$ and $s = (u'_1, \dots, u'_d, v'_1, \dots, v'_d)$ of \mathbb{R}^{2d} , we observe that $\langle t, \Theta s \rangle = -u \cdot v' + v \cdot u'$. Hence, we recover the commutation rule

$$(3.7) \quad e^{i(uA+vB)} e^{i(u'A+v'B)} = e^{-\frac{1}{2}i(u \cdot v' - v \cdot u')} e^{i((u+u')A+(v+v')B)}$$

of [Hal13, Notation 14.10] with $\hbar = 1$ and the one of [NeP20, (4.2)].

We proceed by introducing the Weyl functional calculus which is suitable for our Θ -Weyl tuples.

Definition 3.6 Let (A_1, \dots, A_n) be a Θ -Weyl tuple on a Banach space X . Consider some Schwartz class function $a \in \mathcal{S}(\mathbb{R}^n)$. We define the Θ -Weyl functional calculus by

$$(3.8) \quad a(A)x \stackrel{\text{def}}{=} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{a}(u) e^{iu \cdot A} x \, du, \quad x \in X,$$

where we use the definition of the Fourier transform

$$(3.9) \quad \hat{a}(u) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} a(x) e^{-i\langle u, x \rangle} \, dx.$$

It is clear from the fact that the groups $(e^{itA_k})_{t \in \mathbb{R}}$ are bounded for $k \in \{1, \dots, n\}$ that $a(A)$ defines a bounded operator on X for any function $a \in L^1(\mathbb{R}^n)$ with $\hat{a} \in L^1(\mathbb{R}^n)$.

Remark 3.7 We can define $a(A)$ if a is a tempered distribution on \mathbb{R}^n with $\hat{a} \in L^1(\mathbb{R}^n)$.

Example 3.8 With the notations of Example 3.3 where $n = 2d$, we recover the Weyl calculus

$$(3.10) \quad a(A, B)x \stackrel{\text{def}}{=} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \hat{a}(u, v) e^{i(uA+vB)} x \, du \, dv, \quad x \in X.$$

of [Hal13, Definition 4.2].

4 Twisted transference principle and universal Θ -Weyl tuples

Twisted convolution Let G be a locally compact group equipped with a left Haar measure μ_G and a continuous 2-cocycle $\sigma: G \times G \rightarrow \mathbb{T}$ on G . Following [EdL69a, p. 120], we define the twisted convolution $f *_\sigma g: G \rightarrow \mathbb{C}$ of two functions $f, g: G \rightarrow \mathbb{C}$, when it exists, by

$$(4.1) \quad (f *_\sigma g)(t) \stackrel{\text{def}}{=} \int_G \sigma(s, s^{-1}t) f(s) g(s^{-1}t) d\mu_G(s), \quad t \in G.$$

It is immediate to see that the twisted convolution is associative on $L^1(G)$. If the cocycle σ is normalized, note that

$$(4.2) \quad (f *_\sigma g)(t) \stackrel{(2.7)}{=} \int_G \sigma(t^{-1}, s) f(s) g(s^{-1}t) d\mu_G(s).$$

Example 4.1 In the case of the group $G = \mathbb{R}^d$ equipped with the 2-cocycle defined in (1.2), we obtain and we will use the notation

$$(4.3) \quad (f *_\Theta g)(x) = \int_{\mathbb{R}^d} f(y) g(x - y) e^{-\frac{i}{2}\langle x, \Theta y \rangle} dy.$$

The following result is elementary.

Lemma 4.2 *Let G be a locally compact group equipped with a left Haar measure μ_G and a continuous 2-cocycle $\sigma: G \times G \rightarrow \mathbb{T}$ on G . For any $1 \leq p \leq \infty$ and any Banach space X , the operator $C_{f,\sigma}: L^p(G, X) \rightarrow L^p(G, X)$, $g \mapsto f *_\sigma g$ is a well-defined linear operator on the Bochner space $L^p(G, X)$ for any function $f \in L^1(G)$ and we have*

$$(4.4) \quad \|f *_\sigma g\|_{L^p(G, X)} \leq \|f\|_{L^1(G)} \|g\|_{L^p(G, X)}, \quad g \in L^p(G, X).$$

Proof : It suffices to show that $C_{f,\sigma}$ induces a bounded operator on the Banach spaces $L^1(G)$ and $L^\infty(G)$, because then, it is a regular operator on any $L^p(G)$ by [Pis10, p. 13] (see also [ArK23, Lemma 3.20 p. 36]), and as such extends to a bounded operator on the Bochner space $L^p(\mathbb{R}^n, X)$ for any Banach space X by definition [Pis10, p. 13]. For any $t \in G$, one has

$$|(C_{f,\sigma}g)(t)| \stackrel{(4.1)}{=} \left| \int_G \sigma(s, s^{-1}t) f(s) g(s^{-1}t) d\mu_G(s) \right| \leq \int_G |f(s)| |g(s^{-1}t)| d\mu_G(s) \stackrel{(2.5)}{=} (|f| * |g|)(t).$$

Therefore, by the classical Young inequality [Fol16, Proposition 2.40 p. 57], we obtain the inequalities $\|C_{f,\sigma}g\|_{L^1(G)} \leq \|f\|_{L^1(G)} \|g\|_{L^1(G)}$ and $\|C_{f,\sigma}g\|_\infty \leq \|f\|_{L^1(G)} \|g\|_{L^\infty(G)}$. \blacksquare

Remark 4.3 In particular, if $f \in L^1(G)$, the twisted convolution operator $C_{f,\sigma}: L^p(G) \rightarrow L^p(G)$, $g \mapsto f *_\sigma g$ is regular in the sense of [HvNVW18, p. 154] with $\|C_{f,\sigma}\|_{\text{reg}, L^p(G) \rightarrow L^p(G)} \leq \|f\|_{L^1(G)}$.

Twisted transference The following result is a generalization of [BGM89, Theorem 2.8]. See also [BPW94], [CoW76]. In this statement, for any integrable function $b \in L^1(G)$ the integral $\int_G b(s) \pi_s d\mu_G(s)$ is defined strongly, which means that for any $x \in X$ the integral $\int_G b(s) \pi_s(x) d\mu_G(s)$ is a Bochner integral, since $\int_G \|b(s) \pi_s(x)\|_X d\mu_G(s) \leq M \|b\|_{L^1(G)} \|x\|_X$. In particular, this entails that $\left\| \int_G b(s) \pi_s d\mu_G(s) \right\|_{X \rightarrow X} \leq M \|b\|_{L^1(G)}$.

Proposition 4.4 *Let G be an amenable locally compact group equipped with a normalized continuous 2-cocycle σ . Consider a strongly continuous σ -projective representation $\pi: G \rightarrow \mathbf{B}(X)$ of G on a Banach space X which is uniformly bounded i.e. $M \stackrel{\text{def}}{=} \sup \{ \|\pi_s\|_{X \rightarrow X} : s \in G \} < \infty$. Suppose that $1 \leq p < \infty$. Then for any function $b \in L^1(G)$, we have*

$$(4.5) \quad \left\| \int_G b(s) \pi_s \, d\mu_G(s) \right\|_{X \rightarrow X} \leq M^2 \|C_{b,\sigma}\|_{L^p(G,X) \rightarrow L^p(G,X)}$$

where $C_{b,\sigma}: f \rightarrow b *_{\sigma} f$.

Proof : Note that both sides of the inequality define continuous functions on $L^1(G)$. So by density, we can suppose that the function b is continuous with compact support \mathcal{K} . Let $\varepsilon > 0$. Since the group G is amenable, we can use Leptin's characterization of amenability [Pie84, Theorem 7.9 p. 70]. There exists a measurable subset V of G with $0 < \mu(V) < \infty$ such that $\frac{\mu(\mathcal{K}^{-1}V)}{\mu(V)} \leq 1 + \varepsilon$. For any $x \in X$, we define the vector-valued function $f: G \rightarrow X$ by

$$(4.6) \quad f(s) \stackrel{\text{def}}{=} 1_{V^{-1}\mathcal{K}}(s^{-1}) \pi_{s^{-1}} x.$$

Note that

$$(4.7) \quad \begin{aligned} \|f\|_{L^p(G,X)}^p &\stackrel{(4.6)}{=} \int_G \|1_{V^{-1}\mathcal{K}}(s^{-1}) \pi_{s^{-1}} x\|_X^p \, ds = \int_G \|\pi_{s^{-1}} x\|_X^p 1_{V^{-1}\mathcal{K}}(s^{-1}) \, ds \\ &\leq M^p \int_G \|x\|_X^p 1_{V^{-1}\mathcal{K}}(s^{-1}) \, ds = \mu(\mathcal{K}^{-1}V) M^p \|x\|_X^p. \end{aligned}$$

Introducing the relation $\pi_t \pi_{t^{-1}} \stackrel{(2.8)}{=} \text{Id}$ in the first equality, we observe that for any $t \in V$

$$\begin{aligned} \left\| \int_G b(s) \pi_s x \, ds \right\|_X &= \left\| \pi_t \pi_{t^{-1}} \left(\int_G b(s) \pi_s x \, ds \right) \right\|_X \leq M \left\| \int_G b(s) \pi_{t^{-1}} \pi_s x \, ds \right\|_X \\ &\stackrel{(2.8)}{=} M \left\| \int_G \sigma(t^{-1}, s) b(s) \pi_{t^{-1} s} x \, ds \right\|_X. \end{aligned}$$

Raising to the p^{th} power and averaging over V , we obtain

$$\begin{aligned} \left\| \int_G b(s) \pi_s x \, ds \right\|_X^p &\leq M^p \frac{1}{\mu(V)} \int_V \left\| \int_K \sigma(t^{-1}, s) b(s) \pi_{t^{-1} s} x \, ds \right\|_X^p \, dt \\ &= M^p \frac{1}{\mu(V)} \int_V \left\| \int_G \sigma(t^{-1}, s) b(s) 1_{V^{-1}\mathcal{K}}(t^{-1}s) \pi_{t^{-1} s}(x) \, ds \right\|_X^p \, dt \\ &\stackrel{(4.6)}{\leq} M^p \frac{1}{\mu(V)} \int_G \left\| \int_G \sigma(t^{-1}, s) b(s) f(s^{-1}t) \, ds \right\|_X^p \, dt \\ &\stackrel{(4.2)}{=} M^p \frac{1}{\mu(V)} \int_G \|(b *_{\sigma} f)(t)\|_X^p \, dt \leq M^p \frac{1}{\mu(V)} \|b *_{\sigma}\|_{L^p(G,X) \rightarrow L^p(G,X)}^p \|f\|_{L^p(G,X)}^p \\ &\stackrel{(4.7)}{\leq} M^{2p} (1 + \varepsilon) \|b *_{\sigma}\|_{L^p(G,X) \rightarrow L^p(G,X)}^p \|x\|_X^p. \end{aligned}$$

■

Remark 4.5 It seems to us that we cannot deduce this result from the standard transference result by using the Mackey group and the lifting of the projective representation on this group.

Using the projective representation $t \mapsto e^{it \cdot A}$ provided by Lemma 3.4, we deduce the following result.

Corollary 4.6 *Let $A = (A_1, \dots, A_n)$ be a Θ -Weyl tuple acting on some Banach space X . Set $M_A \stackrel{\text{def}}{=} \sup_{t \in \mathbb{R}^n} \|e^{it \cdot A}\|_{X \rightarrow X}$. Suppose that $1 \leq p < \infty$. Then for any tempered distribution a with $\hat{a} \in L^1(\mathbb{R}^n)$, we have*

$$(4.8) \quad \|a(A)\|_{X \rightarrow X} \leq M_A^2 \|C_a^\Theta\|_{L^p(\mathbb{R}^n, X) \rightarrow L^p(\mathbb{R}^n, X)}$$

where

$$(4.9) \quad (C_a^\Theta g)(x) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{1}{2}i\langle y, \Theta x \rangle} a(y) g(x - y) dy.$$

Universal Θ -Weyl tuple Suppose that $1 \leq p < \infty$. For each $k \in \{1, \dots, n\}$, consider the strongly continuous groups $(e^{isQ_k})_{t \in \mathbb{R}}$ and $(e^{isP_k})_{t \in \mathbb{R}}$ of operators acting on the Banach space $L^p(\mathbb{R}^n)$ defined by

$$(4.10) \quad (e^{isQ_k} f)(x) = e^{isx_k} f(x) \text{ and } (e^{isP_k} f)(x) = f(x + se_k), \quad f \in L^p(\mathbb{R}^n).$$

Note that the infinitesimal generators act on their domains by

$$(4.11) \quad (Q_k f)(x) = x_k f(x) \quad \text{and} \quad (P_k f)(x) = -i\partial_k f(x).$$

We refer to [EnN06, p. 65] for more information. Now, we introduce the universal Θ -Weyl tuple, which plays a special role in the theory of Θ -Weyl tuples.

Proposition 4.7 *Consider the families $Q = (Q_1, \dots, Q_n)$ and $P = (P_1, \dots, P_n)$. The tuple*

$$(4.12) \quad A_{\text{univ}}^\Theta \stackrel{\text{def}}{=} \left(\frac{1}{2} \sum_{l=1}^n \Theta_{1l} Q_l - P_1, \dots, \frac{1}{2} \sum_{l=1}^n \Theta_{kl} Q_l - P_k, \dots, \frac{1}{2} \sum_{l=1}^n \Theta_{nl} Q_l - P_n \right)$$

is a Θ -Weyl tuple. Moreover, its Θ -Weyl calculus satisfies for any function $a \in \mathcal{S}(\mathbb{R}^n)$:

$$(4.13) \quad a(A_{\text{univ}}^\Theta) = C_a^\Theta.$$

More precisely, we have

$$(4.14) \quad (e^{it \cdot A_{\text{univ}}^\Theta} f)(x) = e^{\frac{1}{2}i\langle t, \Theta x \rangle} f(x - t), \quad x, t \in \mathbb{R}^n.$$

The tuple A_{univ}^Θ is said to be the universal Θ -Weyl tuple, also denoted by $A_{\text{univ}}^\Theta = \frac{1}{2}\Theta Q - P$.

Proof : For any $k \in \{1, \dots, n\}$, we have $A_k = \frac{1}{2} \sum_{l=1}^n \Theta_{kl} Q_l - P_k$. Since Q_l and P_k commute for any distinct indices l and k and since $\Theta_{kk} = 0$ by skew-adjointness of Θ , all the operators constituting A_k commute. Thus, we see that

$$(4.15) \quad (e^{it \cdot A_k} f)(x) = \prod_{l=1}^n e^{\frac{1}{2}it\Theta_{kl}Q_l} e^{-itP_k} f(x) \stackrel{(4.10)}{=} \prod_{l=1}^n e^{\frac{1}{2}it\Theta_{kl}x_l} f(x - te_k).$$

We infer that

$$\begin{aligned}
(e^{itA_j} e^{isA_k} f)(x) &\stackrel{(4.15)}{=} e^{itA_j} \prod_{l=1}^n e^{\frac{1}{2}is\Theta_{kl}x_l} f(x - se_k) \\
&\stackrel{(4.15)}{=} \prod_{m=1}^n e^{\frac{1}{2}it\Theta_{jm}x_m} \left(\prod_{l \neq j} e^{\frac{1}{2}is\Theta_{kl}x_l} \right) e^{\frac{1}{2}is\Theta_{kj}(x_j-t)} f(x - se_k - te_j) \\
&= \prod_{m=1}^n e^{\frac{1}{2}it\Theta_{jm}x_m} \prod_{l=1}^n e^{\frac{1}{2}is\Theta_{kl}x_l} e^{-\frac{1}{2}ist\Theta_{kj}} f(x - se_k - te_j).
\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}
(e^{isA_k} e^{itA_j} f)(x) &= \prod_{m=1}^n e^{\frac{1}{2}it\Theta_{jm}x_m} \prod_{l=1}^n e^{\frac{1}{2}is\Theta_{kl}x_l} e^{+\frac{1}{2}ist\Theta_{kj}} f(x - se_k - te_j) \\
&= e^{ist\Theta_{kj}} e^{itA_j} e^{isA_k} f(x).
\end{aligned}$$

By (3.1), we infer that A_{univ}^Θ is a Θ -Weyl tuple.

Next, we note that for any $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ and any $k = 1, \dots, n$, we have

$$(e^{it_k A_k} f)(x) \stackrel{(4.15)}{=} \prod_{l=1}^n e^{\frac{1}{2}it_k \Theta_{kl} x_l} f(x - te_k) = e^{\frac{1}{2}it_k (\Theta x)_k} f(x - te_k).$$

Thus we obtain

$$\begin{aligned}
\left(\prod_{k=1}^n e^{it_k A_k} f \right)(x) &= e^{\frac{1}{2}it_1 (\Theta x)_1} e^{\frac{1}{2}it_2 (\Theta(x-t_1 e_1))_2} \dots e^{\frac{1}{2}it_n (\Theta(x-t_1 e_1 - \dots - t_{n-1} e_{n-1}))_n} f(x-t) \\
&= e^{\frac{1}{2}i \sum_{k=1}^n t_k (\Theta x)_k} e^{-\frac{1}{2}i \sum_{k=2}^n \sum_{l=1}^{k-1} t_k t_l \Theta_{kl}} f(x-t) \\
&= e^{\frac{1}{2}i \langle t, \Theta x \rangle} e^{-\frac{1}{2}i \langle t, \Theta^\perp t \rangle} f(x-t).
\end{aligned}$$

We deduce that

$$(e^{it \cdot A_{\text{univ}}^\Theta} f)(x) \stackrel{(3.2)}{=} e^{-\frac{1}{2}i \langle t, \Theta^\perp t \rangle} e^{-\frac{1}{2}i \langle t, \Theta^\perp t \rangle} e^{\frac{1}{2}i \langle t, \Theta x \rangle} f(x-t) = e^{\frac{1}{2}i \langle t, \Theta x \rangle} f(x-t).$$

We have proven (4.14). Therefore, the Weyl calculus is given by

$$\begin{aligned}
(a(A_{\text{univ}}^\Theta) f)(x) &\stackrel{(3.8)}{=} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{a}(u) e^{iu \cdot A_{\text{univ}}^\Theta} f(x) du \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{a}(u) e^{\frac{1}{2}i \langle u, \Theta x \rangle} f(x-u) du \stackrel{(4.9)}{=} C_{\hat{a}}^\Theta f(x).
\end{aligned}$$

■

Example 4.8 If $n = 2$ and if $\Theta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, we recover the Weyl pair $(A, B) = (-\frac{1}{2}Q_2 - P_1, \frac{1}{2}Q_1 - P_2)$ considered in [NeP20, Lemma 6.2]. Note that by Proposition 2.4, we have $[A, B] = -i$.

5 Associated semigroup and functional calculus of the operator $\sum_{k=1}^n A_k^2$

In this section, we introduce a semigroup generated by the «sum of squares» $\sum_{k=1}^n A_k^2$ associated with a Θ -Weyl tuple $A = (A_1, \dots, A_n)$. In the particular case where A is the universal Θ -Weyl tuple A_{univ} , we can explicitly describe the integral kernel of this semigroup and show the complete boundedness of the $H^\infty(\Sigma_\theta)$ functional calculus of the generator for any angle $\frac{\pi}{2} < \theta < \pi$.

Nilpotent Lie algebras and nilpotent Lie groups We need to briefly review some elementary theory of nilpotent Lie groups. Let \mathfrak{g} be a finite-dimensional Lie algebra over \mathbb{R} . The descending central series of \mathfrak{g} is defined inductively as in [HiN12, Definition 5.2.1 p. 91] by

$$\mathfrak{g}^1 \stackrel{\text{def}}{=} \mathfrak{g}, \quad \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^3 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \quad \text{and} \quad \mathfrak{g}^{k+1} \stackrel{\text{def}}{=} [\mathfrak{g}, \mathfrak{g}^k].$$

We have $\mathfrak{g}^{k+1} \subset \mathfrak{g}^k$ for any integer $k \geq 1$. The Lie algebra \mathfrak{g} is said to be nilpotent if $\mathfrak{g}^{s+1} = \{0\}$ for some $s \in \mathbb{N}$. If s is the smallest integer such that $\mathfrak{g}^{s+1} = 0$, then \mathfrak{g} is said to be nilpotent of step s . Every abelian Lie algebra is nilpotent of step 1. By [HiN12, Theorem 11.2.5 p. 444], a connected Lie group G is nilpotent if and only if its Lie algebra is nilpotent. A connected nilpotent Lie group G is said to be of step s if its Lie algebra has step s .

A stratification [BLU07, Definition 2.2.3 p. 122] (see also [Fir16, Definition 3.1.5 p. 93]) of a finite-dimensional real Lie algebra \mathfrak{g} is a vector space decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s$ such that

$$\mathfrak{g}_{k+1} = [\mathfrak{g}_1, \mathfrak{g}_k], \quad 1 \leq k \leq s-1, \quad [\mathfrak{g}_1, \mathfrak{g}_s] = 0.$$

In other words, we have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus [\mathfrak{g}_1, \mathfrak{g}_1] \oplus [\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]] \oplus [\mathfrak{g}_1, [\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]]] \oplus \dots$$

The subspace \mathfrak{g}_1 is often referred to as the first layer or the horizontal layer and generates \mathfrak{g} as a Lie algebra by [BiB19, Proposition A.17 p. 346]. By [BiB19, Lemma A.20 p. 347], the existence of such a stratification with $\mathfrak{g}_s \neq \{0\}$ implies that the Lie algebra \mathfrak{g} is nilpotent of step s . We say that a simply connected nilpotent Lie group G is stratified when its Lie algebra is equipped with a stratification and we say that it has m generators if $m \stackrel{\text{def}}{=} \dim \mathfrak{g}_1$.

The Mackey group Let $n \geq 1$. We start by describing some information on the Mackey group $\mathbb{H}_\Theta \stackrel{\text{def}}{=} \mathbb{R}^n \times \mathbb{T}$ introduced in (2.11) associated to the 2-cocycle σ_Θ defined in (1.2) on the group \mathbb{R}^n by the skew-symmetric matrix Θ .

Proposition 5.1 *There exists a unique structure of Lie group on the locally compact group \mathbb{H}_Θ . Moreover, this group is connected and nilpotent. Furthermore, its Lie algebra \mathfrak{h}_Θ identifies to $\mathbb{R}^n \oplus \mathbb{R}$ equipped with the Lie bracket*

$$(5.1) \quad [(x_1, z_1), (x_2, z_2)] = (0, \langle x_1, \Theta x_2 \rangle), \quad x_1, x_2 \in \mathbb{R}^n, z_1, z_2 \in \mathbb{R}$$

and if $\Theta \neq 0$, this decomposition defines a 2-step stratification $\mathfrak{h}_\Theta = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of \mathfrak{h}_Θ .

Proof : Recall that by (2.12) the group \mathbb{H}_Θ is a central extension of the group \mathbb{R}^n by \mathbb{T} . Consequently, it admits by Theorem 2.5 a canonical structure of Lie group. Moreover, by (2.14) we deduce that its Lie algebra \mathfrak{h}_Θ is a central extension of the Lie algebra \mathfrak{t} of \mathbb{T} by the

Lie algebra \mathfrak{g} of \mathbb{R}^n . This means that we have an exact sequence $0 \rightarrow \mathfrak{t} \rightarrow \mathfrak{h}_\Theta \rightarrow \mathfrak{g} \rightarrow 0$ with $[\mathfrak{t}, \mathfrak{h}_\Theta] = 0$. By (2.13), the Lie algebra \mathfrak{h}_Θ identifies as a vector space to $\mathfrak{g} \oplus \mathfrak{t} = \mathbb{R}^n \oplus i\mathbb{R}$ and that the Lie bracket has the form

$$[(x_1, iz_1), (x_2, iz_2)] \stackrel{(2.13)}{=} (0, c(x_1, x_2)), \quad x_1, x_2 \in \mathbb{R}^n, z_1, z_2 \in \mathbb{R}$$

where $c: \mathbb{R}^n \times \mathbb{R}^n \rightarrow i\mathbb{R}$ is the associated Lie algebra 2-cocycle. Here, we use the well-known identification of \mathfrak{t} with the real vector space $i\mathbb{R}$, e.g. [HiN12, p. 71]. We can obtain a concrete description of this cocycle by using a suitable differentiation. Indeed, for any $x_1, x_2 \in \mathbb{R}^n$, we have

$$\begin{aligned} c(x_1, x_2) &\stackrel{(2.15)}{=} \frac{d^2}{ds dt} \Big|_{t=0, s=0} [\sigma_\Theta(\exp_{\mathbb{R}^n}(tx_1), \exp_{\mathbb{R}^n}(sx_2)) - \sigma_\Theta(\exp_{\mathbb{R}^n}(sx_2), \exp_{\mathbb{R}^n}(tx_1))] \\ &= \frac{d^2}{ds dt} \Big|_{t=0, s=0} [\sigma_\Theta(tx_1, sx_2) - \sigma_\Theta(sx_2, tx_1)] \stackrel{(1.2)}{=} \frac{d^2}{ds dt} \Big|_{t=0, s=0} [e^{\frac{1}{2}i\langle tx_1, \Theta sx_2 \rangle} - e^{\frac{1}{2}i\langle sx_2, \Theta tx_1 \rangle}] \\ &= \frac{d^2}{ds dt} \Big|_{t=0, s=0} [e^{\frac{1}{2}its\langle x_1, \Theta x_2 \rangle} - e^{\frac{1}{2}ist\langle x_2, \Theta x_1 \rangle}] = \frac{i}{2}\langle x_1, \Theta x_2 \rangle - \frac{i}{2}\langle x_2, \Theta x_1 \rangle = i\langle x_1, \Theta x_2 \rangle. \end{aligned}$$

Now, identifying $i\mathbb{R}$ with \mathbb{R} we obtain (5.1). Furthermore, it is clear that the decomposition $\mathfrak{h}_\Theta = \mathbb{R}^n \oplus \mathbb{R}$ defines a 2-step stratification of the Lie algebra \mathfrak{h}_Θ if $\Theta \neq 0$. \blacksquare

Remark 5.2 We can easily describe some additional properties of this group. Indeed, it is unimodular by [Fol16, p. 53] or [HiN12, Proposition 10.4.11 p. 410] and amenable since any solvable locally compact group is amenable by [Pie84, Corollary 13.5 p. 120].

Recall that by [HiN12, Proposition 9.5.1 p. 335] the Lie group \mathbb{H}_Θ and its universal covering $\tilde{\mathbb{H}}_\Theta$ admit isomorphic Lie algebras. Consequently, the universal covering $\tilde{\mathbb{H}}_\Theta$ is a (simply connected) 2-step stratified nilpotent Lie group. So the previous result leads us to introduce some useful concepts for the study of 2-step stratified Lie groups. It is widely recognized that these groups are non-abelian Lie groups which closely resemble abelian groups, although they display distinct phenomena not present in abelian groups.

Kaplan J -operators on 2-step stratified Lie groups Let G be a 2-step stratified connected Lie group and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the stratification of its Lie algebra. Let \mathfrak{g}_2^* be the dual of \mathfrak{g}_2 . Any element $\mu \in \mathfrak{g}_2^*$ gives rise to a skew-symmetric bilinear form $\omega_\mu: \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathbb{R}$ defined by

$$\omega_\mu(x, y) = \mu([x, y]), \quad x, y \in \mathfrak{g}_1.$$

Following [BLU07, Proposition 3.7.3 p. 174], we say that the group G is called a M etivier group if ω_μ is non-degenerate for all *non-zero* linear forms $\mu \in \mathfrak{g}_2^*$.

