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Check on a nonlocal Maxwellian theory of sound propagation in fluid-saturated rigid-framed porous media

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HIGHLIGHTS

- A nonlocal theory of sound propagation in porous media is validated.
- Porous media are rigid-framed and saturated with a viscothermal fluid.
- The validation geometry is taken to be cylindrical circular tubes.
- Results of nonlocal theory match with those of Kirchhoff's exact solutions.
- Results based on Zwikker and Kosten's local theory are also presented.

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ABSTRACT

A macroscopic nonlocal theory of sound propagation in homogeneous rigid-framed porous media permeated with a viscothermal fluid has been recently proposed in this journal. It accounts for the first time for the full temporal and spatial dispersion effects, independently of the nature of the microgeometry. In this paper this new Maxwellian theory is validated in the case of sound propagation in cylindrical circular tubes, by showing that it matches exactly the long-known direct Kirchhoff–Langevin's solutions.

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1. Introduction

In a recent paper [1] a fundamental and new nonlocal macroscopic acoustic theory has been developed, describing in a typically Maxwellian manner the phenomenon of linear sound propagation and attenuation in a macroscopically homogeneous rigid-framed porous medium permeated with a viscothermal fluid, which is governed by the classical Navier–Stokes–Fourier equations of near-equilibrium fluid mechanics [2] at the pore scale. Assuming for simplicity isotropy or propagation along a principal axis, this macroscopic theory introduces a macroscopic acoustic velocity field $\mathbf{V} = \langle \mathbf{v} \rangle$, where $\langle \rangle$ refers to a macroscopic averaging operation, and a macroscopic pressure stress field, denoted H .¹ These two fields appeared to play roles similar to those of the electric and magnetic macroscopic fields, $\mathbf{E} = \langle \mathbf{e} \rangle$ and \mathbf{H} , in electromagnetic

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¹ In the absence of the mentioned symmetries, H becomes a symmetric tensor. The theory and procedures described in [1] can be generalized without any difficulty. The permittivities become symmetric tensor operators.

wave propagation in a medium [3]. The macroscopic pressure field H , in general different from the direct macroscopic average of the pore-scale pressure field, is defined such as the ‘Umov’ product HV represents the ‘acoustic part of the energy current density’, by analogy with a similar interpretation of its counterpart in electromagnetics [4], the ‘Poynting–Heaviside’ product $\mathbf{E} \times \mathbf{H}$. Gaining in this manner the character of generalized susceptibility functions [5], [1, Section 2.4], the corresponding two acoustic permittivities are determined in terms of two independent action–response problems.

The resulting theory, which must be valid in principle for all microgeometries of the porous medium, and fully takes – for the first time – into account both temporal and spatial dispersion effects, lends itself to direct verifications in various geometries. We give here a check on, in one of the simplest workable test cases: that of the propagation in a medium having parallel cylindrical circular pores of radius R . The possible mode solutions, in the fluid-filled pores, are known since the classical work of Kirchhoff [6] and Langevin [7,8].

We first describe in Section 2, how the general macroscopic acoustic nonlocal theory derived in [1], give in the above simple geometry, definite predictions for the possible wavenumbers and impedances of the axisymmetric modes, irrespective of the frequencies. In Sections 3 and 4, we present the details of the direct Kirchhoff–Langevin’s and alternative nonlocal macroscopic Maxwellian computations of these quantities. In Section 5, we show numerically that the two approaches give identical results, irrespective of the frequency range. This provides a clear validation of the new nonlocal theory.

2. General ideas in nonlocal theory, and application to the special case of cylindrical pores

We note that there are usually two different ways to perform the smoothing–out represented by the averaging symbol $\langle \cdot \rangle$: ensemble averaging or volume averaging, which may be associated to the names of J.W. Gibbs and H.A. Lorentz, respectively. In the Gibbs conception the macroscopic theories do not refer to what happens in a given sample of the medium. They describe what happens, in an averaged sense, in an ensemble of realizations of the medium. The macroscopic mean operation $\langle \cdot \rangle$ then designate expectation values performed at a given time and spatial position, over the complete set of realizations. In the Lorentz conception, the macroscopic theories do refer to a given sample. The averaging operation $\langle \cdot \rangle$ is performed by integrating over an averaging volume, at a given time. This volume should be taken sufficiently large to smooth out the irregularities, which requires in general, satisfying some separation conditions on different length scales. Namely, a ‘Lorentz’ macroscopic wave theory will be well-defined only when the macroscopic medium spatial variations occur at a scale sufficiently large compared to the wavelengths, themselves sufficiently large compared to the volume-averaged scale; for a macroscopically uniform medium the former condition is automatically satisfied and it is sufficient that the wavelengths be large compared to the size of the averaging volume.

We recall now the particular macroscopic acoustic issue we considered in [1], and the way we proposed to solve it. Within Gibbs’s conception the whole space is assumed to be divided in two complementary phase regions which depend on the realization ω : the void (pore) region $\mathcal{V}_f(\omega)$ which is a connected region fully permeated with a single homogeneous viscothermal fluid, and the complementary solid-phase region $\mathcal{V}_s(\omega)$, set to remain motionless and at ambient temperature. The pore-wall region is denoted by $\partial\mathcal{V}(\omega)$. Macroscopic homogeneity is assumed.

A small-amplitude acoustic wave disturbance propagates in the fluid in the macroscopic direction x , a principal axis of the medium. At the pore level and for each realization the following classical Navier–Stokes–Fourier linear fluid-mechanics equations can be written in $\mathcal{V}_f(\omega)$ (see Eqs. (1) and notations in [1]; $b = \rho' / \rho_0$ is the condensation, τ the excess temperature)

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \eta \nabla^2 \mathbf{v} + \left(\zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot \mathbf{v}) \tag{1a}$$

$$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{v} = 0 \tag{1b}$$

$$\gamma \chi_0 p = b + \beta_0 \tau \tag{1c}$$

$$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \nabla^2 \tau \tag{1d}$$

with boundary conditions

$$\mathbf{v} = 0, \quad \tau = 0 \tag{2}$$

on $\partial\mathcal{V}(\omega)$.

Extending to zero the fields in the solid region $\mathcal{V}_s(\omega)$, and introducing the following macroscopic averaged fields (\mathbf{e}_x is the unit vector along axis x):

$$V \mathbf{e}_x \equiv \langle \mathbf{v} \rangle, \quad \text{and} \quad B \equiv \langle b \rangle \tag{3}$$

we argued in [1] that there are ‘Maxwell’ acoustic density fields $\mathbf{D} = D \mathbf{e}_x$ and H , and susceptibilities operators $\hat{\rho}$ and $\hat{\chi}$, such that the following macroscopic nonlocal ‘Maxwellian’ equations hold true:

field equations:

$$\frac{\partial B}{\partial t} = -\frac{\partial V}{\partial x}, \quad \frac{\partial D}{\partial t} = -\frac{\partial H}{\partial x} \tag{4}$$

constitutive relations:

$$D(t, x) = \hat{\rho}V(t, x) = \int_{-\infty}^t dt' \int dx' \rho(t - t', x - x')V(t', x') \tag{5a}$$

$$H(t, x) = \hat{\chi}^{-1}B(t, x) = \int_{-\infty}^t dt' \int dx' \chi^{-1}(t - t', x - x')B(t', x'). \tag{5b}$$

