



HAL
open science

Well-posedness and long-time behavior for self-consistent Vlasov-Fokker-Planck equations with general potentials

Pierre Gervais, Maxime Herda

► To cite this version:

Pierre Gervais, Maxime Herda. Well-posedness and long-time behavior for self-consistent Vlasov-Fokker-Planck equations with general potentials. 2024. hal-04681175

HAL Id: hal-04681175

<https://hal.science/hal-04681175>

Preprint submitted on 29 Aug 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Well-posedness and long-time behavior for self-consistent Vlasov-Fokker-Planck equations with general potentials

Pierre Gervais Maxime Herda

August 29, 2024

Abstract

We study the well-posedness, steady states and long time behavior of solutions to Vlasov-Fokker-Planck equation with external confinement potential and nonlinear self-consistent interactions. Our analysis introduces newly characterized conditions on the interaction kernel that ensure the local asymptotic stability of the unique steady state. Compared to previous works on this topic, our results allow for large, singular and non-symmetric interactions. As a corollary of our main results, we show exponential decay of solutions to the Vlasov-Poisson-Fokker-Planck equation in dimension 3, for low regularity initial data. In the repulsive case, the result holds in strongly nonlinear regimes (*i.e.* for arbitrarily small Debye length). Our techniques rely on the design of new Lyapunov functionals based on hypocoercivity and hypoellipticity theories. We use norms which include part of the interaction kernel, and carefully mix “macroscopic quantities based”-hypocoercivity with “commutators based”-hypocoercivity.

Mathematics Subject Classification (2020): 35Q83, 35Q84, 35B35, 35B65, 82C40.

Keywords: Vlasov-Fokker-Planck; Hypocoercivity; Hypoellipticity; Convolution operator; Coulomb interactions.

Contents

| | | |
|----------|---|----------|
| 1 | Introduction | 2 |
| 2 | Setting and main result | 6 |
| 2.1 | Notations | 6 |
| 2.2 | Confining potential | 6 |
| 2.3 | Interaction potential | 7 |
| 2.4 | Main results | 10 |
| 2.5 | Particular cases and extensions | 12 |
| 2.6 | Outline | 14 |

| | | |
|----------|---|-----------|
| 3 | Existence, uniqueness and regularity of the steady state | 14 |
| 3.1 | Existence of fixed points | 15 |
| 3.2 | Regularity of fixed points | 18 |
| 3.3 | Uniqueness of fixed points | 20 |
| 3.4 | Functional inequalities related to the steady state | 22 |
| 3.5 | Proof of Theorem 3.1 and 2.10 | 23 |
| 4 | Linearized VFP | 23 |
| 4.1 | Preliminary properties | 24 |
| 4.2 | L^2 -hypo-coercivity | 25 |
| 4.3 | Derivatives estimates | 31 |
| 4.4 | Mixed $H_{x,v}^1$ - $L_{x,v}^2$ -hypo-coercivity | 35 |
| 4.5 | Hypo-ellipticity | 36 |
| 4.6 | $H_x^1 L_v^2$ hypo-coercivity | 38 |
| 5 | Non-linear VFP | 40 |
| 5.1 | Functional setting | 41 |
| 5.2 | Nonlinear estimates | 42 |
| 5.3 | Proof of Theorem 2.12 | 44 |
| 6 | Concluding remarks | 44 |

1 Introduction

We are interested in the study of Vlasov-Fokker-Planck (VFP) equation in the presence of an external confining potential and self-consistent interactions. This equation describes the time evolution of the distribution function F in the phase space $\mathbb{R}^d \times \mathbb{R}^d$ with $d \geq 1$, through the kinetic equation

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \nabla_x (\Psi_F + V) \cdot \nabla_v F = \nu \nabla_v \cdot (v F + \nabla_v F), \\ \Psi_F(t, x) = \int_{\mathbb{R}^{2d}} k(x-y) F(t, y, v) dy dv, \\ F|_{t=0} = F_{\text{in}}. \end{cases} \quad (1.1)$$

In this equation the Vlasov transport operator $v \cdot \nabla_x - \nabla_x (\Psi_F + V) \cdot \nabla_v$ accounts for the movement of particles in the phase space under the action of two forces. The first force is driven by an external potential V with confining properties. The second force is self-consistent and derives from the potential Ψ_F which is given by the convolution between a long-range interaction kernel k and the macroscopic density. This makes the model (1.1) nonlinear and non-local. The Fokker-Planck operator $\nu \nabla_v \cdot (v \cdot) + \nu \Delta_v$, with $\nu > 0$, accounts for short range interactions with, typically, a fixed background of particles at constant temperature. In appropriate contexts, it can also be viewed as a toy model for nonlinear collisional operators such as the Dougherty-Fokker-Planck [24] or Landau operator [50]. The Vlasov-Fokker-Planck equation can also be

interpreted as the Kolmogorov-Fokker-Planck equation for the law of a stochastic process following the underdamped Langevin equations, which describes the trajectories of individual particles.

The self-consistent Vlasov-Fokker-Planck equation appears in a variety of physical contexts ranging from astrophysics [17], plasmas [9, 37], relativistic beams [15] or chemical solutions [52]. It even has applications in machine learning [19], through its relation with the underdamped Langevin process. Depending on the physics at hand the interaction potential may take various forms. A widely studied model is the Vlasov-Poisson-Fokker-Planck (VPFP) equation for which the kernel k is given by the Coulomb kernel

$$k_C(x) = \frac{I}{|x|^{d-2}}, \quad (1.2)$$

with $d = 3$. The potential k_C describes repulsive electrostatic interactions with $I > 0$ scaling as the squared inverse Debye length. In the case of attractive gravitational interactions, the Newton potential

$$k_N(x) = -\frac{I}{|x|^{d-2}}, \quad (1.3)$$

with $d = 3$, is used. The Coulomb and Newton interaction kernels are radially symmetric and in particular even functions of their arguments, which is natural for particle systems arising from classical mechanics. However, for other applications, asymmetric interaction kernels can arise.

In particle accelerator physics, the VFP equation (1.1) is used to describe the longitudinal dynamics of relativistic beams of charged particles. In this context, there are several models for describing self-consistent interactions, taking into account synchrotron radiation effects. In the simplest case of a relativistic particle in free space on a circular orbit the interaction potential (related to the wakefield in this context) is given by

$$k_S(x) = 2 \frac{\cosh\left(\frac{5}{3} \sinh^{-1}(x)\right) - \cosh\left(\sinh^{-1}(x)\right)}{\sinh\left(2 \sinh^{-1}(x)\right)} \mathbb{1}_{x>0}(x), \quad (1.4)$$

with $d = 1$. We refer to [15] and references therein for further details on this model. Observe that unlike k_C and k_N , the potential k_S is not an even function of its argument (see [15, Figure 2] for a graphical representation).

Steady state The steady states solutions of the Vlasov-Fokker-Planck equation are of the form (see [25, 22, 9] or Section 3)

$$F_\star(x, v) = \rho_\star(x)M(v), \quad \rho_\star(x) = e^{-V_\star(x)},$$

where the local Maxwellian distribution is given by

$$M(v) = \frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{\frac{d}{2}}},$$

and the Gibbs potential V_\star is such that

$$\nabla_x V_\star = \nabla_x (V + k * \rho_\star),$$

where $*$ denotes the convolution. Observe that $k * \rho_\star = \Psi_{F_\star}$. The macroscopic part of the steady state can be characterized equivalently, under the normalization condition $\int \rho_\star = 1$, as the fixed point of some nonlinear integral mapping

$$\rho_\star = \frac{e^{-V - k * \rho_\star}}{\int_{\mathbb{R}^d} e^{-V - k * \rho_\star} dx}.$$

This steady state equation coincides with the one of the McKean-Vlasov equation

$$\partial_t \rho = \nabla_x \cdot (\nabla_x \rho + \rho \nabla_x k * \rho + \rho \nabla_x V).$$

This equation can be seen as the macroscopic version of (1.1) and can be, at least formally, related to (1.1) in the diffusion limit $\nu \rightarrow \infty$ on the appropriate time scale. We refer to the recent [5] and references therein for more details on diffusion limits, at least in the case of Coulomb interactions. We point out that the existence and stability of steady states for the McKean-Vlasov equation has been considered for instance in [44, 18, 14], as well as the question of bifurcations and phase transitions.

Free energy In some cases there is a macroscopic quantity of interest (aside of the total mass) for the Vlasov-Fokker-Planck equation (1.1). It is usually referred to as free energy functional and writes

$$\begin{aligned} \mathcal{E}[F] &= \int_{\mathbb{R}^{2d}} F(x, v) \log F(x, v) dx dv \\ &+ \int_{\mathbb{R}^{2d}} F(x, v) \left(\frac{|v|^2}{2} + V(x) + \frac{\Psi_F(x)}{2} \right) dx dv, \end{aligned} \quad (1.5)$$

where the first part represents the entropy of the system, and the second one represents the sum of its kinetic and potential energy. One may check (see for instance [15]) that this functional satisfies the differential identity

$$\begin{aligned} \frac{d\mathcal{E}}{dt} + \nu \int_{\mathbb{R}^{2d}} F(x, v) \left| \nabla_v \log \left(\frac{F(x, v)}{M(v)} \right) \right|^2 dx dv \\ = - \int_{\mathbb{R}^{2d}} \nabla_x \Psi_F^o(x) v F(x, v) dx dv, \end{aligned}$$

where Ψ_F^o is the part of the total interaction potential induced by the odd part of k , namely $k^o(x) = (k(x) - k(-x))/2$ (see Section 2.3). In particular, when k is even, the right-hand side of the previous identity vanishes, thus the free energy is a Lyapunov functional. In this case the steady states of VFP described previously coincide with critical points of \mathcal{E} under the constraint of given total mass only if k is even. When k yields enough coercivity and convexity to

the functional \mathcal{E} , it becomes clear that the steady state is actually unique and asymptotically stable. This is the case for instance for Coulomb interactions, for which (1.5) allows to show non-quantitative asymptotic stability of the dynamics, see [9].

However, for asymmetric k , the functional \mathcal{E} does not yield as much information anymore. We shall see in Section 3 that it may still be used to investigate the uniqueness of steady states. For more details on Lyapunov functionals for (1.1) with general asymmetric kernels we refer to [15] and [39].

State of the art and contributions In the present paper, we study the well-posedness of self-consistent VFP equations (1.1) around steady states, which we prove to exist and be unique under appropriate conditions, and derive quantitative estimates of decay and regularization of solutions.

Early works on the well-posedness of VFP include [41, 21, 49, 48, 8]. Concerning the long-time behavior, in linear settings ($k = 0$), quantitative convergence estimates can be established thanks to hypocoercivity methods [34, 51, 23, 10]. In the nonlinear case, with smooth interaction kernels ($k \in W^{2,\infty}$), hypocoercivity methods have been complemented with techniques involving entropy / free energy functionals [51, Section A.21], low regularity norms and probabilistic techniques [7, 29, 4].

The case of singular kernels and more precisely the Coulomb interactions, and correspondingly the Vlasov-Poisson-Fokker-Planck equation has been studied a lot. Notably, [35] provided the first quantitative decay estimates for VPFP with confining potential in dimension 3, though their results were restricted to weakly nonlinear regimes (for k sufficiently small). Subsequent work [37] continued in this direction. Recently, [1] demonstrated that appropriate norms could extend these results beyond weak interaction regimes for the linearized system, leaving the nonlinear case as a future research direction. Improvement were obtained recently in this direction by [46]. Asymmetric kernels have only recently been explored in [15] and [39], with results limited to one dimension and requiring at least Lipschitz regularity for k .

In light of the existing literature, we have identified several gaps that this manuscript starts to address. First, we enhance state-of-the-art results by deriving quantitative decay estimates for VPFP in strongly nonlinear regimes. This goes beyond the restrictions of [35, 37] and addresses part of the open problems posed in [1]. We also improve the regularity requirements on the initial data, compared to the recent [46]. Furthermore, our analysis goes beyond VPFP and includes a large class of VFP models. Indeed we place particular emphasis on considering general interaction potentials k , which may be non-symmetric, singular, or exhibit repulsive and/or mildly attractive behavior. In this regard we establish sufficient criteria for the uniqueness of steady states and their asymptotic stability, noting that such stability is not always guaranteed.

In the following section, we precisely define our setting and present our main results.

2 Setting and main result

In this section, we present the main results of this paper. First we introduce some notation and the general assumptions, under which our main results hold.

2.1 Notations

In the following, for a complex number z , the real and imaginary parts are denoted by $\Re z$ and $\Im z$. The notations $x_+ = \max(0, x)$ and $x_- = \min(0, -x)$ denote the non-negative and non-positive parts respectively. Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we write ∂^α for the partial derivative $\partial^{\alpha_1} \dots \partial^{\alpha_d}$ and $|\alpha| = \sum_i \alpha_i$. Given a function (or a distribution) $k : \mathbb{R}^d \rightarrow \mathbb{R}$, we write $\tilde{k} : x \mapsto k(-x)$ and we denote by $\mathcal{F}(k)(\xi) = \hat{k}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} k(x) dx$ the Fourier transform of k . The symbol $*$ denotes the convolution between a distribution and a function, namely $k * \rho(x) = \int_{\mathbb{R}^d} k(x - y) \rho(y) dy$.

The notation $\langle \cdot, \cdot \rangle$ denotes a duality bracket or a scalar product. In the latter case, a subscript may be added to precise the Hilbert space. We write $[A, B] = AB - BA$ to denote the commutator between two operators. Given $p \in [1, \infty]$, we denote by $p' = (1 - p^{-1})^{-1}$ the conjugate Lebesgue exponent. For weighted Lebesgue space, we denote the norms

$$\|f\|_{L^p(m)}^p = \int_{\Omega} |f(\omega)|^p m(\omega) d\omega, \quad 1 \leq p < \infty$$

$$\|f\|_{L^\infty(m)} = \sup_{\omega \in \mathbb{R}^d} |f(\omega) m(\omega)|.$$

For weighted Sobolev spaces, when $s \in \mathbb{N}$, we denote the squared norm

$$\|f\|_{H^s(m)}^2 = \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2(m)}^2$$

and when $s \in (0, \infty) \setminus \mathbb{N}$, we define $H^s(m)$ by interpolation (through the real or complex method unambiguously, see [16, Theorems 3.3 and 3.5]).

In inequalities C will denote a constant which may change from a line to another and we sometimes use \lesssim instead of $\leq C$. When it is necessary, we shall be more precise and write $C(a, b, c, \dots)$ to denote a constant depending on the parameters a, b, c, \dots .

2.2 Confining potential

Let us first describe our general assumptions on the confinement potential.

Assumption 2.1. The confining potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$ is a $C^2(\mathbb{R}^d)$ function. It is assumed to satisfy that for any $\varepsilon > 0$ there is some constant $C_\varepsilon > 0$, such that

$$\forall x \in \mathbb{R}^d, \quad |\nabla^2 V(x)| \leq \varepsilon |\nabla V(x)| + C_\varepsilon. \quad (2.1)$$

Moreover the following integrability and boundedness conditions hold

$$(1 + |\nabla V|^2) e^{-V} \in L^1 \cap L^\infty \quad \text{and} \quad \int_{\mathbb{R}^d} e^{-V(x)} dx = 1. \quad (2.2)$$

Finally, we assume that the measure e^{-V} admits a Poincaré inequality

$$\int_{\mathbb{R}^d} |u|^2 e^{-V} dx - \left(\int_{\mathbb{R}^d} u e^{-V} dx \right)^2 \leq C_P \int_{\mathbb{R}^d} |\nabla_x u|^2 e^{-V} dx, \quad (2.3)$$

for any $u : \mathbb{R}^d \rightarrow \mathbb{R}$ such that the integrals are finite and some constant $C_P > 0$.

The Poincaré inequality (2.3) is equivalent to the existence of a spectral gap for the Witten Laplacian associated to V , namely $-\Delta_V = -\Delta + \frac{1}{4}|\nabla V|^2 - \frac{1}{2}\Delta V$. A sufficient condition for $-\Delta_V$ to admit a spectral gap is given by Persson's Theorem [42, Theorem 2.1] taking $a(x) = \frac{1}{4}|\nabla V(x)|^2 - \frac{1}{2}\Delta V(x)$, which requires that

$$\inf_{|x| \geq R} a(x) > 0 \quad \text{for } R \text{ large enough.}$$

In virtue of (2.1), the Poincaré assumption (2.3) could then be replaced by the stronger one

$$|\nabla V(x)| \rightarrow \infty \quad \text{when} \quad |x| \rightarrow \infty.$$

Some simple families of such potential are given for any $\alpha \in (0, \infty)$

$$V(x) = \langle x \rangle^{1+\alpha} \quad \text{or} \quad V(x) = \langle x \rangle \log(1 + |x|^2)^\alpha,$$

up to some normalizing additive constant. We refer to [3, 51] for improvements and more criteria leading to a Poincaré inequality.

In the following any norm involving only the confinement potential will be denoted by R_V .

2.3 Interaction potential

Let us now discuss and state our hypotheses for the interaction potential k . We will rather use the corresponding convolution operator \mathcal{K} (in the sense of [27, Definition 2.5.1]) which we assume to be bounded between some Lebesgue spaces. In turn, this yields the existence and uniqueness [27, Theorem 2.5.2] of a unique tempered distribution k realizing the convolution. We have

$$\mathcal{K}\rho = k * \rho \quad \text{and} \quad \mathcal{K}^*\rho = \check{k} * \rho, \quad (2.4)$$

for any Schwartz function $\rho : \mathbb{R}^d \rightarrow \mathbb{R}$, where we denoted by \mathcal{K}^* the formal adjoint of \mathcal{K} . From there we define the symmetric and skew-symmetric parts of \mathcal{K} ,

$$\mathcal{K}^e \rho = k^e * \rho \quad \text{and} \quad \mathcal{K}^o \rho = k^o * \rho, \quad (2.5)$$

where the superscripts e and o refer to the fact that k^e and k^o are respectively the even and odd parts of k , namely

$$k^e = \frac{k + \check{k}}{2} \quad \text{and} \quad k^o = \frac{k - \check{k}}{2}.$$

Observe that the corresponding Fourier multipliers are related by $\widehat{k}^e = \Re \widehat{k}$ and $\widehat{k}^o = \text{i} \Im \widehat{k}$. With obvious notation, we use the superscripts $\alpha = e, o$ to denote the part of the potential Ψ_F^α related to the even or odd part of k . Before stating our assumptions on \mathcal{K} , let us just mention that by [27, Lemma 2.5.3], $\partial^\beta \mathcal{K} = \mathcal{K} \partial^\beta$ for any multi-index $\alpha \in \mathbb{N}^d$.

Assumption 2.2. We assume that for some $p, q \in [2, \infty]$ with $q > d$, the following continuity estimates hold:

$$\|\mathcal{K}^\alpha \rho\|_{L^p} + \|\nabla \mathcal{K}^\alpha \rho\|_{L^q} \leq \bar{\kappa}^\alpha \|\rho\|_{L^1 \cap L^2}, \quad \alpha = e, o, \quad (2.6)$$

and we give ourselves, to clarify the dependencies of constants, some arbitrary $\bar{\kappa}_{\max} > 0$ such that

$$\bar{\kappa} := \bar{\kappa}^e + \bar{\kappa}^o \leq \bar{\kappa}_{\max}.$$

Moreover we assume that there are $\theta \in [0, 1]$ and $\underline{\kappa}^e > 0$ such that

$$\langle \mathcal{K}^e h, h \rangle \geq -\underline{\kappa}^e (\theta \|h\|_{L^2}^2 + (1-\theta) \|h\|_{L^1}^2), \quad \text{for all } h \in L^1 \cap L^2 \text{ s.t. } \int h = 0. \quad (2.7)$$

Finally we assume the monotonicity property

$$\rho \geq 0 \Rightarrow \mathcal{K} \rho \geq 0, \quad \text{for all } \rho \in L^1 \cap L^2. \quad (2.8)$$

Remark 2.3. In the bilinear estimate (2.7), $\langle \mathcal{K}^e h, h \rangle$ is well-defined as the $L^p - L^{p'}$ duality bracket thanks to (2.6). Observe that it involves only the symmetric part of the kernel since $\langle \mathcal{K}^e h, h \rangle = \langle \mathcal{K} h, h \rangle$. Moreover a sufficient condition for (2.7) to hold is that the negative part of the Fourier transform has some integrability. Indeed one can show that for any $\theta \in [0, 1]$,

$$\langle \mathcal{K}^e h, h \rangle \geq -C_\theta \|\widehat{k}_-^e\|_{L^{\frac{1}{1-\theta}}} \|h\|_{L^2}^{2\theta} \|h\|_{L^1}^{2(1-\theta)}.$$

with $C_\theta = \theta^{-\theta} (1-\theta)^{-(1-\theta)}$. This yields (2.7) with $\underline{\kappa}^e = C_\theta \|\widehat{k}_-^e\|_{L^{\frac{1}{1-\theta}}}$. This implies also that if \widehat{k}_-^e is merely $L^1 + L^\infty$ then (2.7) holds with $\underline{\kappa}^e = 2 \|\widehat{k}_-^e\|_{L^1 + L^\infty}$ and $\theta = 1/2$.

