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Laurent Meersseman

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GEOGRAPHY OF THE TEICHMÜLLER STACK

LAURENT MEERSSEMAN

ABSTRACT. In this article, we describe the geography of the Teichmüller stack of [29] and of one of its variants we introduce here, giving some answers to questions as: which points are orbifold points? What are the different local models of special points?... We give a rough description in the general case, and we use the compactness of the cycle spaces to get a much more detailed picture in the Kähler setting.

1. INTRODUCTION.

Let X_0 be a compact complex manifold with underlying C^∞ manifold denoted by M . For such a non-necessarily Kähler manifold, Hodge Theory does not apply and the set of complex structures close to X_0 is controlled by its Kuranishi space through Kodaira-Spencer theory of deformations. Thus to obtain the full moduli space of complex structures on M , it is theoretically enough to glue at most a countable number of Kuranishi spaces, or a suitable quotient of them.

This process cannot however be realized in this degree of generality with classical, GIT, or orbifold quotients. General Artin Stacks are needed. In [29] (see also [30] for a comprehensive presentation), under a very mild hypothesis, we build a moduli stack of complex structures on M , resp. a Teichmüller stack of M , describing explicitly the gluing process involved. Rather than gluing Kuranishi spaces, we glue Kuranishi stacks. These stacks are, roughly speaking, the quotient of Kuranishi base spaces by the automorphism group of the base points. The moduli and Teichmüller stacks thus obtained are an analytic enriched version of the topological moduli and Teichmüller spaces.

This being said, the next step consists of analyzing the geometric structure of these stacks, understanding why they are in general neither analytic spaces nor orbifolds, classifying the different types of points, giving adequate local models of the special points and establishing a cartography of them. This is what we begin to do in this paper.

A first very well known obstruction for the moduli/Teichmüller space being locally an analytic space at some complex manifold X_0 , is the fact that the dimension of the automorphism group of X_0 may differ from the dimension of the automorphism group of close complex manifolds. This dimension is an upper semicontinuous function of the Kuranishi space for

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the Zariski topology [17]¹, so the geography of the points where it jumps is clear. They are located on a strict analytic subspace. Moreover, the corresponding Kuranishi spaces have a foliated structure described in [28] from which local models can be derived. All this analysis can be transposed to the moduli/Teichmüller stack, cf. §3.

Another well known fact is that the action of the mapping class group on the Teichmüller space (whose quotient is the moduli space) can be very wild with dense or locally dense orbits. This is the case for complex tori of dimension at least 2, for K3 surfaces, ... So we avoid this wildness by considering here only the Teichmüller stack $\mathcal{T}(M)$ and not the moduli stack. We also introduce a variant of it, the \mathbb{Z} -Teichmüller stack $\mathcal{T}^{\mathbb{Z}}(M)$, that basically satisfies all the properties of $\mathcal{T}(M)$.

We show in this paper that there exists another subtler phenomenon. In the brief account of the construction of the Teichmüller stack at the beginning of this introduction, we said that we glue Kuranishi stacks to obtain it. This is a slight simplification for it is not always true that the Teichmüller stack is locally isomorphic at some X_0 to the Kuranishi stack of X_0 . Points where this does not happen are called exceptional or \mathbb{Z} -exceptional in the case of $\mathcal{T}^{\mathbb{Z}}(M)$. They exhibit a different local model and some strange properties that are analyzed in §8. Their geography is also not so clear than that of jumping points.

It is not easy to find exceptional or \mathbb{Z} -exceptional points - none of the classical examples admits one. Indeed, this work was strongly delayed because, for a long time, we did not have any example. We finally build a non-Kähler example of a \mathbb{Z} -exceptional point that is presented in §8.4 and Theorem 8.17. We note that it is not exceptional. In the Kähler case, that is when we restrict to the open² substack of Kähler points of the Teichmüller stack, we show in Theorem 10.1 that the closure of exceptional points as well as that of \mathbb{Z} -exceptional points form a strict analytic substack, making use of the compactness of the cycle spaces. It is important to stress that all the arguments using compactness of cycle spaces break completely when X_0 is neither Kähler nor in Fujiki class (\mathcal{C}), so that it is natural to expect a dichotomy between the Kähler and the non-Kähler cases. Pushing forward this analysis, we state in the very polarized Conjecture 6.2 that there does not exist neither exceptional nor \mathbb{Z} -exceptional Kähler points; whereas such points can be dense in a connected component of non-Kähler points of the Teichmüller or the \mathbb{Z} -Teichmüller stack. Theorems 8.17 and 10.1 cited above are the best results we obtained on exceptional points. But they left wide open several assertions of Conjecture 6.2 starting from the existence of an exceptional point as well as several important associated questions, especially whether the set of exceptional points is closed.

As a consequence of this dichotomy, the local structure of $\mathcal{T}(M)$ or $\mathcal{T}^{\mathbb{Z}}(M)$ at a general non-Kähler point may be much more complicated than

¹ Indeed, upper semicontinuity is proven in [24] for the ordinary topology on a smooth base, and the general case can be deduced from the proof of Grauert's direct image theorem in [17].

² By a classical result of Kodaira-Spencer, Kählerianity is a stable property through small deformations.

at a Kähler point. An example of this phenomenon for $\mathcal{T}^{\mathbb{Z}}(M)$ is given in Corollary 8.16. This is somewhat surprising since, at the level of the Kuranishi space (and Kuranishi stack), there is no difference: the Kuranishi space of a Kähler, even of a projective, manifold can exhibit all the pathologies (for example not irreducible [18], not reduced [35], arbitrary singularities [40]) the Kuranishi space of a non-Kähler, non class (\mathcal{C}) one can have. Moreover, due to the possibly wild action of the mapping class group, there is no difference between them at the level of the Riemann moduli stack. *This difference only appears when considering the Teichmüller stack.* Its full complexity is only seen at non-Kähler non class (\mathcal{C}) points *hence its geometry cannot be fully understood without dealing with such manifolds.* We hope to understand this better in the future.

Going back to the paper, we introduce the notion of normal and \mathbb{Z} -normal points in §7.1. They almost coincide with non-jumping, non-exceptional points. They form an open substack of $\mathcal{T}(M)$ and $\mathcal{T}^{\mathbb{Z}}(M)$. We show in Theorem 7.16 that both can be reduced to an étale stack, that is a stack locally isomorphic to an at most discrete quotient of an analytic space. These normal Teichmüller stacks are quite easy to handle. Once again, and for the same reasons, more can be said in the Kähler case: the normal Teichmüller stacks are then orbifolds.

At the end of the day, we obtain the following rough description of the geography of the Teichmüller stacks in the general case (see Theorem 13.3 for a precise statement):

- Jumping points are the most pathological points but are quite well understood and form a strict analytic substack of $\mathcal{T}(M)$ and $\mathcal{T}^{\mathbb{Z}}(M)$.
- Exceptional points exhibit a different strange behaviour but also quite well understood.
- Normal points form an open substack associated to the étale normal Teichmüller stack that is easy to handle.
- Complementary points (if exist) have a non-reduced Kuranishi space with some special property.

In the Kähler case³, making use of the compactness of the cycle spaces allows us to get a much more detailed picture of the geography of $\mathcal{T}(M)$ and $\mathcal{T}^{\mathbb{Z}}(M)$ (see Theorem 13.1 for a precise statement):

- The closure of exceptional points is a strict analytic substack of $\mathcal{T}(M)$ and of $\mathcal{T}^{\mathbb{Z}}(M)$.
- The normal Teichmüller stack is an orbifold.

Here is an outline of the paper. The main protagonists, that is the Teichmüller and Kuranishi stacks, are introduced in §2-3. The material comes essentially from [29] but with some slight differences and additions. Notably, we show in Theorem 3.11 that the germ of Kuranishi stack at a point has a universal property. This generalizes the semi-universality property of the Kuranishi space. At a rough level, this is folklore (see for example [42]), but we never saw a precise statement of such a property, probably because the stack setting developed in [29] is necessary to a get clear formulation. We introduce the \mathbb{Z} -Teichmüller stack in §4. Every statement about the

³Some - but not all - statements are valid under the weaker Fujiki class (\mathcal{C}) hypothesis.

Teichmüller stack given in this paper holds with obvious changes for the \mathbb{Z} -Teichmüller stack. In the sequel, to avoid redundancies, we often content ourselves with stating the full results and definitions only for the Teichmüller stack and with briefly indicating the changes needed for the \mathbb{Z} -Teichmüller stack. The main interest of introducing $\mathcal{T}^{\mathbb{Z}}(M)$ is the already cited Theorem 8.17 of a \mathbb{Z} -exceptional point. It is not an example of an exceptional point but it is a strong evidence that such points exist. Section §5 begins the local analysis of $\mathcal{T}(M)$ and $\mathcal{T}^{\mathbb{Z}}(M)$. It culminates with Theorem 5.3 showing that the natural morphism from the Kuranishi stack to the Teichmüller stack is étale. This gives a complete solution to the problem of comparing these two stacks at a point. Section §6 begins with Definition 6.1 of exceptional points and then states the main Conjecture 6.2 about them. In §7, we define and study normal points and the normal Teichmüller stack. The main result is the already mentioned Theorem 7.16. Of interest also is the characterization of points with Kuranishi stacks being orbifolds in §7.2. We then switch to the analysis of exceptional points in §8. We introduce the cycle spaces that are related to $\mathcal{T}(M)$ and $\mathcal{T}^{\mathbb{Z}}(M)$ but the most important results are Theorem 8.17 on a 3-fold that is a \mathbb{Z} -exceptional point and the associated Theorem 8.15 showing that the subgroup of the automorphism group of these 3-folds formed by elements inducing the identity in cohomology with coefficients in \mathbb{Z} is infinite discrete. Such a phenomenon cannot occur on Kähler manifolds, so we obtain here points whose \mathbb{Z} -Teichmüller stack is not locally isomorphic to that of any Kähler manifold, see Corollary 8.16. The next two sections §9-10 focus on the Kähler case. We first draw all the consequences of the compactness of cycle spaces in the Kähler setting, as Theorem 9.12 showing that the étale morphism from the Kuranishi stack to the Teichmüller stack is indeed finite, allowing to characterize easily points where $\mathcal{T}(M)$ is an orbifold from the analogous results proven for the Kuranishi stack. Then we show in Theorem 10.1 that the closure of exceptional points is an analytic substack of $\mathcal{T}(M)$ and $\mathcal{T}^{\mathbb{Z}}(M)$. As an interlude, §11 gives some variations about exceptionality, for example introducing the notion of exceptional pairs and showing that non-separated pairs of points are exceptional pairs in §11.3. Going back to the Kähler setting, we investigate in §12 the structure of $\mathcal{T}(M)$ from the side of pathological families. One of the main interests in using stacks is to give a dictionary between properties of $\mathcal{T}(M)$ as a moduli space and properties of families of compact complex manifolds diffeomorphic to M . In this way, jumping points are related to jumping families that is families with all fibers biholomorphic except for one. We define several types of pathologies and analyze them from both the family and the moduli space point of view. The philosophy developed in §12 is that, at least in the Kähler case, pathologies only may occur at jumping and/or exceptional points so occur on a strict analytic substack. We also comment in §12.2 on a false statement of [28]. Finally, we gather all the previous results to state in §13 the two main results (Theorems 13.3 and 13.1) on the cartography of $\mathcal{T}(M)$ and $\mathcal{T}^{\mathbb{Z}}(M)$ in the general and in the Kähler case. Many additional comments are included. We also revisit this geography through the concept of holonomy points that sheds a different light on our results. Jumping points are points with continuous holonomy, normal

points have at most discrete holonomy, and exceptional points are points a neighborhood of which is not controlled by the holonomy group. Section §13 ends with some remarks on Teichmüller stacks and GIT quotients.

The notions of exceptional vs. normal points are reminiscent from Catanese question in [9], see also [10], on conditions under which the Teichmüller and Kuranishi spaces are locally homeomorphic. This question has played a central role in the genesis of this work. We turned it into giving conditions under which the Teichmüller and Kuranishi stacks are locally isomorphic. We first thought that this is always the case, before the concept of exceptional points emerges.

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2. THE TEICHMÜLLER STACK: BASIC FACTS

We recollect some facts about the Teichmüller stack of a connected, compact oriented C^∞ manifold M admitting complex structures. We refer to [29] for more details.

The general idea is the following. From the one hand, the Teichmüller stack is the category of analytic families of compact complex manifolds diffeomorphic to M (in $\text{Diff}^0(M)$) together with a functor that sends a family to its base. From the other hand, it has an atlas (which is not unique), that is an analytic space with a smooth and surjective mapping to the Teichmüller stack. Roughly speaking, the Teichmüller stack appears thus as a quotient of the atlas, say T . The morphism from the atlas T to the Teichmüller stack $\mathcal{T}(M)$ is given by the choice of a family of compact complex manifolds above T , thanks to Yoneda's Lemma. It is smooth and surjective if the projections $T \times_{\mathcal{T}(M)} T \rightarrow T$ are smooth and surjective morphisms between analytic spaces. This allows to form the analytic groupoid $T \times_{\mathcal{T}(M)} T \rightrightarrows T$ that contains all the information needed to reconstruct the Teichmüller stack as a category of families. Indeed, starting with T , one recovers a stack isomorphic to $\mathcal{T}(M)$ through a process called stackification. And any other choice of an atlas of $\mathcal{T}(M)$ gives an analytic groupoid that is Morita equivalent to that associated to T .

Let us begin with the categorical viewpoint. Let \mathfrak{S} be the category of analytic spaces and morphisms endowed with the euclidian topology. Given $S \in \mathfrak{S}$, we call M -deformation over S a proper and smooth morphism $\mathcal{X} \rightarrow S$ whose fibers are compact complex manifolds diffeomorphic to M . As C^∞ -object, such a deformation is a bundle over S with fiber M and structural group $\text{Diff}^+(M)$ (diffeomorphisms of M that preserve its orientation). It is called *reduced* if the structural group is reduced to $\text{Diff}^0(M)$. In the same way, a morphism of reduced M -deformations \mathcal{X} and \mathcal{X}' over an analytic morphism $f : S \rightarrow S'$ is a cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}' \\ \downarrow & \square & \downarrow \\ S & \xrightarrow{f} & S' \end{array}$$

such that \mathcal{X} and $f^*\mathcal{X}'$ are isomorphic as $\text{Diff}^0(M)$ -bundles over S .

The Teichmüller stack $\mathcal{T}(M)$ is the stack over the site \mathfrak{S} whose objects are reduced M -deformations and morphisms are morphisms of reduced M -deformations. The natural morphism $\mathcal{T}(M) \rightarrow \mathfrak{S}$ sends a reduced M -deformation onto its base and a morphism of reduced M -deformation to the corresponding morphism between their bases. Alternatively, $\mathcal{T}(M)$ can be seen as a 2-functor from \mathfrak{S} to the category of groupoids such that

- i) $\mathcal{T}(M)(S)$ is the groupoid of isomorphism classes of reduced M -deformations over S .
- ii) $\mathcal{T}(M)(f)$ is the pull-back morphism f^* from $\mathcal{T}(M)(S')$ to $\mathcal{T}(M)(S)$.

A point $X_0 := (M, J_0)$ is an object of $\mathcal{T}(M)(pt)$ that is a complex structure on M up to biholomorphisms smoothly isotopic to the identity.

Remark 2.1. In the definition of analytic stack used in [29], we did not impose that the diagonal is representable, see the discussion in §2.4 of [29]. However, this is indeed true in full generality, hence the definition of an analytic stack as the stackification over \mathfrak{S} of a smooth analytic groupoid given in [29, §2.4] is equivalent to the definition of an analytic stack as a stack over \mathfrak{S} with representable diagonal, see [6, §2.4].

Let us switch to the atlas point of view. Roughly speaking, $\mathcal{T}(M)$ can be considered as an analytic version of the quotient $\mathcal{I}(M)/\text{Diff}^0(M)$. Here, $\mathcal{I}(M)$ is the set of integrable complex operators on M compatible with its orientation (o.c.), that is

$$(2.1) \quad \mathcal{I}(M) = \{J : TM \longrightarrow TM \mid J^2 \equiv -Id, J \text{ o.c.}, [T^{1,0}, T^{1,0}] \subset T^{1,0}\}$$

for

$$T^{1,0} = \{v - iJv \mid v \in TX\}.$$

and $\text{Diff}^0(M)$ is the group of diffeomorphisms of M which are C^∞ -isotopic to the identity. It acts on the right on $\mathcal{I}(M)$ through

$$(2.2) \quad J \cdot f := df^{-1} \circ J \circ df$$

Hence the mapping

$$(2.3) \quad x \in X_{J \cdot f} \longmapsto f(x) \in X_J$$

is an isomorphism (note the order: f sends $J \cdot f$ to J).

In [29], a finite-dimensional atlas T of (a connected component of) $\mathcal{T}(M)$ is described under the hypothesis that the dimension of the automorphism group of the complex manifolds encoded in $\mathcal{T}(M)$ is bounded. Basically, it is given by a (at most countable) disjoint union of Kuranishi spaces⁴. The construction of the Kuranishi space of a compact complex manifold is recalled in Section 3.1. The morphism from T to $\mathcal{T}(M)$ is given by the choice of a family of complex manifolds above T by Yoneda's Lemma. Here we take the Kuranishi families, see §3.1. Then, to finish the construction, we need to compute the fiber product $T \times_{\mathcal{T}(M)} T$ and check both projections to T are smooth and surjective, forming in this way an analytic groupoid that encodes completely $\mathcal{T}(M)$. This is the crux of [29]. In this paper, we will

⁴Indeed, for technical reasons, a fattening process is used to ensure that all components of T have the same dimension.

not make use of this global atlas since we analyse the local models, hence we will skip its precise construction.

Two points have to be emphasized here. Firstly, as a stack, $\mathcal{T}(M)$ also encodes the isotropy groups of the action. Recall that the isotropy group at X_0 is the group

$$(2.4) \quad \text{Aut}^1(X_0) := \text{Aut}(X_0) \cap \text{Diff}^0(M).$$

which may be different from $\text{Aut}^0(X_0)$, the connected component of the identity of the automorphism group $\text{Aut}(X_0)$, see [31] and [11]. Secondly, since $\text{Diff}^0(M)$ acts on the (infinite-dimensional) analytic space $\mathcal{I}(M)$ preserving its connected components and its irreducible components, we may speak in this way of connected components and irreducible components of $\mathcal{T}(M)$. Indeed Kuranishi's Theorem tells us that the set $\mathcal{I}(M)$ is locally the product of a finite-dimensional local analytic section K_0 to the action with an infinite-dimensional manifold, cf. Section 3.2. Hence, locally, the irreducible components of $\mathcal{I}(M)$ correspond to those of the finite-dimensional space K_0 .

We finish this part with some terminology. By Teichmüller *space*, we mean the topological space obtained by endowing the quotient of (2.1) by (2.2) with the quotient topology. It is of course different from the Teichmüller *stack* $\mathcal{T}(M)$, although related as follows. Given $T \rightarrow \mathcal{T}(M)$ an atlas and $T_1 \rightrightarrows T$ the associated groupoid, the Teichmüller space is homeomorphic to the quotient of T by the equivalence relation induced by T_1 . We also introduce the notion of open, resp. Zariski open, resp. analytic substack of $T \rightarrow \mathcal{T}(M)$, that plays an important role in the sequel. It is obtained as the stack induced from $T \rightarrow \mathcal{T}(M)$ by an open, resp. Zariski open subset, resp. analytic subspace of T . In other words, an analytic stack $A \rightarrow \mathcal{A}$ is an open, resp. Zariski open, resp. analytic substack of $T \rightarrow \mathcal{T}(M)$ if there exists an open, resp. Zariski open subset B , resp. analytic subspace B of T such that $A \rightarrow \mathcal{A}$ is isomorphic to the analytic groupoid $B \times_{\mathcal{T}(M)} B \rightrightarrows B$.

Remark 2.2. Such an analytic substack $A \rightarrow \mathcal{A}$ is said to be *proper* or *strict* if B can be chosen so that it is an analytic subspace of positive codimension in the reduction of T .

Note also that we admit analytic subspaces B with a countable number of connected components, cf. Example 13.4.

3. THE KURANISHI STACK

In the first two subsections, we review the construction of the Kuranishi family first from classical deformation theory point of view, then from Kuranishi-Douady's point of view.

We then review the construction of the Kuranishi stack(s) introduced in [29]. They play a fundamental role in the local theory. Especially we prove in Subsection 3.4 that they enjoy the universal property the Kuranishi space does not fulfill.

It is worth pointing out that the classical point of view (which presents Kuranishi family from a formal/algebraic point of view leaving aside the analytic details of the construction) is not enough for our purposes. This is indeed an infinitesimal point of view and even if it gives complete equations

for the Kuranishi space, it fails in describing the properties of the structures close to the base complex structure. Kuranishi-Douady's point of view allows to pass from the infinitesimal point of view to a local one.

3.1. The Kuranishi family. The Kuranishi family $\pi : \mathcal{X}_0 \rightarrow K_0$ of X_0 is a semi-universal deformation of X_0 . It comes with a choice of a marking, that is of an isomorphism i between X_0 and the fiber $\pi^{-1}(0)$ over the base point 0 of K_0 . The semi-universal property means that

- i) Every marked deformation $\mathcal{X} \rightarrow B$ of X_0 is locally isomorphic to the pull-back of the Kuranishi family by a pointed holomorphic map f defined in a small neighborhood of the base point of B with values in a neighborhood of 0 in K_0 .
- ii) Neither the mapping f nor its germ at the base point are unique; but its differential at the base point is.

Two such semi-universal deformations of X_0 are isomorphic up to restriction to a smaller neighborhood of their base points. Hence the germ of deformation $(\mathcal{X}_0, \pi^{-1}(0)) \rightarrow (K_0, 0)$ is unique. This explains why we talk of the Kuranishi family, even if, in many cases, we work with a representative of the germ rather than with the germ itself.

The Zariski tangent space to the Kuranishi space K_0 at 0 identifies naturally with $H^1(X_0, \Theta_0)$, the first cohomology group with values in the sheaf Θ_0 of germs of holomorphic tangent vector fields of X_0 . Indeed, K_0 is locally isomorphic to an analytic subspace of $H^1(X_0, \Theta_0)$ whose equations coincide at order 2 with the vanishing of the Schouten bracket.

The groups $\text{Aut}(X_0)$, $\text{Aut}^0(X_0)$ and $\text{Aut}^1(X_0)$ act on this tangent space. However, this infinitesimal action cannot always be integrated in an action of the automorphism groups of X_0 onto K_0 , see [13]. Still there exists an action of each 1-parameter subgroup and all these actions can be encoded in an analytic groupoid and thus in a stack. To do this, we need to know more about the complex properties of the structures encoded in a neighborhood of 0 in K_0 .

3.2. Kuranishi-Douady's presentation. Let V be an open neighborhood of J_0 in $\mathcal{I}(M)$. Complex structures close to J_0 can be encoded as $(0, 1)$ -forms ω with values in $T^{1,0}$ which satisfy the equation

$$(3.1) \quad \bar{\partial}\omega + \frac{1}{2}[\omega, \omega] = 0$$

Choose an hermitian metric and let $\bar{\partial}^*$ be the L^2 -adjoint of $\bar{\partial}$ with respect to this metric. Let U be a neighborhood of 0 in the space of global smooth sections of $(T^{0,1})^* \otimes T^{1,0}$. Set

$$(3.2) \quad K_0 := \{\omega \in U \mid \bar{\partial}\omega + \frac{1}{2}[\omega, \omega] = \bar{\partial}^*\omega = 0\}$$

Let W an open neighborhood of 0 in the vector space of vector fields L^2 -orthogonal to the vector space of holomorphic vector fields $H^0(X_0, \Theta_0)$. In Douady's setting [14], Kuranishi's Theorem states the existence of a local isomorphism between $\mathcal{I}(M)$ at J_0 and the product of K_0 with W such that every plaque $\{pt\} \times W$ is sent through the inverse of this isomorphism into a

single local $\text{Diff}^0(M)$ -orbit. To be more precise, up to restricting U , V and W , the Kuranishi mapping

$$(3.3) \quad (\xi, J) \in W \times K_0 \longmapsto J \cdot e(\xi) \in V$$

is an isomorphism of infinite-dimensional analytic spaces. As usual, we use the exponential map associated to the chosen metric in order to define the map e which gives a local chart of $\text{Diff}^0(M)$ at Id . And \cdot denotes the natural right action (2.2) of $\text{Diff}^0(M)$ onto $\mathcal{I}(M)$.

Remark 3.1. In the sequel, we will always work with an open set V of $\mathcal{I}(M)$ which is a product through (3.3). Especially, in all statements, the expression ‘‘Reducing V if necessary’’ or ‘‘For V small enough’’ must be understood as taking a smaller V but that still satisfies (3.3) for a smaller K_0 .

3.3. Automorphisms and jumping points. Let us now analyze how automorphisms of X_0 are related to the local geography of the Teichmüller stack and determine a first obstruction for $\mathcal{S}(M)$ to be locally isomorphic to an analytic space or to an orbifold.

We begin with a construction in a slightly more general setting. Let f be an element of $\text{Diff}^0(M)$ such that $J_0 \cdot f$ belongs to V , e.g. f is an automorphism of X_0 . Composing the inverse of (3.3) with the projection onto K_0 gives a retraction map $\Xi : V \rightarrow K_0$. Then, define

$$(3.4) \quad U_f = \{J \in K_0 \mid J \cdot f \in V\}$$

This is an open set since V is open. And it contains J_0 . Now

$$(3.5) \quad \text{Hol}_f : J \in U_f \subset K_0 \longmapsto \Xi(J \cdot f) \in K_0$$

is a well defined analytic map that must be thought of as the action of f onto K_0 . Note however that composition does not work in general, that is $\text{Hol}_{f \circ g}$ may be different from $\text{Hol}_g \circ \text{Hol}_f$. In particular, there is no well defined action of $\text{Aut}^0(X_0)$ onto K_0 , see [13] for a counterexample.

Let $(F_t)_{t \in [0,1]}$ be a continuous path in $\text{Diff}^0(M)$ joining f to the identity. Assume that $J_0 \cdot F_t$ belongs to V for all $t \in [0, 1]$, e.g. $f \in \text{Aut}^0(X_0)$ and (F_t) is any continuous path in $\text{Aut}^0(X_0)$ joining f to the identity. We call *compatible* such a path. Then, U_{F_t} and Hol_{F_t} are well defined for all t and all U_{F_t} contain J_0 .

Lemma 3.2. *The intersection $\bigcap_{t \in [0,1]} U_{F_t}$ contains an open neighborhood of J_0 .*

Proof. For every $t \in [0, 1]$, choose some relatively compact open set $V_t \Subset U_{F_t}$. By continuity, there exists an open interval I_t containing t such that $t' \in I_t$ means $V_t \subset U_{F_{t'}}$. By compactity, there exist t_1, \dots, t_k in $[0, 1]$ such that

$$[0, 1] = I_{t_1} \cup \dots \cup I_{t_k}$$

Hence,

$$\forall t \in [0, 1], \quad \bigcap_{1 \leq i \leq k} V_{t_i} \subset U_{F_t}$$

and we are done, since the lefthand term is a non-empty open set containing J_0 as a finite intersection of such open sets. \square

Thanks to Lemma 3.2, we may define the open set

$$(3.6) \quad \Omega_f := \bigcup_{(F_t)_{t \in [0,1]}} \text{Int} \left(\bigcap_{t \in [0,1]} U_{F_t} \right)$$

where the union is taken on all compatible paths $(F_t)_{t \in [0,1]}$ in $\text{Diff}^0(M)$.

By definition, $J \in \Omega_f$ if and only if there exists a compatible path (F_t) in $\text{Diff}^0(M)$ such that $J \cdot F_t \in V$ for all $t \in [0, 1]$.

Define now the function

$$(3.7) \quad t \in K_0 \mapsto h^0(t) := \dim(\text{Aut}(X_t)) \in \mathbb{N}$$

It is a well known fact that (3.7) may not be constant - the best we can say is that it is upper semicontinuous for the Zariski topology of K_0 , see [17] and footnote 1. When it is constant and K_0 is reduced, a classical Theorem of Wavrik [43] asserts the Kuranishi space is universal, i.e. f is unique in the setting of Subsection 3.1. But we also have

Lemma 3.3. *Assume that (3.7) is constant and K_0 is reduced. Then Hol_f is equal to the identity of K_0 for any $f \in \text{Aut}^0(X_0)$.*

We will come back to this in Section 7 when defining and studying normal points.

Proof. Let $f \in \text{Aut}^0(X_0)$ and let $J \in \Omega_f$. We assume that J is different from J_0 and we let $(F_t)_{t \in [0,1]}$ be a compatible path in $\text{Aut}^0(X_0)$ with $J \cdot F_t$ in V for all t . By definition, $\text{Hol}_{F_t}(J)$ is well defined for all $t \in [0, 1]$, drawing a continuous path in K_0 between J and $\text{Hol}_f(J)$ all of whose points encode the same manifold X_J up to biholomorphism C^∞ -isotopic to the identity. Assume this path is non-constant. Then we can find distinct points of K_0 encoding X_J through a biholomorphism arbitrary close to the identity in $\text{Diff}^0(M)$ -topology. Since (3.7) is constant, this would contradict Theorem 1 of [26]. Hence the path has to be constant so that $\text{Hol}_f(J) = J$ for all $J \in \Omega_f$. Since K_0 is reduced, this is enough to conclude that Hol_f is the identity on Ω_f , thus on U_f by analyticity. But U_f must then be equal to K_0 . \square

In the general case, that is when (3.7) is not constant, the line of arguments used in the proof of Lemma 3.3 can be expanded to show that K_0 acquires a stratified foliated structure as defined and analyzed in [28, §3]. Firstly one decomposes K_0 into strata $(K_0)_a$ with function (3.7) bounded above by a . Then each difference $S_a := (K_0)_a \setminus (K_0)_{a-1}$ admits a holomorphic foliation with non singular leaves. The highest S_a contains at least the base point J_0 and its foliation is a foliation by points. The other S_a admits positive-dimensional leaves. The leaves correspond to the connected components in K_0 of the following equivalence relation: $J \equiv J'$ if and only if both operators belong to the same $\text{Diff}^0(M)$ -orbit.

In other words, when the function (3.7) is not constant, some positive-dimensional connected submanifolds of K_0 encode the same complex manifold up to biholomorphism C^∞ -isomorphic to the identity. Hence the Teichmüller space is not homeomorphic to K_0 . Indeed, it is the leaf space of this stratified foliated structure and is usually non-Hausdorff. We thus set

Definition 3.4. We say that X_0 is a *jumping point* of $\mathcal{T}(M)$ if (3.7) is not locally constant.

Jumping points are the first obstruction for $\mathcal{T}(M)$ to be locally isomorphic to an analytic space or an orbifold and correspond to points where the Teichmüller space has very bad properties. However, as recalled above, from the one hand they form a strict analytic substack of $\mathcal{T}(M)$; and from the other hand, they correspond to points with non trivial foliated structure. Describing them geometrically boils down to describing the foliated structure of K_0 . The first draft is done in [28] but there is still much to do. One of the most challenging question is the following.

Question 3.5. When (3.7) is non constant, does the foliated structure always admit a separatrix?

By separatrix, we mean a positive-dimensional leaf that contains 0 in its closure. The existence of a separatrix at a jumping point implies that there exists a jumping family based at that point, see Example 11.5 and Section 12.