As discussed in [3,9] in the electromagnetic case, such macroscopic difference-kernel constitutive equations are the most general ones that can be written. They only express the spatial and time homogeneity of the material properties, and the fact that, since V and B are not completely independent but related by the first Eq. (4), it is useless to add a convolution B term in (5a) or a convolution V term in (5b). Regarding these Eqs. (4) and (5), the only question then to be asked is: how are the kernel functions $\rho(t, x)$ and $\chi(t, x)$, or their Fourier coefficients $\rho(\omega, k)$ and $\chi(\omega, k)$, determined by the microgeometry? The answer to this question has been conjectured in [1] to be simple, provided the following fundamental thermodynamic definition of the H field is applied:

$$H(\mathbf{v}) = \langle p\mathbf{v} \rangle. \tag{6}$$

It refers to Schoch’s interpretation of the quantity $p\mathbf{v}$ as an ‘acoustic part of energy current density’ in the fluid [10], implying that the product $H\mathbf{V}$ defined by (6) possesses the same interpretation at the macroscopic level. It makes the macroscopic acoustic stress field H akin to the macroscopic magnetic field \mathbf{H} , which defines through the product $\mathbf{E} \times \mathbf{H}$ an ‘electromagnetic part of energy current density’. Making appropriate use of this definition (6) (see [1, Section 4]), the constitutive Fourier coefficients $\rho(\omega, k)$ and $\chi(\omega, k)$ have been conjectured to be directly related to the macroscopic response of the permeating fluid subject to a harmonic fictitious pressure term $\mathcal{P}(t, x) = \mathcal{P}_0 e^{i(k\mathbf{e}_x \cdot \mathbf{x} - \omega t)}$ added to the pressure, either in the Navier–Stokes equation (1a), or the Fourier equation (1d). Once the functions $\rho(\omega, k)$ and $\chi(\omega, k)$ are determined, the normal wave wavenumbers along the x -axis in the medium k_l , with $l = 1, 2, \dots$, can be computed as the possible solutions of the ‘Maxwell’ dispersion equation

$$\rho(\omega, k)\chi(\omega, k)\omega^2 = k^2 \tag{7}$$

directly obtained from Eqs. (4) and (5) by passing over to the Fourier transform. For a given solution k_l , and thanks to the fundamental identification (6) of the H field, the associated characteristic impedance Z_l can be defined by setting $Z_l = H/V$. Again passing over to the Fourier transform of Eqs. (4) and (5), there follows the expression of the characteristic impedance

$$Z_l = \sqrt{\rho(\omega, k_l)\chi^{-1}(\omega, k_l)}. \tag{8}$$

We notice that these normal wave quantities k_l and Z_l describe, within the macroscopic Gibbs viewpoint, ‘coherent fields’ in the language of the multiple-scattering theories; they do not involve a scale separation condition for their definition. As such, they remain meaningful irrespective of the frequencies. In particular, and because the set of random translations of a given periodic medium leads to an homogeneous Gibbs medium, the nonlocal theory is capable to predict the wavenumbers of the periodic medium Bloch modes and define their characteristic impedances, irrespective of the frequencies.

Within Lorentz’s conception the macroscopic wave propagation is formulated in a similar, but restricted manner. Here, we are given a unique sample of the medium, which is macroscopically homogeneous in a volume-averaged sense. The Eqs. (1) and (2) are respectively written in the corresponding \mathcal{V}_f , and on its boundary $\partial\mathcal{V}$. The volume averaging $\langle \rangle$ is best performed in Russakoff’s manner by convolution with a test function—a signal analysis refinement of Lorentz’s ball average [11–13]. The macroscopic equations (3)–(8) and upscaling action–response procedures determining the functions $\rho(\omega, k)$ and $\chi(\omega, k)$ are the same, but these equations and procedures now make sense only as far as the spatial variations of the fields respect the long-wavelength separation condition.

In what follows we consider an especially simple geometry that falls under the scope of both Gibbs’s and Lorentz’s conception. To establish in a simple, yet nontrivial, manner the nonlocal theory’s upscaling procedures, determining $\rho(\omega, k)$ and $\chi(\omega, k)$ and stated in [1, Section 4], we consider a medium having parallel cylindrical circular pores of radius R and axis x . Through the solution of the ‘Maxwell’ dispersion equation (7) and the insertion of the solution in Eq. (8), the nonlocal theory should predict all possible wavenumbers k_l and characteristic impedances Z_l in the medium, without any limitation on frequencies.² As the different pores are equivalent, they carry the same wave fields, and the averaging operation $\langle \rangle$, either Gibbs or Lorentz, is expressed in terms of a cross-section average $\langle \rangle_p$ performed over one single pore: $\langle \rangle = \phi \langle \rangle_p$ with ϕ the porosity. Therefore, the wavenumbers k_l should coincide with those predicted by Kirchhoff–Langevin theory of sound propagation in a fluid-filled cylindrical circular tube. Apart from the porosity factor, the associated Z_l should also coincide with those predicted by Kirchhoff–Langevin theory when substituting $\langle \rangle_p$ for $\langle \rangle$ in Eq. (8) and making the cross-section averaging. In what follows we make abstraction of the irrelevant porosity factor ϕ and omit the index p .

² Here, we have no frequency limitation in the Lorentz conception, because the averaging volume – a cross section – is of zero thickness, ensuring the separation condition. As regards the Gibbs conception it runs as follows. Within the ‘total volume’ normalization of the Gibbs average $\langle \rangle$, as defined in [1] Section 3.1, we imagine that we are given an ensemble Ω of samples ω , each obtained by random translation in two radial directions, of one reference sample ω_0 . This sample presents identical parallel pores of radius R , which are directed in the axial direction x , distributed in periodic manner in the two radial directions, and permeated with the fluid. The ensemble-average operation over the space Ω , becomes the cross-section average $\langle \rangle$ in a single pore, multiplied by the porosity. In fact, the fields relating to the harmonic action–response problems considered in [1] have the form $a(\omega, k, r) e^{-i\omega t + ikx}$, with r the radial coordinate in one pore and with amplitudes $a(\omega, k, r)$ independent of x , and thereby, needed to be averaged only over a cross-section.

3. Kirchhoff–Langevin theory: computation of the wavenumbers and impedances in cylindrical circular tubes

The issue of sound propagation in a hollow solid cylindrical circular tube filled by the air has been solved long ago by G. Kirchhoff [6], accounting for the effects of viscosity and thermal exchanges. The boundary conditions (2) have been applied on the tube wall ($r = R$) and justified by the neglect of solid wall vibrations, and the very large heat capacity and heat conduction coefficient of the solid part compared to the fluid part. In Kirchhoff’s treatment, the air is an ideal gas (the thermal expansion coefficient β_0 is replaced by $1/T_0$) and the second viscosity is neglected. P. Langevin [7,8], later completed Kirchhoff theory by considering the full Navier–Stokes–Fourier linear equations (1). We recall here Kirchhoff–Langevin theory in some detail, in order to use it later for Maxwellian developments.

We first substitute the state Eq. (1c) in the Fourier equation (1d) and use the general thermodynamic identity (Eq. (25) in [1]) to obtain the following alternative form of (1d)

$$\frac{\partial \tau}{\partial t} = \frac{\gamma - 1}{\beta_0} \frac{\partial b}{\partial t} + \frac{\kappa}{\rho_0 c_v} \nabla^2 \tau. \tag{9}$$

Following Rayleigh’s presentation [14], using a temperature variable $\tau' = \beta_0 \tau / (\gamma - 1)$, and introducing the reference adiabatic c_a and isothermal c_i sound speed, which verify $\rho_0 \chi_0 c_a^2 = 1$, $c_a^2 = \gamma c_i^2$, the Eqs. (1c) and (9) are written in simpler forms

$$\frac{p}{\rho_0} = c_i^2 b + (c_a^2 - c_i^2) \tau' \tag{10}$$

and

$$\frac{\partial \tau'}{\partial t} = \frac{\partial b}{\partial t} + \frac{\kappa}{\rho_0 c_v} \nabla^2 \tau'. \tag{11}$$

Assuming that the variables \mathbf{v} , b , p , τ' , vary with time like $e^{-i\omega t}$, and eliminating the pressure and condensation, give rise to the following velocity–temperature equations

$$-i\omega \mathbf{v} - \frac{\eta}{\rho_0} \nabla^2 \mathbf{v} = -\nabla X \tag{12a}$$

$$X = \left[c_a^2 - \frac{\eta}{\rho_0} + \frac{\zeta}{\rho_0} - i\omega \right] \tau' + \frac{\kappa}{\rho_0 c_v i\omega} \left[c_i^2 - \frac{\eta}{\rho_0} + \frac{\zeta}{\rho_0} \right] \nabla^2 \tau' \tag{12b}$$

$$\nabla \cdot \mathbf{v} - i\omega \tau' - \frac{\kappa}{\rho_0 c_v} \nabla^2 \tau' = 0. \tag{12c}$$

