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Proximity guarantees of a lift-and-project approach for the cardinality-constrained Boolean quadric polytope

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Abstract

We consider a lift-and-project approach for the cardinality-constrained Boolean quadric polytope. Some upper bounds for the distance between the polytope and its linear approximation are derived. Unsurprisingly, the distance converges to 0 when the number of variables increases sufficiently.

Keywords: Boolean quadric polytope, linear relaxations, lift-and-project.

1. Introduction

Let $U^n = \text{conv}(\{1_S 1_S^t, S \subseteq [n]\})$ where $[n] = \{1, 2, \dots, n\}$ and 1_S is a vector of \mathbb{R}^n whose i^{th} component equals 1 (resp. 0) if $i \in S$ (resp. otherwise). Since any matrix $X \in U^n$ is symmetric, we can simply consider the values X_{ij} for $j \geq i$, leading to the well-known Boolean quadric polytope $\text{BQP}^n \subseteq \mathbb{R}^{n(n+1)/2}$ that can be seen as a projection of U^n . Since 0-1 quadratic programs can obviously be formulated as linear programs over BQP^n (or U^n), good approximations of these polytopes are sought to better solve this important class of problems. In many applications, a cardinality constraint is added: only k variables can be non-zero. This leads to the study of U_k^n and BQP_k^n where $U_k^n = \text{conv}(\{1_S 1_S^t, S \in [n]_k\})$, $[n]_k = \{S \subseteq [n] : |S| = k\}$ and BQP_k^n is the projection of U_k^n .

We will review some related work in next section. Then the paper contributions and some notation are presented in Section 3. The main results and their proofs are provided in Section 4. Concluding remarks follow in the last section.

2. Related work

The Boolean quadric polytope introduced in [20] is also called the full correlation polytope [22] since $X \in \text{BQP}^n$ is related to some probability space $(\Omega, \mathcal{A}, \mu)$ through the equations $X_{ii} = \mu(A_i)$ and $X_{ij} = \mu(A_i \cap A_j)$ where $(A_i)_{i \in [n]} \in \mathcal{A}$. As a consequence, [22] observed that the typical probability problem (Boole's problem) of minimizing $\mu(A_1 \cup \dots \cup A_n)$ given $\mu(A_i)$ and $\mu(A_i \cap A_j)$ values, can be stated as a linear program involving a variable for each extreme point of BQP^n (to express X as a convex combination).

Given the connection with 0-1 quadratic programming, membership in BQP^n is unsurprisingly shown to be NP-complete [22].

Several families of valid inequalities for BQP^n have been proposed in literature. In addition to the obvious inequalities $0 \leq X_{ij} \leq 1$, $i \leq j$, inequalities $X_{ij} + 1 \geq X_{ii} + X_{jj}$ and $X_{ij} \leq X_{ii}$ are presented in [10, 17]. These are now called RLT inequalities [24]. Triangle inequalities are derived in [20]: $X_{ii} + X_{jj} + X_{kk} \leq X_{ij} + X_{jk} + X_{ki} + 1$ and $X_{ij} + X_{jk} \leq X_{ii} + X_{kk}$. Other families include clique inequalities [20], cut inequalities [20], Boros and Hammer inequalities [5], gap inequalities [9] adapted from [14] and inequalities implied by the positive semidefinite nature of the matrix $X - \text{diag}(X)\text{diag}(X)^T$ [12, 25] where $\text{diag}(X)$ is

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the diagonal vector of X .

The Boolean quadric polytope is equivalent to the cut polytope since one can be obtained from the other through a straightforward affine transformation [6, 20, 3]. More precisely, for $X \in \text{BQP}^n$ one can define a point in $z \in \text{CUT}^{n+1}$ (the convex hull of incidence vectors of cuts in a complete graph of order $n + 1$) by setting $z_{i,n+1} = X_{ii}$ for $i \in [n]$ and $z_{i,j} = X_{ii} + X_{jj} - 2X_{ij}$ for $i < j \in [n]$. Inequalities defined for the cut polytope can then be easily transformed into valid inequalities for BQP^n and vice-versa. More details can be found in [15].

