



HAL
open science

COMPLEX WEIGHING MATRICES AND QUATERNARY CODES

Danni Lu, Minjia Shi, Ronan Egan, Patrick Solé

► **To cite this version:**

Danni Lu, Minjia Shi, Ronan Egan, Patrick Solé. COMPLEX WEIGHING MATRICES AND QUATERNARY CODES. Bulletin of the Australian Mathematical Society, In press. hal-04680614

HAL Id: hal-04680614

<https://hal.science/hal-04680614>

Submitted on 28 Aug 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COMPLEX WEIGHING MATRICES AND QUATERNARY CODES

DANNI LU, MINJIA SHI, RONAN EGAN and PATRICK SOLÉ

Abstract

Weighing matrices with entries in the complex cubic and sextic roots of unity are employed to construct Hermitian self-dual codes and Hermitian LCD codes over the finite field $GF(4)$. The parameters of these codes are explored for small matrix orders and weights.

2020 *Mathematics subject classification*: primary 05B20; secondary 94B05.

Keywords and phrases: Weighing matrices, Quaternary codes, Hermitian self-dual codes, LCD codes.

1. Introduction

Weighing matrices (to be defined below) constitute a far reaching generalization of Hadamard matrices [9]. Several databases are available on the internet [14, 15]. In this note, we consider such matrices over the cubic and sextic complex root of unity [3]. Note that the quotient of the Eisenstein integers by the ideal generated by 2 is equal to $GF(4)$. In view of that well-known arithmetic fact, it is natural to construct quaternary codes from such weighing matrices. In this note, we construct Hermitian quaternary self-dual codes by extending two constructions of the binary self-dual codes [1, 10] to quaternary codes. The two families of codes obtained are called $C_{n,k}$ and $C_{n,k}^*$, where n stands for the order of the weighing matrix and k for its weight. In addition to being of interest in their own right, Hermitian quaternary codes that are self-orthogonal are used in the construction of quantum error-correcting codes, see [5]. Further, since 4 is the smallest square prime power greater than 1, Hermitian quaternary self-dual codes have been the first and the most studied amongst Hermitian self-dual codes, see e.g., [12]. The second construction requires Hermitian weighing matrices. Such objects are not classical and a generation technique, of independent interest, is described.

In the same vein, we construct Hermitian Linear Complementary Dual (shortly LCD) codes over $GF(4)$ in the sense of [7]. The two types of codes obtained are called $L_{n,k}$ and $L_{n,k}^*$, where n (resp. k) stands for the order (resp. the weight) of the weighing matrix used. A new method for constructing Hermitian LCD codes is also introduced in [11]. We also believe that the codes generated using the construction methods in

this paper can serve as invariants of such matrices from a classification perspective. This is the spirit of the classification of designs by their codes in the famous book [2]."

The material is arranged as follows. The next section collects the notions and notations needed for the rest of the paper. Section 3 (resp. Section 4) studies Hermitian quaternary self-dual codes (resp Hermitian LCD codes). In Section 5, an algorithm is presented for finding Hermitian matrices in the equivalence class of $CW(n, k, q)$. Section 6 contains the numerical examples of these constructions.

2. Notations and Definitions

2.1. Codes A $[n, k]$ linear code C over $GF(q)$ is a k -dimensional vector subspace of $GF(q)^n$, where $GF(q)$ denotes the Galois field of order q , with q being a power of a prime p . The elements in C are called codewords and the weight $wt(x)$ of a codeword x is the number of its non-zero coordinates. The distance between two codewords x and y is the weight $wt(x - y)$. The minimum weight of a linear code C is defined as $\min\{wt(x) \mid 0 \neq x \in C\}$. A $[n, k, d]$ code is a linear code with minimum weight d . If there exists an n -order monomial matrix P over $GF(q)$ such that $C' = CP = \{cP \mid c \in C\}$, then the codes C and C' over $GF(q)$ are said to be *equivalent*. If $C = CP$ holds, it is referred to as an automorphism of C , and the set of all automorphisms of C forms the automorphism group of C . Let $GF(4) = \{0, 1, \nu, \nu + 1\}$ be a finite field of order 4. The *Hermitian inner product* between codewords $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ is defined as $x \cdot y = \sum_{i=1}^n x_i y_i^2$. The *Hermitian dual code* C^{\perp_H} is defined as $C^{\perp_H} = \{x \in GF(4)^n \mid x \cdot y = 0, \forall y \in C\}$. If $C \subseteq C^{\perp_H}$, then C is *Hermitian self-orthogonal*, and *Hermitian self-dual* if $C = C^{\perp_H}$. The table of Hermitian self-dual codes over $GF(4)$ is provided in [13], where for a given length, the Highest-Bound is given. If a Hermitian self-dual code meets this bound, it is considered *optimal*. A linear code C over $GF(4)$ is usually called a Hermitian LCD code if $C \cap C^{\perp_H} = \{0\}$.