Eliminating the velocity, results in the excess temperature wave equation

$$\omega^2 \tau' + \left[c_a^2 - i\omega \left(\frac{\kappa}{\rho_0 c_v} + \frac{4\eta}{3\rho_0} + \zeta \right) \right] \nabla^2 \tau' + \frac{\kappa}{\rho_0 c_v i\omega} \left[c_i^2 - i\omega \frac{4\eta}{3\rho_0} + \zeta \right] \nabla^2 \nabla^2 \tau' = 0. \tag{13}$$

Let λ_1 and λ_2 be the two, small and large, solutions of the associated Kirchhoff–Langevin free-fluid characteristic equation

$$\omega^2 + \left[c_a^2 - i\omega \left(\frac{\kappa}{\rho_0 c_v} + \frac{4\eta}{3\rho_0} + \zeta \right) \right] \lambda + \frac{\kappa}{\rho_0 c_v i\omega} \left(c_i^2 - i\omega \frac{4\eta}{3\rho_0} + \zeta \right) \lambda^2 = 0. \tag{14}$$

The small solution – mainly real – describes propagating acoustic waves with small absorption, and the large solution – purely imaginary – highly damped diffusive entropic waves. The temperature variable τ' solution to (13) will write

$$\tau' = A_1 \varphi_1 + A_2 \varphi_2 \tag{15}$$

with the functions φ_1 and φ_2 verifying: $\nabla^2 \varphi_1 = \lambda_1 \varphi_1$ and $\nabla^2 \varphi_2 = \lambda_2 \varphi_2$. The velocity \mathbf{v} will write $\mathbf{v} = \mathbf{v}' + B_1 \nabla \varphi_1 + B_2 \nabla \varphi_2$, with \mathbf{v}' the vortical part, such that

$$\nabla^2 \mathbf{v}' = \frac{-i\omega \rho_0}{\eta} \mathbf{v}', \quad \nabla \cdot \mathbf{v}' = 0. \tag{16}$$

The relations $B_{1,2} = (\kappa / \rho_0 c_v + i\omega / \lambda_{1,2}) A_{1,2}$ between coefficients B and A follow from (12c).

Considering now the axisymmetric fields propagating in the right-going x direction in a cylindrical tube of radius R filled with the visco-thermal fluid, we want to determine normal modes proportional to $e^{+ik_l x}$, as functions of radial coordinate r , where the k_l ’s, $\Im(k_l) > 0$, $l = 1, 2, \dots$, are complex wavenumbers to be specified as functions of frequency and tube radius. In what follows, for convenience, the index l labeling the different axisymmetric mode solutions will be omitted.

For these mode solutions the operator ∇ , resp. ∇^2 , can be replaced by $ik_l \mathbf{e}_x + \mathbf{e}_r \partial / \partial r$ with \mathbf{e}_r the radial unit vector, resp. $\partial^2 / \partial r^2 + \partial / r \partial r - k^2$. The different fields $a(t, \mathbf{x})$ can be replaced by their amplitudes a such that $a(t, \mathbf{x})$

$= a(r)e^{-i\omega t + ikx}$. By elementary calculations, there follows that the corresponding amplitudes φ_1 and φ_2 will be described by Bessel functions $\varphi_{1,2} = J_0\left(r\sqrt{-\lambda_{1,2} - k^2}\right)$. Writing the vortical velocity \mathbf{v}' in the form $\mathbf{v}' = u'\mathbf{e}_x + q'\mathbf{e}_r$, with axial and radial amplitudes u' and q' independent of azimuthal angle, it is easy to see that Eqs. (16) imply u' is the solution to $(\partial^2/\partial r^2 + \partial/r\partial r)u' = (-i\omega/\nu + k^2)u'$, where $\nu \equiv \eta/\rho_0$ is the kinematic viscosity and q' is determined by the relation: $q' = [-ik/(-i\omega/\nu + k^2)](\partial u'/\partial r)$. As a result: $u' = A\varphi$, where φ is the Bessel function $\varphi = J_0\left(r\sqrt{i\omega/\nu - k^2}\right)$, and $q' = A[-ik/(-i\omega/\nu + k^2)](\partial\varphi/\partial r)$. Gathering these expressions and writing the total velocity as $\mathbf{v} = u\mathbf{e}_x + q\mathbf{e}_r$, we find that u and q are given by

$$u = A\varphi + ik\left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1}\right)A_1\varphi_1 + ik\left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2}\right)A_2\varphi_2 \tag{17a}$$

$$q = \frac{-ikA}{\frac{-i\omega}{\nu} + k^2} \frac{\partial\varphi}{\partial r} + \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1}\right)A_1 \frac{\partial\varphi_1}{\partial r} + \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2}\right)A_2 \frac{\partial\varphi_2}{\partial r}. \tag{17b}$$

Because of the vanishing of velocity and excess temperature at the tube wall, the expressions (17a)–(17b) and (15) vanish at $r = R$, giving three homogeneous equations which have nontrivial solutions only if their determinant is zero. In this manner the following dispersion equation is obtained:

$$\left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1}\right) \frac{1}{\varphi_{1w}} \frac{\partial\varphi_1}{\partial r_w} - \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2}\right) \frac{1}{\varphi_{2w}} \frac{\partial\varphi_2}{\partial r_w} - \frac{k^2}{\frac{-i\omega}{\nu} + k^2} \left(\frac{i\omega}{\lambda_1} - \frac{i\omega}{\lambda_2}\right) \frac{1}{\varphi_w} \frac{\partial\varphi}{\partial r_w} = 0. \tag{18}$$

The index w indicates that the functions and derivatives are evaluated at the tube wall. Eq. (18) is the well-known Kirchhoff–Langevin dispersion equation. It has a series of discrete complex wavenumber solutions $k_l(\omega)$, $\Im(k_l) > 0$, $l = 1, 2, \dots$, which can be sorted by convention in ascending order of the values of $\Im(k_l)$: $\Im(k_1) < \Im(k_2) < \dots$. To determine these numbers $k_l(\omega)$, a simple Newton scheme can be used.

In the lossless limit $\eta, \zeta, \kappa \rightarrow 0$, (18) simplifies and tends to the equation $J_1\left(R\sqrt{\omega^2/c_a^2 - k^2}\right) = 0$, yielding the following k_{0l} starting values for the Newton scheme: $k_{0l}^2 = \omega^2/c_a^2 - x_l^2/R^2$, where $x_l \geq 0$ are the successive zeros of the function $J_1(x)$. As $J_1(0) = 0$, the first zero is $x_1 = 0$, and the first starting value is $k_{01} = \omega/c_a$. Without losses, at a given frequency, there are one (k_{01}), or more, real positive solutions, describing right-going propagating waves, followed by an infinite discrete set of purely imaginary solutions with $\Im(k_{0l}) > 0$, describing evanescent waves. In general, a few Newton iterations are sufficient to make these starting lossless purely real or purely imaginary solutions k_{0l} (to which may be added small imaginary or real parts), converge towards the complex solutions k_l . As we consider the waves propagating in the direction $+x$, which can be created by a source placed on the left and thus attenuate on the right, we select the solutions with positive imaginary parts, which dampen like $e^{-\Im(k_l)x}$.