Consider an inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra \mathfrak{g} for which the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is orthogonal. The inner product $\langle \cdot, \cdot \rangle$ induces a norm on the dual \mathfrak{g}_2^* that we denote by $|\cdot|$. Let $J_\mu: \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ be the unique skew-symmetric endomorphism such that

$$(5.2) \quad \langle J_\mu(x), y \rangle = \mu([x, y]), \quad x, y \in \mathfrak{g}_1.$$

Then it is immediate that G is a M etivier group if and only if J_μ is invertible for all non-zero linear forms $\mu \in \mathfrak{g}_2^*$. We call G a Heisenberg type group (or H -type group) if the endomorphisms J_μ are orthogonal for all $\mu \in \mathfrak{g}_2^*$ of norm 1. This is equivalent to say that

$$J_\mu^2 = -|\mu|^2 \text{Id}_{\mathfrak{g}_1} \quad \text{for all } \mu \in \mathfrak{g}_2^*.$$

This class of groups was introduced by Kaplan in [Kap80]. The class of Heisenberg type groups is a strict subclass of that of Métivier groups.

If G is a simply-connected 2-step nilpotent Lie group and equipped with the left invariant metric determined by the inner product on $\mathfrak{g} = T_e G$, all the geometry of G can be described by the maps J_μ , as showed in [Ebe94], at least in the case where \mathfrak{g}_2 is the center of \mathfrak{g} . These operators were introduced by Kaplan in [Kap80], [Kap81] and [Kap83], initially for studying the Heisenberg type groups.

Exponential maps on connected nilpotent Lie groups If \mathfrak{g} is a nilpotent Lie algebra, then by [HiN12, Corollary 11.2.6 p. 445] the Dynkin series defines a polynomial map $*$: $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, $(x, y) \mapsto x + y + \frac{1}{2}[x, y] + \dots$. Moreover, $(\mathfrak{g}, *)$ defines a Lie group whose Lie algebra is \mathfrak{g} with $\exp_{\mathfrak{g}} = \text{Id}_{\mathfrak{g}}$. For nilpotent algebras of step 2, observe that the product is given by

$$(5.3) \quad x * y = x + y + \frac{1}{2}[x, y], \quad x, y \in \mathfrak{g}.$$

Recall that by [HiN12, Corollary 11.2.7 p. 446], the exponential map $\exp_G: (\mathfrak{g}, *) \rightarrow G$ of a connected nilpotent Lie group is the universal covering morphism of G . In particular, the exponential map of G is surjective. If in addition G is simply connected then the exponential map $\exp: \mathfrak{g} \rightarrow G$ is a diffeomorphism and can identify G with $(\mathfrak{g}, *)$. In this case, the Haar measure on G coincides with the Lebesgue measure on \mathfrak{g} by [CoG90, Theorem 1.2.10 p. 19].

The universal covering of \mathbb{H}_Θ Now, we will describe the universal covering of the connected nilpotent Lie group \mathbb{H}_Θ . We can consider the canonical basis (a_1, \dots, a_{n+1}) of $\mathbb{R}^n \oplus \mathbb{R}$. In particular, we obtain

$$(5.4) \quad c(a_i, a_j) = \Theta_{ij}, \quad i, j \in \{1, \dots, n\}.$$

Note that we have an isomorphism $\mathfrak{g}_2^* \rightarrow \mathbb{R}$, $\mu \mapsto \mu(a_{n+1})$. Consider a basis (a_1, \dots, a_n) of \mathfrak{g}_1 where $n = \dim \mathfrak{g}_1$. Let $\langle \cdot, \cdot \rangle$ be the inner product on \mathfrak{g}_1 that turns (a_1, \dots, a_{n+1}) into an orthonormal basis.

Proposition 5.3 *The universal covering $\tilde{\mathbb{H}}_\Theta$ of the Lie group \mathbb{H}_Θ is given by $\tilde{\mathbb{H}}_\Theta = \mathbb{R}^n \times \mathbb{R}$ equipped with the product*

$$(5.5) \quad (s, u) \cdot (s', u') = \left(s + s', u + u' + \frac{1}{2}\langle s, \Theta s' \rangle \right), \quad s, s' \in \mathbb{R}^n, u, u' \in \mathbb{R}$$

and the morphism $\tilde{\mathbb{H}}_\Theta \rightarrow \mathbb{H}_\Theta$ is defined by $(s, u) \mapsto (s, e^{iu})$. Moreover, for any $\mu \in \mathbb{R}$, the skew-symmetric endomorphism $J_\mu: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is determined by $J_\mu = -\mu\Theta$. Finally, the Lie group \mathbb{H}_Θ is a Métivier group if and only if the matrix Θ is invertible and a Heisenberg type group if and only if the matrix Θ is orthogonal.

Proof : The first sentence is a consequence of (5.1) and (5.3). For any $\mu \in \mathbb{R}$, we have

$$\langle J_\mu x, y \rangle_{\mathbb{R}^n} \stackrel{(5.2)}{=} \mu[x, y] \stackrel{(5.1)}{=} \mu \langle x, \Theta y \rangle = -\mu \langle \Theta x, y \rangle_{\mathbb{R}^n}.$$

We deduce that $J_\mu = -\mu\Theta$. The last sentence is obvious. ■

Note that by [BLU07, Theorem 3.2.2 p. 160] the groups defined in (5.5) are *exactly* the $n + 1$ -dimensional stratified groups of step two with n generators.

Partial Fourier transform Let G be a 2-step stratified simply connected Lie group and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the stratification of its Lie algebra. If $f \in L^1(G)$ and $\mu \in \mathfrak{g}_2^*$, then we denote by f^μ the μ -section of the partial Fourier transform of f along \mathfrak{g}_2 , given by

$$(5.6) \quad f_\mu(s) \stackrel{\text{def}}{=} \int_{\mathfrak{g}_2} f(s, u) e^{-i\langle \mu, u \rangle} du, \quad s \in \mathfrak{g}_1.$$

We also define the even meromorphic functions S and R by

$$(5.7) \quad S(z) \stackrel{\text{def}}{=} \frac{z}{\sin(z)} \quad \text{and} \quad R(z) \stackrel{\text{def}}{=} \frac{z}{\tan(z)}, \quad z \in \mathbb{C} - \{k\pi : 0 \neq k \in \mathbb{Z}\}$$

and the square root determined by the principal branch of the logarithm $\log: \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$. Let a_1, \dots, a_n be a basis of \mathfrak{g}_1 and let $\Delta = -\sum_{i=1}^n a_i^2$ be the corresponding sub-Laplacian where $n = \dim \mathfrak{g}_1$, see [BLU07, Definition 2.2.25 p. 144] or [Rob91, p. 296]. For any $t > 0$, we denote by K_t the kernel of the operator $e^{-t\Delta}$. Observe that by [Rob91, Lemma 4.15 p. 327], the semigroup $(e^{-t\Delta})_{t \geq 0}$ on $L^2(G)$ is a bounded holomorphic semigroup of angle $\frac{\pi}{2}$. We denote by K_z its kernel.

Note that $J_\mu: \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ is naturally identified with a skew-symmetric endomorphism of the complexification $(\mathfrak{g}_1)_\mathbb{C}$ of \mathfrak{g}_1 , endowed with the corresponding hermitian inner product, and, for all $z \in \mathbb{C}$, the map zJ_μ is a normal endomorphism of $(\mathfrak{g}_1)_\mathbb{C}$. It is stated in [MaM16, Proposition 4] (see also [Cyg79]) that

$$(5.8) \quad K_{z,\mu}(s) = \frac{1}{(4\pi z)^{n/2}} \sqrt{\det(S(zJ_\mu))} \exp\left(-\frac{1}{4z} \langle R(zJ_\mu)s, s \rangle\right), \quad \Re z > 0$$

for any $\mu \in \mathfrak{g}_2^*$ and any $s \in \mathfrak{g}_1$. We also refer to [CCCFH11, Theorem 10.2.7, p. 249] and [BGG96, Theorem 4, p. 318], which present formulas that are likely equivalent for real times.

The sum of squares $\sum_{k=1}^n A_k^2$ and its associated semigroup Now, we connect the «sum of squares» associated to a Weyl pair to a class of «subelliptic» operators studied in [Rob91, Chapter IV], while a less general setting is investigated in the thesis of Langlands, see [Lan59] and [Lan60]. These operators are constructed from a connected real finite-dimensional unimodular Lie group G , equipped with a Lie algebraic basis (a_1, \dots, a_n) of the Lie algebra \mathfrak{g} , that is a finite sequence of linearly independent elements of \mathfrak{g} generating \mathfrak{g} , and a continuous representation $\pi: G \rightarrow B(X)$ on a Banach space X . Using the classical notation $d\pi(a_i)$ of [Mag92, Definition 3.12 p. 51] for the infinitesimal generator of the strongly continuous group $(\pi(\exp_G(ta_i)))_{t \in \mathbb{R}}$, we can consider the operator

$$(5.9) \quad H \stackrel{\text{def}}{=} -(d\pi(a_1))^2 - \dots - (d\pi(a_n))^2.$$

We warn the reader that the notation A_i is often used for the operator $d\pi(a_i)$ but in our paper, the notation A_i denotes an element of a Θ -Weyl tuple.

The following result is a combination of [Jor75, Proposition 3.2 p. 127], [Rob91, Corollary 4.17 p. 332] and [EIR92, Theorem 2.1].

Theorem 5.4 *1. The operator H defined by (5.9) is closable and its closure \overline{H} generates a continuous bounded holomorphic strongly continuous semigroup $(T_t)_{t \geq 0}$ on the Banach space X of angle $0 < \varphi \leq \frac{\pi}{2}$ with the property that $\text{Ran } T_t$ is a subspace of the domain of any monomial in the operators $d\pi(a_1), \dots, d\pi(a_n)$ for all $t > 0$.*

2. If K_t is the kernel of the operator $e^{-t\Delta}$ where $\Delta = -\sum_{i=1}^n a_i^2$ then

$$(5.10) \quad T_t = \int_G K_t(s)\pi(s) ds, \quad t > 0.$$

Now, we are able to prove the following result.

Proposition 5.5 *Let $A = (A_1, \dots, A_n)$ be a Θ -Weyl tuple on some Banach space X .*

1. *The opposite of the closure \mathcal{A} of the operator $\sum_{k=1}^n A_k^2$ generates a strongly continuous semigroup $(T_t)_{t \geq 0}$ of operators acting on X satisfying*

$$(5.11) \quad T_t = \int_{\tilde{\mathbb{H}}_\Theta} K_t(s, u)\pi(s, u) ds du, \quad t \geq 0$$

where $(K_t)_{t \geq 0}$ is the kernel semigroup associated to the subelliptic Laplacian Δ on the universal covering $\tilde{\mathbb{H}}_\Theta$ of the nilpotent Lie group \mathbb{H}_Θ and where $\pi: \tilde{\mathbb{H}}_\Theta \rightarrow \mathbb{B}(X)$ is a continuous representation.

2. *The semigroup generated by $-\mathcal{A}$ is bounded holomorphic of some angle $\varphi \in (0, \frac{\pi}{2}]$ and can be written as*

$$(5.12) \quad T_z x = \int_{\mathbb{R}^n} p_z^\Theta(s) e^{is \cdot A} x ds, \quad x \in X, |\arg z| < \varphi$$

where the kernel $(p_z^\Theta)_{\Re z > 0}$ is given by

$$(5.13) \quad p_z^\Theta(s) \stackrel{\text{def}}{=} \frac{1}{(4\pi z)^{\frac{n}{2}}} \sqrt{\det S(z\Theta)} \exp\left(-\frac{1}{4z} \langle R(z\Theta)s, s \rangle\right) = \int_{\mathbb{R}} e^{iu} K_z(s, u) du, \quad s \in \mathbb{R}^n.$$

3. *In particular, if A_{univ}^Θ is the universal Θ -Weyl tuple on $L^p(\mathbb{R}^n)$ from (4.12), then*

$$(5.14) \quad (T_z g)(x) = \int_{\mathbb{R}^n} k_z^\Theta(x, y) g(x - y) dy, \quad \text{a.e. } x \in \mathbb{R}^d, |\arg z| < \varphi, g \in L^p(\mathbb{R}^n)$$

where the kernel $(k_z^\Theta)_{\Re z > 0}$ is defined by

$$(5.15) \quad k_z^\Theta(x, y) \stackrel{\text{def}}{=} e^{\frac{1}{2}i \langle y, \Theta x \rangle} p_z^\Theta(y), \quad x, y \in \mathbb{R}^d.$$

Proof : Let (A_1, \dots, A_n) be a Θ -Weyl tuple on a Banach space X . Recall that by Lemma 3.4 we have a projective representation $\mathbb{R}^n \rightarrow \mathbb{B}(X)$, $s \mapsto e^{is \cdot A}$. By Proposition 2.6, we can consider its lifting $\pi_\Theta: \mathbb{H}_\Theta \rightarrow \mathbb{B}(X)$ defined by $\pi_\Theta(s, e^{iu}) = e^{iu} e^{is \cdot A}$ which is a continuous representation. By composition, we obtain a representation $\pi: \tilde{\mathbb{H}}_\Theta \rightarrow \mathbb{B}(X)$ of the universal covering $\tilde{\mathbb{H}}_\Theta$ of the Lie group \mathbb{H}_Θ .

Recall that by [HiN12, Proposition 9.5.1 p. 335] the Lie groups \mathbb{H}_Θ and $\tilde{\mathbb{H}}_\Theta$ admit isomorphic Lie algebras. Then the family (a_1, \dots, a_n) of (5.4) is a Hörmander system. We define $(K_z)_{\Re z > 0}$ to be the heat kernel of the subelliptic Laplacian $\Delta = -\sum_{j=1}^n a_j^2$.

For any $1 \leq j \leq n$ and any $t \in \mathbb{R}$, using [Upm85, Proposition 6.12 p. 99] note that $\pi(\exp(ta_j)) = \exp(\pi_*(ta_j)) = e^{itA_j}$ where the computation of π_* is left to the reader. We deduce that $d\pi(a_j) = iA_j$. Consequently, by Theorem 5.4, the unbounded operator $\sum_{j=1}^n A_j^2 = -\sum_{j=1}^n (d\pi(a_j))^2$ defined on the subspace $\bigcap_{j=1}^n \text{dom } A_j^2$ of the Banach space X is closable, and its closure \mathcal{A} generates a strongly continuous semigroup $(T_t)_{t \geq 0}$ of operators acting on X defined

by (5.11). Furthermore, the same result says that this semigroup is bounded analytic of angle $0 < \varphi \leq \frac{\pi}{2}$. Its extension by $(T_z)_{|\arg z| < \varphi}$ is given by

$$(5.16) \quad T_z = \int_{\tilde{\mathbb{H}}_\Theta} K_z(s, u) \pi(s, u) \, ds \, du, \quad |\arg z| < \varphi.$$

2. For any $x \in X$ and any complex number $z \in \mathbb{C}^*$ with $|\arg z| < \varphi$, we can describe T_z by the formula

$$\begin{aligned} T_z(x) &\stackrel{(5.16)}{=} \int_{\tilde{\mathbb{H}}_\Theta} K_z(s, u) \pi(s, u) x \, ds \, du \stackrel{(5.1)}{=} \int_{\tilde{\mathbb{H}}_\Theta} K_z(s, u) e^{iu} e^{is \cdot A} x \, ds \, du \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}} e^{iu} K_z(s, u) \, du \right) e^{is \cdot A} x \, ds \stackrel{(5.17)}{=} \int_{\mathbb{R}^n} \hat{a}_z(s) e^{is \cdot A} x \, ds \end{aligned}$$

where

$$(5.17) \quad \hat{a}_z(s) \stackrel{\text{def}}{=} \int_{\mathbb{R}} e^{iu} K_z(s, u) \, du.$$

We obtain

$$\hat{a}_z(s) \stackrel{(5.17)}{=} \int_{\mathbb{R}} e^{iu} K_z(s, u) \, du \stackrel{(5.6)}{=} K_{z, -1}(s) \stackrel{(5.8)}{=} \frac{1}{(4\pi z)^{n/2}} \sqrt{\det(S(z\Theta))} \exp\left(-\frac{1}{4z} \langle R(z\Theta)s, s \rangle\right).$$

3. This follows directly from the explicit description of $e^{iu \cdot A}$ from (4.14) in the case where A is the universal Θ -Weyl tuple. \blacksquare

Remark 5.6 Using [DtER03, II.22 p. 25], we see that the local dimension of $(\tilde{\mathbb{H}}_\Theta, (X_1, \dots, X_n))$ is equal to $n + 2$.

Remark 5.7 It might be feasible to replace the angle φ by $\frac{\pi}{2}$ in the previous result by re-examining the method of [Kis76] that establishes holomorphicity at angle $\frac{\pi}{2}$.

We now turn to the functional calculus properties of the operator \mathcal{A} , when A_{univ}^Θ is the universal Θ -Weyl tuple from Proposition 4.7.

Proposition 5.8 *Let X be a UMD Banach space. Consider the operator \mathcal{A} associated to the universal Θ -Weyl tuple A_{univ}^Θ . Then $\mathcal{A} \otimes \text{Id}_X$ admits a bounded $\mathbf{H}^\infty(\Sigma_\omega)$ functional calculus on the Bochner space $L^p(\mathbb{R}^n, X)$ for any $1 < p < \infty$ and any angle $\omega \in (\frac{\pi}{2}, \pi)$.*

Proof : The heat kernel p_t on $\tilde{\mathbb{H}}_\Theta$ from Proposition 5.5 satisfies $\|p_t\|_{L^1(\tilde{\mathbb{H}}_\Theta)} = 1$ for all $t > 0$. Note that by the formula (4.3) defining the twisted convolution, we see the formula (5.14) says that each operator T_t is a twisted convolution operator. Then Remark 4.3 implies that each T_t is contractively regular (and even an absolute contraction).

Consequently, we have a semigroup of contractively regular operators. So, we can use [Fen97, p. 738] to obtain a dilation of the semigroup $(T_t \otimes \text{Id}_X)_{t \geq 0}$. If $p \neq 2$, using the contractive regularity of the operators of the dilation, we conclude with a classical argument [KuW04, Corollary 10.9 p. 214] that $\mathcal{A} \otimes \text{Id}_X$ admits a bounded $\mathbf{H}^\infty(\Sigma_\omega)$ functional calculus on the Bochner space $L^p(\mathbb{R}^n, X)$ for any $\omega \in (\frac{\pi}{2}, \pi)$. Then if $p = 2$, the operator $\mathcal{A} \otimes \text{Id}_X$ still admits a bounded $\mathbf{H}^\infty(\Sigma_\omega)$ functional calculus on $L^2(\mathbb{R}^n, X)$ for any angle $\omega \in (\frac{\pi}{2}, \pi)$, since we can interpolate its spectral multipliers as bounded operators on the Bochner space $L^{p_1}(\mathbb{R}^n, X)$ and $L^{p_2}(\mathbb{R}^n, X)$ to obtain bounded operators on the Banach space $L^2(\mathbb{R}^n, X)$, where $\frac{1}{2} = \frac{\theta}{p_2} + \frac{1-\theta}{p_1}$ for some suitable choice of the parameters p_1, p_2 and θ . \blacksquare

Remark 5.9 Since the strongly continuous groups $(e^{itA_k})_{t \in \mathbb{R}}$ consist in particular of L^2 isometries, the operators A_k are self-adjoint on L^2 by Stone's theorem ($k = 1, \dots, n$). Thus $(T_t)_{t \geq 0}$ is a diffusion semigroup in the sense of [JMX06, Section 5] on a *commutative* von Neumann algebra.

Example 5.10 It is worth noting that the «sum of squares» associated to the Weyl pair $(A, B) = (-\frac{1}{2}Q_2 - P_1, \frac{1}{2}Q_1 - P_2)$ of Example 4.8 identifies to the «twisted Laplacian» L on \mathbb{R}^2 . A formal computation gives

$$\begin{aligned} (-\tfrac{1}{2}Q_2 - P_1)^2 + (\tfrac{1}{2}Q_1 - P_2)^2 &= P_1^2 + P_2^2 + \tfrac{1}{4}Q_1^2 + \tfrac{1}{4}Q_2^2 + Q_2P_1 - Q_1P_2 \\ &\stackrel{(4.11)}{=} -(\partial_1^2 + \partial_2^2) + \tfrac{1}{4}(x_1^2 + x_2^2) - i(x_2\partial_1 - x_1\partial_2). \end{aligned}$$

Note that this linear operator L has been studied by many people, see e.g. [Car20], [DOS02, Section 7.9 p. 477], [JLR23], [KoR07], [Nar03], [StZ98], [Tha98, (1.4.23) p. 19], [Tha12], [Tie06] and [Won05]. It is a perturbation of the operator $-(\partial_1^2 + \partial_2^2) + \frac{1}{4}(x_1^2 + x_2^2)$ by the partial differential operator $i(x_1\partial_2 - x_2\partial_1)$. The twisted Laplacian L emerges in physics as the Hamiltonian representing the movement of a free electron in an infinite two-dimensional plane, subjected to a constant magnetic field oriented perpendicularly to the plane, following the seminal work [Lan30] of Landau. Furthermore, it is well-known that this operator is connected to the subelliptic Laplacian $\Delta \stackrel{\text{def}}{=} -X^2 - Y^2$ of [Won23, (18.1) p. 139] on the Heisenberg group $\mathbb{C} \times \mathbb{R}$. Here, the group law is defined following e.g. [Tha12, p. 6] by

$$(5.18) \quad (s, u) \cdot (s', u') = (s + s', u + u' + \tfrac{1}{2}\Im(ss'\bar{)), \quad (s, u), (s', u') \in \mathbb{C} \times \mathbb{R},$$

and X and Y are the left-invariant vector fields on the Heisenberg group $\mathbb{C} \times \mathbb{R}$ defined by $X \stackrel{\text{def}}{=} \partial_x + \frac{1}{2}y\partial_u$ and $Y \stackrel{\text{def}}{=} \partial_y - \frac{1}{2}x\partial_u$ where the complex variable s identifies to $x + iy$. These vector fields satisfy $[X, Y] = -T$ where $T \stackrel{\text{def}}{=} \partial_u$ and (X, Y, T) is a basis of the Lie algebra. Indeed, by [Tha12, p. 9] for any suitable function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ we have

$$\Delta(e^{it}f) = e^{it}L(f), \quad t \in \mathbb{R}.$$

Finally, replacing the matrix Θ by the matrix $\mathcal{J} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ in the group law (5.5), we recover the group law (5.18) of the Heisenberg group observing that $\langle s, \mathcal{J}s' \rangle = \Im(ss'\bar{)}$ for any elements s, s' of $\mathbb{R}^2 = \mathbb{C}$. Denoting $(K_t)_{t \geq 0}$ the kernel semigroup associated to the subelliptic Laplacian Δ , we have by [Tha04, Theorem 2.8.1 p. 84] the formula

$$(5.19) \quad \int_{\mathbb{R}} e^{i\lambda u} K_t(s, u) du = \frac{\lambda}{4\pi \sinh \lambda t} \exp\left(-\frac{\lambda \coth(t\lambda)}{4}(s_1^2 + s_2^2)\right), \quad (s, u) \in \mathbb{R}^2 \times \mathbb{R}, \lambda \in \mathbb{R}.$$

So, we recover the formula

$$p_t^{\mathcal{J}}(s) \stackrel{(5.13)}{=} \int_{\mathbb{R}} e^{iu} K_t(s, u) du \stackrel{(5.19)}{=} \frac{1}{4\pi \sinh t} \exp\left(-\frac{\coth(t)}{4}(s_1^2 + s_2^2)\right), \quad s \in \mathbb{R}^2, t > 0$$

proved in [Won05, Theorem 4.3 p. 277] using an alternative approach. We will generalize this formula in (6.6) to complex times by a different method.

Remark 5.11 If the skew-symmetric matrix Θ is orthogonal, we have seen in Proposition 5.3 that the Lie group $\tilde{\mathbb{H}}_{\Theta}$ is an H -type Lie group. Then using the description of the heat kernel of [Ran96] (rediscovered in [YaZ08, Theorem 3.2]), we can determine similarly an expression for p_t^{Θ} for any $t > 0$.

Example 5.12 Suppose that $n = 2d$ for some integer $d \geq 1$. Consider the standard Weyl pair $(Q_1, \dots, Q_d, P_1, \dots, P_d)$ considered in [NeP20, Example 3.2] where the operators Q_j and P_j are defined in (4.11). By Example 3.3, this sequence defines a Θ -Weyl tuple on the Banach space $L^p(\mathbb{R}^d)$ for the matrix $\Theta = \begin{bmatrix} 0 & -I_d \\ I_d & 0 \end{bmatrix}$. The «sum of squares» identifies to the classical harmonic oscillator $-\Delta + |x|^2$ acting on a dense subspace of the Banach space $L^p(\mathbb{R}^d)$. By [CCCF11, p. 62] or [Tha12, p. 10], we have Mehler's formula

$$(T_t f)(x) = \frac{1}{(2\pi \sinh 2t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2 \sinh(2t)} [(|x|^2 + |y|^2) \coth(2t) - 2\langle x, y \rangle]\right) f(y) dy, \quad t > 0$$

where $x \in \mathbb{R}^d$, which describes the associated semigroup $(T_t)_{t \geq 0}$ on the space $L^p(\mathbb{R}^d)$. Now, consider the simplest case $d = 1$ with the notations $P \stackrel{\text{def}}{=} P_1$, $Q \stackrel{\text{def}}{=} Q_1$ and $\Theta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. For any $s = (s_1, s_2)$ of \mathbb{R}^2 , note the equality $\langle s, \Theta^\dagger s \rangle = -s_1 s_2$. Using the second part of Lemma 6.1, the action of the operators of the associated semigroup on any function $f \in L^p(\mathbb{R})$ can also be described by

$$\begin{aligned} T_z f &\stackrel{(5.12)}{=} \int_{\mathbb{R}^2} p_z^\Theta(x) e^{is \cdot A} f ds \stackrel{(3.2)}{=} \int_{\mathbb{R}^2} p_z^\Theta(x) e^{i(xP+yQ)} f dx dy \\ &\stackrel{(6.6)}{=} \frac{1}{4\pi \sinh(z)} \int_{\mathbb{R}^2} \exp\left(-\frac{\coth(z)}{4}(x^2 + y^2)\right) e^{i(xP+yQ)} f dx dy, \end{aligned}$$

where $|\arg z| < \varphi$, in the spirit of the formula [Lus06, (3) p. 50] for the operator $T_{\frac{t^2}{2}}$. This formula seems to be new.

Example 5.13 For any integer $n \geq 1$, the choice $\Theta = 0_n$ in Proposition 4.7 gives the universal 0_n -Weyl tuple $A_{\text{univ}}^0 = (-P_1, \dots, -P_n)$ where the operator P_i is defined in (4.11). The «sum of squares» is the Laplacian $-\Delta$ on the Banach space $L^p(\mathbb{R}^n)$.

6 H^∞ calculus for the shifted harmonic oscillator of the univ. Θ -Weyl tuple

Consider a skew-symmetric matrix $\Theta \in M_n(\mathbb{R})$. In the following, it will turn out that the «sum of squares» \mathcal{A} , defined as the closure of the operator $\sum_{k=1}^n A_k^2$ of the universal Θ -Weyl tuple, has a spectral gap at 0. The value of this gap is explicitly given in (6.3).

