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# Event-triggered gain scheduling of $2 \times 2$ hyperbolic PDEs with time and space varying coupling coefficients

Jean Auriol<sup>1</sup>, Nicolas Espitia<sup>2</sup>,

**Abstract**—In this paper, we address the problem of exponential stabilization of  $2 \times 2$  hyperbolic PDEs systems with time- and space-varying in-domain coupling coefficients using event-triggered gain scheduling. More precisely, we sample the coupling terms according to a Lyapunov-based event-triggering condition. At each triggering time, we define the control input as the classical static backstepping control law that would stabilize the system, thereby scheduling the gains of the controller according to the triggering mechanism while solely considering the spatial variation of the coefficients. We prove that we avoid the Zeno phenomenon under the event-triggering policy, provided that the coupling coefficients are slowly time-varying. The closed-loop exponential stability is shown using a Lyapunov analysis. Unlike existing results in the literature, the proposed approach does not require solving time-varying backstepping kernel equations in real-time, which implies a smaller computational burden and better applicability.

## I. INTRODUCTION

Hyperbolic partial differential equations (PDEs) play a crucial role in the mathematical description of complex systems involving transport phenomena with finite propagation speeds (e.g., transport of matter, sound waves, and information). This class of equations finds significant utility in modeling physical networks, including hydraulic [9, Chapter 8], road traffic [25], [47], [46], [45], gas flow pipelines [26], oil drilling [1] among others.

Control design for complex systems modeled by PDEs constitutes a central research focus. A traditional way to act on those systems is through *boundary control*. In this context, *the backstepping method* has been used as a standard and powerful tool to design stabilizing controllers, offering several key advantages. This method enables the expression of explicit control laws for an entire class of systems; it allows for extension to parameter-adaptive use with real-time parameter estimation and demonstrates the potential to achieve the enhanced type of convergence, including the desirable outcomes of *finite-time convergence*. This method initially emerged to deal with 1D reaction-diffusion parabolic PDEs in [12], [39], and since then, it has been employed to deal with the boundary stabilization of broader classes of PDEs (for an overview see [30]). PDE backstepping usually uses Volterra transformations to map the PDE system into a suitable *target PDE* system -with desired stability properties - enabling subsequent Lyapunov stability analysis. For 1D hyperbolic PDEs, the backstepping approach has been instrumental in solving the problem of boundary stabilization in the presence of coupling terms. For example, [40] solves the problem of stabilization of  $2 \times 2$  linear hyperbolic PDEs by output feedback. The result has been further extended to

local exponential stabilization of  $2 \times 2$  quasi-linear hyperbolic systems [17], first order  $n \times n$  inhomogeneous quasi-linear hyperbolic systems [27], robust control design [6], [8], finite-time output regulation [18], adaptive control for hyperbolic PDEs [3], and intricate dynamics, including couplings with ordinary differential equations [20]. Additionally, the backstepping methodology has been the object of further advances, which include the use of Fredholm transformations [13], [15], [35].

Nevertheless, the contributions above primarily address time-invariant coupling parameters (e.g., constant or spatially varying only). Dealing with more general linear hyperbolic PDE systems, which feature time- and space-varying in-domain and boundary coupling coefficients, poses a more significant challenge for control and estimation. Such hyperbolic PDEs with time- and space-varying coefficients may come into play in numerous applications such as physical models of balance laws linearized around time-varying trajectories as plug flow chemical reactors equations, Saint-Venant equations or heat exchangers equations (see, e.g., [10]). They also occur in trajectory planning problems.

Very few contributions exist for backstepping-based control and estimation of hyperbolic systems with time- and space-varying coefficients. In [19], an observer was proposed for a hyperbolic partial integro-differential equation (PIDEs). As for boundary stabilization and tracking for hyperbolic PDEs, [2] and subsequently [4] are the first contributions (to our knowledge) that achieved finite-time stabilization using a time-varying backstepping transformation. More recently, [16] addressed the problem of finite-time boundary stabilization for general hyperbolic systems of balance laws with both time- and space-varying coupling coefficients and transport speeds. The proposed approach employed time-dependent Volterra and Fredholm transformations that enabled transforming the original system into suitable target systems, one in "control normal form" (in the same spirit as in [14], [36], [37]) and the other with simple coupling structure exhibiting a finite-time stability property. They performed a rigorous analysis of the well-posedness of the multi-dimensional hyperbolic PDEs of the resulting kernel equations. However, to implement the resulting controllers on real systems, it is necessary to solve the proposed set of time-varying kernel equations in real-time, which can be computationally expensive and reduce the applicability of the approach.

One potential approach to address the complexity associated with time-varying kernels while achieving exponential stabilization is to schedule the kernel gains at specific time instants, which we will refer to as *triggering times* hereafter. Instead of handling a time-varying kernel capturing the time- and space-varying coefficient, we use a simpler kernel capturing only the spatial variation of the coupling coefficients. This is possible as long as the coupling coefficients

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are sampled in time, thus reducing the kernel-PDEs to a form involving space-varying coefficients only between two successive sampling or triggering times. This fundamental idea was introduced in [28], which solved the problem of exponential stabilization reaction-diffusion PDE with time- and space-varying coefficients using event-triggered gain scheduling. The kernel computation was scheduled aperiodically when needed and relied on the current state information of the closed-loop system and the time- and space-varying reaction coefficient, which was considered as a distributed input disturbance.

