



**HAL**  
open science

# Proximity guarantees of a lift-and-project approach for the cardinality-constrained Boolean quadric polytope

Walid Ben-Ameur

► **To cite this version:**

Walid Ben-Ameur. Proximity guarantees of a lift-and-project approach for the cardinality-constrained Boolean quadric polytope. *Operations Research Letters*, 2024, 57, pp.107166. 10.1016/j.orl.2024.107166 . hal-04680844

**HAL Id: hal-04680844**

**<https://hal.science/hal-04680844v1>**

Submitted on 21 Jan 2025

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Proximity guarantees of a lift-and-project approach for the cardinality-constrained Boolean quadric polytope

Walid Ben-Ameur<sup>a</sup>,

<sup>a</sup>Samovar, Télécom SudParis, Institut Polytechnique de Paris, Palaiseau, France

---

## Abstract

We consider a lift-and-project approach for the cardinality-constrained Boolean quadric polytope. Some upper bounds for the distance between the polytope and its linear approximation are derived. Unsurprisingly, the distance converges to 0 when the number of variables increases sufficiently.

Keywords: Boolean quadric polytope, linear relaxations, lift-and-project.

---

## 1. Introduction

Let  $U^n = \text{conv}(\{1_S 1_S^t, S \subseteq [n]\})$  where  $[n] = \{1, 2, \dots, n\}$  and  $1_S$  is a vector of  $\mathbb{R}^n$  whose  $i^{\text{th}}$  component equals 1 (resp. 0) if  $i \in S$  (resp. otherwise). Since any matrix  $X \in U^n$  is symmetric, we can simply consider the values  $X_{ij}$  for  $j \geq i$ , leading to the well-known Boolean quadric polytope  $\text{BQP}^n \subseteq \mathbb{R}^{n(n+1)/2}$  that can be seen as a projection of  $U^n$ . Since 0-1 quadratic programs can obviously be formulated as linear programs over  $\text{BQP}^n$  (or  $U^n$ ), good approximations of these polytopes are sought to better solve this important class of problems. In many applications, a cardinality constraint is added: only  $k$  variables can be non-zero. This leads to the study of  $U_k^n$  and  $\text{BQP}_k^n$  where  $U_k^n = \text{conv}(\{1_S 1_S^t, S \in [n]_k\})$ ,  $[n]_k = \{S \subseteq [n] : |S| = k\}$  and  $\text{BQP}_k^n$  is the projection of  $U_k^n$ .

We will review some related work in next section. Then the paper contributions and some notation are presented in Section 3. The main results and their proofs are provided in Section 4. Concluding remarks follow in the last section.

## 2. Related work

The Boolean quadric polytope introduced in [20] is also called the full correlation polytope [22] since  $X \in \text{BQP}^n$  is related to some probability space  $(\Omega, \mathcal{A}, \mu)$  through the equations  $X_{ii} = \mu(A_i)$  and  $X_{ij} = \mu(A_i \cap A_j)$  where  $(A_i)_{i \in [n]} \in \mathcal{A}$ . As a consequence, [22] observed that the typical probability problem (Boole's problem) of minimizing  $\mu(A_1 \cup \dots \cup A_n)$  given  $\mu(A_i)$  and  $\mu(A_i \cap A_j)$  values, can be stated as a linear program involving a variable for each extreme point of  $\text{BQP}^n$  (to express  $X$  as a convex combination).

Given the connection with 0-1 quadratic programming, membership in  $\text{BQP}^n$  is unsurprisingly shown to be NP-complete [22].

Several families of valid inequalities for  $\text{BQP}^n$  have been proposed in literature. In addition to the obvious inequalities  $0 \leq X_{ij} \leq 1$ ,  $i \leq j$ , inequalities  $X_{ij} + 1 \geq X_{ii} + X_{jj}$  and  $X_{ij} \leq X_{ii}$  are presented in [10, 17]. These are now called RLT inequalities [24]. Triangle inequalities are derived in [20]:  $X_{ii} + X_{jj} + X_{kk} \leq X_{ij} + X_{jk} + X_{ki} + 1$  and  $X_{ij} + X_{jk} \leq X_{ii} + X_{kk}$ . Other families include clique inequalities [20], cut inequalities [20], Boros and Hammer inequalities [5], gap inequalities [9] adapted from [14] and inequalities implied by the positive semidefinite nature of the matrix  $X - \text{diag}(X)\text{diag}(X)^T$  [12, 25] where  $\text{diag}(X)$  is

---

Email address: walid.benameur@telecom-sudparis.eu (Walid Ben-Ameur )

the diagonal vector of  $X$ .