The Boolean quadric polytope under the cardinality constraint $\sum_i X_{ii} \leq k$ has been studied in [21, 18]. [18] focused on the case where variables X_{ij} are considered only for pairs (i, j) in a proper subset E of the set of all pairs $i \leq j$ in $[n] \times [n]$. This is naturally related to the Boolean quadratic program $\min \sum_{(i,j) \in E} q_{ij} x_i x_j$ where coefficients q_{ij} are non-zero only for $(i, j) \in E$. Some valid inequalities have been proposed there including star inequalities ($\sum_{j \neq i} X_{ij} \leq (k-1)X_{ii}$), tree inequalities, long-cycle inequalities, long-tree inequalities and long-forest inequalities generalizing some of the inequalities proposed in [11]. According to [18] (Section 5), the separation problems for all the inequalities appear to be difficult. [21] considered the complete case and studied the polytope $\text{BQP}_{k \leq}^n$ (the analogue of BQP_k^n where we require $\sum_i X_{ii} \leq k$ instead of equality). They also proposed the star and tree inequalities mentioned above. Moreover, they considered the projection of $\text{BQP}_{k \leq}^n$ obtained by ignoring X_{ii} variables (since some applications do not involve node weights).

BQP_k^n is studied in [8]. Its dimension is shown to be equal to $n(n-1)/2 - 1$ for $2 \leq k \leq n$. Authors recall that BQP_k^n is stable by permutation allowing to build some facets from others by straightforward coefficient permutation. Switching (or complement operation) is also considered in [8] showing that BQP_{n-k}^n can be obtained from BQP_k^n by simple affine transformation. BQP_k^n is proved to be a face of BQP^n and a full description of BQP_k^n is given when $k = 2$

and $k = n - 2$. It is also shown that $\text{BQP}_{k \leq}^n$ is a projection of BQP_k^{n+k} . Some connections with the equicut polytope are exposed [8]. More generally, the affine correspondence mentioned above between the cut polytope and BQP^n leads to a mapping between BQP_k^n and the uniform cut polytope CUT_k^{n+1} (the cut polytope under a cardinality constraint). Inequalities defined for the uniform cut polytope such as those of [7, 19] can be mapped to some valid inequalities for BQP_k^n .

The relaxations proposed in the paper are based on lifting where the set of variables and constraints are extended by multiplying constraints by variables and linearizing by adding new variables representing products of old variables. Lifting is of course a well-known technique proposed in different papers including [23, 16, 2]. Roughly speaking, we know that by using any one of the three variants introduced there, we get the convex hull of integer solutions inside a polytope at the cost of adding an exponential number of variables and constraints [2, 13]. Several studies tried to determine the maximum number of the lift-and-project iterations required to get the true convex-hull (generally called rank) (see, e.g., [13, 1]).

3. Contributions and further notation

In our paper, we aim to analyze the quality of the relaxation obtained by considering some kind of lifting. More precisely, focusing on the special case of U_k^n and considering a relaxation involving a variable λ_S for any set S of size r ($r \leq k$), we show that the normalized distance between U_k^n and the relaxed set is at most $\min(1 - (\frac{r}{k})^2, 1 - \frac{k}{n})$.

The lifting approach proposed in the current paper is somewhat related to the recent work in [4] where the cone of factor-width- k matrices is approximated through lifting. The 0-1 case was not handled there and a different distance (Frobenius distance) is considered in [4].

Let us add some notation. Given a n -square matrix X , let $\|X\|_1 = \sum_{i \in [n]} \sum_{j \in [n]} |X_{ij}|$. Assume \mathcal{T}_1 and \mathcal{T}_2 are subsets of square matrices, then the Hausdorff distance in-

duced by the previous norm is given by: $\text{dist}(\mathcal{T}_1, \mathcal{T}_2) = \max(\sup_{X \in \mathcal{S}_2} \inf_{Y \in \mathcal{S}_1} \|X - Y\|_1, \sup_{X \in \mathcal{S}_1} \inf_{Y \in \mathcal{S}_2} \|X - Y\|_1)$. We will use $\text{tr}(\cdot)$ to denote the trace of a matrix. Given a set \mathcal{H} defined by two sets of variables X and Y , $\mathcal{P}_X(\mathcal{H})$ is the projection of \mathcal{H} considering X variables: $\mathcal{P}_X(\mathcal{H}) = \{X : \exists Y, \text{ such that } (X, Y) \in \mathcal{H}\}$.

