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Bridging the Hybrid High-Order and Hybridizable Discontinuous Galerkin Methods: Summary

Bernardo Cockburn, Daniele A. Di Pietro, Alexandre Ern

Abstract

This paper summarizes results originally published in [16] establishing a link between Hybrid High-Order and Hybridizable Discontinuous Galerkin methods. It also briefly discusses past and ongoing further developments.

1 Introduction

We summarize the main results of the awarded paper [16] and discuss its impact on subsequent, ongoing developments.

The main achievement of the paper in question was establishing a link between two extremely successful paradigms for the discretization of partial differential equations: the Hybridizable Discontinuous Galerkin (HDG) method, originally introduced in [20], and the Hybrid High-Order (HHO) method, originally introduced in [29, 26]. This made it possible to share developments made in the context of both methods, such as the high-order stabilization characteristic of HHO or the $H(\text{div})$ -conforming reconstructions originally developed in the context of HDG. From a broader perspective, the results of [16] fueled several new lines of research, including , among others: the development of compatible discretization methods on general polytopal meshes, as in the Discrete de Rham paradigm; the study of the intimate relations between spaces and stabilizations in hybrid methods, as in M-decompositions; the development of superconvergent post-processings of the approximate solution; the development of unfitted hybrid methods; and the study of hybrid methods for wave propagation. The results of [16] also provided another way, complementing the arguments originally given in [12], to highlight the close links between HDG, HHO, and Weak Galerkin (WG) methods.

Let $d \in \{2, 3\}$ and denote by $\Omega \subset \mathbb{R}^d$ an open, bounded, connected, polytopal (i.e., polygonal or polyhedral) domain with Lipschitz boundary $\partial\Omega$. Our focus is on the numerical approximation of the following pure diffusion PDE: Given $f : \Omega \rightarrow \mathbb{R}$, find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{aligned} -\nabla \cdot (\kappa \nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\kappa : \Omega \rightarrow \mathbb{R}^{d \times d}$ denotes a symmetric and uniformly elliptic diffusion coefficient. For the sake of simplicity, we will assume that κ is piecewise constant on a fixed polytopal partition of the domain.

The rest of the paper is organized as follows. In Section 2 we briefly introduce the discrete setting. Sections 3 and 4 respectively introduce the HHO and HDG methods. The main results of the awarded paper [16] are summarized in Section 5. Further developments of the original results of this paper are briefly discussed in Section 6

2 Setting

Denote by $(\mathcal{T}_h, \mathcal{F}_h)$ a polytopal mesh of Ω in the sense of [21, Chapter 1], with \mathcal{T}_h denoting the set of polytopal elements and \mathcal{F}_h the set of planar faces (see also [10, Section 2.1] and [32] concerning the extension to meshes with asymptotically small faces). Given a mesh element $T \in \mathcal{T}_h$, we denote by $\mathcal{F}_T \subset \mathcal{F}_h$ the set of faces that lie on its boundary and, for all $F \in \mathcal{F}_T$, n_{TF} is the unit normal vector to F pointing out of T .

For any integer $\ell \geq 0$ and any mesh element or face $Y \in \mathcal{T}_h \cup \mathcal{F}_h$, we denote by $\mathcal{P}^\ell(Y)$ the set spanned by the restriction to Y of polynomials in the space variables. The L^2 -orthogonal projector on $\mathcal{P}^\ell(Y)$ will be denoted by π_Y^ℓ . Given a mesh element $T \in \mathcal{T}_h$, we will also need the space $\mathcal{P}^\ell(\mathcal{F}_T)$ spanned by (possibly discontinuous) functions whose restriction to each $F \in \mathcal{F}_T$ is in $\mathcal{P}^\ell(F)$.

We will assume that the mesh is compatible with κ , meaning that jumps of this coefficient can only occur at interfaces. The restriction of κ to $T \in \mathcal{T}_h$ is henceforth denoted by $\kappa_T \in \mathcal{P}^0(T)^{d \times d}$ and, for all $F \in \mathcal{F}_T$, we let $\kappa_{TF} := (\kappa_T n_{TF}) \cdot n_{TF}$.

To avoid the proliferation of constants, we abbreviate whenever possible by $a \lesssim b$ the inequality $a \leq Cb$ with real number $C > 0$ independent of the meshsize, the diffusion coefficient κ , and, for local inequalities, on the mesh element or face.

3 The Hybrid High-Order method

The HHO method hinges on spaces spanned by fully discontinuous polynomial functions on mesh elements and faces and local reconstructions obtained by solving local problems inside each element. In this section we briefly recall the classical HHO discretization of problem (1.1) along with key results from the analysis.