The Boolean quadric polytope is equivalent to the cut polytope since one can be obtained from the other through a straightforward affine transformation [6, 20, 3]. More precisely, for  $X \in \text{BQP}^n$  one can define a point in  $z \in \text{CUT}^{n+1}$  (the convex hull of incidence vectors of cuts in a complete graph of order  $n + 1$ ) by setting  $z_{i,n+1} = X_{ii}$  for  $i \in [n]$  and  $z_{i,j} = X_{ii} + X_{jj} - 2X_{ij}$  for  $i < j \in [n]$ . Inequalities defined for the cut polytope can then be easily transformed into valid inequalities for  $\text{BQP}^n$  and vice-versa. More details can be found in [15].

The Boolean quadric polytope under the cardinality constraint  $\sum_i X_{ii} \leq k$  has been studied in [21, 18]. [18] focused on the case where variables  $X_{ij}$  are considered only for pairs  $(i, j)$  in a proper subset  $E$  of the set of all pairs  $i \leq j$  in  $[n] \times [n]$ . This is naturally related to the Boolean quadratic program  $\min \sum_{(i,j) \in E} q_{ij} x_i x_j$  where coefficients  $q_{ij}$  are non-zero only for  $(i, j) \in E$ . Some valid inequalities have been proposed there including star inequalities ( $\sum_{j \neq i} X_{ij} \leq (k-1)X_{ii}$ ), tree inequalities, long-cycle inequalities, long-tree inequalities and long-forest inequalities generalizing some of the inequalities proposed in [11]. According to [18] (Section 5), the separation problems for all the inequalities appear to be difficult. [21] considered the complete case and studied the polytope  $\text{BQP}_{k \leq}^n$  (the analogue of  $\text{BQP}_k^n$  where we require  $\sum_i X_{ii} \leq k$  instead of equality). They also proposed the star and tree inequalities mentioned above. Moreover, they considered the projection of  $\text{BQP}_{k \leq}^n$  obtained by ignoring  $X_{ii}$  variables (since some applications do not involve node weights).

$\text{BQP}_k^n$  is studied in [8]. Its dimension is shown to be equal to  $n(n-1)/2 - 1$  for  $2 \leq k \leq n$ . Authors recall that  $\text{BQP}_k^n$  is stable by permutation allowing to build some facets from others by straightforward coefficient permutation. Switching (or complement operation) is also considered in [8] showing that  $\text{BQP}_{n-k}^n$  can be obtained from  $\text{BQP}_k^n$  by simple affine transformation.  $\text{BQP}_k^n$  is proved to be a face of  $\text{BQP}^n$  and a full description of  $\text{BQP}_k^n$  is given when  $k = 2$

and  $k = n - 2$ . It is also shown that  $\text{BQP}_{k \leq}^n$  is a projection of  $\text{BQP}_k^{n+k}$ . Some connections with the equicut polytope are exposed [8]. More generally, the affine correspondence mentioned above between the cut polytope and  $\text{BQP}^n$  leads to a mapping between  $\text{BQP}_k^n$  and the uniform cut polytope  $\text{CUT}_k^{n+1}$  (the cut polytope under a cardinality constraint). Inequalities defined for the uniform cut polytope such as those of [7, 19] can be mapped to some valid inequalities for  $\text{BQP}_k^n$ .

The relaxations proposed in the paper are based on lifting where the set of variables and constraints are extended by multiplying constraints by variables and linearizing by adding new variables representing products of old variables. Lifting is of course a well-known technique proposed in different papers including [23, 16, 2]. Roughly speaking, we know that by using any one of the three variants introduced there, we get the convex hull of integer solutions inside a polytope at the cost of adding an exponential number of variables and constraints [2, 13]. Several studies tried to determine the maximum number of the lift-and-project iterations required to get the true convex-hull (generally called rank) (see, e.g., [13, 1]).