Note that the skew-symmetric matrix $\Theta \in M_n(\mathbb{R})$ is orthogonally similar to a suitable block diagonal matrix. More precisely, there exists a decomposition

$$(6.1) \quad \Theta = O^T J O$$

where O is an orthogonal matrix of $M_n(\mathbb{R})$ and J is the 2×2 block diagonal matrix

$$(6.2) \quad J \stackrel{\text{def}}{=} \begin{pmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & \ddots & J_k & 0 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}, \quad \text{where } J_j \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \alpha_j \\ -\alpha_j & 0 \end{pmatrix}, \quad j = 1, \dots, k$$

for some integer $k \in [0, \frac{n}{2}]$ with $\alpha_1, \dots, \alpha_k > 0$. We put

$$(6.3) \quad \alpha \stackrel{\text{def}}{=} \sum_{j=1}^k \alpha_j > 0$$

with the convention that $\alpha = 0$ if $\Theta = 0$. We also set $\beta \stackrel{\text{def}}{=} \min\{\alpha_1, \dots, \alpha_k\} > 0$ in the case where $\Theta \neq 0$. Note that α and β do not depend on the particular choice of O and J in the decomposition $\Theta = O^T J O$ hereabove. Finally, we will use the matrix

$$(6.4) \quad \mathcal{J} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let us now calculate more explicitly the integral kernel of the semigroup from Proposition 5.5 in some particular cases. The kernel of the third point is the heat kernel on \mathbb{R}^n .

Lemma 6.1 *Let p_z^Θ denote the kernel defined in (5.13).*

1. *Decomposing $\Theta = O^T J O$ as in (6.1), we have*

$$(6.5) \quad p_z^\Theta(x) = p_z^J(Ox), \quad \Re z > 0, \quad x \in \mathbb{R}^n.$$

2. *The kernel p_z^J can be written as*

$$(6.6) \quad p_z^J(x) = \frac{1}{2\pi} \frac{1}{e^z - e^{-z}} \exp\left(-\frac{1}{4} \frac{e^z + e^{-z}}{e^z - e^{-z}} (x_1^2 + x_2^2)\right) = \frac{1}{4\pi \sinh(z)} \exp\left(-\frac{\coth(z)}{4} (x_1^2 + x_2^2)\right).$$

3. *The kernel p_z^{0n} is equal to*

$$(6.7) \quad p_z^{0n}(x) = \frac{1}{(4\pi z)^{\frac{n}{2}}} \exp\left(-\frac{1}{4z} \sum_{j=1}^n x_j^2\right).$$

4. *Let $J = \alpha \mathcal{J}$ for some $\alpha > 0$. Then we have*

$$(6.8) \quad p_z^{\alpha \mathcal{J}}(x) = \alpha p_{\alpha z}^{\mathcal{J}}(\sqrt{\alpha}x).$$

Proof : Recall that the meromorphic functions $S(z) = \frac{z}{\sin(z)}$ and $R(z) = \frac{z}{\tan(z)}$ are defined in (5.7).

1. Then $S(z\Theta) \stackrel{(6.1)}{=} S(zO^T J O) = O^T S(zJ)O$. Thus, $\det S(z\Theta) = \det S(zJ)$. On the other hand, $R(z\Theta) \stackrel{(6.1)}{=} R(zO^T J O) = O^T R(zJ)O$, so that $\langle R(z\Theta)x, x \rangle = \langle O^T R(zJ)Ox, x \rangle = \langle R(zJ)Ox, Ox \rangle$. Recalling the formula (5.13) for p_z^Θ , we deduce the first part of the lemma.

2. The matrix \mathcal{J} is skew-adjoint hence normal. Its eigenvalues being i and $-i$, we can write $\mathcal{J} = U^{-1} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} U$ for some invertible U . As S is even, functional calculus of matrices yields

$$S(z\mathcal{J}) = U^{-1} S\left(\begin{bmatrix} iz & 0 \\ 0 & -iz \end{bmatrix}\right) U = U^{-1} \begin{bmatrix} S(iz) & 0 \\ 0 & S(-iz) \end{bmatrix} U = S(iz)U^{-1}\text{Id}U = S(iz)\text{Id}.$$

Therefore,

$$\sqrt{\det(S(z\mathcal{J}))} = \sqrt{S(iz)^2} \stackrel{(5.7)}{=} \frac{iz}{\sin(iz)} = \frac{2z}{e^z - e^{-z}}.$$

Recall that the function R is also even. Thus, we have similarly

$$(6.9) \quad R(z\mathcal{J}) = \begin{bmatrix} R(iz) & 0 \\ 0 & R(iz) \end{bmatrix} \stackrel{(5.7)}{=} \begin{bmatrix} z \frac{e^z + e^{-z}}{e^z - e^{-z}} & 0 \\ 0 & z \frac{e^z + e^{-z}}{e^z - e^{-z}} \end{bmatrix}.$$

Consequently, we deduce that

$$\begin{aligned} \exp\left(-\frac{1}{4z}\langle R(z\mathcal{J})x, x \rangle\right) &\stackrel{(6.9)}{=} \exp\left(-\frac{1}{4z}\left\langle \begin{bmatrix} z \frac{e^z + e^{-z}}{e^z - e^{-z}} & 0 \\ 0 & z \frac{e^z + e^{-z}}{e^z - e^{-z}} \end{bmatrix} x, x \right\rangle\right) \\ &= \exp\left(-\frac{1}{4} \frac{e^z + e^{-z}}{e^z - e^{-z}} (x_1^2 + x_2^2)\right). \end{aligned}$$

3. We have $S(z0_n) = I_n$ and $R(z0_n) = I_n$. Consequently

$$p_z^{0_n}(x) \stackrel{(5.13)}{=} \frac{1}{(4\pi z)^{\frac{n}{2}}} \sqrt{\det I_n} \exp\left(-\frac{1}{4z}\langle I_n x, x \rangle\right) = \frac{1}{(4\pi z)^{\frac{n}{2}}} \exp\left(-\frac{1}{4z} \sum_{j=1}^n x_j^2\right).$$

4. Using the explicit description of p_z^Θ , we have

$$\begin{aligned} p_z^{\alpha\mathcal{J}}(x) &\stackrel{(5.13)}{=} \frac{1}{4\pi z} \sqrt{S(z\alpha\mathcal{J})} \exp\left(-\frac{1}{4z} R(z\alpha\mathcal{J})x, x\right) \\ &= \alpha \cdot \frac{1}{4\pi\alpha z} \sqrt{S(\alpha z\mathcal{J})} \exp\left(-\frac{1}{4\alpha z} R(\alpha z\mathcal{J})\sqrt{\alpha}x, \sqrt{\alpha}x\right) \stackrel{(5.13)}{=} \alpha p_{\alpha z}^{\mathcal{J}}(\sqrt{\alpha}x). \end{aligned}$$

■

In the sequel, we will show that the shifted operator $(\mathcal{A} - \alpha \text{Id}_{L^p(\mathbb{R}^n)}) \otimes \text{Id}_X$ admits a bounded $H^\infty(\Sigma_\omega)$ functional calculus on the Bochner space $L^p(\mathbb{R}^n, X)$, provided X has the UMD property. For that, we will use the following shorthand notation

$$(6.10) \quad \mathcal{A}_\alpha \stackrel{\text{def}}{=} (\mathcal{A} - \alpha \text{Id}_{L^p(\mathbb{R}^n)}) \otimes \text{Id}_X,$$

where $-\mathcal{A}$ is the generator of the semigroup associated with the universal Θ -Weyl tuple for some fixed skew-symmetric matrix $\Theta \in M_n(\mathbb{R})$. Recall that the kernel k_z^Θ of the semigroup $(e^{-z\mathcal{A}})_{z \in \mathbb{C}_+}$ is defined in (5.15).

The semigroup generated by \mathcal{A} naturally splits into tensor factors acting on $L^p(\mathbb{R}^2)$ and on $L^p(\mathbb{R}^{n-2k})$, corresponding to the block diagonal form of the matrix J with non-zero 2×2 blocks and the big vanishing block. This is the content of the next lemma.

Lemma 6.2 *Let $\Theta = J$ be a 2×2 block diagonal skew-symmetric matrix of $\in M_n(\mathbb{R})$ as in (6.2). Let A_{univ}^J be the universal J -Weyl tuple. Then the kernel $k_t^J(x, y)$ of Proposition 5.5 splits accordingly to the space decomposition $\mathbb{R}^n = \mathbb{R}^2 \oplus \dots \oplus \mathbb{R}^2 \oplus \mathbb{R}^{n-2k}$, $x = x_1 + \dots + x_k + x''$ to*

$$k_t^J(x, y) = k_t^{J_1}(x_1, y_1) \cdots k_t^{J_k}(x_k, y_k) k_t^{0_{n-2k}}(x'', y'').$$

Consequently, the semigroup $(e^{-t\mathcal{A}})_{t \geq 0}$ splits into $e^{-t\mathcal{A}} = e^{-t\mathcal{A}^{J_1}} \otimes \dots \otimes e^{-t\mathcal{A}^{J_k}} \otimes e^{-t\mathcal{A}^{0_{n-2k}}}$ on the space $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^2) \otimes_p \dots \otimes_p L^p(\mathbb{R}^2) \otimes_p L^p(\mathbb{R}^{n-2k})$ where $e^{-t\mathcal{A}^{J_j}}$ is associated with the kernel $k_t^{J_j}$. Moreover, the generator \mathcal{A} is the closure of $\mathcal{A}^{J_1} \otimes \text{Id}_{L^p(\mathbb{R}^{n-2})} + \dots + \text{Id}_{L^p(\mathbb{R}^{2k-2})} \otimes \mathcal{A}^{J_k} \otimes \text{Id}_{L^p(\mathbb{R}^{n-2k})} + \text{Id}_{L^p(\mathbb{R}^{2k})} \otimes \mathcal{A}^{0_{n-2k}}$ defined on $\text{dom } \mathcal{A}^{J_1} \otimes \dots \otimes \text{dom } \mathcal{A}^{0_{n-2k}}$.

Proof : For any $t \geq 0$, note that

$$(6.11) \quad \det S(tJ) = \prod_{j=1}^{k+1} \det S(tJ_j)$$

where $J_{k+1} = 0_{n-2k}$. Thus, according to Lemma 6.1, the kernel splits as

$$\begin{aligned} k_t^J(x, y) &\stackrel{(5.15)}{=} e^{\frac{1}{2}i\langle y, Jx \rangle} p_t^J(y) \\ &\stackrel{(5.13)}{=} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{1}{2}i\langle y, Jx \rangle} \sqrt{\det S(tJ)} \exp\left(-\frac{1}{4t}\langle R(tJ)y, y \rangle\right) \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{1}{2}i\langle y, Jx \rangle} \prod_{j=1}^{k+1} \sqrt{\det S(tJ_j)} \exp\left(-\frac{1}{4t}\left\langle \begin{bmatrix} R(tJ_1) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & R(tJ_k) & 0 \\ 0 & & 0 & R(t0_{n-2k}) \end{bmatrix} y, y \right\rangle\right) \\ &= \left[\frac{1}{4\pi t} e^{\frac{1}{2}i\langle y_1, J_1 x_1 \rangle} \sqrt{\det S(tJ_1)} \exp\left(-\frac{1}{4t}\langle R(tJ_1)y_1, y_1 \rangle\right) \right] \cdots \\ &\quad \left[\frac{1}{4\pi t} e^{\frac{1}{2}i\langle y_k, J_k x_k \rangle} \sqrt{\det S(tJ_k)} \exp\left(-\frac{1}{4t}\langle R(tJ_k)y_k, y_k \rangle\right) \right] \\ &\quad \left[\frac{1}{(4\pi t)^{\frac{n-2k}{2}}} \sqrt{\det S(t0_{n-2k})} \exp\left(-\frac{1}{4t}\langle R(t0_{n-2k})y'', y'' \rangle\right) \right] \\ &\stackrel{(5.13)}{=} [e^{\frac{1}{2}i\langle y_1, J_1 x_1 \rangle} p_t^{J_1}(y_1)] \cdots [e^{\frac{1}{2}i\langle y_k, J_k x_k \rangle} p_t^{J_k}(y_k)] p_t^{0_{n-2k}}(y'') \\ &\stackrel{(5.15)}{=} k_t^{J_1}(x_1, y_1) \cdots k_t^{J_k}(x_k, y_k) k_t^{0_{n-2k}}(x'', y''). \end{aligned}$$

By definition, $k_t^J(x, y)$ is the kernel of the semigroup $T_t = e^{-tA}$ on $L^p(\mathbb{R}^n)$. Moreover, it is well-known that $k_t^{0_{n-2k}}(x'', y'')$ is the kernel of the heat semigroup T_t'' on $L^p(\mathbb{R}^{n-2k})$. Denote the operator T_t^j given by the integral kernel $k_t^{J_j}(x_j, y_j)$, $j = 1, \dots, k$. Then $(T_t^j)_{t \geq 0}$ is the semigroup on $L^p(\mathbb{R}^2)$ associated with the universal J_j -Weyl tuple. By identification, we find $T_t(f_1 \otimes f_2 \otimes \cdots \otimes f_k \otimes f'') = T_t^1(f_1) \otimes T_t^2(f_2) \otimes \cdots \otimes T_t^k(f_k) \otimes T_t''(f'')$. The last assertion on the generator is a direct consequence of [AGG86, p. 23]. \blacksquare

Proposition 6.3 *Suppose that $1 < p < \infty$ and $A_{\text{univ}}^{\mathcal{J}}$ be the universal \mathcal{J} -Weyl tuple acting on the Banach space $L^p(\mathbb{R}^2)$. Then for any complex number z with $\Re z > 0$, there exists a smooth even function $h_z: \mathbb{R} \rightarrow \mathbb{R}_+$ which is decreasing on \mathbb{R}_+ such that*

$$(6.12) \quad |k_z^{\mathcal{J}}(x, y)| = |p_z^{\mathcal{J}}(y)| \leq g_z(y) \stackrel{\text{def}}{=} h_z(y_1)h_z(y_2), \quad x, y \in \mathbb{R}^2$$

with

$$(6.13) \quad \sup_{\Re z > 0} \cos(\arg z) e^{\Re z} \|g_z\|_{L^1(\mathbb{R}^2)} < \infty.$$

Proof : We have

$$|k_z^{\mathcal{J}}(x, y)| \stackrel{(5.15)}{=} |e^{\frac{1}{2}i\langle y, \mathcal{J}x \rangle} p_z^{\mathcal{J}}(y)| = |p_z^{\mathcal{J}}(y)|.$$

Taking into account the explicit formula for $p_z^{\mathcal{J}}$ of the second part of Lemma 6.1, we see that it suffices to put

$$(6.14) \quad h_z(y) \stackrel{\text{def}}{=} \frac{1}{\sqrt{4\pi|\sinh(z)|}} \exp\left(-\frac{\Re \coth(z)}{4} y^2\right), \quad y \in \mathbb{R}.$$

Clearly, this function takes positive values and it is smooth, even and decreasing on \mathbb{R}_+ . Then the estimate of the L^1 -norm can be easily achieved. Indeed, we have¹

$$\begin{aligned} \|g_z\|_{L^1(\mathbb{R}^2)} &= \left(\int_{\mathbb{R}} h_z(y) dy \right)^2 \stackrel{(6.14)}{=} \frac{1}{4\pi |\sinh(z)|} \left(\int_{\mathbb{R}} \exp\left(-\frac{\Re \coth(z)}{4} y^2\right) dy \right)^2 \\ &\stackrel{(2.1)}{\cong} \frac{|e^{-z}|}{|1 - e^{-2z}|} \frac{1}{\Re \coth(z)} = \frac{e^{-\Re z}}{|1 - e^{-2z}|} \left[\Re \left(\frac{1 + e^{-2z}}{1 - e^{-2z}} \right) \right]^{-1} \\ &= \frac{e^{-\Re z}}{|1 - e^{-2z}|} \left[\Re \left(\frac{(1 + e^{-2z})(1 - e^{-2\bar{z}})}{|1 - e^{-2z}|^2} \right) \right]^{-1} = e^{-\Re z} \frac{|1 - e^{-2z}|}{1 - e^{-2\Re z}}. \end{aligned}$$

We let $x \stackrel{\text{def}}{=} \Re z > 0$ and $y \stackrel{\text{def}}{=} \Im z$. Observe that

$$1 - e^{-2z} = 1 - e^{-2x} e^{-2yi} = 1 - (1 - 2x + O(x^2))(1 - 2yi + O(y^2)) \sim_0 2x + 2yi.$$

Consequently, we have $|1 - e^{-2z}| \sim_0 |2x + 2yi| = 2\sqrt{x^2 + y^2}$. Since $1 - e^{-2x} \sim_0 2x$, we conclude that

$$\|g_z\|_{L^1(\mathbb{R}^2)} = e^{-x} O\left(\frac{\sqrt{x^2 + y^2}}{x}\right). \quad \blacksquare$$

Lemma 6.4 *Let $(p_t^{\mathcal{J}})_{t \geq 0}$ be the semigroup kernel of the universal \mathcal{J} -Weyl tuple from Proposition 5.5. Then*

$$(6.15) \quad \left\| \frac{d}{dt} p_t^{\mathcal{J}} + p_t^{\mathcal{J}} \right\|_{L^1(\mathbb{R}^2)} \lesssim e^{-3t}, \quad t \in \mathbb{R}^+.$$

Proof : Recall that $\frac{d}{dt} \frac{1}{\sinh t} = -\frac{\coth t}{\sinh t}$ and $\frac{d}{dt} \coth t = -\frac{1}{\sinh^2 t}$ for any $t > 0$. With the formula of the second part of Lemma 6.1, we calculate

$$\begin{aligned} \frac{d}{dt} p_t^{\mathcal{J}}(x) + p_t^{\mathcal{J}}(x) &\stackrel{(6.6)}{=} -\coth(t) p_t^{\mathcal{J}}(x) + \frac{x_1^2 + x_2^2}{4 \sinh^2 t} p_t^{\mathcal{J}}(x) + p_t^{\mathcal{J}}(x) \\ &= (1 - \coth t) p_t^{\mathcal{J}}(x) + \frac{x_1^2 + x_2^2}{4 \sinh^2 t} p_t^{\mathcal{J}}(x) = -\frac{e^{-t}}{\sinh t} p_t^{\mathcal{J}}(x) + \frac{x_1^2 + x_2^2}{4 \sinh^2 t} p_t^{\mathcal{J}}(x). \end{aligned}$$

We estimate the $L^1(\mathbb{R}^2)$ -norms of the two terms separately. On the one hand, we have

$$\begin{aligned} \left\| \frac{e^{-t}}{\sinh t} p_t^{\mathcal{J}} \right\|_{L^1(\mathbb{R}^2)} &\stackrel{(6.6)}{\cong} \frac{e^{-t}}{\sinh^2 t} \int_{\mathbb{R}^2} \exp\left(-\frac{\coth(t)}{4}(x_1^2 + x_2^2)\right) dx \\ &\stackrel{(2.1)}{\cong} \frac{e^{-t}}{\sinh^2 t} \frac{1}{\coth^2 t} = \frac{e^{-t}}{\cosh^2 t} \lesssim e^{-3t}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \left\| \frac{x_1^2 + x_2^2}{4 \sinh^2 t} p_t^{\mathcal{J}} \right\|_{L^1(\mathbb{R}^2)} &\stackrel{(6.6)}{\cong} \frac{1}{\sinh^3 t} \int_{\mathbb{R}^2} (x_1^2 + x_2^2) \exp\left(-\frac{\coth(t)}{4}(x_1^2 + x_2^2)\right) dx \\ &\stackrel{(2.1)}{\cong} \frac{1}{\sinh^3 t} \frac{1}{\coth^3 t} = \frac{1}{\cosh^3 t} \lesssim e^{-3t}. \end{aligned} \quad \blacksquare$$

1. Observe that $\Re \coth(z) > 0$ if $\Re z > 0$ since $\Re \coth(x+iy) = \frac{\coth x(1+\cot^2 y)}{\cosh^2 x + \cot^2 y}$ and that $\Re[(1+e^{-2z})(1-e^{-2\bar{z}})] = \Re[1 - e^{-2\Re z} + e^{-2z} - e^{-2\bar{z}}] = 1 - e^{-2\Re z}$.

Corollary 6.5 *Suppose that $1 < p < \infty$ and let $A_{\text{univ}}^{\mathcal{J}}$ be the universal \mathcal{J} -Weyl tuple acting on the Banach space $L^p(\mathbb{R}^2)$. Consider a Banach space X .*

1. *Then the operator $\mathcal{A}_1 \stackrel{(6.10)}{=} (\mathcal{A} - \text{Id}_{L^p(\mathbb{R}^n)}) \otimes \text{Id}_X$ generates a bounded holomorphic semigroup on the Bochner space $L^p(\mathbb{R}^2, X)$.*
2. *More precisely, we have the following norm estimate*

$$\|t\mathcal{A}_1 \exp(-t\mathcal{A}_1)\|_{L^p(\mathbb{R}^2, X) \rightarrow L^p(\mathbb{R}^2, X)} \lesssim te^{-2t}.$$

3. *In the same way, if $B_{\text{univ}}^{\alpha\mathcal{J}} = (B_1, B_2)$ is the universal $\alpha\mathcal{J}$ -Weyl tuple on $L^p(\mathbb{R}^2)$ for some $\alpha > 0$, and X is any Banach space, then the closure \mathcal{B}_α of $(B_1^2 + B_2^2 - \alpha\text{Id}_{L^p(\mathbb{R}^2)}) \otimes \text{Id}_X$ generates a bounded holomorphic semigroup on $L^p(\mathbb{R}^2, X)$ with estimate*

$$(6.16) \quad \|\mathcal{B}_\alpha \exp(-t\mathcal{B}_\alpha)\|_{L^p(\mathbb{R}^2, X) \rightarrow L^p(\mathbb{R}^2, X)} \lesssim te^{-2\alpha t}.$$

Proof : 1. and 2. Recall that the semigroup $(T_t)_{t \geq 0}$ is bounded holomorphic by the second part of Proposition 5.5 and that $\frac{d}{dt}T_t = \mathcal{A}T_t$ by [Jac01, Lemma 4.2.8 p. 290] for any $t > 0$. Note that by (5.15) the operator T_t is a twisted convolution operator $p_t^{\mathcal{J}} *_{\mathcal{J}} \cdot$ by the kernel $p_t^{\mathcal{J}}$. A standard argument shows that the operator $\frac{d}{dt}T_t$ is the twisted convolution operator $(\frac{d}{dt}p_t^{\mathcal{J}}) *_{\mathcal{J}} \cdot$.

We will use Proposition 2.1. We start with the domain condition. It is clear that if $h \otimes x$ belongs to $L^p(\mathbb{R}^2) \otimes X$ then the element $(T_t \otimes \text{Id}_X)(h \otimes x) = T_t(h) \otimes x$ belongs to $\text{dom}(\mathcal{A}) \otimes X$ by the first part of Theorem 5.4, hence to the domain of the closure $\mathcal{A} \overline{\otimes} \text{Id}_X$ of $\mathcal{A} \otimes \text{Id}_X$. By linearity, $(T_t \otimes \text{Id}_X)(L^p(\mathbb{R}^2) \otimes X)$ is a subset of $\text{dom}(\mathcal{A} \overline{\otimes} \text{Id}_X)$.

Let $g \in L^p(\mathbb{R}^2, X)$. There exists a sequence (g_n) of elements of the space $L^p(\mathbb{R}^2) \otimes X$ such that $g_n \rightarrow g$ in the Banach space $L^p(\mathbb{R}^2, X)$. By continuity, we see that $(T_t \otimes \text{Id}_X)(g_n) \rightarrow (T_t \otimes \text{Id}_X)(g)$. It is easy to check with that the estimate (6.15) and the vector-valued twisted Young inequality from Lemma 4.2 that the operator $\mathcal{A}T_t \otimes \text{Id}_X = \frac{d}{dt}T_t \otimes \text{Id}_X$ induces a bounded operator. Now, we deduce that

$$(\mathcal{A} \overline{\otimes} \text{Id}_X)(T_t \otimes \text{Id})(g_n) = (\mathcal{A}T_t \otimes \text{Id}_X)(g_n) \rightarrow (\mathcal{A}T_t \otimes \text{Id}_X)(g).$$

For any $t > 0$, we conclude that $(T_t \otimes \text{Id}_X)(g)$ belongs to the space $\text{dom} \mathcal{A} \overline{\otimes} \text{Id}_X$. Indeed, we proved that $(T_t \otimes \text{Id}_X)_{t \geq 0}$ is bounded holomorphic.

According to (5.15) and [EnN00, 2.2 p. 60], the operator

$$\exp(-t(\mathcal{A} - \text{Id})) = e^t \exp(-t\mathcal{A})$$

is the twisted convolution operator by the kernel $e^t p_t^{\mathcal{J}}(y)$ for any $t > 0$. For any $g \in L^p(\mathbb{R}^2, X)$ and any $t > 0$, it follows from the previous paragraph that the vector $\exp(-t\mathcal{A}_1)g$ belongs to $\text{dom} \mathcal{A}_1$ and that $t \mapsto \exp(-t\mathcal{A}_1)$ is differentiable. Using [Jac01, Lemma 4.2.8 p. 290] in the first equality, we obtain

$$\mathcal{A}_1 \exp(-t\mathcal{A}_1)g = -\frac{d}{dt} [\exp(-t\mathcal{A}_1)g] = -\left(\frac{d}{dt} [e^t p_t^{\mathcal{J}}] \right) *_{\mathcal{J}} g.$$

Using again the vector-valued twisted Young inequality from Lemma 4.2 in the first inequality, we obtain

$$\begin{aligned} \|t\mathcal{A}_1 \exp(-t\mathcal{A}_1)\|_{L^p(\mathbb{R}^2, X) \rightarrow L^p(\mathbb{R}^2, X)} &= \left\| t \frac{d}{dt} \exp(-t\mathcal{A}_1) \right\|_{L^p(\mathbb{R}^2, X) \rightarrow L^p(\mathbb{R}^2, X)} \\ &\stackrel{(4.4)}{\leq} \left\| t \frac{d}{dt} (e^t p_t^{\mathcal{J}}) \right\|_{L^1(\mathbb{R}^2)} = te^t \left\| \frac{d}{dt} p_t^{\mathcal{J}} + p_t^{\mathcal{J}} \right\|_{L^1(\mathbb{R}^2)} \stackrel{(6.15)}{\lesssim} te^{-2t}. \end{aligned}$$

We conclude with Proposition 2.1.

3. In the same way as for the first two assertions, we have with the fourth part of Lemma 6.1 and the result from the second point,

$$\begin{aligned}
\|t\mathcal{B}_\alpha \exp(-t\mathcal{B}_\alpha)\|_{L^p(\mathbb{R}^2, X) \rightarrow L^p(\mathbb{R}^2, X)} &= \left\| t \frac{d}{dt} \exp(-t\mathcal{B}_\alpha) \right\|_{L^p(\mathbb{R}^2, X) \rightarrow L^p(\mathbb{R}^2, X)} \\
&\leq \left\| t \frac{d}{dt} (e^{\alpha t} p_t^{\alpha \mathcal{J}}) \right\|_{L^1(\mathbb{R}^2)} \stackrel{(6.8)}{=} \left\| t \frac{d}{dt} (e^{\alpha t} \alpha p_{\alpha t}^{\mathcal{J}}(\sqrt{\alpha} \cdot)) \right\|_{L^1(\mathbb{R}^2)} = \left\| t \frac{d}{dt} (e^{\alpha t} p_{\alpha t}^{\mathcal{J}}) \right\|_{L^1(\mathbb{R}^2)} \\
&= \alpha t \left\| \frac{d}{ds} (e^s p_s^{\mathcal{J}}) \Big|_{s=\alpha t} \right\|_{L^1(\mathbb{R}^2)} = \alpha t e^{t\alpha} \left\| \left(p_s^{\mathcal{J}} + \frac{d}{ds} p_s^{\mathcal{J}} \right) \Big|_{s=\alpha t} \right\|_{L^1(\mathbb{R}^2)} \stackrel{(6.15)}{\lesssim} \alpha t e^{-2\alpha t} \cong t e^{-2\alpha t}.
\end{aligned}$$

■

Note that by [EnN00, 2.2 p. 60] the (negative) generator of the semigroup $(e^{t\alpha} \exp(-t\mathcal{A}) \otimes \text{Id}_X)_{t \geq 0}$ is the shifted generator $\mathcal{A}_\alpha = (\mathcal{A} - \alpha \text{Id}_{L^p(\mathbb{R}^n)}) \otimes \text{Id}_X$. To show the boundedness of the H^∞ functional calculus for the shifted operator, we rely on the specific case where $\Theta = J$ takes the block diagonal form as given in (6.2).