There remains to see how to define and compute the associated impedances $Z_l(\omega)$. For a given mode solution l , the fields depend on only one independent arbitrary complex amplitude constant \mathcal{A} . It corresponds to the arbitrary choice of the sound pressure level and phase. Writing the two independent conditions expressing the vanishing of τ' and u at the tube wall, we find, between the amplitudes A, A_1 and A_2 , relations having the form: $A = \mathcal{A}ik(i\omega/\lambda_1 - i\omega/\lambda_2)\varphi_{1w}\varphi_{2w}$, $A_1 = -\mathcal{A}\varphi_w\varphi_{2w}$ and $A_2 = \mathcal{A}\varphi_w\varphi_{1w}$. The fields then write as follows

$$\frac{u}{ik\mathcal{A}} = \left(\frac{i\omega}{\lambda_1} - \frac{i\omega}{\lambda_2}\right)\varphi_{1w}\varphi_{2w}\varphi - \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1}\right)\varphi_w\varphi_{2w}\varphi_1 + \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2}\right)\varphi_w\varphi_{1w}\varphi_2 \tag{19a}$$

$$\frac{q}{\mathcal{A}} = \left(\frac{i\omega}{\lambda_1} - \frac{i\omega}{\lambda_2}\right)\varphi_{1w}\varphi_{2w} \frac{k^2}{\frac{-i\omega}{\nu} + k^2} \frac{\partial\varphi}{\partial r} - \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1}\right)\varphi_w\varphi_{2w} \frac{\partial\varphi_1}{\partial r} + \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2}\right)\varphi_w\varphi_{1w} \frac{\partial\varphi_2}{\partial r} \tag{19b}$$

$$\frac{b}{\mathcal{A}} = -\left(1 + \frac{\kappa}{\rho_0 c_v i\omega}\lambda_1\right)\varphi_w\varphi_{2w}\varphi_1 + \left(1 + \frac{\kappa}{\rho_0 c_v i\omega}\lambda_2\right)\varphi_w\varphi_{1w}\varphi_2 \tag{19c}$$

$$\frac{p}{-\mathcal{A}\rho_0} = \left(c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega}\lambda_1\right)\varphi_w\varphi_{2w}\varphi_1 + \left(c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega}\lambda_2\right)\varphi_w\varphi_{1w}\varphi_2 \tag{19d}$$

$$\frac{\tau'}{\mathcal{A}} = -\varphi_w\varphi_{2w}\varphi_1 + \varphi_w\varphi_{1w}\varphi_2. \tag{19e}$$

The expressions on the right determine normalized modal field patterns which are fixed as soon as one substitutes for k , one of its possible values k_l .

As we have insisted, in the proposed nonlocal macroscopic theory, we define a macroscopic pressure field H by means of the Eq. (6). Here, as we identify the mean operation $\langle \cdot \rangle$ with a cross-section average, for a given mode, the field H is computed by inserting the fields (19a) and (19d) in the equation $H\langle u \rangle = \langle pu \rangle$, where $\langle f \rangle = \frac{1}{\pi R^2} \int_0^R dr \int_0^{2\pi} rd\vartheta f$. In this

manner, the characteristic complex impedance $Z_l(\omega)$ is unambiguously defined by setting

$$H = Z_l \langle u \rangle, \quad \text{i.e. } Z_l = \frac{\langle pu \rangle}{\langle u \rangle^2}. \tag{20}$$

Using the known formulae giving the cross-section average $\langle \cdot \rangle$ of Bessel functions φ , φ_1 and φ_2 and their products, the impedance factor $Z_l(\omega)$ is a function of ω parametrized by $k = k_l$, which writes analytically as a ratio of sums containing 6 terms, each made up of a product of 6 Bessel's functions multiplied by expressions involving ω , k_l^2 , λ_1 and λ_2 .

Knowing the complex wavenumber $k_l(\omega)$ and impedance $Z_l(\omega)$ of a given radial mode solution l , an equivalent-fluid complex density $\rho_l(\omega)$ and bulk modulus $\chi_l^{-1}(\omega)$ might be defined through setting relations having the usual form (see e.g. [15]) $k_l = \omega/c_l = \omega\sqrt{\rho_l\chi_l}$ and $Z_l = \rho_l c_l = \sqrt{\rho_l\chi_l^{-1}}$, i.e.

$$\rho_l(\omega) = \frac{k_l}{\omega} Z_l = \frac{k_l \langle pu \rangle}{\omega \langle u \rangle^2}, \quad \chi_l^{-1}(\omega) = \frac{\omega}{k_l} Z_l = \frac{\omega \langle pu \rangle}{k_l \langle u \rangle^2}. \tag{21}$$

Like the impedance factors $Z_l(\omega)$ in (20), the densities and bulk moduli $\rho_l(\omega)$ and $\chi_l^{-1}(\omega)$ in (21) are the expression of closed-form functions of ω parametrized by $k_l, F(\omega, k_l)$, directly computable once k_l is evaluated.

4. Nonlocal Maxwellian theory: computations of the same wavenumbers and impedances

We describe now, the alternative computation of these same quantities $k_l(\omega)$, $\rho_l(\omega)$, $\chi_l(\omega)$, and $Z_l(\omega)$, that must be possible as explained in Section 2, using the method of the macroscopic nonlocal Maxwellian theory [1]. We need first to determine the nonlocal susceptibility functions $\rho(\omega, k)$ and $\chi(\omega, k)$.

4.1. Determination of the nonlocal density $\rho(\omega, k)$

To compute $\rho(\omega, k)$, we consider the response of the fluid subjected to the action of an external driving bulk force \mathbf{f} (per unit volume), which derives from a fictitious harmonic pressure waveform inserted in the Navier–Stokes equation (see Sections 2.9 and 4 in [1]). Thus we consider solving the action–response problem given by the Eqs. (1) written in the tube $r < R$, with external driving bulk force $\mathbf{f} = -\nabla \mathcal{P} = -ik\mathbf{e}_x \mathcal{P}_0 e^{-i\omega t + ikx}$ inserted in the right-hand side of Eq. (1a), and boundary conditions (2) at the tube wall $r = R$.

With calculations similar to those which have been done before, the velocity–temperature equations (12) and temperature equation (13) are the same, with source terms $+\mathbf{f}/\rho_0$ and $-k^2 \mathcal{P}/\rho_0$ inserted in the right-hand sides (13) and (12a), respectively. A particular solution of the temperature equation is $\tau'_p = Ck^2 \mathcal{P}/\rho_0$, with

$$C = \left\{ \omega^2 - \left[c_a^2 - i\omega \left(\frac{\kappa}{\rho_0 c_v} + \frac{4\eta}{3} + \zeta \right) \right] k^2 + \frac{\kappa}{\rho_0 c_v i\omega} \left(c_i^2 - i\omega \frac{4\eta}{3} + \zeta \right) k^4 \right\}^{-1}.$$

The complete temperature solution is this particular solution added to the general solution (15) of the homogeneous equation (13)

$$\tau' = \tau'_0 + \tau'_p = A_1 \varphi_1 + A_2 \varphi_2 + C \frac{k^2}{\rho_0} \mathcal{P}. \tag{22}$$

Similarly, the complete expression of the velocity is $\mathbf{v} = u\mathbf{e}_x + q\mathbf{e}_r = u_0\mathbf{e}_x + q_0\mathbf{e}_r + \mathbf{v}_p$, where u_0 and q_0 are written as in (17a)–(17b), and $\mathbf{v}_p = u_p\mathbf{e}_x$ is the particular solution determined by (12c) with $\tau' = \tau'_p$, giving, $iku_p = [i\omega - \kappa k^2 / (\rho_0 c_v)] \tau'_p$. Only the x component u is required to compute $\rho(\omega, k)$. However, the radial component q needs also to be written as it is involved in the boundary conditions, by means of which the amplitudes A, A_1 and A_2 are finally fixed. Both components write accordingly

$$u = A\varphi + ik \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \varphi_1 + ik \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \varphi_2 + \left(-i\omega + \frac{\kappa}{\rho_0 c_v} k^2 \right) C \frac{ik}{\rho_0} \mathcal{P} \tag{23a}$$

$$q = \frac{-ikA}{-i\omega + k^2} \frac{\partial \varphi}{\partial r} + \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \frac{\partial \varphi_1}{\partial r} + \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \frac{\partial \varphi_2}{\partial r}. \tag{23b}$$

Now, we seek the excess pressure solution as the last required quantity. It has the complete form: $p = p_0 + p_p$. The particular solution p_p is determined by (1c), written in the form of (10), with $\tau' = \tau'_p, b = b_p$, and $i\omega b_p = iku_p$. Thus

$$\frac{p}{\rho_0} = \left(c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} \lambda_1 \right) A_1 \varphi_1 + \left(c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} \lambda_2 \right) A_2 \varphi_2 + \left(c_a^2 - c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} k^2 \right) C \frac{k^2}{\rho_0} \mathcal{P}. \tag{24}$$