Remark 2.4. The assumption (2.8) could be replaced by

$$\forall \rho \in L^1 \cap L^2, \quad \rho \geq 0 \Rightarrow \mathcal{K} \rho \geq -C \|\rho\|_{L^1}.$$

Indeed if the kernel k has a negative part in L^∞ , then, without loss of generality, it can be translated by a constant, which is C here, to be made positive without changing (1.1).

Let us now give some examples of kernels satisfying, or not, our assumptions.

Example 2.5 (Lipschitz kernels). Given a (possibly non-symmetric) $k \in W^{1, \infty}$, such as (1.4), without loss of generality (see Remark 2.4), we can always assume that $k \geq 0$. Then, as consequences of Young and Holder inequalities the hypotheses are satisfied with $p = q = \infty$, and $\underline{\kappa}^e = \|k^e\|_{L^\infty}$ and $\bar{\kappa}^\alpha = \|k^\alpha\|_{W^{1, \infty}}$ for $\alpha = e, o$.

Example 2.6 (Repulsive Riesz and Coulomb kernel). In dimension d , let us consider the symmetric repulsive Riesz potential

$$k(x) = \frac{I}{|x|^{d-\alpha}}, \quad I \geq 0.$$

Then, as a direct consequence of Hardy-Littlewood-Sobolev inequalities, the hypothesis (2.6) is satisfied under the conditions

$$\frac{d}{2} < \alpha \leq d \quad \text{and} \quad d \geq 2,$$

so that $q > d$. This includes the Coulomb kernel ($\alpha = 2$) in dimension 3. Besides, (2.7) and (2.8) follow from the non-negativity of k and \widehat{k} . In particular

$$\underline{\kappa}^e = \bar{\kappa}^o = 0 \quad \text{and} \quad \bar{\kappa}^e \underset{I \rightarrow 0}{=} O(I).$$

Observe that a smallness condition on I would translate into a condition on $\bar{\kappa}^e$ only. As we shall see there will be no such condition for our results to hold, unlike previous works in the literature.

Example 2.7 (Singular attractive kernels). Singular attractive kernels such as

$$k(x) = -\frac{I}{|x|^{d-\alpha}}, \quad I \geq 0,$$

fail to satisfy hypothesis (2.8), even if they are translated by a constant (see Remark 2.4). This hypothesis is important in our treatment of the steady VFP equation but we discuss an alternative in Section 2.5. Otherwise, in the light of Remark 2.3 and Example 2.6, it is clear that the rest of the assumptions hold with

$$\bar{\kappa}^o = 0, \quad \underline{\kappa}^e \underset{I \rightarrow 0}{=} O(I), \quad \text{and} \quad \bar{\kappa}^e \underset{I \rightarrow 0}{=} O(I).$$

Example 2.8 (More singular kernels). When $d \geq 2$, if $1 \leq \alpha \leq \frac{d}{2}$ (which includes Manev potentials $\alpha = 1$, $d = 3$, see [20, 6]) using Hardy-Littlewood-Sobolev inequalities or Calderon-Zygmund theory only provide $q \in [2, d]$. This situation could be treated with our approach by requiring more regularity on the initial condition, but we decided to restrict this work to H^1 -type regularity at most. We refer to Section 6 for more on this perspective. When $\alpha < 1$, the potential $\nabla \mathcal{K}$ maps L^2 to some Sobolev space which is less regular, as can be seen by the identification $\mathcal{K} = c_\alpha (-\Delta)^{-\alpha/2}$.

Example 2.9 (Other kernels in weak Lebesgue spaces). The boundedness assumptions on \mathcal{K} and $\nabla \mathcal{K}$ are satisfied whenever k and ∇k lie in suitable weak Lebesgue spaces (see [27] for the definition). Namely, if $k \in L^{s,\infty}$ for some $s \in (1, \infty)$ and $\nabla k \in L^{t,\infty}$ for some $t \in \left(\frac{2d}{2+d}, \infty\right)$, Young's generalized convolution inequality [27, Theorem 1.4.25] yield $q = \frac{2t}{2-t} > d$ with

$$\bar{\kappa}^o = C (\|k^o\|_{L^{s,\infty}} + \|\nabla k^o\|_{L^{t,\infty}}) \quad \text{and} \quad \bar{\kappa}^e = C (\|k^e\|_{L^{s,\infty}} + \|\nabla k^e\|_{L^{t,\infty}}),$$

for some universal constant $C = C(s, t)$ related to Young's inequality.

2.4 Main results

Let us first state our main result concerning steady states of (1.1). The following theorem is, for the sake of presentation, a restricted version of a more general result, Theorem 3.1 that we prove in Section 3.

Theorem 2.10. *Under Assumptions 2.1 and 2.2 there is $\delta^e > 0$ such that if*

$$\underline{\kappa}^e < \delta^e(\theta, \bar{\kappa}_{max}, R_V),$$

then the equation (1.1) admits a unique steady state with mass 1. It is of the form

$$F_\star(x, v) = e^{-V_\star(x)} M(v),$$

with $V_\star - V = \Psi_{F_\star} \in W^{2,\infty}$. The probability measure with density F_\star satisfies the Poincaré and weighted Poincaré inequalities stated in Section 3.4 for some constant C_\star depending only on $\|(1 + |\nabla V|^2) e^{-V}\|_{L^1 \cap L^\infty}$, the Poincaré constant of $e^{-V} dx$, as well as $\bar{\kappa}_{max}$ and θ .

Remark 2.11. Uniqueness is expected when \mathcal{K} is symmetric and positive because the steady state is characterized by minimization of a strictly convex functional. When \mathcal{K} is non-symmetric and weakly non-positive, a new geometric argument on the free energy is used to prove the uniqueness of steady states (see Section 3.3).

Our second main result concerns the well-posedness of (1.1), as well as the decay and regularity of solutions around steady states.

Theorem 2.12. *Under Assumptions 2.1 and 2.2, there are constants $\delta^e > 0$ and $\delta^o > 0$ such that if*

$$\underline{\kappa}^e < \delta^e(\theta, \bar{\kappa}_{max}, R_V) \quad \text{and} \quad \bar{\kappa}^o < \delta^o(\bar{\kappa}_{max}, R_V, \underline{\kappa}^e, \nu),$$

the unique steady state of (1.1) is stable in the following sense. For any $s \in [0, 1]$ such that

$$s > s_c := \frac{3}{2} \left(\frac{d}{q} - \frac{1}{3} \right),$$

there is a constant $R > 0$ such that if

$$\|F_{in} - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} < R,$$

then (1.1) has a unique solution $F \in \mathcal{C}([0, \infty); H_x^s L_v^2(F_\star^{-1}))$. Moreover, there are constants $C > 0$ and λ such that for all $t > 0$

$$\|F(t) - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} \leq C \|F_{in} - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} e^{-\lambda t}, \quad (2.9)$$

$$\|F(t) - F_\star\|_{H_x^1 L_v^2(F_\star^{-1})} \leq C \|F_{in} - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} \left(1 + t^{-\frac{3}{2}(1-s)}\right) e^{-\lambda t}, \quad (2.10)$$

$$\|F(t) - F_\star\|_{L_x^2 H_v^{1-s}(F_\star^{-1})} \leq C \|F_{in} - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} \left(1 + t^{-\frac{1}{2}(1-s)}\right) e^{-\lambda t}. \quad (2.11)$$

Finally, $F_{in} \mapsto F$ is Lipschitz continuous from $H_x^s L_v^2(F_\star^{-1})$ to $\mathcal{C}([0, \infty); H_x^s L_v^2(F_\star^{-1}))$.

The current work also bring several novelties and improvements on the Cauchy theory for the nonlinear Vlasov-Fokker-Planck equation close to equilibrium.

1. Assumption 2.2 on the interaction operator \mathcal{K} encompasses many interactions potentials which are dealt with in a unified manner. They include singular, non-symmetric potentials (see the examples of Section 2.3 and consequences of Theorem 2.12 in the next section).
2. Concerning the regularity assumption on the initial data, no regularity is required in the v variable, which can be compared with [35, 46]. This is made possible by performing the standard interpolation argument between the linearized theories in $L_x^2 L_v^2$ and $H_x^1 L_v^2$ (instead of $H_{x,v}^1$), thus allowing to assume regularity for the initial data in position without assuming any in velocity. Both theories are established by applying the so-called L^2 -hypo-coercivity method of [23] to both F and $\nabla_x F$. The price to pay is the extra assumption (2.1) on the confining potential V (see Remark 2.14).
3. We prove nonlinear hypoellipticity (2.10) by proving linear hypoellipticity in the presence of a source term in Proposition 4.16. During the iterative scheme used to construct the solution, this allows to use the fact that both the linear flow and the approximate solution enjoys regularization properties, thus requiring less regularity for the initial condition.
4. Theorem 2.12 allows for strongly nonlinear regimes, in the sense that there is no smallness condition on $\bar{\kappa}^e$. Following the ideas of [1], at the linearized level, we do not consider the even part of \mathcal{K} , namely \mathcal{K}^e , to be a “bad term” which is to be treated perturbatively. As it appears in the free energy functional (1.5) of the equation, it is in some sense part of the natural metric to study the equation. As a consequence, when this even part is positive, it does not need to be small. In the particular case of Coulomb interactions, we thus prove uniqueness and stability of the equilibria, even in the strongly nonlinear regime (see Example 2.6 and Corollary 2.16).

Remark 2.13. Let us point out that, in [46] which deals with Coulomb interactions, the main requirement on the confining potential V is that $|\nabla V|e^{-V} \in L^r$ for some $r > d$. In the present work, the integrability assumption $|\nabla V|^2 e^{-V} \in L^r$ for some $r > \frac{d}{2}$ would have been sufficient. Note that the control on $|\nabla V|^2$ we require is used only to prove Lemmas 4.7 and 4.13, which allow respectively to get rid of any smallness assumption on $\bar{\kappa}^e$ (as introduced in [1]) and any regularity in the velocity variable for the initial data.

Remark 2.14. The assumption (2.1) with $\varepsilon = 1$ is standard in the context of hypo-coercivity with confinement, but the version used here (for $\varepsilon \rightarrow 0$) can be found in [12], as well in works of semi-classical analysis (in the stronger form $\nabla^2 V = o(\nabla V)$, see for instance [32, 31, 33]). This assumption is used for ε small only in the proof of Lemma 4.13, which allows to avoid assuming regularity in

the velocity variable for the initial condition. Without this assumption, as in [46], a similar result can be proved for initial data in $H_{x,v}^s$ (instead of $L_v^2 H_x^s$).

Remark 2.15. When $s_c < 0$, that is to say when $q > 3d$, the standard $H_{x,v}^1$ hypoellipticity strategy is enough, in particular the mild growth assumption (2.1) is unnecessary.

2.5 Particular cases and extensions

The explicit and quantitative stability of equilibria in the case of Poisson interactions was first studied in [35] and improved in [46]. Our framework allows to reduce even more the regularity required for the initial data F_{in} .

Corollary 2.16 (Repulsive VPFP). *Under Assumptions 2.1, consider the Vlasov-Poisson-Fokker-Planck equation with Coulomb interactions, in dimension $d = 3$,*

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \nabla_x (\Psi_F + V) \cdot \nabla_v F = \nu \nabla_v \cdot (vF + \nabla_v F), \\ -\delta^2 \Delta \Psi_F(t, x) = \int_{\mathbb{R}^d} F(t, x, v) dv, \\ F|_{t=0} = F_{\text{in}}, \end{cases} \quad (2.12)$$

for any (arbitrarily small) $\delta > 0$. Then for any $\frac{1}{4} < s \leq 1$, there is a constant $R > 0$ such that if

$$\|F_{\text{in}} - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} < R,$$

then (2.12) has a unique solution $F \in \mathcal{C}([0, \infty); H_x^s L_v^2(F_\star^{-1}))$. Moreover, there are constants $C > 0$ and λ such that for all $t > 0$

$$\|F(t) - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} \leq C \|F_{\text{in}} - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} e^{-\lambda t}.$$

In this result, we improve the regularity assumption on the initial condition compared to previous works. We only require $H_x^{\frac{1}{4}+} L_v^2$ instead of (at least) $H_{x,v}^{\frac{1}{2}+}$ regularity [35, 46]. In particular we do not require any regularity in the velocity variable for the initial condition. Moreover the results hold for any arbitrarily small Debye length, namely out of weakly nonlinear regimes.

As pointed out in Example 2.7, the hypothesis (2.8) does not hold in the case of attractive Newtonian interactions. However, this assumption is necessary only to prove the existence of steady states, which are already known to exist [9]. The remaining assumptions still hold, which allows to prove the stability of said equilibria.

Corollary 2.17 (Attractive VPFP). *Under Assumptions 2.1, consider the Vlasov-Poisson-Fokker-Planck equation with Newton interactions, in dimension $d = 3$,*

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \nabla_x (\Psi_F + V) \cdot \nabla_v F = \nu \nabla_v \cdot (vF + \nabla_v F), \\ \Delta \Psi_F(t, x) = \Gamma \int_{\mathbb{R}^d} F(t, x, v) dv, \\ F|_{t=0} = F_{\text{in}}. \end{cases} \quad (2.13)$$

Then there is $\Gamma_{max} > 0$ such that if

$$\Gamma < \Gamma_{max},$$

then for any $\frac{1}{4} < s \leq 1$, there is a constant $R > 0$ such that if

$$\|F_{in} - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} < R,$$

then (2.12) has a unique solution $F \in \mathcal{C}([0, \infty); H_x^s L_v^2(F_\star^{-1}))$. Moreover, there are constants $C > 0$ and λ such that for all $t > 0$

$$\|F(t) - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} \leq C \|F_{in} - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} e^{-\lambda t}.$$

Remark 2.18 (Coulomb and Newton kernels in small dimensions). Our approach allows to prove the stability of equilibria in dimension $d = 1, 2$ for Coulomb and (small) Newton interactions, up to a minor extra assumption on V . Let us only briefly comment on this as these cases have already been investigated in the literature (see [1] and references therein). Our general framework does not exactly fit these situations because the interaction kernel blows up for large relative distances, which prevents \mathcal{K} from mapping $L^1 \cap L^2$ onto some Lebesgue space. The study of steady states is classical (see [9] and references therein) and their smoothness can be proved following the approach of Section 3.2. This then allows to prove the various functional inequalities of Section 3.4. To prove the stability of these equilibria, the assumption that \mathcal{K} maps in some Lebesgue space is used only in the proof of Lemma 4.4, which still holds up to assuming also that (for instance) $\langle \cdot \rangle^5 \lesssim e^{V/2}$ thanks to the observation that $\|\langle \cdot \rangle^{-3} \mathcal{K}(\rho)\|_{L^\infty} \leq C \|\langle \cdot \rangle^3 \rho\|_{L^2}$. In the end, the same result as Corollaries 2.16 and 2.17 would then hold with $0 \leq s \leq 1$.

In the case of the synchrotron kernel (2.14), we obtain the same result as in [15].

Corollary 2.19 (Synchrotron kernels). *Under Assumptions 2.1 (without (2.1), see Remark 2.15), consider the self-consistent Vlasov-Fokker-Planck equation for a relativistic electron bunch in dimension $d = 1$*

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \nabla_x (\Psi_F + V) \cdot \nabla_v F = \nu \nabla_v \cdot (vF + \nabla_v F), \\ \Psi_F(t, x) = I \int_{\mathbb{R}^d} k_S(x - y) F(t, x, v) dv, \\ F|_{t=0} = F_{in}, \end{cases} \quad (2.14)$$

where k_S is given by (1.4). Then there is $I_{max} > 0$ such that if

$$I < I_{max},$$

then for any $0 \leq s \leq 1$, there is a constant $R > 0$ such that if

$$\|F_{in} - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} < R,$$

then (2.14) has a unique solution $F \in \mathcal{C}([0, \infty); H_x^s L_v^2(F_\star^{-1}))$. Moreover, there are constants $C > 0$ and λ such that for all $t > 0$

$$\|F(t) - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} \leq C \|F_{in} - F_\star\|_{H_x^s L_v^2(F_\star^{-1})} e^{-\lambda t}.$$

2.6 Outline

In Section 3 we prove several results related to the steady state equation (3.2), namely existence, uniqueness and regularity. These results are of independent interest and hold under weaker assumptions than the general Assumptions 2.1 and 2.2. We then proceed to prove the functional inequalities related to the measure induced by this steady state in Section 3.4 necessary to the stability analysis.

In Section 4 we study the linearized equation around the steady state, using the so called H^1 -hypo-coercivity and hypo-ellipticity strategies of [34, 51]. We also use the so called L^2 -hypo-coercivity of [23], and more precisely we take inspiration from [1] which incorporates \mathcal{K}^e into the functional setting so that no smallness assumption on the upper bound $\bar{\kappa}^e$ is required. In turn we consider \mathcal{K} to be almost positive symmetric, that is we assume $\bar{\kappa}^o$ and κ^e small. One novelty of this work is that we apply the L^2 -hypo-coercivity strategy to $\nabla_x(F - F_\star)$ in order to build a linear theory in $H_x^1 L_v^2$ so that no regularity in velocity is required, which requires the mild growth assumption (2.1) on V .

In Section 5 we combine all previous results to prove Theorem 2.12. We first interpolate the hypo-coercivity and hypo-ellipticity results to deduce that the linearized flow maps initial data and source continuously for some appropriate “natural” norm. In Section 5.2, we show that the nonlinearity is bounded in this “natural” norm, so that, in Section 5.3, the proof of Theorem 2.12 reduces to a straightforward application of Banach-Picard’s fixed point theorem.

In Section 6, we present some perspectives, namely some possible improvements regarding the regularity assumptions on $\nabla\mathcal{K}$ and the initial data, and some continuation of the present work.

3 Existence, uniqueness and regularity of the steady state

The first goal of this section is to prove Theorem 3.1 below (and thus Theorem 2.10) concerning the existence, uniqueness and regularity of steady states. From this construction, we will derive various functional inequalities related to the steady state in Section 3.4.

