On the Dynamic Green's Function of the Three-Dimensional Infinite Anisotropic Medium

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Abstract

The propagation of an acoustic wave generated by a δ-pulse unit point force in a three dimensional anisotropic medium is considered. A general expression for the retarded 3×3 *elastodynamic* displacement Green's function is derived for arbitrary anisotropic symmetry. The initial integration problem of four integrations is reduced to only one φ-integration problem combined with the problem of solving three algebraic equations. The evaluation of expressions of the Green's function for the behavior at infinity yields the radiation condition and describes the acoustic wave propagation and gives the possible distinguished propagation directions from the source point for outgoing acoustic waves (high symmetry directions of the crystal beside other distinguished directions). For high symmetry directions of the crystal this condition yields three outgoing singular acoustic waves, one with longitudinal and two with transverse polarization. For elastic isotropy of the medium we obtain the result known from literature [17] which was derived there by using other techniques. For hexagonal symmetry of the medium we derive closed-form expressions of the farfield waves for the high symmetry directions (i.e. for propagation directions parallel and in the plane perpendicular to the hexagonal c-axis).

Keywords. Acoustic Problem, Elastodynamic Green's Function, Dynamic Green's Function, Retarded Green's Function, Causality, Elastic Medium, Elastic Anisotropy, Wave Propagation

Contents

1 Introduction

In recent years the modeling of dynamic processes in composite structures has become of great interest due to widespread applications in ultrasonics technology. In the description of the effective characteristics Green's function methods are powerful tools. Especially useful are closed form representations.

In numerous papers Green's functions of elastostatics have been derived. In the isotropic elasticity the Green's function of statics was derived by Kelvin (Sir Thomson) [4]. For the infinite medium of hexagonal symmetry closed form representations of Green's functions for displacements have been derived by Lifshitz and Rosenzweig [6] and more compactly by Kröner [7]. For the electroelastic infinite medium of hexagonal symmetry the explicit static Green's function has been derived by Michelitsch [9] by employing an integral transformation technique which is outlined in [10]. This closed-form formulation of the electroelastic Green's function has turned out to be a useful tool for the explicit evaluation of the electroelastic analogue of the Eshelby tensor [11] which represents a key quantity in the treatment of inclusion problems. For the semi-infinite piezoelectric medium of hexagonal symmetry closed-form expressions for the static Green's function are derived by Karapetian et al. [12] only recently by utilizing an advanced method in potential theory that was discovered by Fabrikant [8].
A treatment of the space-frequency representation of the dynamic Green's function which represents the solution for the dynamic response of a time harmonic point force was given by Norris [13]. In that paper time-harmonic Green's functions for anisotropic piezoelectric, thermoelastic and poroelastic solids are constructed where the similarity in the theories is outlined. Another representation of the frequency-space-representation of the dynamic Green's function was given by Willis [20] among other authors. The wave propagation in elastic solids was analyzed by Achenbach in [18]. Only for the isotropic elasticity a full closed-form solution for the frequency-space-representation of the dynamic Green's function is known in the literature [17]. In second order gradient electrodynamics retarded Green's functions and related quantities were derived by Lazar [19]. The time-domain dynamic response function which represents the dynamic displacement response to a unit point force having Heaviside step function time dependence in elastic anisotropic media was treated by Every and Kim [15]. In their paper special emphasis is given on singularities and discontinuities that appear in dynamic response functions. A treatment of the dynamic Green's function of the space-frequency representation of the infinite transversely isotropic medium was given by Tverdokhlebov and Rose [16]. In that paper a formula is developed for the case of weak anisotropy that allows especially a convenient treatment of scattering fields in the farfield approximation. An advanced analysis of the dynamic response of transient electroacoustic waves in piezoelectric solids was given by Sosa and Khutoryansky (1999, 2001) [1, 2]. Utilizing this approach general expression for the Green's function was also derived by Daros and Antes (2000) [3].

In the present paper our goal is to develop a general method for the construction of the space-time representation of the causal (retarded) Green's function. Our approach complements the existing literature and is closely related to that introduced by Sosa and Khutoryansky (1999, 2001) for piezoelectric solids, but with focus on farfield expressions for purely elastic solids of arbitrary anisotropic symmetry. We show that this approach yields useful representations especially for the farfield approximation of the dynamic Green's function. The constructed Green's function represents the dynamic displacement-response of a δ-pulse-point-force. We formulate our method for arbitrary symmetry and anisotropy of the infinitely extended medium. The presented formulation of the dynamic Green's function in the space-time domain may be especially useful for a treatment of scattering problems, for instance the scattering of acoustic waves at impurities or inclusions.

2 Equations of motion

We start with the field equations for the stress tensor $\boldsymbol{\sigma}$ and the elastic displacements $\boldsymbol{u}$. The governing equilibrium conditions are given by:

$$\rho_M \frac{d^2}{dt^2} u_i = \partial_j \sigma_{ij} + f_i\,, \;\; \sigma_{ij} = \sigma_{ji} \tag{1}$$

∂_j indicates the spatial derivatives with respect to the Cartesian space coordinates $x_j = (x, y, z)$, $\frac{d}{dt}$ is the time derivative, $\boldsymbol{f}$ the density of body forces and ρ_M the mass density. As constitutive equations we assume the anisotropic Hooke's law:

$$\sigma_{ij} = \mathcal{C}_{ijkl}\epsilon_{kl} \tag{2}$$

where $\mathcal{C}_{ijkl}$ denote the elastic moduli. They have the symmetry properties $\mathcal{C}_{ijkl} = \mathcal{C}_{klij} = \mathcal{C}_{jikl} = \mathcal{C}_{ijlk}$. The elastic displacement field $\boldsymbol{u}$ is related to the tensor $\boldsymbol{\epsilon}$ of elastic strains by the ansatz

$$\epsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \tag{3}$$

Putting ansatz (3) using (2) into the field equations (1) we obtain a 3×3 second-order differential equation (wave equation) for the displacement field $\boldsymbol{u}$ of the form:

$$[\boldsymbol{T}\left(\nabla\right) - \rho_M \frac{d^2}{dt^2}\mathbf{1}]\boldsymbol{u}(\boldsymbol{r},t) + \boldsymbol{f}(\boldsymbol{r},t) = 0 \tag{4}$$

∇ indicates the spatial gradient operator, $\mathbf{1}$ denotes the 3×3 unit tensor and $\boldsymbol{r} = (x, y, z)$ the space vector and t the time coordinate. $\boldsymbol{T}\left(\nabla\right)$ indicates the fundamental 3×3 second-order tensor differential operator of elasticity and is given by:

$$T_{ij}\left(\nabla\right) = \mathcal{C}_{ipjq}\partial_p\partial_q \tag{5}$$

The displacement field $\boldsymbol{u}$ can then be represented by the dynamic 3×3 Green's function $\boldsymbol{G}$ according to

$$\boldsymbol{u}(\boldsymbol{r},t) = \int\int \boldsymbol{G}(\boldsymbol{r}-\boldsymbol{r}^{'}, t-t^{'})\boldsymbol{f}(\boldsymbol{r}^{'}, t^{'})\mathrm{d}^3\boldsymbol{r}^{'}\mathrm{d}t^{'} \tag{6}$$

The requirement that $\boldsymbol{u}$ must be a solution of (4) yields as defining wave equation for the dynamic Green's function

$$[\boldsymbol{T}\left(\nabla\right) - \rho_M \frac{d^2}{dt^2}\mathbf{1}]\boldsymbol{G}\left(\boldsymbol{r},t\right) + \delta^3\left(\boldsymbol{r}\right)\delta(t)\mathbf{1} = 0 \tag{7}$$

where $\delta^3\left(\boldsymbol{r}\right)$ and $\delta(t)$ represent the spatial δ-function and the time dependent δ-function, respectively. Above defined dynamic Green's function has the following physical interpretation:
$G_{ij}(\boldsymbol{r},t)$ gives the elastic displacement in x_i-direction at spacepoint $\boldsymbol{r}$ and time t caused by a unit δ-pulse-point-force at $\boldsymbol{r}^{'} = 0$ and $t^{'} = 0$ in x_j-direction. We furthermore observe the symmetry property $G_{ij} = G_{ji}$.

Here we note an important observation: on the one hand, since equation (7) is a partial second-order differential equation, its solution $\boldsymbol{G}$ has the property of time inversion symmetry being invariant against the replacement $t \to -t$. On the other hand we have the physical requirement of causality for the Green's function to guarantee the fact that the displacement response at spacepoint $\boldsymbol{r}$ and time t can only take place *after* being excited by a source at spacepoint $\boldsymbol{r}^{'}$ and time $t^{'} < t$. As a consequence thereof the causal (retarded) Green's function $\boldsymbol{G}$ must fulfill the property $\boldsymbol{G} \neq 0$ for $t > 0$ and $\boldsymbol{G} = 0$ for $t < 0$. This is an obvious contradiction to the time inversion-symmetry-property of equation (7) that can only be eliminated by introducing an infinitesimal first-order damping term $\epsilon\frac{d}{dt}$ ($\epsilon \to 0+$) into (7) and (4), breaking the time inversion symmetry and guaranteeing causality as we shall show in the subsequent section. Simultaneously this infinitesimal damping term is necessary to regularize our problem as we depict in the next section. The (causal) dynamic Green's function (often denoted as causal propagator) has the form

$$\boldsymbol{G}(\boldsymbol{r},t) = \Theta(t)\boldsymbol{G}(\boldsymbol{r},t) \tag{8}$$

where we introduced the Heaviside Θ-function (step function) which is defined by $\Theta(t) = 1$ if $t > 0$ and $\Theta(t) = 0$ if $t < 0$.

3 Outline of the method

In this section our goal is to derive a general method that allows us to construct the causal dynamic 3×3 Green's function defined by equation (7) for the infinite medium with arbitrary symmetry. To that end we represent the Green's function $\boldsymbol{G}$ by Fourier transformation according to

$$\boldsymbol{G}(\boldsymbol{r},t) = \frac{1}{(2\pi)^4}\int\int \exp\left(i\omega t\right)\exp\left(i\boldsymbol{k}\cdot\boldsymbol{r}\right)\tilde{\boldsymbol{G}}(\boldsymbol{k},\omega)\mathrm{d}^3\boldsymbol{k}\mathrm{d}\omega \tag{9}$$

The 3×3 Fourier transform $\tilde{\boldsymbol{G}}(\boldsymbol{k},\omega)$ of the Green's function can then be obtained by using eq. (7)

$$\tilde{\boldsymbol{G}}(\boldsymbol{k},\omega) = [\boldsymbol{T}(\boldsymbol{k}) - \rho_M\omega^2\mathbf{1}]^{-1} \tag{10}$$

In order to obtain a convenient formulation we now introduce the (orthonormal) eigenvectors $\boldsymbol{v}_\lambda(\boldsymbol{k})$ with the eigenvalues $\rho_M \mu_\lambda^2(\boldsymbol{k})$ ($\lambda = 1, 2, 3$) of $T_{ij}(\boldsymbol{k}) = \mathcal{C}_{ipjq} k_p k_q$ which are defined by

$$\boldsymbol{T}(\boldsymbol{k})\boldsymbol{v}_\lambda(\boldsymbol{k}) = \rho_M \mu_\lambda^2(\boldsymbol{k})\boldsymbol{v}_\lambda(\boldsymbol{k}) \ \ , \lambda = 1, 2, 3 \tag{11}$$