### 3. Contributions and further notation

In our paper, we aim to analyze the quality of the relaxation obtained by considering some kind of lifting. More precisely, focusing on the special case of  $\text{U}_k^n$  and considering a relaxation involving a variable  $\lambda_S$  for any set  $S$  of size  $r$  ( $r \leq k$ ), we show that the normalized distance between  $\text{U}_k^n$  and the relaxed set is at most  $\min(1 - (\frac{r}{k})^2, 1 - \frac{k}{n})$ .

The lifting approach proposed in the current paper is somewhat related to the recent work in [4] where the cone of factor-width- $k$  matrices is approximated through lifting. The 0-1 case was not handled there and a different distance (Frobenius distance) is considered in [4].

Let us add some notation. Given a  $n$ -square matrix  $X$ , let  $\|X\|_1 = \sum_{i \in [n]} \sum_{j \in [n]} |X_{ij}|$ . Assume  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are subsets of square matrices, then the Hausdorff distance in-

duced by the previous norm is given by:  $\text{dist}(\mathcal{T}_1, \mathcal{T}_2) = \max(\sup_{X \in \mathcal{S}_2} \inf_{Y \in \mathcal{S}_1} \|X - Y\|_1, \sup_{X \in \mathcal{S}_1} \inf_{Y \in \mathcal{S}_2} \|X - Y\|_1)$ . We will use  $\text{tr}(\cdot)$  to denote the trace of a matrix. Given a set  $\mathcal{H}$  defined by two sets of variables  $X$  and  $Y$ ,  $\mathcal{P}_X(\mathcal{H})$  is the projection of  $\mathcal{H}$  considering  $X$  variables:  $\mathcal{P}_X(\mathcal{H}) = \{X : \exists Y, \text{ such that } (X, Y) \in \mathcal{H}\}$ .

#### 4. Linear relaxations through lifting

Consider the following polyhedral set  $\mathcal{H}_{k,r}^n$  involving a matrix variable  $X$  in addition to a variable  $\lambda_T$  for each subset  $T$  of size  $r$  where  $2 \leq r \leq k$ .

$$\mathcal{H}_{k,r}^n = \left\{ \begin{array}{ll} X \in \mathcal{S}^n, \text{tr}(X) = k & \\ X_{ij} \geq 0 & \forall i, j \in [n] \\ \lambda_T \geq 0 & \forall T \in [n]_r \\ \sum_{T \in [n]_r: T \ni i} \lambda_T = \binom{k-1}{r-1} X_{ii} & \forall i \in [n] \\ \sum_{T \in [n]_r: T \ni \{i,j\}} \lambda_T = \binom{k-2}{r-2} X_{ij} & \forall i \neq j \in [n]. \end{array} \right.$$

Lemma 1.  $\mathbb{U}_k^n \subseteq \mathcal{P}_X(\mathcal{H}_{k,r}^n)$ .

Proof. Let  $X = 1_S 1_S^t$  for some  $S \in [n]_k$ . Then  $X$  is obviously non-negative, symmetric and its trace equals  $k$ . For each  $T \in [n]_r$ , let  $\lambda_T = \prod_{i \in T} X_{ii}$ . The sum  $\sum_{T \in [n]_r: T \ni i} \lambda_T$  is equal to 0 if  $X_{ii} = 0$  (i.e.,  $i \notin S$ ). If  $i \in S$ , then  $\lambda_T = 1$  only if  $T \subseteq S$  and 0 otherwise, leading to  $\sum_{T \in [n]_r: T \ni i} \lambda_T = \binom{k-1}{r-1} X_{ii}$ . The last inequality  $\sum_{T \in [n]_r: T \ni \{i,j\}} \lambda_T = \binom{k-2}{r-2} X_{ij}$  holds for the same reasons. Consequently, the projection of  $\mathcal{H}_{k,r}^n$  onto the space of  $X$  variables contains the extreme points of  $\mathbb{U}_k^n$  and therefore the whole set  $\mathbb{U}_k^n$  for convexity reasons.  $\square$

Lemma 2.  $\mathcal{P}_X(\mathcal{H}_{k,2}^n) \supseteq \mathcal{P}_X(\mathcal{H}_{k,3}^n) \supseteq \dots \supseteq \mathcal{P}_X(\mathcal{H}_{k,k}^n) = \mathbb{U}_k^n$ .