Theorem 3.1. *Assume that the confinement potential V satisfies, for some $N \geq 2$*

$$\begin{cases} \|e^{-V}\|_{L^1} = 1 \\ \forall 1 \leq n \leq N, \quad \left(1 + |\nabla^n V|^{\frac{N}{n}}\right) e^{-V} \in L^1 \cap L^\infty \end{cases}$$

and that the interaction operator \mathcal{K} satisfies that for some $p, q \in [2, \infty]$,

$$\begin{cases} \|\mathcal{K}\|_{L^1 \cap L^2 \rightarrow L^p} + \|\nabla\mathcal{K}\|_{L^1 \cap L^2 \rightarrow L^q} \leq \bar{\kappa}, \\ \forall \rho \in L^1 \cap L^2, \quad \rho \geq 0 \Rightarrow \mathcal{K}\rho \geq 0, \\ \|\mathcal{K}^*(e^{-V})\|_{L^\infty} \leq \zeta. \end{cases}$$

Then, smooth and rapidly decaying solutions to the stationary Vlasov-Fokker-Planck equation

$$v \cdot \nabla_x F_\star - \nabla_x (\Psi_{F_\star} + V) \cdot \nabla_v F_\star = \nu \nabla_v \cdot (v F_\star + \nabla_v F_\star) \quad (3.1)$$

are given by all functions

$$F_\star(x, v) = \rho_\star(x) M(v),$$

with ρ_\star a solution of the fixed point problem

$$\rho_\star = \mathcal{T}(\rho_\star) := \frac{\mathcal{S}(\rho_\star)}{\|\mathcal{S}(\rho_\star)\|_{L^1}}, \quad \mathcal{S}(\rho) := e^{-V - \mathcal{K}\rho}. \quad (3.2)$$

Under the hypotheses on the potentials, there exists a solution $\rho_\star = e^{-V_\star} > 0$ to (3.2). Moreover, if additionally $q \geq d/2$ and $N > 1 + d/q$, then any solution satisfies $V_\star - V \in W^{2,\infty}$. Finally under the additional assumptions that (2.7) holds with $\underline{\kappa}^e < \delta^e(\theta, \bar{\kappa}, R_V)$, then ρ_\star is uniquely defined, and therefore there is a unique steady state to (3.1).

3.1 Existence of fixed points

From here we consider that the domain of \mathcal{T} is the convex set

$$\mathcal{C}_M := \{\rho \in L^1 \cap L^r : \|\rho\|_{L^1} \leq 1, \|\rho\|_{L^r} \leq M, \rho \geq 0\}$$

with $M > 0$ to be determined. The purpose of the Lebesgue index r is to make the following estimates more general than what is strictly needed to prove Theorem 3.1. In the proof of Theorem 3.1, we shall take $r = 2$.

Lemma 3.2 (Properties of \mathcal{T}). *Assume that the interaction potential satisfies the bound*

$$\|\nabla \mathcal{K}\|_{L^1 \cap L^r \rightarrow L^q} \leq \bar{\kappa}, \quad 1 \leq r \leq q \leq \infty,$$

and it is assumed to be positive in the sense that

$$\forall \rho \in L^1 \cap L^r, \quad \rho \geq 0 \Rightarrow \mathcal{K}\rho \geq 0.$$

The confining potential is assumed to be such that

$$\|e^{-V}\|_{L^1} = 1 \quad \text{and} \quad \|(1 + |\nabla V|) e^{-V}\|_{L^1 \cap L^\infty} \leq R_V. \quad (3.3)$$

Finally, we assume the following relationship between \mathcal{K} and V :

$$\|\mathcal{K}^* (e^{-V})\|_{L^\infty} \leq \zeta. \quad (3.4)$$

Under these assumptions, \mathcal{T} satisfies for any $\rho \in \mathcal{C}_M$

$$\|\mathcal{T}(\rho)\|_{L^s} \leq e^\zeta R_V \quad \text{and} \quad \|\mathcal{T}(\rho)\|_{L^1} = 1, \quad s \in [1, \infty], \quad (3.5)$$

and more precisely the concentration estimate

$$\left\| e^{V/s'} \mathcal{T}(\rho) \right\|_{L^s} \leq e^{\zeta/s'}, \quad s \in [1, \infty], \quad (3.6)$$

as well as the regularity estimate

$$\|\nabla \mathcal{T}(\rho)\|_{L^s} \leq R_V e^\zeta (1 + \bar{\kappa} + \bar{\kappa}M), \quad s \in [1, q]. \quad (3.7)$$

Furthermore, $\mathcal{T}(\mathcal{C}_M)$ is precompact in L^1 , and \mathcal{T} satisfies the continuity estimate

$$\|\mathcal{T}(\rho) - \mathcal{T}(\sigma)\|_{L^s} \leq 2e^\zeta \zeta^{1/s} R_V^{1/s'} \|\rho - \sigma\|_{L^1}^{1/s}, \quad s \in [1, \infty]. \quad (3.8)$$

Proof. We first prove the integrability bound (3.5), then the gradient estimate (3.7). We then deduce the compactness from these bounds, and finally proceed to prove the Hölder continuity in L^s (where $s < \infty$).

Step 1: Integrability estimate. We start by exhibiting a lower bound on the mass of $\mathcal{S}(\rho)$, using Jensen's inequality with respect to the measure $e^{-V(x)} dx$:

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{S}(\rho) dx &= \int_{\mathbb{R}^d} e^{-V - \mathcal{K}\rho} dx \\ &\geq \exp\left(-\int_{\mathbb{R}^d} \mathcal{K}\rho e^{-V} dx\right) \\ &= \exp\left(-\int_{\mathbb{R}^d} \rho \mathcal{K}^*(e^{-V}) dx\right), \end{aligned}$$

from which we deduce using (3.4) and the fact that $\|\rho\|_{L^1} \leq 1$

$$\int_{\mathbb{R}^d} \mathcal{S}(\rho) dx \geq e^{-\zeta}. \quad (3.9)$$

Clearly $\|\mathcal{T}(\rho)\|_{L^1} = 1$ and since $\mathcal{K}\rho \geq 0$, one easily deduces

$$\left\| e^V \mathcal{T}(\rho) \right\|_{L^\infty} \leq e^\zeta$$

which then yields the upper bound (3.5) and the concentration estimate (3.6).

Step 2: Regularity estimate. Starting from the observation (since $\mathcal{K}\rho \geq 0$)

$$|\nabla \mathcal{S}(\rho)| = |\nabla V + \nabla \mathcal{K}(\rho)| \mathcal{S}(\rho) \leq |\nabla V + \nabla \mathcal{K}(\rho)| e^{-V}$$

and denoting $\frac{1}{s} = \frac{1}{q} + \frac{1}{t}$, one obtains the bound

$$\begin{aligned} \|\nabla \mathcal{S}(\rho)\|_{L^s} &\leq \|\nabla V e^{-V}\|_{L^s} + \|\nabla \mathcal{K}(\rho)\|_{L^q} \|e^{-V}\|_{L^t} \\ &\leq R_V (1 + \bar{\kappa} \|\rho\|_{L^1 \cap L^r}) \\ &\leq R_V (1 + \bar{\kappa} + \bar{\kappa}M), \end{aligned}$$

which, combined with (3.9) yields (3.7).

Step 3: Compactness. Let us consider some sequence $(\rho_n) \subset \mathcal{C}_M$ and denote its image $\sigma_n := \mathcal{T}(\rho_n)$. Since (3.6) and (3.7) imply that for some $C > 0$

$$\|e^V \sigma_n\|_{L^\infty} + \|\nabla \sigma_n\|_{L^s} \leq C,$$

the Rellich-Kondrachov theorem states that some subsequence, which we still denote σ_n , converges in L^1_{loc} to some σ_∞ satisfying the same bounds as σ_n . The global convergence follows from the weighted L^∞ bound because for any $R > 0$

$$\|\sigma_n - \sigma_\infty\|_{L^1} \leq 2C \|\mathbf{1}_{|x| \geq R} e^{-V}\|_{L^1} + \|\mathbf{1}_{|x| \leq R} (\sigma_n - \sigma_\infty)\|_{L^1}$$

and $e^{-V} \in L^s$. The right-hand side can be made as small as desired by taking R large and then n large.

Step 4: Continuity estimates. Using the inequality

$$\forall a, b \geq 0, \quad |e^{-a} - e^{-b}| \leq |a - b|$$

with $a = \mathcal{K}(\rho) \geq 0$ and $b = \mathcal{K}(\sigma) \geq 0$, as well as $|\mathcal{K}(\rho - \sigma)| \leq \mathcal{K}(|\rho - \sigma|)$, which are consequences of the non-negativity of \mathcal{K} , one has the control

$$|\mathcal{S}(\rho) - \mathcal{S}(\sigma)| \leq \mathcal{K}(|\rho - \sigma|) e^{-V}.$$

We then deduce that

$$\begin{aligned} \|\mathcal{S}(\rho) - \mathcal{S}(\sigma)\|_{L^1} &\leq \|(\rho - \sigma) \mathcal{K}^*(e^{-V})\|_{L^1} \\ &\leq \|\rho - \sigma\|_{L^1} \|\mathcal{K}^*(e^{-V})\|_{L^\infty} \\ &\leq \zeta \|\rho - \sigma\|_{L^1}, \end{aligned} \tag{3.10}$$

and observing that the following identity holds

$$\mathcal{T}(\rho) - \mathcal{T}(\sigma) = \frac{\mathcal{S}(\rho) - \mathcal{S}(\sigma)}{\|\mathcal{S}(\rho)\|_{L^1}} + \mathcal{T}(\sigma) \frac{\|\mathcal{S}(\rho)\|_{L^1} - \|\mathcal{S}(\sigma)\|_{L^1}}{\|\mathcal{S}(\rho)\|_{L^1}},$$

one deduces from (3.9) and (3.10)

$$\|\mathcal{T}(\rho) - \mathcal{T}(\sigma)\|_{L^1} \leq 2\zeta e^\zeta \|\rho - \sigma\|_{L^1}.$$

Interpolating with the bound

$$\|\mathcal{T}(\rho) - \mathcal{T}(\sigma)\|_{L^\infty} \leq 2e^\zeta R_V,$$

one deduces (3.8). This concludes the proof. \square

The following existence result is a direct consequence of Lemma 3.2 and an application of Schauder's theorem with $M = e^\zeta R_V$ and endowing \mathcal{C}_M with the topology of L^1 (note that \mathcal{T} is indeed Lipschitz continuous in virtue of (3.8)).

Proposition 3.3 (Existence of the steady state). *Under the assumptions of Lemma 3.2, the mapping $\mathcal{T} : L^1 \rightarrow L^1$ admits a fixed point in \mathcal{C}_M with $M = e^\zeta R_V$. Furthermore, any other fixed point satisfies the estimates of Proposition 3.2.*

3.2 Regularity of fixed points

Lemma 3.4 (Regularity of the steady state). *Under the assumptions of Lemma 3.2, and assuming furthermore $q \geq \frac{d}{2}$, as well as for some $N \geq 2$*

$$\forall n \leq N, \quad \left\| |\nabla^n V|^{\frac{N}{n}} e^{-V} \right\|_{L^1 \cap L^\infty} \leq R_V,$$

there exists some $C = C(N, \bar{\kappa}, \zeta, R_V)$ such that any fixed point ρ_\star of \mathcal{T} satisfies the inductive estimate, for $1 \leq n \leq N$,

$$\|\rho_\star\|_{W^{n,1}} + \|\rho_\star\|_{W^{n,q}} \leq C (1 + \|\rho_\star\|_{W^{n-1,1}} + \|\rho_\star\|_{W^{n-1,r}})^n. \quad (3.11)$$

Proof. Let us consider ρ_\star of the form $\rho_\star = e^{-V_\star}$. We first prove a Faà di Bruno-type inequality, and then proceed to prove (3.11).

Step 1: A Faà di Bruno-type inequality. Let us prove that there exists some $C = C(N)$ such that for any $n \in \{1, \dots, N\}$, one has

$$|\nabla_x^n \rho_\star| \leq C \rho_\star \sum_{k=1}^n |\nabla_x^k V_\star|^{\frac{n}{k}}. \quad (3.12)$$

This is true for $n = 1$, let us prove it by induction. Assume it holds for some $n \geq 1$ and consider $\alpha \in \mathbb{N}^d$ such that $|\alpha| = n$, there holds in virtue of Leibniz's formula for partial derivatives

$$\partial^{\alpha + \mathbf{e}_j} \rho_\star = -\partial^\alpha (\partial^{\mathbf{e}_j} V_\star \rho_\star) = -\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^{\beta + \mathbf{e}_j} V_\star) (\partial^{\alpha - \beta} \rho_\star)$$

and thus, letting $k = |\beta| \in [0, n]$, there holds for some $C = C(N) > 0$

$$|\partial^{\alpha + \mathbf{e}_j} \rho_\star| \leq C \sum_{k=1}^{n+1} \left(\rho_\star^{\frac{k}{n+1}} |\nabla^k V_\star| \right) \left(\rho_\star^{-\frac{k}{n+1}} |\nabla^{n+1-k} \rho_\star| \right).$$

Using Young's inequality with the exponents $\frac{k}{n+1} + \frac{n+1-k}{n+1} = 1$, and then re-indexing, one obtains, up to enlarging C

$$|\partial^{\alpha + \mathbf{e}_j} \rho_\star| \leq C \rho_\star \left(\sum_{k=1}^{n+1} |\nabla^k V_\star|^{\frac{n+1}{k}} + \sum_{k=1}^n (\rho_\star^{-1} |\nabla^k \rho_\star|)^{\frac{n+1}{k}} \right)$$

Using the inductive assumption and Minkowski's inequality, we deduce

$$|\partial^{\alpha + \mathbf{e}_j} \rho_\star| \leq C \rho_\star \sum_{k=1}^{n+1} |\nabla^k V_\star|^{\frac{n+1}{k}}$$

Summing over $j = 1, \dots, d$, we deduce (3.12) at rank $n + 1$.

Step 2: Inductive control. Since $\nabla V_\star = \nabla V + \nabla \mathcal{K}(\rho_\star)$, the previous step yields the estimate

$$\|\nabla^n \rho_\star\|_{L^p} \leq C \sum_{k=1}^n \|\rho_\star |\nabla^k V|^{\frac{n}{k}}\|_{L^p} + C \sum_{k=0}^{n-1} \left\| \rho_\star |\nabla^k (\nabla \mathcal{K}(\rho_\star))|^{\frac{n}{k+1}} \right\|_{L^p}.$$

The first term is bounded by $C = C(\zeta, R_V, N)$ because of (3.6), so we focus on the inductive term. Hölder's inequality yield

$$\begin{aligned} \left\| \rho_\star |\nabla^k (\nabla \mathcal{K}(\rho_\star))|^{\frac{n}{k+1}} \right\|_{L^1 \cap L^q} &\leq \|\rho_\star\|_{L^{q'} \cap L^\infty} \left\| |\nabla^k (\nabla \mathcal{K}(\rho_\star))|^{\frac{n}{k+1}} \right\|_{L^q} \\ &= \|\rho_\star\|_{L^{q'} \cap L^\infty} \left\| \nabla^k \nabla \mathcal{K}(\rho_\star) \right\|_{L^{\frac{nq}{k+1}}}^{\frac{n}{k+1}} \end{aligned}$$

One then interpolates for $k = 0, \dots, n-1$ each term between the end points $k = 0$ and $k = n-1$ with $\theta = \frac{k}{n-1}$ using Gagliardo-Nirenberg's inequality :

$$\left\| \nabla^k \nabla \mathcal{K}(\rho_\star) \right\|_{L^{\frac{nq}{k+1}}}^{\frac{n}{k+1}} \leq C \|\nabla \mathcal{K}(\rho_\star)\|_{L^{nq}}^{1-\theta} \|\nabla^{n-1} \nabla \mathcal{K}(\rho_\star)\|_{L^q}^\theta,$$

and then, since $N \geq 2$ and $q \geq \frac{d}{2}$, the first term is controlled using the embedding $W^{n-1, q} \subset L^{qn}$:

$$\|\nabla \mathcal{K}(\rho_\star)\|_{W^{k, \frac{nq}{k+1}}} \leq C \|\nabla \mathcal{K}(\rho_\star)\|_{W^{n-1, q}}.$$

In conclusion, using (3.5), there holds

$$\|\nabla^n \rho_\star\|_{L^1 \cap L^q} \leq C + C \sum_{k=0}^{n-1} \|\nabla \mathcal{K}(\rho_\star)\|_{W^{n-1, q}}^{\frac{n}{k+1}},$$

from which we deduce (3.11) using the boundedness of $\nabla \mathcal{K}$, this concludes the proof. \square

Proposition 3.5 (Regularity of the confinement perturbation). *Under the assumptions of Lemma 3.4, and assuming that $N > 1 + \frac{d}{q}$ and*

$$\|\mathcal{K}\|_{L^1 \cap L^r \rightarrow L^p} \leq \bar{\kappa} \quad \text{for some } p \in [1, \infty],$$

any fixed point $\rho_\star = e^{-V_\star}$ of \mathcal{T} satisfies for some $C = C(N, \bar{\kappa}, \zeta, R_V)$

$$\|\rho_\star\|_{W^{N,1}} + \|\rho_\star\|_{W^{N,q}} + \|V - V_\star\|_{W^{2,\infty}} \leq C.$$

Proof. As a direct consequence of Lemma 3.4, there is some $C_0 = C_0(N, \bar{\kappa}, \zeta, R_V)$ such that

$$\|\rho_\star\|_{W^{N,1}} + \|\rho_\star\|_{W^{N,q}} \leq C_0$$

and therefore, up to enlarging C_0 , and noticing that $\nabla \mathcal{K}(\rho_\star) = \nabla(V_\star - V)$

$$\|\nabla(V_\star - V)\|_{W^{N,q}} \leq C_0,$$

and then by Sobolev embedding

$$\|\nabla(V_\star - V)\|_{W^{1,\infty}} \leq C_0.$$

Finally, noticing that for any smooth enough u and $x \in \mathbb{R}^d$, there holds

$$\begin{aligned} |u(x)| &\leq \frac{1}{\omega_d} \int_{|x-y|\leq 1} |u(x) - u(y)| \, dy + \frac{1}{\omega_d} \int_{|x-y|\leq 1} |u(y)| \, dy \\ &\leq \|\nabla u\|_{L^\infty} + \frac{1}{\omega_d^{1/p}} \|u\|_{L^p}, \end{aligned}$$

where ω_d is the volume of the unit ball, taking $u = V_\star - V = \mathcal{K}(\rho_\star)$, this estimate guarantees that $V_\star - V \in L^\infty$. This concludes the proof. \square

3.3 Uniqueness of fixed points

Let us introduce the macroscopic free energy functional (see (1.5) for its kinetic counterpart)

$$\forall \rho \in X, \quad \mathcal{F}[\rho] := \int_{\mathbb{R}^d} \left(V + \frac{1}{2} \mathcal{K}^e(\rho) \right) \rho \, dx + \int_{\mathbb{R}^d} \rho \log \rho \, dx$$

defined in the convex set

$$X := \left\{ \rho \in L^1 \cap L^2 : 0 < \rho < R_V e^\zeta, \int \rho \, dx = 1 \right\},$$

where the upper bound is that of (3.5) with $s = \infty$. Here we assume that $\mathcal{K}^e : L^1 \cap L^2 \rightarrow L^p$ with $p \in [2, \infty]$. Observe that, compared to the previous section (see Proposition 3.5), the range of indices for p has been shrunk and we take $r = 2$. We also introduce the differential of \mathcal{F} with respect to ρ :

$$\forall h \in Y, \quad d_\rho \mathcal{F}[\rho] \cdot h = \int_{\mathbb{R}^d} (V + \mathcal{K}^e(\rho) + \log \rho) h \, dx,$$

where we denoted the vector space

$$Y = \left\{ h \in L^1 \cap L^2 : \int_{\mathbb{R}^d} h \, dx = 0 \right\}.$$

Proposition 3.6 (Uniqueness of the steady state). *Under the assumptions of Lemma 3.2, the following properties hold.*

- a) *If \mathcal{K}^e is such that \mathcal{F} is strictly convex, then \mathcal{T} has a unique fixed point.*
- b) *If $\mathcal{K}^o = 0$, fixed points of \mathcal{T} coincide exactly with critical points of \mathcal{F} with mass 1. In particular, under the hypothesis of a), the unique fixed point of \mathcal{T} is the unique minimizer of \mathcal{F} under the constraint of unit mass.*

Proof. One observes that any $\rho \in X$ of the form

$$\rho = c e^{-V - \mathcal{K}\rho} \quad \text{for some } c \in (0, \infty)$$

is such that the first order differential of \mathcal{F} writes

$$\forall h \in Y, \quad d_\rho \mathcal{F}[\rho].h = - \int_{\mathbb{R}^d} \mathcal{K}^\circ(\rho) h \, dx,$$

and in particular, the skew-symmetry of \mathcal{K}° implies

$$(d_\rho \mathcal{F}[\rho_1] - d_\rho \mathcal{F}[\rho_0]).(\rho_1 - \rho_0) = 0. \quad (3.13)$$

for any ρ_0 and ρ_1 of the previous form. Therefore, by strict convexity of \mathcal{F} , $\rho_1 = \rho_0$ which proves a). The property b) is an immediate consequence of the expressions of $d_\rho \mathcal{F}$ and \mathcal{T} . \square

Corollary 3.7. *Under the assumptions of Proposition 3.5 and if there is $\underline{\kappa}^e < \delta^e(\theta, R_V, \zeta)$ such that*

$$\langle \mathcal{K}^e h, h \rangle \geq -\underline{\kappa}^e (\theta \|h\|_{L^2}^2 + (1 - \theta) \|h\|_{L^1}^2), \quad \text{for all } h \in L^1 \cap L^2 \text{ s.t. } \int h = 0.$$

then the mapping \mathcal{T} admits a unique fixed point.