The boundary conditions imply that the three quantities, excess temperature (22) and the two components of velocity (23a) and (23b), should vanish at the tube wall $r = R$. This yields a linear system, whose solution uniquely determines the three response amplitudes A , A_1 and A_2 , in terms of the driving pressure amplitude constant \mathcal{P}_0 . Knowing the response fields u in (23a) and p in (24) as functions of ω , k and r , is all we need to compute the density $\rho(\omega, k)$. According to the conjectured upscaling procedure [1] Section 4, we assume that in the present action–response problem, the role of the macroscopic field H in the Eqs. (4) and (5) is played by the field $P + \mathcal{P}$, where P is the macroscopic part of the response pressure field p , which is defined by

$$\langle p\mathbf{v} \rangle = P\langle \mathbf{v} \rangle, \quad \text{i.e.} \langle pu \rangle = P\langle u \rangle. \tag{25}$$

This assumption leads to Eq. (69) in [1], which gives here

$$\rho(\omega, k) = \frac{k(P + \mathcal{P})}{\omega\langle u \rangle}. \tag{26}$$

Putting the expressions (23a) and (24) in (26) and using the known formulae giving the cross-section average $\langle \cdot \rangle$ of Bessel functions φ , φ_1 and φ_2 and their products, the nonlocal density is obtained as a function of ω and k , which writes analytically as a ratio of sums containing 12 terms in the numerator, 10 terms in the denominator, each made up of a product of 6 Bessel’s functions multiplied by factors involving ω , k^2 , λ_1 and λ_2 . To save time, instead of seeking its most compact final form, we have done a direct Matlab programming for this lengthy function expression of $\rho(\omega, k)$, with ω and k as input arguments.

4.2. Determination of the nonlocal bulk modulus $\chi^{-1}(\omega, k)$

The same type of calculations can be performed to compute $\chi^{-1}(\omega, k)$. Here, we consider the response of the fluid subjected to the action of an external driving bulk rate of heat supply \dot{Q} (per unit volume and unit time), which derives from a fictitious harmonic pressure waveform inserted in the Fourier equation (see Sections 2.9 and 4 in [1]). Thus we consider solving the action–response problem given by the Eqs. (1) written in the tube $r < R$, with external driving rate of heat supply given by $\dot{Q} = \beta_0 T_0 (\partial \mathcal{P} / \partial t) = -i\omega \beta_0 T_0 \mathcal{P}_0 e^{-i\omega t + ikx}$ inserted in the right-hand side of Eq. (1d), and boundary conditions (2) at the tube wall $r = R$.

With calculations similar to those which have been done before, the velocity–temperature equations (12) and temperature equation (13) are the same, with source terms $-i\omega \gamma \chi_0 \mathcal{P}$ and $-(i\omega - \nu k^2) \gamma \chi_0 i\omega \mathcal{P}$, respectively inserted in the right-hand sides (12c) and (13). A particular solution of the temperature equation is $\tau'_p = -C(i\omega - \nu k^2) \gamma \chi_0 i\omega \mathcal{P}$, with the same constant C as before. The complete excess temperature solution is this particular solution added to the general solution (15) of the homogeneous equation (13)

$$\tau' = \tau'_0 + \tau'_p = A_1 \varphi_1 + A_2 \varphi_2 - C(i\omega - \nu k^2) \gamma \chi_0 i\omega \mathcal{P}. \tag{27}$$

Likewise, the complete velocity solution is in the form: $\mathbf{v} = u\mathbf{e}_x + q\mathbf{e}_r = u_0\mathbf{e}_x + q_0\mathbf{e}_r + \mathbf{v}_p$, where u_0 and q_0 are written as in (17a)–(17b), and $\mathbf{v}_p = u_p\mathbf{e}_x$ is the particular solution determined by (12c) with the source term $-i\omega \gamma \chi_0 \mathcal{P}$ inserted and $\tau' = \tau'_p$, giving $iku_p = (i\omega - \kappa k^2 / \rho_0 c_v) \tau'_p - i\omega \gamma \chi_0 \mathcal{P}$. We obtain for the two components of the velocity

$$u = A\varphi + ik \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \varphi_1 + ik \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \varphi_2 + \left[\left(-i\omega + \frac{\kappa}{\rho_0 c_v} k^2 \right) C(i\omega - \nu k^2) - 1 \right] \gamma \chi_0 \frac{\omega}{k} \mathcal{P} \tag{28a}$$

$$q = \frac{-ikA}{\frac{-i\omega}{\nu} + k^2} \frac{\partial \varphi}{\partial r} + \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_1} \right) A_1 \frac{\partial \varphi_1}{\partial r} + \left(\frac{\kappa}{\rho_0 c_v} + \frac{i\omega}{\lambda_2} \right) A_2 \frac{\partial \varphi_2}{\partial r}. \tag{28b}$$

The complete pressure solution, similarly, is written as $p = p_0 + p_p$, where p_p is determined by (1c), written in the form of (10), with $\tau' = \tau'_p$ and $b = b_p$, $i\omega b_p = iku_p$. Thus we have

$$\begin{aligned} \frac{p}{\rho_0} = & \left(c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} \lambda_1 \right) A_1 \varphi_1 + \left(c_a^2 + c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} \lambda_2 \right) A_2 \varphi_2 \\ & - C(i\omega - \nu k^2) i\omega \left(c_a^2 - c_i^2 \frac{\kappa}{\rho_0 c_v i\omega} k^2 \right) \gamma \chi_0 \mathcal{P} - c_i^2 \gamma \chi_0 \mathcal{P}. \end{aligned} \tag{29}$$

The condensation solution will now also be required. It may be written from (10) and the expressions (27) and (29), which yield

$$b = \left(1 + \frac{\kappa}{\rho_0 c_v i\omega} \lambda_1 \right) A_1 \varphi_1 + \left(1 + \frac{\kappa}{\rho_0 c_v i\omega} \lambda_2 \right) A_2 \varphi_2 - C(i\omega - \nu k^2) i\omega \left(1 - \frac{\kappa}{\rho_0 c_v i\omega} k^2 \right) \gamma \chi_0 \mathcal{P} - \gamma \chi_0 \mathcal{P}. \tag{30}$$

As before, the boundary conditions imply that the three quantities, excess temperature and the two components of velocity (27), (28a) and (28b), should vanish at the tube wall $r = r_w = R$. This yields a linear system whose solution uniquely determines the three response amplitudes A , A_1 and A_2 , in terms of the driving amplitude constant \mathcal{P}_0 . Knowing the response

Table 1

Fluid properties used in all computations.

ρ_0 (kg/m ³)	T_0 (K)	c_0 (m/s)	η (kg m s ⁻¹)	ζ (kg m s ⁻¹)	κ (W m ⁻¹ K ⁻¹)	χ_0 (Pa ⁻¹)	c_p (J kg ⁻¹ K ⁻¹)	γ
1.205	293.5	340.14	1.84×10^{-5}	0.6η	2.57×10^{-2}	7.17×10^{-6}	997.54	1.4

fields u in (28a), p in (29) and b in (30) as functions of ω , k and r , is what we need to compute the bulk modulus $\chi^{-1}(\omega, k)$. According to the conjectured upscaling procedure [1, Section 4], we assume that in the present action–response problem, the role of the macroscopic field H in the Eqs. (4) and (5) is played by the field $P + \mathcal{P}$, where P is the macroscopic part of the response pressure field p , which is defined as before by (25), and at the same time, the role of the macroscopic field B is played by the averaged field $\langle b + \gamma \chi_0 \mathcal{P} \rangle$. This leads to Eq. (73) in [1], which writes here

$$\chi^{-1}(\omega, k) = \frac{P(\omega, k) + \mathcal{P}}{\langle b(\omega, k, r) \rangle + \gamma \chi_0 \mathcal{P}}. \tag{31}$$

Putting the expressions (28a), (29), (30) in (25) and (31), and using the known formulae giving the cross-section average $\langle \cdot \rangle$ of Bessel functions φ , φ_1 and φ_2 and their products, the nonlocal bulk modulus is obtained in analytical form, as a function of ω and k . It is a very lengthy expression involving a number of terms of the type seen before for $\rho(\omega, k)$. As before to save time, we have done a direct Matlab programming for this function $\chi^{-1}(\omega, k)$, with ω and k as input arguments.