Remark 3.6. Given an automorphism f of X_0 , observe that Hol_f respects the foliation of K_0 . Moreover, for f in $\text{Aut}^0(X_0)$ close to the identity, then Hol_f fixes each leaf of the foliation of K_0 . However, a general element of $\text{Aut}^0(X_0)$ may send a leaf to a different leaf. This is due to the fact that the restriction to V may disconnect the $\text{Aut}^0(X_0)$ -orbits in $\mathcal{I}(M)$.

3.4. The Kuranishi stacks. The Kuranishi stacks encode the maps (3.5) in an analytic groupoid. The first step to do this consists in proving that there is an isomorphism

$$(3.8) \quad (\xi, g) \in W \times \text{Aut}^0(X_0) \longmapsto g \circ e(\xi) \in \mathcal{D}_0$$

with values in a neighborhood \mathcal{D}_0 of $\text{Aut}^0(X_0)$ in $\text{Diff}^0(M)$, see [29, Lemma 4.2].

Let now $\text{Diff}^0(M, \mathcal{K}_0)$ denote the set of C^∞ diffeomorphisms from M to a fiber of the Kuranishi family $\mathcal{K}_0 \rightarrow K_0$. This is an infinite-dimensional analytic space⁵, see [14]. Here by $(J, F) \in \text{Diff}^0(M, \mathcal{K}_0)$, we mean that we consider F as a diffeomorphism from M to the complex manifold X_J .

Definition 3.7. Given (J, F) an element of $\text{Diff}^0(M, \mathcal{K}_0)$, we say it is (V, \mathcal{D}_0) -admissible if there exists a finite sequence (J_i, F_i) (for $0 \leq i \leq p$) of $\text{Diff}^0(M, \mathcal{K}_0)$ such that

- i) $J_0 = J$ and $J_{i+1} = J_i \cdot F_i$ is in K_0 for all $0 \leq i \leq p$, adding the convention $J_{p+1} := J \cdot F$.
- ii) $F = F_0 \circ \dots \circ F_p$.
- iii) Each F_i belongs to \mathcal{D}_0 as well as F_i^{-1} .
- iv) J_i belongs to Ω_{F_i} for all $0 \leq i \leq p$ and to $\Omega_{F_{i-1}^{-1}}$ for all $1 \leq i \leq p+1$.

⁵Strictly speaking, we have to pass to Sobolev L^2_l -structures for a big l to have an analytic space, and $\text{Diff}^0(M, \mathcal{K}_0)$ is the subset of C^∞ points of this analytic set. In the sequel, we automatically make this slight abuse of terminology, cf. Convention 3.2 in [29].

Remark 3.8. It could seem more natural to speak of (K_0, \mathcal{D}_0) -admissible, but, by Remark 3.1, changing V is equivalent to changing K_0 when \mathcal{D}_0 is fixed. We keep this terminology to be coherent with that of [29]. It should be pointed however that the previous definition is a bit more restrictive than that of [29]. We shall see that the additional point *iv*) plays a crucial role in the proof of Lemma 5.2.

Notice that, given (J, F) and $(J \cdot F, F')$ both (V, \mathcal{D}_0) -admissible, then $(J, F \circ F')$ is also (V, \mathcal{D}_0) -admissible, as well as $(J \cdot F, F^{-1})$. We set then

$$(3.9) \quad \mathcal{A}_0 = \{(J, F) \in \text{Diff}^0(M, \mathcal{K}_0) \mid (J, F) \text{ is } (V, \mathcal{D}_0)\text{-admissible}\}$$

This set encodes identifications between structures in K_0 that are given by composing diffeomorphisms in the neighborhood \mathcal{D}_0 .

We also consider the two maps from \mathcal{A}_0 to K_0

$$(3.10) \quad s(J, F) = J \quad \text{and} \quad t(J, F) = J \cdot F$$

and the composition and inverse maps

$$(3.11) \quad m((J, F), (J \cdot F, F')) = (J, F \circ F'), \quad i(J, F) = (J \cdot F, F^{-1})$$

With these structure maps, the groupoid $\mathcal{A}_0 \rightrightarrows K_0$ is an analytic groupoid [29, Prop. 4.6] whose stackification over \mathfrak{S} is called *the Kuranishi stack* of X_0 . We denote it by \mathcal{A}_0 . Note that it depends indeed of the particular choice of V .

As a category, its objects are still reduced M -deformations over bases belonging to \mathfrak{S} . However, the allowed complex structures are those encoded in V ; and the allowed families are those obtained by gluing pull-back families of $\mathcal{K}_0 \rightarrow K_0$ with respect to (V, \mathcal{D}_0) -admissible diffeomorphisms. In the same way, morphisms are those induced by (V, \mathcal{D}_0) -admissible diffeomorphisms. Hence, not only the complex fibers of the families have to be isomorphic to those of \mathcal{K}_0 , but gluings and morphisms of families are restricted.

Of course, the same construction can be carried out for the automorphism groups $\text{Aut}^1(X_0)$, resp. $\text{Aut}(X_0)$, with the following modifications. In (3.8), $\text{Aut}^0(X_0)$ is replaced with $\text{Aut}^1(X_0)$, resp. $\text{Aut}(X_0)$, defining a neighborhood \mathcal{D}_1 of $\text{Aut}^1(X_0)$ in $\text{Diff}^0(M)$, resp. \mathcal{D} of $\text{Aut}(X_0)$ in $\text{Diff}^+(M)$. This allows to speak of (V, \mathcal{D}_1) -admissible, resp. (V, \mathcal{D}) -admissible diffeomorphisms. But in the \mathcal{A} and \mathcal{A}_1 cases, the sets Ω_{F_i} , resp. $\Omega_{F_{i-1}^{-1}}$, in point *iv*) must be replaced with U_{F_i} , resp. $U_{F_{i-1}^{-1}}$ in the definition of admissibility, since $\text{Aut}(X_0)$ and $\text{Aut}^1(X_0)$ may contain elements that are not connected to the identity. Then, replacing \mathcal{D}_0 with \mathcal{D}_1 , resp. \mathcal{D} in (3.9) we obtain the analytic groupoid $\mathcal{A}_1 \rightrightarrows K_0$, resp. $\mathcal{A} \rightrightarrows K_0$. Its stackification over \mathfrak{S} gives a stack \mathcal{A}_1 , resp. \mathcal{A} . The previous description of \mathcal{A}_0 as a category applies to \mathcal{A}_1 , resp. \mathcal{A} with the obvious changes. We also call them *Kuranishi stacks*.

Before analyzing more thoroughly these Kuranishi stacks, we would like to say a little more about automorphisms and K_0 . Given $f \in \text{Aut}(X_0)$, define Hol_f as in (3.5) and set

$$(3.12) \quad \sigma_f : J \in U_f \subset K_0 \longmapsto (J, f \circ e(\chi(J))) \in \mathcal{A}$$

where χ is an analytic mapping from $U_f \subset K_0$ to W with $\chi(0) = 0$ defined by

$$(3.13) \quad \text{Hol}_f(J) = \Xi(J \cdot f) = (J \cdot f) \cdot e(\chi(J))$$

The map σ_f is a local analytic section of the source map $s : \mathcal{A} \rightarrow K_0$ defined on U_f . It satisfies

$$(3.14) \quad t \circ \sigma_f = \text{Hol}_f$$

Moreover, when f is an element of $\text{Aut}^0(X_0)$, it has values in \mathcal{A}_0 , when restricted to Ω_f .

Finally, we prove that \mathcal{A}_0 contains the connected component of the identity of the automorphism group of every fiber.

Lemma 3.9. *Given any $J \in K_0$ and any $f \in \text{Aut}^0(X_J)$, then (J, f) belongs to \mathcal{A}_0 .*

Proof. Let $J \in K_0$ and let $f \in \text{Aut}^0(X_J)$. Assume that f is sufficiently close to the identity to belong to \mathcal{D}_0 . Then, using (3.8), we find $g \in \text{Aut}^0(X_0)$ such that J belongs to U_g and

$$(J, f) = \sigma_g(J)$$

This proves that (J, f) belongs to \mathcal{A}_0 as soon as f is small enough. Since any element in $\text{Aut}^0(X_J)$ is a finite composition of small elements, this is still true for any $f \in \text{Aut}^0(X_J)$. \square

Connexity is crucial here. Lemma 3.9 does not hold true for $\text{Aut}^1(X_J)$.

3.5. Universality of the Kuranishi stacks. Recall that Kuranishi's Theorem asserts the existence of a semi-universal deformation for any compact complex manifold. This is however not a universal deformation when the dimension of the automorphism group varies in the fibers of the Kuranishi family, i.e. in the setting of section 3.1, the germ of mapping f is not unique. Replacing the Kuranishi space with the Kuranishi stack allows to recover a universality property.

To do that, we need to germify the Kuranishi stacks. We replace our base category \mathfrak{S} with the base category \mathfrak{G} of germs of analytic spaces. We turn \mathfrak{G} into a site by considering the trivial coverings. Hence each object of \mathfrak{G} has a unique covering and there is no non trivial descent data.

We then germify the groupoids. Starting with $\mathcal{A} \rightrightarrows K_0$, resp. $\mathcal{A}_0 \rightrightarrows K_0$ and $\mathcal{A}_1 \rightrightarrows K_0$, and using s and t as defined in (3.10), we germify K_0 at 0, \mathcal{A} , resp. \mathcal{A}_0 and \mathcal{A}_1 , at the fiber $(s \times t)^{-1}(0)$ and germify consequently all the structure maps. We thus obtain the groupoids $(\mathcal{A}, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$, resp. $(\mathcal{A}_0, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$ and $(\mathcal{A}_1, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$.

Finally, we stackify $(\mathcal{A}, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$, resp. $(\mathcal{A}_0, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$ and $(\mathcal{A}_1, (s \times t)^{-1}(0)) \rightrightarrows (K_0, 0)$, over \mathfrak{G} . We denote the corresponding stacks by $(\mathcal{A}, 0)$, resp. $(\mathcal{A}_0, 0)$ and $(\mathcal{A}_1, 0)$.

The objects of $(\mathcal{A}, 0)$ over a germ of analytic space $(S, 0)$ are germs of M -deformations $p : \mathcal{X} \rightarrow S$ with fiber at the point 0 of S isomorphic to X_0 . We denote them by $(\mathcal{X}, p^{-1}(0)) \rightarrow (S, 0)$. The morphisms over some analytic mapping $f : S \rightarrow S'$ are germs of morphisms between M deformations $(\mathcal{X}, p^{-1}(0)) \rightarrow (S, 0)$ and $(\mathcal{X}', p'^{-1}(0)) \rightarrow (S', 0')$ over f . Note that $f(0) = 0'$.

Remark 3.10. It is crucial to notice that we deal with germs of *unmarked* deformations. There is obviously a distinguished point (since we deal with germs), but there is no marking of the distinguished fiber.

The following theorem shows that $(\mathcal{A}, 0)$ contains indeed *all* such germs of M -deformations and of morphisms between M -deformations. It is folklore although we never saw a paper stating this in a precise way.

Theorem 3.11. *The stack $(\mathcal{A}, 0)$ is the stack \mathcal{M} over \mathfrak{G} whose objects are the germs of M -deformations of X_0 and whose morphisms are the germs of morphisms between M -deformations.*

Proof. Since the site \mathfrak{G} does not contain any non-trivial covering, there is no gluings of families, and the torsors associated to $(\mathcal{A}, 0)$ are just given by the pull-backs of the germ of Kuranishi family $(\mathcal{K}_0, \pi^{-1}(0)) \rightarrow (K_0, 0)$. Kuranishi's Theorem implies that the natural inclusion of $(\mathcal{A}, 0)$ in the stack \mathcal{M} is essentially surjective.

Morphisms over the identity of some germ $(S, 0)$ of analytic space are thus given by morphisms F between two germs of families $(f^* \mathcal{K}_0, \pi^{-1}(0)) \rightarrow (B, 0)$ and $(g^* \mathcal{K}_0, \pi^{-1}(0)) \rightarrow (B, 0)$ for f and g germs of analytic mappings from $(B, 0)$ to $(K_0, 0)$. Hence F restricted to the central fiber $X_0 \simeq \pi^{-1}(0)$ is an automorphism of the central fiber that is an element of $\text{Aut}(X_0)$. But $(s \times t)^{-1}(0)$ is isomorphic to $\text{Aut}(X_0)$ so such a morphism F is induced by an analytic mapping from $(B, 0)$ to $(\mathcal{A}, (s \times t)^{-1}(0))$ that we still denote by F which satisfies $s \circ F = f$ and $t \circ F = g$. This shows that the natural inclusion of $(\mathcal{A}, 0)$ in the stack \mathcal{M} is fully faithful. \square

This must be thought of as the good property of universality. Indeed, the failure of universality in Kuranishi's theorem comes from the existence of automorphisms of the Kuranishi family fixing the central fiber but not all the fibers. Imposing a marking is an artificial and incomplete solution to this problem because it only kills automorphisms inducing a non-trivial automorphism on the central fiber. Now, the stack $(\mathcal{A}, 0)$ is universal for germs of M -deformations of X_0 , because, thanks to Theorem 3.5, any such germ $(\mathcal{X}, X_0) \rightarrow (B, 0)$ is induced by an analytic map from its base to $(\mathcal{A}, 0)$, yielding a diagram

$$(3.15) \quad \begin{array}{ccc} (\mathcal{X}, X_0) & \xrightarrow{F} & (\mathcal{K}_0, \pi^{-1}(0)) \\ \downarrow & & \downarrow \pi \\ (B, 0) & \xrightarrow{f} & (K_0, 0) \end{array}$$

Moreover the full map (f, F) is unique up to unique isomorphism of family encoded in $(\mathcal{A}, 0)$. So thinking of $(\mathcal{A}, 0)$ as the quotient of the Kuranishi space by the automorphisms of the Kuranishi family, not only the map f but also the full map (f, F) is unique; and this occurs with no extra condition.

In the same way, we have

Corollary 3.12. *The stack $(\mathcal{A}_1, 0)$ is the stack over \mathfrak{G} whose objects are the germs of reduced M -deformations and whose morphisms are the germs of morphisms between reduced M -deformations.*

Here C^∞ -markings of the M -deformations, that is the choice of a C^∞ diffeomorphism from M to the central fiber, can be used to characterize reduced families. Morphisms are required to induce on M a diffeomorphism isotopic to the identity through the markings. Here again, we may rephrase this Corollary as: the stack $(\mathcal{A}_1, 0)$ is universal for germs of reduced M -deformations of X_0 .

We also have

Corollary 3.13. *The stack $(\mathcal{A}_0, 0)$ is the stack over \mathfrak{G} whose objects are the germs of 0-reduced M -deformations and whose morphisms are the germs of morphisms between 0-reduced M -deformations.*

In other words, the stack $(\mathcal{A}_0, 0)$ is universal for germs of 0-reduced M -deformations of X_0 . A 0-reduced M -deformation is just a marked family. We use a different terminology because morphisms are different. A morphism of marked families is required to induce on X_0 the identity through the markings, whereas a morphism of 0-reduced M -deformation is required to induce on X_0 an element of $\text{Aut}^0(X_0)$ through the markings.

With this difference in mind, it is interesting to compare Corollary 3.13 with the already cited classical statement of universality of [43], see also [26]. When the dimension of the automorphism group is constant on the fibers of the Kuranishi family and the Kuranishi space is reduced, every element of $\text{Aut}^0(X_0)$ extends as an automorphism of the Kuranishi family inducing the identity on K_0 . Imposing a marking of the families prevents from reparametrizing with an element of $\text{Aut}(X_0)/\text{Aut}^0(X_0)$, yielding unicity of the pull-back morphism f and universality in the classical sense. However, universality in the stack sense of Corollary 3.13 is

- i) more general because it does not need extra hypotheses.
- ii) more natural because the good condition to impose on the central fiber is to authorize reparametrizations by an element of $\text{Aut}^0(X_0)$, and not to prevent any reparametrization as the classical marking does.
- iii) more precise because it gives unicity of the full mapping (f, F) of (3.15), that is it keeps track of the automorphisms of the Kuranishi family, even if they induce the identity on the base.

4. THE \mathbb{Z} -TEICHMÜLLER STACK

Motivated by [11], we introduce now the \mathbb{Z} -Teichmüller stack as a new stack intermediary between the Teichmüller stack and the moduli stack. It will play an important role when analyzing exceptional points. All that has been said before on the Teichmüller stack can be easily adapted to the \mathbb{Z} -Teichmüller stack.

4.1. Definition and basic facts. Let $\text{Diff}^{\mathbb{Z}}(M)$ be the subgroup of $\text{Diff}^+(M)$ of diffeomorphisms that induce the identity on the singular cohomology groups $H^*(M, \mathbb{Z})$. Recall that the C^∞ -type of a M -deformation, resp. a reduced M -deformation, is a bundle over some base S with fiber M and structural group $\text{Diff}^+(M)$, resp. $\text{Diff}^0(M)$. In the same way, a M -deformation is called \mathbb{Z} -reduced if the structural group is reduced to $\text{Diff}^{\mathbb{Z}}(M)$. And a morphism of \mathbb{Z} -reduced M -deformations \mathcal{X} and \mathcal{X}' over an analytic morphism

$f : S \rightarrow S'$ is a cartesian diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}' \\ \downarrow & \square & \downarrow \\ S & \xrightarrow{f} & S' \end{array}$$

such that \mathcal{X} and $f^*\mathcal{X}'$ are isomorphic as $\text{Diff}^{\mathbb{Z}}(M)$ -bundles over S .

The \mathbb{Z} -Teichmüller stack $\mathcal{T}^{\mathbb{Z}}(M)$ is the stack over the site \mathfrak{S} whose objects are \mathbb{Z} -reduced M -deformations and morphisms are morphisms of \mathbb{Z} -reduced M -deformations. The natural morphism $\mathcal{T}^{\mathbb{Z}}(M) \rightarrow \mathfrak{S}$ sends a \mathbb{Z} -reduced M -deformation onto its base and a morphism of \mathbb{Z} -reduced M -deformation to the corresponding morphism between their bases. By a direct adaptation of [29], it is an analytic stack under the hypothesis that the h^0 -function is bounded on the set $\mathcal{I}(M)$. Indeed, the analytic atlas T of the Teichmüller stack constructed in [29] under the same hypothesis is also an analytic atlas of $\mathcal{T}^{\mathbb{Z}}(M)$. The difference between the two cases occur when computing the fiber product $T \times_{\mathcal{T}^{\mathbb{Z}}(M)} T$ but the projections are still smooth and surjective. A point $X_0 := (M, J_0)$ is an object of $\mathcal{T}^{\mathbb{Z}}(M)(pt)$ that is a complex structure on M up to biholomorphisms inducing the identity in cohomology with \mathbb{Z} -coefficients⁶.

From the natural inclusions $\text{Diff}^0(M) \subset \text{Diff}^{\mathbb{Z}}(M) \subset \text{Diff}^+(M)$, we deduce the natural inclusions

$$(4.1) \quad \mathcal{T}(M) \hookrightarrow \mathcal{T}^{\mathbb{Z}}(M) \hookrightarrow \mathcal{M}(M)$$

meaning that an object, resp. a morphism of $\mathcal{T}(M)$ is also an object, resp. a morphism of $\mathcal{T}^{\mathbb{Z}}(M)$ and that an object, resp. a morphism of $\mathcal{T}^{\mathbb{Z}}(M)$ is also an object, resp. a morphism of $\mathcal{M}(M)$. The isotropy group of X_0 as a point of $\mathcal{T}^{\mathbb{Z}}(M)$ is the group

$$(4.2) \quad \text{Aut}^{\mathbb{Z}}(X_0) := \text{Aut}(X_0) \cap \text{Diff}^{\mathbb{Z}}(M).$$

which contains both $\text{Aut}^0(X_0)$ and $\text{Aut}^1(X_0)$ and is contained in $\text{Aut}(X_0)$. Note that all inclusions may be strict [11], showing in particular that the first inclusion map of (4.1) may also be strict.

4.2. The \mathbb{Z} -Kuranishi stack. The construction of §3.4 can be carried out for the automorphism groups $\text{Aut}^{\mathbb{Z}}(X_0)$ with the following modifications. In (3.8), $\text{Aut}^0(X_0)$ is replaced with $\text{Aut}^{\mathbb{Z}}(X_0)$ defining a neighborhood $\mathcal{D}_{\mathbb{Z}}$ of $\text{Aut}^{\mathbb{Z}}(X_0)$ in $\text{Diff}^{\mathbb{Z}}(M)$. This allows to speak of $(V, \mathcal{D}_{\mathbb{Z}})$ -admissible diffeomorphisms. The sets Ω_{F_i} , resp. $\Omega_{F_{i-1}^{-1}}$, in point iv must be replaced with U_{F_i} , resp. $U_{F_{i-1}^{-1}}$ in the definition of admissibility, since $\text{Aut}^{\mathbb{Z}}(X_0)$ may contain elements that are not connected to the identity. Then, replacing \mathcal{D}_0 with $\mathcal{D}_{\mathbb{Z}}$ in (3.9) we obtain the analytic groupoid $\mathcal{A}_{\mathbb{Z}} \rightrightarrows K_0$. Its stackification over \mathfrak{S} gives a stack $\mathcal{A}_{\mathbb{Z}}$ that we call the \mathbb{Z} -Kuranishi stack of X_0 .

We may now deduce another Corollary to Theorem 3.11 that refers to \mathbb{Z} -reduced M -deformations. As in §9.3, we germify the analytic groupoid $\mathcal{A}_{\mathbb{Z}} \rightrightarrows K_0$ and denote its stackification over \mathfrak{S} by $(\mathcal{A}_{\mathbb{Z}}, 0)$. We then have

⁶Of course, there are natural variants of the \mathbb{Z} -Teichmüller stack by considering other cohomologies or homologies such as singular cohomology with coefficients in \mathbb{Q} .

Corollary 4.1. *The stack $(\mathcal{A}_{\mathbb{Z}}, 0)$ is the stack over \mathfrak{G} whose objects are the germs of \mathbb{Z} -reduced M -deformations and whose morphisms are the germs of morphisms between \mathbb{Z} -reduced M -deformations.*

Here C^∞ -markings of the M -deformations, that is the choice of a C^∞ diffeomorphism from M to the central fiber, can be used to characterize \mathbb{Z} -reduced families. Morphisms are required to induce on M a diffeomorphism inducing the identity in cohomology through the markings. Here again, we may rephrase this Corollary as: the stack $(\mathcal{A}_{\mathbb{Z}}, 0)$ is universal for germs of \mathbb{Z} -reduced M -deformations of X_0 .

5. LOCAL STRUCTURE OF THE TEICHMÜLLER STACKS

A neighborhood of X_0 in $\mathcal{T}(M)$ consists of M -deformations all of whose fibers are close to X_0 , that is can be encoded by structures J living in a neighborhood V of J_0 in $\mathcal{I}(M)$. As in [29], we shall denote it by $\mathcal{T}(M, V)$. The corresponding neighborhood of $\mathcal{T}^{\mathbb{Z}}(M)$ is denoted by $\mathcal{T}^{\mathbb{Z}}(M, V)$. From now on, we assume that V is open, connected and small enough to come equipped with a Kuranishi mapping (3.3).

5.1. Atlas. The main difficulty to construct an atlas in [29] was to describe all the morphisms between the different Kuranishi spaces involved to compute the fiber product. Here, in the local case, we just need to use one Kuranishi space and family as atlas and it is straightforward to give the associated groupoid for $\mathcal{T}(M, V)$. Just consider

$$(5.1) \quad \mathcal{T}_V := \{(J, f) \in \text{Diff}^0(M, \mathcal{K}_0) \mid J \cdot f \in K_0\}$$

and the groupoid $\mathcal{T}_V \rightrightarrows K_0$ with structure maps as in (3.10) and (3.11). And consider

$$(5.2) \quad \mathcal{T}_V^{\mathbb{Z}} := \{(J, f) \in \text{Diff}^{\mathbb{Z}}(M, \mathcal{K}_0) \mid J \cdot f \in K_0\}$$

and $\mathcal{T}_V^{\mathbb{Z}} \rightrightarrows K_0$ for a neighborhood of X_0 in $\mathcal{T}^{\mathbb{Z}}(M)$.

Observe that (5.1), resp. (5.2) is very close to the groupoid $\mathcal{A}_1 \rightrightarrows K_0$ of the Kuranishi stack \mathcal{A}_1 , resp. of $\mathcal{A}_{\mathbb{Z}} \rightrightarrows K_0$. Indeed the points of $\mathcal{T}(M, V)$, resp. $\mathcal{T}^{\mathbb{Z}}(M, V)$ are exactly the same than those of \mathcal{A}_1 , resp. $\mathcal{A}_{\mathbb{Z}}$, but \mathcal{A}_1 , resp. $\mathcal{A}_{\mathbb{Z}}$, have less morphisms, hence also less descent data and thus less objects. To understand how to pass from \mathcal{A}_1 to $\mathcal{T}(M, V)$, resp. from $\mathcal{A}_{\mathbb{Z}}$ to $\mathcal{T}^{\mathbb{Z}}(M, V)$, we need to understand and encode the "missing" morphisms.

5.2. Target Germification. As in the case of the Kuranishi stacks, we would like to germify V and consider only complex structures belonging to the germ of some point J_0 in V . This process is different from the germification process of section 3.5 which was about germifying the base category and thus the base of M -deformations. Here we still want to consider M -deformations over any analytic bases, but need to germify the set of possible fibers. Hence we need a target germification process, as opposed to the source germification process used in Section 3.5. To avoid cumbersome notations and an unreasonable use of resp., we only describe the process for $\mathcal{T}(M, V)$ and let the reader add a \mathbb{Z} at each step in the case of $\mathcal{T}^{\mathbb{Z}}(M, V)$.

To do that, we look at sequences of stacks $\mathcal{T}(M, V_n)$ for (V_n) an inclusion decreasing sequence of neighborhoods of a fixed point J_0 with $V_0 = V$. Corresponding to a nesting sequence

$$(5.3) \quad \dots \subset V_n \subset \dots \subset V \subset \mathcal{I}$$

we obtain the sequence

$$(5.4) \quad \dots \hookrightarrow \mathcal{T}(M, V_n) \hookrightarrow \dots \hookrightarrow \mathcal{T}(M, V)$$

We consider sequences of M -deformations over the same base $(\mathcal{X}_n \rightarrow B)$ such that \mathcal{X}_n is an object of $\mathcal{T}(M, V_n)$ for some decreasing sequence (5.3). We identify two such sequences $(\mathcal{X}'_n \rightarrow B)$ and $(\mathcal{X}_n \rightarrow B)$ if the families $\mathcal{X}'_n \rightarrow B$ and $\mathcal{X}_n \rightarrow B$ are isomorphic as objects of $\mathcal{T}(M, V)$ for every large n . Here are some examples of such sequences

- i) Start with a M -deformation $\mathcal{X} \rightarrow \mathbb{D}$ over the disk with central fiber isomorphic to X_0 . Then consider the pull-back sequence $(\lambda_n^* \mathcal{X} \rightarrow \mathbb{D})$ where (λ_n) is a sequence of homotheties with ratio decreasing from 1 to 0.
- ii) Start with a fiber bundle $E \rightarrow B$ with fiber X_0 and structural group $\text{Aut}^1(X_0)$ and a M -deformation $\pi : \mathcal{X} \rightarrow B \times \mathbb{D}$ which coincides with the bundle E over $B \times \{0\}$. Then pick up some sequence (x_n) in the disk which converges to 0. Then consider the sequence of families $(\pi^{-1}(B \times \{x_n\}) \rightarrow B)$.

Morphisms from $(\mathcal{X}'_n \rightarrow B)$ to $(\mathcal{X}_n \rightarrow B)$ are sequences (f_n) with f_n a family morphism over B from \mathcal{X}'_n to \mathcal{X}_n for every n . Once again, we identify two such sequences (f_n) and (g_n) if there exists some integer k such that $f_n = g_n$ as morphisms of $\mathcal{T}(M, V)$ for $n \geq k$.

We call the resulting category the *target germification* of $\mathcal{T}(M)$ at J_0 and denote it by $(\mathcal{T}(M), J_0)$. Observe that this is not a stack but rather a projective limit of stacks.

5.3. The natural morphism from Kur to Teich. We want to analyse the structure of the analytic space \mathcal{T}_V defined in (5.1) and compare it with \mathcal{A}_1 .

We already observed in Section 5.1 that there is a natural inclusion of groupoids of \mathcal{A}_1 into \mathcal{T}_V . It comes from the fact that \mathcal{T}_V encodes every morphism between fibers of the Kuranishi family, whereas \mathcal{A}_1 encodes *some* morphisms between fibers of the Kuranishi family. This inclusion is just the description at the level of atlases of the natural inclusion of \mathcal{A}_1 into $\mathcal{T}(M, V)$: \mathcal{A}_1 -objects, resp. \mathcal{A}_1 -morphisms, inject in $\mathcal{T}(M, V)$ -objects, resp. $\mathcal{T}(M, V)$ -morphisms. So our final goal here is to give the structure of this inclusion.

There exists also a natural inclusion of \mathcal{A}_0 into $\mathcal{T}(M, V)$. We first relate the morphisms encoded in \mathcal{T}_V to those encoded in \mathcal{A}_0 . Set

Definition 5.1. Let $(J, f) \in \mathcal{T}_V$ and let $(J, g) \in \mathcal{T}_V$. Then they are *s-homotopic* if there exists a compatible path (F_t) in $\text{Diff}^0(M)$ such that $t \mapsto (J, F_t)$ joins (J, f) to (J, g) in \mathcal{T}_V .

In other words, (J, f) to (J, g) are *s-homotopic* if they belong to the same connected component of the *s*-fiber of $\mathcal{T}_V \rightrightarrows K_0$ at J .

Lemma 5.2. *Let $(J, f) \in \mathcal{T}_V$ and let $(J, g) \in \mathcal{T}_V$. Then these two elements are s -homotopic if and only if $(J \cdot f, f^{-1} \circ g)$ belongs to \mathcal{A}_0 .*

Proof. Assume $(J \cdot f, f^{-1} \circ g)$ belongs to \mathcal{A}_0 , that is $(J \cdot f, f^{-1} \circ g)$ is (V, \mathcal{D}_0) -admissible. Then we may decompose it as

$$f^{-1} \circ g = h_1 \circ h_2 \circ \dots \circ h_p$$

with each $h_i \in \mathcal{D}_0$; and

$$J_{i+1} = J_i \cdot h_i \quad \text{for} \quad i = 1, \dots, p-1$$

belongs to K_0 with $J_1 := J \cdot f$. We claim that (J_p, Id) and (J_p, h_p) stay in the same connected component of \mathcal{T}_V . Indeed, it follows from point v) of Definition 3.7 that J_p belongs to Ω_{h_p} . Hence there exists a compatible path (H_t) joining h_p to the identity and (J_p, H_t) is a continuous path in \mathcal{T}_V joining (J_p, Id) and (J_p, h_p) . Since this path has fixed first coordinate, we may compose on the left by $f \circ h_1 \circ h_2 \circ \dots \circ h_{p-1}$ and obtain a continuous path between $(J, g \circ h_{p-1})$ and (J, g) . Repeating the process, we connect (J, f) to (J, g) .