Because of elastic stability the eigenvalues fulfill the property $\mu_\lambda^2(\boldsymbol{k}) > 0$ for $\boldsymbol{k} \neq 0$. The $\mu_\lambda(\boldsymbol{k})$ can be physically interpreted as eigenfrequencies of three harmonic oscillators corresponding to three elastic waves with wave vector $\boldsymbol{k}$ and polarization vector $\boldsymbol{v}_\lambda(\boldsymbol{k})$. We observe that these three waves having wave vector $\boldsymbol{k}$ represent homogeneous solutions of wave equation (4). Obviously there are three distinguished frequency branches $\mu_\lambda(\boldsymbol{k})$. As a consequence of the symmetry property $T_{ij} = T_{ji}$ the normalized eigenvectors $\boldsymbol{v}_\lambda$ represent an orthonormal basis and fulfill

$$\boldsymbol{v}_\lambda(\boldsymbol{k}) \cdot \boldsymbol{v}_\mu(\boldsymbol{k}) = \delta_{\lambda\mu} \tag{12}$$

together with the completeness property

$$\sum_\lambda^3 \boldsymbol{v}_\lambda(\boldsymbol{k}) \otimes \boldsymbol{v}_\lambda(\boldsymbol{k}) = \mathbf{1} \tag{13}$$

where $\otimes$ stands for dyadic multiplication. Using the set $\{\boldsymbol{v}_\lambda \otimes \boldsymbol{v}_\lambda\}$ as tensor basis we can represent the $\boldsymbol{T}$-tensor in its spectral representation

$$\boldsymbol{T}(\boldsymbol{k}) = \sum_\lambda^3 \rho_M \mu_\lambda^2(\boldsymbol{k}) \boldsymbol{v}_\lambda(\boldsymbol{k}) \otimes \boldsymbol{v}_\lambda(\boldsymbol{k}) \tag{14}$$

By employing these properties the Fourier transform (10) can be written as

$$\tilde{\boldsymbol{G}}(\boldsymbol{k}, \omega) = \frac{1}{\rho_M} \sum_\lambda^3 \boldsymbol{v}_\lambda(\boldsymbol{k}) \otimes \boldsymbol{v}_\lambda(\boldsymbol{k}) g_\lambda(\boldsymbol{k}, \omega) \tag{15}$$

where we have introduced the abbreviation

$$g_\lambda(\boldsymbol{k}, \omega) = \frac{1}{\mu_\lambda^2(\boldsymbol{k}) - \omega^2} \tag{16}$$

First we start with the calculation of $\boldsymbol{G}(\boldsymbol{k}, t)$ which is given by

$$\boldsymbol{G}(\boldsymbol{k}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\boldsymbol{G}}(\boldsymbol{k}, \omega) \exp(i\omega t) \mathrm{d}\omega \tag{17}$$

From equation (9) we observe that $G(\boldsymbol{r}, t)$ and $G(\boldsymbol{k}, t)$ are related by

$$\boldsymbol{G}(\boldsymbol{r}, t) = \frac{1}{(2\pi)^3} \int \exp(i\boldsymbol{k} \cdot \boldsymbol{r}) \boldsymbol{G}(\boldsymbol{k}, t) \mathrm{d}^3\boldsymbol{k} \tag{18}$$

We can rewrite eq. (17) as

$$\boldsymbol{G}(\boldsymbol{k}, t) = \frac{1}{\rho_M} \sum_\lambda^3 \boldsymbol{v}_\lambda(\boldsymbol{k}) \otimes \boldsymbol{v}_\lambda(\boldsymbol{k}) g_\lambda(\boldsymbol{k}, t) \tag{19}$$

where we denote

$$g_\lambda(\boldsymbol{k}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g_\lambda(\boldsymbol{k}, \omega) \exp(i\omega t) \mathrm{d}\omega \tag{20}$$

Obviously integral (20) is not well defined because of the singularities of $g_\lambda(\boldsymbol{k}, \omega)$ at $\omega_{1,2} = \pm\mu_\lambda(\boldsymbol{k})$ which are located on the real ω-axis (eq. (16)). In order to obtain a regularization of (20) we introduce an infinitesimal damping $\epsilon \to 0+$ ($\epsilon > 0$). The amplitude $g_\lambda(\boldsymbol{k}, t)$ fulfills then the equation of motion of an infinitesimally damped harmonic oscillator with eigenfrequency $\mu_\lambda > 0$ excited by a δ-pulse

$$\left\{ \frac{d^2}{dt^2} + 2\epsilon \frac{d}{dt} + \mu_\lambda^2(\boldsymbol{k}) \right\} g_\lambda(\boldsymbol{k}, t) = \delta(t) \tag{21}$$

For the regularized $g_\lambda(\boldsymbol{k}, \omega)$ we now obtain

$$g_\lambda(\boldsymbol{k},\omega) = \frac{1}{\mu_\lambda^2(\boldsymbol{k}) - \omega^2 + 2i\epsilon\omega} \tag{22}$$

Expression (20) becomes with (22) a regular well defined integral. The integrand has now singularities at $\omega_{1,2} = i\epsilon \pm \mu_\lambda$ which are infinitesimally displaced from the real ω-axis into the upper complex ω-plane. In order to evaluate integral (20) via residue theorem we have to find a closed integration path in the complex ω-plane which contains on the one hand the real ω-axis and yields on the other hand in a limiting process as only non zero contribution integral (20). At this point we apply a very common method of integration that is described in many introductory textbooks of theoretical physics (see e.g. [21]). We therefore do not repeat its detailed derivation here. Its brief description is as follows: For $t > 0$ we can calculate (20) using residue theorem when we close the integration path by a semicircle around the upper complex ω-plane ($Im\,\omega > 0$) with radius $|\omega| \to \infty$. For $t < 0$ we have to close the integration path by a semicircle around the lower ω-plane ($Im\,\omega < 0$). In the limiting case when the radii of the semicircles tend to infinity ($|\omega| \to \infty$) the contributions of the integrals over the semicircles to (20) are tending to zero faster than $1/|\omega|$. Since the singularities $\omega_{1,2}$ are located in the upper complex ω-plane there are non zero contributions to (20) only for $t > 0$. For $t < 0$ (20) is vanishing. We thus can write

$$\begin{aligned} g_\lambda(\boldsymbol{k},t) &= \frac{1}{2\pi}\oint g_\lambda(\boldsymbol{k},\omega)\exp(i\omega t)\mathrm{d}\omega \\ &= 2\pi i\sum Res[g_\lambda(\boldsymbol{k},\omega)\exp(i\omega t)] \end{aligned} \tag{23}$$

where $Res[g_\lambda(\boldsymbol{k},\omega)\exp(i\omega t)]$ denote the residues of function $g_\lambda(\boldsymbol{k},\omega)\exp(i\omega t)$. By evaluating this expression we obtain

$$g_\lambda(\boldsymbol{k},t) = \Theta(t)\exp(-\epsilon t)\frac{\sin\mu_\lambda t}{\mu_\lambda} \tag{24}$$

This result is well known as *causal* (retarded) Green's function of the damped harmonic oscillator (e.g. [21]). The exponential factor $\exp(-\epsilon t)$ can be omitted since we assume that ϵ is chosen small enough that the limiting case $t\epsilon \to 0+$ is fulfilled for finite t. The Heaviside function (Θ-function) in (24) describes the causality of the Green's function. The causality is automatically fulfilled by introducing the infinitesimal damping into eq. (21). Putting (24) into (19) then gives the causal $\boldsymbol{G}(\boldsymbol{k},t)$. The remaining task is now to perform the $\boldsymbol{k}$-integration of (19) by evaluating integral (18) which takes the form

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{r},t) \;=\; & \frac{\Theta(t)}{(2\pi)^3\rho_M}\sum_\lambda^3\int_0^{2\pi}\mathrm{d}\varphi \\ & \times\int_0^\pi \mathrm{d}\vartheta\sin\vartheta\frac{1}{\mu_\lambda(\hat{\boldsymbol{k}})}\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}})\otimes\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \\ & \times\int_0^\infty \mathrm{d}k\,k\cos(kr\cos(\vartheta))\sin(tk\mu_\lambda(\hat{\boldsymbol{k}})) \end{aligned} \tag{25}$$

Here we have introduced $\boldsymbol{k} = k\hat{\boldsymbol{k}}$ where $\hat{\boldsymbol{k}}$ denotes the unit vector in $\boldsymbol{k}$-direction. We furthermore make use of the homogeneity property

$$\mu_\lambda(k\hat{\boldsymbol{k}}) = k\mu_\lambda(\hat{\boldsymbol{k}}) \tag{26}$$

The unit vector $\hat{\boldsymbol{k}}$ can be represented as

$$\hat{\boldsymbol{k}}(u,\varphi) = \boldsymbol{\xi}(\varphi)\sqrt{(1-u^2)} + \boldsymbol{e}_3 u \tag{27}$$

with ϑ denoting the angle between space vector $\boldsymbol{r}$ and wave vector $\mathbf{k}$ with $u = \cos\vartheta$ and $+\sqrt{(1-u^2)} = \sin\vartheta$ ($0 \le \vartheta \le \pi$). $\boldsymbol{e}_3 = \boldsymbol{r}/r$ denotes the unit vector in the $\boldsymbol{r}$-direction. The unit vector $\boldsymbol{\xi}$ is perpendicular to $\boldsymbol{r}$ and has the form [9, 10]

$$\boldsymbol{\xi}(\varphi) = \boldsymbol{e}_1\cos\varphi + \boldsymbol{e}_2\sin\varphi \tag{28}$$

The vectors $\boldsymbol{e}_i$ form a useful orthonormal basis

$$\boldsymbol{e}_1 = \frac{1}{\rho}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{e}_2 = \frac{1}{\rho r}\begin{pmatrix} -zx \\ -zy \\ \rho^2 \end{pmatrix} \quad , \quad \boldsymbol{e}_3 = \frac{1}{r}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{29}$$

where $\rho^2 = x^2 + y^2$, $r^2 = \rho^2 + z^2$. The basis vectors $\boldsymbol{e}_i$ fulfill $\boldsymbol{e}_i = \frac{1}{2}\epsilon_{ijk}\boldsymbol{e}_j \times \boldsymbol{e}_k$. Here ϵ_{ijk} denotes the anti-symmetric permutation tensor. In order to evaluate (25) it is convenient to use the relation

$$\sin(tk\mu_\lambda(\hat{\boldsymbol{k}})) = \frac{-1}{k\mu_\lambda(\hat{\boldsymbol{k}})}\frac{d}{dt}\cos(tk\mu_\lambda(\hat{\boldsymbol{k}})) \tag{30}$$

Then eq. (25) takes the form

$$\begin{aligned} &\boldsymbol{G}(\boldsymbol{r},t) = \\ &-\frac{\Theta(t)}{(2\pi)^3\rho_M}\frac{d}{dt}\sum_\lambda^3 \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{+1} \mathrm{d}u \frac{1}{\mu_\lambda^2(\hat{\boldsymbol{k}})}\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \otimes \boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \\ &\times \int_0^\infty \mathrm{d}k \cos(kru)\cos(tk\mu_\lambda(\hat{\boldsymbol{k}})) \end{aligned} \tag{31}$$

where $\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}(u,\varphi)$ (eq. (27)). To simplify above equation a consideration of symmetry properties is useful: the operator $T_{ij}(\boldsymbol{k}) = C_{irjs}k_r k_s$ is a bilinear function of k_i depending only on the products $\{k_r k_s\}$ which obviously are invariant under inversion $\boldsymbol{k} \to -\boldsymbol{k}$. Consequently $\mu_\lambda^2(\boldsymbol{k})$ and $\mu_\lambda(\boldsymbol{k}) = +\sqrt{\mu_\lambda^2(\boldsymbol{k})}$ are even functions under inversion

$$\mu_\lambda(-\boldsymbol{k}) = +\mu_\lambda(\boldsymbol{k}) \tag{32}$$

since they are always functions of the set of products $\{k_r k_s\}$ and not affected by the signs of the k_i. As we show below this symmetry property is crucial to obtain a very compact formulation of the Green's function. To that end we note that the eigenvectors fulfill the property

$$\boldsymbol{v}_\lambda(-\boldsymbol{k}) = p\boldsymbol{v}_\lambda(\boldsymbol{k}) \tag{33}$$

with $p^2 = 1$ thus $p = \pm 1$. This follows directly from the orthonormality relation (12). From these observations we can conclude that obviously $\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \otimes \boldsymbol{v}_\lambda(\hat{\boldsymbol{k}})/\mu_\lambda^2(\hat{\boldsymbol{k}})$ is an even function with respect to inversion. We then observe that the surface-integral

$$\begin{aligned} 0 = &\int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{+1} \mathrm{d}u \Big[\frac{1}{\mu_\lambda^2(\hat{\boldsymbol{k}})}\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \otimes \boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \\ &\times \sin(kru)\sin(tk\mu_\lambda(\hat{\boldsymbol{k}})) \Big] \end{aligned} \tag{34}$$

is vanishing since its integrand is an odd function under inversion. Using this property we arrive at [1]