4.3. The matching with Kirchhoff–Langevin theory

Given the functions $\rho(\omega, k)$ and $\chi(\omega, k)$, the possible wavenumbers k_l in the medium are found by solving the ‘Maxwell’ dispersion equation (7). The associated impedances Z_l are found by (8). For the proposed macroscopic nonlocal theory to be correct, this ‘Maxwell’ dispersion equation (7) should be mathematically equivalent to the original Kirchhoff–Langevin dispersion equation (18). Both equations ought to have the same set of solutions k_l ; moreover, the macroscopic impedances Z_l (8) and (20) ought also to be the same. Using the point of view of both approaches, this matching holds if and only if, the following equalities $l = 1, 2, \dots$ are satisfied:

$$\rho(\omega) = \frac{k_l}{\omega} Z_l = \rho(\omega, k_l), \quad \chi_l^{-1}(\omega) = \frac{\omega}{k_l} Z_l = \chi^{-1}(\omega, k_l). \tag{32}$$

The calculations to be performed in order to study the mathematical equivalence of (18) and (7) appear very tedious, because of the large number of terms to be collected and rearranged in order to express the mean term $\langle pu \rangle$. Additionally, it is even not obvious that the mathematical identity between the two forms of the dispersion equation would directly appear, once the calculations are done. Thus in what follows, the easier way to establish the validity of the theory, is to make a direct numerical check on the relations (32). The quantities on the left, computed entirely along Kirchhoff–Langevin theory, will be referred by the index K for Kirchhoff–Langevin; the quantities on the right, computed entirely within the nonlocal Maxwellian theory and its upscaling procedures, will be referred by the index M for Maxwell.

5. A check on the nonlocal Maxwellian theory

We describe in what follows the validation of the nonlocal theory through three frequency regimes referred to as ‘narrow’ tube, ‘wide’ tube, and ‘very wide’ tube, in acoustic literature [16]. In all calculations the parameters of the air are set to the values shown in Table 1. Given the radius R and the frequency f , we proceed as follows to evaluate the different quantities. To evaluate Zwikker and Kosten density and bulk modulus ρ_Z and χ_Z^{-1} , the formulae (A.6), (A.7) and (A.9), reported in the Appendix, are used. As shown in this Appendix, these Zwikker and Kosten’s values correspond to a local Maxwellian description which neglects the phenomenon of spatial dispersion. To evaluate Kirchhoff–Langevin’s density and bulk modulus of a given mode – not necessarily the least-attenuated mode considered in Zwikker and Kosten theory – we first determine Kirchhoff–Langevin’s mode wavenumber k_K by solving, via the Newton scheme, the dispersion equation (18) which is an equation of type $F(\omega, k) = 0$. We have an explicit analytical expression of the function $F(\omega, k)$, which also yields in analytical form the partial k -derivative. Once a wavenumber $k = k_K$ solution to (18) is determined, the density ρ_K and bulk modulus χ_K^{-1} can be analytically computed using the relations (21).

To evaluate anew the density and bulk modulus for a given macroscopic normal wave, entirely within the nonlocal Maxwellian theory, we now first determine Maxwell’s wavenumber k_M by solving Maxwell dispersion equation (7). Because the corresponding function $F(\omega, k)$ is given by a Matlab program, its partial k -derivative is now evaluated numerically. Compared with Kirchhoff–Langevin dispersion equation (18), the Newton scheme for Maxwell dispersion equation (7) is less stable and requires more iterations to converge. Once a possible wavenumber $k = k_M$ is determined, Maxwell’s density $\rho_M = \rho(\omega, k_M)$ and bulk modulus $\chi_M^{-1} = \chi^{-1}(\omega, k_M)$ can be computed analytically for the corresponding wave, using the Matlab programming of the lengthy functions (26) and (31).

Table 2

Narrow tubes—least attenuated plane mode. Density and Bulk modulus.

ρ_Z	$1.6066193 + 23.39190009i$
ρ_K	$1.60661313809 + 23.391900424437i$
ρ_M	$1.60661313809 + 23.391900424437i$
χ_Z^{-1}	$99620.16441 - 1043.52791i$
χ_K^{-1}	$99620.16409534 - 1043.53176900i$
χ_M^{-1}	$99620.16409534 - 1043.53176900i$

5.1. Narrow tubes: $R = 10^{-4}$ m, $f = 100$ Hz

This configuration is the same as the one considered by Stinson [17]. Such radius and frequency are typical regarding pore dimensions of porous materials and frequencies involved in low-frequency noise control applications [18]. By definition, in the low frequency range the viscous skin depth $\delta = (2\nu/\omega)^{1/2}$ and thermal skin depth – of the same order for air – are greater than R . Here, with a viscous skin depth, two times the radius, the fundamental least-attenuated mode is mostly diffusive and the higher order modes are highly attenuated. For the least attenuated mode, which is named the plane wave mode as its pressure profile is quasi flat, the computed values of the density and bulk modulus are given in Table 2.

Zwikker and Kosten local theory is very satisfactory. The main relative discrepancies are in the order of 10^{-6} and concern the real part of the density and the imaginary part of the bulk modulus. Notwithstanding, a clear validation of the upscaling procedures is provided by the exact matching observed in Table 2, between Kirchhoff–Langevin direct mathematical solution and nonlocal Maxwellian solution. Within the local theory, the value 1.6 of the real part of the density is interpreted as the ambient density $\rho_0 = 1.2$, times the visco-inertial factor $\alpha_0 = 4/3$. This value corresponds to the fully developed parabolic ‘Poiseuille’ flow established in this frequency range (see e.g. [19] (21) for the significance and definition of α_0). The imaginary part $\Im(\rho)$ is interpreted as $\eta/(\omega k_0)$, where $k_0 = R^2/8$ represents the tube permeability. The real part of the bulk modulus is interpreted as the ambient atmospheric pressure: as the compression is mostly isothermal, one finds $\Re(\chi^{-1}) = \gamma^{-1}\chi_0 = P_0$. Its imaginary part is interpreted as the quantity $-[(\gamma - 1)/\gamma](\omega k_0/\nu')P_0$ (easily deduced using the low-frequency expansions in [19]), where $\nu' \equiv \nu/Pr \equiv \kappa/(\rho_0 c_p)$.

We note that, in Kirchhoff–Langevin’s calculations, the Newton convergence towards wavenumber k_K is relatively insensitive to the starting value. Convergence is obtained by starting from the lossless case solution $k = \omega/c_a$, the Zwikker–Kosten solution, or the value $k = 6 + 2i$ taken in [17]. On the contrary, in Maxwell’s calculations, the convergence towards the wavenumber k_M is sensitive to the starting value. A meaningless unattenuated value which is not reported here, is found when using as starting value the lossless-relating solution $k = \omega/c_a$. For the first higher order mode, the imaginary part of the wavenumber is already in the order of $5 \times 10^4 i$. This leads to a large imaginary part in the complex arguments of the Bessel functions φ and $\varphi_{1,2}$, which may prevent doing reliable computations with Matlab. The bulk modulus seems very close to zero and is not estimated in a reliable manner.

5.2. Wide tubes: $R = 10^{-3}$ m, $f = 10$ kHz

The same configuration is considered in [17]. With radius R now being more than 50 times of the viscous skin depth δ , the least attenuated plane wave mode is a well-propagating mode. For this least attenuated plane wave mode, the computed values of the density and bulk modulus are reported in Table 3. As expected, Zwikker and Kosten local theory is still very satisfactory in this regime.³ It yields the results corresponding to the Johnson et al. [20] and Champoux–Allard [21] high-frequency asymptotic expansions of the density and bulk modulus, in terms of the viscous and thermal characteristic lengths Λ and Λ' (both equal to the tube radius, here). The relative discrepancies with Kirchhoff–Langevin exact theory are in the order of 10^{-3} for the imaginary part of the density and bulk modulus. Maxwell values are remarkably indistinguishable from Kirchhoff–Langevin’s, which by itself, again provides a validation of the upscaling procedures. Notwithstanding, because of lower imaginary parts in the Bessel functions, a precise numerical verification of the higher order modes is now feasible. It shows the wanted coincidences.