In this paper, we adapt the method of [28] to address the problem of exponential stabilization of  $2 \times 2$  hyperbolic PDEs systems with time- and space-varying in-domain coupling coefficients using event-triggered gain scheduling. It is worth mentioning that, as in [28], this approach draws inspiration from the framework of event-triggered control for PDEs (see, e.g., [23], [21], [43], [42], [22] for hyperbolic PDEs and [38], [29], [24], [34] for parabolic PDEs). Although in this work we do not deal with event-triggered *control* (the state value of the control is not updated on events) but gain scheduling, the main theoretical questions underlying the design of event-triggered mechanisms, such avoidance of the Zeno phenomena, well-posedness issues, and robustness to the sampling schedule while guaranteeing exponential stability, persist within this framework.

In this paper, we propose a scheme wherein we treat the time- and space-varying in-domain coupling coefficients as external (multiplicative) disturbances integrated into the control and event-triggered mechanism; as such, our proposed scheme is a feedforward controlled one (see, e.g., [11]) that compensate for the disturbances. We sample the coupling terms according to a Lyapunov-based event-triggering condition. Therefore, at each triggering time, we recalculate the kernels of the Volterra backstepping transformation, thereby scheduling the gains of the controller according to the triggering mechanism while solely considering the spatial variation of the coefficients. Following the backstepping transformation, the resulting target system exhibits errors due to sampling. However, by ensuring these errors remain sufficiently small, we can still guarantee both exponential stabilization and the well-posedness of the closed-loop system. We prove that under the even-triggering policy, we avoid the Zeno phenomenon, provided that the coupling coefficients are slowly time-varying.

The structure of the present work is as follows. In Section II, we present the system under consideration, the control objectives, and the proposed controller based on an event-triggered gain scheduling mechanism. This triggering condition determines the time instants at which the coefficients must be sampled and, thereby when the kernel computation/updates must be done. It is designed in Section III-A. We also prove the existence of a minimal dwell time (therefore guaranteeing there is no Zeno phenomenon) and the closed-loop stability using a Lyapunov analysis. Finally, we present some simulation results in Section IV.

## II. PROBLEM DESCRIPTION AND CONTROL DESIGN

### A. System under consideration

Consider the following  $2 \times 2$  linear hyperbolic system

$$\partial_t u(t, x) + \lambda(x) \partial_x u(t, x) = \sigma^+(t, x) v(t, x), \quad (1)$$

$$\partial_t v(t, x) - \mu(x) \partial_x v(t, x) = \sigma^-(t, x) u(t, x), \quad (2)$$

with the boundary conditions

$$u(t, 0) = qv(t, 0), \quad v(t, 1) = \rho u(t, 1) + U(t), \quad (3)$$

where  $(u(t, x), v(t, x))^T$  is the state of the system, the different arguments evolving in  $\{(t, x) \text{ s.t. } t > t_0, x \in [0, 1]\}$ , where  $t_0 > 0$  is the initial time. The function  $U(t)$  corresponds to the boundary actuation. The velocities  $\lambda(x) > 0$  and  $\mu(x) > 0$  belong to  $C^1([0, 1])$  and are assumed to satisfy  $-\mu(x) < 0 < \lambda(x)$  for all  $x \in [0, 1]$ . The time- and space-varying coupling coefficients  $\sigma^+$  and  $\sigma^-$  belong to  $C^0([0, \infty) \times [0, 1]) \cap L^\infty((0, \infty) \times [0, 1])$ , while the boundary couplings  $\rho$  (the proximal reflection) and  $q$  (the distal reflection) are considered as constants. We assume that all of the time-varying coupling terms are bounded, i.e., there exists a constant  $M_\sigma$  such that

$$|\sigma^-(t, x)|, |\sigma^+(t, x)| < M_\sigma, \quad (4)$$

Furthermore, we consider the following assumptions:

*Assumption 1:* The boundary coupling terms verify  $|\rho q| < 1$ . We also assume that  $q \neq 0$ .

This assumption means that the boundary conditions are *dissipative*. Such an assumption is necessary to guarantee the delay-robustness of the closed-loop system (i.e., the closed-loop system needs to be robust to the introduction of an arbitrarily small delay in the loop), as shown in [7], [8], [32]. Having  $q \neq 0$  is not necessary but simplifies the backstepping transformation we use later in the paper.

*Assumption 2:* There exist a constant  $\varphi_\sigma > 0$  such that the following inequalities hold for all  $x \in [0, 1]$ , and all  $t, s \geq t_0$

$$|\sigma^+(t, x) - \sigma^+(s, x)| \leq \varphi_\sigma |t - s|, \quad (5)$$

$$|\sigma^-(t, x) - \sigma^-(s, x)| \leq \varphi_\sigma |t - s|. \quad (6)$$

This assumption means that the different time-varying coefficients are Lipschitz with respect to time.

*Remark 1:* Similarly to what has been proposed in [28], we will not consider the system (1)-(3) as a time-varying system but as a time-invariant system subject to the control input  $U(t)$  and (multiplicative) disturbance inputs (i.e., the coupling coefficients  $\sigma^-(t, x)$ ,  $\sigma^+(t, x)$ ). Since we consider (1)-(3) as a time-invariant system, we can assume that the initial time is zero. Therefore, the proposed control scheme may be seen as a feedforward control one that compensates for the effect of the disturbance inputs.