Conversely, let (J, f) and (J, g) be s -homotopic. Then, there exists an isotopy (J, f_t) joining these two points in \mathcal{T}_V . But then we may find by compactness $t_0 = 0 < t_1 < \dots < t_k = 1$ such that

$$(J_i, h_i) := (J \cdot f_{t_i}, f_{t_i}^{-1} \circ f_{t_{i+1}})$$

satisfies that h_i and h_i^{-1} are sufficiently small to belong to an open neighborhood of the identity in \mathcal{D}_0 that maps every $J \cdot f_{t_j}$ and J_0 inside V . As a consequence, we have $J \cdot f_{t_i}$ in Ω_{h_i} and in $\Omega_{h_{i-1}^{-1}}$. Hence,

$$(J_0, h_0 \circ \dots \circ h_k) = (J \cdot f, f^{-1} \circ g)$$

is (V, \mathcal{D}_0) -admissible as needed. \square

We are now in position to state and prove our first main result.

Theorem 5.3. *The natural inclusion of \mathcal{A}_0 into $\mathcal{T}(M, V)$, resp. of \mathcal{A}_1 into $\mathcal{T}(M, V)$, is an étale morphism of analytic stacks.*

Let us make a few comments before proving Theorem 5.3. First of all, the statement may be a bit misleading for readers used to the classical notion of étale morphism of analytic space. Given a discrete group G acting holomorphically onto an analytic space X , then the morphism $X \rightarrow [X/G]$, with $[X/G]$ the quotient stack, is étale even if it has dense orbits or infinite stabilizers.

Then, by étale morphism of analytic stacks, we mean that, given any $B \in \mathfrak{S}$ and any morphism u from B to $\mathcal{T}(M, V)$, the fiber product

$$(5.5) \quad \begin{array}{ccc} B \times_u \mathcal{A}_0 & \xrightarrow{f_2} & \mathcal{A}_0 \\ f_1 \downarrow & \square & \downarrow \text{inclusion} \\ B & \xrightarrow{u} & \mathcal{T}(M, V) \end{array} \quad \text{resp.} \quad \begin{array}{ccc} B \times_u \mathcal{A}_1 & \xrightarrow{f_2} & \mathcal{A}_1 \\ f_1 \downarrow & \square & \downarrow \text{inclusion} \\ B & \xrightarrow{u} & \mathcal{T}(M, V) \end{array}$$

satisfies

- i) $B \times_u \mathcal{A}_0$, resp. $B \times_u \mathcal{A}_1$, is a \mathbb{C} -analytic space.
- ii) The morphism f_1 is an étale morphism between \mathbb{C} -analytic spaces.

In other words, point i) means that the natural inclusion is a representable morphism. Hence it may enjoy any property preserved by arbitrary base change and local at target that a classical morphisms between analytic spaces may enjoy. Then point ii) means that, amongst all these properties, we prove that the natural inclusion is étale. We note that this corresponds to the "strong" notion of étale in the literature on algebraic stacks, e.g. in the local structure theorem of [2], the constructed étale morphism is not representable in general so a weaker notion of étale morphisms of algebraic stacks is used.

Last but not least, Theorem 5.3 must be understood geometrically as follows. A family $\mathcal{X} \rightarrow B$ with all fibers belonging to V can be decomposed as local pull-backs of \mathcal{X}_0 glued together through a cocycle of morphisms (u_{ij}) in \mathcal{T}_V . It is an object of \mathcal{A}_1 if and only we may find an equivalent cocycle living in \mathcal{A}_1 . This is completely similar to the process of reduction of the structural group of a fiber bundle. Theorem 5.3 says that, given such a family, there exists at most a discrete set of non-equivalent reductions. Assume X_0 is rigid with Kuranishi space being a reduced point. Then, any reduced M -family is indeed a locally trivial holomorphic bundle with fiber X_0 and structural group $\text{Aut}^1(X_0)$. There is no difference with families that are objects of \mathcal{A}_1 , however objects of \mathcal{A}_0 are bundles with fiber X_0 and structural group $\text{Aut}^0(X_0)$ this time. So, in this particular case, Theorem 5.3 really describes the set of non-equivalent reductions of the structural group of such a bundle from $\text{Aut}^1(X_0)$ to $\text{Aut}^0(X_0)$. And this set can be easily determined by passing to the associated principal bundles and making use of the following observation. Given a principal $\text{Aut}^1(X_0)$ -bundle E over some base B , let $\text{Aut}^0(X_0)$ act on the fibers of E . The quotient E' has fibers $\text{Aut}^1(X_0)/\text{Aut}^0(X_0)$ and is trivializable if and only if E admits a $\text{Aut}^0(X_0)$ -reduction. Hence the set we are looking for is the set of trivializations of E' and identifies with the set of holomorphic maps from B to the discrete $\text{Aut}^1(X_0)/\text{Aut}^0(X_0)$ set. The proof given below in the general case follows the same strategy.

We give another geometric interpretation of Theorem 5.3 in Section 9.3.

Proof. By Yoneda's lemma, a morphism $u : B \rightarrow \mathcal{T}(M, V)$ corresponds to a family $\mathcal{X} \rightarrow B$. The fiber product $B \times_u \mathcal{A}_0$ encodes the isomorphisms of $\mathcal{T}(M, V)$

$$(5.6) \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\alpha} & \mathcal{X}' \\ & \searrow & \swarrow \\ & & B \end{array}$$

between families $\mathcal{X} \rightarrow \mathcal{X}'$ over B (with \mathcal{X}' in \mathcal{A}_0) modulo isomorphisms β over B

$$(5.7) \quad \begin{array}{ccc} & & \mathcal{X}' \\ & \nearrow \alpha & \downarrow \beta \\ \mathcal{X} & & \mathcal{X}'' \\ & \searrow \alpha' & \end{array}$$

belonging to \mathcal{A}_0 .

Assume B connected. Decompose B as a union of connected open sets $B_1 \cup \dots \cup B_k$ in such a way that the family \mathcal{X} , resp. \mathcal{X}' , is locally isomorphic above B_i to $u_i^* \mathcal{K}_0$ for some $u_i : B_i \rightarrow K_0$, resp. to $(u_i')^* \mathcal{K}_0$ for some $u_i' : B_i \rightarrow K_0$. These local models are glued through a cocycle $u_{ij} : B_i \cap B_j \rightarrow \mathcal{T}_V$, resp. $u'_{ij} : B_i \cap B_j \rightarrow \mathcal{A}_0 \subset \mathcal{T}_V$, satisfying $s(u_{ij}) = u_i$ and $t(u_{ij}) = u_j$, resp. $s(u'_{ij}) = u_i'$ and $t(u'_{ij}) = u_j'$, to obtain a family isomorphic to \mathcal{X} , resp. \mathcal{X}' .

In these models, up to passing to a finer covering, an isomorphism (5.6) corresponds to a collection $F_i : B_i \rightarrow \mathcal{T}_V$ fulfilling

- i) $s \circ F_i = u_i$ and $t \circ F_i = u_i'$
- ii) $m(u_{ij}, F_j) = m(F_i, u'_{ij})$

Then \mathcal{X}'' corresponds to a cocycle $u''_i : B_i \rightarrow K_0$ and α' to a collection $F'_i : B_i \rightarrow \mathcal{T}_V$ satisfying similar relations.

Let β be $\alpha' \circ \alpha^{-1}$. This is a morphism of $\mathcal{T}(M, V)$ which is given in our localisation by the collection

$$(5.8) \quad G_i := m(i(F_i), F'_i) : B_i \longrightarrow \mathcal{T}_V$$

We want to know when β is a morphism of \mathcal{A}_0 , that is when G_i has image in \mathcal{A}_0 for all i .

Since the B_i are connected, the image of each map F_i, F'_i is included in a single connected component of the space \mathcal{S}_i of s -sections of $B_i \times_{u_i} \mathcal{T}_V$ above B_i . By Lemma 5.2, F_i and F'_i land in the same connected component of \mathcal{S}_i if and only if G_i lands in \mathcal{A}_0 .

Choose a point b_i in each B_i . Then α and α' are equivalent through (5.7) if and only if $(b_i, F_i(b_i))$ and $(b_i, F'_i(b_i))$ belong to the same connected component of \mathcal{S}_i for all i .

Now, assume that $(b_1, F_1(b_1))$ and $(b_1, F'_1(b_1))$ belong to the same connected component of \mathcal{S}_1 . Given $i \neq 1$ and taking $c \in B_1 \cap B_i$, it follows from the compatibility relations that

$$(5.9) \quad F_i(c) = m(u_{i1}, m(F_1, u'_{1i}))(c)$$

and

$$(5.10) \quad F'_i(c) = m(u_{i1}, m(F'_1, u''_{1i}))(c)$$

But u'_{1i} and u''_{1i} are mappings with values in \mathcal{A}_0 , hence, applying once again Lemma 5.2, we deduce that $m(F_1, u'_{1i})(c)$ and $m(F'_1, u''_{1i})(c)$ belong to the same connected component of \mathcal{S}_1 , say S , and finally $F_i(c)$ and $F'_i(c)$ to the same connected component of \mathcal{S}_i since both lie in the image of S by $m(u_{i1}, -)$. And so do $F_i(b_i)$ and $F'_i(b_i)$ for all i by connectedness of B .

As a consequence, α and α' are equivalent through (5.7) if and only if $(b_i, F_i(b_i))$ and $(b_i, F'_i(b_i))$ belong to the same connected component of \mathcal{S}_i for some i .

Therefore, the fiber product $B \times_u \mathcal{A}_0$ identifies with a disjoint union of copies of B . On the points of $B \times_u \mathcal{A}_0$ ⁷, this identification is given by the map

$$(5.11) \quad (b, F(b)) \in (B \times_u \mathcal{A}_0) \longmapsto (b, \sharp(F_i(b_0))) \in B \times \sharp S_0$$

⁷that is, for objects above some point $b \in B$.

Here S_0 is the s -fiber of \mathcal{T}_V above $u_i(b_0)$, the set $\sharp S_0$ is the set of connected components of S_0 and the \sharp application maps an element of S_0 to the connected component of S_0 which contains it; the mapping F_i is defined as above as a local expression for α satisfying (5.6), the point b_0 and the index i are fixed with b_0 belonging to B_i . It follows from what precedes that the quantity $F_i(b_0)$ depends only on the class of α modulo (5.7), showing that (5.11) is an isomorphism onto its image. Its image is $B \times E_0$, for E_0 a subset of $\sharp S_0$ which can be a strict subset because some connected components of S_0 may not be compatible with any cocycle of the family \mathcal{X} . Indeed, it may be empty, corresponding to a family \mathcal{X} that is not isomorphic to any family of \mathcal{A}_0 , i.e. corresponding to $B \times_u \mathcal{A}_0$ empty. Finally, when it is not empty, f_1 can be rewritten as the natural projection map

$$(5.12) \quad B \times \{1, \dots, g(\mathcal{X})\} \longrightarrow B$$

for $g(\mathcal{X})$ a number $\mathbb{N}^* \cup \{+\infty\}$ that depends on \mathcal{X} , as the notation suggests, and is the number of connected components of S_0 that can be attained through (5.11). This proves that the inclusion of \mathcal{A}_0 in $\mathcal{T}(M, V)$ is an étale morphism.

Let us deal now with the \mathcal{A}_1 case. We thus consider diagrams (5.6) and (5.7) with \mathcal{X}' and β in \mathcal{A}_1 . We define $u_i, u'_i, u''_i, F_i, F'_i$ and G_i as before. We want to know when G_i has image in \mathcal{A}_1 for all i . Using (5.9) and (5.10), we obtain that G_i is in \mathcal{A}_1 for all i if and only if G_1 is in \mathcal{A}_1 .

Then, by connexity of B_1 , this occurs if and only if $G_1(b_1)$ belongs to \mathcal{A}_1 for some fixed $b_1 \in B_1$. Indeed, letting b be another point of B_1 , we have that $G_1(b_1)$ and $G_1(b)$ belong to the same connected component of S_1 , hence are related through an element of \mathcal{A}_0 by Lemma 5.2. Since the composition of an element of \mathcal{A}_1 with an element of \mathcal{A}_0 belongs to \mathcal{A}_1 , we deduce that $G_1(b)$ is also in \mathcal{A}_1 for all $b \in B_1$.

This is equivalent to $F_1(b_1)$ and $m(F'_1(b_1), h)$ are s -homotopic for some $h \in \mathcal{A}_1$ with $s(h) = t(F_1(b_1))$. Say that $F_1(b_1)$ and $F'_1(b_1)$ are $(s, 1)$ -homotopic when this is true. In other words, we define on S_0 the following equivalence relation: (J, f) and (J, g) are $(s, 1)$ -homotopic (where $J = u_1(b_1)$) if and only if $(J \cdot f, f^{-1} \circ g)$ belongs to \mathcal{A}_1 . It is straightforward to check that it only depends on the connected component of (J, f) and (J, g) in S_0 , that is the $(s, 1)$ -homotopy descends as an equivalence relation on $\sharp S_0$.

Thus, one eventually finds that the fiber product $B \times_u \mathcal{A}_1$ identifies with the disjoint union of an at most countable number of copies of B , say $g_1(\mathcal{X})$, through the map

$$(5.13) \quad (b, F(b)) \in (B \times_u \mathcal{A}_1) \longmapsto (b, \sharp_1 F_i(b_0)) \in B \times \sharp_1 S_0$$

Here we use the same notations and conventions as in (5.11), $\sharp_1 S_0$ is the set of $(s, 1)$ -homotopy classes of $\sharp S_0$, and the \sharp_1 application maps an element of \mathcal{T}_V to the element of \sharp_1 which contains it. Finally f_1 can be rewritten as the natural projection map

$$(5.14) \quad B \times \{1, \dots, g_1(\mathcal{X})\} \longrightarrow B$$

for $g_1(\mathcal{X})$ the number of $\sharp \text{Aut}^1(X_0)$ -orbits of connected components of \mathcal{T}_V that can be attained through (5.13). \square

Of course, a similar statement holds for the \mathbb{Z} -Teichmüller stack. We have indeed

Corollary 5.4. *In the following commutative diagram of natural inclusions,*

$$(5.15) \quad \begin{array}{ccccc} \mathcal{A}_0 & \hookrightarrow & \mathcal{A}_1 & \hookrightarrow & \mathcal{T}(M, V) \\ \parallel & & \downarrow & & \downarrow \\ \mathcal{A}_0 & \hookrightarrow & \mathcal{A}_{\mathbb{Z}} & \hookrightarrow & \mathcal{T}^{\mathbb{Z}}(M, V) \end{array}$$

every arrow is an étale morphism of analytic stacks.

Proof. The top line is given by Theorem 5.3 and the bottom line is proven in the same way. Then, take the fibered product of (5.15) with some $u : B \rightarrow \mathcal{T}^{\mathbb{Z}}(M, V)$ obtaining

$$(5.16) \quad \begin{array}{ccccc} B \times_u \mathcal{A}_0 & \longrightarrow & B \times_u \mathcal{A}_1 & \longrightarrow & B \times_u \mathcal{T}(M, V) \\ \parallel & & \downarrow & & \downarrow \\ B \times_u \mathcal{A}_0 & \longrightarrow & B \times_u \mathcal{A}_{\mathbb{Z}} & \longrightarrow & B \end{array}$$

that is a diagram of analytic spaces with all horizontal lines being étale. Thus the vertical ones are also étale and we are done. \square

Since being étale is a local at base property, we also have

Corollary 5.5. *The natural inclusion of $\mathcal{T}(M)$ in $\mathcal{T}^{\mathbb{Z}}(M)$ is étale analytic.*

6. THE MAIN CONJECTURE

6.1. Exceptional and \mathbb{Z} -exceptional points. In view of Theorem 5.3 and Corollary 5.4, it is natural to single out the following case.

Definition 6.1. We say that X_0 is an *exceptional point of the Teichmüller stack $\mathcal{T}(M)$* or simply that X_0 is *exceptional* if there is no neighborhood V of X_0 such that the étale morphism $\mathcal{A}_1 \rightarrow \mathcal{T}(M, V)$ is an isomorphism.

Analogously, we say that X_0 is an *exceptional point of the \mathbb{Z} -Teichmüller stack $\mathcal{T}^{\mathbb{Z}}(M)$* , or simply that X_0 is *\mathbb{Z} -exceptional* if there is no neighborhood V of X_0 such that the étale morphism $\mathcal{A}_{\mathbb{Z}} \rightarrow \mathcal{T}^{\mathbb{Z}}(M, V)$ is an isomorphism.

The idea behind this definition is of course that these étale morphisms should be isomorphisms at a generic point for a sufficiently small V . The situation is however much more complicated. It turns out that it strongly depends on the existence of a Kähler metric on the manifold X_0 .

6.2. Conjecture on exceptional points. We now state and discuss the main conjecture on exceptional points that will occupy ourselves in Sections 8 to 10.

Conjecture 6.2. (Main conjecture on exceptional points).

- I. *Let X_0 be Kähler. Then X_0 is neither exceptional in $\mathcal{T}(M)$ nor \mathbb{Z} -exceptional in $\mathcal{T}^{\mathbb{Z}}(M)$.*
- II. *There exist some exceptional and \mathbb{Z} -exceptional (non-Kähler) points. They may even be dense in a connected component of $\mathcal{T}(M)$ or $\mathcal{T}^{\mathbb{Z}}(M)$.*

As we already did in the introduction §1, we emphasize that classical deformation theory from an analytic point of view is rather insensible to Kählerianity and that the dichotomy in Conjecture 6.2 is only seen on the Teichmüller and \mathbb{Z} -Teichmüller stacks, not on the moduli stack. Especially, only in the context of Kähler manifolds acted on by $\text{Diff}^0(M)$ or $\text{Diff}^{\mathbb{Z}}(M)$ can we use results on the compactness of cycle spaces.

Point I of Conjecture 6.2 is the most optimistic. Basic reasons to believe it include the already cited compactness results in the Kähler setting such as Lieberman's Theorem recalled in §9; the fact that submanifolds of Kähler manifolds represent non-trivial cohomology classes rigidifying $\text{Aut}^1(X_0)$ and $\text{Aut}^{\mathbb{Z}}(X_0)$; and the difficulties in finding an example of an exceptional point even for non-Kähler manifolds. But this is far from giving a strong evidence for I. It also supposes that the $\text{Diff}^0(M)$ and $\text{Diff}^{\mathbb{Z}}(M)$ -orbits of $\mathcal{I}(M)$ have a very simple topology with no holonomy phenomenon. We shall prove in Section 10 a weaker result: exceptional Kähler points, resp. \mathbb{Z} -exceptional Kähler points, if exist, form a strict analytic substack of $\mathcal{T}(M)$, resp. $\mathcal{T}^{\mathbb{Z}}(M)$, see Theorem 10.1.

Point II of Conjecture 6.2 seems more plausible although not easier to prove. Basic reasons to believe it boil down to the fact that all the techniques used to prove Theorem 10.1 break down completely in the non-Kähler setting. Also we give in §8.4 an example of \mathbb{Z} -exceptional points with a property of local density, see Theorem 8.17. We are unable however to show the existence of exceptional points.

Indeed, Conjecture 6.2 sheds some light on the dichotomy between Kähler and non-Kähler points, pushes forward it to the extreme form of a 0 – 1 conjecture. We hope it will serve as a challenging problem and a source of motivation for studying these questions.

Before looking with more care at exceptional points, we will focus on the points where the Teichmüller stack is locally simpler.

7. STRUCTURE OF THE TEICHMÜLLER STACK OF NORMAL POINTS

Section 6 introduces the notion of exceptional points. At such a point, the local Teichmüller stack includes morphisms that are not close to automorphisms of the central fiber, adding complexity. But jumping points are also bad points, where the Teichmüller space is usually locally non-Hausdorff at X_0 . Roughly speaking, normal points are points of $\mathcal{T}(M)$ that are neither jumping nor exceptional.

7.1. Normal points. We start with the following Lemma

Lemma 7.1. *Assume (3.7) is locally constant at X_0 . If V is small enough, every automorphism of $\text{Aut}^0(X_0)$ extends as an automorphism of the whole Kuranishi family.*

Proof. Assume K_0 reduced. Recall that the Lie algebra of $\text{Aut}^0(X_0)$ is the cohomology group $H^0(X_0, \Theta_0)$. Since (3.7) is locally constant at 0, the projection

$$(7.1) \quad \bigcup_{t \in K_0} H^0(X_t, \Theta_t) \longrightarrow K_0$$

is a locally trivial fiber bundle for V and K_0 small enough. Hence we may extend every element of $H^0(X_0, \Theta_0)$ as a holomorphic vector field tangent to the fibers of the Kuranishi family. Taking the exponential and composing, this means that every element of $\text{Aut}^0(X_0)$ extends as an automorphism of the whole Kuranishi family as wanted.

If K_0 is not reduced, then the previous argument shows that every automorphism of $\text{Aut}^0(X_0)$ extends as an automorphism above K_0^{red} , the reduction of K_0 . Now, given $f \in \text{Aut}^0(X_0)$ and G an extension of f

$$(7.2) \quad \begin{array}{ccc} \mathcal{K}_0^{\text{red}} & \xrightarrow{G} & \mathcal{K}_0^{\text{red}} \\ \downarrow & & \downarrow \\ K_0^{\text{red}} & \xrightarrow{g} & K_0^{\text{red}} \end{array}$$

then g is defined on K_0 through

$$(7.3) \quad g(J) := J \cdot G_J$$

Then, since $\mathcal{K}_0 \rightarrow K_0$ is a smooth morphism, \mathcal{K}_0 is locally isomorphic to $K_0 \times \mathbb{C}^n$. On such an open subset of $K_0 \times \mathbb{C}^n$, we define G as

$$(7.4) \quad (J, z) \mapsto (g(J), G_J(z))$$

and we are done. \square

Notice that, when K_0 is reduced, the proof of Lemma 7.1 shows that every automorphism of $\text{Aut}^0(X_0)$ extends as an automorphism of the nearby fibers of the Kuranishi space of X_0 . Indeed the extension at J is given by the function $f \circ e(\chi(J))$ appearing in (3.12), which is an automorphism of X_J by Lemma 3.3. As a consequence the source and target morphisms of $\mathcal{A}_0 \rightrightarrows K_0$ are equal. If K_0 is not reduced, then the extensions descend as the identity on the reduction of K_0 but not always as the identity of K_0 .

This motivates the following definition.

Definition 7.2. A point X_0 of the Teichmüller stack is a *normal point* if

- i) choosing V small enough, every automorphism of $\text{Aut}^0(X_0)$ extends as an automorphism of the whole Kuranishi family which descends as the identity on K_0 .
- ii) It does not belong to the closure of the set of exceptional points.

Replacing exceptional with \mathbb{Z} -exceptional in Definition 7.2 gives the notion of \mathbb{Z} -normal points.

As a consequence of point i) in Definition 7.2, the source and target maps of $\mathcal{A}_0 \rightrightarrows K_0$ can be assumed to be equal at a normal point. Indeed, it is exactly the analytic space and morphism

$$(7.5) \quad \mathcal{N} := \bigcup_{t \in K_0} \text{Aut}^0(X_t) \longrightarrow K_0$$

constructed by Namba⁸ in [36] when K_0 is reduced. This can be proven as follows. By Lemma 3.9, there is a set theoretic inclusion of \mathcal{N} into \mathcal{A}_0 . The equality $s = t$ yields the reverse inclusion. Taking account that the topology

⁸To be precise, the Namba space is the union of the full groups $\text{Aut}(X_t)$, hence (7.5) is an open subset of it.

in \mathcal{N} is that of uniform convergence, this bijection is indeed a homeomorphism. Finally, both \mathcal{A}_0 and \mathcal{N} being smooth over K_0 reduced - for \mathcal{N} , this is true because (3.7) is constant - with same fibers, this homomorphism can be turned into an analytic isomorphism.

Recall also that point i) in Definition 7.2 is stronger than (3.7) being locally constant in the non-reduced case.

Example 7.3. Consider the case of compact complex tori. Then $\text{Aut}^0(X_0)$ is not trivial, since it contains the translations. So neither $\mathcal{T}(M, V)$ nor \mathcal{A}_1 is an orbifold, since their isotropy groups are not finite. However, if we forget about the stack structure, the Teichmüller space is naturally a complex manifold. Indeed, roughly speaking, the stack is obtained from this complex manifold by attaching a group of translations to each point. This is an example of a stack represented by an analytic groupoid with s and t equal. More precisely, (7.5) is the universal family of tori, see [29], Example 13.1.

At a normal point, resp. a \mathbb{Z} -normal point, the local Teichmüller stack and the Kuranishi stack \mathcal{A}_1 , resp. the local \mathbb{Z} -Teichmüller stack and $\mathcal{A}_{\mathbb{Z}}$, coincide. Lemma 7.1 and (7.5) are however not enough to describe both of them, since $\text{Aut}^1(X_0)$, resp. $\text{Aut}^{\mathbb{Z}}(X_0)$, may have several connected components.

7.2. Kuranishi stack as an orbifold. Before analyzing the Teichmüller stack of normal points, we investigate the important case when the Kuranishi stack(s) is (are) an orbifold. Here by an orbifold, we mean a stack given as the global quotient of an analytic space by an holomorphic action of a finite group with a point fixed by the whole group. We include non-effective actions. We have

Theorem 7.4. *The following two statements are equivalent*

- i) *There exists some open neighborhood $V' \subset V$ of X_0 such that the Kuranishi stack \mathcal{A} restricted to V' is an orbifold*
- ii) *$\text{Aut}(X_0)$ is finite*

Remark 7.5. In order to endow the Kuranishi stack \mathcal{A} with a structure of an orbifold, we need to start with an open set V stable under the action of the automorphisms. This explains the restriction to some V' in the statement. In the proof V' is constructed as such a stable open set.

Proof. Since the isotropy group of X_0 is $\text{Aut}(X_0)$, the condition is obviously necessary. So let us assume that $\text{Aut}(X_0)$ is finite. We start with an arbitrary atlas $\mathcal{A} \rightrightarrows K_0$. We assume that the $\text{Aut}(X_0)$ version of (3.8) is valid.

We show that we may choose $V' \subset V$ so that the corresponding atlas $\mathcal{A}' \rightrightarrows K_0 \cap V'$ of \mathcal{A} is Morita equivalent to the translation groupoid $\text{Aut}(X_0) \times K_0 \cap V' \rightrightarrows K_0 \times V'$.

The proof of Theorem 7.4 consists in the following three lemmas.

Lemma 7.6. *For all $f \in \text{Aut}(X_0)$, the map $\sigma_f : U_f \subset K_0 \rightarrow \mathcal{A}$ constructed in (3.12) is the unique (up to restriction) extension of f .*

By extension of f , we mean a section F of s defined in a neighborhood of 0 and such that $F(0) = f$.

Proof of Lemma 7.6. The map σ_f is obviously an extension of f as desired.

Let now G be another extension of f . Then, for all $J \in K_0$ close to 0, we have a decomposition

$$(7.6) \quad G(J) = f \circ e(\eta(J))$$

using (3.8). Here the factor in $\text{Aut}(X_0)$ is constant equal to f since $\text{Aut}(X_0)$ is discrete.

We have

$$(7.7) \quad J \cdot G(J) = \Xi(J \cdot G(J)) = \Xi(J \cdot f) = \text{Hol}_f(J)$$

so the mapping η also satisfies (3.13). But since (3.3) is an isomorphism, (3.13) is uniquely verified and $\eta = \chi$. Thus $G = \sigma_f$ on a neighborhood of J_0 in K_0 . \square

As a consequence, we have

Lemma 7.7. *For all $f \in \text{Aut}(X_0)$, and $g \in \text{Aut}(X_0)$, we have $\sigma_{g \circ f} = m(\sigma_g, \sigma_f)$ on a neighborhood of J_0 .*

Proof of Lemma 7.7. Define

$$(7.8) \quad m(\sigma_g, \sigma_f) : J \longmapsto m(\sigma_g(J), \sigma_f(J \cdot \sigma_g(J)))$$

This is an extension of $g \circ f$, and thus by Lemma 7.6 is equal to $\sigma_{g \circ f}$ on a neighborhood of J_0 . \square

Let U be the intersection of all U_f for f in the finite group $\text{Aut}(X_0)$. Then all Hol_f are defined on U with values in K_0 . Redefine K_0 as the intersection of all $\text{Hol}_f(U)$ for f in the finite group $\text{Aut}(X_0)$. Observe that

$$(7.9) \quad \text{Hol}_g(\cap_f(\text{Hol}_f(U))) = (\cap_f(\text{Hol}_{f \circ g}(U))) = \cap_f(\text{Hol}_f(U))$$

because the intersection of the $\text{Hol}_f(U)$ is included in U and because of Lemma 7.7. Then all Hol_f map bijectively K_0 to K_0 . Associated to this new K_0 and to (3.8) is some $V' \subset V$. Set

$$(7.10) \quad \mathcal{E}xt = \{\sigma_f : K_0 \rightarrow \mathcal{A} \mid f \in \text{Aut}(X_0)\}$$

We have

Lemma 7.8. *$(\mathcal{E}xt, \circ)$ is a group isomorphic to $\text{Aut}(X_0)$.*

Proof of Lemma 7.8. By Lemma 7.7, $m(\sigma_g, \sigma_f)$ is equal to $\sigma_{g \circ f}$ on a neighborhood of J_0 , hence on K_0 by analyticity.

The same arguments used in Lemma 7.7 show that if $g_1 \circ \dots \circ g_k = \text{Id}$, then the same relation holds for the σ_{g_i} 's. \square

The space K_0 is invariant by the action of the group $(\mathcal{E}xt, \circ)$, which describes all the morphisms of the Kuranishi stack. We may thus take as atlas for \mathcal{A} the translation groupoid $\mathcal{E}xt \times K_0 \rightrightarrows K_0$, or, equivalently the translation groupoid $\text{Aut}(X_0) \times K_0 \rightrightarrows K_0$. \square

Replacing \mathcal{A} with $\mathcal{A}_{\mathbb{Z}}$, resp. \mathcal{A}_1 , resp. \mathcal{A}_0 and $\text{Aut}(X_0)$ with $\text{Aut}^{\mathbb{Z}}(X_0)$, resp. $\text{Aut}^1(X_0)$, resp. $\text{Aut}^0(X_0)$ yields the following immediate corollaries.

Corollary 7.9. *The following two statements are equivalent*

- i) *There exists some open neighborhood $V' \subset V$ of X_0 such that the Kuranishi stack \mathcal{A}_1 restricted to V' is an orbifold*
- ii) *$\text{Aut}^1(X_0)$ is finite*

then,

Corollary 7.10. *The following two statements are equivalent*

- i) *There exists some open neighborhood $V' \subset V$ of X_0 such that the Kuranishi stack $\mathcal{A}_{\mathbb{Z}}$ restricted to V' is an orbifold*
- ii) *$\text{Aut}^{\mathbb{Z}}(X_0)$ is finite*

and finally,

Corollary 7.11. *The following two statements are equivalent*

- i) *There exists some open neighborhood $V' \subset V$ of X_0 such that the Kuranishi stack \mathcal{A}_0 restricted to V' is an orbifold*
- ii) *There exists some open neighborhood $V' \subset V$ of X_0 such that the Kuranishi stack \mathcal{A}_0 restricted to V' is an analytic space*
- iii) *$\text{Aut}^0(X_0)$ is reduced to the identity.*

Proof. Just notice that $\text{Aut}^0(X_0)$ is finite if and only if it is reduced to the identity. \square

7.3. The étale Teichmüller stack of normal points. We are now in position to analyse the structure of the Teichmüller stack restricted to the set of normal points. First note the following result.