Proof. Let us show that under this additional hypothesis, \mathcal{F} is strictly convex. Indeed, one has

$$(d_\rho \mathcal{F}[\rho_1] - d_\rho \mathcal{F}[\rho_0]).(\rho_1 - \rho_0) = \langle \mathcal{K}^e \rho_1 - \rho_0, \rho_1 - \rho_0 \rangle_{L^2} + \int (\rho_1 - \rho_0) \log \left(\frac{\rho_1}{\rho_0} \right).$$

On the one hand by the Csiszár-Kullback-Pinsker inequality, one has

$$\int (\rho_1 - \rho_0) \log \left(\frac{\rho_1}{\rho_0} \right) \geq \|\rho_1 - \rho_0\|_{L^1}^2$$

On the other hand, thanks to the upper bound on functions in X , one also has

$$\int (\rho_1 - \rho_0) \log \left(\frac{\rho_1}{\rho_0} \right) \geq \frac{1}{R_V e^\zeta} \|\rho_1 - \rho_0\|_{L^2}^2$$

where we used the identity

$$\forall x, y > 0, \quad (x - y) \log \left(\frac{x}{y} \right) = (x - y)^2 \int_0^1 \frac{1}{ty + (1 - t)x} dt.$$

By combining these controls of the L^1 and L^2 distances with the assumption, for a small enough $\delta^e(\theta, R_V, \zeta)$, one shows

$$(d_\rho \mathcal{F}[\rho_1] - d_\rho \mathcal{F}[\rho_0]).(\rho_1 - \rho_0) \geq 0$$

with equality only if $\rho_0 = \rho_1$. \square

Remark 3.8. If $\theta = 0$ then clearly $\delta^e = 1$.

Remark 3.9. The lower bound assumption is not a technical one, when the interaction kernel has large negative modes one may observe phase transition and appearance of several equilibria. We refer to [14] in the case of the McKean-Vlasov equation on the torus without confinement.

3.4 Functional inequalities related to the steady state

These properties are an immediate consequence of Lemma 3.4 and the Holley-Strook perturbation principle, and we just point out that (3.17) and (3.18) follow from an integration by part argument.

Lemma 3.10 (Uniform bounds). *Under the assumptions of Proposition 3.5 and assuming that for any $\varepsilon > 0$ there exists some $C_\varepsilon > 0$ such that*

$$|\nabla^2 V| \leq \varepsilon |\nabla V| + C_\varepsilon,$$

then any fixed point ρ_\star of \mathcal{T} is such that, for any $\varepsilon > 0$, there exists a constant $C_{\star, \varepsilon} = C_{\star, \varepsilon}(N, \bar{\kappa}, \zeta, R_V, C_\varepsilon) > 0$ such that

$$|\nabla^2 V_\star| \leq \varepsilon |\nabla V_\star| + C_{\star, \varepsilon}. \quad (3.14)$$

Furthermore, there exists some $C_\star > 0$ such that

$$\|(1 + |\nabla V_\star|^2) \rho_\star\|_{L^1 \cap L^\infty} \leq C_\star. \quad (3.15)$$

Lemma 3.11 (Poincaré inequalities). *Under the assumptions of Lemma 3.10 and assuming that for any*

$$u \in H^1(e^{-V}) \quad \text{such that} \quad \int_{\mathbb{R}^d} u(x) e^{-V(x)} dx = 0$$

the Poincaré inequality holds:

$$\|u\|_{L^2(e^{-V})} \leq C_P \|\nabla u\|_{L^2(e^{-V})},$$

then, for any fixed point $\rho_\star = e^{-V_\star}$ of \mathcal{T} , there exists some constant $C_\star = C_\star(N, \bar{\kappa}, \zeta, R_V, C_P) > 0$ satisfying for any

$$u \in H^1(\rho_\star) \quad \text{such that} \quad \int_{\mathbb{R}^d} u(x) \rho_\star(x) dx = 0$$

the Poincaré inequality:

$$\|u\|_{L^2(\rho_\star)} \leq C_\star \|\nabla u\|_{L^2(\rho_\star)}, \quad (3.16)$$

as well as the following weighted variations:

$$\|u |\nabla V_\star|\|_{L^2(\rho_\star)} \leq C_\star \|\nabla u\|_{L^2(\rho_\star)}, \quad (3.17)$$

$$\|u |\nabla V_\star|^2\|_{L^2(\rho_\star)} \leq C_\star \left(\|\nabla u\|_{L^2(\rho_\star)} + \|\nabla u |\nabla V_\star|\|_{L^2(\rho_\star)} \right). \quad (3.18)$$

Lemma 3.12 (Weighted Sobolev spaces comparison). *Under the assumptions of Proposition 3.11 and for any fixed point $\rho_\star = e^{-V_\star}$ of \mathcal{T} , there exists some $C = C(C_\star)$ such that*

$$\frac{1}{C} \|u \rho_\star^{1/2}\|_{H^s} \leq \|u\|_{H^s(\rho_\star)} \leq C \|u\|_{H^s}, \quad s \in [0, 1], \quad (3.19)$$

furthermore, denoting $\nabla^\star := -\nabla + \nabla V_\star$, there also holds for $s = 1$

$$\frac{1}{C} \|u\|_{H^1(\rho_\star)} \leq \|u\|_{L^2(\rho_\star)} + \|\nabla^\star u\|_{L^2(\rho_\star)} \leq C \|u\|_{H^1(\rho_\star)}. \quad (3.20)$$

Proof. The bounds of (3.19) follow from (3.15) and (3.17) for $s = 0, 1$, which then holds for $s \in [0, 1]$ by interpolation. To prove (3.20), one writes

$$\begin{aligned}\|\nabla u\|_{L^2(\rho_\star)}^2 &= \langle \nabla u, \nabla u \rangle_{L^2(\rho_\star)} \\ &= \langle \nabla^\star u, \nabla^\star u \rangle_{L^2(\rho_\star)} + \langle [\nabla^\star, \nabla] u, u \rangle_{L^2(\rho_\star)}.\end{aligned}$$

Observing that $[\nabla, \nabla^\star] = \Delta V_\star$, we deduce from (3.14) and (3.17) that the second term is in fact a lower order term:

$$\left| \langle [\nabla^\star, \nabla] u, u \rangle_{L^2(\rho_\star)} \right| \leq \frac{1}{2} \|\nabla u\|_{L^2(\rho_\star)}^2 + C \|u\|_{L^2(\rho_\star)}^2$$

and therefore

$$-C \|u\|_{L^2(\rho_\star)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(\rho_\star)}^2 \leq \|\nabla^\star u\|_{L^2(\rho_\star)} \leq C \|u\|_{L^2(\rho_\star)} + \frac{1}{2} \|\nabla u\|_{L^2(\rho_\star)}^2,$$

which allows to conclude. \square

3.5 Proof of Theorem 3.1 and 2.10

Let F_\star solve (1.1). Then observe that $G_\star(x, v) = e^{-|v|^2 - V(x) - \Psi_{F_\star}(x)}$ also solves (1.1). Then by multiplying the equation by F_\star/G_\star one obtains

$$\|\nabla_v(F_\star/G_\star)\|_{L^2(G_\star)}^2 = 0$$

which shows that $F_\star(x, v) = \rho_\star(x)M(v)$. By plugging back this ansatz into (1.1), it shows that ρ_\star solves (3.2). The rest of the proof of Theorem 3.1 is a combination of the results of Proposition 3.2 to Corollary 3.7. For the proof of Theorem 2.10, just observe that Assumption 2.1 and Assumption 2.2 implies all the hypotheses of Theorem 2.10. In particular, since $L^p \cap \dot{W}^{1,q}$ embeds into L^∞ for $q > d$, the assumption on $\mathcal{K}^\star(e^{-V})$ is automatically satisfied and $\zeta = \zeta(\bar{\kappa}, R_V)$. Finally the functional inequalities of Section 3.4 hold and we also point out that, all constants depending on $\bar{\kappa}$ can be considered to depend on the larger bound $\bar{\kappa}_{\max}$.

4 Linearized VFP

In this section, we study decay and regularization properties of the linearized flow of the equation (1.1) around the equilibrium $F_\star = \rho_\star M$ of Section 3, where $M(v) = (2\pi)^{-d/2} \exp(-|v|^2/2)$. For that we introduce a perturbation f satisfying

$$f = \frac{F - F_\star}{F_\star}.$$

The functional framework will rely on the Hilbert space $L^2(F_\star)$. In this space the adjoints of ∇_x and ∇_v are respectively

$$\nabla_x^\star = -\nabla_x + \nabla_x V_\star \quad \text{and} \quad \nabla_v^\star = -\nabla_v + v.$$

We introduce the subspace

$$\mathcal{H}_0 = \left\{ f \in L^2(F_\star) \text{ such that } \int_{\mathbb{R}^{2d}} f F_\star dx dv = 0 \right\},$$

and the orthogonal projection in $L^2(F_\star)$ onto local equilibrium

$$\Pi f(x) = \int_{\mathbb{R}^d} f(x, v) M(v) dv = \frac{\rho_f(x)}{\rho_\star(x)} \quad \text{with} \quad \rho_f(x) = \int_{\mathbb{R}^d} f(x, v) F_\star(x, v) dv.$$

Observe that $\rho_f = \rho_{\Pi f}$. Let us write the linerized equation. By plugging $F = F_\star(1 + f)$ into (1.1), one finds the linearized VFP equation

$$\begin{cases} (\partial_t + \Lambda)f(t) + v \cdot \nabla_x \psi_f(t) = \nabla_v^* \varphi(t), \\ \Lambda = \nu \nabla_v^* \nabla_v + v \cdot \nabla_x - \nabla_x V_\star \cdot \nabla_v \\ \psi_f = \mathcal{K} \rho_f, \\ f(0, x, v) = f_{\text{in}}(x, v), \end{cases} \quad (4.1)$$

with $F_{\text{in}} = F_\star(1 + f_{\text{in}})$. In the original equation the right-hand side is given with $\varphi = f \nabla_x \psi_f$. However, in this section, we will assume that it is a given source. In the following lemmas and propositions, we assume that the general hypotheses of the paper, Assumption 2.1 and Assumption 2.2, are satisfied. We will write C_\star for any positive constant depending only on the steady state and therefore on $\bar{\kappa}_{\text{max}}$, R_V and θ only. Note that $-\Lambda$ generates a \mathcal{C}^0 -semigroup on \mathcal{H} (see [30, Proposition 5.5]), and thus $-(\Lambda + v \cdot \nabla_x \psi_{(\cdot)})$ as well since $v \cdot \nabla \psi_{(\cdot)}$ is a bounded operator in \mathcal{H} in virtue of Lemma 4.1.

4.1 Preliminary properties

The following bounds are a translation of the assumption on \mathcal{K} to the current $L^2(F_\star)$ framework and are a consequence of Hölder's inequality. We use the notation $\psi_f^\alpha = \mathcal{K}^\alpha \rho_f$ with $\alpha = \text{o, e}$.

Lemma 4.1. *The following upper bounds hold for $\alpha = \text{e, o}$*

$$\|\psi_f^\alpha\|_{L^2(F_\star)} \leq C_\star \bar{\kappa}^\alpha \|\Pi f\|_{L^2(F_\star)}, \quad (4.2)$$

$$\|\nabla_x \psi_f^\alpha\|_{L^2(F_\star)} + \|\nabla_x \psi_f^\alpha\|_{L^q} \leq C_\star \bar{\kappa}^\alpha \|\Pi f\|_{L^2(F_\star)}, \quad (4.3)$$

as well as the lower bound:

$$\langle \psi_f, \rho_f \rangle_{L^2} = \langle \psi_f^{\text{e}}, \rho_f \rangle_{L^2} \geq -(C_\star \theta + (1 - \theta)) \underline{\kappa}^{\text{e}} \|\Pi f\|_{L^2(F_\star)}^2. \quad (4.4)$$

The next lemma is a consequence of the Poincaré inequalities and other results of Section 3.4.

Lemma 4.2. *Let $f \in \mathcal{H}_0 \subset L^2(F_\star)$ and $g \in L^2(F_\star)$. Then there is $C = C(C_\star)$ such that*

$$\|\Pi f\|_{L^2(F_\star)} \leq C \|\nabla_x f\|_{L^2(F_\star)}, \quad (4.5)$$

$$\|(\text{Id} - \Pi)g\|_{L^2(F_\star)} \leq \|\nabla_v g\|_{L^2(F_\star)}, \quad (4.6)$$

$$\|\Pi g\|_{L^2(F_\star)} \leq \|g\|_{L^2(F_\star)} \leq C (\|\nabla_x g\|_{L^2(F_\star)} + \|\nabla_v g\|_{L^2(F_\star)}), \quad (4.7)$$

Moreover

$$\|\nabla_v^* g\|_{L^2(F_\star)}^2 = \|g\|_{L^2(F_\star)}^2 + \|\nabla_v g\|_{L^2(F_\star)}^2. \quad (4.8)$$

Finally, for any $\varepsilon > 0$ there is $C_\varepsilon = C_\varepsilon(C_\star)$ such that

$$\|\nabla^2 V_\star |g\rangle\|_{L^2(F_\star)} \leq C_\varepsilon \|g\|_{L^2(F_\star)} + \varepsilon \|\nabla_x g\|_{L^2(F_\star)} \quad (4.9)$$

4.2 L^2 -hypocoercivity

In the following we want to apply the L^2 hypocoercivity techniques of [23]. Taking inspiration from [1], we will introduce a suitable norm which will allow us to include some part the interaction term $v \cdot \nabla \psi_f$ into the dissipative mechanisms of the equation instead of a mere perturbation of the hypocoercive operator Λ . Accordingly, let us rewrite the equation (4.1) as

$$\begin{cases} (\partial_t + T - L)f = -v \cdot \nabla_x \psi_f^e + \nabla_v^* \varphi, \\ f(0, x, v) = f_{\text{in}}(x, v), \end{cases} \quad (4.10)$$

where the operators T and L are defined as

$$L := -\nu \nabla_v^* \nabla_v \quad \text{and} \quad T := v \cdot \nabla_x - \nabla_x V_\star \cdot \nabla_v + v \cdot \nabla_x \psi_f^e.$$

4.2.1 The linearized free energy norm

We endow the subspace \mathcal{H}_0 with a norm for which L and T will be shown to be respectively symmetric and skew-symmetric. Let us consider the twisted inner product

$$\langle\langle f, g \rangle\rangle := \langle f, g \rangle_{L^2(F_\star)} + \langle \psi_f^e, \rho_g \rangle_{L^2} = \langle f + \psi_f^e, g \rangle_{L^2(F_\star)}, \quad (4.11)$$

and the corresponding (squared) norm

$$\|f\|^2 := \langle\langle f, f \rangle\rangle = \iint_{\mathbb{R}^{2d}} f(x, v)^2 F_\star(x, v) dx dv + \int_{\mathbb{R}^d} (\mathcal{K}^e \rho_f)(x) \rho_f(x) dx. \quad (4.12)$$

Remark 4.3. In the case of a purely symmetric kernel $k = k^e$, observe that (4.12) is related to the natural Lyapunov functional of the equation: the free energy (1.5). More precisely, if one plugs in $F = F_\star(1 + \varepsilon f)$ and lets $\varepsilon \rightarrow 0$, (4.12) is the main order contribution. That is why we refer to it as the linearized free energy norm. This observation is a generalization to arbitrary kernels of the one made in [1].

4.2.2 The Dolbeault-Mouhot-Schmeiser functional \mathcal{E}_0

For the new inner product (4.12) we use the superscript \dagger to denote the adjoint of an operator for the inner product (4.11). Following [23], we introduce, for some $\varepsilon \in (0, 1/2)$, the following functional:

$$\mathcal{E}_0[f] := \frac{1}{2} \|f\|^2 + \varepsilon \langle\langle Af, f \rangle\rangle \quad \text{where} \quad A := (\text{Id} + (T\Pi)^\dagger T\Pi)^{-1} (T\Pi)^\dagger.$$

On the one hand, we will show that the functional \mathcal{E}_0 satisfies a Lyapunov inequality, and on the other hand that both quantities $\mathcal{E}_0^{\frac{1}{2}}$ and $\|\cdot\|$ are equivalent to the norm of $L^2(F_\star)$.

Lemma 4.4. *For a small enough $\delta^e = \delta^e(C_\star, \theta) > 0$, assuming that $\underline{\kappa}^e < \delta^e$, one has for some $C = C(\bar{\kappa}^e, C_\star)$ that*

$$\frac{1}{2}\|f\|_{L^2(F_\star)}^2 \leq \|f\|^2 \leq C\|f\|_{L^2(F_\star)}^2.$$

The inner product is compatible with the micro–macro decomposition in the sense that

$$\Pi^\dagger = \Pi \quad \text{and} \quad \ker(L)^{\perp_{\mathcal{H}_0}} = \ker(L)^{\perp_{L^2(F_\star)}},$$

and furthermore

$$\langle\langle \text{Id} - \Pi)f, g \rangle\rangle = \langle (\text{Id} - \Pi)f, g \rangle_{L^2(F_\star)} \quad (4.13)$$

Finally, the diffusion and transport operators L and T are skew-symmetric and symmetric:

$$T^\dagger = -T \quad \text{and} \quad L = L^\dagger.$$

Proof. This follows from Lemma 4.1, and more precisely (4.2) and (4.4). The properties of T, L and Π can be proved exactly as in [1, Lemma 15 and 16]. \square

4.2.3 Properties of T, L, A and Π

We now prove some properties concerning the previously introduced operators. For properties that can be proved exactly as in [1] we sketch the proof at most. We only detail proofs when they differ substantially or when we concern additional properties not proved in [1].

Lemma 4.5. *The operator T , as an operator of \mathcal{H}_0 , satisfies the parabolic dynamics property:*

$$\Pi T \Pi = 0,$$

and, more precisely, there holds for any $f \in \mathcal{H}_0$

$$T \Pi f = (\text{Id} - \Pi) T \Pi f = v \cdot \nabla_x (\Pi f + \psi_f^e)$$

as well as

$$\Pi T f = \Pi T (\text{Id} - \Pi) f = -\nabla_x^* \Pi(vf).$$

Furthermore, assuming $\underline{\kappa}^e < \delta^e(C_\star, \theta)$, there is some $\lambda_M(C_\star, \theta) > 0$ such that the following macroscopic coercivity estimate hold, namely

$$\|T \Pi f\| \geq \lambda_M \|\Pi f\|. \quad (4.14)$$

Proof. Performing the same computations as [1, Lemma 18] with the Poincaré inequality (3.16), one has

$$\|T \Pi f\|_{L^2(F_\star)}^2 \geq \frac{1}{C_\star} \left(\|\Pi f\|_{L^2(\rho_\star)}^2 + 2 \int_{\mathbb{R}^d} \psi_f^e(x) \rho_f(x) dx \right)$$

thus, using the definition of $\|\cdot\|$ and the lower bound (4.4), we have for some $C = C(\theta, C_\star)$

$$\|T\Pi f\|_{L^2(F_\star)}^2 \geq \frac{1}{C_\star} (1 - \underline{\kappa}^e C) \|\Pi f\|_{L^2(\rho_\star)}^2,$$

which allows to conclude. \square

This next proposition can be proved following [1, Lemma 20].