Proposition 6.6 *Let $\Theta = J$ be a 2×2 block diagonal skew-symmetric matrix of $M_n(\mathbb{R})$ as in (6.2). Suppose that $1 < p < \infty$ and let A_{univ}^J be the universal J -Weyl tuple acting on the Banach space $L^p(\mathbb{R}^n)$. Let X be a Banach space. In the first three statements, we suppose that the matrix J is invertible.*

1. Then \mathcal{A}_α generates a bounded holomorphic semigroup on the Bochner space $L^p(\mathbb{R}^n, X)$.
2. More precisely, we have the following norm estimate

$$(6.17) \quad \|t\mathcal{A}_\alpha \exp(-t\mathcal{A}_\alpha)\|_{L^p(\mathbb{R}^n, X) \rightarrow L^p(\mathbb{R}^n, X)} \lesssim t e^{-2\beta t},$$

where $\beta = \min\{\alpha_1, \dots, \alpha_k\}$ is defined at the beginning of this section.

3. The operator $\mathcal{A}_\alpha - 2\beta \text{Id}_{L^p(\mathbb{R}^n, X)}$ is sectorial of angle $\frac{\pi}{2}$ on the subspace $\overline{\text{Ran } \mathcal{A}_\alpha}$ of $L^p(\mathbb{R}^n, X)$.
4. In this part, J is not necessarily invertible. Assume that the Banach space X is UMD. The operator \mathcal{A}_α admits a bounded $H^\infty(\Sigma_\omega)$ functional calculus on the Bochner space $L^p(\mathbb{R}^n, X)$ for any angle $\omega \in (\frac{\pi}{2}, \pi)$.

Proof : 1. and 2. In the case where $J = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$ for some $\alpha > 0$, (6.16) shows the estimate (6.17), with $\beta = \alpha$ in this case, and hence also 1. If J is a bigger 2×2 block diagonal matrix, according to Lemma 6.2,

$$\begin{aligned}
\exp(-t\mathcal{A}_\alpha) &= e^{t\alpha} \exp(-t\mathcal{A}) \otimes \text{Id}_X = \prod_{j=1}^k (e^{t\alpha_j} \exp(-t\mathcal{A}^j) \otimes \text{Id}_{L^p(\mathbb{R}^{n-2}, X)}) \\
&= \prod_{j=1}^k \exp(-t\mathcal{A}_{\alpha_j}^j) \otimes \text{Id}_{L^p(\mathbb{R}^{n-2})}.
\end{aligned}$$

Thus, $\exp(-t\mathcal{A}_\alpha)$ is the product of operators $\exp(-t\mathcal{A}_{\alpha_j}^j) \otimes \text{Id}_{L^p(\mathbb{R}^{n-2})}$ which are as in the first case with α replaced by α_j and X replaced by the space $L^p(\mathbb{R}^{n-2}, X)$. Noting that

$\mathcal{A}_\alpha \exp(-t\mathcal{A}_\alpha) = -\frac{d}{dt} \exp(-t\mathcal{A}_\alpha)$, applying Leibniz rule, and observing that the semigroups $\exp(-t\mathcal{A}_{\alpha_j}^j) \otimes \text{Id}_{L^p(\mathbb{R}^{n-2})}$ are uniformly bounded, we obtain the second part since

$$\begin{aligned} & \|t\mathcal{A}_\alpha \exp(-t\mathcal{A}_\alpha)\|_{L^p(\mathbb{R}^n, X) \rightarrow L^p(\mathbb{R}^n, X)} \\ & \leq \sum_{j=1}^k \left\| t\mathcal{A}_{\alpha_j}^j \exp(-t\mathcal{A}_{\alpha_j}^j) \otimes \text{Id}_{L^p(\mathbb{R}^{n-2})} \right\| \prod_{l \neq j} \left\| \exp(-t\mathcal{A}_{\alpha_l}^l) \otimes \text{Id}_{L^p(\mathbb{R}^{n-2})} \right\| \\ & \stackrel{(6.16)}{\lesssim} t \sum_{j=1}^k e^{-2\alpha_j t} \cong te^{-2\beta t}. \end{aligned}$$

3. By Proposition 2.2, it suffices to check that the norm of $\exp(-t(\mathcal{A}_\alpha - 2\beta\text{Id})) = e^{2\beta t} \exp(-t\mathcal{A}_\alpha)$ on the subspace $\overline{\text{Ran } \mathcal{A}_\alpha}$ is bounded as $t \rightarrow \infty$. If $f = \mathcal{A}_\alpha g$ for some $g \in \text{dom } \mathcal{A}_\alpha$, one has $\lim_{s \rightarrow \infty} \exp(-s\mathcal{A}_\alpha)f = \lim_{s \rightarrow \infty} \mathcal{A}_\alpha \exp(-s\mathcal{A}_\alpha)g = 0$ according to the second part. Therefore, again by the second part, we have for any $t \geq 1$ the estimate

$$\begin{aligned} \|\exp(-t\mathcal{A}_\alpha)f\|_{L^p(\mathbb{R}^n, X)} &= \left\| \int_t^\infty -\frac{d}{ds} \exp(-s\mathcal{A}_\alpha)f \, ds \right\| \leq \int_t^\infty \|\mathcal{A}_\alpha \exp(-s\mathcal{A}_\alpha)f\| \, ds \\ &\stackrel{(6.17)}{\lesssim} \int_t^\infty e^{-2\beta s} \, ds \|f\|_{L^p(\mathbb{R}^n, X)} \lesssim e^{-2\beta t} \|f\|_{L^p(\mathbb{R}^n, X)}. \end{aligned}$$

We obtain the third point.

4. Now, J is no longer necessarily invertible. According to Lemma 6.2, the operator \mathcal{A}_α equals (the closure of) $\mathcal{A}_\alpha^1 \otimes \text{Id}_{L^p(\mathbb{R}^{n-2k})} + \text{Id}_{L^p(\mathbb{R}^{2k}, X)} \otimes \mathcal{A}^{0_{n-2k}}$, where \mathcal{A}^1 is the semigroup generator associated with the universal Weyl tuple of an invertible skew-symmetric matrix, and $\mathcal{A}^{0_{n-2k}}$ is the ordinary Laplacian on $L^p(\mathbb{R}^{n-2k})$. Recalling that the UMD space $Y = L^p(\mathbb{R}^n, X)$ has property (Δ) , according to [LM03, Theorem 1.1], it suffices to establish the boundedness of the $H^\infty(\Sigma_\omega)$ functional calculus for the operators $\mathcal{A}_\alpha^1 \otimes \text{Id}_{L^p(\mathbb{R}^{n-2k})}$ and $\text{Id}_{L^p(\mathbb{R}^{2k})} \otimes \mathcal{A}^{0_{n-2k}}$ on Y separately. Note that when applying Le Merdy's theorem, the angles ω_1 and ω_2 for \mathcal{A}_α^1 and $\mathcal{A}^{0_{n-2k}}$ add together. Therefore, it is important that we are able to choose $\omega_2 > 0$ to be as small as desired. Since the Banach spaces $L^p(\mathbb{R}^{n-2k}, X)$ and $L^p(\mathbb{R}^{2k}, X)$ are UMD spaces, we are reduced to show the boundedness of the $H^\infty(\Sigma_\omega)$ functional calculus for the operators \mathcal{A}_α^1 and $\mathcal{A}^{0_{n-2k}} \otimes \text{Id}_X$ on these spaces for a generic UMD Banach space X . According to Proposition 5.8, the operator

$$\mathcal{A}^1 \otimes \text{Id}_X = (\mathcal{A}^1 - \alpha \text{Id}_{L^p(\mathbb{R}^n)}) \otimes \text{Id}_X + \alpha \text{Id}_{L^p(\mathbb{R}^n, X)} = (\mathcal{A}_\alpha^1 - 2\beta \text{Id}) + (\alpha + 2\beta) \text{Id}$$

has a bounded $H^\infty(\Sigma_\omega)$ functional calculus on the Bochner space $L^p(\mathbb{R}^{2k}, X)$, and thus, on the invariant subspace $\overline{\text{Ran } \mathcal{A}_\alpha^1}$. Since the operator $\mathcal{A}_\alpha^1 - 2\beta \text{Id}$ is sectorial on the space $\overline{\text{Ran } \mathcal{A}_\alpha^1}$ according to the third part, Lemma 2.3 implies that the operator

$$\mathcal{A}_\alpha^1 = (\mathcal{A}_\alpha^1 - 2\beta \text{Id}) + 2\beta \text{Id}$$

also has a bounded $H^\infty(\Sigma_\omega)$ functional calculus on the space $\overline{\text{Ran } \mathcal{A}_\alpha^1}$. One has trivially the boundedness of the $H^\infty(\Sigma_\omega)$ functional calculus of the operator \mathcal{A}_α^1 on $\text{Ker } \mathcal{A}_\alpha^1$. Since the Banach space X is UMD, $L^p(\mathbb{R}^{2k}, X)$ is reflexive and therefore, the sectorial operator \mathcal{A}_α^1 (see the first part) induces a space decomposition

$$L^p(\mathbb{R}^{2k}, X) \stackrel{(2.4)}{=} \overline{\text{Ran } \mathcal{A}_\alpha^1} \oplus \text{Ker } \mathcal{A}_\alpha^1.$$

So the operator \mathcal{A}_α^1 has a bounded $H^\infty(\Sigma_\omega)$ functional calculus on the space $L^p(\mathbb{R}^{2k}, X)$. Finally, $\mathcal{A}^{0_{n-2k}} \otimes \text{Id}_X$ has also a bounded $H^\infty(\Sigma_\omega)$ functional calculus on the space $L^p(\mathbb{R}^{n-2k}, X)$ by [HvNVW18, Theorem 10.2.25 p. 391] since it is ordinary Laplacian on $L^p(\mathbb{R}^{n-2k})$ and since the Banach space X is UMD. ■

We are now able to state the main result of this section for a general skew-symmetric matrix. Recall that α is defined in (6.3).

Corollary 6.7 *Consider a skew-symmetric matrix $\Theta \in M_n(\mathbb{R})$ and the universal Θ -Weyl tuple A_{univ} acting on $L^p(\mathbb{R}^n)$. Let X be a UMD Banach space. Then the operator $\mathcal{A}_\alpha = (\mathcal{A} - \alpha \text{Id}_{L^p(\mathbb{R}^n)}) \otimes \text{Id}_X$ admits a bounded $H^\infty(\Sigma_\omega)$ functional calculus on the Bochner space $L^p(\mathbb{R}^n, X)$ for any $\omega \in (\frac{\pi}{2}, \pi)$.*

Proof : Recall again the formula $\Theta = O^T J O$ from (6.1). Consider the isometry $\tilde{O}: L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$, $f \mapsto f(O^T \cdot)$. Denote the semigroups generated by $\mathcal{A} - \alpha \text{Id}_{L^p(\mathbb{R}^n)}$ associated with the universal J -Weyl tuple and universal Θ -Weyl tuple by $(T_t^J)_{t \geq 0}$ and $(T_t^\Theta)_{t \geq 0}$.

In the same manner, we recall the notation p_t^J resp. p_t^Θ from Proposition 5.5. Then for any $g \in L^p(\mathbb{R}^n)$, we have

$$T_t^J g(x) \stackrel{(5.14)}{=} \int_{\mathbb{R}^n} e^{\frac{1}{2}\langle y, Jx \rangle} p_t^J(y) g(x-y) dy.$$

Moreover, since $p_t^\Theta(x) \stackrel{(6.5)}{=} p_t^J(Ox)$, we have

$$\begin{aligned} \tilde{O}^{-1} T_t^J \tilde{O} g(x) &= \int_{\mathbb{R}^n} e^{\frac{1}{2}i\langle y, JOx \rangle} p_t^J(y) g(x - O^T y) dy \\ &= \int_{\mathbb{R}^n} e^{\frac{1}{2}i\langle Oy, JOx \rangle} p_t^J(Oy) g(x-y) dy = \int_{\mathbb{R}^n} e^{\frac{1}{2}i\langle y, \Theta x \rangle} p_t^\Theta(y) g(x-y) dy \stackrel{(5.14)}{=} T_t^\Theta g(x). \end{aligned}$$

Since $\tilde{O}^{\pm 1}$ is an isometry on all $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, it extends to an isometry on $L^p(\mathbb{R}^n, X)$. As the generator of $(T_t^J \otimes \text{Id}_X)_{t \geq 0}$ has a bounded $H^\infty(\Sigma_\omega)$ functional calculus according to Proposition 6.6, also that of $(T_t^\Theta \otimes \text{Id}_X)_{t \geq 0}$ admits a bounded $H^\infty(\Sigma_\omega)$ functional calculus by [Fac14, Lemma 4.3.3 p. 135]. ■

Proposition 6.8 *Let $\Theta \in M_n(\mathbb{R})$ be any skew-symmetric matrix. Consider any Θ -Weyl tuple A acting on some Banach space X . Let $(T_t)_{t \geq 0}$ be the semigroup from Proposition 5.5 and denote $S_t \stackrel{\text{def}}{=} e^{\alpha t} T_t$ if $t \geq 0$. Then the rescaled semigroup $(S_t)_{t \geq 0}$, with (negative) generator $\mathcal{A} - \alpha \text{Id}_X$, is bounded holomorphic of angle $\frac{\pi}{2}$. Its extension is given by*

$$(6.18) \quad S_z x = \int_{\mathbb{R}^n} b_z(u) e^{iu \cdot A} x du, \quad \Re z > 0, x \in X$$

for some function $b_z \in L^1(\mathbb{R}^n)$ and in addition we have

$$(6.19) \quad \sup_{z \in \Sigma_\omega} \|b_z\|_{L^1(\mathbb{R}^n)} < \infty, \quad \omega \in (0, \frac{\pi}{2}).$$

Finally, $b_z = b_z^\Theta$ does not depend on A .

Proof : According to Proposition 5.5 and Lemma 6.1, $b_z(u) = e^{\alpha z} p_z^\Theta(u) \stackrel{(6.5)}{=} e^{\alpha z} p_z^J(Ou)$ if $|\arg z| < \varphi$. Then part 4 of Lemma 6.1, the tensor product decomposition of Lemma 6.2 and Proposition 6.3 yield the result since for any $x \in X$ we have

$$\left\| \int_{\mathbb{R}^n} b_z(u) e^{iu \cdot A} x du \right\|_X \leq \int_{\mathbb{R}^n} |b_z(u)| \|e^{iu \cdot A} x\|_X du \stackrel{(3.2)}{\lesssim} \int_{\mathbb{R}^n} |b_z(u)| du \|x\|_X.$$

Here we use the identity theorem for analytic functions. ■

7 Maximal estimates in noncommutative L^p -spaces

In this section, we delve into some aspects of harmonic analysis within noncommutative L^p -spaces. Specifically, we give estimates of some convolution type maximal function defined by well-behaving kernels.

ℓ^∞ -valued noncommutative L^p -spaces Let \mathcal{M} be a von Neumann algebra equipped with a normal semifinite faithful trace and let Ω be a measure space. Suppose that $1 \leq p < \infty$. We define $L^p(\mathcal{M}, \ell_I^\infty)$ as the space consisting of all families $x = (x_i)_{i \in I}$ of elements of $L^p(\mathcal{M})$ that admit a factorization

$$x_i = ay_i b, \quad i \in I$$

where $a, b \in L^{2p}(\mathcal{M})$ and where $(y_i)_{i \in I}$ is a bounded family of elements of $L^\infty(\mathcal{M})$. The norm of each element x of $L^p(\mathcal{M}, \ell_I^\infty)$ is defined by

$$\|x\|_{L^p(\mathcal{M}, \ell_I^\infty)} \stackrel{\text{def}}{=} \inf \left\{ \|a\|_{L^{2p}(\mathcal{M})} \sup_{i \in I} \|y_i\|_{L^\infty(\mathcal{M})} \|b\|_{L^{2p}(\mathcal{M})} \right\},$$

where the infimum runs over all factorizations of x as previously. We will follow the classical convention which is that the norm $\|x\|_{L^p(\mathcal{M}, \ell_I^\infty)}$ is denoted by $\|\sup_{i \in I}^+ x_i\|_{L^p(\mathcal{M})}$ or $\|\sup_{i \in I}^+ x_i\|_p$. In the sequel, we will use the fact that if $x_i \geq 0$ for any $i \in I$, we have

$$(7.1) \quad \left\| \sup_{i \in I}^+ x_i \right\|_p = \inf \left\{ \|y\|_{L^p(\mathcal{M})} : x_i \leq y \text{ for any } i \in I \right\}.$$

Let Ω be a measurable space and Z be an index set. We will use the following elementary fact that if $(\mu_z)_{z \in Z}$ is a family of finite positive measures on Ω and $(R_\omega)_{\omega \in \Omega}$ is a family of positive operators $R_\omega : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$ such that $\omega \mapsto R_\omega(u)$ is strongly μ_z -measurable for any $u \in L^p(\mathcal{M})$ and any $z \in Z$, then the following bound holds for any positive element $u \in L^p(\mathcal{M})$

$$(7.2) \quad \left\| \sup_{z \in Z}^+ \left\{ \int_\Omega R_\omega(u) d\mu_z(\omega) \right\} \right\|_p \leq \left(\sup_{z \in Z} \mu_z(\Omega) \right) \left\| \sup_{\omega \in \Omega}^+ R_\omega(u) \right\|_p.$$

Hardy-Littlewood maximal operators Let B be a symmetric convex body in \mathbb{R}^d and let \mathcal{N} be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace. For any $t > 0$, we may consider the operator-valued Hardy-Littlewood operator $\Phi_t : L^p(\mathbb{R}^d, L^p(\mathcal{N})) \rightarrow L^p(\mathbb{R}^d, L^p(\mathcal{N}))$ defined by

$$(7.3) \quad (\Phi_t f)(x) \stackrel{\text{def}}{=} \frac{1}{\mu(B)} \int_B f(x - ty) dy, \quad x \in \mathbb{R}^d.$$

By [HWW23, Theorem 5.13 (3) p. 79], if B is the closed Euclidean unit ball of \mathbb{R}^d , then there exists a positive constant C_p which is independent of the dimension d such that for any $1 < p < \infty$, we have

$$(7.4) \quad \left\| \sup_{t > 0}^+ \Phi_t f \right\|_p \leq C_p \|f\|_{L^p(\mathbb{R}^d, L^p(\mathcal{N}))}, \quad f \in L^p(\mathbb{R}^d, L^p(\mathcal{N})).$$

Using an argument based on direct sums of semifinite von Neumann algebras, one can show that C_p can be chosen independently of the von Neumann algebra \mathcal{N} . We also refer to the

papers [Hon20] and [HLW21] for related facts. For any $t > 0$ and any integer $1 \leq j \leq n$, we introduce the operator $R_t^j: L^p(\mathbb{R}^n, S^p) \rightarrow L^p(\mathbb{R}^n, S^p)$ defined by

$$(7.5) \quad (R_t^j f)(x) \stackrel{\text{def}}{=} \frac{1}{2t} \int_{x_j-t}^{x_j+t} f(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n) dy, \quad \text{a.e. } x \in \mathbb{R}^n.$$

Using [HvNVW16, Proposition 1.2.24 p. 25] together with [ABHN11, Proposition 1.2.2 p. 16], it is easy to check that the map $(0, \infty) \rightarrow L^p(\mathbb{R}^n, S^p)$, $t \mapsto R_t^j f$ is continuous, hence strongly measurable for any $1 \leq j \leq n$ and any vector-valued function $f \in L^p(\mathbb{R}^n, S^p)$.

From the estimate (7.4) applied with $d = 1$, $B = [-1, 1]$ and $L^p(\mathcal{N}) = L^p(\mathbb{R}^{n-1}, S^p)$, we immediately deduce with (7.1) the following result, using the equality $L^p(\mathbb{R}, L^p(\mathcal{N})) = L^p(\mathbb{R}^n, S^p)$.

Proposition 7.1 *Let $1 < p < \infty$ and $j \in \{1, \dots, n\}$. Let u be a positive element of the space $L^p(\mathbb{R}^n, S^p)$. Then there exists some positive element $v \in L^p(\mathbb{R}^n, S^p)$ such that for any $t > 0$*

$$R_t^j(u) \leq v \quad \text{with} \quad \|v\|_{L^p(\mathbb{R}^n, S^p)} \leq C_p \|u\|_{L^p(\mathbb{R}^n, S^p)}.$$

The next result is fundamental for the next section.

Lemma 7.2 *Let Z be an index set. For any integer $j \in \{1, \dots, n\}$ and any $z \in Z$, consider a smooth even integrable function $h_z: \mathbb{R} \rightarrow \mathbb{R}_+$ which is decreasing on \mathbb{R}_+ . Write $g_z(y_1, y_2) \stackrel{\text{def}}{=} h_z(y_1)h_z(y_2)$ for any $y_1, y_2 \in \mathbb{R}$. Assume that $\sup_{z \in Z} \|g_z\|_{L^1(\mathbb{R}^2)} < \infty$. Suppose that $1 < p < \infty$. Let $u \in L^p(\mathbb{R}^2, S^p)$ be a positive element. Then*

$$(7.6) \quad \left\| \sup_{z \in Z}^+ g_z * u \right\|_{L^p(\mathbb{R}^2, S^p)} \leq C_p^2 \sup_{z \in Z} \|g_z\|_{L^1(\mathbb{R}^2)} \|u\|_{L^p(\mathbb{R}^2, S^p)}.$$

Proof : Consider a positive element u of the space $L^p(\mathbb{R}^2, S^p)$. Let $\varepsilon > 0$. By Proposition 7.1, there exists some positive element $u_2 \in L^p(\mathbb{R}^2, S^p)$ such that $R_t^2(u) \leq u_2$ for all $t > 0$ with $\|u_2\|_{L^p(\mathbb{R}^2, S^p)} \leq C_p \|u\|_{L^p(\mathbb{R}^2, S^p)} + \varepsilon$. For any $t \in (0, \infty)$, note that $R_t^1(u_2)$ is a positive element. Using again Proposition 7.1, we see that there exists some positive element $u_1 \in L^p(\mathbb{R}^2, S^p)$ such that $R_t^1(u_2) \leq u_1$ for all $t > 0$ with

$$\|u_1\|_{L^p(\mathbb{R}^2, S^p)} \leq C_p \|u_2\|_{L^p(\mathbb{R}^2, S^p)} + \varepsilon \leq C_p^2 \|u\|_{L^p(\mathbb{R}^2, S^p)} + 2\varepsilon.$$

In particular, the positivity of each operator R_t^1 entails that $R_{t_1}^1 R_{t_2}^2(u) \leq R_{t_1}^1(u_2) \leq u_1$ for any $t_1, t_2 > 0$. By (7.1), we deduce that

$$\left\| \sup_{t_1, t_2 > 0}^+ R_{t_1}^1 R_{t_2}^2(u) \right\|_p \leq C_p^2 \|u\|_{L^p(\mathbb{R}^2, S^p)}.$$

Now, consider a family $(\mu_z)_{z \in Z}$ of positive finite measures on the product $(0, \infty) \times (0, \infty)$ such that $\sup_{z \in Z} \|\mu_z\|_{M((0, \infty)^2)} < \infty$. According to (7.2), we obtain that

$$(7.7) \quad \left\| \sup_{z \in Z}^+ \int_0^\infty \int_0^\infty R_{t_1}^1 R_{t_2}^2(u) d\mu_z(t_1, t_2) \right\|_p \leq C_p^2 \left(\sup_{z \in Z} \|\mu_z\|_{M((0, \infty)^2)} \right) \|u\|_{L^p(\mathbb{R}^2, S^p)}.$$

It suffices now to identify $g_z * u$ with an integral $\int_0^\infty \int_0^\infty R_{t_1}^1 R_{t_2}^2(u) d\mu_z(t_1, t_2)$ for some suitable measure μ_z . Note that the assumptions imply that $\lim_{t \rightarrow \infty} t h_z(t) = 0$ for any $z \in Z$ (this is a

classical exercise). So by integrating by parts, we obtain²

$$\begin{aligned}
(g_z * u)(x) &\stackrel{(2.5)}{=} \int_{\mathbb{R}^2} g_z(y)u(x-y) \, dy = \int_{\mathbb{R}^2} h_z(y_1)h_z(y_2)u(x-y) \, dy \\
&= \int_0^\infty \int_0^\infty h_z(y_1)h_z(y_2) \sum_{v \in \{\pm 1\}^2} u(x+v \cdot y) \, dy_1 \, dy_2 \\
&= \int_0^\infty \int_0^\infty (h_z)'(y_1)(h_z)'(y_2) \left(\int_0^{y_1} \int_0^{y_2} \sum_{v \in \{\pm 1\}^2} u(x+v \cdot w) \, dw_1 \, dw_2 \right) \, dy_1 \, dy_2 \\
&\stackrel{(7.5)}{=} \int_0^\infty \int_0^\infty R_{y_1}^1 u(x) R_{y_2}^2 u(x) \times 4y_1 y_2 (h_z)'(y_1)(h_z)'(y_2) \, dy_1 \, dy_2.
\end{aligned}$$

For any $z \in Z$, we are led to set $\mu_z(y_1, y_2) \stackrel{\text{def}}{=} 4y_1 y_2 (h_z)'(y_1)(h_z)'(y_2) \, dy_1 \, dy_2$. Note that the density of μ_z maintains a constant sign since the function h_z is decreasing on \mathbb{R}_+ . Thus we can easily calculate, again by integration by parts without boundary terms,

$$\begin{aligned}
\mu_z((0, \infty)^2) &= 4 \int_0^\infty \int_0^\infty y_1 y_2 (h_z)'(y_1)(h_z)'(y_2) \, dy_1 \, dy_2 \\
&= 4 \int_0^\infty \int_0^\infty h_z(y_1)h_z(y_2) \, dy_1 \, dy_2 = 4 \int_0^\infty \int_0^\infty g_z(y_1, y_2) \, dy_1 \, dy_2 = \|g_z\|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

Therefore, we have

$$g_z * u = \int_0^\infty \int_0^\infty R_{t_1}^1 R_{t_2}^2 (u) \, d\mu_z(t_1, t_2).$$

The last sentence is a consequence of (7.1). ■

Remark 7.3 Note that Lemma 7.2 has a direct counterpart for convolutions on \mathbb{R}^n for other dimensions n than 2. In the proof of the fourth point of Proposition 10.3, we will use the one-dimensional version.

8 R -boundedness and tensor product of operators

We need Proposition 8.2 in Section 10. First, we establish the following lemma on R -boundedness, which is an immediate consequence of Kahane-Khintchine's inequalities.