Proposition 7.12. *The subset $\mathcal{N} \subset \mathcal{I}$ of normal points is an open set of \mathcal{I} .*

Proof. Let J_0 be a normal point of \mathcal{I} . Then $\mathcal{A}_0 \rightrightarrows K_0$ have equal source and target morphisms. Taking K_0 smaller if necessary, we may assume that K_0 does not meet the closure of exceptional points and that K_0 is complete at any point, see [28]. Let now J_1 belong to K_0 . Set $X_1 = (M, J_1)$. Since (3.7) is locally constant, Proposition 2 of [26] applies and the germ of K_0 at J_1 is universal for families above a reduced base, hence the reduction of K_0 at J_1 in K_0 and the reduction of the Kuranishi space K_1 of X_1 are locally isomorphic. It follows from completeness that K_1 injects in K_0 at X_1 . Hence the Kuranishi stack $\mathcal{A}_0(X_1) \rightrightarrows K_1$ of X_1 injects in $\mathcal{A}_0 \rightrightarrows K_0$ at X_1 : any morphism of $\mathcal{A}_0(X_1)$ is an extension of an automorphism of X_1 , so belong to \mathcal{A}_0 by Lemma 3.9 taking K_1 smaller if necessary. As a consequence its source and target morphisms are equal and X_1 is a normal point. \square

Let X_0 be a normal point. Since the source and target maps of $\mathcal{A}_0 \rightrightarrows K_0$ are equal, the multiplication of the groupoid induces a fibered action of \mathcal{A}_0 onto \mathcal{A}_1 that preserves the source and target maps of $\mathcal{A}_1 \rightrightarrows K_0$. Given $(J, f) \in \mathcal{A}_0$ and $(J, g) \in \mathcal{A}_1$, we set

$$(7.11) \quad (J, f) \cdot (J, g) := m((J, f), (J, g)) = (J, f \circ g)$$

We have

Lemma 7.13. *Assume X_0 is normal. Then, the quotient space $\mathcal{A}_1/\mathcal{A}_0$ is an analytic space and the morphism s , resp. $t : \mathcal{A}_1 \rightarrow K_0$, descends as an étale morphism from $\mathcal{A}_1/\mathcal{A}_0$ to K_0 .*

Proof. Let (J, f) belong to \mathcal{A}_1 . For (J, f') close enough to (J, f) , the diffeomorphism $f' \circ f^{-1}$ belongs to $\text{Aut}^0(X_J)$ hence to \mathcal{A}_0 by Lemma 3.9 and (J, f') equals (J, f) in the quotient space. Choose a local s -, resp. t -section σ from a neighborhood of J in K_0 to a neighborhood of $(J, f) \in \mathcal{A}_1$ with $\sigma(J) = (J, f)$. Such local sections exist since s , resp. t is a smooth morphism. Then the restriction of s , resp. t to this section realizes a local isomorphism between $\mathcal{A}_1/\mathcal{A}_0$ and K_0 .

We are left with proving that $\mathcal{A}_1/\mathcal{A}_0$ is Hausdorff. Let (J_n, f_n) converge to (J, f) in \mathcal{A}_1 and (J'_n, g_n) converge to (J', g) . Assume they are \mathcal{A}_0 -equivalent. By definition, we thus have $J_n = J'_n$ and $g_n \circ f_n^{-1}$ is an element of $\text{Aut}^0(X_{J_n})$ for all n . Since K_0 is Hausdorff, passing to the limit gives $J = J'$. Moreover all $g_n \circ f_n^{-1}$ are holomorphic extensions at J_n of some automorphism of $\text{Aut}^0(X_0)$ by Lemma 7.1, hence the limit $g \circ f^{-1}$ still belongs to $\text{Aut}^0(X_0)$. Hence a convergent sequence in $\mathcal{A}_1/\mathcal{A}_0$ has a unique limit, showing Hausdorffness. \square

Hence we may define an étale quotient groupoid $\mathcal{A}_1/\mathcal{A}_0 \rightrightarrows K_0$. Its stackification over the analytic site describes classes of reduced (M, V) -families up to \mathcal{A}_0 -equivalence.

This stack can be defined over the full open set \mathcal{N} of normal points. In this context, a reduced (M, \mathcal{N}) -family is \mathcal{A}_0 -equivalent to a trivial family if it can be decomposed as local pull-back families glued by a cocycle in \mathcal{A}_0 , cf. the proof of Theorem 5.3; and an isomorphism of a reduced (M, \mathcal{N}) -family is \mathcal{A}_0 -equivalent to the identity, if it is given by local \mathcal{A}_0 -sections once decomposed as local pull-back families. We set

Definition 7.14. The stack over the analytic site of \mathcal{A}_0 -equivalence classes of reduced (M, \mathcal{N}) -families is called *the normal Teichmüller stack* and denoted by $\mathcal{N}\mathcal{T}(M)$.

and

Definition 7.15. We call $\text{Aut}^1(X_0)$ *mapping class group* the group

$$(7.12) \quad \text{Map}^1(X_0) := \text{Aut}^1(X_0)/\text{Aut}^0(X_0)$$

We are in position to give some properties of $\mathcal{N}\mathcal{T}(M)$

Theorem 7.16. *The normal Teichmüller stack $\mathcal{N}\mathcal{T}(M)$ satisfies the following properties*

- i) *It is an analytic étale stack with atlas an (at most) countable union of Kuranishi spaces.*
- ii) *There is a natural morphism from $\mathcal{T}(M, \mathcal{N})$ to $\mathcal{N}\mathcal{T}(M)$. Moreover, these two stacks are associated to the same topological quotient space.*
- iii) *Let X_0 be a normal point. The isotropy group of $\mathcal{N}\mathcal{T}(M)$ at X_0 is the discrete group $\text{Map}^1(X_0)$.*
- iv) *Assume $\text{Map}^1(X_0)$ is finite. Then, there exists some open neighborhood $V' \subset V$ of X_0 such that, reshaping K_0 to V' , the group $\text{Map}^1(X_0)$ acts holomorphically on K_0 fixing J_0 and $\mathcal{N}\mathcal{T}(M)$ is locally isomorphic at X_0 to the orbifold $[K_0/\text{Map}^1(X_0)]$.*
- v) *Epecially, if every normal point belongs to Fujiki class (\mathcal{C}) , then the stack $\mathcal{N}\mathcal{T}(M)$ is an orbifold at every point.*

Remark 7.17. The natural morphism of point ii) is not representable since the isotropy groups of $\mathcal{T}(M)$ do not inject in those of $\mathcal{N}\mathcal{T}(M)$.

Hence, an open substack of the Teichmüller stack behaves as the quotient of an analytic space by a discrete equivalence relation; and as an orbifold around a point in Fujiki class (\mathcal{C}), that is bimeromorphic to a Kähler manifold, so in particular around any Kähler or projective point.

This étale normal Teichmüller stack is closer to the moduli space side than to the family side of the Teichmüller stack. It forgets the family automorphisms that induce the identity on the base but these automorphisms do not induce any identification of points in $\mathcal{I}(M)$ so are not important when analyzing the structure of the set of $\text{Diff}^0(M)$ -orbits in $\mathcal{I}(M)$.

We may reformulate Theorem 7.16 by saying that the restriction of the topological Teichmüller space $\mathcal{N}/\text{Diff}^0(M)$ to the set of normal points acquires complex orbifold charts at the points with finite $\text{Map}^1(X_0)$ group.

Before giving the proof of this result, we give some examples.

Example 7.18. We consider once again the case of complex tori, compare with Example 7.3. All points are normal and passing from the Teichmüller stack to the normal Teichmüller stack consists of forgetting about the translation group and replacing the universal family over the upper half plane \mathbb{H} with \mathbb{H} . Hence the normal Teichmüller stack is the standard Teichmüller space \mathbb{H} and the surjective morphism sends a family of complex tori $\pi : \mathcal{X} \rightarrow B$ to the morphism

$$(7.13) \quad b \in B \longmapsto \tau(b) \in \mathbb{H} \quad \text{with} \quad \pi^{-1}(b) \simeq \mathbb{E}_{\tau(b)}$$

Example 7.19. We consider now the case of K3 surfaces. The function h^0 is constant equal to zero and the Kuranishi spaces of K3 can be glued together to form a 20-dimensional complex manifold called the moduli space of marked K3 surfaces. This is at the same time the Teichmüller stack and the normal Teichmüller stack. As for tori, all points are normal. However, this space is non-Hausdorff because a sequence of $\text{Diff}^0(M)$ -orbits may accumulate onto two disjoint orbits. Non-separated pairs of points encode however the same manifold. In other words, they are in the same orbit of the mapping class group $\text{Diff}^+(M)/\text{Diff}^0(M)$. See [20, §7.2] for more details about all this.

Example 7.20. We consider finally the case of (primary) Hopf surfaces. We concentrate on a single connected component of the Teichmüller stack, cf. the discussion in [29]. These surfaces X_g are quotient of $\mathbb{C}^2 \setminus \{0\}$ by the group generated by a contracting biholomorphism g of \mathbb{C}^2 fixing 0. Here h^0 takes the values 2, 3 and 4. Since the function h^0 is upper semi-continuous, normal points correspond to the smallest value, that is 2. Hopf surfaces with h^0 equal to 2 are those with g either non-linearizable and conjugated to

$$(7.14) \quad (z, w) \longmapsto (\lambda z + w^p, \lambda^p w)$$

for $\lambda \in \mathbb{D}^*$ and $p > 0$ or linearizable with non-resonant eigenvalues, i.e. the two eigenvalues are not of the form (λ, λ^p) for some $p > 0$, see [44] for more details about classification and properties of Hopf surfaces. As a consequence, a normal point X_g is completely determined by the determinant and the trace of the linear part g_{lin} of g and (a connected component of)

the normal Teichmüller stack is the bounded domain $\mathbb{D}^* \times \mathbb{D}$ in \mathbb{C}^2 . Given a family of Hopf surfaces $\pi : \mathcal{X} \rightarrow B$, we associate to it the morphism

$$(7.15) \quad b \in B \mapsto \left(\det g_{lin}(b), \frac{1}{2} \text{Tr } g_{lin}(b) \right) \in \mathbb{D}^* \times \mathbb{D}$$

with $\pi^{-1}(b) \simeq X_{g(b)}$. Note that the Teichmüller stack has a much more complicated structure, far from being a manifold and with non-Hausdorff Teichmüller space, that is analyzed in [15], see also [30], and that we recall and make use of in Example 13.4.

Proof. Start with the atlas of $\mathcal{T}(M, \mathcal{N})$ constructed in [29]. Since h^0 is constant along \mathcal{N} , observe that it is given as an (at most) countable union of Kuranishi spaces, there is no need to make use of the fattening process of [29]. Let us denote the corresponding symmetry groupoid with $\mathcal{T}_1 \rightrightarrows \mathcal{T}_0$. It can be chosen as follows. Cover \mathcal{N} with a(n) (at most) countable union of Kuranishi charts $\Xi_i : V_i \rightarrow K_i$. Then set

$$\mathcal{T}_0 = \bigsqcup K_i$$

and

$$\mathcal{T}_1 = \{(J, f) \in \text{Diff}^0(M, \sqcup \mathcal{K}_i) \mid J \cdot f \in \sqcup K_i\}$$

where $\mathcal{K}_i \rightarrow K_i$ are the Kuranishi families (compare with (5.1)). We may decompose it as

$$(7.16) \quad \bigsqcup \mathcal{T}_{1,j} := \bigsqcup \{(J, f) \in \text{Diff}^0(M, \mathcal{K}_j) \mid J \cdot f \in \sqcup K_i\}$$

Now observe that $\mathcal{A}_{0,j}$ act fiberwise on $\mathcal{T}_{1,j}$ through (7.11); and that Lemma 7.13 generalizes immediately to this context. We may thus define a fiberwise $\sqcup \mathcal{A}_{0,i}$ -action (7.11) on \mathcal{T}_1 decomposed as in (7.16), obtaining an étale quotient groupoid, say $\mathcal{N}_1 \rightrightarrows \mathcal{T}_0$. Its stackification over \mathfrak{S} is the normal Teichmüller stack proving i).

By construction, there is a natural quotient map from each component of \mathcal{T}_1 to a component of \mathcal{N}_1 preserving the structure maps, hence yielding a groupoid homomorphism from $\mathcal{T}_1 \rightrightarrows \mathcal{T}_0$ to $\mathcal{N}_1 \rightrightarrows \mathcal{T}_0$. It induces a natural morphism from $\mathcal{T}(M)$ to $\mathcal{N}\mathcal{T}(M)$. Since the morphism $\mathcal{T}_1 \rightarrow \mathcal{N}_1$ is surjective and commutes with the source and target morphisms, a couple of points of \mathcal{T}_0 is source and target of a morphism of \mathcal{T}_1 if and only if it is source and target of a morphism of \mathcal{N}_1 . This proves that the two stacks are associated to the same topological quotient space.

Still by construction, the isotropy group of a point X_0 is given by the $\text{Aut}^1(X_0)$ -mapping class group of X_0 . Assume now that it is finite. Since X_0 is normal, it is not exceptional and the Kuranishi stack \mathcal{A}_1 and the local Teichmüller stack $\mathcal{T}(M, V)$ coincides. Choose now automorphisms f_1, \dots, f_k of $\text{Aut}^1(X_0)$ so that their classes in $\text{Map}^1(X_0)$ generate $\text{Map}^1(X_0)$. Observe that the f_i 's may not fulfill the $\text{Map}^1(X_0)$ -relations their classes do; but the Hol_{f_i} 's do and the group generated by the Hol_{f_i} 's is isomorphic to $\text{Map}^1(X_0)$. We may thus argue as in the proof of Theorem 7.4, and redefine a smaller open set V and a smaller K_0 such that every Hol_{f_i} is an isomorphism of K_0 . Therefore, $\mathcal{A}_1/\mathcal{A}_0 \rightrightarrows K_0$ is Morita equivalent to the translation groupoid $\text{Map}^1(X_0) \times K_0 \rightrightarrows K_0$. The point X_0 is an orbifold point, proving iv).

If X_0 is bimeromorphic to a Kähler manifold, then $\text{Map}^1(X_0)$ is finite by [16] and we may apply iv). \square

Remark 7.21. In other words, the action of $\text{Diff}^0(M)$ restricted to the set of normal points is a bonafide foliation and the normal Teichmüller stack is nothing else than its étale holonomy groupoid, cf. [32] and [29, §6].

All the previous considerations and results apply to the \mathbb{Z} -normal points. Proposition 7.12 is still true with normal replaced with \mathbb{Z} -normal and Lemma 7.13 applies with $\mathcal{A}_{\mathbb{Z}}$ instead of \mathcal{A}_1 . We may thus define the stack of \mathbb{Z} -normal points as the stackification of the étale groupoid $\mathcal{A}_{\mathbb{Z}}/\mathcal{A}_0 \rightrightarrows K_0$. This is the \mathbb{Z} -normal Teichmüller stack $\mathcal{N}\mathcal{T}^{\mathbb{Z}}(M)$. Calling $\text{Aut}^{\mathbb{Z}}(X_0)$ mapping class group and denoting by $\text{Map}^{\mathbb{Z}}(X_0)$ the quotient of $\text{Aut}^{\mathbb{Z}}(X_0)$ by $\text{Aut}^0(X_0)$, we have

Corollary 7.22. *The \mathbb{Z} -normal Teichmüller stack $\mathcal{N}\mathcal{T}^{\mathbb{Z}}(M)$ satisfies all the properties listed in Theorem 7.16 with the following obvious changes in the statements: $\mathcal{N}\mathcal{T}(M)$ is replaced with $\mathcal{N}\mathcal{T}^{\mathbb{Z}}(M)$, normal with \mathbb{Z} -normal, $\mathcal{T}(M)$ with $\mathcal{T}^{\mathbb{Z}}(M)$, and $\text{Map}^1(X_0)$ with $\text{Map}^{\mathbb{Z}}(X_0)$.*

In Examples 7.18, 7.19 and 7.20, points are normal if and only if they are \mathbb{Z} -normal and there is no difference between $\mathcal{N}\mathcal{T}^{\mathbb{Z}}(M)$ and $\mathcal{N}\mathcal{T}(M)$. This is often the case.

8. EXCEPTIONAL POINTS

We now switch to the analysis of exceptional and \mathbb{Z} -exceptional points. By definition, there exist morphisms belonging to the Teichmüller stack but not to the Kuranishi stack at an exceptional point. We first analyze which type of morphisms they are both from the point of view of sequences of isomorphisms in Section 8.1 and from the point of view of cycle spaces in Section 8.2. Finally we give an example of a \mathbb{Z} -exceptional point in Section 8.4. It should be noted that neither K3 surfaces, Hirzebruch surfaces, nor Hopf surfaces exhibit exceptional points. In fact, for a long time, we did not know any example until we finally manage to construct the one presented in 8.4. It is however only \mathbb{Z} -exceptional, not exceptional. By Proposition 8.12, the converse is not possible.

8.1. Sequences of isomorphic structures in the Kuranishi space.

Let (f_n) be a morphism of $(\mathcal{T}(M), J_0)$, that is a sequence of morphisms from $(\mathcal{X}'_n \rightarrow B)$ to $(\mathcal{X}_n \rightarrow B)$ as explained above in Section 5.2. Since $\mathcal{T}_{V_n} \rightrightarrows K_0 \cap V_n$ is an atlas for $\mathcal{T}(M, V_n)$, each f_n is obtained by gluing a cocycle of morphisms in \mathcal{T}_{V_n} over an open cover of B . Such morphisms are local morphisms of the Kuranishi family. We are not interested in morphisms which act on the base B as the identity. For such morphisms restrict to automorphisms of the fibers, especially of the central fiber, and are thus already encoded in \mathcal{A}_1 , at least for n big enough. Now the existence of a morphism f_n acting non-trivially on the base is subject to the existence of two isomorphic distinct fibers of the Kuranishi family, that is to the existence of two distinct points in K_0 encoding the same complex manifold up to isomorphism. In the same way, the existence of sequences of morphisms

(f_n) acting non-trivially on the base is subject to the existence of sequences (x_n) and (y_n) of points in K_0 such that

- i) Both sequences (x_n) and (y_n) converge to the base point of K_0 .
- ii) For every n , the fibers of the Kuranishi family above x_n and y_n are isomorphic.

In particular, there exists a sequence (ϕ_n) of $\text{Diff}^0(M)$ such that

$$(8.1) \quad \forall n, \quad x_n \cdot \phi_n = y_n$$

We assume that, for all $n \geq 0$, the points x_n and y_n belongs to V_n and the morphism ϕ_n to \mathcal{T}_{V_n} , where (V_n) is a nested sequence as in (5.3). We also set $V = V_0$.

Now, we are looking for *missing* morphisms in the Kuranishi stacks. In other words, we are looking for such sequences (x_n) and (y_n) with the additional property that (x_n, ϕ_n) does not belong to \mathcal{A}_1 , that is (x_n, ϕ_n) is not (V_n, \mathcal{D}_1) -admissible.

In the sequel, we assume that \mathcal{T}_V is reduced, replacing it with its reduction if needed. Assume that the sequence (ϕ_n) belongs to a fixed irreducible component \mathcal{C}_0 of \mathcal{T}_V . We shall see that only special components, that we call exceptional (see Definition 8.5), may contain morphisms that are not in the Kuranishi stack \mathcal{A}_1 . To do that, we need to better understand the different types of \mathcal{T}_V -components.

Let A be a connected component of $\text{Aut}^1(X_0)$. Then A belongs to an irreducible component \mathcal{C}_0 of \mathcal{T}_V . We have

Lemma 8.1. *The component \mathcal{C}_0 is unique, that is there does not exist another irreducible component of \mathcal{T}_V that contains A .*

Proof. Let \mathcal{C}_1 be an irreducible component of \mathcal{T}_V that contains A . Then it contains all the extensions of the automorphisms of $\text{Aut}^1(X_0)$. More precisely it contains all the morphisms of \mathcal{T}_V that are in a sufficiently small neighborhood of A . But this is also the case of \mathcal{C}_0 , so the irreducible components \mathcal{C}_0 and \mathcal{C}_1 intersects on an open set, hence are equal. \square

We may thus define

Definition 8.2. We call A^1 -component the irreducible component of \mathcal{T}_V that contains A .

Remark 8.3. Notice from the proof that the s -image of an A^1 -component covers a neighborhood of an irreducible component of K_0 . Also, if the intersection of a A^1 -component with $\text{Aut}^1(X_0)$ is not connected but has a finite number of connected components, restricting it to V_n for n big enough it becomes connected. In other words, if the A^1 -component contains two connected components of the intersection, its restriction to V_n for n large splits into two disjoint A^1 -components.

However A^1 -components do not contain missing morphisms. Indeed,

Lemma 8.4. *Let (ϕ_n) be a sequence (8.1). Assume it is contained in a A^1 -component. Then for n big enough, the restriction of the A^1 -component to V_n only contains (V_n, \mathcal{D}_1) -admissible morphisms.*

Proof. All extensions of automorphisms of A are (V_n, \mathcal{D}_1) -admissible for n big enough by (3.9) (in fact by the $\text{Aut}^1(X_0)$ version of (3.9)). Since an A^1 -component contains all the morphisms of \mathcal{T}_V that are in a sufficiently small neighborhood of A , we are done. \square

There exists another type of irreducible component of \mathcal{T}_V , namely

Definition 8.5. An irreducible component \mathcal{C}_0 of \mathcal{T}_V is called *exceptional* if

- i) For n large, its restriction to V_n does not intersect $\text{Aut}^1(X_0)$,
- ii) It contains a sequence (8.1).

In other words, (J_0, J_0) does not belong to the $(s \times t)$ -image of an exceptional component restricted to V_n for large n , but belongs to its closure.

We have

Lemma 8.6. *If \mathcal{T}_V contains an exceptional component, then X_0 is an exceptional point of the Teichmüller stack.*

Proof. As discussed above, if \mathcal{T}_V contains an exceptional component, then the associated sequence (8.1) is a sequence of \mathcal{T}_V but not of \mathcal{A}_1 and X_0 is exceptional. \square

The converse to Lemma 8.6 is not true in general. If the number of irreducible components of \mathcal{T}_V is infinite, there may exist wandering sequences (8.1), that is sequences (8.1) such that every irreducible component of \mathcal{T}_V contains at most a finite number of terms of the sequence and is not exceptional.

To have a better understanding of the situation, we make use of cycle spaces.

8.2. Morphisms of the Teichmüller stack and cycle spaces. We consider the Barlet space of (relative) n -cycles of $\mathcal{K}_0^{\text{red}} \times \mathcal{K}_0^{\text{red}}$ for $\mathcal{K}_0^{\text{red}}$ the reduction of the Kuranishi family. Hence a cycle is a finite sum of compact analytic subspaces of some $X_t \times X_s$, for X_t and X_s fibers of the Kuranishi family. We only consider in this space irreducible components

- i) that only contains singular cycles with both projection maps onto X_t and X_s of degree one; we call them *completely singular components*
- ii) that contains at least the graph of a biholomorphism between two fibers which induces the identity in cohomology with coefficients in \mathbb{Z} ; we call them *regular components*.

We denote by \mathcal{C} the union of completely singular and regular components. In the Barlet space of n -cycles of $X_0 \times X_0$, we also consider only the union of completely singular and regular components, that we denote by \mathcal{C}_0 .

Assume \mathcal{T}_V is reduced. Every irreducible component of \mathcal{T}_V injects in an irreducible component of \mathcal{C} . Just send (J, f) to its graph as a cycle of $\mathcal{K}_0 \times \mathcal{K}_0$. Indeed \mathcal{T}_V encodes the regular cycles of \mathcal{C} . So examining \mathcal{C} gives more information about \mathcal{T}_V . Especially, a sequence (8.1), where each ϕ_n is an isomorphism between the fiber $\pi^{-1}(x_n)$ and the fiber $\pi^{-1}(y_n)$ of the Kuranishi family, defines a sequence (γ_n) of \mathcal{C} . Then, several cases may occur

- i) up to passing to a subsequence, (γ_n) converges to the graph of an automorphism g of $\text{Aut}^1(X_0)$.

- ii) up to passing to a subsequence, (γ_n) converges to a singular cycle in \mathcal{C}_0 .
- iii) up to passing to a subsequence, (γ_n) lives in a single component of \mathcal{C} but does not converge to a cycle in \mathcal{C}_0 .
- iv) every irreducible component of \mathcal{C} contains at most a finite number of terms of (γ_n) .

If X_0 is Kähler, it is well known that only cases i) and ii) can occur. It is important however to keep in mind that we also consider the general case. Here is a classical example of case iii).

Example 8.7. Let B be the subset of matrices of type

$$(8.2) \quad \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{with} \quad \lambda_i \in \mathbb{D}^* \text{ for } i = 1, 2$$

We associate to B the family of Hopf surfaces

$$(8.3) \quad \mathcal{X} := (\mathbb{C}^2 \setminus \{(0, 0)\} \times B) / \mathbb{Z}$$

where $p \in \mathbb{Z}$ acts on (v, A) through

$$(8.4) \quad p \cdot (Z, A) := (A^p Z, A)$$

Two Hopf surfaces X_A and $X_{A'}$, corresponding to A and A' of type (8.2), are biholomorphic if and only if the matrices A and A' are conjugated, thus if and only if they have the same eigenvalues. In particular, denoting by \check{A} the matrix obtained from A by inverting its eigenvalues, i.e.

$$(8.5) \quad A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix} \quad \check{A} = \begin{pmatrix} \lambda_2 & 1 \\ 0 & \lambda_1 \end{pmatrix}$$

then X_A and $X_{\check{A}}$ are biholomorphic. Moreover, denoting by B^* the subset of B formed by matrices with distinct eigenvalues, the isomorphism of $\mathbb{C}^2 \setminus \{(0, 0)\} \times B^*$ given by

$$(8.6) \quad (Z, A) \mapsto \left(\begin{pmatrix} (\lambda_1 - \lambda_2)^{-1} & 1 - (\lambda_2 - \lambda_1)^{-2} \\ 1 & (\lambda_1 - \lambda_2)^{-1} \end{pmatrix} Z, \check{A} \right)$$

descends as an isomorphism of the family $\mathcal{X}^* \rightarrow B^*$ obtained by restricting $\mathcal{X} \rightarrow B$ to B^* .

The graphs of biholomorphisms between X_A and $X_{\check{A}}$ given by (8.6) do not converge as $\lambda_1 - \lambda_2$ tends to zero. Indeed, these graphs lift as the restriction to $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^2 \setminus \{0\}$ of complex linear planes in \mathbb{C}^4 . As $\lambda_1 - \lambda_2$ tends to zero, they converge to $\{0\} \times \mathbb{C}^2$ in \mathbb{C}^4 . But this graph is not contained in $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^2 \setminus \{0\}$.

Going back to our four cases, we see that case i) corresponds to a converging sequence in a A^1 -component. Case ii) may occur in a A^1 -component or in an exceptional component. To distinguish the two cases, we set

Definition 8.8. A singular cycle of \mathcal{C}_0 is called *exceptional* if it is the limit of a sequence of regular cycles (γ_n) of \mathcal{C} which are graphs of a sequence (8.1) lying in an exceptional component.

Case iii) may also occur in a A^1 -component or in an exceptional component. To distinguish the two cases, we set

Definition 8.9. A sequence (8.1) satisfying iii) and lying in an exceptional component is called a *vanishing* sequence.

Finally, case iv) is covered by the following definition.

Definition 8.10. A sequence (8.1) such that every irreducible component of \mathcal{C} contains at most a finite number of graphs of terms of (8.1) is called a *wandering sequence*.

We may sum up the previous discussion by the following converse to Lemma 8.6.

Proposition 8.11. *A point X_0 of the Teichmüller stack is exceptional if and only if one of the following statements is fulfilled:*

- i) *There exists an exceptional cycle in \mathcal{C}_0 .*
- ii) *There exists a vanishing sequence in \mathcal{T}_V .*
- iii) *There exists a wandering sequence in \mathcal{T}_V .*

In the case X_0 is Kähler, thanks to classical finiteness properties of the cycle spaces recalled in Section 9.1, we will be able to show much more precise statements about exceptional points in Sections 9 and 10.

We note that it is in no way an exceptional property for \mathcal{C}_0 to contain a connected component of singular cycles. For example, if $\text{Aut}^1(X_0)$ is reduced to zero, then the cycles $X_0 \times pt + pt \times X_0$ form such a component. But they usually do not correspond to exceptional cycles.

8.3. The case of \mathbb{Z} -exceptional points. The contents of §8.1 and §8.2 can be easily adapted to the case of \mathbb{Z} -Teichmüller stack and \mathbb{Z} -exceptional points. We feel free to use the corresponding definitions and results in this context with the obvious changes of notations. The reader should not be affected.

Besides, we note the following interesting comparison Proposition.

Proposition 8.12. *Let X_0 be an exceptional point of $\mathcal{T}(M)$. Then X_0 is also a \mathbb{Z} -exceptional point of $\mathcal{T}^{\mathbb{Z}}(M)$.*

Proposition 8.12 can be equivalently stated as: a \mathbb{Z} -normal point is normal. The example described in §8.4 shows that the converse is false.

Proof. Let X_0 be an exceptional point of $\mathcal{T}(M)$. Let (ϕ_n) be an associated non-convergent sequence of $\text{Diff}^0(M)$ satisfying (8.1). Assume by contradiction that X_0 is not a \mathbb{Z} -exceptional point of $\mathcal{T}^{\mathbb{Z}}(M)$. Then, up to passing to a subsequence, (ϕ_n) must belong to an $A^{\mathbb{Z}}$ -component which is not an A^1 -component. But this means that, for n big enough, ϕ_n belongs to $\text{Diff}^{\mathbb{Z}}(M)$ and not to $\text{Diff}^0(M)$. Contradiction. \square

8.4. An example of a \mathbb{Z} -exceptional point. In this Section, we give an explicit example of a 3-fold which is a \mathbb{Z} -exceptional point of its Teichmüller stack. We start with a Blanchard manifold [5] as revisited in [38] and [8].

Let $a \geq 1$ be an integer and set $W := \mathcal{O}(a) \oplus \mathcal{O}(a) \rightarrow \mathbb{P}^1$. Choose two holomorphic sections σ_i ($i = 0, 1$) of $\mathcal{O}(a)$ with no common zeros. Then the sections of W defined by

$$(8.7) \quad (\sigma_0, \sigma_1), \quad (i\sigma_0, -i\sigma_1), \quad (-\sigma_1, \sigma_0), \quad (-i\sigma_1, -i\sigma_0)$$

trivialize W as a \mathbb{R}^4 -bundle. In each fiber of W , these four sections generates an integer lattice. We denote by $\Gamma(t)$ the lattice above $t \in \mathbb{P}^1$. The group

\mathbb{Z}^4 therefore acts on W by translations along these lattices. The resulting quotient 3-fold $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$ is a deformation of complex 2-tori above the projective line, with lattice $\Gamma(t)$ above t .