Lemma 4.6. *The auxiliary operator A is such that*

$$A = \Pi A(\text{Id} - \Pi) : \ker(L)^\perp \rightarrow \ker(L)$$

and satisfies the following boundedness properties:

$$\|ALf\| \leq \frac{\nu}{2} \|f\|, \quad \|TAf\| \leq \|f\|, \quad \|Af\| \leq \frac{1}{2} \|f\|.$$

Lemma 4.7. *The composition AT satisfies for some $C = C(C_\star, \bar{\kappa}^e)$ the bound*

$$\|AT(\text{Id} - \Pi)f\| \leq C \|(\text{Id} - \Pi)f\|.$$

Proof. In this proof, we denote by $C = C(C_\star, \bar{\kappa}^e)$ a constant that may change from line to line. The proof follows the lines of the proof of [1, Lemma 21], so we focus on the most intricate estimate (4.17). Let us however point out that this boundedness result is proved by establishing it for the adjoint $(AT(\text{Id} - \Pi))^\dagger$ and reducing the problem to

$$\begin{aligned} & \| (AT(\text{Id} - \Pi))^\dagger f \|_{L^2(F_\star)}^2 \\ & \leq C \left(\|g\|_{L^2(F_\star)}^2 + \|f\|_{L^2(F_\star)}^2 + \|\nabla_x w_g |\nabla_x V_\star|\|_{L^2(F_\star)}^2 \right), \end{aligned}$$

where $g \in \ker(L)$ is characterized by the elliptic equation

$$g + \nabla_x^* \nabla_x w_g = f \quad \text{with} \quad w_g := g + \psi_g^e. \quad (4.15)$$

Taking the $L^2(F_\star)$ -inner product against $g + \psi_g^e$, one can prove the control

$$\|g\|^2 + \|\nabla_x g\|_{L^2(F_\star)}^2 + \|\nabla_x w_g\|_{L^2(F_\star)}^2 \leq C \|f\|^2. \quad (4.16)$$

Let us explain how to establish the bound

$$\|\nabla_x w_g |\nabla_x V_\star|\|_{L^2(F_\star)}^2 \leq C \|f\|^2. \quad (4.17)$$

Taking the $L^2(F_\star)$ -inner product of (4.15) with $g |\nabla_x V_\star|^2$,

$$\langle \nabla_x g, \nabla_x (g |\nabla_x V_\star|^2) \rangle_{L^2(F_\star)} + \langle \nabla_x \psi_g^e, \nabla_x (g |\nabla_x V_\star|^2) \rangle_{L^2(F_\star)} = \langle g - f, g |\nabla_x V_\star|^2 \rangle_{L^2(F_\star)},$$

thus, using the Leibniz rule, we get

$$\begin{aligned} \left\| \nabla_x g |\nabla_x V_\star| \right\|_{L^2(F_\star)}^2 &= - \langle \nabla_x g, g \nabla_x (|\nabla_x V_\star|^2) \rangle_{L^2(F_\star)} - \langle \nabla_x \psi_g^e, g \nabla_x (|\nabla_x V_\star|^2) \rangle_{L^2(F_\star)} \\ &\quad - \langle \nabla_x \psi_g^e, \nabla_x g |\nabla_x V_\star|^2 \rangle_{L^2(F_\star)} \\ &\quad + \left\| g |\nabla_x V_\star| \right\|_{L^2(F_\star)}^2 - \langle f, g |\nabla_x V_\star|^2 \rangle_{L^2(F_\star)} =: \sum_{k=1}^5 X_k. \end{aligned}$$

The term X_1 is estimated using that $\nabla_x (|\nabla_x V_\star|^2) \leq C |\nabla_x V_\star|^2 + C$ from (3.14):

$$|X_1| \leq C \int_{\mathbb{R}^d} |\nabla_x g| |g| \rho_\star dx + C \int_{\mathbb{R}^d} |\nabla_x g| |g| |\nabla_x V_\star|^2 \rho_\star dx$$

and thus, for any $\delta \in (0, 1)$

$$\begin{aligned} |X_1| &\leq C \|g\|_{L^2(F_\star)}^2 + C \|\nabla_x g\|_{L^2(F_\star)}^2 \\ &\quad + \frac{C}{\delta} \|g |\nabla_x V_\star|\|_{L^2(F_\star)}^2 + \delta \|\nabla_x g |\nabla_x V_\star|\|_{L^2(F_\star)}^2. \end{aligned} \quad (4.18)$$

Similarly, the term X_2 is controlled using Hölder's inequality for $\frac{1}{2} = \frac{1}{q} + \frac{1}{s}$ followed by (4.3) and (3.15):

$$\begin{aligned} |X_2| &\leq C \int_{\mathbb{R}^d} |\nabla_x \psi_g^e| |g| \rho_\star dx + C \int_{\mathbb{R}^d} |\nabla_x \psi_g^e| \times |g| |\nabla_x V_\star| \rho_\star^{\frac{1}{2}} \times |\nabla_x V_\star| \rho_\star^{\frac{1}{2}} dx \\ &\leq C \|\nabla_x \psi_g^e\|_{L^2(F_\star)} \|g\|_{L^2(F_\star)} + C \|\nabla_x \psi_g^e\|_{L^q} \|g |\nabla_x V_\star|\|_{L^2(F_\star)} \| |\nabla_x V_\star|^2 \rho_\star \|_{L^{s/2}}^{\frac{1}{2}} \\ &\leq C \left(\|g\|_{L^2(F_\star)}^2 + \|g |\nabla_x V_\star|\|_{L^2(F_\star)}^2 \right). \end{aligned} \quad (4.19)$$

The term X_3 is controlled similarly:

$$\begin{aligned} |X_3| &\leq \int_{\mathbb{R}^d} |\nabla_x \psi_g^e| |\nabla_x g| |\nabla_x V_\star|^2 \rho_\star dx \\ &\leq C \|\nabla_x \psi_g^e\|_{L^q} \|\nabla_x g |\nabla_x V_\star|\|_{L^2(F_\star)} \| |\nabla_x V_\star|^2 \rho_\star \|_{L^{s/2}} \\ &\leq \frac{C}{\delta} \|g\|_{L^2(F_\star)}^2 + \delta \|\nabla_x g |\nabla_x V_\star|\|_{L^2(F_\star)}^2. \end{aligned} \quad (4.20)$$

The terms X_4 and X_5 are simply controlled as

$$|X_4| + |X_5| \leq \|g |\nabla_x V_\star|\|_{L^2(F_\star)}^2 + \frac{1}{\delta} \|f\|_{L^2(F_\star)}^2 + \delta \|g |\nabla_x V_\star|^2\|_{L^2(F_\star)}^2. \quad (4.21)$$

Putting together (4.18)–(4.21), using the weighted Poincaré inequalities (3.17)–(3.18) (recall that g has zero mean), and using the bound (4.16) we have

$$\left\| \nabla_x g |\nabla_x V_\star| \right\|_{L^2(F_\star)}^2 \leq C \|f\|^2.$$

Finally, as for (4.19)–(4.20), one proves a similar weighted Sobolev estimate for ψ_g , which together with the bound above yields (4.17). This concludes the proof. \square

4.2.4 Properties of the functional \mathcal{E}_0

Lemma 4.8. *For $\varepsilon \in (0, 1/2]$, the functional \mathcal{E}_0 is equivalent to the norm $\|\!\| \cdot \|\!\|$ in the sense that*

$$\frac{1}{4} \|\!\| f \|\!\|^2 \leq \mathcal{E}_0[f] \leq \frac{3}{4} \|\!\| f \|\!\|^2.$$

Furthermore, under the smallness assumptions

$$\underline{\kappa}^e < \delta^e(C_\star, \theta) \quad \text{and} \quad \bar{\kappa}^o < \delta^o(C_\star, \bar{\kappa}^e, \underline{\kappa}^e, \nu)$$

for some appropriate choice of ε , there exists λ, μ and C depending on $C_\star, \bar{\kappa}^e, \underline{\kappa}^e, \bar{\kappa}^o, \theta$ and ν such that any solution to (4.1) satisfies the differential inequality

$$\frac{d}{dt} \mathcal{E}_0[f] + \lambda \mathcal{E}_0[f] + \mu \|\nabla_v f\|_{L^2(F_\star)}^2 \leq C \|(\text{Id} - \Pi)\varphi\|_{L^2(F_\star)}^2.$$

Proof. The equivalence of norms follows from the bound of A provided by Lemma 4.6 which gives

$$\frac{1 - \varepsilon}{2} \|\!\| f \|\!\|^2 \leq \mathcal{E}_0[f] \leq \frac{1 + \varepsilon}{2} \|\!\| f \|\!\|^2.$$

Differentiating $\mathcal{E}_0[f]$, and using that

$$\begin{aligned} v \cdot \nabla_x \psi_f^o &\in \ker(L)^\perp, \quad Lf \in \ker(L)^\perp, \quad \text{and} \quad \nabla_v^* \varphi \in \ker(L)^\perp, \\ T^\dagger &= -T, \quad \text{and} \quad Af \in \ker(L) \end{aligned}$$

one has the identity

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_0[f] &= \langle\langle \partial_t f, f \rangle\rangle + \varepsilon \langle\langle A \partial_t f, f \rangle\rangle + \varepsilon \langle\langle Af, \partial_t f \rangle\rangle \\ &= \langle\langle Lf, f \rangle\rangle + \langle\langle v \cdot \nabla_x \psi_f^o, f \rangle\rangle \\ &\quad - \varepsilon \langle\langle ATf, f \rangle\rangle + \varepsilon \langle\langle ALf, f \rangle\rangle + \varepsilon \langle\langle A(v \cdot \nabla_x \psi_f^o), f \rangle\rangle \\ &\quad - \varepsilon \langle\langle Af, Tf \rangle\rangle \\ &\quad + \langle\langle f, \nabla_v^* \varphi \rangle\rangle + \varepsilon \langle\langle Af, \nabla_v^* \varphi \rangle\rangle =: \sum_{k=1}^8 X_k. \end{aligned}$$

The term X_1 is estimated using the microscopic coercivity estimate (4.6) and the fact that $\|\cdot\|_{L^2(F_\star)} = \|\!\| \cdot \|\!\|$ on $\ker(L)^\perp$:

$$X_1 \leq -\frac{\nu}{2} \|\!\| (\text{Id} - \Pi)f \|\!\|^2 - \frac{\nu}{2} \|\nabla_v f\|_{L^2(F_\star)}^2,$$

Noting that $(T\Pi)^\dagger T\Pi$ is a self-adjoint operator whose spectrum lies in $[\lambda_M, +\infty)$ by Lemma 4.5, one has by functional calculus

$$\langle\langle AT\Pi f, \Pi f \rangle\rangle \geq \frac{\lambda_M}{1 + \lambda_M} \|\!\| \Pi f \|\!\|^2,$$

thus, the term X_3 is estimated for any $\delta \in (0, 1)$ as

$$\begin{aligned} X_3 &= -\varepsilon \langle \langle ATf, \Pi f \rangle \rangle \leq -\frac{\varepsilon \lambda_M}{1 + \lambda_M} \|\Pi f\|^2 - \varepsilon \langle \langle AT(\text{Id} - \Pi)f, \Pi f \rangle \rangle \\ &\leq -\varepsilon \left(\frac{\lambda_M}{1 + \lambda_M} - \delta \right) \|\Pi f\|^2 + \frac{\varepsilon C}{\delta} \|(\text{Id} - \Pi)f\|^2 \end{aligned}$$

for some constant $C = C(\bar{\kappa}^\varepsilon, C_\star)$ by Lemma 4.7. Similarly, the boundedness of TA from Lemma 4.6 yields

$$\begin{aligned} X_6 &= \varepsilon \langle \langle TA(\text{Id} - \Pi)f, f \rangle \rangle \\ &\leq \varepsilon \|\|TA(\text{Id} - \Pi)f\|\| \|\Pi f\| + \varepsilon \|\|TA(\text{Id} - \Pi)f\|\| \|(\text{Id} - \Pi)f\| \\ &\leq \delta \varepsilon \|\Pi f\|^2 + \varepsilon \left(1 + \frac{1}{4\delta} \right) \|(\text{Id} - \Pi)f\|^2, \end{aligned}$$

and that of AL

$$\begin{aligned} X_4 &= \varepsilon \langle \langle AL(\text{Id} - \Pi)f, \Pi f \rangle \rangle \leq \varepsilon \delta \|\Pi f\|^2 + \frac{\varepsilon}{4\delta} \|AL(\text{Id} - \Pi)f\|^2 \\ &\leq \varepsilon \delta \|\Pi f\|^2 + \frac{\varepsilon \nu^2}{16\delta} \|(\text{Id} - \Pi)f\|^2. \end{aligned}$$

The remaining force terms are controlled using the fact that $\langle \cdot, \cdot \rangle$ coincide with the $L^2(F_\star)$ -inner product on microscopic distributions, as well as the upper bound (4.3) on ψ_f° :

$$X_2 = \langle v \cdot \nabla_x \psi_f^\circ, (\text{Id} - \Pi)f \rangle_{L^2(F_\star)} \leq C \bar{\kappa}^\circ \left(\|\Pi f\|^2 + \|(\text{Id} - \Pi)f\|^2 \right)$$

where we used the equivalence of norms in the last line, and similarly, using that A has operator norm $1/2$ from Lemma 4.6

$$X_5 \leq C \varepsilon \bar{\kappa}^\circ \|\Pi f\|^2.$$

Finally, since $\nabla_v^* \varphi$ is microscopic in the sense that $\Pi(\nabla_v^* \varphi) = 0$, one has in virtue of Lemma 4.6 that $X_8 = 0$ as well as

$$\begin{aligned} X_7 &= \langle \langle f, \nabla_v^* \varphi \rangle \rangle = \langle (\text{Id} - \Pi)f, \nabla_v^* \varphi \rangle_{L^2(F_\star)} \\ &\leq \frac{\nu}{4} \|\nabla_v f\|_{L^2(F_\star)}^2 + \frac{1}{\nu} \|(\text{Id} - \Pi)\varphi\|_{L^2(F_\star)}^2. \end{aligned}$$

Put together, these estimates yield

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_0[f] &\leq -\frac{\nu}{4} \|\nabla_v f\|_{L^2(F_\star)}^2 \\ &\quad - \left\{ \frac{\nu}{2} - \frac{\varepsilon C}{4\delta} - \varepsilon \left(1 + \frac{1}{4\delta} \right) - \frac{\varepsilon \nu^2}{8\delta} - C \bar{\kappa}^\circ \right\} \|(\text{Id} - \Pi)f\|^2 \\ &\quad - \left\{ \varepsilon \left(\frac{\lambda_M}{1 + \lambda_M} - 3\delta \right) - (1 + \varepsilon) C \bar{\kappa}^\circ \right\} \|\Pi f\|^2 \\ &\quad + \frac{1}{\nu} \|(\text{Id} - \Pi)\varphi\|_{L^2(F_\star)}^2. \end{aligned}$$

Choosing $\varepsilon = \delta^2$ and using the upper bound (4.3), this simplifies for some $C = C(\bar{\kappa}^e, C_\star)$ as

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_0[f] &\leq -\frac{\nu}{4} \|\nabla_v f\|_{L^2(F_\star)}^2 \\ &\quad - \left\{ \frac{\nu}{2} - \delta C - 2\delta - \delta\nu - C\bar{\kappa}^o \right\} \|(\text{Id} - \Pi)f\|^2 \\ &\quad - \left\{ \delta^2 \left(\frac{\lambda_M}{1 + \lambda_M} - 3\delta \right) - C\bar{\kappa}^o \right\} \|\Pi f\|^2 \\ &\quad + \frac{1}{\nu} \|(\text{Id} - \Pi)\varphi\|_{L^2(F_\star)}^2. \end{aligned}$$

The result then holds taking δ small enough, depending on C , ν and λ_M , so that the dissipation rates of $(\text{Id} - \Pi)f$ and Πf are of order ν and δ^2 respectively, and then $\bar{\kappa}^o$ small enough, depending on ν and δ^2 . \square

We can now state the main result of this section, which is an immediate consequence of Lemmas 4.8 and 4.4.

Proposition 4.9 ($L^2_{x,v}$ -hypocoercivity). *Under the smallness assumptions*

$$\underline{\kappa}^e < \delta^e(C_\star, \theta) \quad \text{and} \quad \bar{\kappa}^o < \delta^o(C_\star, \bar{\kappa}^e, \underline{\kappa}^e, \nu)$$

there exists λ and C depending on $C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e$ and ν such that any solution to (4.1) satisfies the differential inequality

$$\begin{aligned} \sup_{t \geq 0} e^{2\lambda t} \|f(t)\|_{L^2(F_\star)}^2 + \int_0^\infty e^{2\lambda t} \|\nabla_v f(t)\|_{L^2(F_\star)}^2 dt \\ \leq C \left(\|f_{\text{in}}\|_{L^2(F_\star)}^2 + \int_0^\infty e^{2\lambda t} \|\varphi(t)\|_{L^2(F_\star)}^2 dt \right). \end{aligned}$$

From Proposition 4.9, one can derive a well-posedness and asymptotic stability result for the nonlinear VFP equation (1.1) if Assumption 2.2 is strengthened by assuming that $\nabla \mathcal{K} : L^1 \cap L^2 \rightarrow L^\infty$ is a bounded operator, namely $q = \infty$. For kernels providing less integrability/regularity, one need to resort to higher order estimates and regularization properties of the equation, which is the purpose of the rest of Section 4.

4.3 Derivatives estimates

In this section, we derive preliminary estimates on the derivatives of f .

Lemma 4.10. *Let f be a solution to (4.1), its velocity gradient satisfies the equation*

$$\partial_t(\nabla_v f) + \Lambda(\nabla_v f) + \nu \nabla_v f = -\nabla_x f - \nabla_x \psi_f + \nabla_v \nabla_v^* \varphi.$$

as well as the differential inequality

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla_v f\|_{L^2(F_\star)}^2 + \frac{\nu}{2} \|\nabla_v^2 f\|_{L^2(F_\star)}^2 \\ & \leq \delta \|\nabla_x f\|_{L^2(F_\star)}^2 + C \left(\delta^{-1} \|\nabla_v f\|_{L^2(F_\star)}^2 + \|\varphi\|_{L^2(F_\star)}^2 + \|\nabla_v \varphi\|_{L^2(F_\star)}^2 \right) \end{aligned} \quad (4.22)$$

where $\delta \in (0, 1)$, and for some $C = C(\bar{\kappa}^e, \bar{\kappa}^o, C_\star, \nu)$.