2.2. Combinatorial matrices A complex *weighing matrix* $W \in CW(n, k, q)$ is a matrix of order n and weight k . Its elements are 0 and the q -th roots of unity ζ_q , and it satisfies $WW^* = kI_n$, where $k \leq n$. Here, W^* denotes the conjugate transpose of W and I_n is the $n \times n$ identity matrix. The set $CW(n, n, q)$ corresponds to the set of *Butson Hadamard matrix* $BH(n, q)$. For $q = 3$, the complex weighing matrix $W \in CW(n, k, 3)$ includes elements 0, 1, ζ_3 and ζ_3^2 . For $q = 6$, the complex weighing matrix $W \in CW(n, k, 6)$ includes elements 0, 1, ζ_6 , ζ_6^2 , -1 , $-\zeta_6$ and $-\zeta_6^2$. In this paper, we focus on the codes generated by these two types of matrices over $GF(4)$. It is natural to study complex weighing matrices in the context of constructing codes rather than the full weight Butson Hadamard matrices for two reasons. The first is that when interpreted over $GF(4)$, it is natural to allow for entries equal to 0. But more importantly, non-trivial complex weighing matrices may exist when Butson matrices cannot. This is particularly apparent in the case of $CW(n, k, 3)$ as it is not a requirement that $3 \mid n$ when $k < n$. Additionally, for any n , one can construct at least one $CW(n, k, 6)$ with $2 \leq k \leq n$. See [9] and the references contained therein for further details on complex weighing matrices and their existence.

If there exist two monomial matrices P and Q , where the non-zero entries of P and Q are q -th roots of unity, such that $W' = PWQ^*$, then we say that the two complex weighing matrices W and W' , both of order n and weight k , with non-zero entries that are q -th roots of unity, are *equivalent*, and we write $W \equiv W'$. If $W = PWQ^*$ holds, it is referred to as an automorphism of W . The set of all automorphisms of W forms the automorphism group of W .

3. Hermitian self-dual codes over $GF(4)$

In this section, we present two methods for constructing Hermitian self-dual codes over $GF(4)$ using complex weighing matrices.

THEOREM 3.1. *Let $W \in CW(n, k, q)$ be a complex weighing matrix satisfying $k \equiv 1 \pmod{2}$. If α is a nonzero element in $GF(4)$, then the matrix $G = \begin{bmatrix} \alpha I_n & W \end{bmatrix}$ generates a Hermitian self-dual $[2n, n]$ code $C_{n,k}$ over $GF(4)$. The matrix $G' = \begin{bmatrix} \alpha I_n & W' \end{bmatrix}$ also generates a Hermitian self-dual $[2n, n]$ code over $GF(4)$, where W' is equivalent to W .*

PROOF. Consider the equation $GG^* = \alpha\bar{\alpha}I_n + WW^* = I_n + kI_n = (k+1)I_n = 0$, where $\bar{\alpha}$ denotes the complex conjugate of α . If W' is equivalent to W , then $W'W'^* = kI_n$, which implies $G'G'^* = 0$. Therefore, G' also generates a Hermitian self-dual code, as required. \square

THEOREM 3.2. *Let $W \in CW(n, k, q)$ be a complex weighing matrix that satisfies $k \equiv 0 \pmod{2}$ and $W = W^*$. If α is a nonzero element in $GF(4)$, then the matrix $G = \begin{bmatrix} \alpha I_n & I_n + W \end{bmatrix}$ generates a Hermitian self-dual $[2n, n]$ code $C_{n,k}^*$ over $GF(4)$. The matrix $G' = \begin{bmatrix} \alpha I_n & I_n + W' \end{bmatrix}$ also generates a Hermitian self-dual $[2n, n]$ code over $GF(4)$, where $W' = W'^*$ and W' is equivalent to W .*