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{r},t) = &-\frac{\Theta(t)}{(2\pi)^3\rho_M}\frac{d}{dt}\sum_\lambda^3 \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{+1} \mathrm{d}u \frac{1}{\mu_\lambda^2(\hat{\boldsymbol{k}})} \\ &\times \boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \otimes \boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \int_0^\infty \mathrm{d}k \cos(k[ru - t\mu_\lambda(\hat{\boldsymbol{k}})]) \end{aligned} \tag{35}$$

Taking into account that the k-integration yields a δ-function we obtain

[1] We observe from eq. (34) that the term $\cos(k[ru - t\mu_\lambda(\hat{\boldsymbol{k}})])$ could also be replaced by $\cos(k[ru + t\mu_\lambda(\hat{\boldsymbol{k}})])$. Because of the inversion symmetry $\mu_\lambda(-\hat{\boldsymbol{k}}) = \mu_\lambda(\hat{\boldsymbol{k}})$ both variants lead to the same result for the Green's function (eq. (43) below).

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{r},t) &= -\frac{\Theta(t)}{8\pi^2\rho_M r}\frac{d}{dt}\sum_\lambda^3\int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{+1}\mathrm{d}u \\ &\quad\times\Big[\frac{1}{\mu_\lambda^2(\hat{\boldsymbol{k}})}\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}})\otimes\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \\ &\quad\times\delta\left(u-\frac{t\mu_\lambda(u,\varphi)}{r}\right)\Big]\end{aligned} \tag{36}$$

The (u,φ)-dependencies of μ_λ and $\boldsymbol{v}_\lambda$ depend on the orientation of $\boldsymbol{r}$ (compare eq. (27)). We note here that the eigenvalues $\mu_\lambda(\hat{\boldsymbol{k}})=\mu_\lambda(u,\varphi)$ are functions defined on the unit sphere $|\hat{\boldsymbol{k}}|=1$. By choosing a coordinate free representation we can write (36) alternatively as integral over the unit sphere $|\hat{\boldsymbol{k}}|=1$: [2]

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{r},t) &= -\frac{\Theta(t)}{8\pi^2\rho_M}\frac{d}{dt}\sum_\lambda^3\int_{|\hat{\boldsymbol{k}}|=1}\mathrm{d}\Omega(\hat{\boldsymbol{k}}) \\ &\quad\times\Big[\frac{1}{\mu_\lambda^2(\hat{\boldsymbol{k}})}\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}})\otimes\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}})\delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{r}-t\mu_\lambda(\hat{\boldsymbol{k}}))\Big]\end{aligned} \tag{37}$$

The contributions to (37) are determined by the zeros of the equations

$$\hat{\boldsymbol{k}}\cdot\boldsymbol{r}-t\mu_\lambda(\hat{\boldsymbol{k}})=0\ \ ,\lambda=1,2,3 \tag{38}$$

The solutions $\hat{\boldsymbol{k}}_\lambda(\varphi,t/r)=\hat{\boldsymbol{k}}(u_\lambda(\varphi,t/r),\varphi)$ of (38) represent curves on the unit sphere $|\hat{\boldsymbol{k}}|=1$ which can be parameterized as follows:

$$\begin{aligned}\hat{\boldsymbol{k}}_\lambda\left\{\varphi,\frac{t}{r}\right\} &= \hat{\boldsymbol{k}}\left\{u_\lambda(\varphi,\frac{t}{r}),\varphi\right\} \\ \hat{\boldsymbol{k}}_\lambda\left\{\varphi,\frac{t}{r}\right\} &= \sqrt{1-u_\lambda^2(\varphi,\frac{t}{r})}\boldsymbol{\xi}(\varphi)+\boldsymbol{e}_3u_\lambda(\varphi,\frac{t}{r})\end{aligned} \tag{39}$$

where the $u_\lambda(\varphi,t/r)$ denote only *those* zeros of the equations

$$u-\frac{t\mu_\lambda(u,\varphi)}{r}=0,\quad\lambda=1,2,3 \tag{40}$$

which fulfill additionally the condition

$$1-u^2\geq 0 \tag{41}$$

The solutions u_λ of (40) which fulfill additionally (41) represent the zeros of the arguments in the δ-functions in eq. (36). Condition (41) takes into account that the integration limits are $u=\pm1$. For further evaluation of (36) we have to determine these solutions.

To discuss the solutions of (40) it is convenient to introduce the minimum and maximum values μ_λ^{Min} and μ_λ^{Max} with $0<\mu_\lambda^{Min}\leq\mu_\lambda(\hat{\boldsymbol{k}})\leq\mu_\lambda^{Max}$. We observe, because of the property $\mu_\lambda(u,\varphi)t/r>0$, that the solutions of (40) always fulfill $u_\lambda(\varphi,t/r)>0$. However, only those solutions of (40) affect (36) which fulfill additionally (41). In order to determine them three distinguished t/r-ranges have to be considered for each fixed λ:

Range I:
$t\mu_\lambda^{Min}/r\leq t\mu_\lambda/r\leq t\mu_\lambda^{Max}/r\leq 1$; $u_\lambda(t/r,\varphi)<1$ for all φ.

Range II:
$\mu_\lambda^{Min}t/r\leq 1\leq\mu_\lambda^{Max}t/r$; $u_\lambda(t/r,\varphi)<1$ is fulfilled for distinguished φ-ranges only.

Range III:

[2] Because of the inversion symmetry of the integrand of (37) we observe that both signs in $\delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{r}\pm t\mu_\lambda(\hat{\boldsymbol{k}}))$ lead to the same result.

$1 < \mu_\lambda^{Min} t/r \leq \mu_\lambda t/r \leq \mu_\lambda^{Max} t/r$; $u_\lambda(t/r, \varphi) > 1$ thus (40) has no solutions which fulfill (41).

In the general case of arbitrary symmetry of the medium the solutions of (40) which fulfill (41) for fixed λ's consist in several steady solution segments $u_\lambda^{(s_\lambda)}(\varphi, t/r)$ ($s_\lambda = 1, .., m_\lambda$) where $m_\lambda = m_\lambda(t/r)$ denotes the number of steady solution segments. When the entire φ interval $[0, 2\pi]$ is passed, the solutions passe all m_λ segments $u_\lambda^{(s_\lambda)}(\varphi, t/r)$ sequentially. Each segment $u_\lambda^{(s_\lambda)}$ represents the unique solution of (40) fulfilling (41) for a certain φ-interval. The introduction of steady segments takes into account that the solutions of (40) fulfilling (41) may behave discontinuously, jumping from one segment $u_\lambda^{(p)}(\varphi, t/r)$ to the next $u_\lambda^{(p+1)}(\varphi, t/r)$ when φ is continuously passed. In our subsequent considerations segment index s_λ is omitted whenever possible.

In order to derive a compact formulation for the Green's function from (36) it is convenient to introduce characteristic functions $\Theta_{s_\lambda}(\varphi, t/r)$ with $\Theta_{s_\lambda} = 1$ only in that very φ-interval where the solution assumes segment $u_\lambda^{(s_\lambda)}$ and $\Theta_{s_\lambda} = 0$ outside that φ-interval. Above introduced characteristic functions then fulfill the condition

$$\sum_{s_\lambda}^{m_\lambda} \Theta_{s_\lambda}(\varphi, \frac{t}{r}) = \Theta_\lambda(t/r) \tag{42}$$

with $\Theta_\lambda(t/r) = 1$ for t/r in above defined range I, $0 < \Theta_\lambda(t/r) < 1$ for t/r in range II, and $\Theta_\lambda(t/r) = 0$ for t/r in range III.

Thus we finally obtain for the Green's function (36) the representation

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{r}, t) &= -\frac{\Theta(t)}{8\pi^2 \rho_M r} \frac{d}{dt} \sum_\lambda^3 \sum_{s_\lambda}^{m_\lambda} \int_0^{2\pi} \mathrm{d}\varphi \, \Theta_{s_\lambda}(\varphi, \frac{t}{r}) \\ &\times \Big[\frac{\Theta\{1 - u_\lambda^{(s_\lambda)}(\varphi, \frac{t}{r})\}}{B_\lambda^{(s_\lambda)}[\hat{\boldsymbol{k}}_\lambda^{(s_\lambda)}(\varphi, \frac{t}{r})] \mu_\lambda^2[\hat{\boldsymbol{k}}_\lambda^{(s_\lambda)}(\varphi, \frac{t}{r})]} \\ &\times \boldsymbol{v}_\lambda[\hat{\boldsymbol{k}}_\lambda^{(s_\lambda)}(\varphi, \frac{t}{r})] \otimes \boldsymbol{v}_\lambda[\hat{\boldsymbol{k}}_\lambda^{(s_\lambda)}(\varphi, \frac{t}{r})] \Big] \end{aligned} \tag{43}$$

The $B_\lambda^{(s_\lambda)}$ are given by

$$\begin{aligned} B_\lambda^{(s_\lambda)}[\hat{\boldsymbol{k}}_\lambda^{(s_\lambda)}(\varphi, \tfrac{t}{r})] &= B_\lambda^{(s_\lambda)}(u_\lambda^{(s_\lambda)}, \varphi) \\ &= |1 - \frac{t}{r} \frac{d\mu_\lambda}{du}[\hat{\boldsymbol{k}}(u = u_\lambda^{(s_\lambda)}, \varphi)]| \end{aligned} \tag{44}$$

Above representation (43) of the Green's function reduces our initial integration problem (9) considerably, namely to only one single φ-integration-problem combined with the algebraic problem which consists in finding the zeros u_λ of the three equations (40) fulfilling (41). The $\Theta(1 - u_\lambda)$-functions in (43) guarantee that only those φ-ranges with $u_\lambda(\varphi, \frac{t}{r}) < 1$ contribute to (43).

Expression (43) generally holds for infinite media with arbitrary elastic anisotropy and symmetry. In view of (40) follows that for fixed r and $t \to \infty$ the step functions in (43) vanish in the entire φ-range $[0, 2\pi]$. As a consequence thereof the Green's function is vanishing in the limiting case of $t \to \infty$ for each fixed space-point $\boldsymbol{r}$. This can physically be interpreted by runtime effects.

Subsequently we show that above representation (43) of the dynamic Green's function is highly convenient in order to obtain in a direct way the full closed-form solutions for the isotropic medium (section 5) and for the farfield solution in the case of hexagonal symmetry of the medium (section 6). In the next section we derive general asymptotic expressions for the Green's function from eq. (43) at infinity $r \to \infty$ holding for arbitrary symmetry of the medium.