Proof. The equation satisfied by $\nabla_v f$ is obtained by noticing that the following commutator identity holds:

$$[\nabla_v, \Lambda] = \nabla_x + \nu \nabla_v.$$

Integrating against $\nabla_v f$, there holds

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla_v f\|_{L^2(F_\star)}^2 + \nu \|\nabla_v^2 f\|_{L^2(F_\star)}^2 + \nu \|\nabla_v f\|_{L^2(F_\star)}^2 \\ & = -\langle \nabla_x f, \nabla_v f \rangle_{L^2(F_\star)} - \langle \nabla_x \psi_f, \nabla_v f \rangle_{L^2(F_\star)} + \langle \nabla_v^* \varphi, \nabla_v^* \nabla_v f \rangle_{L^2(F_\star)} \\ & \leq \delta \left(\|\nabla_x f\|_{L^2(F_\star)}^2 + \|\nabla_x \psi_f\|_{L^2(F_\star)}^2 \right) + \varepsilon \|\nabla_v^* \nabla_v f\|_{L^2(F_\star)}^2 \\ & \quad + \frac{1}{\delta} \|\nabla_v f\|_{L^2(F_\star)}^2 + \frac{1}{\varepsilon} \|\nabla_v^* \varphi\|_{L^2(F_\star)}^2. \end{aligned}$$

From there on the one hand one uses (4.8) with $g = \nabla_v f$. On the other hand, one combines (4.3) with the Poincaré inequality (4.5) to get for some $C = C(C_\star, \bar{\kappa})$

$$\|\nabla_x \psi_f\|_{L^2(F_\star)} \leq C \|\nabla_x f\|_{L^2(F_\star)}.$$

In turn, one concludes to (4.22) by taking ε small enough with respect to ν . \square

Lemma 4.11. *Let f be a solution to (4.1), its spatial gradient satisfies the equation*

$$\partial_t(\nabla_x f) + \Lambda(\nabla_x f) = (\nabla_x^2 V_\star) \nabla_v f + (v \cdot \nabla_x \psi [\partial_{x_i}^* f])_{i=1}^d + \nabla_v^* \nabla_x \varphi,$$

as well as the differential inequality

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla_x f\|_{L^2(F_\star)}^2 + \frac{\nu}{2} \|\nabla_v \nabla_x f\|_{L^2(F_\star)}^2 \\ & \leq C \left(\|\nabla_x f\|_{L^2(F_\star)}^2 + \|\nabla_v f\|_{L^2(F_\star)}^2 + \|\nabla_x \varphi\|_{L^2(F_\star)}^2 \right), \end{aligned} \quad (4.23)$$

for some $C = C(\bar{\kappa}^e, \bar{\kappa}^o, C_\star, \nu)$.

Proof. The differential equation satisfied by $\nabla_x f$ is immediately obtained by noticing

$$[\nabla_x, \Lambda] = -(\nabla_x^2 V_\star) \nabla_v \quad \text{and} \quad \nabla_x \psi[f] = -\psi[\nabla_x^* f].$$

integrating against $\nabla_x f$, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla_x f\|_{L^2(F_\star)}^2 + \nu \|\nabla_v \nabla_x f\|_{L^2(F_\star)}^2 \\ &= \langle (\nabla_x^2 V_\star) \nabla_v f, \nabla_x f \rangle_{L^2(F_\star)} + \sum_{i=1}^d \langle v \cdot \nabla_x \psi [\partial_{x_i}^* f], \partial_{x_i} f \rangle_{L^2(F_\star)} \\ & \quad + \langle \nabla_x \varphi, \nabla_x \nabla_v f \rangle_{L^2(F_\star)}. \end{aligned}$$

Concerning the first source term, thanks to the weighted Poincaré inequality (4.9) with (say) $\varepsilon = 1$, one has for some $C = C(C_\star)$ which may change from line to line

$$\begin{aligned} & \langle (\nabla_x^2 V_\star) \nabla_v f, \nabla_x f \rangle_{L^2(F_\star)} \\ & \leq C (\|\nabla_v f\|_{L^2(F_\star)} + \|\nabla_x \nabla_v f\|_{L^2(F_\star)}) \|\nabla_x f\|_{L^2(F_\star)} \\ & \leq C \left(\varepsilon \|\nabla_x \nabla_v f\|_{L^2(F_\star)}^2 + \frac{1}{\varepsilon} \|\nabla_x f\|_{L^2(F_\star)}^2 + \|\nabla_v f\|_{L^2(F_\star)}^2 \right). \end{aligned}$$

Concerning the second source term, we have from the bound (4.3) and the Poincaré inequality (4.5) that for some $C = C(C_\star, \bar{\kappa}^e, \bar{\kappa}^o)$

$$\sum_{i=1}^d \langle v \cdot \nabla_x \psi [\partial_{x_i}^* f], \partial_{x_i} f \rangle_{L^2(F_\star)} \leq C \|\nabla_x f\|_{L^2(F_\star)}^2$$

Concerning the third term, we have that

$$\begin{aligned} \langle \nabla_x \varphi, \nabla_x \nabla_v f \rangle_{L^2(F_\star)} & \leq \|\nabla_x \varphi\|_{L^2(F_\star)} \|\nabla_x \nabla_v f\|_{L^2(F_\star)} \\ & \leq \frac{1}{2\nu} \|\nabla_x \varphi\|_{L^2(F_\star)}^2 + \frac{\nu}{2} \|\nabla_x \nabla_v f\|_{L^2(F_\star)}^2. \end{aligned}$$

This concludes the proof. \square

Lemma 4.12. *Let f be a solution to (4.1), the inner product of its partial gradients satisfies the differential inequality*

$$\begin{aligned} & \frac{d}{dt} \langle \nabla_x f, \nabla_v f \rangle_{L^2(F_\star)} + \frac{1}{2} \|\nabla_x f\|_{L^2(F_\star)}^2 \leq \delta \|\nabla_x \nabla_v f\|_{L^2(F_\star)}^2 + C \|f\|_{L^2(F_\star)}^2 \\ & \quad + \frac{C}{\delta} \left(\|\nabla_v f\|_{L^2(F_\star)}^2 + \|\nabla_v^2 f\|_{L^2(F_\star)}^2 + \|\varphi\|_{L^2(F_\star)}^2 + \|\nabla_v \varphi\|_{L^2(F_\star)}^2 \right), \end{aligned} \tag{4.24}$$

for any $\delta \in (0, \delta_1)$, where the constants C and δ_1 depend on $\bar{\kappa}^e, \bar{\kappa}^o, C_\star$ and ν .

Proof. On the one hand, there holds

$$\begin{aligned}
& \frac{d}{dt} \langle \nabla_v f, \nabla_x f \rangle_{L^2(F_\star)} \\
&= \langle \partial_t (\nabla_v f), \nabla_x f \rangle_{L^2(F_\star)} + \langle \partial_t (\nabla_x f), \nabla_v f \rangle_{L^2(F_\star)} \\
&= -\langle \Lambda (\nabla_v f), \nabla_x f \rangle_{L^2(F_\star)} - \|\nabla_x f\|_{L^2(F_\star)}^2 - \nu \langle \nabla_v f, \nabla_x f \rangle_{L^2(F_\star)} \\
&\quad - \langle \Lambda (\nabla_x f), \nabla_v f \rangle_{L^2(F_\star)} + \langle (\nabla_x^2 V_\star) \nabla_v f, \nabla_v f \rangle_{L^2(F_\star)} \\
&\quad - \langle \nabla_x \psi_f, \nabla_x f \rangle_{L^2(F_\star)} + \sum_{i=1}^d \langle v \cdot \nabla_x \psi [\partial_{x_i}^* f], \partial_{v_i} f \rangle_{L^2(F_\star)} \\
&\quad + \langle \nabla_v \nabla_v^* \varphi, \nabla_x f \rangle_{L^2(F_\star)} + \langle \nabla_x \nabla_v^* \varphi, \nabla_v f \rangle_{L^2(F_\star)} \\
&= -\|\nabla_x f\|_{L^2(F_\star)}^2 + X_1 + X_2 + X_3 + X_4,
\end{aligned}$$

where we denoted (and using that the symmetric part of Λ is $-L = \nu \nabla_v^* \nabla_v$)

$$\begin{aligned}
X_1 &= -\langle \Lambda (\nabla_v f), \nabla_x f \rangle_{L^2(F_\star)} - \langle \Lambda (\nabla_x f), \nabla_v f \rangle_{L^2(F_\star)} + \langle (\nabla_x^2 V_\star) \nabla_v f, \nabla_v f \rangle_{L^2(F_\star)} \\
&= -\nu \langle \nabla_v \nabla_x f, \nabla_v^2 f \rangle_{L^2(F_\star)} + \langle (\nabla_x^2 V_\star) \nabla_v f, \nabla_v f \rangle_{L^2(F_\star)}
\end{aligned}$$

as well as

$$\begin{aligned}
X_2 &:= -\nu \langle \nabla_v f, \nabla_x f \rangle_{L^2(F_\star)}, \\
X_3 &:= -\langle \nabla_x \psi_f, \nabla_x f \rangle + \sum_{i=1}^d \langle v \cdot \nabla_x \psi [\partial_{x_i}^* f], \partial_{v_i} f \rangle_{L^2(F_\star)}, \\
X_4 &:= \langle \nabla_v^* \varphi, \nabla_v^* \nabla_x f \rangle_{L^2(F_\star)} + \langle \nabla_v^* \varphi, \nabla_x^* \nabla_v f \rangle_{L^2(F_\star)}.
\end{aligned}$$

As in the proof of (4.23), one has for the term X_1 for some $C = C(C_\star, \nu)$ that may change from line to line

$$\begin{aligned}
X_1 &\lesssim C \|\nabla_v \nabla_x f\|_{L^2(F_\star)} (\|\nabla_v f\|_{L^2(F_\star)} + \nu \|\nabla_v^2 f\|_{L^2(F_\star)}) + C \|\nabla_v f\|_{L^2(F_\star)}^2 \\
&\lesssim \delta \|\nabla_v \nabla_x f\|_{L^2(F_\star)}^2 + \frac{1}{\delta} (\|\nabla_v f\|_{L^2(F_\star)}^2 + \|\nabla_v^2 f\|_{L^2(F_\star)}^2)
\end{aligned}$$

For the term X_2 , one has

$$X_2 \leq \nu \|\nabla_v f\|_{L^2(F_\star)} \|\nabla_x f\|_{L^2(F_\star)} \leq \nu^2 \|\nabla_v f\|_{L^2(F_\star)}^2 + \frac{1}{4} \|\nabla_x f\|_{L^2(F_\star)}^2.$$

As in the proof of (4.23), one has for the term X_3 for some $C = C(C_\star, \bar{\kappa}^e, \bar{\kappa}^o)$ that may change from line to line

$$X_3 \lesssim C (\|f\|_{L^2(F_\star)} \|\nabla_x f\|_{L^2(F_\star)} + \|\nabla_v f\|_{L^2(F_\star)} \|\nabla_x f\|_{L^2(F_\star)}),$$

therefore, there holds

$$X_3 \lesssim \varepsilon \|\nabla_x f\|_{L^2(F_\star)}^2 + \frac{C}{\varepsilon} (\|f\|_{L^2(F_\star)}^2 + \|\nabla_v f\|_{L^2(F_\star)}^2).$$

Finally, one can show for X_4

$$\begin{aligned} X_4 &\lesssim (\|\varphi\|_{L^2(F_\star)} + \|\nabla_v \varphi\|_{L^2(F_\star)}) (\|\nabla_x f\|_{L^2(F_\star)} + \|\nabla_v f\|_{L^2(F_\star)} + \|\nabla_v \nabla_x f\|_{L^2(F_\star)}) \\ &\lesssim \delta \left(\|\nabla_x f\|_{L^2(F_\star)}^2 + \|\nabla_v f\|_{L^2(F_\star)}^2 + \|\nabla_v \nabla_x f\|_{L^2(F_\star)}^2 \right) + \delta^{-1} \left(\|\varphi\|_{L^2(F_\star)}^2 + \|\nabla_v \varphi\|_{L^2(F_\star)}^2 \right). \end{aligned}$$

Gathering all estimates and taking δ and ε small enough, we conclude the proof. \square

4.4 Mixed $H^1_{x,v}$ - $L^2_{x,v}$ -hypoocoercivity

In this section, we introduce a Lyapunov functional for the linearized equation. It involves a combination of the Dolbeault-Mouhot-Schmeiser L^2 hypoocoercivity functional and the Hérau-Nier / Villani H^1 hypoocoercivity functional.

Define for some small parameters $a, b, c \in (0, 1)$ and the following functional:

$$\mathcal{E}_{1,1}[f] := \mathcal{E}_0[f] + a \|\nabla_v f(t)\|_{L^2(F_\star)}^2 + b \langle \nabla_v f, \nabla_x f \rangle_{L^2(F_\star)} + c \|\nabla_x f\|_{L^2(F_\star)}^2.$$

Lemma 4.13. *Under the assumption $b^2 \leq ac$, the functional $\mathcal{E}_{1,1}$ is equivalent to the $H^1(F_\star)$ -norm in the sense that, for some $M = M(C_\star, \bar{\kappa}^e)$ and $m = m(a, c)$, one has*

$$m \|f\|_{H^1(F_\star)}^2 \leq \mathcal{E}_{1,1}[f] \leq M \|f\|_{H^1(F_\star)}^2. \quad (4.25)$$

Furthermore, under the smallness assumptions

$$\underline{\kappa}^e < \delta^e(C_\star, \theta) \quad \text{and} \quad \bar{\kappa}^o < \delta_2(C_\star, \bar{\kappa}^e, \underline{\kappa}^e, \nu),$$

for some appropriate choice of a, b and c , there exists λ, μ and C depending on $C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e$ and ν such that any solution to (4.1) satisfies the differential inequality

$$\frac{d}{dt} \mathcal{E}_{1,1}[f] + \lambda \mathcal{E}_{1,1}[f] + \mu \|\nabla_v f\|_{H^1(F_\star)}^2 \leq C \|\varphi\|_{H^1(F_\star)}^2.$$

Proof. First of all, note that

$$\begin{aligned} \mathcal{E}_0[f] + \frac{a}{2} \|\nabla_v f\|_{L^2(F_\star)}^2 + \frac{c}{2} \|\nabla_x f\|_{L^2(F_\star)}^2 \\ \leq \mathcal{E}_{1,1}[f] \leq \mathcal{E}_0[f] + (a+b) \|\nabla_v f\|_{L^2(F_\star)}^2 + (b+c) \|\nabla_x f\|_{L^2(F_\star)}^2, \end{aligned}$$

which implies the equivalence of $\mathcal{E}_{1,1}$ with the $H^1(F_\star)$ -norm as stated by (4.25). Gathering Lemma 4.8, (4.22) with some δ_v , (4.24) with some δ_{xv} and (4.23), one gets the energy estimate for some $\mu = \mu(C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e)$ and $K = K(C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e)$

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_{1,1}[f] + K \mathcal{E}_0[f] (\mu - b) + K \|\nabla_v f\|_{L^2(F_\star)}^2 \left(\mu - \frac{a}{\delta_v} - \frac{b}{\delta_{xv}} - c \right) \\ + K \|\nabla_x f\|_{L^2(F_\star)}^2 (\mu b - a \delta_v - c) \\ + K \|\nabla_v^2 f\|_{L^2(F_\star)}^2 \left(\mu a - \frac{b}{\delta_{xv}} \right) \\ + K \|\nabla_x \nabla_v f\|_{L^2(F_\star)}^2 (\mu c - b \delta_{xv}) \leq C \|\varphi\|_{H^1(F_\star)}^2. \end{aligned}$$

Considering $c \ll b \ll 1$, it is enough to choose parameters such that

$$\frac{b}{a} \ll \delta_{xv} \ll \frac{c}{b} \quad \text{and} \quad a \ll \delta_v \ll \frac{b}{a}$$

and one checks that the scaling

$$c = \varepsilon^{21}, \quad b = \varepsilon^{20}, \quad a = \varepsilon^{16}, \quad \delta_v = \varepsilon^8, \quad \text{and} \quad \delta_{xv} = \varepsilon^2$$

fits the requirement. Taking ε small enough, we conclude this proof by norm equivalence (4.25). \square

The following hypocoercivity result is a direct consequence of Lemma 4.13.

Proposition 4.14. *Under the smallness assumptions*

$$\underline{\kappa}^e < \delta^e(C_\star, \theta) \quad \text{and} \quad \bar{\kappa}^o < \delta^o(C_\star, \bar{\kappa}^e, \underline{\kappa}^e, \nu)$$

there exists λ and C depending on $C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e$ and ν such that any solution to (4.1) satisfies the differential inequality

$$\begin{aligned} \sup_{t \geq 0} e^{2\lambda t} \|f(t)\|_{H^1(F_\star)}^2 + \int_0^\infty e^{2\lambda t} \|\nabla_v f(t)\|_{H^1(F_\star)}^2 dt \\ \leq C \left(\|f_{\text{in}}\|_{H^1(F_\star)}^2 + \int_0^\infty e^{2\lambda t} \|\varphi(t)\|_{H^1(F_\star)}^2 dt \right). \end{aligned}$$

4.5 Hypocoercivity

In this section we modify the functional of the previous section with time weights to capture regularization properties of the VFP equation.

Define for some small parameters $a, b, c > 0$ the time-weighted homogeneous functional

$$\mathcal{G}^{\text{hom}}[t, f] = at \|\nabla_v f\|_{L^2(F_\star)}^2 + bt^2 \langle \nabla_v f, \nabla_x f \rangle_{L^2(F_\star)} + ct^3 \|\nabla_x f\|_{L^2(F_\star)}^2,$$

as well as the inhomogeneous one:

$$\mathcal{G}[t, f] = \mathcal{E}_0[f] + \mathcal{G}^{\text{hom}}[t, f].$$

Lemma 4.15. *Assuming $b^2 \leq ac$, there holds*

$$\frac{1}{2} \mathcal{G}^{\text{hom}}[t, f] \leq at \|\nabla_v f\|_{L^2(F_\star)}^2 + ct^3 \|\nabla_x f\|_{L^2(F_\star)}^2 \leq 2\mathcal{G}^{\text{hom}}[t, f]. \quad (4.26)$$

Furthermore, under the smallness assumptions

$$\underline{\kappa}^e < \delta^e(C_\star, \theta) \quad \text{and} \quad \bar{\kappa}^o < \delta^o(C_\star, \bar{\kappa}^e, \underline{\kappa}^e, \nu),$$

for some appropriate choice of a, b and c , there exists μ and C depending on $C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e$ and ν such that any solution to (4.1) satisfies the differential inequality, for $t \in (0, 1]$,

$$\begin{aligned} \frac{d}{dt} \mathcal{G}[t, f] + \mu \left(\mathcal{E}_0[f] + \|\nabla_v f\|_{L^2(F_\star)}^2 + t \|\nabla_v^2 f\|_{L^2(F_\star)} \right. \\ \left. + t^2 \|\nabla_x f\|_{L^2(F_\star)}^2 + t^3 \|\nabla_x \nabla_v f\|_{L^2(F_\star)}^2 \right) \\ \leq C \left(\|\varphi\|_{L^2(F_\star)}^2 + t \|\nabla_v \varphi\|_{L^2(F_\star)}^2 + t^3 \|\nabla_x \varphi(t)\|_{L^2(F_\star)}^2 \right). \end{aligned} \quad (4.27)$$

Proof. The comparison (4.26) follows from the observation that

$$|bt^2 \langle \nabla_v f, \nabla_x f \rangle_{L^2(F_\star)}| \leq \frac{at}{2} \|\nabla_v f\|_{L^2(F_\star)}^2 + \frac{ct^3}{2} \|\nabla_x f\|_{L^2(F_\star)}^2.$$

Combined with Lemma 4.8, (4.22) with $\delta = \delta_v t$, (4.24) with $\delta = \delta_{xv} t$ and (4.23), one gets the energy estimate for some $\mu = \mu(C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e)$ and $K = K(C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e)$,

$$\begin{aligned} \frac{d}{dt} \mathcal{G}[f] + K \mathcal{E}_0[f] (\mu - bt^2) + K \|\nabla_v f\|_{L^2(F_\star)}^2 \left(\mu - \frac{a}{\delta_v} - \frac{bt}{\delta_{xv}} - ct^3 - 2a \right) \\ + K \|\nabla_x f\|_{L^2(F_\star)}^2 (\mu bt^2 - at^2 \delta_v - ct^3 - ct^2 - 3ct^2) \\ + K \|\nabla_v^2 f\|_{L^2(F_\star)}^2 \left(\mu at - \frac{bt}{\delta_{xv}} \right) \\ + K \|\nabla_x \nabla_v f\|_{L^2(F_\star)}^2 (\mu ct^3 - bt^3 \delta_{xv}) \\ \leq C_{a,b,c,\delta_v,\delta_{xv}} \left(\|\varphi\|_{L^2(F_\star)}^2 + t \|\varphi\|_{L^2(F_\star)}^2 + t^3 \|\nabla_x \varphi\|_{L^2(F_\star)}^2 \right). \end{aligned}$$

The same choice of parameters as in the proof of Lemma 4.13 allows to conclude. \square

Proposition 4.16 (H^1 -hypoellipticity). *Under the smallness assumptions*

$$\underline{\kappa}^e < \delta^e(C_\star, \theta) \quad \text{and} \quad \bar{\kappa}^o < \delta^o(C_\star, \bar{\kappa}^e, \underline{\kappa}^e, \nu),$$

there exists C depending on $C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e$ and ν such that any solution to (4.1) satisfies the short time regularization estimates:

$$\begin{aligned} \sup_{0 < t \leq 1} \left\{ t^3 \|\nabla_x f(t)\|_{L^2(F_\star)}^2 + t \|\nabla_v f(t)\|_{L^2(F_\star)}^2 \right\} \\ + \int_0^1 \left\{ t^2 \|\nabla_x f(t)\|_{L^2(F_\star)}^2 + \|\nabla_v f(t)\|_{L^2(F_\star)}^2 \right\} dt \\ + \int_0^1 \left\{ t^3 \|\nabla_x \nabla_v f(t)\|_{L^2(F_\star)}^2 + t \|\nabla_v^2 f(t)\|_{L^2(F_\star)}^2 \right\} dt \\ \leq C \left(\|f_{\text{in}}\|_{L^2(F_\star)}^2 + \int_0^1 \left(\|\varphi(t)\|_{L^2(F_\star)}^2 + t \|\nabla_v \varphi(t)\|_{L^2(F_\star)}^2 + t^3 \|\nabla_x \varphi(t)\|_{L^2(F_\star)}^2 \right) dt \right). \end{aligned}$$

Remark 4.17. Note that the pointwise estimate imply, in the case of spatial derivatives (and $\varphi = 0$), that

$$\|t^2 \nabla_x f\|_{L_t^r L_{x,v}^2(F_\star)} \leq C_r \|f_{\text{in}}\|_{L_{x,v}^2}, \quad 1 \leq r < 2,$$

however, the integral estimate from Proposition 4.16 indicates that this holds for the endpoint $r = 2$ as well.