*Remark 2:* The approach we present in this paper could be extended to the case of time-varying boundary coupling terms ( $\rho(t)$  and  $q(t)$ ). However, the computations become more involved in this case. For the sake of clarity and concision, we chose to consider these coupling terms as constants in this paper.

### B. Control objective

It is worth recalling that for hyperbolic systems with time-independent coefficients, a stabilizing boundary control law was designed in [40], [17] using the backstepping approach and appropriate invertible Volterra transformations. The extension to time-dependent coefficients has been proposed in [16] using time-varying backstepping transformations. In this case, the resulting kernel PDE equations contain a time derivative of the kernel and involve the time- and space-varying coefficients. The resulting theoretical proof of well-posedness becomes involved and technical. Moreover, contrary to the case of time-independent backstepping kernels

that can be computed offline, having time-varying kernels raises critical issues regarding the practical implementation of the corresponding time-varying feedback control law. Indeed, it implies solving this set of time-varying kernel equations in real time, which may be computationally challenging. Note that such an implementation problem has been partially considered for time-varying parabolic systems by employing specific numerical strategies such as the method of successive approximations (see, e.g., [33]-[41]).

Our objective is to stabilize the system (1)-(3) exponentially in the sense of the  $L^2$ -norm using a suitable backstepping-based boundary controller while avoiding solving time-varying kernel equations as those given in [16]. This is possible by scheduling the backstepping kernel gains at a specific increasing sequence of times. We adjust the approach proposed in [28] to design an **event-triggered gain scheduling controller**. More precisely, let us choose a given time instants sequence  $t_j$  ( $j \in \mathbb{N}$ ). To design our feedback controller, we will consider that all the time-varying coefficients are constant on the time interval  $[t_j, t_{j+1})$ , their values being set to their respective values at time  $t_j$ . We then apply the backstepping methodology presented in [17] and solve the corresponding kernel equations at  $t = t_j$  to obtain the candidate control law.

### C. Backstepping control design

Let  $\{t_j\}_{j \in \mathbb{N}}$  be an increasing sequence of times with  $t_0 = 0$ . For all  $x \in [0, 1]$ , we denote

$$\sigma_j^+(x) := \sigma^+(t_j, x), \quad \sigma_j^-(x) := \sigma^-(t_j, x), \quad (7)$$

which are the sampled versions of the different coupling coefficients that appear in equations (1)-(3). We also define the errors when sampling as

$$\delta_{\sigma_j^+}(t, x) := \sigma(t, x) - \sigma^+(t_j, x), \quad (8)$$

$$\delta_{\sigma_j^-}(t, x) := \sigma^-(t, x) - \sigma^-(t_j, x). \quad (9)$$

For all  $t \in [t_j, t_{j+1})$ , we can rewrite the system (1)-(3) as

$$\begin{aligned} \partial_t u(t, x) + \lambda(x) \partial_x u(t, x) &= \sigma_j^+(x) v(t, x) \\ &\quad + \delta_{\sigma_j^+}(t, x) v(t, x), \end{aligned} \quad (10)$$

$$\begin{aligned} \partial_t v(t, x) - \mu(x) \partial_x v(t, x) &= \sigma_j^-(x) u(t, x), \\ &\quad + \delta_{\sigma_j^-}(t, x) u(t, x), \end{aligned} \quad (11)$$

with the boundary conditions

$$u(t, 0) = qv(t, 0), \quad (12)$$

$$v(t, 1) = \rho u(t, 1) + U(t), \quad (13)$$

Consider the following Volterra integral change of coordinates for  $j \geq 0$ :

$$\begin{aligned} \alpha_j(t, x) &= u(t, x) + \int_0^x K_j^{uu}(x, y) u(t, y) dy \\ &\quad + \int_0^x K_j^{uv}(x, y) v(t, y) dy, \end{aligned} \quad (14)$$

$$\begin{aligned} \beta_j(t, x) &= v(t, x) + \int_0^x K_j^{vu}(x, y) u(t, y) dy \\ &\quad + \int_0^x K_j^{vv}(x, y) v(t, y) dy, \end{aligned} \quad (15)$$

which can be rewritten in the following compact form:

$$(\alpha_j(t, x), \beta_j(t, x)) := (\mathcal{T}_j(u(t, \cdot), v(t, \cdot)))(x), \quad (16)$$

where the kernels  $K_j^{\cdot\cdot}$  are continuous functions defined on the triangular domain  $\mathcal{D} = \{(x, y) \in [0, 1]^2, 0 \leq y \leq x \leq 1\}$ . They satisfy [40], [17]

$$\begin{aligned} \lambda(x) \partial_x K_j^{uu}(x, y) + \partial_y (\lambda(y) K_j^{uu}(x, y)) &= \\ - \sigma_j^-(y) K_j^{uv}(x, y), \end{aligned} \quad (17)$$

$$\begin{aligned} \lambda(x) \partial_x K_j^{uv}(x, y) - \partial_y (\mu(y) K_j^{uv}(x, y)) &= \\ - \sigma_j^+(y) K_j^{uu}(x, y), \end{aligned} \quad (18)$$

$$\begin{aligned} \mu(x) \partial_x K_j^{vu}(x, y) - \partial_y (\lambda(y) K_j^{vu}(x, y)) &= \\ \sigma_j^-(y) K_j^{vv}(x, y), \end{aligned} \quad (19)$$