4 Asymptotic expressions for the Green's function

In this section we derive asymptotic expressions for the Green's function (43) far away from the source point $\boldsymbol{r}' = 0$ for $r \to \infty$ and $t > 0$. To that end we have to consider the $1/r$-expansion of (43) which is obtained by evaluating the expression

$$S_\lambda = \frac{d}{dt}\left\{\frac{\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}_\lambda)\otimes\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}_\lambda)\Theta(1-u_\lambda)}{\mu_\lambda^2(\hat{\boldsymbol{k}}_\lambda)[B_\lambda(\hat{\boldsymbol{k}}_\lambda)]}\right\} \tag{45}$$

to arrive at

$$S_\lambda = -\frac{du_\lambda}{dt}\left\{\frac{\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}_\lambda)\otimes\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}_\lambda)\delta(1-u_\lambda)}{\mu_\lambda^2(\hat{\boldsymbol{k}}_\lambda)[B_\lambda(\hat{\boldsymbol{k}}_\lambda)]}\right\} + O(\frac{1}{r}) \tag{46}$$

where the δ-function comes into play from the time derivative of the Θ-function and $\boldsymbol{O}(1/r)$ denotes remaining terms of at least first-order in $1/r$ corresponding to a $1/r^2$-order term of (43). This is a consequence of the property

$$\frac{du_\lambda}{dt} = \frac{\mu_\lambda}{r} + \frac{t}{r}\frac{d\mu_\lambda}{du}\frac{du_\lambda}{dt} \tag{47}$$

Collecting terms $\frac{du_\lambda}{dt}$ we obtain an expression of order $1/r$:

$$\begin{aligned}\frac{du_\lambda}{dt} &= \frac{\mu_\lambda}{r\left(1-\frac{t}{r}\frac{d\mu_\lambda}{du}\right)} \\ &= \frac{\mu_\lambda}{r}\frac{p_\lambda}{B_\lambda}\end{aligned} \tag{48}$$

Here we have introduced $p_\lambda = sign(1-\frac{t}{r}\frac{d\mu_\lambda}{du})$ that can take the values ± 1.
Equation (46) can now be evaluated by using (48). The factor μ_λ/r can be absorbed into the δ-function:

$$\frac{d}{dt}\Theta(1-u_\lambda) = -\delta(t-\frac{r}{\mu_\lambda})\frac{p_\lambda}{B_\lambda} \tag{49}$$

To obtain the farfield approximation in the evaluation of (43) only the δ-term of (46) has to be taken into account as dominant term for $r\to\infty$. Thus (43) yields then an expression of the form

$$\boldsymbol{G}_\infty(\boldsymbol{r},t) = \sum_{\lambda=1}^{3}\boldsymbol{G}_\infty^{(\lambda)}(\boldsymbol{r},t) \tag{50}$$

where

$$\boldsymbol{G}_\infty^{(\lambda)}(\boldsymbol{r},t) = \frac{p_\lambda}{8\pi^2\rho_M r}\frac{\delta(t-\frac{r}{\mu_\lambda})}{\mu_\lambda^2}\int_0^{2\pi}\boldsymbol{v}_\lambda\otimes\boldsymbol{v}_\lambda\frac{\mathrm{d}\varphi}{B_\lambda^2} \tag{51}$$

The quantities $B_\lambda = B_\lambda(u=1,\varphi)$ in general depend on φ whereas $\mu_\lambda(u=1) = \mu_\lambda(\boldsymbol{e}_3)$ in (51) do not. The eigenvectors $\boldsymbol{v}_\lambda$ depend on φ under certain circumstances as will be discussed. To evaluate (51) we have to consider the expressions $\frac{d\mu_\lambda}{du}$ for $u\to 1$ since the δ-functions are contributing when $u_\lambda = \mu_\lambda t/r = 1$. To that end we write

$$\hat{\boldsymbol{k}} = \boldsymbol{e}_3 + \delta\hat{\boldsymbol{k}} \tag{52}$$

where $\delta\hat{\boldsymbol{k}}$ is a small deviation given by (compare (27))

$$\delta\hat{\boldsymbol{k}} = \boldsymbol{e}_3(u-1) + \boldsymbol{\xi}(\varphi)\sqrt{1-u^2} \tag{53}$$

where $\boldsymbol{e}_3 = \boldsymbol{r}/r$. We now assume that μ_λ can be expanded around $\boldsymbol{e}_3$ according to

$$\mu_\lambda(\boldsymbol{e}_3+\delta\hat{\boldsymbol{k}}) = \mu_\lambda(\boldsymbol{e}_3) + \boldsymbol{\alpha}\cdot\delta\hat{\boldsymbol{k}} + \frac{1}{2}\delta\hat{\boldsymbol{k}}^T\boldsymbol{\gamma}\delta\hat{\boldsymbol{k}} + \ldots \tag{54}$$

Here we denote

$$\begin{aligned}\boldsymbol{\alpha} &= \nabla_{\boldsymbol{k}}\mu_\lambda(\boldsymbol{k}=\boldsymbol{e}_3) \\ \boldsymbol{\gamma} &= \nabla_{\boldsymbol{k}}\otimes\nabla_{\boldsymbol{k}}\mu_\lambda(\boldsymbol{k}=\boldsymbol{e}_3)\end{aligned} \tag{55}$$

where $\nabla_{\boldsymbol{k}}$ indicates the gradient operator with respect to $\boldsymbol{k}$. By taking into account the u-dependence of $\delta\hat{\boldsymbol{k}}$ of (53) we obtain the representation

$$\begin{aligned}\mu_\lambda(\boldsymbol{e}_3+\delta\hat{\boldsymbol{k}}) &= \mu_\lambda(\boldsymbol{e}_3)+(u-1)\boldsymbol{\alpha}\cdot\boldsymbol{e}_3+\sqrt{1-u^2}\boldsymbol{\alpha}\cdot\boldsymbol{\xi}\\ &\quad+\tfrac{1}{2}\boldsymbol{e}_3\boldsymbol{\gamma}\boldsymbol{e}_3(u-1)^2\\ &\quad+\tfrac{1}{2}(\boldsymbol{\xi}\boldsymbol{\gamma}\boldsymbol{e}_3+\boldsymbol{e}_3\boldsymbol{\gamma}\boldsymbol{\xi})\sqrt{1-u^2}(u-1)\\ &\quad+\tfrac{1}{2}\boldsymbol{\xi}\boldsymbol{\gamma}\boldsymbol{\xi}(1-u^2)+O(1-u)^{3/2}\end{aligned} \tag{56}$$

where $O(1-u)^{3/2}$ denotes higher order terms containing the common factor $(1-u)^{3/2}$. The derivative with respect to u of (56) yields [3]

$$\begin{aligned}\frac{d\mu_\lambda}{du}(\boldsymbol{e}_3+\delta\hat{\boldsymbol{k}}) &= \boldsymbol{\alpha}\cdot\boldsymbol{e}_3-\tfrac{u}{\sqrt{1-u^2}}\boldsymbol{\alpha}\cdot\boldsymbol{\xi}\\ &\quad+\boldsymbol{e}_3\boldsymbol{\gamma}\boldsymbol{e}_3(u-1)\\ &\quad+(\boldsymbol{\xi}\boldsymbol{\gamma}\boldsymbol{e}_3+\boldsymbol{e}_3\boldsymbol{\gamma}\boldsymbol{\xi})\tfrac{1}{2}\sqrt{\tfrac{1-u}{1+u}}(1+2u)\\ &\quad-\boldsymbol{\xi}\boldsymbol{\gamma}\boldsymbol{\xi}u+O(1-u)^{1/2}\end{aligned} \tag{57}$$

For $u\to 1$ the term $\frac{u}{\sqrt{(1-u^2)}}\boldsymbol{\alpha}\cdot\boldsymbol{\xi}$ tends to infinity if $\boldsymbol{\alpha}\cdot\boldsymbol{\xi}\neq 0$. For arbitrary elastic anisotropy there are only few distinguished $\boldsymbol{r}$-directions where $\boldsymbol{\alpha}\cdot\boldsymbol{\xi}=0$. Only for these distinguished directions (57) yields finite values. As a consequence thereof, in view of (51), propagation of farfield waves is possible only in these directions. For all other $\boldsymbol{r}$-directions where $\boldsymbol{\alpha}\cdot\boldsymbol{\xi}\neq 0$ the integral (51) is vanishing due the singular behavior of (44). Then the propagation of farfield waves (and the corresponding energy flux) is suppressed. This is an ultimate consequence of the elastic anisotropy which causes the u-dependence of μ_λ.

Before considering the possible distinguished propagation directions $\boldsymbol{r}/r=\boldsymbol{e}_S$ fulfilling $\boldsymbol{\alpha}\cdot\boldsymbol{\xi}=0$ we discuss the physical meaning of the term $\boldsymbol{\alpha}\cdot\boldsymbol{\xi}$: This term expresses (since $\boldsymbol{\xi}$ is perpendicular to $\boldsymbol{r}$) the derivative in the $\boldsymbol{\xi}$-direction perpendicular to $\boldsymbol{r}$. To clarify this we consider the gradient

$$\nabla_{\boldsymbol{k}}\mu_\lambda(\boldsymbol{k})=\hat{\boldsymbol{k}}\mu_\lambda(\hat{\boldsymbol{k}})+\tilde{\nabla}_{\boldsymbol{k}}\mu_\lambda(\hat{\boldsymbol{k}}) \tag{58}$$

which has the physical meaning of *group velocity*. This representation is obtained by using $\nabla_{\boldsymbol{k}}=\hat{\boldsymbol{k}}\frac{d}{dk}+\frac{1}{k}\tilde{\nabla}_{\boldsymbol{k}}$ together with the homogeneity property $\mu_\lambda(k\hat{\boldsymbol{k}})=k\mu_\lambda(\hat{\boldsymbol{k}})$. Note furthermore that the first term in above equation $\hat{\boldsymbol{k}}\mu_\lambda(\hat{\boldsymbol{k}})$ has the physical meaning of *phase velocity*. When we now put $\hat{\boldsymbol{k}}=\boldsymbol{e}_3=\boldsymbol{r}/r$ we see that the term $\boldsymbol{\alpha}\cdot\boldsymbol{e}_3=\mu_\lambda(\boldsymbol{e}_3)$ $(u=1)$ coincides with the phase velocity and $\boldsymbol{\alpha}\cdot\boldsymbol{\xi}$ measures the deviation between phase- and group velocity. Thus we can state (where we assume the source point being located in the origin): *In $\boldsymbol{r}$-directions, where group- and phase velocities do not coincide, no radiation, i.e. wave propagation occurs*. This is caused by the singularity of B_λ at $u=1$ for these points as a consequence of $\boldsymbol{\alpha}\cdot\boldsymbol{\xi}\neq 0$ (compare eq. (57)). Physically this can be interpreted as follows: if group and phase velocity does not coincide *dispersion* occurs, that is, a wave package cannot propagate stably and "survive" until infinity ($r\to\infty$). Hence we can formulate the *radiation condition* in the anisotropic elasticity as follows: the possible (distinguished) $\boldsymbol{r}$-directions $\boldsymbol{e}_3$ from the source point in which propagation of farfield waves is possible are determined by the condition

$$\boldsymbol{\alpha}\cdot\boldsymbol{\xi}=\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{k}}\mu_\lambda(\boldsymbol{k}=\boldsymbol{e}_3)=0 \tag{59}$$

For these directions $\hat{\boldsymbol{k}}=\boldsymbol{r}/r$ then group- and phase velocity coincide and, as a consequence, a wave package can stably propagate without dispersion in that directions without decay. The notation $\boldsymbol{e}_3=\hat{\boldsymbol{e}}_S$ indicates that there are only special distinguished directions which fulfill (59), namely high symmetry directions beside a few other distinguished directions. For such directions (57) yields for $u=1$ (corresponding to $\hat{\boldsymbol{k}}=\boldsymbol{e}_3=\hat{e}_S$) finite expressions of the form

[3] Here superscripts λ of $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are omitted.