Proof. Let us show that  $\mathcal{P}_X(\mathcal{H}_{k,r}^n) \supseteq \mathcal{P}_X(\mathcal{H}_{k,r+1}^n)$ . Let  $(X, \lambda)$  belong to  $\mathcal{H}_{k,r+1}^n$  where  $\lambda_T$  is defined for  $T \in [n]_{r+1}$ . One can define a point  $(X, \lambda')$  of  $\mathcal{H}_{k,r}^n$  as follows. For each set  $S \in [n]_r$ , let  $\lambda'_S = \frac{1}{k-r} \sum_{T \supset S, |T|=r+1} \lambda_T$ . Observe that

$$\begin{aligned} \sum_{S \in [n]_r: S \ni i} \lambda'_S &= \frac{1}{k-r} \sum_{S \in [n]_r: S \ni i} \sum_{T \supset S, |T|=r+1} \lambda_T \\ &= \frac{r}{k-r} \sum_{T \in [n]_{r+1}: T \ni i} \lambda_T = \frac{r}{k-r} \binom{k-1}{r} X_{ii} = \binom{k-1}{r-1} X_{ii}. \end{aligned}$$

We can check in the same way that  $\sum_{S \in [n]_r: S \ni \{i,j\}} \lambda'_S = \binom{k-2}{r-2} X_{ij}, \forall i \neq j \in [n]$ . As a consequence,  $(X, \lambda') \in \mathcal{H}_{k,r}^n$  implying that  $X \in \mathcal{P}_X(\mathcal{H}_{k,r}^n)$  and proving that  $\mathcal{P}_X(\mathcal{H}_{k,r}^n) \supseteq \mathcal{P}_X(\mathcal{H}_{k,r+1}^n)$ .

By summing up constraints  $\sum_{T \in [n]_r: T \ni i} \lambda_T = \binom{k-1}{r-1} X_{ii}$  and using the trace constraint we get that  $\sum_{T \in [n]_r} \lambda_T = \binom{k}{r}$ . Hence, for  $k = r$  the sum is just equal to 1 and  $X$  is a convex combination of matrices of type  $1_T 1_T^t$  where  $T \in [n]_k$  implying that  $X \in \mathbb{U}_k^n$ . This leads to  $\mathcal{P}_X(\mathcal{H}_{k,k}^n) = \mathbb{U}_k^n$ .  $\square$

Let us consider the bijective linear mapping  $\mathcal{L}^{k,r}$  defined as follows:

$$\mathcal{L}^{k,r} : (X, \lambda) \mapsto (X', \lambda') \left\{ \begin{array}{l} X'_{ii} = \frac{k+1}{k} X_{ii}, \forall i \in [n] \\ X'_{ij} = \frac{k+1}{k-1} X_{ij}, \forall i \neq j \in [n] \\ \lambda'_S = \frac{k+1}{k-r+1} \lambda_S, \forall S \in [n]_r. \end{array} \right.$$

Lemma 3.  $\mathcal{L}^{k,r}(\mathcal{H}_{k,r}^n) = \mathcal{H}_{k+1,r}^n$ .

Proof. Let  $(X, \lambda) \in \mathcal{H}_{k,r}^n$  and  $(X', \lambda') = \mathcal{L}^{k,r}(X, \lambda)$ . Then  $X'$  is symmetric and non-negative, and  $\text{tr}(X') = k+1$ . We also have:

$$\begin{aligned} \sum_{T \in [n]_r: T \ni i} \lambda'_T &= \frac{k+1}{k-r+1} \sum_{T \in [n]_r: T \ni i} \lambda_T = \frac{k+1}{k-r+1} \binom{k-1}{r-1} X_{ii} \\ &= \frac{k+1}{k-r+1} \binom{k-1}{r-1} \frac{k}{k+1} X'_{ii} = \binom{k}{r-1} X'_{ii}. \end{aligned}$$