$$\begin{aligned} \mu(x) \partial_x K_j^{vv}(x, y) + \partial_y (\mu(y) K_j^{vv}(x, y)) &= \\ \sigma_j^+(y) K_j^{vu}(x, y), \end{aligned} \quad (20)$$

with the boundary conditions

$$K_j^{uu}(x, 0) = \frac{\mu(0)}{\lambda(0)q_j} K_j^{uv}(x, 0), \quad (21)$$

$$K_j^{uv}(x, x) = \frac{-\sigma_j^+(x)}{\lambda(x) + \mu(x)}, \quad (22)$$

$$K_j^{vv}(x, 0) = \frac{\lambda(0)q_j}{\mu(0)} K_j^{vu}(x, 0), \quad (23)$$

$$K_j^{vu}(x, x) = \frac{\sigma_j^-(x)}{\lambda(x) + \mu(x)}. \quad (24)$$

The backstepping boundary control design is performed by mapping the system (10)-(13) to a target system which will reflect the errors when sampling. The transformation (14)-(15) is a Volterra transformation and is invertible [44]. Consequently, there exist bounded functions  $L_j^{\cdot\cdot}$  defined on the triangular domain  $\mathcal{T}$  such that for all  $t \in [t_j, t_{j+1})$ , all  $x \in [0, 1]$ , we have

$$\begin{aligned} u(t, x) &= \alpha_j(t, x) + \int_0^x L_j^{\alpha\alpha}(x, y) \alpha_j(t, y) dy \\ &\quad + \int_0^x L_j^{\alpha\beta}(x, y) \beta_j(t, y) dy, \end{aligned} \quad (25)$$

$$\begin{aligned} v(t, x) &= \beta_j(t, x) + \int_0^x L_j^{\beta\alpha}(x, y) \alpha_j(t, y) dy \\ &\quad + \int_0^x L_j^{\beta\beta}(x, y) \beta_j(t, y) dy. \end{aligned} \quad (26)$$

The inverse transformation is denoted  $\mathcal{T}_j^{-1}$  such that for all  $t \in [t_j, t_{j+1})$  we have

$$(u(t, x), v(t, x)) := (\mathcal{T}_j^{-1}(\alpha_j(t, \cdot), \beta_j(t, \cdot)))(x), \quad (27)$$

Let us introduce the concatenation matrices  $K_j$  and  $L_j$  defined by:

$$K_j := \begin{pmatrix} K_j^{uu} & K_j^{uv} \\ K_j^{vu} & K_j^{vv} \end{pmatrix}, \quad L_j := \begin{pmatrix} L_j^{\alpha\alpha} & L_j^{\alpha\beta} \\ L_j^{\beta\alpha} & L_j^{\beta\beta} \end{pmatrix},$$

and their respective norms

$$\|K_j\|_\infty := \sup_{(x, y) \in \mathcal{D}} |K_j(x, y)|,$$

$$\|L_j\|_\infty := \sup_{(x,y) \in \mathcal{D}} |L_j(x,y)|.$$

Applying [17, Proposition A.6], we can show that

$$\max\{\|K_j\|_\infty, \|L_j\|_\infty\} < \bar{\phi} e^{\bar{D}_j \kappa}, \quad (28)$$

where

$$\begin{aligned} \kappa &= \max_{(x,y) \in \mathcal{T}} \left\{ \frac{1}{\lambda(x)}, \frac{1}{\mu(x)} \right\}, \quad \bar{q} = \max\left\{ \left| \frac{\lambda(0)q}{\mu(0)} \right|, \left| \frac{\mu(0)}{\lambda(0)q} \right| \right\}, \\ \bar{C}_j &= \max\left\{ \max_{(x,y) \in \mathcal{T}} |\lambda'(x)|, \max_{(x,y) \in \mathcal{T}} |\mu'(x)|, \max_{(x,y) \in \mathcal{T}} |\sigma_j^+(x)|, \right. \\ &\quad \left. \max_{(x,y) \in \mathcal{T}} |\sigma_j^-(x)| \right\}, \quad \bar{D}_j = 16(1 + 2\bar{q})\bar{C}_j, \\ \bar{\phi} &= \bar{q} \max\left\{ \max_{(x,y) \in \mathcal{T}} |\sigma_j^+(x)|, \max_{(x,y) \in \mathcal{T}} |\sigma_j^-(x)| \right\}. \end{aligned}$$

Due to equation (4), we obtain that all the kernels are uniformly bounded by some constant that does not depend on  $j$ , i.e., there exists  $M_1 > 0$  such that for all  $j \geq 0$ :

$$\max\{\|K_j\|_\infty, \|L_j\|_\infty\} < M_1. \quad (29)$$

Moreover, using Cauchy Schwartz inequality on equations (16) and (27) imply the following estimates for  $j \geq 0$ :

$$\begin{aligned} \|\mathcal{T}_j^{-1}(\alpha_j(t, \cdot), \beta_j(t, \cdot))\|_{L^2} &\leq (1 + M_1) \|\alpha_j(t, \cdot), \beta_j(t, \cdot)\|_{L^2}, \\ \|\mathcal{T}_j(u(t, \cdot), v(t, \cdot))\|_{L^2} &\leq (1 + M_1) \|u(t, \cdot), v(t, \cdot)\|_{L^2}. \end{aligned} \quad (30)$$