Given a holomorphic section τ of W , we define an automorphism ϕ_τ of \mathcal{X} by translating in each fiber along τ . We thus define a map

$$(8.8) \quad \tau \in H^0(\mathbb{P}^1, W) \longmapsto \phi_\tau \in \text{Aut}^0(\mathcal{X})$$

Let G be the additive subgroup of $H^0(\mathbb{P}^1, W)$ generated by the four sections (8.7). We have

Lemma 8.13. *The map (8.8) induces a monomorphism from $H^0(\mathbb{P}^1, W)/G$ to $\text{Aut}^0(\mathcal{X})$.*

Proof. We have

$$\phi_\sigma \circ \phi_\tau = \phi_\tau \circ \phi_\sigma = \phi_{\sigma+\tau}$$

hence (8.8) is a group morphism. Its kernel is given by the linear combinations over \mathbb{Z} of the four sections (8.7) so is equal to G . \square

Choose now $a + 1$ distinct points in the base \mathbb{P}^1 . We denote them by t_0, \dots, t_a . We assume that there is no automorphism of \mathcal{X} permuting non-trivially the t_i -fibers. This is achieved for a generic choice of points since moving slightly the points t_i yields that any pair of t_i -fibers are non-isomorphic. We fix one point P_i in each t_i -fiber. We call $\hat{\mathcal{X}}$ the manifold obtained from \mathcal{X} by blowing up these $a + 1$ points. Define

$$(8.9) \quad \Sigma := \{ \tau \in H^0(\mathbb{P}^1, W) \mid \tau(t_i) \in \Gamma(t_i) \text{ for } i = 0, \dots, a \}$$

We prove

Lemma 8.14. *The map (8.8) induces an isomorphism*

$$(8.10) \quad [\tau] \in \Sigma/G \longmapsto \hat{\phi}_\tau \in \text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}})$$

and we have

$$(8.11) \quad \text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}}) \simeq \mathbb{Z}^{4a} \quad \text{and} \quad \text{Aut}^1(\hat{\mathcal{X}}) = \{Id\}.$$

Proof. An automorphism $\hat{\phi}$ of $\hat{\mathcal{X}}$ corresponds to an automorphism ϕ of \mathcal{X} that permutes the points P_i . Now, an automorphism of \mathcal{X} lifts to an automorphism of W so sends fibers to fibers, and especially t_i -fibers to t_i -fibers. Since we assume that there is no automorphism of \mathcal{X} permuting non-trivially the t_i -fibers, then both ϕ and $\hat{\phi}$ induce the identity on the base and fix the points P_i . Now, ϕ inducing the identity in cohomology with coefficients in \mathbb{Z} means that it induces a translation in every fiber hence lies in the image of (8.8). Since it fixes the points P_i , it is the identity in every t_i -fiber, hence lies in the image of Σ . So we have an epimorphism $\tau \in \Sigma \mapsto \hat{\phi}_\tau \in \text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}})$ where $\hat{\phi}_\tau$ is the automorphism of $\hat{\mathcal{X}}$ corresponding to ϕ_τ . As above in Lemma 8.13, its kernel is given by the linear combinations over \mathbb{Z} of the four sections (8.7) so is equal to G . This proves that (8.10) is an isomorphism.

The space $H^0(\mathbb{P}^1, W)$ has dimension $2a + 2$, each section being given by a pair of elements of $\mathbb{C}[X]$ of degree $a + 1$. To belong to Σ , every polynomial must satisfy $a + 1$ equations. More precisely, given any $(a + 1)$ -uple (Q_0, \dots, Q_a) with each Q_i belonging to $\Gamma(t_i)$, there exists a unique section of W passing through Q_i at t_i . It is given as a pair of Lagrange interpolation

polynomials. Hence Σ identifies with the product $\Gamma(t_0) \times \dots \times \Gamma(t_a)$ so is isomorphic to \mathbb{Z}^{4a+4} . The action of G is equivalent to a transitive action on $\Gamma(t_0)$ and Σ/G identifies with the product $\Gamma(t_1) \times \dots \times \Gamma(t_a)$ so is isomorphic to \mathbb{Z}^{4a} . The same occurs therefore for $\text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}})$.

Finally, we note that the universal covering \hat{W} of $\hat{\mathcal{X}}$ is W blown up at each vertex of the lattices $P_i + \Gamma(t_i)$. Hence any non-trivial element $\hat{\phi}$ of $\text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}})$ lifts to an automorphism of \hat{W} that permutes non-trivially the blown-up points. Such an automorphism is not C^∞ -isotopic to the identity yielding that $\hat{\phi}$ is not C^∞ -isotopic to the identity. This achieves the proof of (8.11). \square

Choose a point P in $\hat{\mathcal{X}}$. Denote by $\hat{\mathcal{X}}_P$ the blow up of $\hat{\mathcal{X}}$ at P . Let $s = \pi(P)$ and define

$$(8.12) \quad \Sigma'_s = \{\tau \in \Sigma \mid \tau(s) \in \Gamma(s)\}$$

As an immediate corollary, we obtain the following important result.

Theorem 8.15. *The 3-fold $\hat{\mathcal{X}}_P$ satisfies*

$$(8.13) \quad \text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}}_P) \simeq \Sigma'_s/G \quad \text{and} \quad \text{Aut}^1(\hat{\mathcal{X}}_P) = \{Id\}$$

In particular,

i) *We have*

$$(8.14) \quad \text{Map}^{\mathbb{Z}}(\hat{\mathcal{X}}_P) \simeq \text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}}_P) \simeq \mathbb{Z}^{4a}$$

and

$$(8.15) \quad \text{Map}^1(\hat{\mathcal{X}}_P) \simeq \text{Aut}^1(\hat{\mathcal{X}}_P) = \{Id\}$$

for $s = t_a$.

ii) *We have*

$$(8.16) \quad \text{Map}^{\mathbb{Z}}(\hat{\mathcal{X}}_Q) \simeq \text{Map}^1(\hat{\mathcal{X}}_Q) \simeq \text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}}_Q) = \text{Aut}^1(\hat{\mathcal{X}}_Q) = \{Id\}$$

for $\pi(Q)$ a generic small deformation of s in \mathbb{P}^1 .

Proof. The proof follows from that of Lemma 8.14. An element of $\text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}}_P)$ is given by a section τ of Σ such that ϕ_τ is the identity on the s -fiber, that is τ belongs to Σ'_s . This defines an epimorphism from Σ'_s to $\text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}}_P)$. As in the previous cases, the kernel is G proving the first equality of (8.13). The second one is obtained as in the proof of Lemma 8.14 by arguing that a non-trivial element of $\text{Aut}^{\mathbb{Z}}(\hat{\mathcal{X}}_P)$ lifts to an automorphism of the universal covering that permutes non-trivially blown-up points, so is not C^∞ -isotopic to the identity. This also proves (8.15). If $s = t_a$, we have $\Sigma'_s = \Sigma$ since every ϕ_τ with $\tau \in \Sigma$ induces the identity in the t_a -fiber and (8.14) is a direct application of Lemma 8.14. Finally, define the set

$$(8.17) \quad \Gamma'_s := \{\tau(s) \in \pi^{-1}(s) \mid \tau \in \Sigma \setminus G\}$$

We claim that, at a generic point, Γ'_s does not intersect $\Gamma(s)$, so Σ'_s is reduced to G yielding (8.16). Indeed, choose $4a$ sections τ_i ($i = 1, \dots, 4a$) which generates Σ as a \mathbb{Z} -module together with the four sections (8.7). In particular Σ/G identifies with the group generated by the τ_i . It is enough to prove that no non-trivial linear combination over \mathbb{Z} of the $\tau_i(s)$ belong to $\Gamma(s)$ for a generic s . Assume by contradiction that, for every s in a small

disk of \mathbb{P}^1 disjoint from the set of points t_i , we can find some i with $\tau_i(s)$ belonging to $\Gamma(s)$. By continuity, we may assume that this is the same i for all s , restricting our disk if necessary. But still by continuity this implies that τ_i coincides over a disk with an element of G , that is, is an element of G . Contradiction. \square

Let us put some more context around Theorem 8.15. In [31], we gave examples of non-Kähler 3-folds with non trivial finite $\text{Aut}^1(X_0)$ -mapping class group. Examples of surfaces, including projective ones, having this property or having non trivial finite $\text{Aut}^{\mathbb{Z}}(X_0)$ -mapping class group were given in [11], see also [34] for $\text{Aut}^{\mathbb{Z}}(X_0)$. We ask in the last section of [31] for examples with non trivial *infinite* $\text{Aut}^1(X_0)$ -mapping class group, noting that, by Lieberman's result [27], see also §9.1, they must be non-Kähler. At that time, we were already motivated by questions about the geography of the Teichmüller stack and were looking for points with non-trivial holonomy, that is points with non trivial $\text{Aut}^1(X_0)$ -mapping class group that admits arbitrary small deformations with trivial $\text{Aut}^1(X_0)$ -mapping class group, see [31, §2] and §13.3 for the notion of holonomy in this context. The 3-folds of [31] enjoy this property but they only have finite holonomy.

The manifolds $\hat{\mathcal{X}}$ and $\hat{\mathcal{X}}_P$ for P on the t_a -fiber are examples with infinite discrete $\text{Aut}^{\mathbb{Z}}$ automorphism group, more precisely with h^0 equal to zero but with infinite $\text{Aut}^{\mathbb{Z}}(X_0)$ -mapping class group. Moreover, it follows from Theorem 8.15 that an arbitrary small deformation of $\hat{\mathcal{X}}_P$ has no non-trivial automorphisms inducing the identity in cohomology with coefficients in \mathbb{Z} so that such a point has infinite \mathbb{Z} -holonomy. Notice in particular that this is not an orbifold point of the \mathbb{Z} -Teichmüller stack although it has no non-zero global holomorphic vector fields. Once again, there is no such Kähler points. This shows that these non-Kähler $\hat{\mathcal{X}}_P$ have a more complicated local \mathbb{Z} -Teichmüller stack than Kähler points. More precisely,

Corollary 8.16. *The \mathbb{Z} -Teichmüller stacks of both $\mathbb{S}^2 \times (\mathbb{S}^1)^4$ at a point \mathcal{X} and of $\mathbb{S}^2 \times (\mathbb{S}^1)^4 \# \mathbb{P}^3 \# \dots \# \mathbb{P}^3$ at a point $\hat{\mathcal{X}}_P$ satisfying (8.14) are not locally isomorphic to the \mathbb{Z} -Teichmüller stack of any Kähler manifold.*

Proof. The sections (8.7) trivialize W over the reals, hence the smooth model of \mathcal{X} is $\mathbb{S}^2 \times (\mathbb{S}^1)^4$ and that of $\hat{\mathcal{X}}_P$ is obtained from it by doing the connected sum with \mathbb{P}^3 for every blown-up point. Then, the stabilizer of \mathcal{X} and of $\hat{\mathcal{X}}_P$ are infinite discrete by Lemma 8.14 and Theorem 8.15, whereas the stabilizer of a Kähler point has a finite number of connected component by Lieberman, proving the statement. \square

But we have more.

Theorem 8.17. *Let $P \in \hat{\mathcal{X}}$ such that $s = \pi(P)$ is generic. Then, the 3-fold $\hat{\mathcal{X}}_P$ is an exceptional point of its \mathbb{Z} -Teichmüller stack.*

Proof. Let $P \in \hat{\mathcal{X}}$ such that $s = \pi(P)$ is different from t_a and such that Γ'_s does not intersect $\Gamma(s)$. We have already shown in the proof of Theorem 8.15 that a generic s satisfies $\Gamma'_s \cap \Gamma(s) = \emptyset$. Choose a section τ in $\Sigma \setminus G$. Choose also a real linear map that sends the four sections (8.7) evaluated at s to the canonical basis of \mathbb{R}^4 . Moving slightly s if necessary, we may assume

that the image of the vector $\tau(s)$ through this linear map has coordinates \mathbb{Z} -linearly independent and \mathbb{Z} -linearly independent with 1. Then, the classical Kronecker's Theorem asserts that it generates a dense subgroup in the torus $\mathbb{R}^4/\mathbb{Z}^4$. In other words, given any point Q in the s -fiber, the translates of Q by the \mathbb{Z} -multiples of $\tau(s)$ form a dense orbit of the s -fiber. Choose R in the s -fiber such that $R - P$ is not a \mathbb{Z} -multiple of $\tau(s)$. Note that R can be chosen arbitrarily close to the initial point P . Consider the trivial deformation family $\hat{\mathcal{X}} \times \pi^{-1}(s) \rightarrow \pi^{-1}(s)$. It has a tautological section that sends a point M in the s -fiber to (M, M) , the first M being considered as living in the 3-fold $\hat{\mathcal{X}}$. Blow up this section. This gives a family $\mathcal{Y} \rightarrow \pi^{-1}(s)$ whose fiber above a point M is $\hat{\mathcal{X}}_M$. The section τ defines an automorphism of the trivial family

$$(8.18) \quad \begin{array}{ccc} \hat{\mathcal{X}} \times \pi^{-1}(s) & \xrightarrow{\phi_\tau \times \tau(s)} & \hat{\mathcal{X}} \times \pi^{-1}(s) \\ \downarrow & & \downarrow \\ \pi^{-1}(s) & \xrightarrow{\tau(s)} & \pi^{-1}(s) \end{array}$$

where $\tau(s)$ means translation by $\tau(s)$ in the torus $\pi^{-1}(s)$. It commutes with the tautological section, hence induces a family automorphism

$$(8.19) \quad \begin{array}{ccc} \mathcal{Y} & \xrightarrow{\phi_\tau \times \tau(s)} & \mathcal{Y} \\ \downarrow & & \downarrow \\ \pi^{-1}(s) & \xrightarrow{\tau(s)} & \pi^{-1}(s) \end{array}$$

Especially, this shows that all the fibers above the points of the $\tau(s)$ -translation orbit of R are isomorphic through a map inducing the identity in cohomology. By density, we may in particular extract from this a sequence of points (R_n) in $\pi^{-1}(s)$ that converges to P together with a family (ϕ_n) of diffeomorphisms inducing the identity in cohomology with ϕ_n inducing a biholomorphism between the R_n -fiber and the R_{n+1} -fiber. Look at the corresponding sequence of graphs. Even if it converges, its limit cannot belong to the closure of the $\text{Aut}^{\mathbb{Z}}$ -component of $\hat{\mathcal{X}}_P$ since this group is reduced to the identity by Theorem 8.15. Hence, $\hat{\mathcal{X}}_P$ is a \mathbb{Z} -exceptional point of its Teichmüller stack. \square

Remark 8.18. As already mentioned, it follows from Theorem 8.15 that none of these points are exceptional. This shows in particular that the converse to Proposition 8.12 is false. But it also left wide open the existence part of Conjecture 6.2, point II.

Remark 8.19. It follows from the proof of Theorem 8.17 that, given any $P \in \pi^{-1}(s)$, the set of points of $\pi^{-1}(s)$ above which the fiber of the family $\mathcal{Y} \rightarrow \pi^{-1}(s)$ is isomorphic to $\hat{\mathcal{X}}_P$ through a biholomorphism inducing the identity in cohomology is a countable dense subset of the base $\pi^{-1}(s)$. This is a weak version of the situation discussed in [29], Remark 11.8 and Problem 11.9.

It is interesting to have a closer look at the repartition of these \mathbb{Z} -exceptional points.

Corollary 8.20. *The subset*

$$(8.20) \quad E_{\hat{\mathcal{X}}} := \{M \in \hat{\mathcal{X}} \mid \hat{\mathcal{X}}_M \text{ is } \mathbb{Z}\text{-exceptional}\}$$

satisfies the following properties

- i) *It is dense in $\hat{\mathcal{X}}$.*
- ii) *If M belongs to $E_{\hat{\mathcal{X}}}$, then every point of the fiber $\pi^{-1}(\pi(M))$ belongs to $E_{\hat{\mathcal{X}}}$.*

Proof. Point i) is a direct consequence of Theorem 8.17. To prove ii), consider, as a slight variation of the construction used in the proof of Theorem 8.17, the family $\mathcal{Z} \rightarrow \hat{\mathcal{X}}$ obtained by blowing up the trivial family $\hat{\mathcal{X}} \times \hat{\mathcal{X}} \rightarrow \hat{\mathcal{X}}$ along the section $M \in \hat{\mathcal{X}} \mapsto (M, M) \in \hat{\mathcal{X}}$. The fiber at $M \in \hat{\mathcal{X}}$ is $\hat{\mathcal{X}}_M$.

Let $M \in E_{\hat{\mathcal{X}}}$. Then, there exists an exceptional sequence (Q_n, R_n, ϕ_n) in $\pi^{-1}(P) \times \pi^{-1}(P) \times \text{Diff}^{\mathbb{Z}}(M)$ with ϕ_n inducing a biholomorphism between $\hat{\mathcal{X}}_{Q_n}$ and $\hat{\mathcal{X}}_{R_n}$. A direct adaptation of Theorem 8.15 shows that ϕ_n is given by a translation in the π -fibers along elements of Σ with the factor of translation at $\pi(M)$ going to infinity. Thus, letting P belong to the same π -fiber as M , the sequence $(P - M + Q_n, P - M + R_n, \phi_n)$ is also exceptional, showing that P is \mathbb{Z} -exceptional. □

Remark 8.21. We do not know if the density property of \mathbb{Z} -exceptional points in the family \mathcal{Z} used in the proof of Theorem 8.20 is true for the Kuranishi family of one of these points. If yes, this would give a local positive answer to part II of Conjecture 6.2 for \mathbb{Z} -exceptional points.

9. FINITENESS PROPERTIES OF THE LOCAL TEICHMÜLLER STACK IN THE KÄHLER SETTING

9.1. Consequences of Lieberman's compactness result. In this section, we recall and apply a basic result on cycle spaces in the Kähler case, which is due to Lieberman [27]. We state the relative version, which is adapted to our purposes.

Proposition 9.1. *Let $\pi_i : \mathcal{X}_i \rightarrow B_i$ be smooth morphisms with compact Kähler fibers over reduced analytic spaces B_i for $i = 0, 1$. Let $\mathcal{Z} \rightarrow E$ be a continuous family of relative cycles of $\mathcal{X}_0 \times \mathcal{X}_1 \rightarrow B_0 \times B_1$. Assume that the projection of E is included in a compact of $B_0 \times B_1$. Assume moreover that all cycles of \mathcal{Z} are smooth, i.e are graphs of a biholomorphism from some fiber $(X_0)_t$ onto some fiber $(X_1)_{t'}$. Assume finally that they are graphs of biholomorphisms that induce the identity in cohomology with coefficients in \mathbb{Z} . Then,*

- i) *E has compact closure in the space of cycles of $\mathcal{X}_0 \times \mathcal{X}_1 \rightarrow B_0 \times B_1$ hence only meets a finite number of irreducible components of this space.*
- ii) *Let \mathcal{C} be such a component. Then \mathcal{C} contains a Zariski open subset \mathcal{C}_0 all of whose members are graphs of a biholomorphism inducing the identity in cohomology with coefficients in \mathbb{Z} between a fiber of \mathcal{X}_0 and a fiber of \mathcal{X}_1 .*

Proof. i) Let $(\omega_t^i)_{t \in B_i}$ be a continuous family of Kähler forms on the π_i -fibers ($i = 0, 1$). Let M be the smooth model of X_0 and let $(J_t^i)_{t \in B_i}$ be a continuous family of integrable almost complex operators on M such that $(X_i)_t = (M, J_t^i)$. For every $e \in E$, call $f_e : M \rightarrow M$ the biholomorphism from some fiber $(X_0)_t$ onto some fiber $(X_1)_{t'}$ corresponding to the cycle Z_e . We compute the volume of these cycles using the ω_t . We have

$$\text{Vol}(Z_e) = \int_M (\omega_t^0 + f_e^* \omega_{t'}^1)^n = \int_M (\omega_t^0 + \omega_{t'}^1)^n$$

since f_e induces the identity in cohomology hence $f_e^* \omega_{t'}^1$ and ω_t^0 differs from an exact form. Since the projection of E is included in a compact of $B_0 \times B_1$, we obtain that the volume of the Z_e is uniformly bounded. It follows from [27, Theorem 1] that E has compact closure in the cycle space of $\mathcal{X}_0 \times \mathcal{X}_1$. Hence E only meets a finite number of irreducible components of this cycle space.

ii) Consider the family of cycles $\tilde{\mathcal{C}} \subset \mathcal{X}_0 \times \mathcal{X}_1 \rightarrow \mathcal{C}$. Since this map is proper and surjective, it is smooth on a Zariski open subset. Since some fibers are non singular, the generic fiber is non singular. The cycles above E are submanifolds of some $(X_0)_t \times (X_1)_{t'}$ with projections pr_i being bijective onto both factors. Hence, on a Zariski open subset of \mathcal{C} , every cycle enjoys such properties. So is the graph of a biholomorphism between a fiber of \mathcal{X}_0 and a fiber of \mathcal{X}_1 . Finally all these graphs are smoothly isotopic hence induce the identity in cohomology with coefficients in \mathbb{Z} since at least one of them has this property. \square

Remark 9.2. There exist relative versions of Lieberman's result in the class (\mathcal{C}) . However, they do not apply to smooth morphisms with class (\mathcal{C}) fibers but to morphisms that are equivalent to Kähler morphisms in some sense, see [16, Def. 2.3] or [4, Def 4.1.9]. For that reason, they are not suited to our purposes and we stick to the Kähler setting.

Setting $X_0 = (M, J_0)$, considering the $\text{Diff}^{\mathbb{Z}}(M)$ -orbit \mathcal{O} of J_0 in $\mathcal{I}(M)$ and viewing K_0 as a local transverse section, we obtain a first interesting Corollary.

Corollary 9.3. *If V is small enough then K_0 intersects \mathcal{O} only at J_0 .*

Proof. We assume that V is small enough so that K_0 only contains Kähler points, cf. footnote 2. We also assume that K_0 is reduced, replacing it with its reduction if necessary. Take $\mathcal{X}_1 = \mathcal{X}_0$ and $\mathcal{X}_0 = X_0$ seen as a family over the point $\{J_0\}$. Let E' be the subset of K_0 corresponding to complex structures J in the orbit \mathcal{O} . Now, \mathcal{O} intersects transversally K_0 at J_0 by (3.3) but also at any intersection point. Since $\text{Diff}^{\mathbb{Z}}(M)$ has a countable topology, this intersection contains at most a countable number of points. Since we are only interested in what happens close to J_0 , we may replace E' with its intersection with a compact neighborhood of J_0 in K_0 . Then for each $J \in E'$, choose some element f_J of $\text{Diff}^{\mathbb{Z}}(M)$ mapping J_0 onto it. Set $E = \{J_0\} \times E'$ and let \mathcal{Z} be the cycles corresponding to the graphs of the f_J . Apply Proposition 9.1. We conclude that E meets a finite number of irreducible components of the cycle space of $X_0 \times \mathcal{X}_0$, say $\mathcal{C}_1, \dots, \mathcal{C}_p$.

Still by Proposition 9.1, it follows that a Zariski open subset of each \mathcal{C}_i only contains graph of biholomorphisms between X_0 and some X_J with $J \in E'$. Hence each of these components only contains cycles in a fixed product $X_0 \times X_J$ and E' is a finite subset. Reducing V if necessary, we may assume that E' is just $\{J_0\}$ as wanted. \square

Remark 9.4. Recall that we work with the $\text{Diff}^{\mathbb{Z}}(M)$ -orbit and not with the $\text{Diff}^+(M)$ -orbit. Corollary 9.3 does not say that, in the Kähler setting, there is no infinite sequence of points in K_0 converging to J_0 and all encoding X_0 . It just says that, if it happens, only a finite number of the involved biholomorphisms with X_0 can be chosen inducing the identity in cohomology.

As for the intersection of the $\text{Diff}^{\mathbb{Z}}(M)$ -orbit of an arbitrary $J \in K_0$ with K_0 , we have

Corollary 9.5. *Let $J \in K_0$. If K_0 is small enough, then K_0 intersects the $\text{Diff}^{\mathbb{Z}}(M)$ -orbit of J into a finite number of leaves of the foliation of K_0 .*

Proof. This is completely analogous to the proof of Corollary 9.3. Take $\mathcal{X}_1 = \mathcal{X}_0$ and $\mathcal{X}_0 = X_J$ and apply Proposition 9.1. This proves that the set of graphs of biholomorphisms between X_J and another fiber of \mathcal{X}_0 is the union of the Zariski open subsets of regular cycles of a finite number of irreducible components of the cycle space of relative cycles of $X_J \times \mathcal{X}_0$. By definition (see Subsection 3.2), they project exactly onto a finite number of leaves of the foliation of K_0 . \square

We now analyse further the structure of \mathcal{C} and \mathcal{C}_0 and compare them with $\mathcal{T}_V^{\mathbb{Z}}$ from the one hand and with $\text{Aut}^{\mathbb{Z}}(X_0)$ from the other hand. As before, we assume that K_0 is reduced, replacing it with its reduction if necessary.

Let us begin with \mathcal{C}_0 . By Proposition 9.1 applied to $\mathcal{X}_0 = \mathcal{X}_1 = X_0$, it has only a finite number of irreducible components and it is compact. Moreover, if a component contains a non singular cycle, it must be the graph of an automorphism and a Zariski open subset of it contains such graphs, hence it contains a connected component of $\text{Aut}^{\mathbb{Z}}(X_0)$ as Zariski open set. So the picture to have in mind is the following.

- i) $\text{Aut}^{\mathbb{Z}}(X_0)$ consists of a finite number of connected components. It always includes $\text{Aut}^0(X_0)$ and $\text{Aut}^1(X_0)$. The other connected components, if exist, are connected components of $\text{Aut}(X_0)$ and are all isomorphic to $\text{Aut}^0(X_0)$.
- ii) Either these connected component are all compact and each of them forms an irreducible component of \mathcal{C}_0 ; or they admit an analytic compactification in \mathcal{C}_0 by adding an analytic space of strictly lower dimension of singular cycles, each compactified connected component of $\text{Aut}^{\mathbb{Z}}(X_0)$ becoming a compact irreducible component of \mathcal{C}_0 .
- iii) There may exist in \mathcal{C}_0 a finite number of additional irreducible components which contain only singular cycles.

Notice that two such irreducible components may intersect. For example, two distinct connected components of $\text{Aut}^{\mathbb{Z}}(X_0)$ may intersect once compactified in \mathcal{C}_0 .

As for \mathcal{C} , applying Proposition 9.1 to $\mathcal{X}_0 = \mathcal{X}_1 = \mathcal{X}_0$ yields that it has only a finite number of irreducible components and that its restriction above any

compact subset of $K_0 \times K_0$ is compact. Moreover, if a component contains a non singular cycle, it must be the graph of a biholomorphism between two fibers of the Kuranishi family and a Zariski open subset of it contains such graphs, hence it contains an irreducible component of (the reduction of) $\mathcal{T}_V^{\mathbb{Z}}$ as Zariski open set. And if an irreducible component contains a graph of an automorphism of $\text{Aut}^{\mathbb{Z}}(X_0)$, then it must contain an irreducible component of the closure of $\text{Aut}^{\mathbb{Z}}(X_0)$ in \mathcal{C}_0 . So the picture to have in mind is the following.

- i) The reduction of $\mathcal{T}_V^{\mathbb{Z}}$ consists of a finite number of irreducible components. Each of them either contains a connected component of $\text{Aut}^{\mathbb{Z}}(X_0)$ or does not contain any automorphism of X_0 .
- ii) Each of these irreducible components injects as a Zariski open subset of an irreducible component of \mathcal{C} .
- iii) There may exist in \mathcal{C} a finite number of additional irreducible components which contain only singular cycles.

Notice that two such irreducible components may intersect, that is, two distinct connected components of $\mathcal{T}_V^{\mathbb{Z}}$ may intersect once compactified in \mathcal{C} .

Consider now a sequence (8.1). We may state our second Corollary.

Corollary 9.6. *Assume X_0 is Kähler. Then,*

- i) *Up to passing to a subsequence, we may assume that the graphs of the ϕ_n belong to a fixed irreducible component \mathcal{C} of \mathcal{C} and converges to a cycle γ .*
- ii) *The cycle γ belongs to \mathcal{C} .*
- iii) *The cycle γ and thus the component \mathcal{C} are exceptional if and only if γ does not belong to the closure of an $\text{Aut}^{\mathbb{Z}}(X_0)$ -component of \mathcal{C}_0 .*
- iv) *Let \mathcal{C}_0 be the connected component of the intersection $\mathcal{C} \cap \mathcal{C}_0$ containing γ . The cycle γ and thus the component \mathcal{C} are exceptional if and only if every irreducible component of \mathcal{C}_0 consist only of singular cycles.*

Proof. Since there are only finitely many components in \mathcal{C} , we may assume, up to passing to a subsequence, that the graphs of the ϕ_n belong to a fixed irreducible component \mathcal{C} . The component \mathcal{C} restricted to any compact neighborhood of (J_0, J_0) in $K_0 \times K_0$ is compact, hence the sequence converges, up to passing to another subsequence, and the limit cycle γ belongs to \mathcal{C} . This proves i) and ii). If γ is also obtained as a limit of graphs of elements of $\text{Aut}^{\mathbb{Z}}(X_0)$, then the distance between ϕ_n and $\text{Aut}^{\mathbb{Z}}(X_0)$ tends to zero when n goes to infinity. Hence, for n big enough, (x_n, ϕ_n) is (V, \mathcal{D}_1) -admissible and so is analytically isotopic to an element of $\text{Aut}^{\mathbb{Z}}(X_0)$. Therefore \mathcal{C} contains a Zariski open subset formed by the graphs of a connected component of $\text{Aut}^{\mathbb{Z}}(X_0)$ and their extensions, so is an A^1 -component. Conversely, if γ does not belong to the closure of an $\text{Aut}^{\mathbb{Z}}(X_0)$ -component, the Zariski open subset of regular cycles of \mathcal{C} does not intersect $\text{Aut}^{\mathbb{Z}}(X_0)$. This proves iii). Then iii) implies that \mathcal{C}_0 contains only singular cycles, since an irreducible component of \mathcal{C}_0 which is not completely singular is the closure of an $\text{Aut}^{\mathbb{Z}}(X_0)$ -component. Conversely, if \mathcal{C}_0 is completely singular, then γ is not in the closure of an $\text{Aut}^{\mathbb{Z}}(X_0)$ -component, otherwise, arguing as above, the corresponding $\text{Aut}^{\mathbb{Z}}(X_0)$ -component would belong to \mathcal{C}_0 . Hence, by iii), \mathcal{C} is exceptional. \square

Let \mathcal{C}' be the union of irreducible components of \mathcal{C} containing a sequence (8.1).