The other constraint  $\sum_{T \in [n]_r: T \ni \{i,j\}} \lambda'_T = \binom{k-1}{r-2} X'_{ij}$  can be proved in the same way. We consequently have  $\mathcal{L}^{k,r}(\mathcal{H}_{k,r}^n) \subseteq \mathcal{H}_{k+1,r}^n$ . The other inclusion is also straightforward since  $\mathcal{L}^{k,r}$  is bijective and  $(\mathcal{L}^{k,r})^{-1}(\mathcal{H}_{k+1,r}^n) \subseteq \mathcal{H}_{k,r}^n$ .  $\square$

Let  $\mathcal{M}^{k,r} = \mathcal{L}^{k-1,r} \circ \mathcal{L}^{k-2,r} \circ \dots \circ \mathcal{L}^{r,r}$ . Note that  $\mathcal{M}^{k,r}$  can be expressed explicitly as follows:

$$\mathcal{M}^{k,r} : (X, \lambda) \mapsto (X', \lambda') \left\{ \begin{array}{l} X'_{ii} = \frac{k}{r} X_{ii}, \forall i \in [n] \\ X'_{ij} = \frac{k(k-1)}{r(r-1)} X_{ij}, \forall i \neq j \in [n] \\ \lambda'_S = \binom{k}{r} \lambda_S, \forall S \in [n]_r. \end{array} \right.$$

Let  $\mathcal{M}_X^{k,r}$  be the mapping obtained from  $\mathcal{M}^{k,r}$  by considering only  $X$  variables:

$$\mathcal{M}_X^{k,r} : X \mapsto X' \left\{ \begin{array}{l} X'_{ii} = \frac{k}{r} X_{ii}, \forall i \in [n] \\ X'_{ij} = \frac{k(k-1)}{r(r-1)} X_{ij}, \forall i \neq j \in [n]. \end{array} \right.$$

Lemma 4.  $\mathcal{M}^{k,r}(\mathcal{H}_{k,r}^n) = \mathcal{H}_{k,r}^n$  and  $\mathcal{M}_X^{k,r}(\mathcal{U}_r^n) = \mathcal{P}_X(\mathcal{H}_{k,r}^n)$ .

Proof.  $\mathcal{M}^{k,r}(\mathcal{H}_{k,r}^n) = \mathcal{H}_{k,r}^n$  is obtained by iteratively applying Lemma 3. The second part follows from projection and the fact that  $\mathcal{P}_X(\mathcal{H}_{k,r}^n) = \mathcal{U}_r^n$ .  $\square$

Proposition 1.  $\text{dist}(\mathcal{U}_k^n, \mathcal{P}_X(\mathcal{H}_{k,r}^n)) \leq 2(k^2 - r^2)$ .

Proof. Since  $\mathcal{P}_X(\mathcal{H}_{k,r}^n) \supseteq \mathcal{U}_k^n$ , we only have to prove that  $\sup_{X \in \mathcal{P}_X(\mathcal{H}_{k,r}^n)} \inf_{Z \in \mathcal{U}_k^n} \|X - Z\|_1 \leq 2(k^2 - r^2)$ . Let  $X \in \mathcal{P}_X(\mathcal{H}_{k,r}^n)$ . From Lemma 4, there exists  $Y \in \mathcal{U}_r^n$  such that  $X = \mathcal{M}_X^{k,r}(Y)$ . Observe that the diagonal terms of the matrix  $X - Y$  are  $X_{ii}(1 - r/k)$  while the non-diagonal terms are  $X_{ij}(1 - \frac{r(r-1)}{k(k-1)})$  implying that  $X - Y$  is non-negative and  $\|X - Y\|_1 = \sum_{i,j} |X_{ij} - Y_{ij}| = \sum_{i,j} X_{ij} - Y_{ij} = \|X\|_1 - \|Y\|_1 = k^2 - r^2$ . We hence proved that:

$$\sup_{X \in \mathcal{P}_X(\mathcal{H}_{k,r}^n)} \inf_{Y \in \mathcal{U}_r^n} \|X - Y\|_1 \leq k^2 - r^2. \quad (1)$$