Lemma 8.1 *Let $\{S_z : z \in Z\}$ be an R -bounded family of operators acting on some Banach space X . Consider a σ -finite measure space (Ω, μ) and $1 \leq p < \infty$. Then for any $z \in Z$, the map $\text{Id}_{L^p(\Omega)} \otimes S_z$ extends to a bounded operator on the Bochner space $L^p(\Omega, X)$. Moreover, the set $\{\text{Id}_{L^p(\Omega)} \otimes S_z : z \in Z\}$ is R -bounded over the space $L^p(\Omega, X)$.*

Proof : The first assertion is [HvNVW16, Proposition 2.1.2 p. 69]. Consider some elements

2. Use $\lim_{t \rightarrow \infty} t h_z(t) = 0$ and Hölder's inequality for showing that the bracket of the integration by parts converges to zero.

z_1, \dots, z_n of Z and some functions $f_1, \dots, f_n \in L^p(\Omega, X)$. We have

$$\begin{aligned} \mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j \otimes (\text{Id}_{L^p(\Omega)} \otimes S_{z_j})(f_j) \right\|_{L^p(\Omega, X)}^p &= \mathbb{E} \int_{\Omega} \left\| \sum_{j=1}^n \varepsilon_j \otimes (\text{Id}_{L^p(\Omega)} \otimes S_{z_j})(f_j)(\omega) \right\|_X^p d\mu(\omega) \\ &= \int_{\Omega} \mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j \otimes S_{z_j}(f_j(\omega)) \right\|_X^p d\mu(\omega) \lesssim R(\{S_z : z \in Z\})^p \int_{\Omega} \mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j \otimes f_j(\omega) \right\|_X^p d\mu(\omega) \\ &= R(\{S_z : z \in Z\})^p \mathbb{E} \left\| \sum_{j=1}^n \varepsilon_j \otimes f_j \right\|_{L^p(\Omega, X)}^p. \end{aligned}$$

We conclude with Kahane-Khintchine's inequalities [HvNVW18, Theorem 6.2.4 p. 21 and Remark 8.1.2 p. 165]. \blacksquare

We deduce the following result on R -boundedness of tensor products of operators.

Proposition 8.2 *Let $(\Omega_1, \mu_1), \dots, (\Omega_n, \mu_n)$ be σ -finite measure spaces. Consider the product measure space $(\Omega, \mu) \stackrel{\text{def}}{=} (\Omega_1 \times \Omega_2 \times \dots \times \Omega_n, \mu_1 \otimes \mu_2 \otimes \dots \otimes \mu_n)$. Let $1 \leq p < \infty$ and let X be a Banach space. For $k = 1, \dots, n$ and $z \in Z$, assume that $S_z^k : L^p(\Omega_k) \rightarrow L^p(\Omega_k)$ is a bounded operator such that the family $\{S_z^k \otimes \text{Id}_X : z \in Z\}$ is R -bounded over the Bochner space $L^p(\Omega_k, X)$. Then for any $z \in Z$ the tensor product $S_z \stackrel{\text{def}}{=} S_z^1 \otimes S_z^2 \otimes \dots \otimes S_z^n \otimes \text{Id}_X$ extends to a bounded operator on $L^p(\Omega, X)$ and the family $\{S_z : z \in Z\}$ is R -bounded over the Bochner space $L^p(\Omega, X)$.*

Proof : We proceed by induction over n . If $n = 1$, the statement follows directly from the assumptions. Now, assume that the lemma holds for $n - 1$, that is, $S'_z \stackrel{\text{def}}{=} S_z^2 \otimes \dots \otimes S_z^n \otimes \text{Id}_X$ define an R -bounded family over the space $L^p(\Omega', X)$, where $\Omega' \stackrel{\text{def}}{=} \Omega_2 \times \dots \times \Omega_n$ is equipped with the measure $\mu' \stackrel{\text{def}}{=} \mu_2 \otimes \dots \otimes \mu_n$. Then for any $z \in Z$ we have

$$S_z = (S_z^1 \otimes \text{Id}_{L^p(\Omega')}) \cdot (\text{Id}_{L^p(\Omega_1)} \otimes S'_z).$$

The family $\{S_z^1 \otimes \text{Id}_X : z \in Z\}$ is R -bounded over the space $L^p(\Omega_1, X)$ by assumption. Then Lemma 8.1 and Fubini's theorem yield that the set $\{S_z^1 \otimes \text{Id}_{L^p(\Omega')} \otimes \text{Id}_X : z \in Z\}$ is R -bounded over the space $L^p(\Omega, X)$. Moreover, according to the induction hypothesis, the family $\{S'_z : z \in Z\}$ is R -bounded over the space $L^p(\Omega', X)$. Then again Lemma 8.1 yields that the set $\{\text{Id}_{L^p(\Omega_1)} \otimes S'_z : z \in Z\}$ is R -bounded over the Bochner space $L^p(\Omega, X)$. We conclude since the product of R -bounded families is again R -bounded by [HvNVW18, Proposition 8.1.19 p. 178]. \blacksquare

We also will use in Section 10 the following well-known result [HvNVW18, Proposition 8.4.1 p. 211]. Recall that by [HvNVW18, Theorem 7.4.23 p. 124] a Banach space is K -convex if and only if it has non-trivial type. So a noncommutative L^p -space is K -convex by [PiX03, Corollary 5.5 p. 1481] for any $1 < p < \infty$.

Proposition 8.3 *Let X be a K -convex Banach space. Then a family $\{T_z : z \in Z\}$ of operators acting on X is R -bounded if and only if the family $\{T_z^* : z \in Z\}$ of adjoints is R -bounded over the dual space X^* .*

9 R -boundedness and square-max decompositions

As a result of independent interest, we establish in this section a connection between R -boundedness and some appropriate variants of square-max decompositions introduced in the paper [GJP17].

If $1 < p < \infty$, note that a sufficient condition for the R -boundedness of a family $(C_k)_{k \in \mathcal{K}}$ of integral operators $C_k: L^p(\mathbb{R}^d, X) \rightarrow L^p(\mathbb{R}^d, X)$, $f \mapsto k * f$ on the Bochner space $L^p(\mathbb{R}^d, X)$ is well-known if X is a UMD Banach function space. Indeed, if M is the Hardy-Littlewood maximal operator and if

$$\mathcal{K} \stackrel{\text{def}}{=} \{k \in L^1(\mathbb{R}^d) : |k| * |f| \leq Mf \text{ a.e. for all simple } f: \mathbb{R}^d \rightarrow \mathbb{C}\}$$

then the family $(C_k)_{k \in \mathcal{K}}$ of operators is R -bounded on the Bochner space $L^p(\mathbb{R}^d, X)$, see e.g. [Lor19, Proposition 3.1 p. 371]. As pointed in [Lor19, p. 371], this class of kernels contains any radially decreasing $k \in L^1(\mathbb{R}^d)$ with $\|k\|_{L^1(\mathbb{R}^d)} \leq 1$.

Unfortunately for us, the UMD Banach space S^p is not a Banach function space. Indeed, the proof of the previous result relies on the proof of an estimate of « ℓ^∞ -boundedness» $\left\| \sup_{1 \leq j \leq n} |C_{k_j} f_j| \right\|_{L^p(\mathbb{R}^d, X)} \lesssim \left\| \sup_{1 \leq j \leq n} |f_j| \right\|_{L^p(\mathbb{R}^d, X)}$ for any $k_1, \dots, k_n \in \mathcal{K}$, which has no meaning in the noncommutative setting. So we need to introduce and to use some kind of substitute of the supremum. This substitute will be an estimate of the kind

$$\left\| \sup_{z \in Z}^+ C_{k_z} f \right\|_{L^p(\mathbb{R}^d, S^p)} \lesssim \|f\|_{L^p(\mathbb{R}^d, S^p)}, \quad f \in L^p(\mathbb{R}^d, S^p)$$

where $(k_z)_{z \in Z}$ is a suitable family of kernels.

Warm-up: the scalar case Suppose that $1 < p < \infty$. Let Ω be a measure space. For a measurable function $k: \Omega \times \Omega \rightarrow \mathbb{C}$ such that for any $f \in L^p(\Omega)$ the function $y \mapsto k(x, y)f(y)$ belongs to $L^1(\Omega)$ for almost all $x \in \Omega$ we define the integral operator

$$(9.1) \quad (C_k f)(x) \stackrel{\text{def}}{=} \int_{\Omega} k(x, y) f(y) dy, \quad \text{for almost all } x \in \Omega.$$

We refer to [AbA02, Chapter 5] for a nice discussion of these operators. Note that the adjoint of a well-defined bounded integral operator $C_k: L^p(\Omega) \rightarrow L^p(\Omega)$ is again an integral operator, namely

$$(9.2) \quad (C_k)^* = C_{\check{k}} \quad \text{with} \quad \check{k}(x, y) \stackrel{\text{def}}{=} k(y, x), \quad x, y \in \Omega.$$

Inspired by the L^p -square-max decompositions suitable for family of operators affiliated to group von Neumann algebras, as introduced in [GJP17, Definition 1. p. 893], we propose the following variant.

Definition 9.1 *Let Ω be a measure space. Consider an index set Z . Suppose that $1 \leq p < \infty$. We say that a family $(k_z)_{z \in Z}$, where $k_z: \Omega \times \Omega \rightarrow \mathbb{C}$, admits an L^p -square-max decomposition if there exists a decomposition $k_z = s_z m_z$ with measurable functions $s_z, m_z: \Omega \times \Omega \rightarrow \mathbb{C}$ such that*

$$(9.3) \quad \sup_{z \in Z} \sup_{x \in \Omega} \|s_z(x, \cdot)\|_{L^2(\Omega)} < \infty, \quad \text{and} \quad \left\| \sup_{z \in Z}^+ \mathcal{C}_{|m_z|^2}(f) \right\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}, \quad f \in L^p(\Omega).$$

Equipped with this definition, we are now in a position to prove the following result, which establishes a new sufficient condition for R -boundedness. Notice that $(\frac{p}{p-2})^* = \frac{p}{2}$ if $2 \leq p < \infty$.

Proposition 9.2 *Let Ω be a measure space. Consider an index set Z . Suppose that $2 \leq p < \infty$. For any $z \in Z$, consider a bounded integral operator $\mathcal{C}_{k_z} : L^p(\Omega) \rightarrow L^p(\Omega)$. Suppose that the family $(k_z)_{z \in Z}$ has an $L^{\frac{p}{p-2}}$ -square-max decomposition. Then the family $(\mathcal{C}_{k_z})_{z \in Z}$ is R -bounded.*

Proof : Consider an integer $n \geq 1$. Let f_1, \dots, f_n be some functions belonging to the space $L^p(\Omega)$ and let z_1, \dots, z_n be some elements of Z . We have

$$(9.4) \quad \left\| \left(\sum_{j=1}^n |\mathcal{C}_{k_{z_j}}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} = \left\| \sum_{j=1}^n |\mathcal{C}_{k_{z_j}}(f_j)|^2 \right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{1}{2}}.$$

By duality, there exists a unique positive element $u \in L^{\frac{p}{p-2}}(\Omega)$ of norm 1 such that

$$(9.5) \quad \left\| \sum_{j=1}^n |\mathcal{C}_{k_{z_j}}(f_j)|^2 \right\|_{L^{\frac{p}{2}}(\Omega)} = \int_{\Omega} u \sum_{j=1}^n |\mathcal{C}_{k_{z_j}}(f_j)|^2.$$

Now, we have to estimate the term inside the sum. Using Cauchy-Schwarz inequality

$$(9.6) \quad \begin{aligned} |\mathcal{C}_{k_{z_j}}(f_j)|^2 &\stackrel{(9.1)}{=} \left| \int_{\Omega} k_{z_j}(x, y) f_j(y) \, dy \right|^2 \\ &= \left| \int_{\Omega} s_{z_j}(x, y) m_{z_j}(x, y) f_j(y) \, dy \right|^2 \\ &\leq \int_{\Omega} |s_{z_j}(x, y)|^2 \, d\xi \int_{\Omega} |m_{z_j}(x, y) f_j(y)|^2 \, d\xi \\ &\stackrel{(9.1)}{\leq} \sup_{z \in Z} \sup_{x \in \Omega} \|s_z(x, \cdot)\|_{L^2(\Omega)}^2 \mathcal{C}_{|m_{z_j}|^2}(|f_j|^2) \\ &\stackrel{(9.3)}{\lesssim} \mathcal{C}_{|m_{z_j}|^2}(|f_j|^2). \end{aligned}$$

Since $u \in L^{\frac{p}{p-2}}(\Omega)$ and since $|f_j|^2 \in L^{\frac{p}{2}}(\Omega)$, we have

$$\begin{aligned} &\left\| \left(\sum_{j=1}^n |\mathcal{C}_{k_{z_j}}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}^2 \stackrel{(9.4)(9.5)}{=} \sum_{j=1}^n \int_{\Omega} u |\mathcal{C}_{k_{z_j}}(f_j)|^2 \stackrel{(9.6)}{\leq} \sum_{j=1}^n \int_{\Omega} u (\mathcal{C}_{|m_{z_j}|^2}(|f_j|^2)) \\ &= \sum_{j=1}^n \int_{\Omega} \mathcal{C}_{|m_{z_j}|^2}(u) |f_j|^2 \leq c_{|m_{z_j}|^2}(u) \leq h \int_{\Omega} h \sum_{j=1}^n |f_j|^2 \\ &\leq \left(\inf_{|m_{z_j}|^2}(u) \leq h} \|h\|_{L^{\frac{p}{p-2}}(\Omega)} \right) \left\| \sum_{j=1}^n |f_j|^2 \right\|_{L^{\frac{p}{2}}(\Omega)} \\ &\stackrel{(7.1)}{=} \left\| \sup_{z \in Z} \mathcal{C}_{|m_z|^2}(u) \right\|_{L^{\frac{p}{p-2}}(\Omega)} \left\| \sum_{j=1}^n |f_j|^2 \right\|_{L^{\frac{p}{2}}(\Omega)} \\ &\stackrel{(9.3)}{\lesssim} \|u\|_{L^{\frac{p}{p-2}}(\Omega)} \left\| \sum_{j=1}^n |f_j|^2 \right\|_{L^{\frac{p}{2}}(\Omega)} = \left\| \sum_{j=1}^n |f_j|^2 \right\|_{L^{\frac{p}{2}}(\Omega)}. \end{aligned}$$

Taking the square roots, we conclude that

$$\left\| \left(\sum_{j=1}^n |\mathcal{C}_{k_{z_j}}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \lesssim \left\| \left(\sum_{j=1}^n |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}.$$

So the family $(\mathcal{C}_{k_z})_{z \in Z}$ of operators is R -bounded. \blacksquare

The vector-valued case The following is a variant of Definition 9.1 for Bochner spaces with values in the Schatten space S^p .

Definition 9.3 *Suppose that $1 < p < \infty$. Consider an index set Z and a measure space Ω . We say that a family $(k_z)_{z \in Z}$, where $k_z: \Omega \times \Omega \rightarrow \mathbb{C}$ admits an $L^p(S^p)$ -square-max decomposition if there exists a decomposition $k_z = s_z m_z$ such that*

$$(9.7) \quad \sup_{z \in Z} \sup_{x \in \Omega} \|s_z(x, \cdot)\|_{L^2(\Omega)} < \infty$$

and

$$(9.8) \quad \left\| \sup_{z \in Z}^+ (\mathcal{C}_{|m_z|^2})^*(u) \right\|_{L^p(\Omega, S^p)} \stackrel{(9.2)}{=} \left\| \sup_{z \in Z}^+ \mathcal{C}_{|\check{m}_z|^2}(u) \right\|_{L^p(\Omega, S^p)} \lesssim \|u\|_{L^p(\Omega, S^p)}.$$

By replacing the Cauchy-Schwarz inequality in the proof of Proposition 9.2 by the operator-valued inner product [Lan95, Proposition 1.1 p. 3], we can prove the following result. For the standard notions of Col-boundedness and Row-boundedness which are related to R -boundedness, we refer to [JMX06, (4.2), (4.3)]. Recall that $(\frac{p}{p-2})^* = \frac{p}{2}$ if $2 \leq p < \infty$.

Theorem 9.4 *Let Z be an index set. Let Ω be a measure space. Suppose that $2 \leq p < \infty$. For any $z \in Z$, consider a bounded integral operator $\mathcal{C}_{k_z}: L^p(\Omega) \rightarrow L^p(\Omega)$. Suppose that the family $(k_z)_{z \in Z}$ has an $L^{\frac{p}{p-2}}(S^{\frac{p}{p-2}})$ -square-max decomposition. Then the family $(\mathcal{C}_{k_z} \otimes \text{Id}_{S^p})_{z \in Z}$ is Col-bounded, Row-bounded and R -bounded on the Bochner space $L^p(\Omega, S^p)$.*

Proof : Let $N \in \mathbb{N}$ and $f_1, \dots, f_N \in L^p(\Omega, S^p)$. Consider some $z_1, \dots, z_N \in Z$ and write in short $\mathcal{C}_l \stackrel{\text{def}}{=} \mathcal{C}_{z_l}$, $k_l \stackrel{\text{def}}{=} k_{z_l}$, $s_l \stackrel{\text{def}}{=} s_{z_l}$, $m_l \stackrel{\text{def}}{=} m_{z_l}$, where we denote the $L^{(\frac{p}{2})^*}(S^{(\frac{p}{2})^*})$ -square-max decomposition $k_z = s_z m_z$. So in order to prove Col-boundedness, we are left to show that for some uniform constant C , we have

$$(9.9) \quad \left\| \left(\sum_l |(\mathcal{C}_l \otimes \text{Id}_{S^p})(f_l)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega, S^p)} \leq C \left\| \left(\sum_l |f_l|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega, S^p)}$$

We have

$$(9.10) \quad \left\| \left(\sum_l |(\mathcal{C}_l \otimes \text{Id}_{S^p})(f_l)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega, S^p)}^2 = \left\| \sum_l |(\mathcal{C}_l \otimes \text{Id}_{S^p})(f_l)|^2 \right\|_{L^{\frac{p}{2}}(\Omega, S^{\frac{p}{2}})}.$$

By duality, there exists a unique positive element $u \in L^{(\frac{p}{2})^*}(\Omega, S^{(\frac{p}{2})^*})$ of norm one such that

$$(9.11) \quad \left\| \sum_l |(\mathcal{C}_l \otimes \text{Id}_{S^p})(f_l)|^2 \right\|_{L^{\frac{p}{2}}(\Omega, S^{\frac{p}{2}})} = \left(\int_{\Omega} \otimes \text{Tr} \right) \left(\sum_l |(\mathcal{C}_l \otimes \text{Id}_{S^p})(f_l)|^2 u \right).$$

We shall give an upper bound of $|(\mathcal{C}_l \otimes \text{Id}_{S_m^p})(f_l)|^2$ in the ordered space $L^{\frac{p}{2}}(\Omega, S^{\frac{p}{2}})$. Note that by a standard approximation argument it suffices to consider the case $f_l \in L^p(\Omega, S_m^p)$ where S_m^p is a finite-dimensional Schatten space if we obtain a bound in (9.9) that is independent of the size $m \in \mathbb{N}$. We obtain

$$\begin{aligned} (\mathcal{C}_l \otimes \text{Id}_{S_m^p})(f_l) &= \sum_{i,j=1}^m \mathcal{C}_l(f_{l,ij}) \otimes e_{ij} \stackrel{(9.1)}{=} \sum_{i,j=1}^m \int_{\Omega} k_l(\cdot, y) f_{l,ij}(y) dy \otimes e_{ij} \\ &\stackrel{\text{Def. 9.3}}{=} \sum_{i,j=1}^m \int_{\Omega} s_l(\cdot, y) m_l(\cdot, y) f_{l,ij}(y) dy \otimes e_{ij} \\ &= \left(\int_{\Omega} \otimes \text{Id}_{L^\infty(\Omega)} \otimes \text{Id}_{S_m^\infty} \right) \left((s_l(\cdot, y) \otimes 1_{S_m^\infty})(m_l(\cdot, y) \otimes 1_{S_m^\infty}) \left(\sum_{i,j} f_{l,ij} \otimes 1_{L^\infty(\Omega)} \otimes e_{ij} \right) \right). \end{aligned}$$

Now, we can consider the operator-valued inner product $\langle X, Y \rangle \stackrel{\text{def}}{=} (\text{Id}_{L^\infty(\Omega)} \otimes \int_{\Omega} \otimes \text{Id}_{S_m^\infty})(X^*Y)$ on $L^2(\Omega, L^\infty(\Omega, S_m^\infty))$. It satisfies according to [Lan95, Proposition 1.1 p. 3]

$$(9.12) \quad |\langle X, Y \rangle|^2 \leq \| \langle X, X \rangle \|_{L^\infty(\Omega, S_m^\infty)} \langle Y, Y \rangle.$$

So with $X: y \mapsto \overline{s_l}(\cdot, y) \otimes 1_{S_m^\infty}$ and $Y: y \mapsto (m_l(\cdot, y) \otimes 1_{S_m^\infty}) \left(\sum_{i,j} f_{l,ij} \otimes 1_{L^\infty(\Omega)} \otimes e_{ij} \right)$, we obtain

$$\begin{aligned} |(\mathcal{C}_l \otimes \text{Id}_{S_m^p})(f_l)|^2 &= |\langle X, Y \rangle|^2 \stackrel{(9.12)}{\leq} \| \langle X, X \rangle \|_{L^\infty(\Omega, S_m^\infty)} \langle Y, Y \rangle \\ &= \left(\sup_{x \in \Omega} \int_{\Omega} |s_l(x, y)|^2 dy \right) \times \\ &\quad \left(\text{Id}_{L^\infty(\Omega)} \otimes \int_{\Omega} \otimes \text{Id}_{S_m^\infty} \right) \left(\left| (m_l(\cdot, y) \otimes 1_{S_m^\infty}) \left(\sum_{i,j} f_{l,ij} \otimes 1_{L^\infty(\Omega)} \otimes e_{ij} \right) \right|^2 \right). \end{aligned}$$

We estimate the first factor by

$$(9.13) \quad \sup_{x \in \Omega} \int_{\Omega} |s_l(x, y)|^2 dy \stackrel{(9.7)}{\leq} C.$$

For the second factor, we see that

$$\begin{aligned} (9.14) \quad &\left(\text{Id}_{L^\infty(\Omega)} \otimes \int_{\Omega} \otimes \text{Id}_{S_m^\infty} \right) \left(\left| (m_l(\cdot, y) \otimes 1_{S_m^\infty}) \left(\sum_{i,j} f_{l,ij} \otimes 1_{L^\infty} \otimes e_{ij} \right) \right|^2 \right) \\ &= \int_{\Omega} |m_l(\cdot, y)|^2 \left| \sum_{i,j} f_{l,ij}(y) \otimes e_{ij} \right|^2 dy = \int_{\Omega} |m_l(\cdot, y)|^2 |f_l(y)|^2 dy \\ &\stackrel{(9.1)}{=} \mathcal{C}_{|m_l|^2} \otimes \text{Id}_{S_m^{p/2}}(|f_l|^2). \end{aligned}$$

Thus we can estimate the expression at the beginning of the proof by

$$\begin{aligned}
& \left\| \left(\sum_l |(\mathcal{C}_l \otimes \text{Id}_{S_m^p})(f_l)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega, S^p)}^2 \stackrel{(9.10)(9.11)}{=} \left(\int_{\Omega} \otimes \text{Tr} \right) \left(\sum_l |(\mathcal{C}_l \otimes \text{Id}_{S_m^p})(f_l)|^2 u \right) \\
& \stackrel{(9.13)(9.14)}{\leq} C \left(\int_{\Omega} \otimes \text{Tr} \right) \left(\sum_l \mathcal{C}_{|m_l|^2} \otimes \text{Id}_{S_m^{p/2}}(|f_l|^2) u \right) \\
& \stackrel{(9.2)}{=} C \left(\int_{\Omega} \otimes \text{Tr} \right) \left(\sum_l |f_l|^2 (\mathcal{C}_{|\bar{m}_l|^2} \otimes \text{Id}_{S_m^{(p/2)^*}}(u)) \right) \\
& \leq C \inf \left\{ \left(\int_{\Omega} \otimes \text{Tr} \right) \left(\sum_l |f_l|^2 B \right) : B \geq 0, \mathcal{C}_{|\bar{m}_l|^2} \otimes \text{Id}_{S_m^{(p/2)^*}}(u) \leq B, l = 1, \dots, N \right\}.
\end{aligned}$$

According to (7.1) and (9.8), we can find such $B \geq 0$ as in the previous infimum with the property $\|B\|_{(\frac{p}{2})^*} \leq C \|u\|_{(\frac{p}{2})^*} = C$ (up to $\varepsilon > 0$). We infer that

$$\begin{aligned}
& \left\| \left(\sum_l |(\mathcal{C}_l \otimes \text{Id}_{S^p})(f_l)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega, S^p)}^2 \lesssim \left(\int_{\Omega} \otimes \text{Tr} \right) \left(\sum_l |f_l|^2 B \right) \\
& \lesssim \left\| \sum_l |f_l|^2 \right\|_{\frac{p}{2}} = \left\| \left(\sum_l |f_l|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega, S^p)}^2.
\end{aligned}$$

Thus (9.9) is proved with constant

$$\sup_{z \in Z} \sup_{x \in \Omega} \|s_z(x, \cdot)\|_{L^2(\Omega)} \cdot \left\| \sup_{z \in Z} \mathcal{C}_{|\bar{m}_z|^2}(\cdot) \right\|_{L^{(p/2)^*}(\Omega, S^{(p/2)^*}) \rightarrow L^{(p/2)^*}(\Omega, S^{(p/2)^*})}^{\frac{1}{2}}.$$

Next, we quickly deduce Row-boundedness of $(\mathcal{C}_{k_z} \otimes \text{Id}_{S^p})_{z \in Z}$ from Col-boundedness of that family. In order to obtain Row-boundedness, we have to show that

$$(9.15) \quad \left\| \left(\sum_l |[(\mathcal{C}_l \otimes \text{Id}_{S^p})(f_l)]^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega, S^p)} \leq C \left\| \left(\sum_l |f_l^*|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega, S^p)}.$$

Note that $[(\mathcal{C}_l \otimes \text{Id}_{S^p})(f_l)]^* = \mathcal{C}_{\bar{k}_l} \otimes \text{Id}_{S^p}(f_l^*)$. Thus Row-boundedness follows from Col-boundedness once we know that the family $(\bar{k}_z)_{z \in Z}$ has also an $L^{(\frac{p}{2})^*}(S^{(\frac{p}{2})^*})$ -square-max decomposition. But this is immediate from $\bar{k}_z = \bar{s}_z \bar{m}_z$ and the identities $\left\| \bar{s}_z(x, \cdot) \right\|_{L^2(\Omega)} = \|s_z(x, \cdot)\|_{L^2(\Omega)}$ and $|\bar{m}_z|^2 = |m_z|^2$, together with (9.7) and (9.8).

Finally, Row-boundedness in conjunction with Col-boundedness implies R -boundedness according to [JMX06, (2.21) and p. 31]. \blacksquare

10 R -sectoriality of the harmonic oscillator of the universal Θ -Weyl tuple

In this section, we apply the maximal estimate from Lemma 7.2 of Section 7 to show the R -sectoriality (in a controlled way as the angle increases to $\frac{\pi}{2}$) of the operator $\mathcal{A} - \alpha$ on the

Bochner space $L^p(\mathbb{R}^n, S^p)$. The R -boundedness in such a noncommutative L^p -space can be expressed, thanks to (noncommutative) Khintchine inequalities, by square function estimates.