Corollary 9.7. *Assume X_0 is Kähler. Then,*

- i) *The number of irreducible components of the reduction of $\mathcal{T}_V^{\mathbb{Z}}$ is finite.*
- ii) *If V is a sufficiently small neighborhood of J_0 in $\mathcal{I}(M)$, then there exists a natural bijection between the set of irreducible components of the reduction of $\mathcal{T}_V^{\mathbb{Z}}$ and that of \mathcal{C}' . In particular, every component of \mathcal{C}' is either an A^1 -component or an exceptional one.*
- iii) *If V is a sufficiently small neighborhood of J_0 in $\mathcal{I}(M)$, then the intersection of an irreducible component of \mathcal{C}' with \mathcal{C}_0 is connected.*
- iv) *If V is a sufficiently small neighborhood of J_0 in $\mathcal{I}(M)$, and $V' \subset V$ contains also J_0 , then the natural inclusion of $\mathcal{T}_{V'}$ in $\mathcal{T}_V^{\mathbb{Z}}$ is a bijection between the corresponding sets of irreducible components.*

Proof. Assume $\mathcal{T}_V^{\mathbb{Z}}$ is reduced. Since every irreducible component of $\mathcal{T}_V^{\mathbb{Z}}$ injects in an irreducible component of \mathcal{C} and \mathcal{C} has only a finite number of components by Kählerianity, this proves i). By finiteness, for a sufficiently small neighborhood V of J_0 , the point J_0 is adherent to the s -image and the t -image of every such component. Hence they contain a sequence (8.1). So the irreducible components of $\mathcal{T}_V^{\mathbb{Z}}$ are in fact in 1:1 correspondence with the irreducible components of \mathcal{C}' , proving ii). By compactness of the components of \mathcal{C}_0 and finiteness of their number, the intersection of an irreducible component \mathcal{C} with \mathcal{C}_0 has at most a finite number of connected components. Taking V smaller if needed, the irreducible component \mathcal{C} disconnects as a finite union of irreducible components of \mathcal{C}' that intersects \mathcal{C}_0 in a single connected component. Finally iv) follows from ii) and from the finiteness of components. When restriction to a smaller $V' \subset V$, the number of components of both $\mathcal{T}_{V'}$ and \mathcal{C}' decrease. By finiteness, for a small enough V , this number stays constant when restricting to smaller $V' \subset V$. \square

As a consequence of Corollary 9.7, we do not need to consider the full target germification in the Kähler case. It is enough to look at $\mathcal{S}(M, V)$ for a fixed small enough V since restricting to smaller neighborhoods of 0 will not change the number of components of \mathcal{T}_V .

Moreover, we have

Corollary 9.8. *Assume X_0 is Kähler. Then,*

- i) *There is no wandering sequence (8.1) with each ϕ_n belonging to a different component of \mathcal{T}_V .*
- ii) *There is no vanishing cycle.*

and

Corollary 9.9. *Assume X_0 is Kähler. Then, the following statements are equivalent*

- i) *X_0 is exceptional.*
- ii) *There exists an exceptional component.*
- iii) *There exists an exceptional cycle.*

Proof of Corollaries 9.8 and 9.9. This is essentially a reformulation of what precedes. Corollary 9.8 is a direct consequences of point i) of Corollary 9.6. Corollary 9.9 follows then from Proposition 8.11. \square

In the sequel, we assume that V is small enough so that

- i) It only contains Kähler structures.
- ii) Items ii), iii) and iv) of Corollary 9.7 are valid.

Remark 9.10. Let \mathcal{C}_0 be an exceptional component of \mathcal{C}_0 . Then, although every cycle of \mathcal{C}_0 is singular, not every cycle of \mathcal{C}_0 is exceptional. First of all, we must remove the possibly non-empty intersection of \mathcal{C}_0 with the closure of the components of $\text{Aut}^{\mathbb{Z}}(X_0)$. But in the remaining Zariski open set \mathcal{U} of \mathcal{C}_0 , the exceptional cycles are those lying in the intersection with the finite union of irreducible components of \mathcal{C}' . This intersection forms a subanalytic space of \mathcal{U} that can be strict.

Remark 9.11. A singular cycle of \mathcal{C}_0 is either the graph of a bimeromorphic mapping of X_0 or the sum of two cycles $\gamma_1 + \gamma_2$, such that γ_i maps bimeromorphically onto X_0 through the i -th projection of $X_0 \times X_0$ to X_0 . This may include the case $X_0 \times \{*\} + \{*\} \times X_0$.

9.2. The morphism \mathcal{A}_1 into $\mathcal{T}(M, V)$ in the Kähler setting. The following result is a refinement of Theorem 5.3 in the Kähler case.

Theorem 9.12. *Assume that X_0 is Kähler. Then, the natural inclusion of \mathcal{A}_0 into $\mathcal{T}(M, V)$, resp. of \mathcal{A}_1 into $\mathcal{T}(M, V)$, as well as that of \mathcal{A}_0 , resp. $\mathcal{A}_{\mathbb{Z}}$ into $\mathcal{T}^{\mathbb{Z}}(M, V)$, are finite and étale morphisms of analytic stacks.*

Proof. Just combine Theorem 5.3 and Corollary 9.6, noting that the cardinal of the fiber at X_0 of these étale morphisms is less than the number of connected components of $\mathcal{T}_V^{\mathbb{Z}}$. \square

Let us compute the fibers of the inclusion of \mathcal{A}_0 into $\mathcal{T}(M, V)$, resp. of \mathcal{A}_1 into $\mathcal{T}(M, V)$ for X_0 Kähler. Observe that this means computing $g(\mathcal{X})$, resp. $g_1(\mathcal{X})$, for \mathcal{X} a family $X_J \rightarrow \{J\}$ for $J \in K_0$. For $J = J_0$, it follows from Corollary 9.3 that there is no point J in K_0 such that X_J is $\text{Diff}^0(M)$ -biholomorphic to X_0 . Hence, we just have to consider isomorphisms of the family \mathcal{X} , that is automorphisms of X_0 . We deduce from that the equalities

$$(9.1) \quad g(\mathcal{X}) = \text{Card } \sharp \text{Aut}^1(X_0) = \text{Card} (\text{Aut}^1(X_0)/\text{Aut}^0(X_0))$$

and

$$(9.2) \quad g_1(\mathcal{X}) = \text{Card} (\text{Aut}^1(X_0)/\text{Aut}^1(X_0)) = 1$$

Both computations (9.1) and (9.2) remain the same for $\mathcal{X} \rightarrow B$ a \mathcal{A}_1 -family⁹ with connected base and at least one fiber biholomorphic to X_0 . Indeed, let $b \in B$ such that the b -fiber is isomorphic to X_0 . Then, the family $\mathcal{X} \rightarrow B$ is locally isomorphic at b to $u^* \mathcal{K}_0$ for u a holomorphic map defined on a neighborhood of $b \in B$ with values in K_0 . Every isomorphism of the family induces an automorphism of the b -fiber, that is of X_0 . Using this point b as

⁹that is a family isomorphic to a family obtained by gluing local pull-backs of the Kuranishi family $\mathcal{K}_0 \rightarrow K_0$ through a cocycle of morphisms belonging to \mathcal{A}_1 .

b_0 in (5.11) and (5.13), and noting that, given any $J \in K_0$, there exists a morphism starting from J in any connected component of \mathcal{A}_1 , we are done.

We go back to the case of \mathcal{X} being a family $X_J \rightarrow \{J\}$ for $J \in K_0$. This time, we assume that J is not J_0 . The intersection of the $\text{Diff}^0(M)$ -orbit of J with K_0 may be positive dimensional, but it consists of a finite number of leaves of the foliation of K_0 by Corollary 9.5. Each leaf corresponds to a connected subset of \mathcal{T}_V . Now, since the maps Hol_f defined in (3.5) preserves the foliation of K_0 , there is an action of $\sharp\text{Aut}^1(X_0)$ on the finite set of leaves corresponding to J . Then $g(\mathcal{X})$ is given by the number of $\sharp\text{Aut}^0(X_0)$ -orbits of leaves and $g_1(\mathcal{X})$ by the number of $\sharp\text{Aut}^1(X_0)$ -orbits of leaves (recall Remark 3.6).

Let us say that a point $J \in K_0$ is *generic* if the finite set of leaves corresponding to J has the same cardinal than $\sharp\mathcal{T}_V$. Equivalently, it is generic if every connected component of \mathcal{T}_V contains a morphism with source J . Then $g(\mathcal{X})$ is given by the number of $\sharp\text{Aut}^0(X_0)$ -orbits in $\sharp\mathcal{T}_V$ and $g_1(\mathcal{X})$ by the number of $\sharp\text{Aut}^1(X_0)$ -orbits in $\sharp\mathcal{T}_V$, that is are maximal. This remains true for a family over a connected base with all fibers generic.

We recollect the previous computations in the following proposition. Of course similar calculations hold for $\mathcal{A}_{\mathbb{Z}}$ -families.

Proposition 9.13. *Assume X_0 is Kähler. Let $\mathcal{X} \rightarrow B$ be a \mathcal{A}_1 -family over a connected base. Then,*

- i) *Assume that at least one fiber of \mathcal{X} is biholomorphic to X_0 . Then (9.1) and (9.2) hold.*
- ii) *Assume that all the fibers are generic. Then $g(\mathcal{X})$ is given by the number of $\sharp\text{Aut}^0(X_0)$ -orbits in $\sharp\mathcal{T}_V$ and $g_1(\mathcal{X})$ by the number of $\sharp\text{Aut}^1(X_0)$ -orbits in $\sharp\mathcal{T}_V$, that is are maximal.*

9.3. Universality of the Kuranishi stacks revisited. By definition, the Teichmüller stack $\mathcal{T}(M)$ is universal for reduced M -deformations, that is any such family is obtained through an analytic mapping from the base of the family to $\mathcal{T}(M)$ and this mapping is unique up to unique isomorphism. The same is true for $\mathcal{T}(M, V)$ and V -families, for $\mathcal{T}^{\mathbb{Z}}(M)$ and \mathbb{Z} -reduced M -deformations, for $\mathcal{T}^{\mathbb{Z}}(M, V)$ and \mathbb{Z} -reduced V -families. This is one of the main interests in using the stack formalism.

By germifying as in Section 3.5, we may consider the germ of \mathcal{T} , resp. of $\mathcal{T}^{\mathbb{Z}}(M)$, at a point $J \in \mathcal{I}(M)$. Then $(\mathcal{T}(M), J)$, resp. $(\mathcal{T}^{\mathbb{Z}}(M), J)$, is universal for germs of reduced, resp. \mathbb{Z} -reduced, M -deformations of X_J (compare with Corollary 3.12).

Theorem 9.12 allows us to prove similar results for the Kuranishi stack of a Kähler, non-exceptional point. In return, this gives a geometric interpretation of these results.

Proposition 9.14. *Assume X_0 Kähler. Then, the Kuranishi stack \mathcal{A}_1 is universal for reduced, V -families if and only if the base point X_0 is not exceptional.*

Also, given $J \in K_0$, the germ of Kuranishi stack (\mathcal{A}_1, J) is universal for germs of reduced, M -deformations of X_J if and only if the isotropy group at J is $\text{Aut}^1(X_J)$.

In particular, if the base point X_0 is not exceptional, for all $J \in K_0$, the germ of Kuranishi stack (\mathcal{A}_1, J) is universal for germs of reduced, M -deformations of X_J .

The previous statements still hold true replacing \mathcal{A}_1 with $\mathcal{A}_{\mathbb{Z}}$, reduced with \mathbb{Z} -reduced, $\text{Aut}^1(X_0)$ with $\text{Aut}^{\mathbb{Z}}(X_0)$.

Let us make precise that, in the second statement, (\mathcal{A}_1, J) is the Kuranishi stack of X_0 but germified at X_J , for $J \in K_0$. It is not the Kuranishi stack of X_J .

Now, Proposition 9.14 tells us that an exceptional point is a point whose Kuranishi stack lacks of universality. Firstly, this lack of universality is a lack of completeness. It is possible that some reduced, V -families do not belong to \mathcal{A}_1 . Secondly, it is a lack of unicity. Indeed, the number $g_1(\mathcal{X})$ gives exactly the number of different ways for obtaining \mathcal{X} from \mathcal{A}_1 . It is interesting however to observe from Proposition 9.13 that $g_1(\mathcal{X})$ is equal to one for all \mathcal{A}_1 -families with a fiber biholomorphic to X_0 . Hence we automatically have universality for this restricted type of families.

As for the second statement, we already know from Corollary 3.12 that (\mathcal{A}_1, J_0) is universal for germs of reduced, M -deformations of X_0 . Proposition 9.14 tells us that this property may be lost for nearby points of K_0 when X_0 is exceptional.

In other words, the geometric meaning of exceptionality is that the universal properties of the Kuranishi stack at the base point fail to be true in a neighborhood of this point.

Proof. Since the Teichmüller stack $\mathcal{T}(M, V)$ enjoys both universal properties of Proposition 9.14, so does \mathcal{A}_1 when X_0 is Kähler and not exceptional.

If X_0 is exceptional, then some fibers of the morphism $\mathcal{A}_1 \rightarrow \mathcal{T}(M, V)$ contain several points. In other words, the uniqueness property of universality is not true for some families $X_J \rightarrow \{J\}$.

Finally, since $\mathcal{K}_0 \rightarrow K_0$ is complete at any point $J \in K_0$, the only requirement to have universality of (\mathcal{A}_1, J) is to have all the morphisms of germs of reduced M -deformations of X_J , that is that the isotropy group at J of \mathcal{A}_1 is $\text{Aut}^1(X_J)$ (compare with the proof of Corollary 3.12).

The proof in the $\mathcal{A}_{\mathbb{Z}}$ case follows exactly the same line of arguments. \square

To finish with, it is interesting to compare the second statement of Proposition 9.14 with the \mathcal{A}_0 -case. Here there is no condition for the universality property at each point. To be more precise,

Proposition 9.15. *Assume X_0 Kähler. If V is small enough, then, for all $J \in K_0$, the germ of Kuranishi stack (\mathcal{A}_0, J) is universal for germs of 0-reduced, M -deformations of X_J .*

Proof. Arguing as in the proof of Proposition 9.14, we obtain that, given $J \in K_0$, the germ of Kuranishi stack (\mathcal{A}_0, J) is universal for germs of 0-reduced, M -deformations of X_J if and only if the isotropy group at J is $\text{Aut}^0(X_J)$.

We claim that this is always the case, reducing V to a smaller neighborhood of J_0 if necessary. In other words, given $J \in K_0$, every automorphism

of $\text{Aut}^0(X_J)$ is (\mathcal{D}_0, V) -admissible. This can be proven as follows. Reducing V if necessary, we may assume that, given any $J \in K_0$, there exists an open neighborhood of the identity in $\text{Aut}^0(X_J)$ that belongs to \mathcal{D}_0 . Since any element of $\text{Aut}^0(X_J)$ can be written as a finite product of elements in a fixed neighborhood of the identity, any element of $\text{Aut}^0(X_J)$ is (\mathcal{D}_0, V) -admissible. \square

9.4. The Teichmüller stack as an orbifold. As second application of Theorem 9.12, we deal with the orbifold case.

Theorem 9.16. *Assume X_0 Kähler. Then, the following statements are equivalent*

- i) *The exists some open set V of $\mathcal{I}(M)$ such that $\mathcal{T}(M, V)$ is an orbifold.*
- ii) *The exists some open set V of $\mathcal{I}(M)$ such that \mathcal{A}_1 is an orbifold and X_0 is not exceptional.*
- iii) *$\text{Aut}^1(X_0)$ is finite and X_0 is not exceptional.*
- iv) *$\text{Aut}^0(X_0)$ is trivial and X_0 is not exceptional.*

Remark 7.5 applies also here for the choice of V . We also have

Corollary 9.17. *Assume X_0 Kähler. Then, the following statements are equivalent*

- i) *The exists some open set V of $\mathcal{I}(M)$ such that $\mathcal{T}^{\mathbb{Z}}(M, V)$ is an orbifold.*
- ii) *The exists some open set V of $\mathcal{I}(M)$ such that $\mathcal{A}_{\mathbb{Z}}$ is an orbifold and X_0 is not \mathbb{Z} -exceptional.*
- iii) *$\text{Aut}^{\mathbb{Z}}(X_0)$ is finite and X_0 is not \mathbb{Z} -exceptional.*
- iv) *$\text{Aut}^0(X_0)$ is trivial and X_0 is not \mathbb{Z} -exceptional.*

Proof. We only prove Theorem 9.16. Assume ii). Then, $\mathcal{T}(M, V)$ is isomorphic to \mathcal{A}_1 , so is an orbifold and i) is proved. Assume i). Then a finite group acts on K_0 stabilizing J_0 and \mathcal{T}_V encodes the orbits of this action. As a consequence, the leaves of the foliation of K_0 are 0-dimensional, hence, by Theorem 2 of [28], the dimension of the automorphism group of the fibers of the Kuranishi family is constant. Combining Proposition 9.13 and Remark 3.6, this implies that this group has the same cardinal as $\sharp\mathcal{T}_V$. Now, if X_0 is exceptional, the finite group $\sharp\mathcal{T}_V$ is not the stabilizer of the base structure X_0 , since the exceptional components do not yield automorphisms at 0. So X_0 is not exceptional, and ii) follows. This proves that i) and ii) are equivalent. Then, ii) and iii) are equivalent by Corollary 3.12. Finally, iii) and iv) are equivalent because of the fact that $\text{Aut}^0(X_0)$ has finite index in $\text{Aut}^1(X_0)$ in the Kähler case. \square

10. DISTRIBUTION OF EXCEPTIONAL POINTS

Our next task is to understand how the exceptional points are distributed, that is in which sense they are rare. Let V_K be the open set of Kähler points of $\mathcal{I}(M)$ (recall footnote 2). We shall prove

Theorem 10.1. *The closure of exceptional points of the Teichmüller stack $\mathcal{T}(M, V_K)$ of Kähler structures form a strict analytic substack $\mathcal{E}(M)$ of $\mathcal{T}(M, V_K)$.*

The same is true for \mathbb{Z} -exceptional points that form a strict analytic substack $\mathcal{E}^{\mathbb{Z}}(M)$ of $\mathcal{T}^{\mathbb{Z}}(M, V_K)$.

and its immediate Corollary

Corollary 10.2. *Normal Kähler points fill a Zariski open substack of the Teichmüller stack $\mathcal{T}(M, V_K)$ of Kähler structures.*

\mathbb{Z} -normal Kähler points fill a Zariski open substack of the \mathbb{Z} -Teichmüller stack $\mathcal{T}^{\mathbb{Z}}(M, V_K)$ of Kähler structures.

Theorem 10.1 and Corollary 10.2 form the best results we are able to prove in relation with main Conjecture 6.2, point I.

Note that the closure of exceptional points, resp. the set of normal points, form a strict analytic subspace, resp. a Zariski open set of any atlas of $\mathcal{T}(M, V_K)$. Since any Kuranishi space K_0 of a point in V_K is a local atlas of $\mathcal{T}(M, V_K)$, this means that the closure of exceptional points, resp. the set of normal points, form a strict analytic subspace, resp. a Zariski open set of any K_0 .

Remark 10.3. The closure of exceptional points, resp. \mathbb{Z} -exceptional points, also form a strict analytic subspace of V_K .

Remark 10.4. In particular, the set of normal points is dense in the Kähler Teichmüller space $V_K/\text{Diff}^0(M)$ and contains an open set. However, due to the non-Hausdorff topology this space may have, this may be a misleading statement. For example, if M is $\mathbb{S}^2 \times \mathbb{S}^2$, then the (Kähler) Teichmüller space of M , as a set, is \mathbb{Z} , a point $a \in \mathbb{Z}$ encoding the Hirzebruch surface \mathbb{F}_{2a} ¹⁰. Now, the topology to put on \mathbb{Z} has for (non trivial) open sets $\{0\}$, $\{0, 1\}$, $\{-1, 0\}$ and so on, cf. [29], Examples 5.14 and 12.6. Hence 0 is an open and dense subset of the Teichmüller space.

Proof. We only prove the result for exceptional points. The proof can be easily modified to treat the case of \mathbb{Z} -exceptional points.

The atlas (5.1) being an atlas of a neighborhood of X_0 in the Teichmüller stack, it contains all the information we need to decide which points close to X_0 are exceptional.

Indeed, pick a point X_J in K_0 . Assume it is exceptional. By our assumptions on V ¹¹, every irreducible component of \mathcal{C} contains a cycle of $X_0 \times X_0$, hence the set \mathcal{C}_J of cycles of $X_J \times X_J$ which are limits of cycles of \mathcal{C} is included in \mathcal{C}' . As a consequence, J is exceptional if and only if there exists a component of \mathcal{C}' whose subset of cycles above J is non-empty and contains a connected component with only singular cycles by Corollary 9.6. Let S be the analytic set of singular cycles of \mathcal{C} . Let p denote the projection of \mathcal{C}' to $K_0 \times K_0$. This is a proper map, since we are in the Kähler setting. We thus have that the set of exceptional points is equal to

$$(10.1) \quad E = \bigcup_{\mathcal{C} \in \text{Irr } \mathcal{C}'} \{J \in K_0 \mid p^{-1}(J, J) \cap \mathcal{C} \neq \emptyset \text{ and } (p^{-1}(J, J) \cap \mathcal{C})_0 \subset S\}$$

where Irr denotes the set of irreducible components and $_0$ means that some connected component of $p^{-1}(J, J) \cap \mathcal{C}$ is included in S . We notice that, by our assumptions on V , the intersection $p^{-1}(J_0, J_0) \cap \mathcal{C}$ is connected. However,

¹⁰The surfaces \mathbb{F}_{2a} and \mathbb{F}_{-2a} are isomorphic, but not through a biholomorphism isotopic to the identity.

¹¹see the end of Section 9.1.

we cannot ensure that this is still true for any point J . Let E^c be the closure of (10.1).

We claim that E^c is an analytic subspace of K_0 . To see that, we first embed E in $K_0 \times K_0$ through the diagonal embedding of K_0 . We still call E the image. Let \mathcal{C} be a component of \mathcal{C}' and let $E_{\mathcal{C}}$ denote the \mathcal{C} -component of (10.1). Let (J_1, J_1) be a point of $K_0 \times K_0$. Let W be an open neighborhood of (J_1, J_1) in $K_0 \times K_0$. If $p^{-1}(J_1, J_1) \cap \mathcal{C}$ is empty, then $p^{-1}(W) \cap \mathcal{C}$ is empty for W small enough. So let us assume that $p^{-1}(J_1, J_1) \cap \mathcal{C}$ is not empty. We decompose $p^{-1}(J_1, J_1) \cap \mathcal{C}$ into connected components $(p^{-1}(J_1, J_1) \cap \mathcal{C})_i$ for i between 0 and k . Let A_i be open neighborhoods of $(p^{-1}(J_1, J_1) \cap \mathcal{C})_i$ in \mathcal{C} such that the union of all A_i is exactly $p^{-1}(W) \cap \mathcal{C}$. We assume that (J_1, J_1) is generic in the sense that the intersection $A_i \cap p^{-1}(J, J)$ is still connected for $(J, J) \in W$ if non empty. Notice that the restriction of p to some A_i is still a proper map. We may decompose $E_{\mathcal{C}} \cap W$ as follows.

$$\begin{aligned}
 (10.2) \quad E_{\mathcal{C}} \cap W &= \bigcup_{0 \leq i \leq k} \{(J, J) \in W \mid p^{-1}(J, J) \cap A_i \neq \emptyset \\
 &\quad \text{and } p^{-1}(J, J) \cap A_i \subset S\} \\
 &= \bigcup_{0 \leq i \leq k} \Delta \cap (p(A_i) \setminus p(A_i \setminus S))
 \end{aligned}$$

where Δ is the diagonal of $K_0 \times K_0$. Hence its closure (10.2) is an analytic set of $\Delta \cap W \cap \mathcal{C}$. But its closure is just $E_{\mathcal{C}}^c \cap W$. So we obtain a chart of analytic subspace for E^c at every generic point. Assume now that (J_1, J_1) is not generic. Then we may perform exactly the same construction but the resulting constructible set (10.2) may forget some exceptional points. Now, let a be a positive integer. The set of points (J, J) of W where $p^{-1}(J, J) \cap A_i$ has exactly a connected components is constructible, see [39], Lemma 37.28.6 in an algebraic context. So is its pull-back by p in A_i . Looking at its connected components, we may decompose A_i into a finite set of constructible sets A_{ij} on which p has connected fibers. Then we obtain the correct decomposition

$$(10.3) \quad E_{\mathcal{C}} \cap W = \bigcup_{i,j} \Delta \cap (p(A_{ij}) \setminus p(A_{ij} \setminus S))$$

making of $E \cap W$ a constructible set and of $E^c \cap W$ an analytic one.

We claim that E^c is a *strict* analytic subspace of K_0 ¹². Assume the contrary. Then there exists an irreducible component \mathcal{C} of \mathcal{C}' such that $E_{\mathcal{C}}^c$ is a union of irreducible components of Δ . For simplicity, let us assume that K_0 and thus Δ are irreducible. By Corollary 9.6, for each $J \in K_0$, there exists a connected component of $\mathcal{C} \cap p^{-1}(J, J)$ that contains only singular cycles. Now \mathcal{C} contains a Zariski open subset of graphs of biholomorphisms between fibers of the Kuranishi family. Hence, $p(\mathcal{C})$ is an analytic set of $K_0 \times K_0$ strictly containing Δ and $p(\mathcal{C})$ has dimension strictly greater than n , the dimension of K_0 . It follows that, at a generic point J of K_0 , the intersection of $p(\mathcal{C})$ with $\{J\} \times K_0$ is positive-dimensional. We may thus find a Zariski open subset of K_0 , say U , such that, for any $J \in U$, the intersection $\mathcal{C} \cap p^{-1}(\{J\} \times K_0)$ contains a

¹²As above in Section 9.1, we replace K_0 with its reduction if necessary, so strict means that E^c is not a whole irreducible component of K_0 .

Zariski open subset of graphs of biholomorphisms. In other words, for those J in U , there exists a path of biholomorphisms between X_J and some $X_{J'_t}$ with J'_t distinct from J . Hence K_0 has a non trivial foliated structure in the sense of [28]. But this implies that the dimension of $\text{Aut}^0(X_0)$ jumps at 0, that is is not constant in a neighborhood of 0 in K_0 . Since $\mathcal{K}_0 \rightarrow K_0$ is complete at every point J of K_0 , denoting its Kuranishi space K_J , then the set of exceptional points in K_J is also the full K_J . Hence the same argument tells that the dimension of the automorphism group also jumps at X_J . But it cannot jump at every point of K_0 . Contradiction. The set E^c is a strict analytic subspace of K_0 .

So we may define a strict analytic substack of $\mathcal{T}(M, V)$ as the stackification of the full subgroupoid of $\mathcal{T}_V \rightrightarrows K_0$ above $E^c \subset K_0$. Since the notion of exceptional point is an intrinsic notion, this substack is just a neighborhood of X_0 of an analytic substack of $\mathcal{T}(M, V_K)$. \square

Remark 10.5. If the intersection of an exceptional component of X_0 and \mathcal{C}_J is non-empty but contains regular cycles, then the corresponding morphisms

- i) either form a component of $\text{Aut}^1(X_J)$ which is not induced by $\text{Aut}^1(X_0)$.
- ii) or send J to points that are distinct from J and not adherent to it.

Hence, when restricting to a sufficiently small neighborhood W of X_J , this component disappears from $\mathcal{T}(M, W)$.

11. VARIATIONS ON EXCEPTIONALITY

In this Section, we explore variants of exceptional points. To avoid redundancies, we only deal with exceptionality but all can be immediatly adapted to \mathbb{Z} -exceptionality.

11.1. Relative exceptionality. The notion of exceptional points and cycles introduced in Definitions 6.1 and 8.8 is an intrinsic notion, in the sense that it depends only on the complex manifold X_0 . In this Section, we elaborate on a relative version, which depends on a family with fiber X_0 .

Definition 11.1. Let $\mathcal{X} \rightarrow B$ be a reduced M -deformation over B a reduced analytic space with fiber X_0 above $0 \in K_0$. Then X_0 is \mathcal{X} -exceptional if there exists a singular cycle γ of the cycle space \mathcal{C}_0 such that

- i) The cycle γ does not belong to the closure of a connected component of $\text{Aut}^1(X_0)$.
- ii) The cycle γ belongs to an irreducible component of $\mathcal{C}_\mathcal{X}$ whose generic member is non singular.

The cycle γ is called \mathcal{X} -exceptional.

Here $\mathcal{C}_\mathcal{X}$ denotes the the union of completely singular and regular components¹³ of the Barlet space of relative cycles of $\mathcal{X} \times \mathcal{X}$.

A direct rephrasing of Corollary 9.6 shows that a Kähler manifold X_0 is exceptional if and only if it is $\mathcal{K}_0^{\text{red}}$ -exceptional. However, in the non-Kähler case, the notion of exceptional point is more general. Notice also that X_0 is \mathcal{X} -exceptional if and only if there exists a sequence (8.1) with (x_n) and (y_n) sequences of B converging to 0 and the graphs of ϕ_n belonging to $\mathcal{C}_\mathcal{X}$

¹³in the sense of §8.2.

such that the limit cycle γ is singular and does not belong to the closure of a connected component of $\text{Aut}^1(X_0)$.

Now, the notion of \mathcal{X} -exceptionality depends strongly on \mathcal{X} . When \mathcal{X} is not the reduction of the Kuranishi family, it may have nothing to do with exceptionality. A simple example is given by a trivial family $X_0 \times B \rightarrow B$. Then X_0 is never $X_0 \times B$ -exceptional.

Consider a cartesian diagram

$$(11.1) \quad \begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & \square & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

Then

Lemma 11.2. *Let $b' \in B'$ and set $b = f(b')$. Assume that $X'_{b'}$, that is the fiber of $\mathcal{X}' \rightarrow B'$ over b' , is \mathcal{X}' -exceptional. Then X_b , the fiber of $\mathcal{X} \rightarrow B$ over b , is \mathcal{X} -exceptional.*

Proof. Just consider a \mathcal{X}' -exceptional cycle γ above b' and its direct image $f_*\gamma$ above b and observe that points i) and ii) of Definition 11.1 are preserved through direct images. \square

As an immediate consequence of Lemma 11.2, if $\mathcal{X} \rightarrow B$ is complete at 0 and X_0 Kähler, then X_0 is \mathcal{X} -exceptional if and only if X_0 is exceptional. To prove the statement, just apply Lemma 11.2 to the following two cartesian diagrams of germs of families

$$(11.2) \quad \begin{array}{ccc} (\mathcal{X}, X_0) & \longrightarrow & (\mathcal{X}_0, X_0) \\ \downarrow & \square & \downarrow \\ (B, 0) & \longrightarrow & (K_0, 0) \end{array} \quad \text{and} \quad \begin{array}{ccc} (\mathcal{X}_0, X_0) & \longrightarrow & (\mathcal{X}, X_0) \\ \downarrow & \square & \downarrow \\ (K_0, 0) & \longrightarrow & (B, 0) \end{array}$$

The left diagram, resp. right diagram, is given by completeness of the Kuranishi family, resp. of $\mathcal{X} \rightarrow B$.

As in the case of exceptional points, we have

Proposition 11.3. *Let $\mathcal{X} \rightarrow B$ be a reduced M -deformation with Kähler fibers and reduced base B . Then, the closure of \mathcal{X} -exceptional points is a strict analytic subspace of B .*

Proof. A straightforward adaptation of the proof of Theorem 10.1 shows that the closure of \mathcal{X} -exceptional points is an analytic subspace of B . We have to prove it is strict. Assume the contrary. Then at least a complete irreducible component of B contains an open and dense subset of \mathcal{X} -exceptional points. For simplicity, assume B irreducible. Let \mathcal{C} be an \mathcal{X} -exceptional component at some point $b \in B$ and let p the natural projection map from \mathcal{C} to $B \times B$. Since there are only a finite number of components in $\mathcal{C}_{\mathcal{X}}$, we may assume that \mathcal{C} is also exceptional for all the points in a neighborhood of b . The component \mathcal{C} intersects $p^{-1}(x, x)$ in a completely singular component \mathcal{C}_x for all x in this neighborhood. Arguing as in the proof of Theorem 10.1, this implies that $p(\mathcal{C})$ is an analytic subspace of $B \times B$ strictly containing the diagonal. Hence, at generic points x of B it intersects $\{x\} \times B$ in a

positive-dimensional subspace. And the cycles above a generic point of this intersection include graphs of biholomorphisms. That means that one can find a path (b_t) in B ending at x such that all X_{b_t} are biholomorphic. Thus one can find sequences of graphs of biholomorphisms between X_{b_t} and X_x that converge to a singular cycle of \mathcal{C}_x . Since a Zariski open subset of \mathcal{C}_x is formed by exceptional cycles, cf. Remark 9.10, we may assume that such a sequence converges to an exceptional cycle. Now that would mean that this exceptional cycle belongs to the closure of a component of the automorphism group of X_x . Contradiction with the definition of exceptionality. \square

Such a result is of course completely false for exceptional points. Letting $f : B \rightarrow K_0$ lands in the subspace of exceptional points, then every point of $f^*\mathcal{K}_0 \rightarrow B$ is exceptional.