For all  $t \in [t_j, t_{j+1})$ , we now define the boundary control (at  $x = 1$ ) as

$$\begin{aligned} U(t) &= \int_0^1 (\rho K_j^{uu}(1, y) - K_j^{vu}(1, y)) u(t, y) dy \\ &\quad + \int_0^1 (\rho K_j^{uv}(1, y) - K_j^{vv}(1, y)) v(t, y) dy. \end{aligned} \quad (31)$$

Using the backstepping transformation (16), we map the original system (10)-(13) into the following target system, defined for all  $t \in [t_j, t_{j+1})$  by:

$$\partial_t \alpha_j(t, x) + \lambda(x) \partial_x \alpha_j(t, x) = (\mathcal{T}_j(f_j(t, \cdot)))_1(x) \quad (32)$$

$$\partial_t \beta_j(t, x) - \mu(x) \partial_x \beta_j(t, x) = (\mathcal{T}_j(f_j(t, \cdot)))_2(x), \quad (33)$$

with the boundary conditions

$$\alpha_j(t, 0) = q\beta_j(t, 0), \quad \beta_j(t, 1) = \rho\alpha_j(t, 1), \quad (34)$$

where

$$f_j(t, x) = (\delta_{\sigma_j^+}(t, x)v(t, x), \delta_{\sigma_j^-}(t, x)u(t, x)), \quad (35)$$

and where  $(\mathcal{T}_j(f_j(t, \cdot)))_i$  ( $i \in \{1, 2\}$ ) denotes the  $i^{\text{th}}$  component of  $(\mathcal{T}_j(f_j(t, \cdot)))$ .

*Remark 3:* Notice that the target system (32)-(35) exhibits the errors when sampling after transformation. If all the time-varying coefficients were equal to their sampled values, the right-hand side of equations (32)-(33) would equal zero. Consequently, the control law (31) would exponentially stabilize the closed-loop system (1)-(3) in the sense of the  $L^2$ -norm [17], [5, Lemma 2]. In the case of time-varying coefficients, we look for an even-triggered mechanism that keeps the errors when sampling relatively small, hence guaranteeing the exponential stability of the target system.

In what follows, we will denote  $w(t, x) := (u(t, x), v(t, x))$  and  $\gamma_j(t, x) := (\alpha_j(t, x), \beta_j(t, x))$ .

#### D. Well-posedness aspects

Inspired by [16], we consider in this paper the concept of *solution along the characteristics or broad solution*. It has been shown in [16] that for any  $F_1 \in L^\infty((0, \infty) \times (0, 1))$  and  $F_2 \in L^\infty((0, \infty) \times (0, 1))$ , for any  $u_j \in L^2(0, 1)$  and any  $v_j \in L^2(0, 1)$ , the system (1)-(3) with the initial condition  $(u(t_j, \cdot) = u_j, v(t_j, \cdot) = v_j)$  and the control input

$$U(t) = \int_0^1 F_1(t, y)u(t, y) + F_2(t, y)v(t, y)dy, \quad (36)$$

admits a unique solution  $(u, v) \in \mathcal{C}^0([t_j, t_{j+1}); L^2(0, 1)^2)$ . Therefore, if we can prove that  $\lim_{j \rightarrow \infty} t_j = +\infty$ , then we would obtain that the system (1)-(3) with the initial condition  $(u(0, \cdot) = u_0, v(0, \cdot) = v_0)$  and the control law (36) admits a unique solution  $(u_j, v_j) \in \mathcal{C}^0([0, \infty); L^2(0, 1)^2)$ . This will be done when showing the existence of a minimum dwell time in Section III-B.

### III. EVENT-TRIGGERED GAIN SCHEDULING STRATEGY AND MAIN RESULTS

#### A. Event-triggered gain scheduling with a static triggering condition

In this section, we introduce an event-triggering mechanism for gain scheduling. The triggering condition determines the time instants at which the coefficients must be sampled and, thereby, when the kernel computation and updating must be done. The triggering condition relies on the evolution of the errors when sampling and a Lyapunov function [10]  $V(t)$ , defined for all  $t \in [t_j, t_{j+1})$  by

$$\begin{aligned} V(w) &= \int_0^1 \frac{e^{\int_0^x \frac{-\nu dr}{\lambda(r)}}}{\lambda(x)} \alpha_j^2(t, x) + a \frac{e^{\int_0^x \frac{\nu dr}{\mu(r)}}}{\mu(x)} \beta_j^2(t, x) dx, \\ &= \langle g(\cdot) \gamma_j, \gamma_j \rangle = \langle g(\cdot) \mathcal{T}_j(w[t]), \mathcal{T}_j(w[t]) \rangle. \end{aligned} \quad (37)$$

with  $g$  being defined as

$$g(x) := \left( \frac{e^{\int_0^x \frac{-\nu dr}{\lambda(r)}}}{\lambda(x)}, a \frac{e^{\int_0^x \frac{\nu dr}{\mu(r)}}}{\mu(x)} \right). \quad (38)$$

The parameters  $\nu$  and  $a$  are chosen such that

$$q^2 - a < 0, \quad a\rho^2 e^{\nu \int_0^1 \frac{dr}{\lambda(r)}} + \int_0^1 \frac{dr}{\mu(r)} < 1. \quad (39)$$