$$\frac{d}{du}\mu_\lambda(\hat{\boldsymbol{e}}_S) = \mu_\lambda(\hat{\boldsymbol{e}}_S) - \boldsymbol{\xi}\boldsymbol{\gamma}^{(\lambda)}\boldsymbol{\xi} \tag{60}$$

In the following we consider only *high symmetry directions* $\hat{\boldsymbol{e}}_S$ of the crystal. In that case all three μ_λ ($\lambda = 1, 2, 3$) *simultaneously* fulfill (59). Thus we obtain for high symmetry propagation directions from the source point *three* corresponding outgoing farfield waves. [4] Relation (60) can also be interpreted as follows: the surfaces $\mu_\lambda(\boldsymbol{k}) = const$ have the property of being perpendicular to $\boldsymbol{r}$ at stationary points (extremal values) in directions $\hat{e}_S$. Without loss of generality we can choose the $\boldsymbol{e}_i$ such ($\boldsymbol{\xi}(\varphi) = \boldsymbol{e}_1 \cos\varphi + \boldsymbol{e}_2 \sin\varphi$ thus $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \hat{e}_S$) that the $\boldsymbol{e}_i$ are eigenvectors of matrix $\boldsymbol{\gamma}^{(\lambda)}$ with corresponding eigenvalues $\gamma_i^{(\lambda)}$. Taking this into account we can write for (60):

$$\frac{d}{du}\mu_\lambda(\boldsymbol{e}_S) = \mu_\lambda(\boldsymbol{e}_S) - (\gamma_1^{(\lambda)} \cos^2\varphi + \gamma_2^{(\lambda)} \sin^2\varphi) \tag{61}$$

and correspondingly we obtain for B_λ by making use of the fact $u = \mu_\lambda(\boldsymbol{e}_S)t/r = 1$ thus $t/r = 1/\mu_\lambda(\boldsymbol{e}_S)$ and using equation (44)

$$B_\lambda(\boldsymbol{e}_S, \varphi) = \frac{1}{\mu_\lambda(\boldsymbol{e}_S)} |(\gamma_1^{(\lambda)} \cos^2\varphi + \gamma_2^{(\lambda)} \sin^2\varphi)| \tag{62}$$

Here we have to note that there are two types of points (directions) $\boldsymbol{e}_S$. Such directions where $\gamma_i^{(\lambda)}$ are both of the same sign (elliptic points on the surface $\mu_\lambda(\hat{\boldsymbol{k}}) = const$) and others having opposite signs of $\gamma_i^{(\lambda)}$ (hyperbolic points of $\mu_\lambda = const$). Equation (51) takes the form

$$\boldsymbol{G}_\infty(\boldsymbol{r}, t) = \frac{1}{8\pi^2 \rho_M r} \sum_{\lambda=1}^{3} p_\lambda \delta(t - \frac{r}{\mu_\lambda}) \boldsymbol{I}_\lambda \tag{63}$$

where it is denoted

$$\begin{aligned} \boldsymbol{I}_\lambda &= \int_0^{2\pi} \frac{\boldsymbol{v}_\lambda \otimes \boldsymbol{v}_\lambda}{B_\lambda^2 \mu_\lambda^2} \mathrm{d}\varphi \\ &= \int_0^{2\pi} \frac{\boldsymbol{v}_\lambda \otimes \boldsymbol{v}_\lambda}{|\gamma_1^{(\lambda)} \cos^2\varphi + \gamma_2^{(\lambda)} \sin^2\varphi|^2} \mathrm{d}\varphi \end{aligned} \tag{64}$$

Above integral is well defined only if the $\gamma_i^{(\lambda)}$ have the same signs (elliptic case). In the case that the $\gamma_i^{(\lambda)}$ have opposite signs (hyperbolic case) integral (64) is not well defined since there appear zeros in the denominator of the integrand of (64) for certain φ. This case requires a regularization technique by introducing an infinitesimal imaginary part $i\epsilon$ ($\epsilon \to 0+$) into the denominator of the integrand of (64). Such "catastrophic" singular behavior of the corresponding farfield amplitude can be classified by catastrophe theory [14] and lies beyond the scope of this paper. Corresponding singularities and “sharp“ features occurring also in the dynamic response function[5] which represents the displacements caused by a point force with Heaviside step function time dependence are treated elsewhere [15].

In this paper we confine ourselves on *elliptic* points, i.e. we assume $sign(\gamma_1^{(\lambda)}) = sign(\gamma_2^{(\lambda)}) = p_\lambda$. As already mentioned there may be two distinguished cases in the evaluation of integral (64):

Case I: The eigenvectors $\boldsymbol{v}_\lambda$ in (64) are independent on angle φ. This is caused by equation (27) which shows for $u = 1$ no φ-dependence of the eigenvectors $\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}})$.

Case II (degenerate case): The sound velocities $\mu_{1,2} = \mu_t$ of the transverse polarized waves are degenerate since $\boldsymbol{e}_S = \boldsymbol{v}_3$ is an axis of rotational symmetry. As a consequence the curvatures coincide, i.e. $\gamma_1^\lambda = \gamma_2^\lambda = H_\lambda$. Therefore the transverse polarization vectors $\boldsymbol{v}_{1,2}$ are only determined by spanning the degenerate transverse subspace according to

$$\boldsymbol{v}_1 \otimes \boldsymbol{v}_1 + \boldsymbol{v}_2 \otimes \boldsymbol{v}_2 = \mathbf{1} - \hat{\boldsymbol{e}}_S \otimes \hat{\boldsymbol{e}}_S \tag{65}$$

[4] Additionally there are distinguished *non high symmetry* directions for which one single μ_λ may fulfill (59), corresponding to only one single outgoing wave.

[5] The dynamic Green’s function represents the time derivative of the dynamic response function.

For $u = 1$ the $\boldsymbol{v}_{1,2}$ actually have indefinite φ-dependence. But in order to define integral (64) *uniquely* for this case it is required to continue the vector functions $\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}_\lambda)$ ($\lambda = 1, 2$) analytically for $u_\lambda \to 1$ in such a way that integral (64) is a steady and unique function of u_λ. This assumption leads to a (non-unique) φ-dependence of the transverse polarization vectors. Without loss of generality they can be chosen as $\boldsymbol{v}_1 = \boldsymbol{\xi}(\varphi)$ and $\boldsymbol{v}_2 = \frac{d}{d\varphi}\boldsymbol{\xi}(\varphi)$.

Case I. We first discuss the case where the $\boldsymbol{v}_\lambda$ do not depend on φ: then we can write $\boldsymbol{I}_\lambda = \boldsymbol{v}_\lambda \otimes \boldsymbol{v}_\lambda \mathcal{A}_\lambda$ with $\mathcal{A}_\lambda$ given by

$$\begin{aligned} \mathcal{A}_\lambda &= \int_0^{2\pi} \frac{1}{(\alpha_\lambda + \beta_\lambda \cos 2\varphi)^2} \mathrm{d}\varphi \\ &= -\frac{d}{d\alpha_\lambda} \int_0^{2\pi} \frac{1}{(\alpha_\lambda + \beta_\lambda \cos 2\varphi)} \mathrm{d}\varphi \end{aligned} \tag{66}$$

where here it is denoted $\alpha_\lambda = \frac{1}{2}(\gamma_1^{(\lambda)} + \gamma_2^{(\lambda)})$ and $\beta_\lambda = \frac{1}{2}(\gamma_1^{(\lambda)} - \gamma_2^{(\lambda)})$. Only for $|\beta_\lambda| < |\alpha_\lambda|$ the integral is regular, which corresponds to our assumption of elliptic points. Evaluation of $\mathcal{A}_\lambda$ then yields

$$\begin{aligned} \mathcal{A}_\lambda &= 2\pi \frac{|\alpha_\lambda|}{(\alpha_\lambda^2 - \beta_\lambda^2)^{\frac{3}{2}}} \\ &= \pi \frac{|\gamma_1^{(\lambda)} + \gamma_2^{(\lambda)}|}{(\gamma_1^{(\lambda)} \gamma_2^{(\lambda)})^{\frac{3}{2}}} \\ &= 2\pi \frac{|H_\lambda|}{K_\lambda^{\frac{3}{2}}} \end{aligned} \tag{67}$$

In the last equation we have introduced the average curvature $H_\lambda = (\gamma_1^{(\lambda)} + \gamma_2^{(\lambda)})/2$ and the Gaussian curvature $K_\lambda = \gamma_1^{(\lambda)} \gamma_2^{(\lambda)}$. With this result we finally obtain by using equation (63), when we assume that $\hat{\boldsymbol{e}}_S$ corresponds *for all* $\lambda = 1, 2, 3$ to *elliptic* points (directions of propagation) the farfield approximation:

$$\begin{aligned} \boldsymbol{G}_\infty(r\hat{\boldsymbol{e}}_S, t) &= \boldsymbol{G}_\infty^t(r\hat{\boldsymbol{e}}_S, t) + \boldsymbol{G}_\infty^l(r\hat{\boldsymbol{e}}_S, t) \\ &= \frac{1}{4\pi\rho_M r} \sum_{\lambda=1}^{3} \frac{H_\lambda}{K_\lambda^{\frac{3}{2}}} \delta(t - \frac{r}{\mu_\lambda}) \hat{\boldsymbol{e}}_\lambda \otimes \hat{\boldsymbol{e}}_\lambda \end{aligned} \tag{68}$$

where $\hat{\boldsymbol{e}}_S$ indicates a high symmetry direction of the crystal from the source point. Here we have put $\hat{\boldsymbol{e}}_\lambda = \boldsymbol{v}_\lambda$. Without loss of generality we have $\boldsymbol{v}_3 = \hat{\boldsymbol{e}}_S$ denoting the longitudinal polarization vector corresponding to the wave propagating with sound velocity μ_3. Equation (68) describes three outgoing waves, one ($\boldsymbol{G}_\infty^l$) being longitudinally polarized ($\lambda = 3$) and two ($\boldsymbol{G}_\infty^t$) with transverse polarization ($\lambda = 1, 2$).

Case II. We now consider the case of $\boldsymbol{v}_\lambda$ being φ-dependent. Integral (64) then yields an expression of the form

$$\boldsymbol{G}_\infty(r\hat{\boldsymbol{e}}_S, t) = \boldsymbol{G}_\infty^t(r\hat{\boldsymbol{e}}_S, t) + \boldsymbol{G}_\infty^l(r\hat{\boldsymbol{e}}_S, t) \tag{69}$$

with

$$\begin{aligned} \boldsymbol{G}_\infty^t(r\hat{\boldsymbol{e}}_S, t) &= \frac{1}{8\pi\rho_M r} (\frac{1}{H_1^2} + \frac{1}{H_2^2}) [\mathbf{1} - \boldsymbol{e}_S \otimes \boldsymbol{e}_S] \delta(t - \frac{r}{\mu_t}) \\ \boldsymbol{G}_\infty^l(r\hat{\boldsymbol{e}}_S, t) &= \frac{1}{4\pi\rho_M r} \frac{1}{H_3^2} \boldsymbol{e}_S \otimes \boldsymbol{e}_S \delta(t - \frac{r}{\mu_l}) \end{aligned} \tag{70}$$

where we have again assumed that $\hat{\boldsymbol{e}}_S$ corresponds to elliptic points for all $\lambda = 1, 2, 3$. Expression $\boldsymbol{G}_\infty^t(r\hat{\boldsymbol{e}}_S, t)$ represents two transverse waves propagating with twofold degenerate sound velocity $\mu_t = \mu_1 = \mu_2$ and $\boldsymbol{G}_\infty^l(r\hat{\boldsymbol{e}}_S, t)$ describes one longitudinal polarized wave propagating with sound velocity $\mu_l = \mu_3$.