Let us show, as an example, the results relating to the first higher-order axisymmetric mode. Its cutoff frequency, around 207 kHz, is well beyond the computation frequency 10 kHz. While this mode is still mostly evanescent, its macroscopic characteristics ρ , χ^{-1} , shown in Table 4, are precisely obtained. The relative discrepancies, in the order of $3 \cdot 10^{-9}$ on the real part of the density and $7 \cdot 10^{-9}$ on real part of the bulk modulus, are not meaningful. Their origin must be traced back to the loss of precision in the Matlab Bessel functions. It is interesting to notice, for these higher order modes, the negative real part of the bulk modulus. This is the type of behavior described for metamaterials made of daisy chained Helmholtz resonators [25], a negative real part of the wavenumber also being present and associated with the phenomenon of negative group velocity. Spatial nonlocality plays an essential role in this behavior. Indeed, without spatial nonlocality the local Zwikker and Kosten description would be obtained. No higher-order modes would exist.

³ The radius is still a small fraction – 1/34 – of the wavelength, which justifies Zwikker and Kosten’s simplifying assumptions, see Appendix.

Table 3
Wide tubes—least attenuated plane mode. Density and Bulk modulus.

ρ_Z	$1.2315315 + 0.0274330i$
ρ_K	$1.231530808362 + 0.027453005128i$
ρ_M	$1.231530808362 + 0.027453005128i$
χ_Z^{-1}	$137957.879 - 1406.1i$
χ_K^{-1}	$137957.823561287 - 1407.920077798i$
χ_M^{-1}	$137957.823561287 - 1407.920077798i$

Table 4
Wide tubes—first higher-order axisymmetric mode. Density and Bulk modulus.

ρ_K	$3662717.006 - 109385009.002i$
ρ_M	$3662717.005 - 109385009.001i$
χ_K^{-1}	$-323505053.99 + 28854278537.01i$
χ_M^{-1}	$-323505053.93 + 28854278536.78i$

Table 5
Very wide tubes—least attenuated mode. Density and Bulk modulus.

ρ_Z	$1.2054 + 0.0004i$
ρ_K	$0.7210323319 - 4.5486444405i$
ρ_M	$0.7210323319 - 4.5486444405i$
χ_Z^{-1}	$139391.4 - 20.6i$
χ_K^{-1}	$82793.27306 - 526992.34910i$
χ_M^{-1}	$82793.27306 - 526992.34910i$

Table 6
Very wide tubes—first five modes. Densities and Bulk moduli.

ρ (kg/m ⁻³)	χ^{-1} (Pa)
$0.72103233188 - 4.54864444048i$	$82793.273058 - 526992.34910i$
$0.389255008 - 80.473941133i$	$31495.2156 - 9359957.04375i$
$-0.371923523 - 389.4963732483i$	$-111665.822 - 45594436.85207i$
$-4.537814 - 2808.960059i$	$-1053940.56 - 335497383.40i$
$-15.689167 - 10386.487842i$	$-3933561.2 - 1278741768.38i$
$-25.61790 - 17487.62974i$	$-6706462. - 2194904861.7i$

5.3. Very wide tubes: $R = 10^{-2}$ m, $f = 500$ kHz

Here, the radius is about fifteen times greater than the wavelength. Thus, the separation condition assumed in Zwikker and Kosten theory, $\lambda \gg R$, is no longer satisfied. In this regime the least attenuated mode is not a plane mode; the sound energy tends to be concentrated near the walls [16]. The computed values of the density and bulk modulus for this mode are indicated in Table 5.

The considerable differences compared to Zwikker and Kosten values indicate the crucial effects of the spatial nonlocality in this regime. We recall that in Zwikker and Kosten local theory, the pressure is viewed as a cross-sectional constant and no essential difference is established between the pressure, the mean pressure, and the field H (see Appendix). Zwikker and Kosten local theory also leads to considering, here, that the velocity profile is nearly flat: the boundary layers at the tube wall have a micrometric extent which is a tiny fraction ($\approx 3 \times 10^{-4}$) of the radius. Thus, the Zwikker and Kosten real part of the density, necessarily tends to the ambient value $\rho_0 = 1.205$. The actual, significantly lower value $\Re(\rho) = 0.72$ of the real part of the effective density, is a direct signature of the breakdown of the assumptions incorporated in this Zwikker and Kosten treatment. As the pressure and the axial velocity are now both variable across the section, the macroscopic pressure field H is actually to be carefully distinguished from the mean pressure field.

In these circumstances, the precise matching obtained between Kirchhoff–Langevin and Maxwell values in Table 5, is a clear validation of the upscaling. This matching is precisely obtained, also, for the higher order modes. For example, we report in Table 6 the densities and bulk moduli of the first six least attenuated modes, indicating the decimals which are found to be the same, in the Kirchhoff–Langevin and Maxwellian computations. It can be noted that, from the second higher order mode, negative–negative densities and bulk moduli are found.

6. Conclusion

In this paper, considering the easily workable case of a material having cylindrical circular identical pores, we have given a simple validation of a new nonlocal macroscopic theory, recently developed in [1] with the aim of providing a general description of sound propagation in homogeneous rigid-framed porous materials permeated with a viscothermal fluid. It has been observed that this theory, thanks to its fundamental Umov–Heaviside–Poynting–Schoch definition (6) of the macroscopic acoustic stress field H , properly takes into account the nonlocal behavior of the macroscopic fields. It describes all different macroscopic waves (cross-sectional averaged axisymmetric modes, here), in various physical regimes of the wave propagation classified in [16].

After presenting briefly the general method to determine the complex effective densities and bulk moduli, or wavenumbers and impedances through nonlocal theory, we have shown how to compute them for the particular case of axisymmetrical sound wave modes in circular tubes. In order to conveniently and easily present the method of calculating these quantities in this geometry, we have taken advantage of the way in which Kirchhoff–Langevin dispersion equation could be obtained. The validation of the nonlocal theory was based on comparing the results of these calculations, according to this theory on one hand, and the Kirchhoff equation on the other hand. In addition, using the Zwikker and Kosten local theory, complex density and bulk modulus associated with the least attenuated wave, the only mode which can be described by this method, have been computed. This allowed to observe the discrepancies appearing between the results coming from Zwikker and Kosten local theory, and the results coming from nonlocal theory or Kirchhoff equation, especially, in the very wide tubes regime.

It may be highlighted that the new nonlocal-relating upscaling procedures are non-perturbative procedures, fundamentally different from the perturbative procedures derived through the conventional two-scale asymptotic homogenization process [22–24]. In forthcoming papers, applications of these precise acoustic nonlocal homogenization procedures will be made for the case of nontrivial geometries, including metamaterials. Indeed, while in the case of cylindrical ducts the present complete non-perturbative inclusion of spatial and temporal dispersion effects leads only to relatively small or marginal extensions of the local treatment, it will be seen that it leads, in the case of nontrivial geometries, to the interesting metamaterial behaviors [25].

Appendix. Zwikker and Kosten local theory

Kirchhoff–Langevin theory of sound, describe the possible wave field solutions to the Navier–Stokes–Fourier linear equations (1) in the fluid-filled cylindrical circular tube $r < R$, subject to the boundary conditions (2) at $r = R$. It has been shown in this paper that, once suitably averaged over the cross section, Kirchhoff–Langevin axisymmetric wave fields can be described in terms of the nonlocal macroscopic Maxwellian theory developed in [1]. Kirchhoff–Langevin theory is relatively heavy, however; in acoustic applications, the much simpler Zwikker and Kosten theory [15] is most often sufficient. In a long-wavelength limit, $\lambda \gg R$, it gives results indistinguishable, in practice, from those of the full nonlocal theory.