These conditions are always feasible since  $|\rho q| < 1$ . The functional  $V$  is equivalent to the  $L^2$ -norm of the state  $(\alpha_j, \beta_j)$  (and consequently to the  $L^2$ -norm of the state  $(u, v)$ ), i.e. there exists  $k_u$  and  $k_\ell$  that do not depend on  $j$  (due to equation (30)) such that for all  $t \in [t_j, t_{j+1})$

$$k_\ell \|w(t, \cdot)\|_{L^2}^2 \leq V(t) \leq k_u \|w(t, \cdot)\|_{L^2}^2. \quad (40)$$

More precisely, we can choose

$$\begin{aligned} k_u &= \max\left\{ \frac{ae^{\int_0^1 \frac{\nu dr}{\mu(r)}}}{\min_{x \in [0,1]} \mu(x)}, \frac{1}{\min_{x \in [0,1]} \lambda(x)} \right\}, \\ k_\ell &= \min\left\{ \frac{e^{-\int_0^1 \frac{\nu dr}{\lambda(r)}}}{\max_{x \in [0,1]} \lambda(x)}, \frac{a}{\max_{x \in [0,1]} \mu(x)} \right\}. \end{aligned}$$

Let  $R \in (0, 1)$  be a positive design parameter. The static event-triggered gain scheduler is defined as follows: The times of events  $t_j \geq t_0$  with  $t_0 = 0$  form a finite or countable set of times, which is determined by the following rules for some  $j \geq 0$ :

- 1) if  $\{t > t_j : 2\langle g(\cdot)\mathcal{T}_j(w[t]), \mathcal{T}_j(f_j[t]) \rangle > \nu RV(t)\} = \emptyset$ , then the set of the times of the events is  $\{t_0, \dots, t_j\}$ .
- 2) if  $\{t > t_j : 2\langle g(\cdot)\mathcal{T}_j(w[t]), \mathcal{T}_j(f_j[t]) \rangle > \nu RV(t)\} \neq \emptyset$ , then the next event time is given by

$$t_{j+1} = \inf\{t > t_j : 2\langle g(\cdot)\mathcal{T}_j(w[t]), \mathcal{T}_j(f_j[t]) \rangle > \nu RV(t)\}, \quad (41)$$

where the function  $g$  is defined by equation (38).

### B. Avoidance of the Zeno phenomenon

To be able to implement our controller on digital platforms, we need to guarantee that there are no infinite triggering times that occur in a finite time interval (which would imply being able to sample infinitely fast). In other words, we need to guarantee there is no Zeno phenomenon. We have the following result

*Lemma 1:* Under the event-triggered condition (41), there exists a minimal dwell time between two triggering times, i.e., there exists a constant  $\tau > 0$  (independent of the initial condition of the state) such that  $t_{j+1} - t_j \geq \tau$  for all  $j \in \mathbb{N}$ .

*Proof:* Assume that an event occurred at  $t = t_{j+1}$ . Then, from (41), we have

$$\nu RV(t_{j+1}) \leq 2\langle g(\cdot)\mathcal{T}_j(w[t_{j+1}]), \mathcal{T}_j(f_j[t_{j+1}]) \rangle.$$

Using Cauchy-Schwarz inequality, equation 30 and Assumption 2, we obtain

$$\nu RV(t_{j+1}) \leq M_2 \varphi_\sigma |t_{j+1} - t_j| \|w[t_{j+1}]\|^2,$$

where

$$M_2 = 2\|g\|_\infty (1 + M_1)^2.$$

Due to equation (40), we obtain

$$0 \leq (M_2 \varphi_\sigma |t_{j+1} - t_j| - \nu R k_\ell) \|w[t_{j+1}]\|^2,$$

Since  $|t_{j+1} - t_j| \geq 0$ , we obtain

$$|t_{j+1} - t_j| \geq \frac{\nu R k_\ell}{M_2 \varphi_\sigma} = \tau, \quad (42)$$

with  $\tau$  being the minimal dwell time, which is independent of the initial condition.  $\blacksquare$

Lemma 1 implies that  $\lim_{j \rightarrow \infty} t_j = +\infty$  and consequently the proposed control law rewrites as in equation (36), which implies the existence of solutions in  $C^0([0, \infty); L^2(0, 1)^2)$  of the closed-loop system under the static event-triggered gain scheduler (41). The minimal dwell time depends on the rate of change of the different coefficients. Larger values of  $\varphi_\sigma$  induce a smaller minimal dwell time, corresponding to a more frequent update of the control law.

### C. Stability analysis

We now show the closed-loop stability under the proposed event-triggered gain scheduling algorithm.

*Theorem 1:* Let us assume that  $\varphi_\sigma$  verifies

$$\ln\left(\frac{k_u}{k_\ell}\right) < \nu^2(1 - R)R \frac{k_\ell}{M_2 \varphi_\sigma}, \quad (43)$$

where  $M_2$ , is defined in the proof of Lemma 1. Then the closed loop system (1)-(3) with the control law (31) and the event-triggered gain scheduler (41) is uniformly

exponentially stable, that is there exists  $\kappa_{\text{CL}} > 0$  and  $\nu_{\text{CL}} > 0$  such that for all  $t > 0$  we have

$$\|(u(t, \cdot), v(t, \cdot))\|_{L^2}^2 \leq \kappa_{\text{CL}} e^{-\nu_{\text{CL}} t} \|(u_0, v_0)\|_{L^2}^2 \quad (44)$$