4. Linear relaxations through lifting

Consider the following polyhedral set $\mathcal{H}_{k,r}^n$ involving a matrix variable X in addition to a variable λ_T for each subset T of size r where $2 \leq r \leq k$.

$$\mathcal{H}_{k,r}^n = \left\{ \begin{array}{ll} X \in \mathcal{S}^n, \text{tr}(X) = k & \\ X_{ij} \geq 0 & \forall i, j \in [n] \\ \lambda_T \geq 0 & \forall T \in [n]_r \\ \sum_{T \in [n]_r: T \ni i} \lambda_T = \binom{k-1}{r-1} X_{ii} & \forall i \in [n] \\ \sum_{T \in [n]_r: T \ni \{i,j\}} \lambda_T = \binom{k-2}{r-2} X_{ij} & \forall i \neq j \in [n]. \end{array} \right.$$

Lemma 1. $\mathbb{U}_k^n \subseteq \mathcal{P}_X(\mathcal{H}_{k,r}^n)$.

Proof. Let $X = 1_S 1_S^t$ for some $S \in [n]_k$. Then X is obviously non-negative, symmetric and its trace equals k . For each $T \in [n]_r$, let $\lambda_T = \prod_{i \in T} X_{ii}$. The sum $\sum_{T \in [n]_r: T \ni i} \lambda_T$ is equal to 0 if $X_{ii} = 0$ (i.e., $i \notin S$). If $i \in S$, then $\lambda_T = 1$ only if $T \subseteq S$ and 0 otherwise, leading to $\sum_{T \in [n]_r: T \ni i} \lambda_T = \binom{k-1}{r-1} X_{ii}$. The last inequality $\sum_{T \in [n]_r: T \ni \{i,j\}} \lambda_T = \binom{k-2}{r-2} X_{ij}$ holds for the same reasons. Consequently, the projection of $\mathcal{H}_{k,r}^n$ onto the space of X variables contains the extreme points of \mathbb{U}_k^n and therefore the whole set \mathbb{U}_k^n for convexity reasons. \square

Lemma 2. $\mathcal{P}_X(\mathcal{H}_{k,2}^n) \supseteq \mathcal{P}_X(\mathcal{H}_{k,3}^n) \supseteq \dots \supseteq \mathcal{P}_X(\mathcal{H}_{k,k}^n) = \mathbb{U}_k^n$.

Proof. Let us show that $\mathcal{P}_X(\mathcal{H}_{k,r}^n) \supseteq \mathcal{P}_X(\mathcal{H}_{k,r+1}^n)$. Let (X, λ) belong to $\mathcal{H}_{k,r+1}^n$ where λ_T is defined for $T \in [n]_{r+1}$. One can define a point (X, λ') of $\mathcal{H}_{k,r}^n$ as follows. For each set $S \in [n]_r$, let $\lambda'_S = \frac{1}{k-r} \sum_{T \supset S, |T|=r+1} \lambda_T$. Observe that

$$\begin{aligned} \sum_{S \in [n]_r: S \ni i} \lambda'_S &= \frac{1}{k-r} \sum_{S \in [n]_r: S \ni i} \sum_{T \supset S, |T|=r+1} \lambda_T \\ &= \frac{r}{k-r} \sum_{T \in [n]_{r+1}: T \ni i} \lambda_T = \frac{r}{k-r} \binom{k-1}{r} X_{ii} = \binom{k-1}{r-1} X_{ii}. \end{aligned}$$

We can check in the same way that $\sum_{S \in [n]_r: S \ni \{i,j\}} \lambda'_S = \binom{k-2}{r-2} X_{ij}, \forall i \neq j \in [n]$. As a consequence, $(X, \lambda') \in \mathcal{H}_{k,r}^n$ implying that $X \in \mathcal{P}_X(\mathcal{H}_{k,r}^n)$ and proving that $\mathcal{P}_X(\mathcal{H}_{k,r}^n) \supseteq \mathcal{P}_X(\mathcal{H}_{k,r+1}^n)$.