3.1 HHO spaces and component energy norm

Given an integer $k \geq 0$, the HHO space is

$$\begin{aligned} \underline{U}_h^k := \left\{ \underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h}) : \right. \\ \left. v_T \in \mathcal{P}^k(T) \text{ for all } T \in \mathcal{T}_h \text{ and } v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_h \right\}. \end{aligned}$$

The meaning of the polynomial components in \underline{U}_h^k is provided by the interpolator $\underline{I}_h^k : H^1(\Omega) \rightarrow \underline{U}_h^k$ such that

$$\underline{I}_h^k v := ((\pi_T^k(v))_{T \in \mathcal{T}_h}, (\pi_F^k(v))_{F \in \mathcal{F}_h}) \quad \forall v \in H^1(\Omega).$$

Given a mesh element $T \in \mathcal{T}_h$, we respectively denote by $\underline{U}_T^k, \underline{v}_T \in \underline{U}_T^k$, and $\underline{I}_T^k : H^1(T) \rightarrow \underline{U}_T^k$ the restrictions to T of $\underline{U}_h^k, \underline{v}_h \in \underline{U}_h^k$, and \underline{I}_h^k , obtained by collecting the components on T and its faces.

We equip \underline{U}_h^k with the energy component seminorm such that, for all $\underline{v}_h \in \underline{U}_h^k$,

$$\|\underline{v}_h\|_{1,\kappa,h} := \left(\sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{1,\kappa,T}^2 \right)^{\frac{1}{2}},$$

$$\|\underline{v}_T\|_{1,\kappa,T}^2 := \|\kappa \nabla v_T\|_{L^2(T)^d}^2 + \sum_{F \in \mathcal{F}_T} \frac{\kappa_{TF}}{h_T} \|v_F - v_T\|_{L^2(F)}^2.$$

3.2 Local reconstruction

Let $T \in \mathcal{T}_h$. The local reconstruction $p_T^{k+1} : \underline{U}_T^k \rightarrow \mathcal{P}^{k+1}(T)$ associates to an element $\underline{v}_T \in \underline{U}_T^k$ the unique function $p_T^{k+1}(\underline{v}_T) \in \mathcal{P}^{k+1}(T)$ such that, for all $w \in \mathcal{P}^{k+1}(T)$,

$$\int_T \kappa_T \nabla p_T^{k+1}(\underline{v}_T) \cdot \nabla w = - \int_T v_T \nabla \cdot (\kappa_T \nabla w) + \sum_{F \in \mathcal{F}_T} \int_F v_F (\kappa_T \nabla w \cdot n_{TF}) \quad (3.1a)$$

and

$$\int_T p_T^{k+1}(\underline{v}_T) = \int_T v_T. \quad (3.1b)$$

It can be proved that $p_T^{k+1} \circ \underline{I}_T^k : H^1(T) \rightarrow \mathcal{P}^{k+1}(T)$ is a bounded projector with optimal approximation properties; see [29, Lemma 3] and [21, Chapter 3].

3.3 HHO scheme and convergence

Denote by $\underline{U}_{h,0}^k$ the subspace of \underline{U}_h^k that has vanishing polynomial components on boundary faces. The HHO discretization of problem (1.1) hinges on the bilinear form $a_h : \underline{U}_{h,0}^k \times \underline{U}_{h,0}^k \rightarrow \mathbb{R}$ such that, for all $(\underline{w}_h, \underline{v}_h) \in \underline{U}_{h,0}^k \times \underline{U}_{h,0}^k$,

$$a_h(\underline{w}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{w}_T, \underline{v}_T),$$

$$a_T(\underline{w}_T, \underline{v}_T) := \int_T \kappa_T \nabla p_T^{k+1}(\underline{w}_T) \cdot \nabla p_T^{k+1}(\underline{v}_T) + s_T(\underline{w}_T, \underline{v}_T).$$

Above, for all $T \in \mathcal{T}_h$, $s_T : \underline{U}_T^k \times \underline{U}_T^k \rightarrow \mathbb{R}$ is a symmetric, positive semi-definite stabilization bilinear form such that

- There is a real number $\eta > 0$ independent of h , T , and κ , such that

$$(\alpha_T \eta)^{-1} \|\underline{v}_T\|_{1,\kappa,T}^2 \lesssim \|\underline{v}_T\|_{a,T}^2 \lesssim (\alpha_T \eta) \|\underline{v}_T\|_{1,\kappa,T}^2,$$

with $\|\underline{v}_T\|_{a,T}^2 := a_T(\underline{v}_T, \underline{v}_T)$ and α_T denoting the ratio between the largest and smallest eigenvalues of κ_T ;