Let  $Y = 1_S 1_S^t$  for  $S \in [n]_r$  be an extreme point of  $\mathcal{U}_r^n$ . Consider any subset  $T \in [n]_k$  containing  $S$  and let  $Z = 1_T 1_T^t$ . The matrix  $Y - Z$  has exactly  $k^2 - r^2$  non-zero coefficients implying that  $\|Y - Z\|_1 = k^2 - r^2$ . Since this holds for any extreme point of  $\mathcal{U}_r^n$ , it also holds for each matrix of  $\mathcal{U}_r^n$ . Indeed, given a convex combination  $\tilde{Y}$  of vertices  $Y$  of  $\mathcal{U}_r^n$ , and given the same combination  $\tilde{Z}$  of vertices  $Z$  of  $\mathcal{U}_k^n$  with  $\|Y - Z\|_1 \leq k^2 - r^2$ , then by convexity of the norm  $\|\tilde{Y} - \tilde{Z}\|_1 \leq k^2 - r^2$ , leading to:

$$\sup_{Y \in \mathcal{U}_r^n} \inf_{Z \in \mathcal{U}_k^n} \|Y - Z\|_1 \leq k^2 - r^2. \quad (2)$$

Combination of (1) and (2) proves the wanted result.  $\square$

Notice the obvious valid inequalities  $X_{ij} \leq 1$  are not considered in  $\mathcal{H}_{k,r}^n$ . It is easy to check that they are not dominated by the other constraints defining  $\mathcal{H}_{k,r}^n$ . An example is given below where  $k = n = 3$ ,  $r = 2$ ,  $\lambda_{ij} = X_{ij}$  for  $i < j \in [n]$

$$\text{and } X = \begin{pmatrix} 3/2 & 3/2 & 3/2 \\ 3/2 & 3/4 & 0 \\ 3/2 & 0 & 3/4 \end{pmatrix}.$$

Let  $\overline{\mathcal{H}}_{k,r}^n$  be the subset obtained from  $\mathcal{H}_{k,r}^n$  by adding the constraints  $X_{ij} \leq 1$  for  $i, j \in [n]$ . Let  $\mathcal{G}_k^n$  be defined as

follows:

$$\mathcal{G}_k^n = \left\{ \begin{array}{l} X \in \mathcal{S}^n \\ \sum_{i \in [n]} X_{ii} = k \\ \sum_{1 \leq i < j \leq n} X_{ij} = \frac{1}{2}(k^2 - k) \\ 0 \leq X_{ij} \leq 1 \quad \forall i, j \in [n]. \end{array} \right.$$

Lemma 5.  $\mathcal{G}_k^n \supseteq \mathcal{P}_X(\overline{\mathcal{H}}_{k,r}^n)$ , and all extreme points of  $\mathcal{G}_k^n$  are binary.

Proof. We already observed that  $\sum_{T \in [n]_k} \lambda_T = \binom{k}{r}$  for  $(X, r) \in \mathcal{H}_{k,r}^n$ . By summing up all equations of type  $\sum_{T \in [n]_r: T \supseteq \{i,j\}} \lambda_T = \binom{k-2}{r-2} X_{ij}$ , we get that  $\sum_{1 \leq i < j \leq n} X_{ij} = \frac{1}{\binom{k-2}{r-2}} \frac{r(r-1)}{2} \binom{k}{r} = \frac{1}{2}(k^2 - k)$ . Since the other inequalities defining  $\mathcal{G}_k^n$  are obviously satisfied,  $\mathcal{G}_k^n \supseteq \overline{\mathcal{H}}_{k,r}^n$  holds. Moreover, since each variable  $X_{ii}$  (resp.  $X_{ij}$ ,  $i < j$ ) appears in only one equality, all extreme points of  $\mathcal{G}_k^n$  can only have 0 - 1 values.  $\square$

Proposition 2.  $\text{dist}(\mathcal{U}_k^n, \mathcal{G}_k^n) \leq 2k^2(1 - \frac{k}{n})$ .