Proposition 10.1 *For any complex number z such that $\Re z > 0$, consider the function $G_z: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}_+$, $(x, y) \mapsto g_z(x - y)$ where g_z is defined in Proposition 6.3. Suppose that $1 < p < \infty$. Then the family $(\cos(\arg z)e^{\Re z}G_z)_{\Re z > 0}$ admits an L^p -square-max decomposition.*

Proof : If $\Re z > 0$ then with the functions $s_z \stackrel{\text{def}}{=} m_z \stackrel{\text{def}}{=} \sqrt{\cos(\arg z)e^{\Re z}g_z}$ we can write $\cos(\arg z)e^{\Re z}g_z = s_z m_z$. Now, decompose $\cos(\arg z)e^{\Re z}G_z(x, y) = S_z(x, y)M_z(x, y)$, with $S_z(x, y) \stackrel{\text{def}}{=} s_z(x - y)$ and $M_z(x, y) \stackrel{\text{def}}{=} m_z(x - y)$. Now, we check that this decomposition is an L^p -square-max decomposition. We have

$$\begin{aligned} \sup_{\Re z > 0} \sup_{x \in \mathbb{R}^2} \|S_z(x, \cdot)\|_{L^2(\mathbb{R}^2)} &= \sup_{\Re z > 0} \sup_{x \in \mathbb{R}^2} \|s_z(x - \cdot)\|_{L^2(\mathbb{R}^2)} \\ &= \sup_{\Re z > 0} \|s_z\|_{L^2(\mathbb{R}^2)} = \sup_{\Re z > 0} \|s_z^2\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} = \sup_{\Re z > 0} \|\cos(\arg z)e^{\Re z}g_z\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}} \stackrel{(6.13)}{<} \infty. \end{aligned}$$

Recall that the function $g_z: \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is even. For any function $u \in L^p(\mathbb{R}^2)$, Lemma 7.2 combined with (6.13) give the estimate

$$\begin{aligned} \left\| \sup_{\Re z > 0} {}^+(\mathcal{C}_{|M_z|^2})^*(u) \right\|_{L^p(\mathbb{R}^2)} &= \left\| \sup_{\Re z > 0} {}^+(\cos(\arg z)e^{\Re z}g_z) * u \right\|_{L^p(\mathbb{R}^2)} \\ &\lesssim \sup_{\Re z > 0} \cos(\arg z)e^{\Re z} \|g_z\|_{L^1(\mathbb{R}^2)} \|u\|_{L^p(\mathbb{R}^2)} \stackrel{(6.13)}{\lesssim} \|u\|_{L^p(\mathbb{R}^2)}. \end{aligned}$$

■

Now, we can obtain an R -bound for the family $(T_z)_{\Re z > 0}$ of operators from Proposition 5.5 using a standard domination trick in the spirit of [KuW04, Example 2.10 p. 89] and [HvNVW18, Proposition 8.1.10 p. 171]. Recall that $\mathcal{J} \stackrel{(6.4)}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proposition 10.2 *Suppose that $1 < p < \infty$. Let $A_{\text{univ}}^{\mathcal{J}}$ be the universal \mathcal{J} -Weyl tuple on the Banach space $L^p(\mathbb{R}^2)$. The family*

$$\{\cos(\arg z)e^{\Re z}T_z : \Re z > 0\}$$

is R -bounded over the Banach space $L^p(\mathbb{R}^2)$.

Proof : First note that since R -boundedness is preserved under taking adjoints by Proposition 8.3 and due to self-adjointness of the operator \mathcal{A} , it suffices to prove the statement for $p \geq 2$. Moreover, since for $p = 2$, R -boundedness is just boundedness, essentially proved in Proposition 6.3, we can restrict to the case $p > 2$. Consider an integer $n \geq 1$ and $f_1, \dots, f_n \in L^p(\mathbb{R}^2)$. Suppose that the complex numbers z_1, \dots, z_n satisfy $\Re z_1 > 0, \dots, \Re z_n > 0$. Using Proposition

9.2 combined with Proposition 10.1 in the last inequality, we obtain that

$$\begin{aligned}
& \left\| \left(\sum_{j=1}^n |\cos(\arg z_j) e^{\Re z_j} T_{z_j}(f_j)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \\
& \stackrel{(5.14)}{=} \left\| \left(\sum_{j=1}^n \left| \int_{\mathbb{R}^2} \cos(\arg z_j) e^{\Re z_j} k_{z_j}^{\mathcal{J}}(\cdot, y) f_j(\cdot - y) dy \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \\
& \leq \left\| \left(\sum_{j=1}^n \left(\int_{\mathbb{R}^2} \cos(\arg z_j) e^{\Re z_j} |k_{z_j}^{\mathcal{J}}(\cdot, y)| |f_j(\cdot - y)| dy \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \\
& \stackrel{(6.12)}{\leq} \left\| \left(\sum_{j=1}^n \left(\int_{\mathbb{R}^2} \cos(\arg z_j) e^{\Re z_j} g_{z_j}(y) |f_j(\cdot - y)| dy \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \\
& \stackrel{(9.1)}{=} \left\| \left(\sum_{j=1}^n \left[\mathcal{C}_{\cos(\arg z) e^{\Re z} G_z}(|f_j|) \right]^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \lesssim \left\| \left(\sum_{j=1}^n |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)}.
\end{aligned}$$

■

For an arbitrary matrix Θ , we could obtain from this result an R -bound. However, we do not prove it here since we will obtain in Theorem 10.6 a more general vector-valued result.

Proposition 10.3 *Suppose that $1 < p < \infty$.*

1. *For any complex number z such that $\Re z > 0$, let $G_z : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}_+$, $(x, y) \mapsto g_z(x - y)$ where g_z is defined in Proposition 6.3. Then the family $(\cos(\arg z) e^{\Re z} G_z)_{\Re z > 0}$ admits an $L^p(S^p)$ -square-max decomposition.*
2. *Let $K_z^{\mathcal{J}} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}_+$ be the integral kernel such that the semigroup of the universal \mathcal{J} -Weyl tuple $A_{\text{univ}}^{\mathcal{J}}$ on the Banach space $L^p(\mathbb{R}^2)$ is given by $\mathcal{C}_{K_z^{\mathcal{J}}}$. Then the family $(\cos(\arg z) e^{\Re z} K_z^{\mathcal{J}})_{\Re z > 0}$ admits an $L^p(S^p)$ -square-max decomposition.*
3. *In the same manner, if $\alpha > 0$ and if $K_z^{\alpha \mathcal{J}}$ is the integral kernel associated with the semigroup of the universal $\alpha \mathcal{J}$ -Weyl tuple $A_{\text{univ}}^{\alpha \mathcal{J}}$ on the space $L^p(\mathbb{R}^2)$, then the family $(\cos(\arg z) e^{\alpha \Re z} K_z^{\alpha \mathcal{J}})_{\Re z > 0}$ admits an $L^p(S^p)$ -square-max decomposition.*
4. *Let p_z be the standard heat kernel on \mathbb{R} . Then the family $((x, y) \mapsto (\cos \arg z)^{\frac{1}{2}} p_z(x - y))_{\Re z > 0}$ admits an $L^p(S^p)$ -square-max decomposition.*

Proof : 1. For any $x, y \in \mathbb{R}^2$, we write

$$(10.1) \quad \cos(\arg z) e^{\Re z} G_z(x, y) = S_z(x, y) M_z(x, y)$$

as in the proof of Proposition 10.1 (in particular $M_z(x, y) = \sqrt{\cos(\arg z) e^{\Re z} g_z(x - y)}$). Then this proof actually does not only show an L^p -square-max decomposition, but even an $L^p(S^p)$ -square-max decomposition using Lemma 7.2.

2. According to (5.15) and (6.12), we have for all $x, y \in \mathbb{R}^2$ and $\Re z > 0$, the inequality

$$(10.2) \quad |K_z^{\mathcal{J}}(x, y)| \stackrel{(5.14)}{=} |p_z^{\mathcal{J}}(x - y)| \stackrel{(6.12)}{\leq} g_z(x - y) = G_z(x, y).$$

Now, we decompose

$$(10.3) \quad \cos(\arg z) e^{\Re z} K_z^{\mathcal{J}} = S'_z M_z$$

with M_z as in the first part of the proof and $S'_z \stackrel{\text{def}}{=} \frac{1}{M_z} (\cos \arg z) e^{\Re z} |K_z^{\mathcal{J}}|$ (note that $g_z > 0$ and hence also $M_z > 0$ everywhere). Then we obtain the estimate

$$|S'_z(x, y)| \leq \frac{(\cos \arg z) e^{\Re z} G_z(x, y)}{M_z(x, y)} \stackrel{(10.1)}{=} S_z(x, y).$$

Hence the first part shows that $K_z^{\mathcal{J}}$ admits an $L^p(S^p)$ -square-max decomposition.

3. Since we have the decomposition (10.3), we obtain here (note that $\arg(z) = \arg(\alpha z)$)

$$(10.4) \quad \begin{aligned} \cos(\arg z) e^{\alpha \Re z} K_z^{\alpha \mathcal{J}}(x, y) &= \cos(\arg z) e^{\alpha \Re z} e^{-\frac{1}{2}i(y, \alpha \mathcal{J} x)} p_z^{\alpha \mathcal{J}}(x - y) \\ &\stackrel{(6.8)}{=} e^{-\frac{1}{2}i(y, \alpha \mathcal{J} x)} \alpha \cos(\arg z) e^{\Re(\alpha z)} p_{\alpha z}^{\mathcal{J}}(\sqrt{\alpha}(x - y)) \\ &\stackrel{(5.15)}{=} e^{i\phi(x, y)} \alpha \cos(\arg z) e^{\Re(\alpha z)} K_{\alpha z}^{\mathcal{J}}(\sqrt{\alpha}x, \sqrt{\alpha}y) \\ &\stackrel{(10.3)}{=} e^{i\phi(x, y)} \alpha S'_{\alpha z}(\sqrt{\alpha}x, \sqrt{\alpha}y) M_{\alpha z}(\sqrt{\alpha}x, \sqrt{\alpha}y) \stackrel{\text{def}}{=} S_z^{\alpha}(x, y) M_z^{\alpha}(x, y), \end{aligned}$$

for some measurable phase function ϕ with

$$S_z^{\alpha}(x, y) \stackrel{\text{def}}{=} e^{i\phi(x, y)} \sqrt{\alpha} S'_{\alpha z}(\sqrt{\alpha}x, \sqrt{\alpha}y) \quad \text{and} \quad M_z^{\alpha}(x, y) \stackrel{\text{def}}{=} \sqrt{\alpha} M_{\alpha z}(\sqrt{\alpha}x, \sqrt{\alpha}y).$$

Now, we show that this is an $L^p(S^p)$ -square-max decomposition. Indeed, we have

$$\|S_z^{\alpha}(x, \cdot)\|_{L^2(\mathbb{R}^2)} = \|\sqrt{\alpha} S'_{\alpha z}(\sqrt{\alpha}x, \sqrt{\alpha} \cdot)\|_{L^2(\mathbb{R}^2)} \approx \|S'_{\alpha z}(\sqrt{\alpha}x, \cdot)\|_{L^2(\mathbb{R}^2)} \leq C.$$

Moreover, it is not difficult to see that we have the factorization $\mathcal{C}_{|M_z^{\alpha}|^2} = D_{\sqrt{\alpha}} \mathcal{C}_{|M_{\alpha z}|^2} D_{\sqrt{\alpha}^{-1}}$, where $D_{\beta}: L^p(\mathbb{R}^2, S^p) \rightarrow L^p(\mathbb{R}^2, S^p)$, $u \mapsto u(\beta \cdot)$ is a positive map such that $\|D_{\beta} u\|_p = \beta^{-\frac{2}{p}} \|u\|_p$. Suppose that u is a positive element of $L^p(\mathbb{R}^2, S^p)$. Then $D_{\sqrt{\alpha}^{-1}} u$ is again a positive element of $L^p(\mathbb{R}^2, S^p)$. By the maximal estimate from the first part, there exists some positive element B of $L^p(\mathbb{R}^2, S^p)$ such that $\mathcal{C}_{|M_{\alpha z}|^2} D_{\sqrt{\alpha}^{-1}} u \leq B$ for all $\Re z > 0$ and in addition $\|B\|_p \leq C \|D_{\sqrt{\alpha}^{-1}} u\|_p$ (up to $\varepsilon > 0$). Thus, we have $\mathcal{C}_{|M_z^{\alpha}|^2} u = D_{\sqrt{\alpha}} \mathcal{C}_{|M_{\alpha z}|^2} D_{\sqrt{\alpha}^{-1}} u \leq D_{\sqrt{\alpha}} B$ and $\|D_{\sqrt{\alpha}} B\|_p = \alpha^{-\frac{1}{p}} \|B\|_p \leq C \alpha^{-\frac{1}{p}} \|D_{\sqrt{\alpha}^{-1}} u\|_p = C \|u\|_p$. Consequently, we obtain

$$\left\| \sup_{\Re z > 0} \mathcal{C}_{|M_z^{\alpha}|^2} u \right\|_{L^p(\mathbb{R}^2, S^p)} \lesssim \|u\|_{L^p(\mathbb{R}^2, S^p)}.$$

4. Using the formula $\Re(\frac{1}{z}) = \frac{\cos \arg z}{|z|}$, we can write for $x, y \in \mathbb{R}$

$$\cos(\arg z)^{\frac{1}{2}} p_z(x - y) \stackrel{(6.7)}{=} e^{i\phi(x, y, z)} \cos(\arg z)^{\frac{1}{2}} \frac{1}{\sqrt{4\pi|z|}} e^{-\cos(\arg z) \frac{(x-y)^2}{4|z|}} = s_z(x, y) m_z(x, y)$$

with some measurable real-valued phase function $\phi: \mathbb{R}^2 \times \mathbb{C}_+ \rightarrow \mathbb{R}$, the maximal part $m_z(x, y) \stackrel{\text{def}}{=} \cos(\arg z)^{\frac{1}{4}} \frac{1}{\sqrt{4\pi|z|}} e^{-\cos(\arg z) \frac{(x-y)^2}{8|z|}}$, and the square part $s_z(x, y) \stackrel{\text{def}}{=} e^{i\phi(x, y, z)} m_z^0(x, y)$. Thus,

$|s_z(x, y)|^2 = |m_z(x, y)|^2 = \cos(\arg z)^{\frac{1}{2}} |p_z(x - y)|$. Then with the classical estimate of [ABHN11, Remark 3.7.10 p. 153] (see also [HvNVW18, p. 543]), we obtain for any $x \in \mathbb{R}$

$$\|s_z(x, \cdot)\|_{L^2(\mathbb{R})}^2 = \|\cos(\arg z)^{\frac{1}{2}} p_z(x - \cdot)\|_{L^1(\mathbb{R})} \leq C,$$

which implies the square estimate (9.7). Moreover, the 1-dimensional version of Lemma 7.2 described in Remark 7.3 together with the L^1 -norm estimate

$$\| |m_z(x, \cdot)|^2 \|_{L^1(\mathbb{R})} = \|\cos(\arg z)^{\frac{1}{2}} p_z(x - \cdot)\|_{L^1(\mathbb{R})} \leq C, \quad x \in \mathbb{R}$$

implies the maximal estimate (9.8). \blacksquare

Corollary 10.4 *Let $\alpha > 0$. Suppose that $1 < p < \infty$ and let $A_{\text{univ}}^{\alpha\mathcal{J}}$ be the universal $\alpha\mathcal{J}$ -Weyl tuple on the Banach space $L^p(\mathbb{R}^2)$. If $(T_z)_{\Re z > 0}$ denotes the complex time semigroup of (5.14) generated by $-\mathcal{A}$ associated with $A_{\text{univ}}^{\alpha\mathcal{J}}$ then the set*

$$\{\cos(\arg z)e^{\alpha\Re z}T_z \otimes \text{Id}_{S^p} : \Re z > 0\}$$

is R -bounded over the Bochner space $L^p(\mathbb{R}^2, S^p)$.

Proof : Let us first consider the case where $p \geq 2$. According to Proposition 10.3, if the integral kernel of T_z is denoted by $K_z^{\alpha\mathcal{J}}$ (that is, $T_z = \mathcal{C}_{K_z^{\alpha\mathcal{J}}}$), then the family $(\cos(\arg z)e^{\alpha\Re z}K_z^{\alpha\mathcal{J}})_{\Re z > 0}$ admits an $L^q(S^q)$ -square-max decomposition for $q = (\frac{p}{2})^*$. Then Theorem 9.4 implies that the family

$$\{\cos(\arg z)e^{\alpha\Re z}T_z \otimes \text{Id}_{S^p} : \Re z > 0\} = \{\cos(\arg z)e^{\alpha\Re z}\mathcal{C}_{K_z^{\alpha\mathcal{J}}} \otimes \text{Id}_{S^p} : \Re z > 0\}$$

is R -bounded over the Bochner space $L^p(\mathbb{R}^2, S^p)$.

If $1 < p < 2$, then the R -boundedness is obtained from a duality argument in the following way. By Proposition 8.3, it suffices to show that the family $\{\cos(\arg z)e^{\alpha\Re z}T_z^* \otimes \text{Id}_{S^{p^*}} : \Re z > 0\}$ is R -bounded over the Banach space $L^{p^*}(\mathbb{R}^2, S^{p^*})$. According to (9.2), we have $T_z^* = \mathcal{C}_{\check{K}_z}$. Now, we will observe that the family $\{\cos(\arg z)e^{\alpha\Re z}\check{K}_z : \Re z > 0\}$ also has an $L^q(S^q)$ -square-max decomposition for any $1 < q < \infty$. Indeed, using the notation introduced in the proof of Proposition 10.3, we see that

$$\begin{aligned} \cos(\arg z)e^{\alpha\Re z}\check{K}_z^{\alpha\mathcal{J}}(x, y) &\stackrel{(9.2)}{=} \cos(\arg z)e^{\alpha\Re z}K_z^{\alpha\mathcal{J}}(y, x) \\ &\stackrel{(10.4)}{=} S_z^\alpha(y, x)M_z^\alpha(y, x) = S_z''(x, y)M_z''(x, y) \end{aligned}$$

with $S_z''(x, y) \stackrel{\text{def}}{=} S_z^\alpha(y, x)$ and $M_z''(x, y) \stackrel{\text{def}}{=} M_z^\alpha(y, x)$. With (10.2) it is easy to see that (9.7) holds for S_z'' . Since $M_z''(x, y) = M_z^\alpha(x, y)$, we can confirm that (9.8) holds for M_z'' . \blacksquare

Arguing in the same manner, but using the fourth point of Proposition 10.3 instead of the third, we obtain the following result. In [HvNVW18, Example 8.3.6 p. 196], the R -boundedness of the set $\{T_z \otimes \text{Id}_X : |\arg z| \leq \theta\}$ is proved with R -bound less than $1 + \frac{2}{\cos \theta}$ up to a constant, for any UMD Banach space X .

Corollary 10.5 *Suppose that $1 < p < \infty$. Let $(T_z)_{\Re z > 0}$ denote the standard heat semigroup on \mathbb{R} . Then the family*

$$\left\{ \cos(\arg z)^{\frac{1}{2}} T_z \otimes \text{Id}_{S^p} : \Re z > 0 \right\}$$

is R -bounded over the Bochner space $L^p(\mathbb{R}, S^p)$.

In the following statement, recall that the value $\alpha \stackrel{\text{def}}{=} \sum_{j=1}^k \alpha_j > 0$ is defined in (6.3) and that A_{univ}^Θ is defined in (4.12).

Theorem 10.6 *Let $\Theta \in M_n(\mathbb{R})$ be a skew-adjoint matrix and $1 < p < \infty$. Consider the universal Θ -Weyl tuple A_{univ}^Θ on the space $L^p(\mathbb{R}^n)$ and denote $(T_z)_{\Re z > 0}$ the complex time semigroup of (5.14) associated with A_{univ}^Θ . Then the set*

$$\{\cos(\arg z)^{\frac{\alpha}{2}} e^{\alpha \Re z} T_z \otimes \text{Id}_{S^p} : \Re z > 0\}$$

is R -bounded over the Bochner space $L^p(\mathbb{R}^n, S^p)$.

Proof : Assume first that $\Theta = J$ has the block diagonal form with matrices $\alpha_j \mathcal{J}$ on the diagonal, see (6.2). Then the operators of the semigroup $(T_z)_{\Re z > 0}$ split into tensor products

$$T_z = T_z^1 \otimes \cdots \otimes T_z^k \otimes T_z' \otimes \cdots \otimes T_z', \quad \Re z > 0,$$

acting on the space $L^p(\mathbb{R}^2) \otimes_p \cdots \otimes_p L^p(\mathbb{R}^2) \otimes_p L^p(\mathbb{R}) \otimes_p \cdots \otimes_p L^p(\mathbb{R})$, where T_z^j is the semigroup associated with the universal $\alpha_j \mathcal{J}$ -Weyl tuple on $L^p(\mathbb{R}^2)$, and T_z' is the standard heat semigroup on the space $L^p(\mathbb{R})$. This follows from Lemma 6.2 for real positive z , and then by uniqueness of holomorphic extensions for an arbitrary complex number z with $\Re z > 0$. Put now $S_z \stackrel{\text{def}}{=} \cos(\arg z)^{\frac{\alpha}{2}} e^{\alpha \Re z} T_z \otimes \text{Id}_{S^p}$, $S_z^j \stackrel{\text{def}}{=} \cos(\arg z) e^{\alpha_j \Re z} T_z^j$ for any $j = 1, \dots, k$ and $S_z' \stackrel{\text{def}}{=} \cos(\arg z)^{\frac{1}{2}} T_z'$. Then

$$S_z = S_z^1 \otimes S_z^2 \otimes \cdots \otimes S_z^k \otimes S_z' \otimes \cdots \otimes S_z' \otimes \text{Id}_{S^p}, \quad \Re z > 0.$$

Moreover, the families $\{S_z^k \otimes \text{Id}_{S^p} : \Re z > 0\}$ are R -bounded over the space $L^p(\mathbb{R}^2, S^p)$ according to Corollary 10.4. The family $\{S_z' \otimes \text{Id}_{S^p} : \Re z > 0\}$ is also R -bounded over the space $L^p(\mathbb{R}, S^p)$ according to Corollary 10.5. Then Lemma 8.2 implies that the family $\{S_z : \Re z > 0\}$ is R -bounded over the Bochner space $L^p(\mathbb{R}^n, S^p)$.

Consider the case of general skew-adjoint $\Theta \in M_n(\mathbb{R})$. Let $O \in M_n(\mathbb{R})$ be the orthogonal matrix such that $\Theta = O^T J O$ (see (6.1)), where J is a block diagonal matrix as in the first case. Denote the semigroups associated with Θ and J by $(T_z^\Theta)_{\Re z > 0}$ and $(T_z^J)_{\Re z > 0}$. Then according to the proof of Corollary 6.7 (first for real z , then by uniqueness of holomorphic extensions for complex z), $T_z^\Theta = \tilde{O}^{-1} T_z^J \tilde{O}$ for the isometry $\tilde{O} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$, $f \mapsto f(O^T \cdot)$. Since \tilde{O} is an isometry for all $1 \leq p \leq \infty$, it extends to an isometry $\tilde{O} \otimes \text{Id}_{S^p}$ on $L^p(\mathbb{R}^n, S^p)$. Then the set

$$\begin{aligned} & \{\cos(\arg z)^{\frac{\alpha}{2}} e^{\alpha \Re z} T_z^\Theta \otimes \text{Id}_{S^p} : \Re z > 0\} \\ &= (\tilde{O}^{-1} \otimes \text{Id}_{S^p}) \{\cos(\arg z)^{\frac{\alpha}{2}} e^{\alpha \Re z} T_z^J \otimes \text{Id}_{S^p} : \Re z > 0\} (\tilde{O} \otimes \text{Id}_{S^p}) \end{aligned}$$

is R -bounded over the Bochner space $L^p(\mathbb{R}^n, S^p)$ by the first case and stability of R -boundedness under products [HvNVW18, Proposition 8.1.19 p. 178]. \blacksquare

Finally, we establish the principal result of this section, Theorem 10.8, by employing the following result as a foundational basis.

Proposition 10.7 *Suppose that $1 < p < \infty$ and $\beta \geq 0$. Let $-A$ be the generator of a bounded holomorphic semigroup $(T_z)_{\Re z > 0}$ acting on a noncommutative L^p -space $L^p(\mathcal{M})$. Suppose that A has a bounded $H^\infty(\Sigma_\omega)$ functional calculus for some angle $\omega \in (0, \pi)$. Assume in addition that the family*

$$(10.5) \quad \{(\cos \arg z)^\beta T_z : \Re z > 0\}$$

of operators is R -bounded over the space $L^p(\mathcal{M})$. Then the operator A admits a bounded Hörmander $H_2^s(\mathbb{R}_+^*)$ Hörmander calculus of any order $s > \beta + \frac{1}{2}$.

Proof : This is an immediate consequence of [KrW18, Theorem 7.1 (2) p. 424]. Indeed, note first that $\frac{1}{\text{type } X} - \frac{1}{\text{cotype } X} = |\frac{1}{2} - \frac{1}{p}| < \frac{1}{2}$ where $X = L^p(\mathcal{M})$ by [PiX03, Corollary 5.5 p. 1481], so we can take $r = 2$ in [KrW18, Theorem 7.1]. If the family (10.5) is R -bounded, then the subset $\{(\cos \theta)^\beta T_{e^{i\theta} 2^{n_t}} : n \in \mathbb{Z}\}$ is also R -bounded, with uniformly controlled R -bound in $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $t > 0$. Then [KrW18, (7.2)] follows by dividing the aforementioned expression by $(\cos \theta)^\beta$ and observing that this expression is equivalent to $(\frac{\pi}{2} - |\theta|)^\beta$. ■

We conclude with the following theorem.