PROOF. The product GG^* yields $\alpha\bar{\alpha}I_n + I_n + W^* + W + WW^*$. Since W is a complex weighing matrix, it satisfies $WW^* = kI_n$. Considering $k \equiv 0 \pmod{2}$ and $W = W^*$, this simplifies to $GG^* = 0$. Similarly, when $W' = W'^*$ and W' is equivalent to W , the same conclusion can be drawn. \square

PROPOSITION 3.3. *If W and W' are equivalent complex weighing matrices of order n and weight k , then the Hermitian self-dual codes constructed from W and W' by Theorem 3.1 are also equivalent.*

PROOF. Since W and W' are equivalent, there exist monomial matrices P and Q such that $W' = PWQ^*$. Therefore,

$$\begin{bmatrix} \alpha I_n & W' \end{bmatrix} = \begin{bmatrix} \alpha I_n & PWQ^* \end{bmatrix} = P \begin{bmatrix} \alpha I_n & W \end{bmatrix} \begin{bmatrix} P^{-1} & \mathbf{0} \\ \mathbf{0} & Q^* \end{bmatrix},$$

where the matrix $\begin{bmatrix} P^{-1} & \mathbf{0} \\ \mathbf{0} & Q^* \end{bmatrix}$ is a $2n \times 2n$ monomial matrix and $\mathbf{0}$ denotes the $n \times n$ zero matrix. This completes the proof. \square

4. Hermitian LCD codes over $GF(4)$

In this section, two methods are presented for constructing Hermitian LCD codes over $GF(4)$ using complex weighing matrices.

PROPOSITION 4.1 ([6], Proposition 2). *If G is a generator matrix for the $[n, k]$ linear code C , then the $k \times k$ matrix GG^* is nonsingular if and only if C is a Hermitian LCD code.*

THEOREM 4.2. *Let $W \in CW(n, k, q)$ be a complex weighing matrix with k being even. If α is a nonzero element in $GF(4)$, then the matrix $G = \begin{bmatrix} \alpha I_n & W \end{bmatrix}$ generates a Hermitian LCD code $L_{n,k}$ of length $2n$ over $GF(4)$.*

PROOF. From the proof of Theorem 3.1, it is known that $GG^* = (k+1)I_n$, hence $\det(GG^*) = (k+1)^n$. Then the claim follows from Proposition 4.1. \square

THEOREM 4.3. *Let $W \in CW(n, k, q)$ be a complex weighing matrix with k being odd and $W = W^*$. If α is a nonzero element in $GF(4)$, then the matrix $G = \begin{bmatrix} \alpha I_n & I_n + W \end{bmatrix}$ generates a Hermitian LCD code $L_{n,k}^*$ of length $2n$ over $GF(4)$.*

PROOF. From the proof of Theorem 3.2, it follows that $GG^* = (k+2)I_n$, hence $\det(GG^*) = (k+2)^n$. Then the result is obtained from Proposition 4.1. \square

REMARK 4.4. *In Theorem 4.2, if W' is equivalent to W , then the same construction will also generate a Hermitian LCD code of length $2n$ over $GF(4)$. Furthermore, the Hermitian LCD code generated by W' is equivalent to the Hermitian LCD code generated by W . Similarly, in Theorem 4.3, if W' is an equivalent Hermitian matrix to W , then same construction method will generate a Hermitian LCD code over $GF(4)$.*

5. Finding Hermitian matrices in the equivalence class of a $CW(n, k, q)$

Let $\text{Mon}_n(q)$ be the group of $n \times n$ monomial matrices with non-zero entries in the q -th roots of unity. The group $\text{Mon}_n(q)^2$ acts on $CW(n, k, q)$ via

$$W(M, N) = MWN^*.$$

The orbits under this action are the equivalence classes. Restricting to the action of the group $\text{Mon}_n(1)^2$, the orbits are permutation equivalence classes. The stabilisers of a matrix W under these actions are the automorphism and permutation automorphism groups respectively. Our next goal is to describe an algorithm for searching through the equivalence class of a given $CW(n, k, q)$ for Hermitian members. That is, given $W \in CW(n, k, q)$, we search for a matrix $H \equiv W$ such that $H = H^*$.