11.2. Exceptional pairs. Let X_0 and X'_0 be two compact complex manifolds diffeomorphic to M . Let K_0 , resp. K'_0 , the Kuranishi space of X_0 , resp. X'_0 . As a second variation on the theme of exceptionality, we define

Definition 11.4. We say that $\{X_0, X'_0\}$ is an *exceptional pair* of $\mathcal{T}(M)$ if

- i) X_0 and X'_0 are not biholomorphic
- ii) There exists a sequence (8.1) with (x_n) , resp. (y_n) , sequence of K_0 , resp. K'_0 , converging to X_0 , resp. X'_0 , such that the graphs of ϕ_n converge to a singular cycle γ in the cycle space of $X_0 \times X'_0$.

The cycle γ is called *exceptional*.

Let us make a few comments on Definition 11.4. Firstly, since X_0 and X'_0 are not biholomorphic, there is no need to add that γ is not the limit of a sequence of biholomorphisms between X_0 and X'_0 . Secondly, as the notion of exceptional points, Definition 11.4 does not depend on $\{X_0, X'_0\}$ up to biholomorphisms smoothly isotopic to the identity of the pair; so the notion of exceptional pairs is intrinsic and this justifies that we speak of exceptional pairs of $\mathcal{T}(M)$. Thirdly, as in the case of relative exceptionality, we do not include in the notion of exceptional pairs the analogues of wandering and vanishing sequences. This is mainly because we focus on the Kähler case where such phenomena do not appear.

Example 11.5. Let $\mathcal{X} \rightarrow B$ be a *jumping family*, that is there is a point $0 \in B$ such that

- i) For all $b \in B$ different from 0, the fiber X_b is biholomorphic to a fixed complex manifold, say X_1 .
- ii) The 0-fiber X_0 is not biholomorphic to X_1 .

Then the pair $\{X_0, X_1\}$ is an exceptional pair. Just take as (x_n) the image through a map $f : B \rightarrow K_0$ such that $f^*\mathcal{K}_0$ is locally isomorphic to \mathcal{X} of a sequence of points in $B \setminus \{0\}$ converging to 0; and for (y_n) the constant sequence X_0 .

Start with (X_0, X'_0) and their Kuranishi spaces (K_0, K'_0) . As usual, we assume both Kuranishi spaces reduced, replacing them with their reduction if necessary. We consider the space of relative cycles of $\mathcal{K}_0 \times \mathcal{K}'_0 \rightarrow K_0 \times K'_0$. Let \mathcal{C}^{pair} be the union of the irreducible components of this relative cycle space that contains at least the graph of a biholomorphism between some

fiber of \mathcal{K}_0 and some fiber of \mathcal{K}'_0 which is smoothly isotopic to the identity. Let S denote the subset of singular cycles of \mathcal{C}^{pair} and let p denote the natural projection of \mathcal{C}^{pair} to $K_0 \times K'_0$.

The set of exceptional couples in $K_0 \times K'_0$ is thus equal to

$$(11.3) \quad \mathcal{P} := \{(J, J') \in K_0 \times K'_0 \mid p^{-1}(J, J') \neq \emptyset \text{ and } p^{-1}(J, J') \subset S\}$$

Analogously to Theorem 10.1, we have

Proposition 11.6. *The closure of the subset of exceptional couples is a strict analytic substack of $\mathcal{T}(M, V_K) \times \mathcal{T}(M, V_K)$.*

Proof. We deduce from (11.3) that \mathcal{P} is a constructible set in $K_0 \times K'_0$, hence its closure is an analytic set. Assume now that the closure contains a full component of $K_0 \times K'_0$. Then a full component of \mathcal{C}^{pair} consists of singular cycles, in contradiction with its definition. \square

However, notice that the projection onto K_0 , resp. K'_0 , may contain a full component of K_0 , resp. K'_0 .

Example 11.7. Let X_0 be the product of projective lines $\mathbb{P}^1 \times \mathbb{P}^1$ and let X'_0 be the second Hirzebruch surface. Then K_0 is a point since $\mathbb{P}^1 \times \mathbb{P}^1$ is rigid; and K'_0 is (the germ of) a unit disk with 0 encoding \mathbb{F}_2 and $t \neq 0$ encoding $\mathbb{P}^1 \times \mathbb{P}^1$ (this is a jumping family in the sense of Example 11.5).

The set of exceptional pairs is the pair formed by the unique point of K_0 and 0. This is a strict analytic subset of the disk but not of the point.

11.3. Non-Hausdorff points. As a consequence of what precedes, we obtain that the set of non-Hausdorff Kähler points is included in a strict analytic substack.

To be more precise, first recall that the topological Teichmüller space - that is the quotient $\mathcal{I}(M)/\text{Diff}^0(M)$ endowed with the quotient topology - is also given as the geometric quotient of the Teichmüller stack, that is, given an atlas $\mathcal{T}_0 \rightarrow \mathcal{T}(M)$ and given the associated groupoid $\mathcal{T}_1 \rightrightarrows \mathcal{T}_0$, the quotient space of \mathcal{T}_0 by the equivalence relation generated by the morphisms encoded in \mathcal{T}_1 . By extension we define

Definition 11.8. We say that X_0 and X_1 are *non-Hausdorff points* in $\mathcal{T}(M)$, or form a *non-Hausdorff pair* of $\mathcal{T}(M)$ if they correspond to non-separated points $[J_0]$ and $[J_1]$ of $\mathcal{I}(M)/\text{Diff}^0(M)$.

Using the local atlases (5.1), if a pair $\{[J_0], [J_1]\}$ of $\mathcal{I}(M)/\text{Diff}^0(M)$ is a pair of non-separated points of $\mathcal{I}(M)/\text{Diff}^0(M)$ then, setting $X_0 := (M, J_0)$, $X_1 := (M, J_1)$, and letting K_0 , resp. K_1 , be the Kuranishi space of X_0 , resp. X_1 , we may find a sequence (8.1) with (x_n) in K_0 converging to X_0 and (y_n) in K_1 converging to X_1 . In other words, two non-separated points of $\mathcal{I}(M)/\text{Diff}^0(M)$ define an exceptional pair of $\mathcal{T}(M)$.

As an immediate consequence to Proposition 11.6, we thus have

Corollary 11.9. *The set of couples of non-Hausdorff Kähler points in $\mathcal{T}(M)$ is contained in the set of exceptional couples of $\mathcal{T}(M, V_K)$, hence contained in a strict analytic substack of $\mathcal{T}(M, V_K) \times \mathcal{T}(M, V_K)$.*

As an example, $\mathbb{P}^1 \times \mathbb{P}^1$ and \mathbb{F}_2 are non-Hausdorff points of $\mathcal{T}(\mathbb{S}^2 \times \mathbb{S}^2)$, cf. Example 11.7, as well as the pair $\{X_0, X_1\}$ in a jumping family, cf. Example 11.5.

12. PATHOLOGIES OF THE TEICHMÜLLER STACK

As explained in the Introduction §1, making use of a Teichmüller stack rather than a Teichmüller space allows to pass easily from the point of view of moduli space to the point of view of families. They are two faces of the same mathematical object.

As a consequence, we want to explore in this Section the pathologies of the Teichmüller stack as pathologies of families and to relate them to pathologies of space. We notice that we already investigated a topological pathology in Section 11.3. But here we would like to search for analytic pathologies. As in §11, to avoid redundancies, we only deal with exceptionality but all can be immediatly adapted to \mathbb{Z} -exceptionality.

12.1. Analytically non-separated, ambiguous and undistinguishable points. Let us begin with some definitions.

Definition 12.1. Two distinct points X_0 and X_1 of the Teichmüller stack $\mathcal{T}(M)$ are *analytically non-separated* if there exist two reduced M -deformations $\mathcal{X}_i \rightarrow B$ of X_i (for $i = 0, 1$) above a reduced positive dimensional base B such that \mathcal{X}_0 and \mathcal{X}_1 are isomorphic above $B \setminus \{0\}$.

Taking sequences in $B \setminus \{0\}$ that converge to 0, we immediatly obtain that $\{X_0, X_1\}$ is an exceptional pair; and a non-Hausdorff pair. However Definition 12.1 is stronger. Observe that

- i) Any couple (X_0, X_1) formed by the base point, resp. the generic point, of a jumping family (cf. Examples 11.7 and 11.5) are analytically non-separated.
- ii) We can always assume that B is a disk.

We pass now to the notions of analytic ambiguity and undistinguishability. In the following definitions, given X_0 and X_1 , we set $X_i = (M, J_i)$ and let V_i be an open neighborhood of J_i in $\mathcal{T}(M)$. Finally, V_i^* denotes V_i minus the $\text{Diff}^0(M)$ -orbit of J_i .

Definition 12.2. Two distinct points X_0 and X_1 of the Teichmüller stack $\mathcal{T}(M)$ are *formally ambiguous* if, for any choice of a small enough neighborhood V_0 , resp. V_1 , there exists a neighborhood V_1 , resp. V_0 , such that $\mathcal{T}(M, V_0^*)$ is equivalent to $\mathcal{T}(M, V_1^*)$, that is there exists a fully faithful and essentially surjective stack morphism from $\mathcal{T}(M, V_0^*)$ to $\mathcal{T}(M, V_1^*)$. They are *analytically ambiguous* if they are formally ambiguous and the corresponding equivalence between $\mathcal{T}(M, V_0^*)$ and $\mathcal{T}(M, V_1^*)$ sends a reduced V_0^* -family to an *isomorphic* reduced V_1^* -family.

In other words, X_0 and X_1 are formally ambiguous iff they have arbitrary small isomorphic punctured neighborhoods in $\mathcal{T}(M)$. To understand the difference with analytically ambiguous, notice that two compact surfaces of genus $g > 1$ are always formally ambiguous, but never analytically ambiguous. Indeed, the Teichmüller stack of compact surfaces of fixed genus $g > 1$ is a bounded domain in \mathbb{C}^{3g-3} ; in particular, it is a manifold, every automorphism group $\text{Aut}^1(X_0)$ is the identity and two distinct points are not biholomorphic through a biholomorphism smoothly isotopic to the identity. Thus, a punctured neighborhood of some X in this Teichmüller

stack is isomorphic to a punctured neighborhood of 0 in \mathbb{C}^{3g-3} proving the first point. However, given X_0 and X_1 distinct, and disjoint neighborhoods V_0 and V_1 with $\mathcal{T}(M, V_0^*)$ isomorphic to $\mathcal{T}(M, V_1^*)$, every complex structure encoded in V_0^* is distinct from any complex structure encoded in V_1^* , hence the previous stack isomorphism cannot send a family to an isomorphic family. Analogously, we have

Definition 12.3. Two distinct points X_0 and X_1 of the Teichmüller stack $\mathcal{T}(M)$ are *formally undistinguishable* if, for any choice of a small enough neighborhood V_0 , resp. V_1 , there exists a neighborhood V_1 , resp. V_0 , such that $\mathcal{T}(M, V_0)$ is equivalent to $\mathcal{T}(M, V_1)$. We ask these equivalences to send X_0 to X_1 .

They are *analytically undistinguishable* if they are formally undistinguishable and the corresponding equivalence between $\mathcal{T}(M, V_0)$ and $\mathcal{T}(M, V_1)$ sends a reduced V_0^* -deformation of X_0 to an *isomorphic* reduced V_1^* -deformation of X_1 .

That is, X_0 and X_1 have arbitrary small isomorphic neighborhoods in $\mathcal{T}(M)$ and there is no way to distinguish them by looking locally at the Teichmüller stack if they satisfy Definition 12.3. Any two compact surfaces of genus $g > 1$ are indeed formally undistinguishable. Besides, there is a simple way to produce analytically undistinguishable points. Just take a Teichmüller stack with a non-trivial action of the mapping class group on the geometric quotient that fixes a connected component. For example, given $\tau \in \mathbb{H}$ and

$$(12.1) \quad \tau' = \frac{p + q\tau}{r + s\tau}, \quad \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

then the two complex tori of moduli τ and τ' are analytically undistinguishable in the Teichmüller stack of $\mathbb{S}^1 \times \mathbb{S}^1$. The same occurs for \mathbb{F}_{2a} and \mathbb{F}_{-2a} in the Teichmüller stack of $\mathbb{S}^2 \times \mathbb{S}^2$. Indeed the flip $(x, y) \mapsto (y, x)$ of $\mathbb{S}^2 \times \mathbb{S}^2$ defines an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ that exchanges \mathbb{F}_{2a} and \mathbb{F}_{-2a} .

Formally, resp. analytically, undistinguishable points are of course formally, resp. analytically ambiguous; but they also have same automorphism group $\mathrm{Aut}^1(X_0)$, which may not be the case of ambiguous points. In particular, the following Lemma gives examples of analytically ambiguous but distinguishable points.

Lemma 12.4. *Let X_0 and X_1 be rigid complex manifolds, i.e. any small enough deformation of X_0 , resp. X_1 , is $\mathrm{Diff}^0(M)$ -biholomorphic¹⁴ to X_0 , resp. X_1 . Then*

- i) X_0 and X_1 are analytically separated.
- ii) X_0 and X_1 are analytically ambiguous.
- iii) X_0 and X_1 are analytically undistinguishable if and only if they have same Aut^1 group.

Proof. If X_0 and X_1 are rigid, given any reduced M -deformations $\mathcal{X}_i \rightarrow B$ of X_i (for $i = 0, 1$), there exists an open neighborhood V of 0 in B above which the restriction of \mathcal{X}_i is isomorphic to $X_i \times V$. Thus both families cannot be

¹⁴that is biholomorphic through a biholomorphism smoothly isotopic to the identity.

isomorphic above $B \setminus \{0\}$ and X_0 and X_1 are analytically separated. This proves i).

Two rigid manifolds satisfy Definition 12.2, since the corresponding neighborhoods V_i^* are empty. This proves ii).

To satisfy 12.3, the existence of an isomorphism between small neighborhoods of X_0 and X_1 is needed. It must send X_0 to X_1 thus they must have same isotropy groups, that is same Aut^1 groups. Conversely if they have same Aut^1 groups, they have isomorphic neighborhoods, since a small enough neighborhood of a rigid manifold is only composed by it. And there is no V_0^* -deformation so the last part of the definition is trivially fulfilled. This proves iii). \square

Undistinguishable points enjoy several interesting properties. We have

Lemma 12.5. *If X_0 and X_1 are formally undistinguishable, then their Kuranishi spaces K_0 and K_1 are isomorphic as germs of analytic spaces.*

Proof. Consider the equivalence F between $\mathcal{T}(M, V_0)$ and $\mathcal{T}(M, V_1)$. It sends every cartesian diagram

$$(12.2) \quad \begin{array}{ccc} (\mathcal{X}, X_0) & \longrightarrow & (\mathcal{X}_0, X_0) \\ \downarrow & \square & \downarrow \\ (B, 0) & \longrightarrow & (K_0, 0) \end{array}$$

to the cartesian diagram

$$(12.3) \quad \begin{array}{ccc} (F(\mathcal{X}), X_1) & \longrightarrow & (F(\mathcal{X}_0), X_1) \\ \downarrow & \square & \downarrow \\ (B, 0) & \longrightarrow & (K_0, 0) \end{array}$$

Since F is essentially surjective, every family of $\mathcal{T}(M, V_1)$ is isomorphic to a family in the image of F and (12.3) expresses that the image of the Kuranishi family $\mathcal{X}_0 \rightarrow K_0$ of X_0 is complete for X_1 . This implies that the dimension of the Zariski tangent space of K_0 at 0 is greater than the dimension of that of K_1 .

Similarly, still because F is essentially surjective, there exists a reduced deformation \mathcal{X}_1 of X_1 over K_1 such that $F(\mathcal{X}_1)$ is isomorphic to \mathcal{X}_1 hence semi-universal for K_1 . Given any reduced deformation $\mathcal{X} \rightarrow B$ of X_0 then any cartesian diagram

$$(12.4) \quad \begin{array}{ccc} (F(\mathcal{X}), X_1) & \longrightarrow & (F(\mathcal{X}_1), X_1) \\ \downarrow & \square & \downarrow \\ (B, 0) & \longrightarrow & (K_1, 0) \end{array}$$

is the image through F of a family morphism $\mathcal{X} \rightarrow \mathcal{X}_1$ over $B \rightarrow K_1$ because F is fully faithful. But this implies that $\mathcal{X}_1 \rightarrow K_1$ is complete for X_0 . In particular, the dimension of the Zariski tangent space of K_1 at 0 is greater than the dimension of that of K_0 . So they are equal and $\mathcal{X}_1 \rightarrow K_1$ is indeed semi-universal for X_0 . As a consequence K_0 and K_1 are isomorphic as germs of analytic spaces at the base point. \square

We also have

Lemma 12.6. *If X_0 and X_1 are analytically undistinguishable, not rigid and Kähler, they are analytically non-separated.*

Proof. Since X_0 and X_1 are not rigid, we can find a non trivial deformation of X_0 over the disk, say $\mathcal{X} \rightarrow \mathbb{D}$. Since X_0 is Kähler, we may assume that 0 is the only point of \mathbb{D} encoding X_0 up to biholomorphisms smoothly isotopic to the identity, cf. the proof of Corollary 9.3. So, in the associated V_0^* -family $\mathcal{X}^* \rightarrow \mathbb{D}^*$, the base is in fact $\mathbb{D} \setminus \{0\}$. By Definition 12.3, to $\mathcal{X} \rightarrow \mathbb{D}$ we may associate a deformation of X_1 over the disk such that the two families are isomorphic outside 0. This is exactly saying that X_0 and X_1 are analytically non-separated. \square

Remark 12.7. Notice however that the line of arguments of the proof of Lemma 12.6 does not work if we assume X_0 and X_1 analytically ambiguous and not undistinguishable. Indeed, under this milder assumption, we only may associate to $\mathcal{X} \rightarrow \mathbb{D}$ a V_1^* -family over the punctured disk and there is no reason it extends to a deformation of X_1 over the disk. The same comment applies to Lemma 12.5.

We now show that all the previous analytic pathologies on families concern only a strict analytic substack of $\mathcal{T}(M, V_K)$.

Proposition 12.8. *Let X_0 and X_1 be distinct non-rigid, Kähler points of the Teichmüller stack. Assume that*

- i) X_0 and X_1 are analytically undistinguishable,
- or ii) X_0 and X_1 are analytically ambiguous,
- or iii) X_0 and X_1 are analytically non-separated..

Then X_0 and X_1 are non-Hausdorff points and form an exceptional pair.

In particular, the set of analytically distinguishable, or ambiguous, or non-separated Kähler pairs is included in a strict analytic substack of $\mathcal{T}(M, V_K) \times \mathcal{T}(M, V_K)$.

Proof. We already observed after Definition 12.1 that analytically non-separated points are non-Hausdorff. Thus this is also true for analytically undistinguishable points thanks to Lemma 12.6. We are left with the case of analytically ambiguous points. For $i = 0, 1$, consider the Kuranishi space K_i of X_i . Its reduction is positive-dimensional by hypothesis, and by Corollary 9.3, we may assume, taking a smaller Kuranishi space if necessary, that K_i^* is equal to K_i minus the base point. By Definition 12.2, both Kuranishi families are isomorphic when restricted to K_i^* . Thus, taking corresponding sequences in $K_i \setminus \{0\}$ that converge to 0, we immediately obtain that $\{X_0, X_1\}$ is an exceptional pair; and a non-Hausdorff pair.

Finally, making use of Proposition 11.6 proves the last sentence. \square

We sum up some of the properties in the following table.

Analytically	Teichmüller	Families	Examples
non-separated	isomorphic punctured slices	existence of isomorphic local punctured deformations	jumping families
ambiguous	isomorphic punctured neighborhoods	duality between local punctured deformations	rigid manifolds
undistinguishable	isomorphic neighborhoods	duality between local deformations	Hirzebruch surfaces \mathbb{F}_a and \mathbb{F}_{-a}

TABLE 1. Pathologies for X_0 and X_1 distinct and Kähler

12.2. The local isomorphism property. Following [28], we say that a compact complex manifold X_0 has the local isomorphism property if any two pointwise isomorphic deformations of X_0 are locally isomorphic at the base point 0.

In [28], we claim that X_0 has the local isomorphism property for deformations over a reduced base if and only if the function (3.7) is constant on the fibers X_t of the Kuranishi family $\mathcal{X}_0 \rightarrow K_0$.

This is however not true and counterexamples exist on K3 surfaces as explained in [22]. This comes from the fact that the numbers of chambers in the positive cone of a K3 surface may jump above at the base point of a deformation, see the main Lemma of [7] or [3]. Start with a K3 surface X_0 whose positive cone is subdivided in several chambers. Choose a (-2) -curve C on X_0 such that the Hodge isometry of $H^2(X_0, \mathbb{C})$ given by reflection around the hyperplane normal to the class of C , say σ , is non-effective, that is does not respect the Kähler chamber. Consider a deformation $\mathcal{X} \rightarrow B$ of X_0 whose generic fiber does not admit such a subdivision of its positive cone. Apply the Hodge isometry σ fiberwise to the punctured family, giving another family isomorphic to the first one above the punctured base B^* . Then it can be checked that this new family extends as a deformation $\mathcal{X}' \rightarrow B$ of X_0 . Now \mathcal{X} and \mathcal{X}' are not locally isomorphic at 0, since a local isomorphism would induce an automorphism on the central fiber X_0 acting on $H^2(X_0, \mathbb{C})$ as the non-effective Hodge isometry σ we start with.

Notice that in such counterexamples, we may assume the families to be reduced families (the construction process is local) but the two families are not pointwise isomorphic as reduced families. Indeed the central fiber of \mathcal{X} and \mathcal{X}' are common, hence identified through the identity, whereas the other fibers are identified through a biholomorphism acting as the reflection σ on the local system of second cohomology groups.

Definition 12.9. We say that a compact complex manifold X_0 has the *local $\text{Diff}^0(M)$ -isomorphism property* if any two pointwise $\text{Diff}^0(M)$ -isomorphic

reduced deformations of X_0 are locally $\text{Diff}^0(M)$ -isomorphic at the base point 0.

We do not know of an example of a compact complex manifold X_0 not having the $\text{Diff}^0(M)$ -local isomorphism property and such that (3.7) is constant. However, we shall prove

Theorem 12.10. *Let X_0 be Kähler. Assume that X_0 has not the $\text{Diff}^0(M)$ -local isomorphism property over reduced bases. Then, either the h^0 -function (3.7) is not locally constant, or X_0 is an exceptional point.*

and its immediate corollary

Corollary 12.11. *The subset of points of $\mathcal{T}(M, V_K)$ not having the $\text{Diff}^0(M)$ -local isomorphism property over reduced bases is included in a strict analytic substack of $\mathcal{T}(M, V_K)$.*

Proof of Corollary 12.11. Apply Theorem 12.10. The conclusion follows then from the fact that the h^0 -function is upper semi-continuous for the Zariski topology on K_0 . \square

Proof of Theorem 12.10. Let X_0 be Kähler and not having the $\text{Diff}^0(M)$ -local isomorphism property over reduced bases. By [28, p.513–514], X_0 has not the $\text{Diff}^0(M)$ -local isomorphism property over disks. So let $\mathcal{X} \rightarrow \mathbb{D}$ and $\mathcal{X}' \rightarrow \mathbb{D}$ be two pointwise $\text{Diff}^0(M)$ -isomorphic but not locally $\text{Diff}^0(M)$ -isomorphic reduced deformations of X_0 . Let $f : \mathbb{D} \rightarrow K_0$, resp. $g : \mathbb{D} \rightarrow K_0$ be holomorphic mappings such that $f^*\mathcal{X}_0$ is isomorphic to \mathcal{X} , resp. $g^*\mathcal{X}_0$ is isomorphic to \mathcal{X}' . As usual, $\mathcal{X}_0 \rightarrow K_0$ is the Kuranishi family of X_0 and the existence of f and g comes from its semi-universality property, cf. Section 3.1. Fix a sequence (t_n) of \mathbb{D}^* converging to zero. Set

$$(12.5) \quad \forall n \in \mathbb{N}, \quad x_n = f(t_n) \quad \text{and} \quad y_n = g(t_n)$$

Thus (x_n) and (y_n) are sequences of points in K_0 converging to 0. Since \mathcal{X} and \mathcal{X}' are pointwise $\text{Diff}^0(M)$ -isomorphic, we may choose a sequence (ϕ_n) in $\text{Diff}^0(M)$ satisfying (8.1). Up to passing to a subsequence, we may assume that the graphs of the ϕ_n all belong to the same irreducible component of \mathcal{C} and converge in this cycle space to a γ_0 .

Assume now that (3.7) is constant and that X_0 is not exceptional. Then γ_0 is singular, otherwise its extension would induce a local isomorphism between the two families \mathcal{X} and \mathcal{X}' . Indeed, assume that γ_0 is the graph of an automorphism H of X_0 , and still denote by H its extension as an isomorphism of the Kuranishi family:

$$(12.6) \quad \begin{array}{ccc} (\mathcal{X}_0, X_0) & \xrightarrow{H} & (\mathcal{X}_0, X_0) \\ \downarrow & & \downarrow \\ (K_0, 0) & \xrightarrow{h} & (K_0, 0) \end{array}$$

then $H^{-1} \circ \phi_n$ tends to zero uniformly in every chart and in every L_k^2 -norm. In particular, since (3.7) is constant, this implies that $H^{-1} \circ \phi_n$ fixes x_n for n big enough, see [26]. Hence h induces a local isomorphism of K_0 making

the following diagram commutative

$$(12.7) \quad \begin{array}{ccc} (\mathbb{D}, 0) & \xrightarrow{f} & (K_0, 0) \\ & \searrow g & \downarrow h \\ & & (K_0, 0) \end{array}$$

because it satisfies this diagram on the convergent sequence (t_n) of the disk.

Since X_0 is not exceptional, the singular cycle γ_0 belongs to an $\text{Aut}^1(X_0)$ -component of \mathcal{C}_0 , hence is the limit of graphs of automorphisms (H_n) of X_0 . We need the following lemma.

Lemma 12.12. *Assume (3.7) is constant and X_0 is not exceptional. Then, there exists an open neighborhood U of 0 in K_0 such that any element of $\text{Aut}^1(X_0)$ admits an extension as isomorphism of the Kuranishi family above U .*

Proof of Lemma 12.12. Let U be an open neighborhood U of 0 in K_0 such that every holomorphic vector field on X_0 extends as a holomorphic vector field on the Kuranishi family above U . This is possible because the vector space of holomorphic vector fields on X_0 is finite-dimensional. Since (3.7) is constant, these extensions are tangent to the fibers of the family. Hence, every element of $\text{Aut}^0(X_0)$, being a composition of flows of such vertical vector fields, is defined above U . Since X_0 is Kähler, $\text{Aut}^1(X_0)$ has a finite number of connected components, say p . Fix one element g_i in each component. Reducing U if necessary, we may assume that the g_i admit a holomorphic extension above U . Now any element of $\text{Aut}^1(X_0)$ is a composition $g_i \circ f$ with $f \in \text{Aut}^0(X_0)$, hence admits an extension over U . \square

Then, reducing K_0 if necessary, we have a commutative diagram (12.6) for every (H_n, h_n) , and all of them are defined above the whole K_0 . Arguing as above, we see that, for n big enough, the sequence (h_n) stabilizes to some mapping h which satisfies (12.7). Hence \mathcal{X} and \mathcal{X}' are locally $\text{Diff}^0(M)$ -isomorphic at 0. Contradiction that proves the Theorem. \square

12.3. Double, split and ubiquitous points. In Section 12.1, we deal with distinct points of the Teichmüller stack admitting isomorphic neighborhoods (or punctured neighborhoods or slices in the milder versions). Here distinct points cannot be recognized by looking at their neighborhoods.

In this section, we deal with a phenomenon which is in a sense opposite: a single point of the Teichmüller stack admitting non-isomorphic neighborhoods or slices. Here a single point does not determine its neighborhood.

Analogously to Definition 12.1, we set

Definition 12.13. A point X_0 of the Teichmüller stack $\mathcal{T}(M)$ is a *double point* if there exist two reduced M -deformations $\mathcal{X}_i \rightarrow B$ of X_0 (for $i = 0, 1$) above a reduced positive dimensional base B such that \mathcal{X}_0 and \mathcal{X}_1 are isomorphic above $B \setminus \{0\}$ but non-isomorphic over B .

We note that a double point X_0 has not the local $\text{Diff}^0(M)$ -isomorphism property. Analogously to Definitions 12.2, we set

Definition 12.14. A point X_0 of the Teichmüller stack $\mathcal{T}(M)$ is *split* if, for any choice of a small enough neighborhood V_0 of X_0 , there exists a neighborhood V_1 of X_0 and an equivalence between $\mathcal{T}(M, V_0^*)$ and $\mathcal{T}(M, V_1^*)$ such that

- i) It sends every reduced V_0^* -deformation of X_0 to an *isomorphic* reduced V_1^* -deformation of X_0 .
- ii) There exist *non-isomorphic* reduced V_i -deformation $\mathcal{X}_i \rightarrow B$ (for $i = 0, 1$) whose restrictions over B^* are image through the equivalence, hence *isomorphic*.

and

Definition 12.15. A point X_0 of the Teichmüller stack $\mathcal{T}(M)$ is *ubiquitous* if, for any choice of a small enough neighborhood V_0 of X_0 , there exists a neighborhood V_1 of X_0 and an equivalence between $\mathcal{T}(M, V_0)$ and $\mathcal{T}(M, V_1)$ such that

- i) It sends every reduced V_0^* -deformation of X_0 to an *isomorphic* reduced V_1^* -deformation of X_0 .
- ii) There exists a V_0 -deformation which is *not isomorphic* to its image.

We note that Definition 12.14, resp. 12.15, is similar to the notion of *analytic* ambiguity, resp. *analytic* undistinguishability. However, formally split, resp. formally ubiquitous, is obviously automatically satisfied so we drop the adjective analytic in Definition 12.15.

Example 12.16. We claim that the Hirzebruch surface \mathbb{F}_2 is split but not ubiquitous. Recall that its Kuranishi space is a jumping family with generic point $\mathbb{P}^1 \times \mathbb{P}^1$. Hence, for V_0 small enough, the objects of $\mathcal{T}(M, V_0)$ are reduced families with every fiber isomorphic either to \mathbb{F}_2 or to $\mathbb{P}^1 \times \mathbb{P}^1$; and the objects of $\mathcal{T}(M, V_0^*)$ are reduced families with every fiber isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, that is locally trivial $\mathbb{P}^1 \times \mathbb{P}^1$ -bundles by Fischer-Grauert Theorem.