In this Appendix we show that Zwikker and Kosten theory, allowing for temporal dispersion only, is a ‘local’ simplification of the complete Maxwellian theory. The aim of this theory is to find frequency-dependent density and compressibility functions, $\rho_Z(\omega)$ and $\chi_Z(\omega)$, such that in harmonic regime

$$\rho_Z(\omega) \frac{\partial V}{\partial t} = -\frac{\partial P}{\partial x}, \quad \chi_Z(\omega) \frac{\partial P}{\partial t} = -\frac{\partial V}{\partial x} \tag{A.1}$$

where V and P are the cross-section averages of velocity and pressure. Knowing $\rho_Z(\omega)$ and $\chi_Z(\omega)$ would entirely specify the tube propagation characteristics. It would give one propagation constant, $k_Z = \omega \sqrt{\rho_Z(\omega) \chi_Z(\omega)}$, and one characteristic impedance, $Z_Z = \sqrt{\rho_Z(\omega) \chi_Z^{-1}(\omega)}$, where the index Z stands for ‘Zwikker and Kosten’. These refer in fact to the least attenuated wave, which is named in this context the ‘plane wave’ mode (the pressure profile is quasi plane over the cross section). We note that, in the framework of these simplifications, one also admits the existence of a relation having the form

$$\rho'_Z(\omega) \frac{\partial \langle \tau \rangle}{\partial t} = \beta_0 T_0 \frac{\partial P}{\partial t} \tag{A.2}$$

with $\langle \tau \rangle$ the cross-section average of the excess temperature, and $\rho'_Z(\omega)$ another complex frequency-dependent response function. The necessity of this relation, simply comes from the similarity between (1a) and (1d). It is easy to find the relation between $\rho'_Z(\omega)$ and $\chi_Z(\omega)$. Combining (1b) and (1c) and averaging over a cross-section, an exact general relation between $\partial P/\partial t$, $\partial \langle \tau \rangle/\partial t$, and $\partial V/\partial x$ is obtained: $\gamma \chi_0 (\partial P/\partial t) = -\partial V/\partial x + \beta_0 (\partial \langle \tau \rangle/\partial t)$. Putting in this equation the relation (A.2) and using the general thermodynamic identity (Eq. (25) in [1]), the relation between the two functions is found to be

$$\chi_Z(\omega) = \chi_0 \left[\gamma - (\gamma - 1) \frac{\rho_0 c_p}{\rho'_Z(\omega)} \right]. \tag{A.3}$$

Now we observe that the Eqs. (A.1) can be put in exactly the same form as the Maxwellian equations (4)–(5) with, however, the notable difference that they do not include the phenomenon of spatial dispersion.

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field equations:

$$\frac{\partial B}{\partial t} = -\frac{\partial V}{\partial x}, \quad \frac{\partial D}{\partial t} = -\frac{\partial H}{\partial x} \tag{A.4}$$

constitutive relations:

$$D(t, x) = \int_{-\infty}^t dt' \rho_Z(t-t')V(t', x) \tag{A.5a}$$

$$H(t, x) = \int_{-\infty}^t dt' \chi_Z^{-1}(t-t')B(t', x), \quad H \equiv P \tag{A.5b}$$

(the real kernel functions $\rho_Z(t)$ and $\chi_Z(t)$ are defined to have the complex Fourier amplitudes $\rho_Z(\omega)$ and $\chi_Z(\omega)$ involved in Eqs. (A.1)).

The interpretation of the equivalence $H \equiv P$ is that the pressure profile is assumed to be flat. This is addressed more fully below. For the moment, we note that, as the Eqs. (A.5) are purely local in space, the Eqs. (A.4) and (A.5) are incompatible with the Eqs. (1) and (2). Indeed, as we have seen, Eqs. (1)–(2) consistently lead to Maxwellian macroscopic acoustic equations (4)–(5) which are nonlocal in space. Thus, what Zwikker and Kosten theory is doing to arrive at macroscopic equations having the local form (A.1), is not to solve the complete Eqs. (1)–(2), but some truncated simplified versions of the latter. These simplified equations, determining the functions ρ_Z and χ_Z , can be obtained by applying the classic two-scale homogenization theory and retaining the leading terms only (see e.g. [24, Appendix A]). We can guess here the correct Zwikker and Kosten functions even more easily, on the simple physical ground that the necessary simplified equations will have to neglect the spatial dispersion.

Because spatial dispersion is discarded, the spatial variation of the pressure gradient term in Eq. (1a) should be neglected for the purpose of determining the fluid velocity pattern. This suggests solving the fictitious action–response problem: $\partial \mathbf{v} / \partial t = \nu \nabla^2 \mathbf{v} + \mathbf{f} / \rho_0$ and $\nabla \cdot \mathbf{v} = 0$, for $r < R$, with $\mathbf{v} = 0$ at $r = R$, and driving force given by $\mathbf{f} = -\nabla \mathcal{P} = \mathbf{e}_x f_0 e^{-i\omega t}$. The incompressibility condition arises here, because, in considering f_0 as a spatial constant, we are actually making abstraction of the dilatation–compression of the fluid. The corresponding density $\rho_Z(\omega)$, such that, $-i\omega \rho_Z(\omega) \langle \mathbf{v} \rangle = \mathbf{f}$, or, to compare with (26), such that (with $\mathbf{v} = \mathbf{e}_x u_0 e^{-i\omega t}$), $\rho_Z(\omega) = f_0 / (-i\omega \langle u_0 \rangle)$, is the wanted Zwikker and Kosten density. It is found by elementary calculations to be given by

$$\frac{1}{\rho_Z(\omega)} = \frac{1}{\rho_0} [1 - \xi_Z(\omega)] \tag{A.6}$$

where $\xi_Z(\omega)$ is the following Zwikker and Kosten relaxation function

$$\xi_Z(\omega) = \frac{2J_1\left(\left(\frac{i\omega}{\nu} R^2\right)^{1/2}\right)}{\left(\frac{i\omega}{\nu} R^2\right)^{1/2} J_0\left(\left(\frac{i\omega}{\nu} R^2\right)^{1/2}\right)}. \tag{A.7}$$

In a similar manner, the spatial variation of the time derivative pressure term in Eq. (1d) should be neglected for the purpose of determining the excess temperature pattern. This suggests solving the fictitious action–response problem: $\partial \tau / \partial t = \nu \nabla^2 \tau + \dot{Q} / (\rho_0 c_p)$ for $r < R$, with $\tau = 0$ at $r = R$, and driving rate of heat supply $\dot{Q} = \beta_0 T_0 (\partial \mathcal{P} / \partial t) = \dot{Q}_0 e^{-i\omega t}$. The corresponding function $\rho'_Z(\omega)$, such that, $-i\omega \rho'_Z(\omega) \langle \tau \rangle = \dot{Q}$, by elementary calculations, is found to be

$$\frac{1}{\rho'_Z(\omega)} = \frac{1}{\rho_0 c_p} [1 - \xi_Z(\omega \text{Pr})] \tag{A.8}$$

where $\xi_Z(\omega)$ is the previous relaxation function (A.7) and $\text{Pr} = \eta c_p / \kappa$ is the Prandtl number. Finally, since ρ'_Z is related to the compressibility χ_Z by the general relation (A.3), Zwikker and Kosten's end result for $\chi_Z(\omega)$ is

$$\chi_Z(\omega) = \chi_0 [1 + (\gamma - 1)\xi_Z(\omega \text{Pr})]. \tag{A.9}$$

As long as one neglects spatial dispersion, viscous and thermal effects are separated. The former determine the density, and the latter the compressibility.

In conclusion we note that, since the pressure gradient term in Eq. (1a) is represented by the spatially constant term \mathbf{f} , and the pressure time derivative term in Eq. (1d) is represented by the spatially constant term \dot{Q} , the excess pressure p itself is assimilable to a constant over the cross-section. Thus, when writing the fundamental definition $\langle up \rangle = H \langle u \rangle$ of the H field, the excess pressure p can be extracted from the averaging symbol and it turns out that $H = p = P$ (since a constant coincides with its mean). As such, no distinction appears between the mean wave pressure P and the effective macroscopic wave pressure H .

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