By summing up constraints $\sum_{T \in [n]_r: T \ni i} \lambda_T = \binom{k-1}{r-1} X_{ii}$ and using the trace constraint we get that $\sum_{T \in [n]_r} \lambda_T = \binom{k}{r}$. Hence, for $k = r$ the sum is just equal to 1 and X is a convex combination of matrices of type $1_T 1_T^t$ where $T \in [n]_k$ implying that $X \in \mathbb{U}_k^n$. This leads to $\mathcal{P}_X(\mathcal{H}_{k,k}^n) = \mathbb{U}_k^n$. \square

Let us consider the bijective linear mapping $\mathcal{L}^{k,r}$ defined as follows:

$$\mathcal{L}^{k,r} : (X, \lambda) \mapsto (X', \lambda') \left\{ \begin{array}{l} X'_{ii} = \frac{k+1}{k} X_{ii}, \forall i \in [n] \\ X'_{ij} = \frac{k+1}{k-1} X_{ij}, \forall i \neq j \in [n] \\ \lambda'_S = \frac{k+1}{k-r+1} \lambda_S, \forall S \in [n]_r. \end{array} \right.$$

Lemma 3. $\mathcal{L}^{k,r}(\mathcal{H}_{k,r}^n) = \mathcal{H}_{k+1,r}^n$.

Proof. Let $(X, \lambda) \in \mathcal{H}_{k,r}^n$ and $(X', \lambda') = \mathcal{L}^{k,r}(X, \lambda)$. Then X' is symmetric and non-negative, and $\text{tr}(X') = k+1$. We also have:

$$\begin{aligned} \sum_{T \in [n]_r: T \ni i} \lambda'_T &= \frac{k+1}{k-r+1} \sum_{T \in [n]_r: T \ni i} \lambda_T = \frac{k+1}{k-r+1} \binom{k-1}{r-1} X_{ii} \\ &= \frac{k+1}{k-r+1} \binom{k-1}{r-1} \frac{k}{k+1} X'_{ii} = \binom{k}{r-1} X'_{ii}. \end{aligned}$$

The other constraint $\sum_{T \in [n]_r: T \ni \{i,j\}} \lambda'_T = \binom{k-1}{r-2} X'_{ij}$ can be proved in the same way. We consequently have $\mathcal{L}^{k,r}(\mathcal{H}_{k,r}^n) \subseteq \mathcal{H}_{k+1,r}^n$. The other inclusion is also straightforward since $\mathcal{L}^{k,r}$ is bijective and $(\mathcal{L}^{k,r})^{-1}(\mathcal{H}_{k+1,r}^n) \subseteq \mathcal{H}_{k,r}^n$. \square

Let $\mathcal{M}^{k,r} = \mathcal{L}^{k-1,r} \circ \mathcal{L}^{k-2,r} \circ \dots \circ \mathcal{L}^{r,r}$. Note that $\mathcal{M}^{k,r}$ can be expressed explicitly as follows:

$$\mathcal{M}^{k,r} : (X, \lambda) \mapsto (X', \lambda') \left\{ \begin{array}{l} X'_{ii} = \frac{k}{r} X_{ii}, \forall i \in [n] \\ X'_{ij} = \frac{k(k-1)}{r(r-1)} X_{ij}, \forall i \neq j \in [n] \\ \lambda'_S = \binom{k}{r} \lambda_S, \forall S \in [n]_r. \end{array} \right.$$

Let $\mathcal{M}_X^{k,r}$ be the mapping obtained from $\mathcal{M}^{k,r}$ by considering only X variables:

$$\mathcal{M}_X^{k,r} : X \mapsto X' \left\{ \begin{array}{l} X'_{ii} = \frac{k}{r} X_{ii}, \forall i \in [n] \\ X'_{ij} = \frac{k(k-1)}{r(r-1)} X_{ij}, \forall i \neq j \in [n]. \end{array} \right.$$

Lemma 4. $\mathcal{M}^{k,r}(\mathcal{H}_{k,r}^n) = \mathcal{H}_{k,r}^n$ and $\mathcal{M}_X^{k,r}(\mathcal{U}_r^n) = \mathcal{P}_X(\mathcal{H}_{k,r}^n)$.

Proof. $\mathcal{M}^{k,r}(\mathcal{H}_{k,r}^n) = \mathcal{H}_{k,r}^n$ is obtained by iteratively applying Lemma 3. The second part follows from projection and the fact that $\mathcal{P}_X(\mathcal{H}_{k,r}^n) = \mathcal{U}_r^n$. \square

Proposition 1. $\text{dist}(\mathcal{U}_k^n, \mathcal{P}_X(\mathcal{H}_{k,r}^n)) \leq 2(k^2 - r^2)$.