- The following holds:

$$s_T(\underline{I}_T^k(w), \underline{v}_T) = 0 \quad \forall (w, \underline{v}_T) \in \mathcal{P}^{k+1}(T) \times \underline{U}_T^k. \quad (3.2)$$

It can be proved that the above conditions enforce a specific dependence of s_T on its arguments [21, Lemma 2.11], so that this bilinear form is in fact a least-squares penalty on the components of $\underline{v}_T - \underline{I}_T^k(p_T^{k+1}(\underline{v}_T)) =: (\delta_T^k \underline{v}_T, (\delta_{TF}^k \underline{v}_T)_{F \in \mathcal{F}_T})$. A classical example is given by

$$s_T(\underline{w}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} \frac{\kappa_{TF}}{h_T} \int_F (\delta_{TF}^k \underline{w}_T - \delta_T^k \underline{w}_T)(\delta_{TF}^k \underline{v}_T - \delta_T^k \underline{v}_T). \quad (3.3)$$

Notice that $\delta_{TF}^k \underline{v}_T - (\delta_T^k \underline{v}_T)|_F = \pi_F^k(v_F - v_{T|F} - ((I - \pi_T^k)p_T^{k+1}(\underline{v}_T))|_F)$, as originally written in [29].

The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ such that

$$a_h(\underline{u}_h, \underline{v}_h) = \sum_{T \in \mathcal{T}_h} \int_T f v_T \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k. \quad (3.4)$$

Theorem 3.1 (Error estimate for the HHO scheme) *Denote by $u \in H_0^1(\Omega)$ the weak solution to (1.1), and assume that $u|_T \in H^{k+2}(T)$ for all $T \in \mathcal{T}_h$. Then, denoting by $\bar{\kappa}_T$ the largest eigenvalue of κ_T ,*

$$\|\underline{u}_h - \underline{I}_h^k u\|_{a,h}^2 \lesssim \sum_{T \in \mathcal{T}_h} \bar{\kappa}_T \alpha_T h_T^{2(k+1)} |u|_{H^{k+2}(T)}^2,$$

with $\|\underline{v}_h\|_{a,h}^2 := a_h(\underline{v}_h, \underline{v}_h)$.

Remark 3.2 (Superconvergence of the potential reconstruction) *If full elliptic regularity holds (κ is constant over Ω and the latter is convex), it can be proved that the broken polynomial function equal to $p_T^{k+1}(\underline{u}_T)$ in each $T \in \mathcal{T}_h$ converges to u in h^{k+2} in the L^2 -norm.*

Remark 3.3 (Variations) *In the original paper [16], we considered variations of the HHO methods with depleted or enriched element components in $\mathcal{P}^l(T)$ with $l \in \{k-1, k, k+1\}$. For the sake of simplicity we have only presented the equal-order case $l = k$ here. Over the years, many other variations of the HHO methods have been studied; we refer, e.g., to [28, 1, 23, 9], and [21, Chapter 5] and [10, Chapter 3] for a broader discussion on this subject.*

4 Hybridizable Discontinuous Galerkin methods

The HDG approximation of problem (1.1) is formulated as the three field problem: Find $(\mathbf{q}_h, u_h, \hat{u}_h) \in \mathbf{V}_h \times W_h \times M_{h,0}$, approximation to $(-\kappa \nabla u|_\Omega, u|_\Omega, u_{\partial \mathcal{T}_h})$, such that

$$\begin{aligned} (\kappa^{-1} \mathbf{q}_h, \mathbf{v})_T - (u_h, \nabla \cdot \mathbf{v})_T + (\hat{u}_h, \mathbf{v} \cdot \mathbf{n}_T)_{\partial T} &= 0 & \forall \mathbf{v} \in \mathbf{V}(T), \\ -(\mathbf{q}_h, \nabla w)_T + (\hat{\mathbf{q}}_h \cdot \mathbf{n}_T, w)_{\partial T} &= (f, w)_T & \forall w \in W(T), \\ \hat{\mathbf{q}}_h \cdot \mathbf{n}_T &:= \mathbf{q}_h \cdot \mathbf{n}_T + \alpha(u_h - \hat{u}_h) & \text{on } \partial T, \\ \sum_{T \in \mathcal{T}_h} (\hat{\mathbf{q}}_h \cdot \mathbf{n}_T, \hat{w})_{\partial T} &= 0 & \forall \hat{w} \in M_{h,0}, \end{aligned}$$