Denote for λ the smallest rate of Propositions 4.14 and 4.16

$$w_\sigma(t) := e^{\lambda t} \min\{1, t\}^{\frac{\sigma}{2}}, \quad \sigma \in [0, 3].$$

The following is a combination of Proposition 4.14 for $t \in (0, 1]$ and Proposition 4.16 for $t \geq 1$.

Proposition 4.18. *Under the smallness assumptions*

$$\underline{\kappa}^e < \delta^e(C_\star, \theta) \quad \text{and} \quad \bar{\kappa}^o < \delta^o(C_\star, \bar{\kappa}^e, \underline{\kappa}^e, \nu),$$

there exists C and λ depending on $C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e$ and ν such that any solution to (4.1) satisfies

$$\begin{aligned} & \sup_{t>0} \left\{ w_3(t)^2 \|\nabla_x f(t)\|_{L^2(F_\star)}^2 + w_1(t)^2 \|\nabla_v f(t)\|_{L^2(F_\star)}^2 + w_0(t)^2 \|f(t)\|_{L^2(F_\star)}^2 \right\} \\ & \quad + \int_0^\infty \left\{ w_2(t)^2 \|\nabla_x f(t)\|_{L^2(F_\star)}^2 + w_0(t)^2 \|\nabla_v f(t)\|_{L^2(F_\star)}^2 \right\} dt \\ & \leq C \left(\|f_{\text{in}}\|_{L^2(F_\star)}^2 + \int_0^\infty \left(w_0(t)^2 \|\varphi(t)\|_{L^2(F_\star)}^2 + w_1(t)^2 \|\nabla_v \varphi(t)\|_{L^2(F_\star)}^2 \right. \right. \\ & \quad \left. \left. + w_3(t)^2 \|\nabla_x \varphi(t)\|_{L^2(F_\star)}^2 \right) dt \right). \end{aligned}$$

From here one could interpolate the results of Proposition 4.18 with those of Propositions 4.14 to quantify the regularization and decay of a solution to the nonlinear equation (1.1) with an initial data in $H_{x,v}^s(F_\star)$, for large enough $s \in [0, 1]$. This is even possible under a slightly weaker assumption than Assumption 2.1 on the confining potential, namely (2.1) holding only for some given ε . At the price of (2.1) holding for ε arbitrarily small, we will be able to lower the regularity assumptions on the initial data to $H_x^s L_v^2(F_\star)$.

4.6 $H_x^1 L_v^2$ hypocoercivity

In this section, we apply the Dolbeault-Mouhot-Schmeiser L^2 -hypocoercivity techniques, in the linearized free energy norm, to a first order x derivative of the solution. More precisely we shall consider $\nabla_x^* f$ instead of $\nabla_x f$ as only the former belongs to \mathcal{H}_0 .

Lemma 4.19. *Let f be a solution to (4.1), its twisted spatial gradient $\nabla_x^* f$ satisfies the equation*

$$\partial_t(\nabla_x^* f) + (T - L)(\nabla_x^* f) = -(v \cdot \nabla_x) \psi_{\nabla_x^* f}^o + \nabla_v^* \nabla_x^* \varphi + X_1 + X_2,$$

where we denoted

$$X_1 := (\nabla_x^2 V_\star) \nabla_v^* f, \quad X_2 := -\nabla_x V_\star (v \cdot \nabla_x \psi_f).$$

Furthermore, under the smallness assumptions

$$\underline{\kappa}^e < \delta^e(C_\star, \theta) \quad \text{and} \quad \bar{\kappa}^o < \delta^o(C_\star, \bar{\kappa}^e, \underline{\kappa}^e, \nu),$$

there exists λ, μ and C depending on $C_\star, \underline{\kappa}^e, \bar{\kappa}^e, \bar{\kappa}^o$ and ν such that f satisfies the differential inequality

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}_0 [\nabla_x^* f] + \lambda \mathcal{E}_0 [\nabla_x^* f] + \mu \|\nabla_v \nabla_x^* f\|_{L^2(F_\star)}^2 \\ & \leq C \left(\|\varphi\|_{L^2(F_\star)}^2 + \|\nabla_x \varphi\|_{L^2(F_\star)}^2 + \|f\|_{L^2(F_\star)}^2 + \|\nabla_v f\|_{L^2(F_\star)}^2 \right). \end{aligned}$$

Proof. The equation satisfied by $\nabla_x^* f$ is obtained from the identities

$$\begin{aligned} & [\nabla_x^*, T] f = -(\nabla_x^2 V_\star) \nabla_v^* + (\nabla_x V_\star) (v \cdot \nabla_x \psi_f^e), \\ & \left[\nabla_x^*, (v \cdot \nabla_x \psi_{(\cdot)}^o) \right] f = (\nabla_x V_\star) (v \cdot \nabla_x \psi_f^o) \quad \text{and} \quad [\nabla_x^*, L] = 0. \end{aligned}$$

Since up to the remainder terms X_1 and X_2 , $\nabla_x^* f$ follows the same equation as f , with the source $\nabla_x^* \varphi$, one can follow the proof of Lemma 4.8 to get (for the same constants) C, λ and μ depending on $\underline{\kappa}^e, \bar{\kappa}^o, \bar{\kappa}^e, C_\star$ and ν that

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}_0 [\nabla_x^* f] + \lambda \mathcal{E}_0 [\nabla_x^* f] + \mu \|\nabla_x^* \nabla_v f\|_{L^2(F_\star)}^2 \\ & \leq C \left(\|\nabla_x^* \varphi\|_{L^2(F_\star)}^2 + Y_1 + Y_2 \right), \end{aligned}$$

where, using the boundedness of the operator A and the norm equivalence between $\|\cdot\|$ and $\|\!\|\!\|$, the remainder terms can be estimated by

$$\begin{aligned} Y_1 & \lesssim \langle \nabla_x^* f, X_1 \rangle \lesssim \|\nabla_x^* f\|_{L^2(F_\star)} \|(\nabla_x^2 V_\star) \nabla_v^* f\|_{L^2(F_\star)}, \\ Y_2 & \lesssim \langle \nabla_x^* f, X_2 \rangle \lesssim \|\nabla_x^* f\|_{L^2(F_\star)} \|(\nabla V_\star) (v \cdot \nabla_x \psi_f)\|_{L^2(F_\star)}. \end{aligned}$$

Then using Young's inequality and (4.9), one obtains

$$Y_1 \lesssim \delta \|\nabla_x^* f\|_{L^2(F_\star)}^2 + \frac{1}{\delta} \left(\varepsilon \|\nabla_x \nabla_v^* f\|_{L^2(F_\star)}^2 + C_\varepsilon \|\nabla_v^* f\|_{L^2(F_\star)}^2 \right).$$

Then up to changing the constants, we have by the equivalence (3.20) and (4.8)

$$\begin{aligned} Y_1 & \lesssim \delta \|\nabla_x^* f\|_{L^2(F_\star)}^2 + \frac{\varepsilon}{\delta} \left(\|\nabla_x^* f\|_{L^2(F_\star)}^2 + \|\nabla_x^* \nabla_v f\|_{L^2(F_\star)}^2 \right) \\ & \quad + \frac{C_\varepsilon}{\delta} \left(\|f\|_{L^2(F_\star)}^2 + \|\nabla_v f\|_{L^2(F_\star)}^2 \right). \end{aligned}$$

For the second remainder term one can use Hölder's inequality for some $\frac{1}{2} = \frac{1}{q} + \frac{1}{s}$ followed by (4.3):

$$\begin{aligned} Y_2 &\lesssim \delta \|\nabla_x^* f\|_{L^2(F_\star)}^2 + \frac{1}{\delta} \|\nabla_x V_\star\|^2 \rho_\star \|_{L^{\frac{s}{2}}} \|\nabla_x \psi_f\|_{L^q}^2, \\ &\lesssim \delta \|\nabla_x^* f\|_{L^2(F_\star)}^2 + \frac{\bar{\kappa}}{\delta} \|\Pi f\|_{L^2(F_\star)}^2. \end{aligned}$$

Putting these estimates together with $\varepsilon = \delta^2$, and using the fact that \mathcal{E}_0 is equivalent to $\|\cdot\|_{L^2(F_\star)}$, we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_0[\nabla_x^* f] + (\lambda - C(\delta + \delta^2)) \mathcal{E}_0[\nabla_x^* f] + (\mu - C\delta) \|\nabla_x^* \nabla_v f\|_{L^2(F_\star)}^2 \\ \leq C\delta \left(\|\nabla_x^* \varphi\|_{L^2(F_\star)}^2 + \bar{\kappa} \|\Pi f\|_{L^2(F_\star)}^2 + \|f\|_{L^2(F_\star)}^2 + \|\nabla_v f\|_{L^2(F_\star)}^2 \right), \end{aligned}$$

thus, taking δ small enough, we conclude the proof. \square

Define $\mathcal{E}_{1,0}$ for some small $\varepsilon \in (0, 1)$ as

$$\mathcal{E}_{1,0}[f] = \mathcal{E}_0[f] + \varepsilon \mathcal{E}_0[\nabla_x^* f].$$

In virtue of (3.20), it is clear that $\mathcal{E}_{1,0}$ defines an equivalent norm to $H_x^1 L_v^2(F_\star)$, thus, the following hypocoercivity result follows from Lemmas 4.8 and 4.19 for a small enough choice of ε .

Proposition 4.20 ($H_x^1 L_v^2$ -hypocoercivity). *Under the smallness assumptions*

$$\underline{\kappa}^e < \delta^e(C_\star, \theta) \quad \text{and} \quad \bar{\kappa}^o < \delta^o(C_\star, \bar{\kappa}^e, \underline{\kappa}^e, \nu)$$

there exists λ and C depending on $C_\star, \bar{\kappa}^e, \bar{\kappa}^o, \underline{\kappa}^e$ and ν such that any solution to (4.1) satisfies the differential inequality

$$\begin{aligned} \sup_{t \geq 0} e^{2\lambda t} \|f(t)\|_{H_x^1 L_v^2(F_\star)}^2 + \int_0^\infty e^{2\lambda t} \|\nabla_v f(t)\|_{H_x^1 L_v^2(F_\star)}^2 dt \\ \leq C \left(\|f_{\text{in}}\|_{H_x^1 L_v^2(F_\star)}^2 + \int_0^\infty e^{2\lambda t} \|\varphi(t)\|_{H_x^1 L_v^2(F_\star)}^2 dt \right). \end{aligned}$$

5 Non-linear VFP

Now we turn to the proof of our main result Theorem 2.12. First, based on the analysis of the linearized equation, we introduce norms which will encompass the gain of regularity of the solution and the exponential decay towards equilibrium. Then we turn to the estimate of the nonlinear quadratic contribution

$$\varphi = f \nabla \psi_f,$$

in these functional spaces. From there we prove our main results of perturbative well-posedness and asymptotic stability using a contraction argument.

5.1 Functional setting

As in the previous section, we consider the time weights

$$w_\sigma(t) = e^{\lambda t} \min\{1, t\}^{\frac{\sigma}{2}}, \quad \sigma \in [0, 3]$$

and, for any $s \in [0, 1]$, we introduce the norms

$$\begin{aligned} \|\varphi\|_{\mathcal{H}^s}^2 = & \int_0^\infty \left(w_{1-s}(t)^2 \|\varphi(t)\|_{L_{x,v}^2(F_\star)}^2 + w_{1-s}(t)^2 \|\varphi(t)\|_{L_x^2 H_v^{1-s}(F_\star)}^2 \right. \\ & \left. + w_{3(1-s)}(t)^2 \|\varphi(t)\|_{H_x^1 L_v^2(F_\star)}^2 \right) dt \end{aligned}$$

as well as

$$\begin{aligned} \|f\|_{\mathcal{X}^s}^2 := & \sup_{t>0} \left(w_0(t)^2 \|f(t)\|_{H_x^s L_v^2(F_\star)}^2 + w_{3(1-s)}(t)^2 \|f(t)\|_{H_x^1 L_v^2(F_\star)}^2 \right. \\ & \left. + w_{1-s}(t)^2 \|f(t)\|_{L_x^2 H_v^{1-s}(F_\star)}^2 \right) + \|f\|_{\mathcal{H}^s}^2 \end{aligned}$$

where we denoted

$$\|f\|_{\mathcal{H}^s}^2 := \int_0^\infty \left(w_{2(1-s)}(t)^2 \|f(t)\|_{H_x^1 L_v^2(F_\star)}^2 + w_0(t)^2 \|f(t)\|_{L_x^2 H_v^{1-s}(F_\star)}^2 \right) dt.$$

We can now reformulate Proposition 4.20 and Proposition 4.16, by the fact that any solution f to (4.1) satisfies

$$\|f\|_{\mathcal{X}^s} \leq C \|(f_{\text{in}}, \varphi)\|_{H_x^s L_v^2(F_\star) \times \mathcal{H}^s} \quad \text{for } s = 0 \text{ or } 1.$$

It is possible to extend these bounds to any $s \in [0, 1]$. Indeed, according to [38, Theorem 1.1], this follows from the fact that \mathcal{H}^s (resp. \mathcal{H}^s) is the geometric interpolation of order $s \in [0, 1]$ between \mathcal{H}^0 and \mathcal{H}^1 (resp. \mathcal{H}^0 and \mathcal{H}^1). This can be seen by writing

$$\|\varphi\|_{\mathcal{H}^{1-s}} = \|(\varphi, \varphi, \varphi)\|_{[\mathcal{H}]^s} = \|A^s(\varphi, \varphi, \varphi)\|_{[\mathcal{H}]^0}$$

where we denoted the Hilbert norm $[\mathcal{H}]^0$

$$\|(\varphi_1, \varphi_2, \varphi_3)\|_{[\mathcal{H}]^0} = \|\varphi_1\|_{L_{t,x,v}^2(F_\star e^{2\lambda t})} + \|\varphi_2\|_{L_{t,x,v}^2(F_\star e^{2\lambda t})} + \|\varphi_3\|_{L_{t,v}^2 H_x^1(F_\star e^{2\lambda t})}$$

and the operator A

$$A(\varphi_1, \varphi_2, \varphi_3) = \left(\varphi_1, \min\{1, t\} (\nabla_v^* \nabla_v)^{1/2} \varphi_2, \min\{1, t\}^3 \varphi_3 \right).$$

Arguing similarly for \mathcal{H}^s , one gets by interpolation using [38, Theorem 1.1]

$$\|f\|_{\mathcal{H}^s} \leq C \|(f_{\text{in}}, \varphi)\|_{H_x^s L_v^2(F_\star) \times \mathcal{H}^s} \quad \text{for } s \in [0, 1].$$

The pointwise bounds are obtained through the same argument, but applied to any given time $t \in (0, \infty)$. For instance, to obtain the x -derivatives control, we have to check that for $s \in [0, 1]$ and $t \in (0, \infty)$

$$\|f(t)\|_{H_x^1 L_v^2(F_\star)} \leq C_s(t) \|(f_{\text{in}}, \varphi)\|_{H_x^s L_v^2(F_\star) \times \mathcal{H}^s},$$

where we denoted

$$C_s(t) = C w_{3(1-s)}(t)^{-1} = C_0(t)^{1-s} C_1(t)^s .$$

We know this holds for $s = 0$ and $s = 1$, thus, by interpolation, this holds for $s \in [0, 1]$. The same argument provides the v -derivatives estimate and the $L_{x,v}^2$ -exponential decay. The following proposition sums up the bounds established by interpolation in this paragraph.

Lemma 5.1. *For any given $s \in [0, 1]$, any solution f to (4.1) satisfies*

$$\|f\|_{\mathcal{X}^s} \leq C \left(\|f_{\text{in}}\|_{H_x^s L_v^2(F_\star)} + \|\varphi\|_{\mathcal{H}^s} \right) .$$

5.2 Nonlinear estimates

We now show that these norms are adapted to the nonlinear problem.

Lemma 5.2. *For any $s \in [0, 1]$ such that*

$$s > s_c := \frac{3}{2} \left(\frac{d}{q} - \frac{1}{3} \right) ,$$

there exists some $C > 0$ such that the following bilinear estimate holds:

$$\|f \nabla_x \psi_g\|_{\mathcal{H}^s} \leq C \|f\|_{\mathcal{X}^s} \|g\|_{\mathcal{X}^s} .$$

Remark 5.3. Note that $s_c < 1 \Leftrightarrow q > d$, and $s = 0$ is allowed as soon as $q > 3d$.