*Proof:* Let us differentiate  $V(t)$  with respect to time and integrate by parts for  $t \in [t_j, t_{j+1}]$ ,  $j \in \mathbb{N}$ . We obtain

$$\begin{aligned} \dot{V}(t) = & -\nu V(t) + 2 \int_0^1 \frac{e^{\int_0^x \frac{-\nu dr}{\lambda(r)}}}{\lambda(x)} \alpha_j(t, x) (\mathcal{T}_j(f_j[t]))_1(x) dx \\ & + 2a \int_0^1 \frac{e^{\int_0^x \frac{\nu dr}{\mu(r)}}}{\mu(x)} \beta_j(t, x) (\mathcal{T}_j(f_j[t]))_2(x) dx + \beta_j^2(t, 0) (q^2 \\ & - a) + (ae^{\nu \int_0^1 \frac{dr}{\mu(r)}} \rho^2 - e^{-\nu \int_0^1 \frac{dr}{\lambda(r)}}) \alpha_j^2(t, 1). \end{aligned}$$

Using equation (41) and equation (39), we obtain

$$\dot{V}(t) \leq -\nu_1 V(t), \quad (45)$$

where  $\nu_1 = \nu(1 - R) > 0$ . Consequently, for all  $t \in [t_j, t_{j+1}]$ , we have

$$V(t) \leq e^{-\nu_1(t-t_j)} V(t_j), \quad (46)$$

which implies, due to equation (40), that for all  $t \in [t_j, t_{j+1}]$ ,

$$\|(u(t, \cdot), v(t, \cdot))\|_{L^2}^2 \leq \frac{k_u}{k_\ell} e^{-\nu_1(t-t_j)} \|(u(t_j, \cdot), v(t_j, \cdot))\|_{L^2}^2,$$

Since  $(u, v) \in C^0([0, \infty); L^2(0, 1)^2)$ , the previous equation holds for  $t = t_{j+1}$  and we get

$$\|(u(t_j, \cdot), v(t_j, \cdot))\|_{L^2}^2 \leq \frac{k_u}{k_\ell} e^{-\nu_1(t-t_j)} \|(u(t_j, \cdot), v(t_j, \cdot))\|_{L^2}^2.$$

Therefore, for all  $t \geq 0$ , we can recursively obtain an estimate of  $\|(u(t, \cdot), v(t, \cdot))\|_{L^2}^2$ , since we know that there have been  $j$  events and that  $j\tau$  units of time have (at least) been passed until  $t$  (where  $\tau$  is defined by equation (42) in the proof of Lemma 1). For all  $j \in \mathbb{N}$ , we have

$$\|(u(t_j, \cdot), v(t_j, \cdot))\|_{L^2}^2 \leq \left(\frac{k_u}{k_\ell}\right)^j e^{-\nu_1 t_j} \|(u_0, v_0)\|_{L^2}^2.$$

For  $t \in [t_j, t_{j+1}]$ , since  $j \leq \frac{t}{\tau}$ , we get

$$\begin{aligned} \|(u(t, \cdot), v(t, \cdot))\|_{L^2}^2 & \leq \left(\frac{k_u}{k_\ell}\right)^{j+1} e^{-\nu_1 t} \|(u_0, v_0)\|_{L^2}^2 \\ & \leq \frac{k_u}{k_\ell} e^{-(\nu_1 - \frac{\ln(\frac{k_u}{k_\ell})}{\tau})t} \|(u_0, v_0)\|_{L^2}^2. \end{aligned}$$

Consequently, we obtain

$$\|(u(t, \cdot), v(t, \cdot))\|_{L^2}^2 \leq \frac{k_u}{k_\ell} e^{-\nu_2 t} \|(u_0, v_0)\|_{L^2}^2, \quad (47)$$

with  $\nu_2 = \nu_1 - \frac{\ln(\frac{k_u}{k_\ell})}{\tau}$ . Due to Lemma 1, and under condition (43), we obtain  $\nu_2 > 0$ , which implies the exponential stability of the state  $(u, v)$ . This concludes the proof.  $\blacksquare$

*Remark 4:* We emphasize that Condition (43) of Theorem 1 is a sufficient condition and that closed-loop stability could be achieved even if this condition is not verified. The possible conservatism of this condition mostly comes from the coefficients  $k_\ell$  and  $k_u$  that depend on the uniform bound of the backstepping kernels  $M_1$  defined in equation (29). In this context, defining  $M_1$  as the uniform bound of the  $\|\cdot\|_\infty$ -norm of the kernels may not be the best choice. Indeed, we believe that our computations could be adjusted

using a uniform bound for the  $\|\cdot\|_{L^2}$ -norm of the kernels (that would be smaller than the  $\|\cdot\|_{\infty}$ -norm). The choice of  $\nu, a$  also impacts the constants  $k_u$  and  $k_\ell$  (due to the definition of the Lyapunov function), while also modifying equation (43). Finally, the tuning parameter  $R$  can be tuned to alter the minimum dwell time, but this also impacts the proposed sufficient condition for stability.