where $\mathbf{V}_h := \prod_{T \in \mathcal{T}_h} \mathbf{V}(T)$, $W_h \times M_h := \left\{ \prod_{T \in \mathcal{T}_h} W(T) \right\} \times \left\{ \prod_{F \in \mathcal{F}_h} M(F) \right\}$, and $M_{h,0} := \{ \mu \in M_h : \mu|_{\partial \Omega} = 0 \}$. An approximation to the normal component of the flux on the boundaries of the element T , $\mathbf{n}_T \cdot \kappa \nabla u|_{\partial T}$, is provided by $\hat{\mathbf{q}}_h \cdot \mathbf{n}_T$.

The different HDG methods are obtained by selecting

- The *local spaces*: $\mathbf{V}(T)$ for the approximation to the flux $-\kappa \nabla u|_T$, $W(T)$ for the approximation to the potential $u|_T$, $M(\partial T)$ for the approximation of the traces of the potential $u|_{\partial T}$;
- The linear *stabilization function* α .

As we see next, the main result of the paper we are summarizing identifies the local spaces and the stabilization for which the above HDG general formulation gives the HHO method. As a byproduct, it identifies the numerical flux of the HHO method using a different approach with respect to [27].

5 Bridging the methods

Our main result is the following reformulation of the HHO method (3.4). It is the key for establishing a link with HDG methods.

Theorem 5.1 (Numerical trace formulation of the HHO method) *Let $\underline{u}_h \in \underline{U}_{h,0}^k$ and, for all $T \in \mathcal{T}_h$, let $u_{\partial T} : \partial T \rightarrow \mathbb{R}$ be such that $(u_{\partial T})|_F := u_F$ for all $F \in \mathcal{F}_T$. Define*

$$\Phi_T(\underline{u}_T) := -\kappa_T \nabla p_T^{k+1} \underline{u}_T \cdot \mathbf{n}_{TF} + \tilde{r}_{\partial T}^k \tau_{\partial T}^k r_{\partial T}^k (u_T - u_{\partial T}).$$

Above, $r_{\partial T}^k : \mathcal{P}^k(\mathcal{F}_T) \rightarrow \mathcal{P}^k(\mathcal{F}_T)$ is such that, for all $\lambda \in \mathcal{P}^k(\mathcal{F}_T)$, setting $\underline{v}_{\lambda,T} := (0, (\lambda|_F)_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$,

$$(r_{\partial T}^k \lambda)|_F := \pi_F^k (\lambda - p_T^{k+1}(\underline{v}_{\lambda,T}) + \pi_T^k p_T^{k+1}(\underline{v}_{\lambda,T}))|_F,$$

while $\tilde{r}_{\partial T}^k$ is its adjoint with respect to the L^2 -product on ∂T and $\tau_{\partial T} : \partial T \rightarrow \mathbb{R}$ is such that $(\tau_{\partial T})|_F := \frac{\kappa_{TF}}{h_T}$ for all $F \in \mathcal{F}_T$. Then, $\underline{u}_h \in \underline{U}_{h,0}^k$ solves (3.4) with local stabilization bilinear forms given by (3.3) if and only if it satisfies, for all $T \in \mathcal{T}_h$,

$$\int_T \kappa_T \nabla p_T^{k+1}(\underline{u}_T) \cdot \nabla v_T + \int_{\partial T} \Phi_T(\underline{u}_T) v_T = \int_T f v_T \quad \forall v_T \in \mathcal{P}^k(T),$$

where p_T^{k+1} is defined by (3.1), and, for every interface $F \in \mathcal{F}_h$ such that $F \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$ with $T_1 \neq T_2$ elements of \mathcal{T}_h , we have

$$(\Phi_{T_1}(\underline{u}_{T_1}))|_F + (\Phi_{T_2}(\underline{u}_{T_2}))|_F = 0.$$

It is now not difficult to see that the numerical trace formulation of the HHO fits into the general form of the HDG method when we take the local spaces as

$$\mathbf{V}(T) := \kappa \nabla \mathcal{P}^{k+1}(T), \quad W(T) := \mathcal{P}^k(T), \quad M(F) := \mathcal{P}^k(F),$$

and the stabilization function as

$$\alpha(\mu)|_{\partial T} := \tilde{r}_{\partial T}^k(\tau_{\partial T} r_{\partial T}^k(\mu)) \quad \forall \mu \in \mathcal{P}^k(\mathcal{F}_T), \quad \forall T \in \mathcal{T}_h.$$