Proof. Let us denote in this proof $\sigma := 1 - s$ and define

$$\delta = \frac{2}{3}s - \left(\frac{d}{q} - \frac{1}{3} \right) > 0 \quad \text{so that} \quad \frac{d}{q} + \delta = s + \frac{\sigma}{3} .$$

We start with the control

$$\|f \nabla_x \psi_g\|_{H_x^1 L_v^2(F_\star)} \lesssim A + B .$$

where we denoted

$$A := \|f \nabla_x \psi_g\|_{L_x^2 L_v^2(F_\star)} + \|\nabla_x f \nabla_x \psi_g\|_{L_x^2 L_v^2(F_\star)}, \quad B := \|f \nabla_x^2 \psi_g\|_{L_x^2 L_v^2(F_\star)} .$$

On the one hand, one has from the bound (4.3) on $\nabla_x \psi$

$$\|\nabla_x \psi_g\|_{L^q} \lesssim \|g\|_{L^2(F_\star)} \quad \text{and} \quad \|\nabla_x \psi_g\|_{W^{1,q}} \lesssim \|g\|_{H_x^1 L_v^2(F_\star)} ,$$

therefore, since we have by real interpolation with $\theta = \frac{d}{q} + \delta = s + \frac{\sigma}{3} \in [0, 1]$ (see for instance [45, Lemma 22.3])

$$[L^q, W^{1,q}]_{\theta,2} = B_{q,2}^\theta \quad \text{and} \quad [L^2(F_\star), H_x^1 L_v^2(F_\star)]_{\theta,2} = H_x^\theta L_v^2(F_\star) ,$$

where $B_{q,2}^\theta$ denoted the suitable Besov space, the following interpolated bound holds:

$$\|\nabla_x \psi g\|_{B_{q,2}^\theta} \lesssim \|g\|_{H_x^\theta L_v^2(F_\star)}.$$

Using the chain of embeddings (see [2, Propositions 2.71 and then 2.39])

$$B_{q,2}^\theta = B_{q,2}^{\frac{d}{q}+\delta} \subset B_{\infty,2}^\delta \subset B_{\infty,1}^0 \subset L^\infty,$$

these two facts yield the estimate

$$A \lesssim \|f\|_{H_x^1 L_v^2(F_\star)} \|\nabla_x \psi g\|_{L^\infty} \lesssim \|f\|_{H_x^1 L_v^2(F_\star)} \|g\|_{H_x^{s+\frac{\sigma}{3}} L_v^2(F_\star)}.$$

On the other hand, using Hölder's inequality with $\frac{1}{2} = \frac{1}{q} + \frac{1}{m}$ followed by the bound on $\nabla_x \psi$, one has

$$B = \|f \nabla_x \psi [\nabla_x^* g]\|_{L_x^2 L_v^2(F_\star)} \lesssim \|F_\star^{1/2} f\|_{L_x^m L_v^2} \|\nabla_x^* g\|_{L_x^2 L_v^2(F_\star)}.$$

Using next the Sobolev embedding

$$H^{s+\frac{\sigma}{3}} = H^{\frac{d}{q}+\delta} \subset H^{\frac{d}{q}} \subset L^m \quad \text{since} \quad \frac{d}{q} = \frac{d}{2} - \frac{d}{m},$$

we thus have thanks to the norm comparison (3.19) that

$$B \lesssim \|f\|_{H_x^{s+\frac{\sigma}{3}} L_v^2(F_\star)} \|g\|_{H_x^1 L_v^2(F_\star)}.$$

Putting together the estimates on A and B , using that $s + \sigma = 1$, we conclude that

$$\begin{aligned} t^{3\sigma} \|f \nabla_x \psi g\|_{H_x^{s+\sigma} L_v^2(F_\star)}^2 &\lesssim \left(t^{2\sigma} \|f\|_{H_x^{s+\sigma} L_v^2(F_\star)}^2 \right) \left(t^\sigma \|g\|_{H_x^{s+\frac{\sigma}{3}} L_v^2(F_\star)}^2 \right) \\ &\quad + \left(t^\sigma \|f\|_{H_x^{s+\frac{\sigma}{3}} L_v^2(F_\star)}^2 \right) \left(t^{2\sigma} \|g\|_{H_x^{s+\sigma} L_v^2(F_\star)}^2 \right), \end{aligned}$$

from which we deduce, interpolating $H_x^{s+\frac{\sigma}{3}}$ between H_x^s and $H_x^{s+\sigma} = H_x^1$

$$\begin{aligned} &\int_0^\infty w_{3\sigma}(t)^2 \|f(t) \nabla_x \psi g(t)\|_{H_x^{s+\sigma} L_v^2(F_\star)}^2 dt \\ &\lesssim \left(\int_0^\infty w_{2\sigma}(t)^2 \|f\|_{H_x^{s+\sigma} L_v^2(F_\star)}^2 dt \right) \sup_{t>0} \left(w_\sigma(t)^2 \|g\|_{H_x^{s+\frac{\sigma}{3}} L_v^2(F_\star)}^2 \right) \\ &\quad + \sup_{t>0} \left(w_\sigma(t)^2 \|f\|_{H_x^{s+\frac{\sigma}{3}} L_v^2(F_\star)}^2 \right) \left(\int_0^\infty w_{2\sigma}(t)^2 \|g\|_{H_x^{s+\sigma} L_v^2(F_\star)}^2 dt \right) \\ &\lesssim \|f\|_{\mathcal{X}^s}^2 \|g\|_{\mathcal{X}^s}^2 \end{aligned}$$

Similarly, one shows

$$t^\sigma \|f \nabla_x \psi g\|_{L_x^2 H_v^\sigma(F_\star)}^2 \lesssim \|f\|_{L_x^2 H_v^\sigma(F_\star)}^2 \left(t^\sigma \|g\|_{H_x^{s+\frac{\sigma}{3}} L_v^2(F_\star)}^2 \right)$$

from which we deduce

$$\int_0^\infty w_\sigma(t)^2 \|f(t) \nabla_x \psi g(t)\|_{L_x^2 H_v^\sigma(F_\star)}^2 dt \lesssim \|f\|_{\mathcal{X}^s}^2 \|g\|_{\mathcal{X}^s}^2.$$

This concludes the proof. \square

5.3 Proof of Theorem 2.12

Fix some $f_{\text{in}} \in H_x^s L_v^2(F_\star)$ and denote $\Phi : \mathcal{X}^s \rightarrow \mathcal{X}^s$ the mapping defined by

$$f := \Phi g \quad \text{where} \quad \begin{cases} (\partial_t + \Lambda)f(t) + v \cdot \nabla_x \psi_f(t) = \nabla_v^*(g \nabla_x \psi_g), \\ f(0, x, v) = f_{\text{in}}(x, v). \end{cases}$$

As a consequence of Lemmas 5.1 and 5.2, there exists some $C > 0$ such that

$$\begin{aligned} \|\Phi(g_1) - \Phi(g_2)\|_{\mathcal{X}^s} &\leq C (\|g_1\|_{\mathcal{X}^s} + \|g_2\|_{\mathcal{X}^s}) \|g_2 - g_1\|_{\mathcal{X}^s}, \\ \|\Phi(g)\|_{\mathcal{X}^s} &\leq C (\|f_{\text{in}}\|_{H_x^s L_v^2(F_\star)} + \|g\|_{\mathcal{X}^s}^2). \end{aligned}$$

It then follows from a classical Picard argument that for $\|f_{\text{in}}\|_{H_x^s L_v^2(F_\star)}$ small enough, the mapping Φ has a unique fixed point, which proves the theorem.

6 Concluding remarks

Let us end this work by stating some perspectives and natural continuations. We first present possible improvements on the regularity assumptions.

Weakly regularizing interaction potential We have assumed the interaction potential to be regularizing enough, namely $\nabla \mathcal{K} : L^1 \cap L^2 \rightarrow L^q$ with $q > d$, so that the initial condition only needs to have (at most) H_x^1 regularity. In order to extend Theorem 2.12 to the case when we only assume $q \geq 2$ (for instance, Manev interactions, see Example 2.7), one would need to consider an initial data H_x^s with s big enough. The strategy to derive $H_x^1 L_v^2$ estimates from Section 4.6 would need to be adapted to $H_x^k L_v^2$ for any integer $k \in \mathbb{N}$, thus requiring suitable adaptation of the hypothesis (2.1). The nonlinear estimate would need to be modified, for instance in the non-regularizing case $q = 2$, the Sobolev algebra inequality [2, Corollary 2.86] yields for $s > \frac{d}{2} - \frac{1}{3}$

$$t^3 \|f \nabla_x \psi_f\|_{H_x^{s+1} L_v^2(F_\star)} \lesssim t^2 \|f\|_{H_x^{s+1} L_v^2(F_\star)}^2 \times t \|f\|_{H_x^{s+\frac{1}{3}} L_v^2(F_\star)}^2.$$

Rough initial data Similarly, we have only considered initial data with at least 0 regularity, in the sense that we excluded H_x^s spaces for $s < 0$. However, when $q > 3d$, the critical regularity index is $s_c < 0$, which suggests that initial data with regularity H_x^s for $s_c < s < 0$ could be considered. This would require a duality argument in the spirit of [34, 36, 13, 26].

Let us now present some natural continuations.

Non-perturbative setting For VPFP, one would be interested in showing exponential convergence to equilibrium starting from an initial data $F(0)$ assuming only physical bounds, namely finite entropy and energy. The strategy would then be to use the free energy, together with regularization estimates in order to perform a trapping argument and show that after a (possibly non-explicit) time $t_0 > 0$, the solution is close to the steady state, allowing to conclude using Theorem 2.12 with $F_{\text{in}} = F(t_0)$. Similar strategies have been considered in [40, 47, 28] for spatially homogeneous kinetic equations. Let us also mention works concerning the global convergence to equilibrium in the case of Vlasov-Fokker-Planck equation [9, 29].

Phase transitions and instabilities It is known, in the case of the torus [14], that negative Fourier modes of the interaction potential may lead to phase transitions, and more precisely non-uniqueness of steady states. In particular, in the case of the synchrotron model (2.14), experimental and numerical evidence suggest that instabilities may arise, and an asymptotically periodic behavior seem to appear (see [43]). In the linearized setting, it has been shown that when the size of k_S is large, oscillating modes appear [11]. This is to be opposed to the symmetric positive setting (such as Poisson interactions) for which global convergence to equilibrium is known to hold.

Acknowledgements

The authors acknowledge support from the LabEx CEMPI (ANR-11-LABX-0007) and CPER WaveTech@HdF.

References

- [1] Lanoir Addala, Jean Dolbeault, Xingyu Li, and M. Lazhar Tayeb. L^2 -hypocoercivity and large time asymptotics of the linearized Vlasov-Poisson-Fokker-Planck system. *J. Stat. Phys.*, 184(1):34, 2021. Id/No 4.
- [2] Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren Math. Wiss.* Berlin: Springer, 2011.
- [3] Dominique Bakry, Franck Barthe, Patrick Cattiaux, and Arnaud Guillin. A simple proof of the Poincaré inequality for a large class of probability measures. *Electron. Commun. Probab.*, 13:60–66, 2008.
- [4] Erhan Bayraktar, Qi Feng, and Wuchen Li. Exponential entropy dissipation for weakly self-consistent Vlasov-Fokker-Planck equations. *J. Nonlinear Sci.*, 34(1):42, 2024. Id/No 7.

- [5] Alain Blaustein. Diffusive limit of the Vlasov-Poisson-Fokker-Planck model: quantitative and strong convergence results. *SIAM J. Math. Anal.*, 55(5):5464–5482, 2023.
- [6] A. V. Bobylev, P. Dukes, R. Illner, and H. D. jun. Victory. On Vlasov-Manev equations. I: Foundations, properties, and nonglobal existence. *J. Stat. Phys.*, 88(3-4):885–911, 1997.
- [7] François Bolley, Arnaud Guillin, and Florent Malrieu. Trend to equilibrium and particle approximation for a weakly selfconsistent Vlasov-Fokker-Planck equation. *ESAIM, Math. Model. Numer. Anal.*, 44(5):867–884, 2010.
- [8] François Bouchut. Existence and uniqueness of a global smooth solution for the Vlasov- Poisson-Fokker-Planck system in three dimensions. *J. Funct. Anal.*, 111(1):239–258, 1993.
- [9] François Bouchut and J. Dolbeault. On long time asymptotics of the Vlasov-Fokker-Planck equation and of the Vlasov-Poisson-Fokker-Planck system with Coulombic and Newtonian potentials. *Differ. Integral Equ.*, 8(3):487–514, 1995.
- [10] Emeric Bouin, Jean Dolbeault, Stéphane Mischler, Clément Mouhot, and Christian Schmeiser. Hypocoercivity without confinement. *Pure Appl. Anal.*, 2(2):203–232, 2020.
- [11] Yunhai Cai. Linear theory of microwave instability in electron storage rings. *Physical Review Special Topics-Accelerators and Beams*, 14(6):061002, 2011.
- [12] Kleber Carrapatoso, Jean Dolbeault, Frédéric Hérau, Stéphane Mischler, Clément Mouhot, and Christian Schmeiser. Special macroscopic modes and hypocoercivity. *arXiv preprint arXiv:2105.04855*, 2021.
- [13] Kleber Carrapatoso, Mohamad Rachid, and Isabelle Tristani. Regularization estimates and hydrodynamical limit for the Landau equation. *Journal de Mathématiques Pures et Appliquées*, 163:334–432, 2022.
- [14] J. A. Carrillo, R. S. Gvalani, G. A. Pavliotis, and A. Schlichting. Long-time behaviour and phase transitions for the McKean-Vlasov equation on the torus. *Arch. Ration. Mech. Anal.*, 235(1):635–690, 2020.
- [15] Ludovic Cesbron and Maxime Herda. On a Vlasov-Fokker-Planck equation for stored electron beams. *J. Differ. Equations*, 404:316–353, 2024.
- [16] S. N. Chandler-Wilde, D. P. Hewett, and A. Moiola. Interpolation of Hilbert and Sobolev spaces: quantitative estimates and counterexamples. *Mathematika*, 61(2):414–443, 2015.

- [17] S. Chandrasekhar. Stochastic problems in physics and astronomy. *Rev. Mod. Phys.*, 15:1–89, 1943.
- [18] Bernard Chazelle, Quansen Jiu, Qianxiao Li, and Chu Wang. Well-posedness of the limiting equation of a noisy consensus model in opinion dynamics. *J. Differ. Equations*, 263(1):365–397, 2017.
- [19] Xiang Cheng, Niladri S Chatterji, Peter L Bartlett, and Michael I Jordan. Underdamped Langevin MCMC: A non-asymptotic analysis. In *Conference on learning theory*, pages 300–323. PMLR, 2018.
- [20] Young-Pil Choi and In-Jee Jeong. Global-in-time existence of weak solutions for Vlasov-Manev-Fokker-Planck system. *Kinet. Relat. Models*, 16(1):41–53, 2023.
- [21] Pierre Degond. Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in 1 and 2 space dimensions. *Ann. Sci. Éc. Norm. Supér. (4)*, 19(4):519–542, 1986.
- [22] J. Dolbeault. Stationary states in plasma physics: Maxwellian solutions of the Vlasov- Poisson system. *Math. Models Methods Appl. Sci.*, 1(2):183–208, 1991.
- [23] Jean Dolbeault, Clément Mouhot, and Christian Schmeiser. Hypocoercivity for linear kinetic equations conserving mass. *Trans. Am. Math. Soc.*, 367(6):3807–3828, 2015.
- [24] JP Dougherty. Model Fokker-Planck equation for a plasma and its solution. *The Physics of Fluids*, 7(11):1788–1799, 1964.
- [25] K. Dressler. Stationary solutions of the Vlasov-Fokker-Planck equation. *Math. Methods Appl. Sci.*, 9:169–176, 1987.
- [26] Pierre Gervais and Bertrand Lods. Hydrodynamic limits for kinetic equations preserving mass, momentum and energy: a spectral and unified approach in the presence of a spectral gap, 2024.
- [27] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Grad. Texts Math.* New York, NY: Springer, 3rd ed. edition, 2014.
- [28] Maria Pia Gualdani, Stéphane Mischler, Clément Mouhot, and Stéphane Mischler. *Factorization of non-symmetric operators and exponential H-theorem*. Société Mathématique de France, 2017.
- [29] Arnaud Guillin, Pierre Le Bris, and Pierre Monmarché. Convergence rates for the Vlasov-Fokker-Planck equation and uniform in time propagation of chaos in non convex cases. *Electron. J. Probab.*, 27:44, 2022. Id/No 124.
- [30] Bernard Helffer and Francis Nier. *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, volume 1862 of *Lect. Notes Math.* Berlin: Springer, 2005.

- [31] Frédéric Hérau, Michael Hitrik, and Johannes Sjöstrand. Tunnel effect for Kramers-Fokker-Planck type operators. *Ann. Henri Poincaré*, 9(2):209–274, 2008.
- [32] Frédéric Hérau, Michael Hitrik, and Johannes Sjöstrand. Tunnel effect for Kramers-Fokker-Planck type operators: return to equilibrium and applications. *Int. Math. Res. Not.*, 2008:48, 2008. Id/No rnn057.
- [33] Frédéric Hérau, Michael Hitrik, and Johannes Sjöstrand. Tunnel effect and symmetries for Kramers-Fokker-Planck type operators. *J. Inst. Math. Jussieu*, 10(3):567–634, 2011.
- [34] Frédéric Hérau and Francis Nier. Isotropic hypoelliptic and trend to equilibrium for the Fokker-Planck equation with a high-degree potential. *Arch. Ration. Mech. Anal.*, 171(2):151–218, 2004.
- [35] Frédéric Hérau and Laurent Thomann. On global existence and trend to the equilibrium for the Vlasov-Poisson-Fokker-Planck system with exterior confining potential. *J. Funct. Anal.*, 271(5):1301–1340, 2016.
- [36] Frédéric Hérau, Daniela Tonon, and Isabelle Tristani. Regularization estimates and Cauchy theory for inhomogeneous Boltzmann equation for hard potentials without cut-off. *Commun. Math. Phys.*, 377(1):697–771, 2020.
- [37] Maxime Herda and L. Miguel Rodrigues. Large-time behavior of solutions to Vlasov-Poisson-Fokker-Planck equations: from evanescent collisions to diffusive limit. *J. Stat. Phys.*, 170(5):895–931, 2018.
- [38] John E. McCarthy. Geometric interpolation between Hilbert spaces. *Ark. Mat.*, 30(2):321–330, 1992.
- [39] Pierre Monmarché. A note on a vlasov-fokker-planck equation with non-symmetric interaction. *arXiv preprint arXiv:2311.05747*, 2023.
- [40] Clément Mouhot. Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials. *Commun. Math. Phys.*, 261(3):629–672, 2006.
- [41] H. Neunzert, M. Pulvirenti, and L. Triolo. On the Vlasov-Fokker-Planck equation. *Math. Methods Appl. Sci.*, 6:527–538, 1984.
- [42] A. Persson. Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator. *Math. Scand.*, 8:143–153, 1960.
- [43] Eléonore Roussel. *Spatio-temporal dynamics of relativistic electron bunches during the microbunching instability: study of the Synchrotron SOLEIL and UVSOR storage rings*. PhD thesis, Université Lille1-Sciences et Technologies, 2014.
- [44] Yozo Tamura. On asymptotic behavior of the solution of a non-linear diffusion equation. *J. Fac. Sci., Univ. Tokyo, Sect. I A*, 31:195–221, 1984.

- [45] Luc Tartar. *An introduction to Sobolev spaces and interpolation spaces*, volume 3 of *Lect. Notes Unione Mat. Ital.* Berlin: Springer, 2007.
- [46] Gayrat Toshpulatov. Well-posedness and trend to equilibrium for the vlasov-poisson-fokker-planck system with a confining potential, 2024.
- [47] Isabelle Tristani. Exponential convergence to equilibrium for the homogeneous Boltzmann equation for hard potentials without cut-off. *J. Stat. Phys.*, 157(3):474–496, 2014.
- [48] Harold Dean jun. Victory. On the existence of global weak solutions for Vlasov-Poisson-Fokker-Planck systems. *J. Math. Anal. Appl.*, 160(2):525–555, 1991.
- [49] Harold Dean jun. Victory and Brian P. O’Dwyer. On classical solutions of Vlasov-Poisson Fokker-Planck systems. *Indiana Univ. Math. J.*, 39(1):105–156, 1990.
- [50] Cédric Villani. A review of mathematical topics in collisional kinetic theory. *Handbook of mathematical fluid dynamics*, 1(71-305):3–8, 2002.
- [51] Cédric Villani. *Hypocoercivity*, volume 950 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 2009.
- [52] Hao Wu, Tai-Chia Lin, and Chun Liu. Diffusion limit of kinetic equations for multiple species charged particles. *Archive for Rational Mechanics and Analysis*, 215:419–441, 2015.

PIERRE GERVAIS

Univ. Lille, CNRS, Inria, UMR 8524 - Laboratoire Paul Painlevé,
F-59000 Lille, France

E-mail: pierre.gervais@univ-lille.fr

MAXIME HERDA

Univ. Lille, Inria, CNRS, UMR 8524 - Laboratoire Paul Painlevé,
F-59000 Lille, France

E-mail: maxime.herda@inria.fr