Theorem 10.8 *Suppose that $1 < p < \infty$. Consider a skew-symmetric matrix $\Theta \in M_n(\mathbb{R})$. Let $-\mathcal{A}$ be the infinitesimal generator of the semigroup of operators acting on $L^p(\mathbb{R}^n)$ associated with the universal Θ -Weyl tuple $A_{\text{univ}}^\Theta = (A_1, \dots, A_n)$ from Proposition 4.7, that is, \mathcal{A} is the closure of $\sum_{j=1}^n A_j^2$. Then the operator $(\mathcal{A} - \alpha \text{Id}) \otimes \text{Id}_{S^p}$ admits a bounded Hör $_2^s(\mathbb{R}_+^*)$ Hörmander functional calculus of order $s > \frac{n}{2} + \frac{1}{2}$ on the Bochner space $L^p(\mathbb{R}^n, S^p)$.*

Proof : According to Corollary 6.7, the operator $(\mathcal{A} - \alpha \text{Id}) \otimes \text{Id}_{S^p}$ has a bounded $H^\infty(\Sigma_\omega)$ functional calculus on the Bochner space $L^p(\mathbb{R}^n, S^p)$ for any angle $\omega \in (\frac{\pi}{2}, \pi)$. Moreover, according to Theorem 10.6, the semigroup $(S_z \otimes \text{Id}_{S^p})_{\Re z > 0}$ of operators acting on the space $L^p(\mathbb{R}^n, S^p)$ of (6.18), generated by $(\mathcal{A} - \alpha \text{Id}) \otimes \text{Id}_{S^p}$, is R -bounded in the following sense:

$$R\left(\left\{(\cos \arg z)^{\frac{n}{2}} S_z \otimes \text{Id}_{S^p} : \Re z > 0\right\}\right) < \infty.$$

Recall that $\frac{1}{\text{type } X} - \frac{1}{\text{cotype } X} = |\frac{1}{2} - \frac{1}{p}| < \frac{1}{2}$ for the Banach space $X = L^p(\mathbb{R}^n, S^p)$. Then according to Proposition 10.7, the operator $(\mathcal{A} - \alpha \text{Id}) \otimes \text{Id}_{S^p}$ admits a bounded Hör $_2^s(\mathbb{R}_+^*)$ Hörmander calculus of order $s > \frac{n}{2} + \frac{1}{2}$ on $L^p(\mathbb{R}^n, S^p)$. ■

11 Functional calculus for the harmonic oscillator associated to Θ -Weyl tuples

Consider a skew-symmetric matrix $\Theta \in M_n(\mathbb{R})$. Thanks to the transference result from Corollary 4.6, we can generalize the Hörmander functional calculus of the semigroup associated with the universal Θ -Weyl tuple A_{univ}^Θ to semigroups associated with any Θ -Weyl tuple acting on a noncommutative L^p -space associated with an injective (=approximate finite-dimensional) semifinite von Neumann algebra. We will use the notion of completely bounded maps between operator spaces and we refer to the books [BLM04], [Efr00] and [Pis98] for background. If $1 \leq p < \infty$, it is known [Pis98, Lemma 1.7 p. 23] that a linear map $T: E \rightarrow F$ between operator spaces is completely bounded if and only if it induces a bounded operator $\text{Id}_{S^p} \otimes T: S^p(E) \rightarrow S^p(F)$ on the vector-valued Schatten space $S^p(E)$ of [Pis98, p. 18]. In this case, we have

$$\|T\|_{\text{cb}, E \rightarrow F} = \|\text{Id}_{S^p} \otimes T\|_{S^p(E) \rightarrow S^p(F)}.$$

Moreover, for any injective von Neumann algebra \mathcal{M} equipped with a normal semifinite faithful trace, a completely bounded map $T: E \rightarrow F$ between operator spaces induces by [Pis98, (3.1) p. 39] a completely bounded map $\text{Id}_{L^p(\mathcal{M})} \otimes T: L^p(\mathcal{M}, E) \rightarrow L^p(\mathcal{M}, F)$ with

$$(11.1) \quad \|\text{Id}_{L^p(\mathcal{M})} \otimes T\|_{\text{cb}, L^p(\mathcal{M}, E) \rightarrow L^p(\mathcal{M}, F)} \leq \|T\|_{\text{cb}, E \rightarrow F}.$$

Proposition 11.1 *Let \mathcal{M} be an injective von Neumann algebra equipped with a normal semifinite faithful trace. Suppose that $1 < p < \infty$. Let A be a Θ -Weyl tuple on the noncommutative*

L^p -space $L^p(\mathcal{M})$. Let us introduce the constant $M_A \stackrel{\text{def}}{=} \sup_{t \in \mathbb{R}^n} \|e^{it \cdot A}\|_{L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})}$. Let A_{univ}^Θ be the universal Θ -Weyl tuple on $L^p(\mathbb{R}^n)$ from Proposition 4.7. For any tempered distribution a with $\hat{a} \in L^1(\mathbb{R}^n)$, we have

$$(11.2) \quad \|a(A)\|_{L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})} \leq M_A^2 \|a(A_{\text{univ}}^\Theta)\|_{\text{cb}, L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)}.$$

Proof : Using the transference result of Corollary 4.6 in the first estimate and Fubini's theorem [Pis98, (3.6')] in the second equality, we obtain

$$\begin{aligned} \|a(A)\|_{L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})} &\stackrel{(4.8)}{\leq} M_A^2 \|C_{\hat{a}}^\Theta\|_{L^p(\mathbb{R}^n, L^p(\mathcal{M})) \rightarrow L^p(\mathbb{R}^n, L^p(\mathcal{M}))} \\ &\stackrel{(4.13)}{=} M_A^2 \|a(A_{\text{univ}}^\Theta) \otimes \text{Id}_{L^p(\mathcal{M})}\|_{L^p(\mathbb{R}^n, L^p(\mathcal{M})) \rightarrow L^p(\mathbb{R}^n, L^p(\mathcal{M}))} \\ &= M_A^2 \|\text{Id}_{L^p(\mathcal{M})} \otimes a(A_{\text{univ}}^\Theta)\|_{L^p(\mathcal{M}, L^p(\mathbb{R}^n)) \rightarrow L^p(\mathcal{M}, L^p(\mathbb{R}^n))} \\ &\stackrel{(11.1)}{\leq} M_A^2 \|a(A_{\text{univ}}^\Theta)\|_{\text{cb}, L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)}. \end{aligned}$$

■

Remark 11.2 In this paper, we only use the case where the von Neumann algebra \mathcal{M} is injective. However, we could replace in Proposition 11.1 the word «injective» by the much more general assumption QWEP. Indeed, we can use [Jun04, (iii) p. 984], whose proof is unfortunately unpublished, instead of [Pis98, (3.1)]. The same remark also applies to Corollary 11.5. We refer to [Oza04] and [AHW16] for more information on QWEP von Neumann algebras.

Let $M(\mathbb{R}_+)$ be the space of all bounded Borel measures on \mathbb{R}_+ . Then $M(\mathbb{R}_+)$ is a Banach algebra under convolution of measures. The Laplace transform $\mathcal{L}\mu$ of a bounded Borel measure μ on $[0, \infty)$ is defined by

$$(\mathcal{L}\mu)(z) \stackrel{\text{def}}{=} \int_0^\infty e^{-sz} d\mu(s), \quad \Re z > 0.$$

This is a bounded holomorphic function on \mathbb{C}_+ . Let $\mathcal{LM} \stackrel{\text{def}}{=} \{\mathcal{L}\mu : \mu \in M(\mathbb{R}_+)\}$. The map $M(\mathbb{R}_+) \rightarrow H^\infty(\mathbb{C}_+)$, $\mu \mapsto \mathcal{L}\mu$ is an injective algebra homomorphism. Thus \mathcal{LM} becomes a Banach algebra for the norm $\|\mathcal{L}\mu\|_{\text{HP}} \stackrel{\text{def}}{=} \|\mu\|_{M(\mathbb{R}_+)}$ and the pointwise product.

Consider a strongly continuous bounded semigroup $(T_t)_{t \geq 0}$ of operators acting on a Banach space X with infinitesimal generator $-A$. The Hille-Phillips functional calculus [HiP74, Chapter XV] [Haa06, Section 3.3] for the operator A is defined as follows. If $f = \mathcal{L}\mu$ is the Laplace transform of a bounded Borel measure μ on \mathbb{R}_+ , we can define the bounded linear operator $f(A)_{\text{HP}} : X \rightarrow X$ by

$$(11.3) \quad f(A)_{\text{HP}} x \stackrel{\text{def}}{=} \int_0^\infty T_t x d\mu(t), \quad x \in X$$

where the integral converges in the strong operator topology. In this context, we use the subscript HP to avoid confusion with the notation (3.8) introduced for the Θ -Weyl functional calculus. The map $\mathcal{LM} \rightarrow B(X)$, $f \mapsto f(A)_{\text{HP}}$ is a continuous algebra homomorphism with norm less than $\sup_{t \geq 0} \|T_t\|$, known as the Hille-Phillips calculus. It is well-known that the function $e_t : \mathbb{C}_+ \rightarrow \mathbb{C}$, $z \mapsto e^{-tz}$ belongs to \mathcal{LM} for any $t > 0$ and that $e_t(A) = \exp(-tA)$ for any $t > 0$.

The Hille-Phillips calculus is compatible with the sectorial functional calculus, i.e. any f of $H_0^\infty(\Sigma_\omega)$, where $\omega \in (\frac{\pi}{2}, \pi)$ belongs to \mathcal{LM} and we have $f(A)_{\text{HP}} = f(A)_{\text{sect}}$ where $f(A)_{\text{sect}}$ is defined by the sectorial functional calculus. We refer to [Haa06, Remark 3.3.3 p. 74].

The following lemma addresses the technical difficulty to exploit the transference from Proposition 11.1 when we are faced with an operator $f(A)$ provided by the functional calculus.

Lemma 11.3 *Consider a strongly continuous projective representation π of \mathbb{R}^n on a Banach space X and a strongly continuous semigroup $(T_t)_{t \geq 0}$ of operators on X such that there exists a family $(p_t)_{t > 0}$ of integrable functions over \mathbb{R}^n such that*

$$(11.4) \quad T_t = \int_{\mathbb{R}^n} p_t(s) \pi_s \, ds, \quad t > 0 \quad \text{with} \quad \sup_{t > 0} \|p_t\|_{L^1(\mathbb{R}^n)} < \infty.$$

Let $f \in \mathcal{LM}$ be the Laplace transform of some bounded Borel measure μ on \mathbb{R}_+ . Then there exists a function $k_f \in C_0(\mathbb{R}^n)$ such that $\hat{k}_f \in L^1(\mathbb{R}^n)$ and $f(A)_{\text{HP}} x = \int_{\mathbb{R}^n} \hat{k}_f(s) \pi_s x \, ds$ for any $x \in X$. Here \hat{k}_f is the Fourier transform of the tempered distribution k_f .

Proof : The Hille-Phillips calculus yields

$$(11.5) \quad f(A)_{\text{HP}} x \stackrel{(11.3)}{=} \int_0^\infty T_t x \, d\mu(t) \stackrel{(6.18)}{=} \int_0^\infty \left(\int_{\mathbb{R}^n} p_t(s) \pi_s x \, ds \right) d\mu(t).$$

According to (11.4), we have

$$\int_0^\infty \left(\int_{\mathbb{R}^n} |p_t(s)| \, ds \right) d\mu(t) \lesssim \sup_{t > 0} \|p_t\|_{L^1(\mathbb{R}^n)} \|\mu\|_{M(\mathbb{R}_+)} < \infty.$$

This shows that the function $g: \mathbb{R}^n \rightarrow \mathbb{C}$, $s \mapsto \int_0^\infty p_t(s) \, d\mu(t)$ belongs to the space $L^1(\mathbb{R}^n)$. So we can introduce the inverse Fourier transform $k_f \stackrel{\text{def}}{=} \mathcal{F}^{-1} g$ which belongs to $C_0(\mathbb{R}^n)$. Now

$$\int_0^\infty \left(\int_{\mathbb{R}^n} |p_t(s)| \|\pi_s x\|_X \, ds \right) d\mu(t) \lesssim \sup_{t > 0} \|p_t\|_{L^1(\mathbb{R}^n)} \|\mu\|_{M(\mathbb{R}_+)} < \infty.$$

Thus we can change the order of integration in (11.5) and we obtain

$$f(A)_{\text{HP}} x = \int_{\mathbb{R}^n} \left(\int_0^\infty p_t(s) \, d\mu(t) \right) \pi_s x \, ds = \int_{\mathbb{R}^n} \hat{k}_f(s) \pi_s x \, ds.$$

■

We need the following approximation lemma.

Lemma 11.4 *Consider a sectorial operator A of type $\leq \frac{\pi}{2}$ on a Banach space X and $s > 0$. Suppose that for some angle $\omega \in (\frac{\pi}{2}, \pi)$ and any function f of the space $H_0^\infty(\Sigma_\omega)$ we have the estimate*

$$(11.6) \quad \|f(A)\|_{X \rightarrow X} \lesssim \|f\|_{\text{Hör}_2^s(\mathbb{R}_+^*)}.$$

Then the operator A admits a bounded Hörmander $\text{Hör}_2^s(\mathbb{R}_+^*)$ functional calculus, in the sense of [KrW18, Definition 3.8].

Proof : Let $f \in H^\infty(\Sigma_\omega)$ and $g(z) \stackrel{\text{def}}{=} z(1+z)^{-2}$. For any integer $j \geq 1$, note that $g^{\frac{1}{j}}$ is a function of $H_0^\infty(\Sigma_\omega)$ by [Haa06, Example 2.2.5 p. 29]. Since $H_0^\infty(\Sigma_\omega)$ is an ideal of the algebra $H^\infty(\Sigma_\omega)$, the function $f_j \stackrel{\text{def}}{=} fg^{\frac{1}{j}}$ belongs to $H_0^\infty(\Sigma_\omega)$. Moreover, the sequence (f_j) converges pointwise on the sector Σ_ω to f and is bounded for the norm of $H^\infty(\Sigma_\omega)$ since $\|g^{\frac{1}{j}}\|_{H^\infty(\Sigma_\omega)} = \|g\|_{H^\infty(\Sigma_\omega)}^{\frac{1}{j}}$. The estimate (11.6) implies that the operator A has a bounded $H^\infty(\Sigma_\omega)$ functional calculus. The well-known convergence lemma of the H^∞ functional calculus [Haa06, Proposition 5.1.4 p. 104], [HvNVW18, Theorem 10.2.13 p. 381] says that the sequence $(f_j(A)_{\text{HP}})_{j \geq 1}$ converges to $f(A)_{\text{HP}}$ for the strong operator topology of the space $B(\overline{\text{Ran } A})$ of bounded operators on $\overline{\text{Ran } A}$. Using the fact that $\text{Hör}_2^s(\mathbb{R}_+^*)$ is a Banach algebra into which $H^\infty(\Sigma_\omega)$ injects continuously [HaP23, Proposition 9.1], we deduce that

$$\begin{aligned} \|f(A)_{\text{HP}}\|_{X \rightarrow X} &\leq \liminf_j \|f_j(A)_{\text{HP}}\|_{X \rightarrow X} \lesssim \liminf_j \|fg^{\frac{1}{j}}\|_{\text{Hör}_2^s(\mathbb{R}_+^*)} \\ &\lesssim \|f\|_{\text{Hör}_2^s(\mathbb{R}_+^*)} \liminf_j \|g^{\frac{1}{j}}\|_{H^\infty(\Sigma_\omega)} = \|f\|_{\text{Hör}_2^s(\mathbb{R}_+^*)} \end{aligned}$$

where we used the strong lower semicontinuity of the norm. Thus, the operator A admits a bounded $\text{Hör}_2^s(\mathbb{R}_+^*)$ functional calculus on the Banach space X as claimed. \blacksquare

Recall that $\alpha \geq 0$ is defined in (6.3).

Corollary 11.5 *Let $A = (A_1, \dots, A_n)$ be a Θ -Weyl tuple on a noncommutative L^p -space associated with an injective von Neumann algebra \mathcal{M} , where $1 < p < \infty$. Then the closure $\mathcal{A} - \alpha \text{Id}$ of $\sum_{k=1}^n A_k^2 - \alpha \text{Id}$ admits a bounded Hörmander $\text{Hör}_2^s(\mathbb{R}_+^*)$ functional calculus of order $s > \frac{n}{2} + \frac{1}{2}$ on the Banach space $L^p(\mathcal{M})$.*

Proof : We claim that for any angle $\omega \in (\frac{\pi}{2}, \pi)$ and any function f of the space $H_0^\infty(\Sigma_\omega)$ we have the estimate

$$(11.7) \quad \|f(\mathcal{A} - \alpha \text{Id})\|_{L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})} \leq C \|f\|_{\text{Hör}_2^s(\mathbb{R}_+^*)}.$$

Indeed, according to [Haa06, Lemma 3.3.1 p. 72], [LM99, Lemma 2.12], the function f belongs to the space \mathcal{LM} . Using Lemma 11.3 and Proposition 6.8, we can write $f(\mathcal{A} - \alpha \text{Id})_{\text{HP}} = k_f(A)$ for some function $k_f \in C_0(\mathbb{R}^n)$ with $\hat{k}_f \in L^1(\mathbb{R}^n)$. Thus, by Proposition 11.1 and Theorem 10.8 applied to the universal Θ -Weyl tuple $A_{\text{univ}}^\Theta = (A_{\text{univ},1}, \dots, A_{\text{univ},n})$ acting on the space $L^p(\mathbb{R}^n)$ and the closure \mathcal{B} of the unbounded operator $\sum_{k=1}^n A_{\text{univ},k}^2$, we obtain

$$\begin{aligned} \|f(\mathcal{A} - \alpha \text{Id})_{\text{HP}}\|_{L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})} &= \|k_f(A)\|_{L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})} \stackrel{(11.2)}{\leq} M_A^2 \|k_f(A_{\text{univ}})\|_{\text{cb}, L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \\ &= M_A^2 \|f(\mathcal{B} - \alpha \text{Id})_{\text{HP}}\|_{\text{cb}, L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \lesssim \|f\|_{\text{Hör}_2^s(\mathbb{R}_+^*)}. \end{aligned}$$

Consequently (11.7) is proved. We conclude using Lemma 11.4. \blacksquare

Let us mention that parts of the previous program also work if the Θ -Weyl tuple A acts on a UMD Banach space X instead of a noncommutative L^p -space.

Proposition 11.6 *Let $A = (A_1, \dots, A_n)$ be a Θ -Weyl tuple acting on a UMD Banach space X . Then the closure $\mathcal{A} - \alpha \text{Id}$ of the operator $\sum_{k=1}^n A_k^2 - \alpha \text{Id}$ acting on X admits a bounded $H^\infty(\Sigma_\omega)$ functional calculus for any angle $\omega \in (\frac{\pi}{2}, \pi)$.*

Proof : Assume $1 < p < \infty$. Consider the universal Θ -Weyl tuple $A_{\text{univ}}^\Theta = (A_{\text{univ},1}, \dots, A_{\text{univ},n})$ acting on the Banach space $L^p(\mathbb{R}^n)$ and the closure \mathcal{B} of the operator $\sum_{k=1}^n A_{\text{univ},k}^2$. By Corollary

6.7, the tensor extension of $\mathcal{B}_\alpha = (\mathcal{B} - \alpha \text{Id}_{L^p(\mathbb{R}^n)}) \otimes \text{Id}_X$ admits a bounded $H^\infty(\Sigma_\omega)$ functional calculus on the Bochner space $L^p(\mathbb{R}^n, X)$ for any angle $\omega \in (\frac{\pi}{2}, \pi)$.

If $\omega \in (\frac{\pi}{2}, \pi)$ and $f \in H_0^\infty(\Sigma_\omega)$ then using Lemma 11.3 together with Theorem 10.8, we can write $f(\mathcal{A} - \alpha \text{Id})_{\text{HP}} = k_f(A)$ for some function k_f with $\hat{k}_f \in L^1(\mathbb{R}^n)$. We obtain

$$\begin{aligned} \|f(\mathcal{A} - \alpha \text{Id})_{\text{HP}}\|_{X \rightarrow X} &= \|k_f(A)\|_{X \rightarrow X} \stackrel{(4.13), (4.8)}{\leq} M_A^2 \|k_f(A_{\text{univ}}^\Theta)\|_{L^p(\mathbb{R}^n, X) \rightarrow L^p(\mathbb{R}^n, X)} \\ &= M_A^2 \|f(\mathcal{B} - \alpha \text{Id})_{\text{HP}}\|_{L^p(\mathbb{R}^n, X) \rightarrow L^p(\mathbb{R}^n, X)} \lesssim \|f\|_{H^\infty(\Sigma_\omega)}. \end{aligned}$$

In other words, the operator $\mathcal{A} - \alpha \text{Id}$ admits a bounded $H^\infty(\Sigma_\omega)$ functional calculus. \blacksquare

A prominent example of Hörmander spectral multipliers are the Bochner-Riesz means. If $\nu, R > 0$, we can define the function

$$\sigma_R^\nu(\lambda) \stackrel{\text{def}}{=} \left(1 - \frac{\lambda}{R}\right)_+^\nu = \begin{cases} (1 - \frac{\lambda}{R})^\nu & \text{if } 0 < \lambda < R \\ 0 & \text{if } \lambda \geq R \end{cases}.$$

One of the motivations of the Hörmander functional calculus is to determine for which (small) ν the spectral multipliers associated with σ_R^ν are bounded and then to deduce strong convergence to the identity as $R \rightarrow \infty$. With this respect, we obtain the following consequence of Corollary 11.5.

Proposition 11.7 *Suppose that $1 < p < \infty$. Let $A = (A_1, \dots, A_n)$ be a Θ -Weyl tuple on a noncommutative L^p -space associated with an injective von Neumann algebra \mathcal{M} , where $1 < p < \infty$. Let $\mathcal{A} - \alpha \text{Id}$ be the closure of $\sum_{k=1}^n A_k^2 - \alpha \text{Id}$ as in Corollary 11.5. Let $\nu > \frac{n}{2}$. Then the Bochner-Riesz means $\sigma_R^\nu(\mathcal{A} - \alpha \text{Id})$ are uniformly bounded on $L^p(\mathcal{M})$, i.e.*

$$\sup_{R>0} \|\sigma_R^\nu(\mathcal{A} - \alpha \text{Id})\|_{L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})} < \infty.$$

Consequently, for any $f \in L^p(\mathcal{M})$ we have $\sigma_R^\nu(\mathcal{A} - \alpha \text{Id})f \rightarrow f$ in $L^p(\mathcal{M})$ as $R \rightarrow \infty$.

Proof : First note that the function σ_R^ν belongs to the space $\text{Hör}_2^s(\mathbb{R}_+^*)$ for any $\nu > s - \frac{1}{2}$ by [COSY16, p. 234], see also [DeK23, proof of Corollary 3.11]. Thus according to Corollary 11.5, the operator $\sigma_R^\nu(\mathcal{A} - \alpha \text{Id})$ is bounded on the space $L^p(\mathcal{M})$ for any $\nu > \frac{n}{2} + \frac{1}{2} - \frac{1}{2} = \frac{n}{2}$. Then since the $\text{Hör}_2^s(\mathbb{R}_+^*)$ norm is invariant under dilations $m \mapsto m(\cdot/R)$ [KrW18, Lemma 3.2 (3)], we conclude the stated uniform boundedness. We turn to the strong convergence as $R \rightarrow \infty$.

Let $(\dot{\varphi}_n)_{n \in \mathbb{Z}}$ be a dyadic partition of unity in the sense of [KrW16]. For $N \in \mathbb{N}$, let $\Phi_N \stackrel{\text{def}}{=} \sum_{n=-N}^N \dot{\varphi}_n$. According to [KrW16, Proposition 3.4], we have $\Phi_N(\mathcal{A} - \alpha \text{Id})f \rightarrow f$ as $N \rightarrow \infty$, for any $f \in L^p(\mathcal{M})$. Now, choose for $R \geq 4$ the integer N_R such that $\frac{1}{4}R \leq 2^{N_R+1} \leq \frac{1}{2}R$. Note that Φ_{N_R} has its support in $[2^{-N_R-1}, 2^{N_R+1}]$, so that the only point $0 < \lambda = R$ where σ_R^ν is non-smooth does not belong to this support. We want to apply [KrW16, Proposition 3.4] to the spectral multipliers $\sigma_R^\nu \Phi_{N_R}$, $R \rightarrow \infty$ with limit 1. Condition (b) there, $\sigma_R^\nu(\lambda) \Phi_{N_R}(\lambda) \rightarrow 1$ for all $\lambda > 0$, is easily checked. For condition (a), if $R \geq 4$, we note that for any $\beta > 0$, any $K \in \mathbb{N}$, with $K > \beta$,

$$\begin{aligned} \|(\sigma_R^\nu \Phi_{N_R})_e\|_{\mathcal{B}_{\infty, \infty}^\beta} &\lesssim \max_{k=0, \dots, K} \sup_{\lambda > 0} \left| \lambda^k \frac{d^k}{d\lambda^k} (\sigma_R^\nu \Phi_{N_R})(\lambda) \right| \\ &\lesssim \max_{k=0, \dots, K} \sup_{\lambda \in [2^{-N_R-1}, 2^{N_R+1}]} \left| \lambda^k \frac{d^k}{d\lambda^k} \sigma_R^\nu(\lambda) \right| \leq C. \end{aligned}$$

Then by uniform boundedness, for any $f \in L^p(\mathcal{M})$,

$$\begin{aligned} \|\sigma_R^\nu(\mathcal{A} - \alpha\text{Id})f - f\|_{L^p(\mathcal{M})} &\leq \|\sigma_R^\nu(\mathcal{A} - \alpha\text{Id})\Phi_{N_R}(\mathcal{A} - \alpha\text{Id})f - f\|_{L^p(\mathcal{M})} \\ &\quad + \|\sigma_R^\nu(\mathcal{A} - \alpha\text{Id})\Phi_{N_R}(\mathcal{A} - \alpha\text{Id})f - \sigma_R^\nu(\mathcal{A} - \alpha\text{Id})f\|_{L^p(\mathcal{M})} \\ &\lesssim \|\sigma_R^\nu(\mathcal{A} - \alpha\text{Id})\Phi_{N_R}(\mathcal{A} - \alpha\text{Id})f - f\|_{L^p(\mathcal{M})} + \|\Phi_{N_R}(\mathcal{A} - \alpha\text{Id})f - f\|_{L^p(\mathcal{M})} \xrightarrow{R \rightarrow \infty} 0, \end{aligned}$$

according to [KrW16, Proposition 3.4] applied to both summands. \blacksquare

Remark 11.8 In case that $A = (A_1, \dots, A_n)$ form a Θ -Weyl tuple not only on a single Banach space, but consistently on the whole scale of noncommutative L^p -spaces associated with an injective von Neumann algebra \mathcal{M} with $1 < p < \infty$, such that on $L^2(\mathcal{M})$, the closure \mathcal{A} of $\sum_{k=1}^n A_k^2$ is self-adjoint, we can slightly improve on the differentiation order of the Hörmander functional calculus by complex interpolation in Corollary 11.5.

Let $p \in (1, \infty)$ be fixed. Choosing p_0 close to 1 (if $p \leq 2$) or close to ∞ (if $p \geq 2$), we get by Corollary 11.5 that $\mathcal{A} - \alpha\text{Id}$ has a bounded $\text{Hör}_2^s(\mathbb{R}_+^*)$ calculus on L^{p_0} with $s > \frac{n}{2} + \frac{1}{2}$. On L^2 , $\mathcal{A} - \alpha\text{Id}$ has a bounded $\text{Hör}_2^s(\mathbb{R}_+^*)$ calculus with $s > \frac{1}{2}$ since then, $\text{Hör}_2^s(\mathbb{R}_+^*)$ is contained in the space of bounded Borel functions (classical self-adjoint functional calculus). Then by complex interpolation of the functional calculus bilinear mapping $\Phi : L^p \times \text{Hör}_2^s(\mathbb{R}_+^*) \rightarrow L^p$, $(x, m) \mapsto m(\mathcal{A} - \alpha\text{Id})x$ between levels L^{p_0} (with p_0 close to 1 resp. ∞) and L^2 , we deduce that on L^p (with $p < 2$ resp. $p > 2$), $\mathcal{A} - \alpha\text{Id}$ has a bounded $\text{Hör}_2^s(\mathbb{R}_+^*)$ calculus on L^p with

$$s > \begin{cases} \left(\frac{2}{p} - 1\right)\left(\frac{n}{2} + \frac{1}{2}\right) + \frac{1}{p'} & : p \leq 2 \\ \left(\frac{2}{p'} - 1\right)\left(\frac{n}{2} + \frac{1}{2}\right) + \frac{1}{p} & : p \geq 2. \end{cases}$$

See e.g. [Kri09, Proposition 4.83 (2)].