Proof. Since $\mathcal{P}_X(\mathcal{H}_{k,r}^n) \supseteq \mathcal{U}_k^n$, we only have to prove that $\sup_{X \in \mathcal{P}_X(\mathcal{H}_{k,r}^n)} \inf_{Z \in \mathcal{U}_k^n} \|X - Z\|_1 \leq 2(k^2 - r^2)$. Let $X \in \mathcal{P}_X(\mathcal{H}_{k,r}^n)$. From Lemma 4, there exists $Y \in \mathcal{U}_r^n$ such that $X = \mathcal{M}_X^{k,r}(Y)$. Observe that the diagonal terms of the matrix $X - Y$ are $X_{ii}(1 - r/k)$ while the non-diagonal terms are $X_{ij}(1 - \frac{r(r-1)}{k(k-1)})$ implying that $X - Y$ is non-negative and $\|X - Y\|_1 = \sum_{i,j} |X_{ij} - Y_{ij}| = \sum_{i,j} X_{ij} - Y_{ij} = \|X\|_1 - \|Y\|_1 = k^2 - r^2$. We hence proved that:

$$\sup_{X \in \mathcal{P}_X(\mathcal{H}_{k,r}^n)} \inf_{Y \in \mathcal{U}_r^n} \|X - Y\|_1 \leq k^2 - r^2. \quad (1)$$

Let $Y = 1_S 1_S^t$ for $S \in [n]_r$ be an extreme point of \mathcal{U}_r^n . Consider any subset $T \in [n]_k$ containing S and let $Z = 1_T 1_T^t$. The matrix $Y - Z$ has exactly $k^2 - r^2$ non-zero coefficients implying that $\|Y - Z\|_1 = k^2 - r^2$. Since this holds for any extreme point of \mathcal{U}_r^n , it also holds for each matrix of \mathcal{U}_r^n . Indeed, given a convex combination \tilde{Y} of vertices Y of \mathcal{U}_r^n , and given the same combination \tilde{Z} of vertices Z of \mathcal{U}_k^n with $\|Y - Z\|_1 \leq k^2 - r^2$, then by convexity of the norm $\|\tilde{Y} - \tilde{Z}\|_1 \leq k^2 - r^2$, leading to:

$$\sup_{Y \in \mathcal{U}_r^n} \inf_{Z \in \mathcal{U}_k^n} \|Y - Z\|_1 \leq k^2 - r^2. \quad (2)$$

Combination of (1) and (2) proves the wanted result. \square

Notice the obvious valid inequalities $X_{ij} \leq 1$ are not considered in $\mathcal{H}_{k,r}^n$. It is easy to check that they are not dominated by the other constraints defining $\mathcal{H}_{k,r}^n$. An example is given below where $k = n = 3$, $r = 2$, $\lambda_{ij} = X_{ij}$ for $i < j \in [n]$

$$\text{and } X = \begin{pmatrix} 3/2 & 3/2 & 3/2 \\ 3/2 & 3/4 & 0 \\ 3/2 & 0 & 3/4 \end{pmatrix}.$$

Let $\overline{\mathcal{H}}_{k,r}^n$ be the subset obtained from $\mathcal{H}_{k,r}^n$ by adding the constraints $X_{ij} \leq 1$ for $i, j \in [n]$. Let \mathcal{G}_k^n be defined as

follows:

$$\mathcal{G}_k^n = \left\{ \begin{array}{l} X \in \mathcal{S}^n \\ \sum_{i \in [n]} X_{ii} = k \\ \sum_{1 \leq i < j \leq n} X_{ij} = \frac{1}{2}(k^2 - k) \\ 0 \leq X_{ij} \leq 1 \quad \forall i, j \in [n]. \end{array} \right.$$

Lemma 5. $\mathcal{G}_k^n \supseteq \mathcal{P}_X(\overline{\mathcal{H}}_{k,r}^n)$, and all extreme points of \mathcal{G}_k^n are binary.

Proof. We already observed that $\sum_{T \in [n]_k} \lambda_T = \binom{k}{r}$ for $(X, r) \in \mathcal{H}_{k,r}^n$. By summing up all equations of type $\sum_{T \in [n]_r: T \supseteq \{i,j\}} \lambda_T = \binom{k-2}{r-2} X_{ij}$, we get that $\sum_{1 \leq i < j \leq n} X_{ij} = \frac{1}{\binom{k-2}{r-2}} \frac{r(r-1)}{2} \binom{k}{r} = \frac{1}{2}(k^2 - k)$. Since the other inequalities defining \mathcal{G}_k^n are obviously satisfied, $\mathcal{G}_k^n \supseteq \overline{\mathcal{H}}_{k,r}^n$ holds. Moreover, since each variable X_{ii} (resp. X_{ij} , $i < j$) appears in only one equality, all extreme points of \mathcal{G}_k^n can only have 0 - 1 values. \square

Proposition 2. $\text{dist}(\mathcal{U}_k^n, \mathcal{G}_k^n) \leq 2k^2(1 - \frac{k}{n})$.