#### IV. NUMERICAL SIMULATIONS

In this section, we illustrate our theoretical results with numerical simulations. We consider the system (1)-(3) with the following coefficients

$$\lambda(x) = 1, \quad \mu(x) = 2, \quad q = 1.2, \quad \rho = 0.3,$$

$$\sigma^+(t, x) = 2 + 0.25xe^{2\sin^2(t)},$$

$$\sigma^-(t, x) = 1 + \frac{5}{\cosh^2((t-8)^2)} + 0.7 \cos(\pi t) + \frac{5}{\cosh^2(5x)}$$

The time- and space-varying coefficient  $\sigma^-(t, x)$  is pictured in Figure 1. The backstepping kernels are computed using an iterative algorithm based on the method of successive approximations [6]. We simulate the evolution of the system using a classical finite volume method based on a Godunov scheme [31] with a space step of 0.025 and a Courant–Friedrichs–Lewy condition equal to 1. We run our simulations on a frame of 15s. The parameters of the triggering condition are chosen as  $a = 1.5$ ,  $\nu = 1.2$ , and  $R = 0.5$ . One can easily verify that equation (39) is satisfied. The kernels are updated aperiodically using equation (41). Therefore, the proposed approach does not require solving time- and space-varying kernel equations, resulting in a simplified procedure. We have pictured in Figure 2 the time-evolution of the  $L^2$ -norm of the closed-loop system (1)-(3) using a control law with constant gains (i.e., the gains are computed based on the parameter values at  $t = 0$  and are never updated) and using the proposed event-triggered controller. The open-loop behavior is not pictured here, but the open-loop system is unstable. The corresponding control efforts are pictured in Figure 3. As can be seen, the nominal controller with constant gains is not robust to the time variations of the coefficients. In the meantime, under the static event-triggered scheduler (41), we have closed-loop stability. We have also pictured in Figure 4 the sampled version of the profile of the time- and space-varying coefficient  $\sigma^-(t_j, x)$  for all  $\{t_j\}_{j \in \mathbb{N}}$  according to the static event-triggered scheduler (41). The number of events generated by our static event-triggered gain scheduler is equal to 44 for this test case. Solving the kernel equations at each event requires 0.1s, which can be done in real time, while simulating the system.

#### V. CONCLUDING REMARKS

In this paper, we have designed an event-triggered gain scheduling to exponentially stabilize a  $2 \times 2$  hyperbolic PDEs systems with time- and space-varying in-domain coupling coefficients. The control design combines the backstepping design with an appropriate event-triggering condition. Unlike existing results in the literature, the proposed approach does not require solving time-varying backstepping kernel equations in real time, which implies a smaller computational burden and better applicability. More precisely, we sample the coupling terms according to a Lyapunov-based event-triggering condition and update the backstepping gains

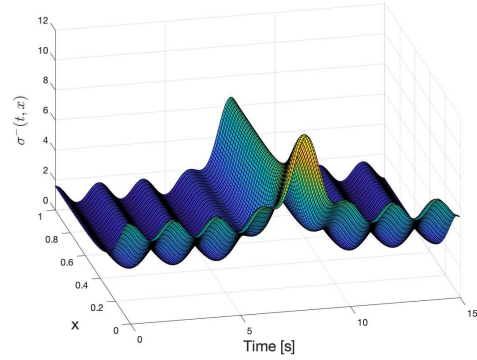


Fig. 1. Profile of the time- and space-varying coefficient  $\sigma^-(t, x)$ .

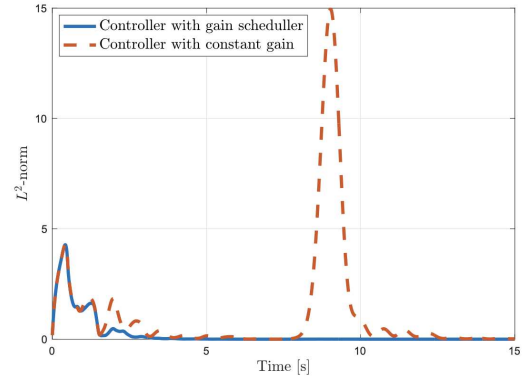


Fig. 2. Time evolution of the  $L^2$  norm of the closed-loop system with a nominal controller with constant backstepping gains and with the proposed event-triggered controller (41).

accordingly. In future contributions, we will consider time-varying boundary couplings and velocities. We will also try to propose a dynamic-triggering condition.

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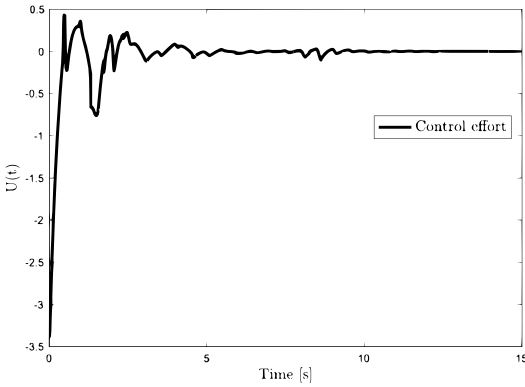


Fig. 3. Time-evolution of the boundary control input under the static event-triggered scheduler (41).

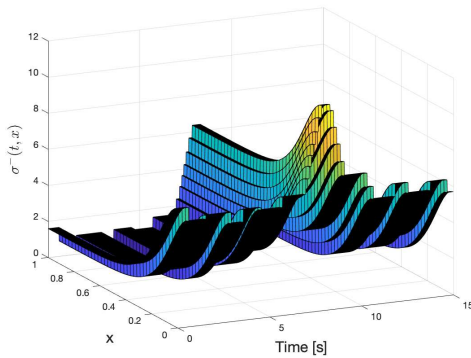


Fig. 4. Sampled version of the profile of the time- and space-varying coefficient  $\sigma^-(t_j, x)$  for all  $\{t_j\}_{j \in \mathbb{N}}$  according to the static event-triggered scheduler (41).

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