Note that under the previous assumptions, it is even possible to lower the differentiation order still a bit by interpolating $\text{Hör}_2^s(\mathbb{R}_+^*)$ calculus on $L^{1+\varepsilon}$ or L^{p_0} with large p_0 and $\text{Hör}_\infty^\varepsilon(\mathbb{R}_+^*)$ on L^2 . Here one defines the space $\text{Hör}_q^s(\mathbb{R}_+^*)$ in the same way as the space $\text{Hör}_2^s(\mathbb{R}_+^*)$, but over the Sobolev space $W_q^s(\mathbb{R})$. One gets that the operator $\mathcal{A} - \alpha\text{Id}$ has a $\text{Hör}_q^s(\mathbb{R}_+^*)$ functional calculus on L^p for $\frac{1}{q} > \left|\frac{1}{p} - \frac{1}{2}\right|$ and $s = \frac{2}{q}\left(\frac{n}{2} + \frac{1}{2}\right)$.

Example 11.9 Let \mathcal{M} be a von Neumann algebra acting on some Hilbert space H , equipped with a normal semifinite faithful trace. Suppose that $1 \leq p < \infty$. For any $a \in \mathcal{M}$, we can define a bounded linear operator $\mathcal{L}_a : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$ $x \mapsto ax$, called left multiplication by a on $L^p(\mathcal{M})$. In [JMX06, Section 8.A], it is explained how to define a left multiplication operator \mathcal{L}_a in the more general case where $a : \text{dom } a \subset H \rightarrow H$ is a closed densely defined operator with $\rho(a) \neq \emptyset$ and affiliated with \mathcal{M} . If $z \in \rho(a)$, this closed densely defined operator is defined by

$$(11.8) \quad \mathcal{L}_a \stackrel{\text{def}}{=} z - \mathcal{L}_{R(z,a)}^{-1} : D \rightarrow L^p(\mathcal{M})$$

where D is the range of the injective operator $\mathcal{L}_{R(z,a)}$ with dense range. This definition does not depend on z . Moreover $\rho(a) \subset \rho(\mathcal{L}_a)$ and we have

$$R(z, \mathcal{L}_a) = \mathcal{L}_{R(z,a)}, \quad z \in \rho(a).$$

Similarly, we can define the right multiplication operator \mathcal{R}_a by a .

If a and b are two self-adjoint operators affiliated with \mathcal{M} , it is proved in [JMX06, p. 86] that the intersection $\text{dom } \mathcal{L}_a \cap \text{dom } \mathcal{R}_b$ is a dense subspace of the Banach space $L^p(\mathcal{M})$ and that the difference operator

$$\mathcal{L}_a - \mathcal{R}_b: \text{dom } \mathcal{L}_a \cap \text{dom } \mathcal{R}_b \rightarrow L^p(\mathcal{M}), x \mapsto \mathcal{L}_a(x) - \mathcal{R}_b(x)$$

is closable. Following [JMX06, p. 87], we define the closed operator $\text{Ad}_{(a,b)}$ as the closure of $\mathcal{L}_a - \mathcal{R}_b$, i.e.

$$(11.9) \quad \text{Ad}_{(a,b)} \stackrel{\text{def}}{=} \overline{\mathcal{L}_a - \mathcal{R}_b}.$$

Note that by [JMX06, Lemma 8.10 p. 87], the operator $i\text{Ad}_{(a,b)}$ generates a strongly continuous group of isometries $(U_t)_{t \in \mathbb{R}}$ acting on the Banach space $L^p(\mathcal{M})$, where $U_t: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$, $x \mapsto e^{ita} x e^{-itb}$ for any $t \in \mathbb{R}$.

Now, consider two finite commuting families (a_1, \dots, a_n) and (b_1, \dots, b_n) of self-adjoint operators affiliated with \mathcal{M} , i.e. $a_i a_j = a_j a_i$ and $b_i b_j = b_j b_i$ for any $1 \leq i, j \leq n$. Suppose that $1 < p < \infty$. For any $1 \leq j \leq n$, we consider the closed operator $A_j \stackrel{\text{def}}{=} \text{Ad}_{(a_j, b_j)}$ acting on the Banach space $L^p(\mathcal{M})$. As observed in the proof of [JMX06, Theorem 8.12 p. 87], the operators (A_1, \dots, A_n) are pairwise commuting in the resolvent sense, i.e.

$$R(z_1, A_i)R(z_2, A_j) = R(z_2, A_j)R(z_1, A_i), \quad z_1 \in \rho(A_i), z_2 \in \rho(A_j), i, j \in \{1, \dots, n\}.$$

With the results stated in [Dav07, p. 176], it is easy to deduce that the strongly continuous groups $(e^{itA_1})_{t \in \mathbb{R}}, \dots, (e^{itA_n})_{t \in \mathbb{R}}$ commute. So, we have a 0_n -Weyl tuple in the sense of Definition 3.1. If \mathcal{M} is *injective*, we deduce by Corollary 11.5 that the «sum of squares» $\mathcal{A} \stackrel{\text{def}}{=} A_1^2 + \dots + A_n^2$ admits a bounded Hörmander $\text{Hör}_2^s(\mathbb{R}_+^*)$ functional calculus of order $s > \frac{n}{2} + \frac{1}{2}$ on the Banach space $L^p(\mathcal{M})$, recovering and improving the boundedness of the $H^\infty(\Sigma_\theta)$ functional calculus for any angle $\theta > 0$ provided by [JMX06, Theorem 8.12 p. 87].

We obtain another generalization if we assume that the strongly continuous groups of unitaries $(e^{ita_1})_{t \in \mathbb{R}}, \dots, (e^{ita_n})_{t \in \mathbb{R}}$ admit a commutation relation

$$e^{ita_j} e^{isa_k} = e^{its\Theta^{(1)}} e^{isa_k} e^{ita_j}$$

and similarly,

$$e^{itb_j} e^{isb_k} = e^{its\Theta^{(2)}} e^{isb_k} e^{itb_j}.$$

In this case, the strongly continuous groups $(e^{itA_1})_{t \in \mathbb{R}}, \dots, (e^{itA_n})_{t \in \mathbb{R}}$ form a $(\Theta^{(1)} - \Theta^{(2)})$ -Weyl tuple over the Banach space $L^p(\mathcal{M})$.

Example 11.10 The preceding example applies in particular to Schur multipliers. Namely, we consider the case $\mathcal{M} = B(L^2(\mathbb{R}))$. Recall that the affiliated operators with this von Neumann algebra are precisely the closed densely defined operators on the Hilbert space $L^2(\mathbb{R})$. If $a, b: \mathbb{R} \rightarrow \mathbb{R}$ are some real-valued measurable functions, then by the multiplication operators $\text{Mult}_a: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $f \mapsto af$ and $\text{Mult}_b: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $f \mapsto bf$ are self-adjoint operators affiliated with the von Neumann algebra $B(L^2(\mathbb{R}))$. We obtain that the operator $A_1 \stackrel{\text{def}}{=} \text{Ad}_{(a,b)}$ is a 0_1 -Weyl tuple over the Schatten space $S_{\mathbb{R}}^p \stackrel{\text{def}}{=} S^p(B(L^2(\mathbb{R})))$ in the sense of Definition 3.1 for any $1 < p < \infty$. Moreover, by an «unbounded version» of [Arh24, Lemma 5.1], the closed unbounded operator A_1 is a Schur multiplier with symbol $a - b: \mathbb{R} \rightarrow \mathbb{R}$. So Corollary 11.5 yields a Hörmander $\text{Hör}_2^s(\mathbb{R}_+^*)$ functional calculus of order $s > 1$ for the Schur multiplier with symbol $(a - b)^2$.

In particular, with the choice $a(x) = x$ and $b(y) = y$, Corollary 11.5 implies that if $f \in \text{Hör}_2^s(\mathbb{R}_+^*)$ then the symbol $\psi: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$, $(x, y) \mapsto f((x - y)^2)$ induces a bounded Schur multiplier $M_\psi: S_{\mathbb{R}}^p \rightarrow S_{\mathbb{R}}^p$ on the Schatten class $S_{\mathbb{R}}^p$. Actually, as stated in [Kri09, Proposition 4.8 (4)], the mapping $f \mapsto f((\cdot)^2)$ is an isomorphism of $\text{Hör}_2^s(\mathbb{R}_+^*)$ for any $s > \frac{1}{2}$. So the preceding assertion implies that if $s > 1$ each $f \in \text{Hör}_2^s(\mathbb{R}_+^*)$ induces a bounded Schur multiplier $M_\psi: S_{\mathbb{R}}^p \rightarrow S_{\mathbb{R}}^p$ with $\psi: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$, $(x, y) \mapsto f(|x - y|)$. Moreover, if $f: \mathbb{R} \rightarrow \mathbb{C}$ is a function such that both $f1_{\mathbb{R}_+}$ and $f(-\cdot)1_{\mathbb{R}_+}$ belong to $\text{Hör}_2^s(\mathbb{R}_+^*)$, then $\psi(x, y) = f(x - y) = 1_{x-y \geq 0}f(|x - y|) + 1_{x-y < 0}f(-|x - y|)$ induces a bounded Schur multiplier $M_\psi: S_{\mathbb{R}}^p \rightarrow S_{\mathbb{R}}^p$. Indeed, the symbols $1_{x-y \geq 0}$ and $1_{x-y < 0}$ induce S^p (completely) bounded operators by the transference [CdS15] to the Fourier multiplier of the Hilbert transform which is indeed completely bounded on $L^p(\mathbb{R})$ since S^p has the UMD property [HvNVW16, Proposition 5.4.2]. Note that this is a particular case of the recent result [CGPT23, Theorem A'] on bounded Schur multipliers on $S_{\mathbb{R}^n}^p \stackrel{\text{def}}{=} S^p(\mathcal{B}(L^2(\mathbb{R}^n)))$ associated to non-necessarily Toeplitz symbols as we consider hereabove.

The same method applies to discrete Schur multipliers. If $\mathcal{M} = \mathcal{B}(\ell^2)$, consider some diagonal matrices $a_1, \dots, a_n, b_1, \dots, b_n$ with real entries $a_{j,k}$ and $b_{j,k}$ on the diagonal. Then these matrices are self-adjoint operators affiliated with the von Neumann algebra $\mathcal{B}(\ell^2)$. We obtain that the operators $A_j = \text{Ad}_{(a_j, b_j)}$ form a 0_n -Weyl tuple over the Schatten space $S^p \stackrel{\text{def}}{=} S^p(\ell^2)$ in the sense of Definition 3.1 for $1 < p < \infty$. Moreover, the operator A_j is a Schur multiplier with symbol $(a_{j,k} - b_{j,l})_{kl}$. So Corollary 11.5 yields a Hörmander $\text{Hör}_2^s(\mathbb{R}_+^*)$ functional calculus of order $s > \frac{n}{2} + \frac{1}{2}$ for the Schur multiplier with symbol

$$\left(\sum_{j=1}^n (a_{j,k} - b_{j,l})^2 \right)_{kl}.$$

In particular, with $n = 1$ and $a_{1,k} = b_{1,k} = k$, we obtain that a Toeplitz Schur multiplier with symbol $f((k - l)^2)$ induces a bounded operator on S^p provided that $f \in \text{Hör}_2^s(\mathbb{R}_+^*)$ with $s > 1$. As in the continuous case, we obtain that a (Toeplitz) Schur multiplier with symbol $f(k - l)$ is bounded on S^p for any function $f \in \text{Hör}_2^s(\mathbb{R}_+^*)$ if $s > 1$.

12 Functional calculus for the harmonic oscillator on the Moyal plane

Let $d \in \mathbb{N}$. Consider a skew-symmetric matrix $\Theta \in M_d(\mathbb{R})$. Recall that the d -dimensional Quantum Euclidean space can be defined as the twisted group von Neumann algebra $\text{VN}(\mathbb{R}^d, \sigma_\Theta)$ where $\sigma_\Theta: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{T}$ is the 2-cocycle defined in (1.2). This von Neumann algebra, denoted \mathbb{R}_Θ^d or $L^\infty(\mathbb{R}_\Theta^d)$, is generated by the unitary operators $\lambda_{\Theta, s}: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ where $s \in \mathbb{R}^d$ defined by

$$(12.1) \quad (\lambda_{\Theta, s} \xi)(t) \stackrel{(2.9)}{=} e^{\frac{1}{2}i\langle s, \Theta t \rangle} \xi(t - s), \quad s, t \in \mathbb{R}^d, \xi \in L^2(\mathbb{R}^d).$$

We have the following relation

$$(12.2) \quad \lambda_{\Theta, s} \lambda_{\Theta, t} \stackrel{(2.10)}{=} e^{\frac{1}{2}i\langle s, \Theta t \rangle} \lambda_{\Theta, s+t}, \quad s, t \in \mathbb{R}^d.$$

This twisted group von Neumann algebra is also generated by the operators $\lambda_\Theta(f): L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ where $f \in L^1(\mathbb{R}^d)$ is defined by $\lambda_\Theta(f) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} f(s) \lambda_{\Theta, s} ds$. The Schwartz space $\mathcal{S}(\mathbb{R}_\Theta^d)$

is defined as the subspace of $L^\infty(\mathbb{R}_\Theta^d)$ which is the image of the classical Schwartz space $\mathcal{S}(\mathbb{R}^d)$ under the map λ_Θ , that is

$$\mathcal{S}(\mathbb{R}_\Theta^d) \stackrel{\text{def}}{=} \{\lambda_\Theta(f) : f \in \mathcal{S}(\mathbb{R}^d)\}.$$

Recall that by [GJM22, Proposition 2.4] and [GJP21, Proposition 1.1] the functional $\tau_\Theta : \mathcal{S}(\mathbb{R}_\Theta^d) \rightarrow \mathbb{C}$, $\lambda_\Theta(f) \mapsto f(0)$ extends uniquely to a normal semifinite faithful trace on the von Neumann algebra $L^\infty(\mathbb{R}_\Theta^d)$. This trace allows us to define the associated noncommutative L^p -spaces $L^p(\mathbb{R}_\Theta^d)$.

For any $t \in \mathbb{R}^d$, we consider the multiplication operator $\mathcal{T}_t : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$, $f \mapsto e^{i\langle t, \cdot \rangle} f(\cdot)$.

Proposition 12.1 *For any $s, t \in \mathbb{R}^d$, we have*

$$\mathcal{T}_t \lambda_{\Theta, s} \mathcal{T}_t^* = e^{i\langle t, s \rangle} \lambda_{\Theta, s}.$$

Proof : For any $r, s, t \in \mathbb{R}^d$ and any $\xi \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned} (\mathcal{T}_t \lambda_{\Theta, s} \mathcal{T}_t^*(\xi))(r) &= e^{i\langle t, r \rangle} (\lambda_{\Theta, s} \mathcal{T}_t^*(\xi))(r) \stackrel{(12.1)}{=} e^{i\langle t, r \rangle} e^{\frac{1}{2}i\langle s, \Theta r \rangle} \mathcal{T}_t^*(\xi)(r - s) \\ &= e^{i\langle t, r \rangle} e^{\frac{1}{2}i\langle s, \Theta r \rangle} e^{-i\langle t, r - s \rangle} \xi(r - s) = e^{i\langle t, s \rangle} e^{\frac{1}{2}i\langle s, \Theta r \rangle} \xi(r - s) \stackrel{(12.1)}{=} e^{i\langle t, s \rangle} \lambda_{\Theta, s}(\xi)(r). \end{aligned}$$

■

Consequently, we have a spatial $*$ -automorphism $\mathcal{T}_t : L^\infty(\mathbb{R}_\Theta^d) \rightarrow L^\infty(\mathbb{R}_\Theta^d)$, $\lambda_{\Theta, s} \mapsto e^{i\langle t, s \rangle} \lambda_{\Theta, s}$ of $L^\infty(\mathbb{R}_\Theta^d)$, hence a weak $*$ continuous group of $*$ -automorphisms by [Tak03, p. 238]. Note that this map is trace preserving since

$$\tau_\Theta(\mathcal{T}_t(\lambda_\Theta(f))) = \tau_\Theta(\lambda_\Theta(e^{i\langle t, \cdot \rangle} f)) = f(0) = \tau_\Theta(\lambda_\Theta(f)).$$

In particular, we obtain a strongly continuous group $(\mathcal{T}_{t,p})_{t \in \mathbb{R}^d}$ of isometries on each noncommutative L^p -space $L^p(\mathbb{R}_\Theta^d)$. We denote by $i\partial_k$ the generator of the strongly continuous group $(\mathcal{T}_{(0, \dots, t, \dots, 0), p})_{t \in \mathbb{R}}$. Note that for any $s \in \mathbb{R}^d$ and any $t \in \mathbb{R}$ we have

$$(12.3) \quad e^{it\partial_k} \lambda_{\Theta, s} = \mathcal{T}_{te_k} \lambda_{\Theta, s} = e^{i\langle te_k, s \rangle} \lambda_{\Theta, s} = e^{its_k} \lambda_{\Theta, s}.$$

For any $1 \leq k \leq d$ and any $t \in \mathbb{R}$ we let $u_k(t) \stackrel{\text{def}}{=} \lambda_{\Theta, (0, 0, \dots, t, \dots, 0)}$. Then $(u_k(t))_{t \in \mathbb{R}}$ is a one-parameter strongly continuous group of unitaries of $L^\infty(\mathbb{R}_\Theta^d)$. We denote by x_k the self-adjoint operator satisfying $u_k(t) = e^{itx_k}$ for any $t \in \mathbb{R}$. By [JMX06, Proposition 8.5 (1)], the unbounded operator x_j is affiliated with the von Neumann algebra $L^\infty(\mathbb{R}_\Theta^d)$. Using [JMX06, Proposition 8.5 (2)], we see that the family of maps $L^p(\mathbb{R}_\Theta^d) \rightarrow L^p(\mathbb{R}_\Theta^d)$, $y \mapsto e^{itx_k} y$ (for $t \in \mathbb{R}$) defines a strongly continuous bounded group of operators on $L^p(\mathbb{R}_\Theta^d)$. Since the operators e^{itx_k} are unitaries, the group consists in fact of isometries on $L^p(\mathbb{R}_\Theta^d)$. We denote by X_k the unbounded operator acting on $L^p(\mathbb{R}_\Theta^d)$ such that

$$(12.4) \quad e^{itX_k}(y) = e^{itx_k} y, \quad y \in L^p(\mathbb{R}_\Theta^d), t \in \mathbb{R}.$$

Using again [JMX06, Proposition 8.5 (2)], we note that the generator of the group $(e^{itX_k})_{t \in \mathbb{R}}$ is the left multiplication operator by ix_k in the sense of [JMX06, (8.3)]. Note that the formula (12.4) implies that

$$(12.5) \quad e^{itX_k}(y) = \lambda_{\Theta, te_k} y, \quad y \in L^p(\mathbb{R}_\Theta^d), t \in \mathbb{R}$$

where e_k is the k^{th} -vector of the canonical basis of the space \mathbb{R}^d .

The groups of operators associated with noncommutative spatial variables and momentum variables satisfy the following commutation relations.

Proposition 12.2 For any $s, t \in \mathbb{R}$ and any $j, k \in \{1, \dots, d\}$, we have

$$(12.6) \quad e^{isX_j} e^{itX_k} = e^{ist\Theta_{jk}} e^{itX_k} e^{isX_j},$$

$$(12.7) \quad e^{is\partial_j} e^{it\partial_k} = e^{it\partial_k} e^{is\partial_j},$$

$$(12.8) \quad e^{it\partial_k} e^{isX_j} = e^{its\delta_{j=k}} e^{isX_j} e^{it\partial_k}.$$

Proof : We start with (12.6). For any $y \in L^p(\mathbb{R}_\Theta^d)$, we have on the one hand

$$e^{isX_j} e^{itX_k} y \stackrel{(12.5)}{=} \lambda_{\Theta, se_j} \lambda_{\Theta, te_k} y \stackrel{(12.2)}{=} e^{\frac{1}{2}i\langle se_j, \Theta te_k \rangle} \lambda_{\Theta, se_j + te_k} y = e^{\frac{1}{2}ist\Theta_{jk}} \lambda_{\Theta, se_j + te_k} y.$$

On the other hand, we have

$$\begin{aligned} e^{ist\Theta_{jk}} e^{itX_k} e^{isX_j} y &\stackrel{(12.5)}{=} e^{ist\Theta_{jk}} \lambda_{\Theta, te_k} \lambda_{\Theta, se_j} y \stackrel{(12.2)}{=} e^{ist\Theta_{jk}} e^{\frac{1}{2}i\langle te_k, \Theta se_j \rangle} \lambda_{\Theta, te_k + se_j} y \\ &= e^{ist\Theta_{jk}} e^{\frac{1}{2}ist\Theta_{kj}} \lambda_{\Theta, te_k + se_j} y = e^{\frac{1}{2}ist\Theta_{jk}} \lambda_{\Theta, se_j + te_k} y. \end{aligned}$$

Let us turn to the equality (12.7). For any $r \in \mathbb{R}^d$ we have

$$e^{is\partial_j} e^{it\partial_k} \lambda_{\Theta, r} \stackrel{(12.3)}{=} e^{itr_k} e^{is\partial_j} \lambda_{\Theta, r} \stackrel{(12.3)}{=} e^{itr_k} e^{itr_j} \lambda_{\Theta, r} \stackrel{(12.3)}{=} e^{it\partial_k} e^{is\partial_j} \lambda_{\Theta, r},$$

from which we deduce (12.7). Finally, we shall prove (12.8). It suffices by density and linearity to check the commutation relations on each $\lambda_{\Theta, r}$ where $r \in \mathbb{R}^d$. Then on the one hand

$$e^{isX_j} e^{it\partial_k} \lambda_{\Theta, r} \stackrel{(12.3)}{=} e^{itr_k} e^{isX_j} \lambda_{\Theta, r} \stackrel{(12.5)}{=} e^{itr_k} \lambda_{\Theta, se_j} \lambda_{\Theta, r} \stackrel{(12.2)}{=} e^{itr_k} e^{\frac{1}{2}i\langle se_j, \Theta r \rangle} \lambda_{\Theta, se_j + r}.$$

On the other hand, a last computation gives

$$\begin{aligned} e^{it\partial_k} e^{isX_j} \lambda_{\Theta, r} &\stackrel{(12.5)}{=} e^{it\partial_k} \lambda_{\Theta, se_j} \lambda_{\Theta, r} \\ &\stackrel{(12.2)}{=} e^{\frac{1}{2}i\langle se_j, \Theta r \rangle} e^{it\partial_k} \lambda_{\Theta, se_j + r} \stackrel{(12.3)}{=} e^{\frac{1}{2}i\langle se_j, \Theta r \rangle} e^{it(r_k + s\delta_{k=j})} \lambda_{\Theta, se_j + r}. \end{aligned}$$

■

Remark 12.3 It is obvious that the relation (12.6) is equivalent to the commutation rule $e^{isx_j} e^{itx_k} = e^{ist\Theta_{jk}} e^{itx_k} e^{isx_j}$ for any $s, t \in \mathbb{R}$ and any $j, k \in \{1, \dots, d\}$. By Proposition 2.4 we deduce that each subspace $\text{dom}(x_j x_k) \cap \text{dom}(x_k x_j)$ is dense in the Banach space $L^p(\mathbb{R}_\Theta^d)$ and the commutation relation (1.3) is true on $\text{dom}(x_j x_k) \cap \text{dom}(x_k x_j)$.

Theorem 12.4 Let $d \in \mathbb{N}$. Consider some skew-symmetric matrix $\Theta \in M_d(\mathbb{R})$. Then the family $A = (X_1, \dots, X_d, \partial_1, \dots, \partial_d)$ defines a $\begin{pmatrix} \Theta & -I_d \\ I_d & 0 \end{pmatrix}$ -Weyl tuple on the Banach space $L^p(\mathbb{R}_\Theta^d)$ for any $1 < p < \infty$. Consequently, the closure $\mathcal{A} - \alpha \text{Id}$ of the operator $\sum_{k=1}^d (X_k^2 + \partial_k^2) - \alpha \text{Id}$ admits a bounded Hörmander $\text{Hör}_2^s(\mathbb{R}_+^*)$ functional calculus of order $s > d + \frac{1}{2}$ on the space $L^p(\mathbb{R}_\Theta^d)$, where $\alpha \geq 0$ is defined in (6.3).

Proof : With Proposition 12.2, the verification of (3.1) is obvious. Then the Hörmander functional calculus is a direct consequence of Corollary 11.5, observing that the von Neumann algebra \mathbb{R}_Θ^d is injective (see e.g. [LSZ20, Section 6]).

■

Remark 12.5 Note that the family $A = (X_1, \dots, X_d, \partial_1, \dots, \partial_d)$ from Theorem 12.4 forms a consistent Weyl tuple on $L^p(\mathbb{R}_\Theta^d)$ for any $1 < p < \infty$ such that on $L^2(\mathbb{R}_\Theta^d)$, $\mathcal{A} - \alpha \text{Id}$ is self-adjoint. Thus Remark 11.8 applies and we can improve the order of the Hörmander functional calculus by means of complex interpolation. We get for $\mathcal{A} - \alpha \text{Id}$ a $\text{Hör}_2^s(\mathbb{R}_+^*)$ calculus on $L^p(\mathbb{R}_\Theta^d)$, $1 < p < \infty$ with

$$s > \begin{cases} (\frac{2}{p} - 1)(d + \frac{1}{2}) + \frac{1}{p'} : p \leq 2 \\ (\frac{2}{p'} - 1)(d + \frac{1}{2}) + \frac{1}{p} : p \geq 2. \end{cases}$$

Magnetic Weyl system The concept of a gauge-covariant Magnetic Weyl calculus, essential for analysing magnetic systems, was initially introduced in a preliminary form by Müller [Mul99]. This framework gained a more solid foundation through the independent efforts of Mantoiu and Purice [MaP04]. This mathematical framework proves to be invaluable in modelling both classical and quantum particles within the classical Euclidean space \mathbb{R}^d , when considering the impact of a magnetic field B . For a more comprehensive understanding, the survey [MaP06], along with the papers [MPR05], [Lei11], and [LeL22] and their respective references, offer in-depth insights.

In this setting, the self-adjoint position operators X_j and momentum operators ∂_j^B for magnetic Weyl calculus are characterized by the commutation relations

$$[X_j, X_k] = 0, \quad i[\partial_j^B, \partial_k^B] = B_{jk}, \quad i[X_j, \partial_k^B] = \delta_{j=k}, \quad 1 \leq j, k \leq d,$$

see [MaP04, (2.26)]. Here $B = \sum_{j,k=1}^d B_{jk} dx_j \wedge dx_k$ is a *constant* magnetic field on \mathbb{R}^d . The difference to non-magnetic systems is that momenta along different directions no longer commute. Of course, we can state a version of Theorem 12.4 for the harmonic oscillator associated with this system using the matrix $\begin{pmatrix} 0 & -I_d \\ I_d & B \end{pmatrix}$.

Magnetic Weyl system on the Moyal planes More generally, the authors of the paper [GJM22] consider unbounded self-adjoint operators $\partial_1^B, \dots, \partial_d^B$ on $L^p(\mathbb{R}_\Theta^d)$ satisfying the conditions $i[\partial_j^B, \partial_k^B] = \delta_{j=k}$ and $i[\partial_j^B, \partial_k^B] = B_{jk}$, where B is some skew-symmetric matrix of $M_n(\mathbb{R})$.

Consequently, the set $A = (X_1, \dots, X_d, \partial_1^B, \dots, \partial_d^B)$ constitutes a $\begin{pmatrix} \Theta & -I_d \\ I_d & B \end{pmatrix}$ -Weyl tuple. The clarification of the construction of these operators is out of the scope of this paper. However, one can obviously state a version of Theorem 12.4 for the harmonic oscillator associated with such system.

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