Proof. Let us prove that for each extreme point X of \mathcal{G}_k^n , there exists a matrix $Y \in \mathcal{U}_k^n$ such that $\|Y - X\|_1 \leq 2k^2(1 - \frac{k}{n})$. We know from the previous lemma that X is binary. X is then a symmetric matrix with exactly k^2 non-zero coefficients. We will show by induction on p ($p \in [n]$) that we can extract from X a principal p -square submatrix containing at least $k^2 p/n$ non-zero coefficients. The result holds for $p = n$ (the submatrix is the whole matrix X). Assume the result holds for p and let us prove it for $p-1$. By induction hypothesis, let Y be a principal p -square submatrix of X containing at least $k^2 p/n$ non-zero coefficients. Let i be the index of the row of Y containing the smallest number of non-zero coefficients. This number is less than or equal to $1/p$ of the total number of non-zero coefficients of Y . Then by considering the $p-1$ -square matrix obtained from Y by eliminating the i^{th} row (and column), we get a matrix Z where the number of 1-coefficients is at least $(1 - 1/p)k^2 p/n$ which is nothing other than $k^2(p-1)/n$. Now using the proved result for $p = k$, we deduce that there is a set $S \in [n]_k$ such that $\|X_S\|_1 \geq k^3/n$ where X_S is

the principal submatrix induced by S . Then $\|X - 1_S 1_S^t\|_1 \leq k^2 + k^2 - 2k^3/n$ ending the proof. \square

Theorem 1. $\text{dist}\left(\mathcal{U}_k^n, \mathcal{P}_X(\overline{\mathcal{H}_{k,r}^n})\right) \leq 2k^2 \min\left(1 - \left(\frac{r}{k}\right)^2, 1 - \frac{k}{n}\right)$.

Proof. The result follows from the inclusion $\mathcal{P}_X(\overline{\mathcal{H}_{k,r}^n}) \subseteq \mathcal{G}_k^n \cap \mathcal{P}_X(\mathcal{H}_{k,r}^n)$ and Propositions 1 and 2. \square

It might be convenient to normalize the distance to get a better idea about the quality of the approximation. One can, for example, define $\text{nd}(X, Y) = \frac{\|X - Y\|_1}{\|X\|_1 + \|Y\|_1}$. While nd is not really a distance, it has some nice features: $\text{nd}(X, Y) = 0$ only if $X = Y$, and $0 \leq \text{nd}(X, Y) \leq 1$. Since all elements of $\mathcal{P}_X(\overline{\mathcal{H}_{k,r}^n})$ and \mathcal{U}_k^n are of norm k^2 , the previous theorem can be written as $\text{nd}\left(\mathcal{U}_k^n, \mathcal{P}_X(\overline{\mathcal{H}_{k,r}^n})\right) \leq \min\left(1 - \left(\frac{r}{k}\right)^2, 1 - \frac{k}{n}\right)$. Note that $1 - k/n$ is a better bound for “relatively small” values of r , namely when r is at most a $\sqrt{k/n}$ fraction of k . The bound of Theorem 1 does not improve with r until r is sufficiently large.

As might be suggested by the proof of Lemma 2, starting from $\mathcal{H}_{k,r}^n$ and performing one iteration of the Sherali-Adams lifting, one can obtain all constraints defining $\mathcal{H}_{k,r+1}^n$ (in addition to others). In other words, starting from $\mathcal{H}_{k,2}^n$, the $r - 1$ -level Sherali-Adams relaxation would be at least as tight as $\mathcal{H}_{k,r}^n$. As a consequence, the upper bound of Theorem 1 applies also to the Sherali-Adams relaxation of level $r - 1$.

5. Concluding remarks

Although the results presented in this paper are specific to the cardinality-constrained Boolean quadric polytope, we believe that they deserve to be adapted and/or generalized to other polytopes.

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