Moreover, with the HDG notation, we have that

$$\widehat{\mathbf{q}}_h \cdot \mathbf{n}_{TF} := \Phi_T(\underline{u}_T), \quad \mathbf{q}_h|_T := -\kappa_T \nabla p_T^{k+1}(\underline{u}_T) \quad \text{and} \quad (u_h, \widehat{u}_h) := \underline{u}_T.$$

Now that we have shown that HHO fits into the HDG framework, we can compare it with the previously defined HDG methods. Let us begin by noting that the HHO can achieve the optimal order of convergence of $k+2$ for the potential and of $k+1$ for the flux for meshes made of general polytopes. All other known HDG methods, with the same M_h space, notably, the LDG-H method [8], namely the HDG method for which

$$\mathbf{V}(T) := [\mathcal{P}^k(T)]^d, \quad W(T) := \mathcal{P}^k(T), \quad M(F) := \mathcal{P}^k(F),$$

and

$$\alpha(\mu)|_{\partial T} := \eta_{\partial T} \mu \quad \forall \mu \in L^2(\partial T) \quad \forall T \in \mathcal{T}_h,$$

can only achieve (with $\eta_{\partial T}$ independent of h) the order $k+1$ for the potential and $k+\frac{1}{2}$ for the flux, even though the local space for the flux is significantly larger than for the HHO method. In contrast with the HHO method, where the function α does not act pointwise but couples point values all over ∂T , the LDG-H stabilization function seems to be, roughly speaking, *too damping*. Thus, this result allows the incorporation of the special stabilization function of the HHO method into other HDG methods. A couple of new HDG methods were thus obtained in [16]. On the other hand, we have been able to identify the numerical trace of the HHO method, which means that divergence-conforming post-processings used for the HDG methods [13] can now be used for the HHO method applied to standard meshes.

6 Further developments

The bridging of the methods in this work has had a sizeable impact on modern hybrid and nonconforming approximation methods for partial differential equations.

Hybrid High-Order methods have inherited and expanded ideas from Discontinuous Galerkin (DG) [3, 25] and modern finite volume [33] methods, and are

nowadays considered a landmark in the context of polytopal methods (i.e., methods that support much more general meshes than standard finite elements). Over the years, they have been successfully applied to a variety of linear and nonlinear problems, integrated into efficient solvers [5, 30], and have made the object of two research monographs [21, 10]. Recent developments of the ideas underlying HHO methods have led to the Discrete de Rham (DDR) paradigm [24, 22]; see also [4] for a recent extension using the language of differential forms. In 2023, Daniele Di Pietro received a prestigious ERC grant for the development of an integrated computational chain based on DDR and related methods. Other developments include unfitted HHO methods for interface problems [6] and HHO methods for wave propagation problems in various settings [7].

Hybridizable Galerkin methods [20] are part of the development of the DG methods for partial differential equations started back in 1973, see the 2018 review [13], as well as part of the development of the hybridizable mixed methods, see the 2004 paper [19], and the 2023 review on HDG methods [14]. The bridge established between the HHO and the HDG methods fueled two ongoing and intertwined lines of research on HDG methods. The first is the transformation of their local spaces into stabilizations [17, 14]. This was first carried out by using the theory of M-decompositions, see [18] and the reference therein, which provides a systematic way of constructing optimally convergent HDG and hybridizable mixed methods. The above-mentioned transformation was later used to show that the hybridizable mixed methods can be rewritten as HDG methods to improve their implementation [2]. The second line of research is the converse of the first one, namely, the transformation of the stabilizations of the HDG methods into local spaces. As recently shown in [15], this allows the introduction of superconvergent post-processings of the approximate solution.

The results of [16] also provided another way to highlight the close links between HDG, HHO, and WG methods, shedding further light on the discussion of [12]. HHO and WG methods were developed independently and share a common devising viewpoint based on reconstruction (called weak gradient in WG) and stabilization (stabilization-free variants exist for both methods). The equal-order HHO stabilization (3.3) is key to achieving the higher-order consistency property (3.2), whereas, in the mixed-order case where the degree of the cell unknowns is one order higher than that of the face unknowns, the key idea for higher-order stabilization can be traced back to the work of Lehrenfeld and Schöberl in the context of HDG methods [34]. We refer the reader to [31] for a recent comparison between HHO and WG methods in the context of biharmonic problems. Finally, the results of [16], combined with the connection between HHO and the Multiscale Hybrid Mixed (MHM) method uncovered in [11], can be leveraged to bridge the HDG and MHM methods.

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