For any Hermitian matrix H , it must be true that $H_{ij} = 0$ if and only if $H_{ji} = 0$. Given any $W \in CW(n, k, q)$, let W_c denote the matrix obtained from W by letting the $[i, j]$ entry be 1 if $W_{ij} = c$, and 0 otherwise. Adhering to this notation,

$$W = \sum_{c \in \{0\} \cup \langle \zeta_q \rangle} cW_c.$$

In particular, the matrix W_0 is a $(0, 1)$ -matrix of weight $n - k$, and if W is Hermitian, then W_0 is symmetric. The following proposition is immediate.

PROPOSITION 5.1. *If there exists a Hermitian matrix in the equivalence class of W , then the matrix W_0 is permutation equivalent to a symmetric matrix.*

We now make another simple observation. Suppose that $H \in CW(n, k, q)$ is Hermitian. For any $M \in \text{Mon}_n(q)$, we observe that

$$(MHM^*)^* = MH^*M^* = MHM^*. \quad (5.1)$$

Hence MHM^* is also Hermitian for any choice of M . Suppose now that W is any matrix in $CW(n, k, q)$, not necessarily Hermitian, that is equivalent to H . Then there exist matrices $S, T \in \text{Mon}_n(q)$ such that $SWT^* = H$. It follows that $T^*SW = T^*HT$, which is Hermitian by (5.1). The next proposition follows immediately.

PROPOSITION 5.2. *Let $W \in CW(n, k, q)$ and suppose that there is a Hermitian matrix H in the equivalence class of W . Then there is a monomial matrix $M \in \text{Mon}_n(q)$ such that MW is Hermitian.*

It follows that if there is a Hermitian $H \equiv W$, we need only search for M such that MW is Hermitian. Now, we may write any matrix $M \in \text{Mon}_n(q)$ uniquely in the form $M = DP$, where D is diagonal and P is a permutation matrix. If MW is Hermitian, then it follows that PW_0 is symmetric.

Suppose now that a matrix $W \in CW(n, k, q)$ is given. We first want to determine whether or not there exists a Hermitian matrix in the equivalence class of W , and find them in the case that they exist. Proposition 5.2 allows us to consider only the orbit of W under the action of $\text{Mon}_n(q)$ via left multiplication. A matrix in $CW(n, k, q)$ is *normalized* if the first non-zero entry in every row and column is 1. Any matrix is diagonally equivalent to a normalized matrix.

PROPOSITION 5.3. *Let $H \in CW(n, k, q)$ be Hermitian. Then there exists a diagonal matrix D such that DHD^* is normalized and Hermitian.*

PROOF. Let D_j be the diagonal matrix with $\overline{H_{ij}}$ in the i^{th} position of the diagonal, if H_{ij} is the first non-zero entry in row j , and 1 elsewhere on the diagonal. Then letting $D = D_n D_{n-1} \cdots D_2 D_1$, the matrix DHD^* is normalized. The matrix DHD^* is Hermitian by Equation (5.1). \square

It follows then from Proposition 5.3 that if $W \in CW(n, k, q)$ is diagonally equivalent to a Hermitian matrix, then there exist diagonal matrices D and E such that DWE^* is both normalized and Hermitian.

Combining the details of the section up to now, we describe a simple computational algorithm for finding a Hermitian matrix, if it exists, in the equivalence class of a given matrix $W \in CW(n, k, q)$.

- (1) Given W , construct W_0 .
- (2) Find a single permutation matrix $Q \in \text{Mon}_n(1)$ such that QW_0 is symmetric.

- (3) Find all permutation matrices $P \in \text{Mon}_n(1)$ such that PW_0 is symmetric.
- (4) For each P found in the previous step, find pairs of diagonal matrices D and E so that the matrix $H = D(PW)E^*$ is normalized. If H is Hermitian, then exit the algorithm.

Steps (1) and (4) of the algorithm above are straightforward, requiring no significant computational effort. Step (2) is computationally difficult, and represents the most time consuming aspect of the algorithm. Fortunately, searching through all of $\text{Mon}_n(1)$ for the matrix Q is not necessary. This is because under any assumption that row i of W_0 is permuted to row j immediately restricts the search space, as the remaining rows must be permuted so as to preserve symmetry. For example, if the first row of W_0 is fixed, then the 1s on the first column must be preserved. Any subsequent assumptions have a similar effect. For the values of n considered in this work, very few assumptions are required before complete searches through remaining search spaces are computationally easy, and running through all possible assumptions is feasible.