In both cases I and II the expressions (68) and (70) describe three singular outgoing waves (one longitudinal and two transverse) propagating with velocities $\mu_\lambda(\hat{\boldsymbol{e}}_S)$ and polarization-directions $\boldsymbol{v}_\lambda(\hat{\boldsymbol{e}}_S)$. Above results (68) and (70) represent the *radiation* of outgoing waves for the general anisotropic elasticity.

In the case of elastic isotropy expression (68) simplifies considerably since the velocities μ_λ then are constant and independent on the direction of propagation $\boldsymbol{e}_3$. Consequently then is $B_\lambda = 1$. When we perform the limiting transition to elastic isotropy, above expressions (68) and (69) coincide with the corresponding terms of δ-function type in the expression for the isotropic Green's function (eq. (75) below).

From above expressions (68) and (70) of the asymptotic elastodynamic Green's function in the space-time-domain we obtain immediately its representation in the space-frequency-domain

$$\boldsymbol{G}_\infty(\boldsymbol{r},\omega) = \int_{-\infty}^{+\infty} \exp(-i\omega t)\boldsymbol{G}_\infty(\boldsymbol{r},t)\mathrm{d}t \tag{71}$$

Thus we obtain for $\boldsymbol{G}_\infty(\boldsymbol{r},\omega)$ corresponding to (68) ff.

$$\boldsymbol{G}_\infty(r\boldsymbol{e}_S,\omega) = \frac{1}{4\pi\rho_M r}\sum_{\lambda=1}^{3}\frac{H_\lambda}{K_\lambda^{\frac{3}{2}}}\exp\left(-\frac{i\omega r}{\mu_\lambda[\boldsymbol{e}_3]}\right)\hat{\boldsymbol{e}}_\lambda\otimes\hat{\boldsymbol{e}}_\lambda \tag{72}$$

This expression represents the leading order in $1/r$ (farfield-approximation) of the Green's function in the space-frequency domain and the analogue result is obtained from eq. (70).

5 Application to the elastic isotropic medium

In this section we evaluate (43) for the case of an elastic isotropic medium. It is the only case that can be evaluated in full closed-form.

In order to determine the eigenvalues and eigenvectors of the tensor $T_{ij}(\boldsymbol{k}) = \mathcal{C}_{ipjq}k_pk_q$ we obtain for (14)

$$\frac{1}{\rho_M}\boldsymbol{T}(\boldsymbol{k}) = k^2\left[\frac{\mu}{\rho_M}(\mathbf{1}-\hat{\boldsymbol{k}}\otimes\hat{\boldsymbol{k}}) + \frac{(\lambda+2\mu)}{\rho_M}\hat{\boldsymbol{k}}\otimes\hat{\boldsymbol{k}}\right] \tag{73}$$

where λ,μ denote the Lame-constants of the medium. From equation (73) we observe one longitudinal eigenvector $\boldsymbol{v}_3(\boldsymbol{k}) = \hat{\boldsymbol{k}}$ with eigenvalue $\mu_3^2(\hat{\boldsymbol{k}}) = (\lambda+2\mu)/\rho_M$ and two transverse eigenvectors $\boldsymbol{v}_{1,2}(\boldsymbol{k})$ with the twofold degenerate eigenvalue $\mu_1^2(\hat{\boldsymbol{k}}) = \mu_2^3(\hat{\boldsymbol{k}}) = \mu/\rho_M$. As a consequence of (13) the transverse eigenvectors $\boldsymbol{v}_{1,2}(\boldsymbol{k})$ fulfill the property

$$\boldsymbol{v}_1(\boldsymbol{k})\otimes\boldsymbol{v}_1(\boldsymbol{k}) + \boldsymbol{v}_2(\boldsymbol{k})\otimes\boldsymbol{v}_2(\boldsymbol{k}) = \mathbf{1}-\hat{\boldsymbol{k}}\otimes\hat{\boldsymbol{k}} \tag{74}$$

where the right hand side of this equation denotes the projection operator into the two-dimensional transverse subspace. At this point we state that one longitudinal and two transverse eigenvectors of (73) correspond to the polarization directions of acoustic waves: The longitudinal polarized wave is propagating with phase-velocity $v_L = \sqrt{(\lambda+2\mu)/\rho_M}$ and its polarization-direction $\boldsymbol{v}_3 = \hat{\boldsymbol{k}}$ is parallel to its propagation-direction $\boldsymbol{k}$. Both two transverse polarized acoustic waves propagate with phase velocity $v_T = \sqrt{\mu/\rho_M}$ and have polarization vectors $\boldsymbol{v}_{1,2}$ in the plane perpendicular to the direction $\hat{\boldsymbol{k}}$ of propagation. In other words, for each wave vector $\boldsymbol{k}$ exist one longitudinal polarized and two transverse polarized acoustic waves. We emphasize an important property of the isotropic medium: the eigenvalues (wave propagation velocities) are independent on φ and u thus $\mu_\lambda(\hat{\boldsymbol{k}}) = const.$ Thus phase- and group velocities coincide for any $\boldsymbol{k}$-direction. Consequently no dispersion occurs as in the case of elastic anisotropy. This is the crucial point that allows us to perform the φ-integration of (43) to obtain the Green's function in full closed-form:

$$
\begin{aligned}
\mathbf{G}(\mathbf{r},t) &= -\frac{\Theta(t)}{4\pi\rho_M}\frac{t}{r^3}\left[1-3\frac{1}{r^2}\mathbf{r}\otimes\mathbf{r}\right] \\
&\quad \times\left\{\Theta(1-\frac{v_T t}{r})-\Theta(1-\frac{v_L t}{r})\right\} \\
&\quad +\frac{\Theta(t)}{4\pi\rho_M r}\left[\frac{\delta(t-\frac{r}{v_T})}{v_T^2}[\mathbf{1}-\frac{1}{r^2}\mathbf{r}\otimes\mathbf{r}]\right. \\
&\quad \left.+\frac{\delta(t-\frac{r}{v_L})}{v_L^2}\frac{1}{r^2}\mathbf{r}\otimes\mathbf{r}\right]
\end{aligned}
\tag{75}
$$

Here $v_L = \sqrt{(\lambda+2\mu)/\rho_M}$ and $v_T = \sqrt{\mu/\rho_M}$ indicate the longitudinal and transverse sound velocities, respectively. The Θ-term only contributes for $r/v_L < t < r/v_T$. The right-hand side term proportional to $\delta(r - v_T t)$ describes a singular wave surface having its displacement perpendicular to the direction of propagation (transverse polarized wave). The term proportional to $\delta(r - v_L t)$ describes a singular wave surface having its displacement parallel to the direction of propagation (longitudinal polarized wave). These two δ-terms correspond to the asymptotic expression of eq. (69) as $r \to \infty$ indicating the radiation of three waves. The Θ-term describes the contributions for space points $\boldsymbol{r}$ on a sphere r during the time period $r/v_L < t < r/v_T$ between the faster wave having longitudinal polarization and the slower waves with transverse polarization.

Expression (75) coincides with the known expression in the literature [17] which was derived there by employing other techniques.

6 Application to a medium with hexagonal symmetry

In this section we apply equation (43) to hexagonal symmetry (transversely isotropy) of the medium with special emphasis to the farfield solutions (eqs. (68) ff.). To that end we consider the wave propagation along the hexagonal high symmetry directions, namely along the c-axis and in the plane perpendicular to the c-axis. The wave propagation in the plane perpendicular to the c-axis is described by equation (68) whereas for wave propagation along the c-axis hold equations (69) ff. For hexagonal symmetry the tensor operator $T_{ij}(\boldsymbol{k}) = \mathcal{C}_{irjs}k_r k_s$ can conveniently be written as [9]

$$
\begin{aligned}
\boldsymbol{T}(\boldsymbol{k}) &= T_{b\perp}\boldsymbol{e}_{b\perp}\otimes\boldsymbol{e}_{b\perp} + T_b\boldsymbol{e}_b\otimes\boldsymbol{e}_b \\
&\quad + T_{bc}(\boldsymbol{e}_b\otimes\boldsymbol{e}_c + \boldsymbol{e}_c\otimes\boldsymbol{e}_b) + T_c\boldsymbol{e}_c\otimes\boldsymbol{e}_c
\end{aligned}
\tag{76}
$$

with

$$
\boldsymbol{k} = k_b\boldsymbol{e}_b + k_c\boldsymbol{e}_c \tag{77}
$$

The components $k_b = \sqrt{k_x^2+k_y^2}$ and $k_c = k_z$ of $\boldsymbol{k}$ denote the $\boldsymbol{k}$-components perpendicular and parallel to the hexagonal c-axis, respectively. The unit vectors $\boldsymbol{e}_b$, $\boldsymbol{e}_{b\perp}$ are perpendicular and $\boldsymbol{e}_c$ is parallel to the c-direction and are determined by the orientation of the $\boldsymbol{k}$-vector with respect to the c-axis:

$$
\begin{aligned}
\boldsymbol{e}_b &= \frac{1}{k_b}\begin{pmatrix} k_x \\ k_y \\ 0 \end{pmatrix} \qquad \boldsymbol{e}_{b\perp} = \frac{1}{k_b}\begin{pmatrix} -k_y \\ k_x \\ 0 \end{pmatrix} \\
\boldsymbol{e}_c &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
\tag{78}
$$

In (76) the quantities

$$
T_{b\perp} = \mathcal{C}_{66}k_b^2 + \mathcal{C}_{44}k_c^2 \tag{79}
$$

$$T_b = \mathcal{C}_{11}k_b^2 + \mathcal{C}_{44}k_c^2 \tag{80}$$

$$T_{bc} = (\mathcal{C}_{13} + \mathcal{C}_{44})\, k_b k_c \tag{81}$$

$$T_c = \mathcal{C}_{44}k_b^2 + \mathcal{C}_{33}k_c^2, \tag{82}$$

have been introduced. $\mathcal{C}_{11}, \mathcal{C}_{44}, \mathcal{C}_{66}, \mathcal{C}_{13}, \mathcal{C}_{33}$ denote the 5 independent hexagonal elastic constants in Voigt's notation.

The evaluation of (43) requires the determination of the eigenvalues and eigenvectors of $\boldsymbol{T}(\boldsymbol{k})$. From (76) we observe that obviously one eigenvalue $\mu_1^2(\boldsymbol{k})$ is given by

$$\mu_1^2(\boldsymbol{k}) = \frac{T_{b\perp}}{\rho_M} \tag{83}$$

with the corresponding eigenvector

$$\boldsymbol{v}_1(\boldsymbol{k}) = \boldsymbol{e}_{b\perp} \tag{84}$$

The remaining problem consists in determining the eigenvalues and eigenvectors of a 2×2 problem. The eigenvalues of this problem are given by the roots of the characteristic equation

$$\mu^4 - \mu^2 \frac{(T_b + T_c)}{\rho_M} + \frac{T_b T_c - T_{bc}^2}{\rho_M^2} = 0 \tag{85}$$

and we obtain the eigenvalues $\mu_{2,3}^2(\boldsymbol{k})$

$$\begin{aligned} \mu_2^2 &= \frac{T_b + T_c - \alpha}{2\rho_M} \\ \mu_3^2 &= \frac{T_b + T_c + \alpha}{2\rho_M} \end{aligned} \tag{86}$$