Proof. Let us prove that for each extreme point  $X$  of  $\mathcal{G}_k^n$ , there exists a matrix  $Y \in \mathcal{U}_k^n$  such that  $\|Y - X\|_1 \leq 2k^2(1 - \frac{k}{n})$ . We know from the previous lemma that  $X$  is binary.  $X$  is then a symmetric matrix with exactly  $k^2$  non-zero coefficients. We will show by induction on  $p$  ( $p \in [n]$ ) that we can extract from  $X$  a principal  $p$ -square submatrix containing at least  $k^2 p/n$  non-zero coefficients. The result holds for  $p = n$  (the submatrix is the whole matrix  $X$ ). Assume the result holds for  $p$  and let us prove it for  $p-1$ . By induction hypothesis, let  $Y$  be a principal  $p$ -square submatrix of  $X$  containing at least  $k^2 p/n$  non-zero coefficients. Let  $i$  be the index of the row of  $Y$  containing the smallest number of non-zero coefficients. This number is less than or equal to  $1/p$  of the total number of non-zero coefficients of  $Y$ . Then by considering the  $p-1$ -square matrix obtained from  $Y$  by eliminating the  $i^{\text{th}}$  row (and column), we get a matrix  $Z$  where the number of 1-coefficients is at least  $(1 - 1/p)k^2 p/n$  which is nothing other than  $k^2(p-1)/n$ . Now using the proved result for  $p = k$ , we deduce that there is a set  $S \in [n]_k$  such that  $\|X_S\|_1 \geq k^3/n$  where  $X_S$  is

the principal submatrix induced by  $S$ . Then  $\|X - 1_S 1_S^t\|_1 \leq k^2 + k^2 - 2k^3/n$  ending the proof.  $\square$

Theorem 1.  $\text{dist}\left(\mathcal{U}_k^n, \mathcal{P}_X(\overline{\mathcal{H}_{k,r}^n})\right) \leq 2k^2 \min\left(1 - \left(\frac{r}{k}\right)^2, 1 - \frac{k}{n}\right)$ .

Proof. The result follows from the inclusion  $\mathcal{P}_X(\overline{\mathcal{H}_{k,r}^n}) \subseteq \mathcal{G}_k^n \cap \mathcal{P}_X(\mathcal{H}_{k,r}^n)$  and Propositions 1 and 2.  $\square$

It might be convenient to normalize the distance to get a better idea about the quality of the approximation. One can, for example, define  $\text{nd}(X, Y) = \frac{\|X - Y\|_1}{\|X\|_1 + \|Y\|_1}$ . While  $\text{nd}$  is not really a distance, it has some nice features:  $\text{nd}(X, Y) = 0$  only if  $X = Y$ , and  $0 \leq \text{nd}(X, Y) \leq 1$ . Since all elements of  $\mathcal{P}_X(\overline{\mathcal{H}_{k,r}^n})$  and  $\mathcal{U}_k^n$  are of norm  $k^2$ , the previous theorem can be written as  $\text{nd}\left(\mathcal{U}_k^n, \mathcal{P}_X(\overline{\mathcal{H}_{k,r}^n})\right) \leq \min\left(1 - \left(\frac{r}{k}\right)^2, 1 - \frac{k}{n}\right)$ . Note that  $1 - k/n$  is a better bound for “relatively small” values of  $r$ , namely when  $r$  is at most a  $\sqrt{k/n}$  fraction of  $k$ . The bound of Theorem 1 does not improve with  $r$  until  $r$  is sufficiently large.

As might be suggested by the proof of Lemma 2, starting from  $\mathcal{H}_{k,r}^n$  and performing one iteration of the Sherali-Adams lifting, one can obtain all constraints defining  $\mathcal{H}_{k,r+1}^n$  (in addition to others). In other words, starting from  $\mathcal{H}_{k,2}^n$ , the  $r - 1$ -level Sherali-Adams relaxation would be at least as tight as  $\mathcal{H}_{k,r}^n$ . As a consequence, the upper bound of Theorem 1 applies also to the Sherali-Adams relaxation of level  $r - 1$ .

## 5. Concluding remarks

Although the results presented in this paper are specific to the cardinality-constrained Boolean quadric polytope, we believe that they deserve to be adapted and/or generalized to other polytopes.

## References

[1] Yu Hin Au and Levent Tunçel. Stable set polytopes with high lift-and-project ranks for the lovász–schrijver sdp operator. *Mathematical Programming*, 2024.

[2] Egon Balas, Sebastián Ceria, and Gérard Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0–1 programs. *Mathematical Programming*, 58(1):295–324, 1993.