Assuming that Step (2) is complete, Step (3) can be implemented rather efficiently as follows. Suppose we have found a single permutation matrix Q such that $QW_0 = X$ is symmetric. Now consider the equivalent problem of finding all P such that PX is symmetric. In this case, the symmetry of PX implies that

$$PX = (PX)^T = X^T P^T = XP^T.$$

It follows that $PXP = X$, and so (P, P^T) is a permutation automorphism of X . Since X is a symmetric $(0, 1)$ -matrix of weight $n - k$, it is an incidence structure. Finding the matrices P such that $PXP = X$ is an incidence structure automorphism problem, for which there are efficient algorithms available in MAGMA [4] that can be taken advantage of. This speeds up the process of finding all of the permutation matrices P such that PW_0 is symmetric significantly.

EXAMPLE 5.4. The matrix

$$W = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \zeta_3 & \zeta_3^2 \\ 1 & 1 & 0 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 & 0 & 1 \\ 1 & \zeta_3^2 & \zeta_3 & 1 & 0 \end{bmatrix}$$

is a $CW(5, 4, 3)$ that is not Hermitian (but it is symmetric). Note that $W_0 = I_5$, which is already symmetric. However, W is already normalized and is not Hermitian, so we must proceed with Step (2) of the algorithm, and find another permutation matrix P such that PW_0 is symmetric. Any symmetric P is a candidate in this case. Let P be the permutation matrix that swaps the first two rows. Clearly, the matrix PW is no longer normalized. Letting $E = \text{diag}(1, 1, 1, \zeta_3^2, \zeta_3)$, the matrix $H = PWE$ is normalized. This

matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & \zeta_3^2 & \zeta_3 \\ 1 & 1 & 0 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3 & \zeta_3^2 & 0 & \zeta_3 \\ 1 & \zeta_3^2 & \zeta_3 & \zeta_3^2 & 0 \end{bmatrix},$$

which is Hermitian. We can now exit the algorithm.

6. Numerical examples

Using the construction methods provided in Section 3, Table 1 presents some optimal Hermitian self-dual codes. Here, n, k and q are the three parameters of the complex weighing matrices, and the parameters of the codes are $[2n, n, d]$.

DEFINITION 6.1. The *direct sum* of an n -order square matrix and an m -order square matrix is defined as

$$A \oplus B = \begin{bmatrix} A & 0_{n \times m} \\ 0_{m \times n} & B \end{bmatrix}.$$

TABLE 1. Optimal Hermitian self-dual codes

n	k	q	d	Sources or constructions of the matrices
3	3	3	4	[14]
5	4	3	4	[3]
6	3	3	4	$CW(3, 3, 3) \oplus CW(3, 3, 3)$
6	5	6	4	[3]
6	4	6	4	[3]
8	7	3	6	[9]
8	7	6	6	[8]
12	6	3	8	$CW(6, 6, 3) \oplus CW(6, 6, 3)$

REMARK 6.2. Among the codes presented in Table 1, the three codes with parameters $[12, 6, 4]$ are non-equivalent, as well as the two codes with parameters $[16, 8, 6]$.

REMARK 6.3. The three matrices $W_1 \in CW(5, 4, 3)$, $W_2 \in CW(6, 4, 6)$, and $W_3 \in CW(12, 6, 3)$ in Table 1 are derived as equivalent Hermitian matrices using the algorithm in Section 5 and are subsequently used to construct Hermitian self-dual codes via Theorem 3.2. The Hermitian matrix for W_1 is provided in Example 5.4, while the Hermitian matrices W'_2 for W_2 and W'_3 for W_3 will be presented in the appendix. The remaining matrices can be directly used to construct Hermitian self-dual codes via Theorem 3.1.