Here we have put

$$\alpha^2 = (T_c - T_b)^2 + 4T_{bc}^2 \tag{87}$$

The eigenvectors $\boldsymbol{v}_{2,3}$ can then be written as

$$\begin{aligned} \boldsymbol{v}_2 &= \frac{1}{\sqrt{2\alpha}}\{-\sqrt{\alpha - \beta}\boldsymbol{e}_b + \sqrt{\alpha + \beta}\boldsymbol{e}_c\} \\ \boldsymbol{v}_3 &= \frac{1}{\sqrt{2\alpha}}\{\sqrt{\alpha + \beta}\boldsymbol{e}_b + \sqrt{(\alpha - \beta)}\boldsymbol{e}_c\} \end{aligned} \tag{88}$$

where

$$\beta = (T_b - T_c) \tag{89}$$

Let us consider (43) for the case where the direction of the spatial vector $\boldsymbol{r}$ is *parallel* to the hexagonal c-axis:

$$\boldsymbol{r} = r\boldsymbol{e}_c \tag{90}$$

where $r = |z|$ and $\boldsymbol{e}_c = \boldsymbol{e}_z$. Note that in this case the quantities (79)-(82) are only functions of $u = \cos\vartheta$ but due to the hexagonal symmetry independent on angle φ in (64). We then have $\alpha + \beta = 0$ and $\alpha = T_c - T_b$. The sound velocities $\mu_1 = \mu_2$ coincide and correspond to the degenerate eigenvectors $\boldsymbol{v}_{1,2} = (\boldsymbol{e}_{b\perp}, -\boldsymbol{e}_b)$ which indicate transverse polarization. The longitudinal polarization direction $\boldsymbol{e}_c$ then corresponds to the sound velocity μ_3. This allows us to rewrite (43):

$$\boldsymbol{G}(\boldsymbol{r}, t) = -\Theta(t)\frac{1}{8\pi^2\rho_M r}\frac{d}{dt}\sum_{\lambda}^{3}\frac{\boldsymbol{I}_\lambda(u_\lambda)\Theta[1 - u_\lambda]}{[B_\lambda(u_\lambda)\mu_\lambda^2(u_\lambda)]} \tag{91}$$

where B_λ is given by (44). The basis vectors (78) assume the form $\boldsymbol{e}_b = (\cos\varphi, \sin\varphi, 0)$, $\boldsymbol{e}_{b\perp} = (-\sin\varphi, \cos\varphi, 0)$ and $\boldsymbol{e}_c = (0, 0, 1)$. To evaluate (91) we have to determine the tensors $\boldsymbol{I}_\lambda$ ($\lambda = 1, 2, 3$) that are given by the integrals

$$\boldsymbol{I}_\lambda = \int_0^{2\pi} \mathrm{d}\varphi \boldsymbol{v}_\lambda(u_\lambda, \varphi) \otimes \boldsymbol{v}_\lambda(u_\lambda, \varphi) \tag{92}$$

where the eigenvectors $\boldsymbol{v}_\lambda$ are given by (84) and (88), respectively. To obtain (91) we have used the following properties which are fulfilled for hexagonal symmetry only if $\boldsymbol{r}$ is parallel to the c-axis (i.e. $r = |z|$): only the eigenvectors $\boldsymbol{v}_\lambda$ for $\lambda = 1, 2$ depend on φ. The zeros of equations (40) $u_\lambda = u_\lambda(t/r)$ are only functions of t/r and the hexagonal material characteristics $\{\mathcal{C}_{ab}/\rho_M\}$ but are independent on φ. A derivation of a closed-form expression then is feasible. In that case the existing φ-integration problem consists only in evaluating the integrals (92) to arrive at

$$\begin{aligned}
\boldsymbol{I}_1 &= \pi[\mathbf{1} - \boldsymbol{e}_c \otimes \boldsymbol{e}_c] \\
\boldsymbol{I}_2 &= \frac{\pi}{2\alpha(u_\lambda)}[(\alpha(u_\lambda) - \beta(u_\lambda))\mathbf{1} + (\alpha(u_\lambda) + 3\beta(u_\lambda))\boldsymbol{e}_c \otimes \boldsymbol{e}_c] \\
\boldsymbol{I}_3 &= \frac{\pi}{2\alpha(u_\lambda)}[(\alpha(u_\lambda) + \beta(u_\lambda))\mathbf{1} + (\alpha(u_\lambda) - 3\beta(u_\lambda))\boldsymbol{e}_c \otimes \boldsymbol{e}_c]
\end{aligned} \tag{93}$$

Farfield Solutions. Our next interest is focused on the wave propagation at infinity ($r \to \infty$) along high symmetry directions of the hexagonal medium, i.e. the c-axis and the plane perpendicular to the c-axis. First we treat the propagation along the hexagonal c-axis ($\boldsymbol{r} = r\boldsymbol{e}_c$). Expression (69) writes

$$\boldsymbol{G}_\infty(r\boldsymbol{e}_c, t) = \boldsymbol{G}^t_\infty(r\boldsymbol{e}_c, t) + \boldsymbol{G}^l_\infty(r\boldsymbol{e}_c, t) \tag{94}$$

where the terms $\boldsymbol{G}^{t,l}_\infty(r\boldsymbol{e}_c, t)$ stand for transverse and longitudinal polarization directions, respectively. They are obtained from eq. (70)

$$\begin{aligned}
\boldsymbol{G}^t_\infty(r\boldsymbol{e}_c, t) &= \frac{1}{8\pi\rho_M r}(\frac{1}{H_1^2} + \frac{1}{H_2^2})[\mathbf{1} - \boldsymbol{e}_c \otimes \boldsymbol{e}_c]\delta(t - \frac{r}{\mu_t}) \\
\boldsymbol{G}^l_\infty(r\boldsymbol{e}_c, t) &= \frac{1}{4\pi\rho_M r}\frac{1}{H_3^2}\boldsymbol{e}_c \otimes \boldsymbol{e}_c\delta(t - \frac{r}{\mu_l})
\end{aligned} \tag{95}$$

where the H_λ denote the curvatures of the sound velocities μ_λ given in (67). The sound velocities μ_λ (where $\mu_t = \mu_1 = \mu_2$ and $\mu_l = \mu_3$) are obtained as

$$\begin{aligned}
\mu_t &= \sqrt{\frac{\mathcal{C}_{44}}{\rho_M}} \\
\mu_l &= \sqrt{\frac{\mathcal{C}_{33}}{\rho_M}}.
\end{aligned} \tag{96}$$

To determine H_λ we put

$$\boldsymbol{\xi} = \boldsymbol{e}_b \tag{97}$$

By using the representation of the gradient operator

$$\nabla_{\boldsymbol{k}} = \boldsymbol{e}_b\frac{\partial}{\partial k_b} + \frac{1}{k_b}\boldsymbol{e}_{b\perp}\frac{\partial}{\partial\varphi} + \boldsymbol{e}_c\frac{\partial}{\partial k_c} \tag{98}$$

we obtain (compare eqs. (60) and (27) and (28)) $\gamma_1^{(\lambda)} = \gamma_2^{(\lambda)} = H_\lambda$ which are defined by the expression $\boldsymbol{\xi}\boldsymbol{\gamma}^{(\lambda)}\boldsymbol{\xi}$. The average curvatures H_λ are obtained by performing the second order derivatives

$$H_\lambda = \frac{\partial^2}{\partial k_b^2}\mu_\lambda(k_b = 0, k_c = 1) \tag{99}$$

to arrive at

$$
\begin{aligned}
H_1 &= \frac{\mathcal{C}_{66}}{\sqrt{\rho_M \mathcal{C}_{44}}} \\
H_2 &= \frac{1}{\sqrt{\rho_M \mathcal{C}_{44}}}(\mathcal{C}_{11} - \frac{(\mathcal{C}_{13}+\mathcal{C}_{44})^2}{\mathcal{C}_{33}-\mathcal{C}_{44}}) \\
H_3 &= \frac{1}{\sqrt{\rho_M \mathcal{C}_{33}}}(\mathcal{C}_{44} + \frac{(\mathcal{C}_{13}+\mathcal{C}_{44})^2}{\mathcal{C}_{33}-\mathcal{C}_{44}})
\end{aligned}
\tag{100}
$$

Expression (95) describes outgoing waves propagating with sound velocities μ_i and being polarized perpendicular (for $i = 1, 2$) and parallel (for $i = 3$) to the hexagonal c-axis (propagation direction) corresponding to the transverse (for $i = 1, 2$) and longitudinal (for $i = 3$) polarization directions.
When performing the limiting transition to the isotropic medium $\mathcal{C}_{66}, \mathcal{C}_{44} \to \mu$, $\mathcal{C}_{11}, \mathcal{C}_{33} \to \lambda + 2\mu$ and $\mathcal{C}_{13} \to \lambda$ (λ, μ indicate Lame's constants), we see that expression (94) yields the δ-terms of the isotropic Green's function (75).

We now consider the wave propagation in the plane perpendicular to the hexagonal c-axis for $r \to \infty$. The space vector then is $\boldsymbol{r} = r\boldsymbol{e}_b$ where $\boldsymbol{e}_b$ is a unit vector in that plane defined by (78). In this case we put $\alpha - \beta = 0$ and $\beta = T_b - T_c$. The farfield expression now can be written as (eq. (68))

$$
\boldsymbol{G}_\infty(r\boldsymbol{e}_b, t) = \boldsymbol{G}^t_\infty(r\boldsymbol{e}_b, t) + \boldsymbol{G}^l_\infty(r\boldsymbol{e}_b, t) \tag{101}
$$

with $= \boldsymbol{G}^{t,l}_\infty(r\boldsymbol{e}_b, t)$ denoting the parts of transverse and longitudinal polarizations, respectively. Equation (101) now indicates the representations

$$
\begin{aligned}
\boldsymbol{G}^t_\infty(r\boldsymbol{e}_b, t) = & \\
& \frac{1}{4\pi\rho_M r}\left(\frac{H_1}{K_1^{\frac{3}{2}}}\delta(t - \frac{r}{\mu_1})\boldsymbol{e}_{b\perp} \otimes \boldsymbol{e}_{b\perp}\right. \\
& \left. + \frac{H_2}{K_2^{\frac{3}{2}}}\delta(t - \frac{r}{\mu_2})\boldsymbol{e}_c \otimes \boldsymbol{e}_c\right)
\end{aligned}
\tag{102}
$$

and

$$
\boldsymbol{G}^l_\infty(r\boldsymbol{e}_b, t) = \frac{1}{4\pi\rho_M r}\frac{H_3}{K_3^{\frac{3}{2}}}\delta(t - \frac{r}{\mu_3})\boldsymbol{e}_b \otimes \boldsymbol{e}_b \tag{103}
$$

where it is denoted (compare eqs. (67))

$$
\begin{aligned}
H_\lambda &= \tfrac{1}{2}(\gamma_1^{(\lambda)} + \gamma_2^{(\lambda)}) \\
K_\lambda &= \gamma_1^{(\lambda)}\gamma_2^{(\lambda)}
\end{aligned}
\tag{104}
$$

Now we only have to determine the quantities $\gamma_1^{(\lambda)}$ for $\boldsymbol{r} = r\boldsymbol{e}_b$. To that end we put (compare eqs. (60) and (27) and (28))

$$
\boldsymbol{\xi} = \xi_1\boldsymbol{e}_{b\perp} + \xi_2\boldsymbol{e}_c \tag{105}
$$

denoting the unit vectors perpendicular to $\boldsymbol{r}$ ($\xi_1^2 + \xi_2^2 = 1$). By using (98) we obtain for the γ_i (for $\lambda = 1, 2, 3$)

$$
\begin{aligned}
\gamma_1^{(\lambda)} &= \frac{1}{k_b}\frac{\partial}{\partial k_b}\mu_\lambda(k_b = 1, k_c = 0) \\
\gamma_2^{(\lambda)} &= \frac{\partial^2}{\partial k_c^2}\mu_\lambda(k_b = 1, k_c = 0)
\end{aligned}
\tag{106}
$$

to arrive at

$$\gamma_1^{(1)} = \sqrt{\frac{\mathcal{C}_{66}}{\rho_M}}$$
$$\gamma_2^{(1)} = \frac{\mathcal{C}_{44}}{\sqrt{\mathcal{C}_{66}\rho_M}} \tag{107}$$

$$\gamma_1^{(2)} = \sqrt{\frac{\mathcal{C}_{44}}{\rho_M}}$$
$$\gamma_2^{(2)} = \frac{1}{\sqrt{\rho_M \mathcal{C}_{44}}}(\mathcal{C}_{33} - \frac{(\mathcal{C}_{13}+\mathcal{C}_{44})^2}{\mathcal{C}_{11}-\mathcal{C}_{44}}) \tag{108}$$

$$\gamma_1^{(3)} = \sqrt{\frac{\mathcal{C}_{11}}{\rho_M}}$$
$$\gamma_2^{(3)} = \frac{1}{\sqrt{\rho_M \mathcal{C}_{11}}}(\mathcal{C}_{44} + \frac{(\mathcal{C}_{13}+\mathcal{C}_{44})^2}{\mathcal{C}_{11}-\mathcal{C}_{44}}) \tag{109}$$

The quantities $\gamma_i^{(1,2)}$ correspond to two transverse polarized waves which propagate with sound velocities $\mu_1 = \sqrt{\mathcal{C}_{66}/\rho_M}$ and $\mu_2 = \sqrt{\mathcal{C}_{44}/\rho_M}$ having polarization vectors $\boldsymbol{e}_{b\perp}$ and $\boldsymbol{e}_c$, respectively. $\gamma_i^{(3)}$ corresponds to one longitudinal polarized wave propagating with sound velocity $\mu_3 = \sqrt{\mathcal{C}_{11}/\rho_M}$ with polarization vector $\boldsymbol{e}_b$.