[3] Francisco Barahona, Michael Jünger, and Gerhard Reinelt. Experiments in quadratic 0–1 programming. *Mathematical Programming*, 44(1):127–137, 1989.

[4] Walid Ben-Ameur. Subset selection and the cone of factor-width- $k$  matrices. *SIAM Journal on Optimization*, 34(1):817–843, 2024.

[5] Endre Boros and Peter L. Hammer. Cut-polytopes, boolean quadric polytopes and nonnegative quadratic pseudo-boolean functions. *Mathematics of Operations Research*, 18(1):245–253, 1993.

[6] Caterina De Simone. The cut polytope and the boolean quadric polytope. *Discrete Mathematics*, 79(1):71–75, 1990.

[7] Michel Deza, Komei Fukuda, and Monique Laurent. The inequicut cone. *Discrete Mathematics*, 119(1):21–48, 1993.

[8] Alain Faye and Quoc an Trinh. A polyhedral approach for a constrained quadratic 0–1 problem. *Discrete Applied Mathematics*, 149(1):87–100, 2005. *Boolean and Pseudo-Boolean Functions*.

[9] Laura Galli, Konstantinos Kaparis, and Adam N. Letchford. Gap inequalities for non-convex mixed-integer quadratic programs. *Operations Research Letters*, 39(5):297–300, 2011.

[10] Fred Glover and Eugene Woolsey. Converting the 0-1 polynomial programming problem to a 0-1 linear program. *Operations Research*, 22(1):180–182, 1974.

[11] Ellis L. Johnson, Anuj Mehrotra, and George L. Nemhauser. Min-cut clustering. *Mathematical Programming*, 62(1):133–151, 1993.

[12] Frank Körner. A tight bound for the boolean quadratic optimization problem and its use in a branch and bound algorithm 1. *Optimization*, 19(5):711–721, 1988.

[13] Monique Laurent. A comparison of the sherali-adams, lovász–schrijver and lasserre relaxations for 0-1 programming. *Mathematics of Operations Research*, 28(3):470–496, 2003. *Pagination*: 27.

[14] Monique Laurent and Svatopluk Poljak. Gap inequalities for the cut polytope. *European Journal of Combinatorics*, 17(2):233–254, 1996.

[15] Adam N. Letchford. The boolean quadric polytope. In Abraham P. Punnen, editor, *The Quadratic Unconstrained Binary Optimization Problem: Theory, Algorithms, and Applications*, pages 97–120. Springer International Publishing, 2022.

[16] L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0–1 optimization. *SIAM Journal on Optimization*, 1(2):166–190, 1991.

[17] Garth P. McCormick. Computability of global solutions to factorable nonconvex programs: Part i—convex underestimating

- problems. *Mathematical Programming*, 10(1):147–175, 1976.
- [18] Anuj Mehrotra. Cardinality constrained boolean quadratic polytope. *Discrete Applied Mathematics*, 79(1):137–154, 1997.
- [19] José Neto. On the polyhedral structure of uniform cut polytopes. *Discrete Applied Mathematics*, 175:62–70, 2014.
- [20] Manfred Padberg. The boolean quadric polytope: Some characteristics, facets and relatives. *Mathematical Programming*, 45(1):139–172, 1989.
- [21] Kyungchul Park, Kyungsik Lee, and Sungsoo Park. An extended formulation approach to the edge-weighted maximal clique problem. *European Journal of Operational Research*, 95(3):671–682, 1996.
- [22] Itamar Pitowsky. Correlation polytopes: Their geometry and complexity. *Mathematical Programming*, 50(1):395–414, 1991.
- [23] Hanif D. Sherali and Warren P. Adams. A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. *SIAM Journal on Discrete Mathematics*, 3(3):411–430, 1990.
- [24] Hanif D. Sherali and Warren P. Adams. *A Reformulation-Linearization Technique for Solving Discrete and Continuous Nonconvex Problems*. Springer, 1998.
- [25] Hanif D. Sherali and Barbara M. P. Fraticelli. Enhancing rlt relaxations via a new class of semidefinite cuts. *Journal of Global Optimization*, 22(1):233–261, 2002.