A. Appendix

$$W'_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \zeta_3^2 & \zeta_3 \\ 0 & 0 & 1 & 1 & \zeta_3 & \zeta_3^2 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & \zeta_3 & \zeta_3^2 & 0 & -1 & 0 \\ 1 & \zeta_3^2 & \zeta_3 & 0 & 0 & -1 \end{bmatrix}$$

$$W'_3 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & \zeta_3 & 0 & 1 & 0 & \zeta_3^2 \\ 1 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & 1 & 0 \\ 0 & 1 & 0 & \zeta_3 & 0 & 1 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & \zeta_3^2 \\ 1 & 0 & \zeta_3^2 & 0 & 1 & 0 & \zeta_3 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 \\ 0 & 1 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & 1 \\ 1 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & 1 & 0 & \zeta_3 & 0 \\ 0 & 1 & 0 & \zeta_3^2 & 0 & \zeta_3^2 & 0 & 1 & 0 & \zeta_3 & 0 & \zeta_3 \\ 1 & 0 & 1 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & \zeta_3 & 0 \\ 0 & 1 & 0 & 1 & 0 & \zeta_3 & 0 & \zeta_3^2 & 0 & \zeta_3^2 & 0 & \zeta_3 \\ 1 & 0 & \zeta_3 & 0 & \zeta_3 & 0 & 1 & 0 & \zeta_3^2 & 0 & \zeta_3^2 & 0 \end{bmatrix}$$

References

- [1] J. A. Armario and M. D. Frau, ‘Self-dual codes from $(-1, 1)$ -matrices of skew symmetric type’, arXiv: 1311.2637.
- [2] E. F. Assmus and J. D. Key, ‘*Designs and their codes*’, Cambridge University Press, Cambridge, 1992.
- [3] D. Best, H. Kharaghani and H. Ramp, ‘On unit weighing matrices with small weight’, *Discrete Mathematics*, 2013, **313**(7): 855–864.
- [4] W. Bosma, J. Cannon and C. Playoust, ‘The Magma algebra system. I. The user language’, *Journal of Symbolic Computation*, 1997, **24**: 235–265.
- [5] R. A. Calderbank, E. M. Rains, P. W. Shor and N. J. Sloane, ‘Quantum error correction via codes over $\text{GF}(4)$ ’, *Institute of Electrical and Electronics Engineers Transactions on Information Theory*, 1998, **44**(4): 1369–1387.
- [6] C. Carlet, S. Mesnager, C. Tang and Y. Qi, ‘Euclidean and Hermitian LCD MDS codes’, *Designs, Codes and Cryptography*, 2018, **86**(11): 2605–2618.
- [7] D. Crnković, R. Egan, B. G. Rodrigues and A. Švob, ‘LCD codes from weighing matrices’, *Applicable Algebra in Engineering, Communication and Computing*, 2021, **32**(2): 175–189.
- [8] R. Egan, ‘Generalizing pairs of complementary sequences and a construction of combinatorial structures’, *Discrete Mathematics*, 2020, **343**(5): 111795.
- [9] R. Egan, ‘A survey of complex generalized weighing matrices and a construction of quantum error-correcting codes’, arXiv: 2309.07522.
- [10] M. Harada, ‘New extremal ternary self-dual codes’, *The Australasian Journal of Combinatorics*, 1998, **17**: 133–146.
- [11] S. Li, M. Shi and H. Liu, ‘Several constructions of optimal LCD codes over small finite fields’, *Cryptography and Communications*, 2024, **16**: 779–800.

- [12] F. J. MacWilliams, A. M. Odlyzko, N. J. A. Sloane and H. N. Ward, 'Self-dual codes over $GF(4)$ ', *Journal of Combinatorial Theory Series A*, 1978, **25**: 288–318.
- [13] https://www.unilim.fr/pages_perso/philippe.gaborit/SD/GF4H/GF4H.htm
- [14] <https://chaos.if.uj.edu.pl/~karol/hadamard/index.html>
- [15] <http://www.math.is.tohoku.ac.jp/~munemasa/research/matrices/weighingmatrices.htm>

Danni Lu, Key Laboratory of Intelligent Computing and Signal Processing, Ministry of Education, School of Mathematical Sciences, Anhui University, Hefei 230601, China. ludanni_in@163.com

Minjia Shi, Key Laboratory of Intelligent Computing and Signal Processing, Ministry of Education, School of Mathematical Sciences, Anhui University, Hefei 230601, China. smjwcl.good@163.com

Ronan Egan, School of Mathematical Sciences, Dublin City University, Ireland. ronan.egan@dcu.ie

Patrick Solé, I2M (Aix Marseille Univ, CNRS, Centrale Marseille), Marseilles, France. sole@enst.fr