We observe again in the limiting transition to isotropy that expression (101) coincides with the δ-terms of the isotropic Green's function (75).

7 Conclusions

A general method for the construction of the dynamic 3×3 Green's function for the anisotropic elasticity for an infinitely extended medium has been derived and applied to the cases of elastic isotropy and hexagonal symmetry of the medium. The resulting representation of the elastodynamic Green's function of the anistropic infinite body (eq. (43)) reduces the initial integration problem of four integrations (compare eq. (9)) to *only one φ-integration* problem (eq. (43)) combined with the solution problem of three algebraic equations ((40) or (38)). As a consequence thereof the mathematical effort is considerably reduced and makes the method useful for many applications. The method leads in a direct and immediate way to closed-form farfield solutions for the transversely isotropic medium, as well as to the well known exact solution of the elastic isotropic medium. The obtained expression (43) is highly convenient for applications to various acoustic problems, e.g. acoustic wave propagation or scattering of acoustic waves at inhomogeneities. Especially useful for many applications are the derived expressions (68) ff. for the asymptotic behavior at infinity ($r \to \infty$) valid for any symmetry of the medium.

Appendix

Here we consider the important relation between the elastodynamic Green's function $\boldsymbol{G}(\boldsymbol{r},t)$ which represents the displacements due to a unit point force with δ-time dependence and the response function (susceptibility function) $\boldsymbol{H}(\boldsymbol{r},t)$ which represents the displacement response caused by a unit point force with Θ-time dependence. We furthermore consider the relation between the dynamic response function and the *elastostatic* Green's function. The dynamic response function $\boldsymbol{H}$ is defined by

$$[\boldsymbol{T}(\nabla) - \rho_M \frac{d^2}{dt^2}\mathbf{1}]\boldsymbol{H}(\boldsymbol{r},t) + \delta^3(\boldsymbol{r})\Theta(t)\mathbf{1} = 0 \tag{A.1}$$

We observe in view of eq. (7) when we perform the time derivative of above equation the relation

$$\boldsymbol{H}(\boldsymbol{r},t) = \frac{d\boldsymbol{G}}{dt}(\boldsymbol{r},t) \tag{A.2}$$

To work out the relation between $\boldsymbol{H}$ and the elastostatic Green's function $\boldsymbol{G}^{static}(\boldsymbol{r})$ we remember eq. (24)

$$g_\lambda(\boldsymbol{k},t) = \Theta(t)\exp(-\epsilon t)\frac{\sin\mu_\lambda t}{\mu_\lambda} \tag{A.3}$$

which fulfills equation (21)

$$\left\{\frac{d^2}{dt^2} + 2\epsilon\frac{d}{dt} + \mu_\lambda^2(\boldsymbol{k})\right\} g_\lambda(\boldsymbol{k},t) = \delta(t) \tag{A.4}$$

Integration of this equation with respect to time yields

$$\left\{\frac{d^2}{dt^2} + 2\epsilon\frac{d}{dt} + \mu_\lambda^2(\boldsymbol{k})\right\} h_\lambda(\boldsymbol{k},t) = \Theta(t) \tag{A.5}$$

where $h_\lambda(\boldsymbol{k},t)$ then is given by

$$h_\lambda(\boldsymbol{k},t) = \int_{-\infty}^{t} g_\lambda(\boldsymbol{k},\tau)\mathrm{d}\tau \tag{A.6}$$

and yields

$$h_\lambda(\boldsymbol{k},t) = \Theta(t)\frac{(1-\cos(\mu_\lambda t)\exp(-\epsilon t))}{\mu_\lambda^2} \tag{A.7}$$

For finite times t the exponential factor $\exp(-\epsilon t)$ can be put equal to 1. This factor becomes important only when we perform the limiting transition $t \to \infty$. Then the infinitesimal small but constant finite damping ϵ becomes important and causes the physically required relaxation for $t >> 1/\epsilon$. The functions h_λ which fulfill relation (A.5) and represent dynamic response functions (susceptibilities) of harmonic oscillators with eigenfrequencies μ_λ. Using eq. (A.7) we can construct the complete dynamic response function by time integration of (19) to arrive at

$$\boldsymbol{H}(\boldsymbol{k},t) = \frac{1}{\rho_M}\sum_{\lambda}^{3}\boldsymbol{v}_\lambda(\boldsymbol{k})\otimes\boldsymbol{v}_\lambda(\boldsymbol{k})h_\lambda(\boldsymbol{k},t) \tag{A.8}$$

with

$$\boldsymbol{H}(\boldsymbol{r},t) = \frac{1}{(2\pi)^3}\int\exp(i\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{H}(\boldsymbol{k},t)\mathrm{d}^3\boldsymbol{k} \tag{A.9}$$

The dynamic response function $\boldsymbol{H}$ fulfills (A.1) what can be seen also by time integration of eq. (7). Let us consider now the relation between the dynamic response function $\boldsymbol{H}$ and the static Green's function. We have the postulation that the relation (see also Norris [13])

$$\lim_{t\to\infty}\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{G}^{static}(\boldsymbol{r}) \tag{A.10}$$

is fulfilled due to physical relaxation. We show now by using (A.7) that (A.10) is indeed fulfilled: From equation (A.7) follows immediately (the exponential factor has now to be taken into account since it is assumed that the damping ϵ is a infinitesimal small but finite constant)

$$\lim_{t\to\infty} h_\lambda\left(\boldsymbol{r},t\right) = h_{\lambda\infty} = \frac{1}{\mu_\lambda^2} \tag{A.11}$$

This result also corresponds to the static solution of equation (A.5) and represents the static solution of the harmonic oscillator with eigenfrequency μ_λ which we obtain when we put all time derivatives in that equation equal to zero. Correspondingly the limiting case $t \to \infty$ of (A.8) becomes

$$\boldsymbol{H}(\boldsymbol{k},\infty) = \frac{1}{\rho_M}\sum_\lambda^3 \boldsymbol{v}_\lambda(\boldsymbol{k}) \otimes \boldsymbol{v}_\lambda(\boldsymbol{k}) \frac{1}{\mu_\lambda^2} = \boldsymbol{T}^{-1}(\boldsymbol{k}) \tag{A.12}$$

This expression coincides with the Fourier transform of the *static* Green's function. Hence we observe the demanded property (A.10) that $\boldsymbol{H}(\boldsymbol{r},\infty) = \boldsymbol{G}^{static}(\boldsymbol{r})$ by performing (A.9). Obviously $\boldsymbol{H}(\boldsymbol{r},\infty)$ fulfills the static equation

$$\boldsymbol{T}\left(\nabla\right)\boldsymbol{H}(\boldsymbol{r},\infty) + \delta^3\left(\boldsymbol{r}\right)\boldsymbol{1} = 0 \tag{A.13}$$

In view of the limiting case (A.12) we decompose the response function $\boldsymbol{H}$ into a dynamic and a static contribution

$$\boldsymbol{H}\left(\boldsymbol{r},t\right) = \boldsymbol{G}^{static}(\boldsymbol{r}) + \tilde{\boldsymbol{H}}(\boldsymbol{r},t) \tag{A.14}$$

with vanishing dynamic part $\tilde{\boldsymbol{H}}(\boldsymbol{r},t) \to 0$ for $t \to \infty$. This representation corresponds to those obtained by Norris [13], (eq. (3.23) therein), with

$$\tilde{\boldsymbol{H}}(\boldsymbol{r},t) = \frac{1}{(2\pi)^3}\int \exp\left(i\boldsymbol{k}\cdot\boldsymbol{r}\right)\tilde{\boldsymbol{H}}(\boldsymbol{k},t)\mathrm{d}^3\boldsymbol{k} \tag{A.15}$$

and in view of (A.7) and (A.8) with

$$\tilde{\boldsymbol{H}}(\boldsymbol{k},t) = -\Theta(t)\exp\left(-\epsilon t\right)\frac{1}{\rho_M}\sum_\lambda^3 \boldsymbol{v}_\lambda(\boldsymbol{k}) \otimes \boldsymbol{v}_\lambda(\boldsymbol{k}) \frac{\cos\left(\mu_\lambda t\right)}{\mu_\lambda^2} \tag{A.16}$$

The contribution $\tilde{\boldsymbol{H}}(\boldsymbol{r},t)$ yields (compare eq. (37))

$$\begin{aligned}\tilde{\boldsymbol{H}}(\boldsymbol{r},t) \;\; = \;\; & -\frac{\Theta(t)}{8\pi^2\rho_M}\sum_\lambda^3 \int_{|\hat{\boldsymbol{k}}|=1} \mathrm{d}\Omega(\hat{\boldsymbol{k}})\Big[\frac{1}{\mu_\lambda^2(\hat{\boldsymbol{k}})}\boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \otimes \boldsymbol{v}_\lambda(\hat{\boldsymbol{k}}) \\ & \times\delta(\hat{\boldsymbol{k}}\cdot\boldsymbol{r} - t\mu_\lambda(\hat{\boldsymbol{k}}))\Big]\end{aligned} \tag{A.17}$$

where we have now omitted the exponential damping factor. This leads to the final expression (compare (43)) for the dynamic part of the response function

$$\begin{aligned}\tilde{\boldsymbol{H}}(\boldsymbol{r},t) \;\; = \;\; & -\frac{\Theta(t)}{8\pi^2\rho_M r}\sum_\lambda^3\sum_{s_\lambda}^{m_\lambda}\int_0^{2\pi}\mathrm{d}\varphi\,\Theta_{s_\lambda}(\varphi) \\ & \times\Big[\frac{\Theta\{1-u_\lambda^{(s_\lambda)}(\varphi,\frac{t}{r})\}}{B_\lambda^{(s_\lambda)}[\hat{\boldsymbol{k}}_\lambda^{(s_\lambda)}(\varphi,\frac{t}{r})]\mu_\lambda^2[\hat{\boldsymbol{k}}_\lambda^{(s_\lambda)}(\varphi,\frac{t}{r})]} \\ & \times\boldsymbol{v}_\lambda[\hat{\boldsymbol{k}}_\lambda^{(s_\lambda)}(\varphi,\frac{t}{r})] \otimes \boldsymbol{v}_\lambda[\hat{\boldsymbol{k}}_\lambda^{(s_\lambda)}(\varphi,\frac{t}{r})]\Big]\end{aligned} \tag{A.18}$$

where the quantities herein are defined next to eq. (43). We note that above expression is related to eq. (43) by time integration.

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