Consider the stack morphism induced by the map $z \mapsto z^2$ of the unit disk. It sends the pull-back of the Kuranishi family by some mapping $f : B \rightarrow \mathbb{D}$ to the pull-back of the Kuranishi family by $z \in B \mapsto (f(z)^2) \in \mathbb{D}$. It is an equivalence on V_0^* , that is on locally trivial $\mathbb{P}^1 \times \mathbb{P}^1$ -bundles so \mathbb{F}_2 is split. Nevertheless, it is not essentially surjective on V_0 since it maps reduced deformations of \mathbb{F}_2 to reduced deformations of \mathbb{F}_2 with Kodaira-Spencer map zero at the base point. Thus, no family in the image is isomorphic to the Kuranishi family and \mathbb{F}_2 is not ubiquitous.

Once again, all these pathologies only occur on a strict analytic substack of $\mathcal{T}(M, V_K)$.

Theorem 12.17. *Let X_0 be a point of $\mathcal{T}(M)$. Then,*

- i) *We have*

$$(12.8) \quad \begin{aligned} X_0 \text{ ubiquitous} &\implies X_0 \text{ split} \implies X_0 \text{ double} \\ &\implies X_0 \text{ is exceptional or (3.7) is not constant.} \end{aligned}$$

- ii) *In particular, the subset of ubiquitous, resp. split, resp. double points of $\mathcal{T}(M, V_K)$ is included in a strict analytic substack of $\mathcal{T}(M, V_K)$.*

Proof. The first two implications are obvious from the definitions. Then we already noticed that a double point has not the local $\text{Diff}^0(M)$ -isomorphism property. Hence we may apply Theorem 12.10 and Corollary 12.11. \square

We sum up some of the properties in the following table, to be compared with table 12.1.

	Teichmüller	Families	Examples
double	isomorphic punctured slices	existence of isomorphic local punctured deformations	Base point of a jumping family
split	isomorphic punctured neighborhoods	duality between local punctured deformations	Hirzebruch surface \mathbb{F}_2
ubiquitous	isomorphic neighborhoods	duality between local deformations	

TABLE 2. Pathologies for X_0 Kähler

We do not know of an example of ubiquitous point.

13. CARTOGRAPHY OF THE TEICHMÜLLER STACK OF KÄHLER STRUCTURES

In this final section, we gather all the previous results to describe the geography of the Teichmüller stack, that is to give a geometric picture of the different strata of points (normal, exceptional, ...) in the Teichmüller stack. We first deal with the set of Kähler points for which this cartography is much more precise. Then we deal with the general case. At the end, we discuss the notion of holonomy points and their repartition and add a few remarks on the geography of bad points in $\mathcal{T}(M)$ vs. in GIT quotients.

13.1. Kähler points.

Theorem 13.1 (Structure Theorem for Kähler points). *Let M be a connected, compact, oriented C^∞ manifold admitting complex Kähler structures. Then the Teichmüller stack of Kähler points $\mathcal{T}(M, V_K)$ admits the following structure:*

- i) *Jumping points form a strict analytic substack \mathcal{J} . At a generic jumping point X_0 , we have that $\mathcal{T}(M, V_K)$ is locally isomorphic to the Kuranishi stack, hence locally homeomorphic to the quotient of the Kuranishi space by an equivalence relation induced by $\text{Aut}^1(X_0)$. It is however far from being a manifold or an orbifold: the equivalence classes are analytic submanifolds of K_0 but that of X_0 is the base point, whereas others are positive dimensional, hence it is not locally Hausdorff.*

- ii) *If non-empty - see Conjecture 6.2, I -, the closure of exceptional points form another strict analytic substack \mathcal{E} . At an exceptional point, we have that the Kuranishi stack only admits a finite étale projection onto $\mathcal{T}(M, V_K)$. At a generic exceptional point, $\mathcal{T}(M, V_K)$ is locally homeomorphic to a finite quotient of its Kuranishi space, but is not an orbifold.*
- iii) *These two analytic substacks may intersect. Generic points referred to in points i) and ii) are points which are not in the intersection $\mathcal{J} \cap \mathcal{E}$. Points in the intersection combine the two pathologies: $\mathcal{T}(M, V_K)$ is neither locally Hausdorff nor isomorphic to the Kuranishi stack.*
- iv) *The complementary Zariski open substack \mathcal{O} contains the open substack of normal Kähler points $\mathcal{T}(M, \mathcal{N}_K)$. At a normal point, we have that $\mathcal{T}(M, \mathcal{N}_K)$ is locally isomorphic to the Kuranishi stack and locally homeomorphic to an orbifold. Moreover, there exists a global analytic morphism from $\mathcal{T}(M, \mathcal{N}_K)$ to a finite étale analytic stack with same topological quotient.*
- v) *The left points of \mathcal{O} are points whose Kuranishi space is not reduced and admitting an automorphism of $\text{Aut}^0(X_0)$ with no extension as a local isomorphism of the Kuranishi family inducing the identity on the base.*

The previous cartography also holds for the \mathbb{Z} -Teichmüller stack of Kähler points $\mathcal{T}^{\mathbb{Z}}(M, V_K)$ with the obvious changes in the statements.

Some additional remarks complete the statement of Theorem 13.1. They also hold in the case of the \mathbb{Z} -Teichmüller stack of Kähler points $\mathcal{T}^{\mathbb{Z}}(M, V_K)$ with the obvious changes in the statements.

- a) Recall that the analytic substacks of jumping points, resp. closure of exceptional points, are given locally by analytic subsets of Kuranishi spaces that glue when identifying two distinct Kuranishi spaces via a local isomorphism. Hence these abstract statements on stacks can be seen locally as classical statements about the geometry of Kuranishi spaces. However, this point of view, though more concrete and geometric, gets rid of the global behaviour, which is the crux of [29] and of this paper.
- b) The construction and the nature of a local moduli space in the classical sense (i.e. a set with some structure such that every isomorphism class of complex structures on M close to X_0 is encoded in a unique point of it) can be deduced from Theorem 13.1. It is always a quotient of the Kuranishi space. If X_0 is normal Kähler, it is a finite quotient fixing the base point, hence a complex orbifold; indeed it is the quotient of K_0 by $\text{Map}^1(X_0)$. If X_0 is exceptional Kähler and not jumping, or in the closure of exceptional points but not jumping it is also a finite quotient of K_0 but not given by a proper group action so it is not a complex orbifold. If it is jumping non exceptional point, it is given as the leaf space of the foliation of K_0 described in [28], see point e). Finally if it is jumping and in the closure of exceptional points, it is a finite quotient of the leaf space of this foliation.
- c) Points where $\mathcal{T}(M, V_K)$ is locally isomorphic to the Kuranishi stack are points where the action of $\text{Diff}^0(M)$ onto V_K is proper. The neighborhood of such a point only depends on two data, both encoding in X_0 : the Kuranishi space K_0 and the extension of $\text{Aut}^1(X_0)$ to the Kuranishi family.

- d) Recall that, by Theorem 12.17, ubiquitous, split and double points are jumping or exceptional points. Hence, the worst pathologies of the Teichmüller stack occur in these two substacks.
- e) The foliation of K_0 of [28] is given as follows. First decompose K_0 into strata

$$(K_0)_a := \{J \in K_0 \mid h^0(J) \leq a\}$$

Each $(K_0)_a$ is a Zariski open set of K_0 and the differences $(K_0)_{a+1} \setminus (K_0)_a$ are analytic subsets of K_0 . Then each stratum admits a holomorphic regular foliation¹⁵ whose dimension is given on $(K_0)_a$ by the difference between a and the minimal value of h^0 on K_0 .

Proof. Theorem 13.1 gathers results proved in the previous sections. Point i) is a mixing of the well known upper semi continuity property of h^0 for the Zariski topology, of the foliated structure of the Kuranishi space proved in [28] and of Corollary 9.3. Point ii) is a rephrasing of Corollary 9.9 and Theorem 9.12 and point iii) follows from the two previous ones. Point iv) is an adaptation of Theorem 7.16, taking into the finiteness properties in the Kähler setting. Point v) is a rephrasing of the gap between normal and non-exceptional and non-jumping points when K_0 is not reduced. \square

Example 13.2. Take $M = \mathbb{S}^2 \times \mathbb{S}^2$ corresponding to the complex structures of Hirzebruch surfaces \mathbb{F}_{2a} for $a \in \mathbb{Z}$, cf. Remark 10.4. We note that $\mathcal{T}(M)$ is not an analytic stack but an inductive limit of analytic stacks, see [28]. Now, all points are Kähler. There is no exceptional points, but a normal point, \mathbb{F}_0 , that is $\mathbb{P}^1 \times \mathbb{P}^1$, and all others points are jumping points. The fact that normal points fill a Zariski open substack can be seen by looking at the Kuranishi space of \mathbb{F}_{2a} which always contains a Zariski open subset of points corresponding to $\mathbb{P}^1 \times \mathbb{P}^1$.

13.2. The general case.

Theorem 13.3 (Structure Theorem – the general case). *Let M be a connected, compact, oriented C^∞ manifold admitting complex structures. Then the Teichmüller stack $\mathcal{T}(M)$ admits the following structure:*

- i) *Jumping points form a strict analytic substack \mathcal{J} . At a generic jumping point X_0 , we have that $\mathcal{T}(M)$ is locally isomorphic to the Kuranishi stack, hence locally homeomorphic to the quotient of the Kuranishi space by an equivalence relation induced by $\text{Aut}^1(X_0)$. It is however far from being a manifold or an orbifold: the equivalence classes are analytic submanifolds of K_0 but that of X_0 is a the base point or at worst a sequence of points accumulating onto it, whereas others are positive dimensional, hence it is not locally Hausdorff.*
- ii) *The closure of exceptional points form a substack \mathcal{E} of special importance. At an exceptional point, we have that the Kuranishi stack only admits an étale projection onto $\mathcal{T}(M, V_K)$. At a generic exceptional point, $\mathcal{T}(M, V_K)$ is locally homeomorphic to an at most discrete quotient of its Kuranishi space, but is not an orbifold. Exceptional points*

¹⁵ The leaves are complex manifolds but the transversals may be singular, see the discussion in [28].

divides into points with an exceptional cycle, vanishing point and wandering points.

- iii) *These two substacks may intersect. Generic points referred to in points i) and ii) are points which are not in the intersection $\mathcal{J} \cap \mathcal{E}$. Points in the intersection combine the two pathologies: $\mathcal{T}(M, V_K)$ is neither locally Hausdorff nor isomorphic to the Kuranishi stack.*
- iv) *The complementary open substack \mathcal{O} contains the open substack of normal Kähler points $\mathcal{T}(M, \mathcal{N}_K)$. At a normal point, we have that $\mathcal{T}(M)$ is locally isomorphic to the Kuranishi stack and locally homeomorphic to the quotient of K_0 by the action of the discrete group $\text{Map}^1(X_0)$ that fixes the base point. Moreover, there exists a global analytic morphism from $\mathcal{T}(M, \mathcal{N}_K)$ to a étale analytic stack with same topological quotient.*
- v) *The left points of \mathcal{O} are points whose Kuranishi space is not reduced and admitting an automorphism of $\text{Aut}^0(X_0)$ with no extension as a local isomorphism of the Kuranishi family inducing the identity on the base.*

The previous cartography also holds for the \mathbb{Z} -Teichmüller stack $\mathcal{T}^{\mathbb{Z}}(M)$ with the obvious changes in the statements.

We state Theorem 13.3 in parallel to Theorem 13.1. It is important to notice that points i) and iii) are identical in both Theorems whereas the statements in the other two points are much weaker in the general case. Indeed,

- a) In point ii), the closure of exceptional points is just a substack, a priori not an analytic substack. In other words, it is given locally by (closed) subsets of Kuranishi spaces that glue when identifying two distinct Kuranishi spaces via a local isomorphism, but not by analytic subsets. Conjecture 6.2, II asserts that nothing more precise can be said. Also exceptional points are of three types, following Proposition 8.11, compare with Corollary 9.9. We do not know what can be said about the geography of each type of exceptional point.
- b) Similarly, in point iv), the Zariski open substack of Theorem 13.1 is simply an open substack in Theorem 13.3. And the finite quotient is replaced with a discrete one.
- c) Here again, the construction and the nature of a local moduli space in the classical sense can be deduced from Theorem 13.3 and this gives a good idea of what we lost in the statement of the general case. The local moduli space is always a quotient of the Kuranishi space. If X_0 is normal, it is an at most discrete quotient by $\text{Map}^1(X_0)$ fixing the base point. So it is not always a complex orbifold. If X_0 is exceptional and not jumping, or in the closure of exceptional points but not jumping it is also a discrete quotient of K_0 but not given by a proper group action. If it is jumping non exceptional point, it is given as the leaf space of the foliation of K_0 described in [28]. Finally if it is jumping and in the closure of exceptional points, it is a discrete quotient of the leaf space of this foliation.
- d) As in Theorem 13.1, points where $\mathcal{T}(M)$ is locally isomorphic to the Kuranishi stack are points where the action of $\text{Diff}^0(M)$ onto V_K is proper.

The neighborhood of such a point only depends on two data, both encoding in X_0 : the Kuranishi space K_0 and the extension of $\text{Aut}^1(X_0)$ to the Kuranishi family.

- e) Recall that, by Theorem 12.17, ubiquitous, split and double points are jumping or exceptional points. Hence, the worst pathologies of the Teichmüller stack occur in these two substacks.

Proof. This is completely similar to the proof of Theorem 13.1. Point i) is a mixing of the well known upper semi continuity property of h^0 for the Zariski topology, of the foliated structure of the Kuranishi space proved in [28] and of Corollary 9.5. Point ii) is a rephrasing of Proposition 8.11 and Theorem 5.3 and point iii) follows from the two previous ones. Point iv) is essentially Theorem 7.16 and point v) just makes explicit the gap between normal and non-exceptional and non-jumping points when K_0 is not reduced. \square

Example 13.4. We go back to Hopf surfaces. We already saw in Example 7.20 that the function h^0 varies from 2 to 4 and that (a connected component of) the normal Teichmüller stack identifies with the bounded domain $\mathbb{D}^* \times \mathbb{D}$ of \mathbb{C}^2 . For the Teichmüller stack, we make use of the results of C. Fromenteau [15].

Let \mathbb{M} be the product $\text{GL}_2^c(\mathbb{C}) \times \mathbb{C}$, where the superscript c stands for contracting, that is $\text{GL}_2^c(\mathbb{C})$ only contains invertible matrices with eigenvalues of modulus strictly less than one. Denoting by λ_1 and λ_2 the eigenvalues of a matrix, we define, for each $n \in \mathbb{N}^*$, the n -resonances detection function

$$(13.1) \quad (M, t) \in \mathbb{M} \longmapsto R_n(M) := (\lambda_1^n - \lambda_2)(\lambda_2^n - \lambda_1) \in \mathbb{C}$$

Observe that R_n is holomorphic as a symmetric function of eigenvalues. The zero set of R_n is exactly the analytic subset of matrices with a resonance of order n .

Let S_n be the open subset of \mathbb{M} consisting of couples (M, t) with M non-resonant or resonant of order n . By definition, the union of all S_n is \mathbb{M} . Notice that

$$(13.2) \quad S := S_i \cap S_j \quad i \neq j$$

is independent of the choice of $i \neq j$ and corresponds to matrices with no resonances. On S_n , consider the \mathbb{Z} -action on $\mathbb{M} \times \mathbb{C}^2 \setminus \{0\}$ generated by

$$(13.3) \quad \left((M, t), \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) \longmapsto \left((M, t), M \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + t \begin{pmatrix} z_2^n \\ z_1^n \end{pmatrix} \right)$$

This defines a reduced family \mathcal{X}_i of Hopf surfaces above S_i . When M has no resonance of order n , the contracting biholomorphism of (13.3) is equivalent to a linear diagonal one, and the corresponding fiber is biholomorphic to a linear diagonal Hopf surface, regardless of the value of t . However, if $R_n(M)$ is zero, and t is not zero, then the contracting biholomorphism of (13.3) cannot be linearized but is equivalent to (7.14). In particular, we can glue \mathcal{X}_i and \mathcal{X}_j above S for every $i \neq j$, thus producing a family of Hopf surfaces over \mathbb{M} . This family is complete at every point, making of \mathbb{M} a connected atlas of a connected component of $\mathcal{T}(\mathbb{S}^3 \times \mathbb{S}^1)$.

The associated groupoid structure is described as follows. Let G be the Lie group biholomorphic to $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}$ as a complex manifold but with the following product rule

$$(13.4) \quad (A, t) * (B, s) = (AB, t + s \det A)$$

Then one may define

a) a holomorphic action \cdot of G onto M .

b) a holomorphic injection ι of M into G

such that the Lie groupoid $(G \times M)/\mathbb{Z} \rightrightarrows M$ is the desired groupoid. Here the \mathbb{Z} -action is defined as

$$(13.5) \quad (p, g, m) \in \mathbb{Z} \times G \times M \mapsto (\iota(m)^p g, m)$$

and the source and target maps are the projections of the maps¹⁶

$$(13.6) \quad (g, m) \in G \times M \mapsto m \quad \text{and} \quad (g, m) \mapsto m \cdot g$$

Remark 13.5. In [12], a connected family of Hopf surfaces containing a copy of every Hopf surface and complete at each point, that is a connected atlas of a connected component of $\mathcal{T}(\mathbb{S}^1 \times \mathbb{S}^3)$, is also constructed. However, the associated groupoid has a more complicated structure than (13.5). The simple form of (13.5) is used in a crucial way in [15] to compute some de Rham cohomology groups of $\mathcal{T}(\mathbb{S}^1 \times \mathbb{S}^3)$, showing in particular the existence of a very particular class in dimension 2 that plays the role of the Euler class of the \mathbb{Z} -gerbe of footnote 16.

We may now describe the geography of $\mathcal{T}(\mathbb{S}^1 \times \mathbb{S}^3)$ by looking at the geometry of $\mathbb{M} = \mathrm{GL}_2^c(\mathbb{C}) \times \mathbb{C}$. Recall that there is no exceptional point.

- a) The function h^0 equals 4 on the analytic subset $\{R_1 = 0\} \cap \{t = 0\}$ of couples $(\lambda Id, 0)$.
- b) It equals 3 on the countable union of analytic subsets $\{R_p = 0\} \cap \{t = 0\}$ of couples $(M, 0)$ with eigenvalues (λ, λ^p) for $p > 1$.
- c) It is equal to 2 at every other point.
- d) Every point with h^0 equal to 3 or 4 is a double point but is not split.
- e) The holomorphic map

$$(13.7) \quad (M, t) \in \mathbb{M} \mapsto (\det M, (\mathrm{Tr} M)/2) \in \mathbb{D}^* \times \mathbb{D}$$

descends as the mapping from the open substack of normal points to the normal Teichmüller stack.

13.3. Holonomy points. It is interesting to revisit Theorems 13.1 and 13.3 through the concept of holonomy points.

Definition 13.6. A point X_0 of $\mathcal{T}(M)$ is called a *holonomy point* if

- i) It is not exceptional.
- ii) There exists $f \in \mathrm{Aut}^1(X_0)$ which does not admit an extension as an isomorphism of the germ of Kuranishi family at X_0 inducing the identity on the base.

¹⁶ The action groupoid $G \times M \rightrightarrows M$ with source and target maps defined in (13.6) is an atlas for the stack of reduced $\mathbb{S}^3 \times \mathbb{S}^1$ -deformations admitting a covering $\mathbb{C}^2 \setminus \{(0, 0)\}$ -deformation plus a choice of a base point in the covering family. Together with ι , this forms a gerbe with band \mathbb{Z} .

Replacing exceptional with \mathbb{Z} -exceptional and $\text{Aut}^1(X_0)$ with $\text{Aut}^{\mathbb{Z}}(X_0)$ gives rise to a \mathbb{Z} -holonomy point.

Closely related is the notion of holonomy group. To define it, note the following lemma.

Lemma 13.7. *Let $E^1(X_0)$ be the subset of $\text{Aut}^1(X_0)$ consisting of elements $f \in \text{Aut}^1(X_0)$ that admits an extension as an isomorphism of the germ of Kuranishi family at X_0 inducing the identity on the base.*

Then $E^1(X_0)$ is a normal Lie subgroup of $\text{Aut}^1(X_0)$.

Replacing $\text{Aut}^1(X_0)$ with $\text{Aut}^{\mathbb{Z}}(X_0)$, one obtains a normal Lie subgroup $E^{\mathbb{Z}}(X_0)$ of $\text{Aut}^{\mathbb{Z}}(X_0)$.

Proof. The composition of two extensions inducing the identity on the base, and the inverse of such an extension, still induce the identity on the base. Closedness is immediate. Let $f \in E^1(X_0)$ and $g \in \text{Aut}^1(X_0)$. Let F be an extension of f as an isomorphism of the germ of Kuranishi family at X_0 inducing the identity on the base. Let G be an extension of g as an isomorphism of the germ of Kuranishi family at X_0 . Then G induces some map h on the germ of Kuranishi space at 0 and h has no reason to be the identity. Now, $G \circ F \circ G^{-1}$ is an extension of $g \circ f \circ g^{-1}$ that induces $h \circ \text{Id} \circ h^{-1}$, that is the identity, on the base. \square

We then define.

Definition 13.8. Let X_0 be a point of the Teichmüller stack. The *holonomy group* $\text{Hol}(X_0)$ of X_0 is defined as the quotient group $\text{Aut}^1(X_0)/E^1(X_0)$. The *\mathbb{Z} -holonomy group* $\text{Hol}^{\mathbb{Z}}(X_0)$ of X_0 is defined as the quotient group $\text{Aut}^{\mathbb{Z}}(X_0)/E^{\mathbb{Z}}(X_0)$.

Thus, a (\mathbb{Z})-holonomy point is a point

- i) that is not (\mathbb{Z})-exceptional,
- ii) and has a non-trivial (\mathbb{Z})-holonomy group.

To understand why we exclude (\mathbb{Z})-exceptional points, recall that they correspond to points where the $\text{Diff}^0(M)$ -orbits, resp. the $\text{Diff}^{\mathbb{Z}}(M)$ -orbits in K_0 are not controlled by $\text{Aut}^1(X_0)$, resp. $\text{Aut}^{\mathbb{Z}}(X_0)$. More precisely,

Lemma 13.9. *A point X_0 of $\mathcal{T}(M)$ is not exceptional if and only if it satisfies the following property:*

For V small enough, two distinct points J_1 and J_2 of K_0 belong to the same $\text{Diff}^0(M)$ -orbit if and only if there exists a local isomorphism of the Kuranishi family $\mathcal{X}_0 \rightarrow K_0$ sending J_1 to J_2 and acting as an automorphism of the central fiber X_0 .

The same characterization holds for X_0 being not \mathbb{Z} -exceptional if we replace $\text{Diff}^0(M)$ with $\text{Diff}^{\mathbb{Z}}(M)$.

Proof. Assume K_0 satisfies the property of Lemma 13.9. Then, to any sequence (8.1), corresponds a sequence of local isomorphisms of the Kuranishi family fixing the central fibers, hence a sequence of automorphisms of X_0 . By Lemma 8.4, the sequence (8.1) is a sequence of morphisms of the Kuranishi stack and X_0 is not exceptional.

Conversely, if X_0 is not exceptional, then, for V small enough, every morphism between two points J_1 and J_2 of K_0 is (V, \mathcal{D}_1) -admissible, hence is

the evaluation at J_1 of a local isomorphism of the Kuranishi family $\mathcal{K}_0 \rightarrow K_0$ sending J_1 to J_2 and acting as an automorphism of the central fiber X_0 . \square

Thus, with Definition 13.6, holonomy points are points such that non-trivial repetitions in the Kuranishi space¹⁷ exist but can be determined by computing the extensions of the elements of $\text{Aut}^1(X_0)$ as isomorphisms of the germ of Kuranishi family. Hence they are induced by the central fiber and not by the geometry of the $\text{Diff}^0(M)$ -orbits in $\mathcal{I}(M)$.

Remark 13.10. Examples $\mathcal{X}_{a,b}$ of [31] have holonomy group \mathbb{Z}_a , since an arbitrary small deformation of them obtained by moving generically the a -th roots of unity has no $\text{Aut}^1(X_0)$ -mapping class group. In the same way, Examples $\hat{\mathcal{X}}_P$ (with P in the t_a -fiber) have \mathbb{Z} -holonomy group \mathbb{Z}^{4a} by Theorem 8.15. However, we do not know if these examples are (\mathbb{Z}) -holonomy points. To answer this question, we should decide whether they are exceptional or not. This supposes to know all their small deformations. Notice also that, if they $\hat{\mathcal{X}}_P$ is not exceptional, this would be an example of a non-exceptional point in the closure of the set of exceptional points by Theorem 8.17 and Corollary 8.20.

Then, we may characterize jumping points by their holonomy group.

Proposition 13.11. *Let X_0 be a point of $\mathcal{T}(M)$. Then, X_0 is a jumping point if and only if it is a holonomy point with non-discrete holonomy group $\text{Hol}(X_0)$.*

Proof. Jumping points have an automorphism group whose dimension is strictly greater than $E^1(X_0)$, hence their holonomy group is non-trivial and positive-dimensional.

Conversely, if $\text{Hol}(X_0)$ is not discrete, it is a positive dimensional Lie group, hence taking a non-trivial element in its Lie algebra, the exponential flow of this element induces a 1-dimensional submanifold of K_0 all of whose points correspond to the same complex structure up to $\text{Diff}^0(M)$ -action. Hence the foliation of K_0 is not trivial and the h^0 function jumps. \square

Let us analyze what happens at normal points. Let X_0 be a normal point. Let \mathcal{E}^1 be the subset of elements f in \mathcal{A}_1 such that $s(f) = t(f)$. Since X_0 is normal, \mathcal{E}^1 contains \mathcal{A}_0 . We may mimic Section 7.3 and define an étale quotient groupoid $\mathcal{A}_1/\mathcal{E}^1 \rightrightarrows K_0$. Its stackification over the analytic site describes classes of reduced (M, V) -families up to \mathcal{E}^1 -equivalence.

This stack can be defined over the full open set \mathcal{N} of normal points. In this context, a reduced (M, \mathcal{N}) -family is \mathcal{E}^1 -equivalent to a trivial family if it can be decomposed as local pull-back families glued by a cocycle in \mathcal{E}^1 , cf. the proof of Theorem 5.3; and an isomorphism of a reduced (M, \mathcal{N}) -family is \mathcal{E}^1 -equivalent to the identity, if it is given by local \mathcal{E}^1 -sections once decomposed as local pull-back families. We set

Definition 13.12. The stack over the analytic site of \mathcal{E}^1 -equivalence classes of reduced (M, \mathcal{N}) -families is called *the holonomy normal Teichmüller stack* and denoted by $\mathcal{HT}(M)$.

¹⁷that is pairs of distinct points in the Kuranishi space that encode the same complex structure up to C^∞ -isotopy.

This is based on the result

Lemma 13.13. *Assume X_0 is normal. Then, the quotient space $\mathcal{A}_1/\mathcal{E}^1$ is an analytic space and the morphism s , resp. $t : \mathcal{A}_1 \rightarrow K_0$, descends as an étale morphism from $\mathcal{A}_1/\mathcal{E}^1$ to K_0 .*

whose statement and proof are identical to those of Lemma 7.13. We then have

Theorem 13.14. *The normal holonomy Teichmüller stack $\mathcal{H}\mathcal{T}(M)$ satisfies the following properties*

- i) *It is an analytic étale stack with atlas an (at most) countable union of Kuratowski spaces.*
- ii) *The isotropy group of $\mathcal{N}\mathcal{T}(M)$ at X_0 is the discrete group $\text{Hol}(X_0)$. It is finite if X_0 is in Fujiki class (\mathcal{C}) .*
- iii) *Assume that $\text{Hol}(X_0)$ is finite. Then, we may assume that it acts effectively on K_0 and the normal holonomy Teichmüller stack is locally isomorphic at X_0 to the effective orbifold $[K_0/\text{Hol}(X_0)]$.*
- iv) *In particular, if X_0 is a normal point with no holonomy, then the normal holonomy Teichmüller stack is locally isomorphic to K_0 at X_0 .*
- v) *There is a natural étale morphism from $\mathcal{N}\mathcal{T}(M)$ to $\mathcal{H}\mathcal{T}(M)$. Moreover, these two stacks are associated to the same topological quotient space.*
- vi) *Locally at X_0 , the morphism from $\mathcal{N}\mathcal{T}(M)$ to $\mathcal{H}\mathcal{T}(M)$ is a gerbe with band $E^1(X_0)/\text{Aut}^0(X_0)$.*

Of course all this applies to the \mathbb{Z} -Teichmüller stack and we may thus define a \mathbb{Z} -normal holonomy Teichmüller stack $\mathcal{H}\mathcal{T}^{\mathbb{Z}}(M)$ that satisfies the properties listed in Theorem (13.14) with the obvious changes.

Remark 13.15. Theorem 13.14 has of course to be compared with Theorem 7.16. The main difference appears in item iii): assuming finiteness of the holonomy group of X_0 , the local form of $\mathcal{H}\mathcal{T}(M)$ is an effective orbifold whereas, assuming finiteness of the $\text{Aut}^1(X_0)$ mapping class group, the local form of $\mathcal{N}\mathcal{T}(M)$ is a possibly non-effective orbifold. The explanation is given in item v): the $\text{Hol}(X_0)$ -action is the effective action induced by the $\text{Map}^1(X_0)$ -action.

Hence, the normal holonomy Teichmüller stack is the closest to a geometric moduli space. But it forgets about the automorphism group $\text{Aut}^1(X_0)$.

Proof. Point i) follows from Lemma 13.13 and the first of point ii) from the mere definition of $\text{Hol}(X_0)$. If X_0 is in Fujiki class (\mathcal{C}) , its automorphism group is finite, hence also its holonomy group. Point iii) can be proved along the same lines that the corresponding statement in Theorem 7.16. An automorphism f of $\text{Aut}^1(X_0)$ acts as the identity on K_0 if and only if its extension Hol_f is the identity. But this means that the class of f in $\text{Hol}(X_0)$ is the class of the identity, showing effectiveness and finishing the proof of point iii). If X_0 has no holonomy, the effective orbifold chart of point iii) is a local isomorphism with K_0 , proving iv). Since $\text{Aut}^0(X_0)$ is included in $E^1(X_0)$ at a normal point, there is a forgetful functor from $\mathcal{N}\mathcal{T}(M)$ to $\mathcal{H}\mathcal{T}(M)$. Its fiber at X_0 is given by the discrete group $E^1(X_0)/\text{Aut}^0(X_0)$

which acts trivially on K_0 making of $\mathcal{N}\mathcal{T}(M)$ to $\mathcal{H}\mathcal{T}(M)$ a gerbe with band $E^1(X_0)/\text{Aut}^0(X_0)$. This proves iv) and v). \square

13.4. Teichmüller stack and GIT quotients. In this last subsection, we want to say a few words in the case of the Teichmüller stack being isomorphic to a quotient stack $[X/G]$ with X affine, resp. projective, and G reductive, see [21] for an example. In such a situation, we can also form the GIT quotient $X//G$, which has its own geography of stable points, resp. unstable, semistable and stable points. We would like to compare both type of quotients. We note that a thorough study of this question is done in [1] in an algebraic context. Here, we content ourselves with some naive geometric remarks. We refer to [19] for basics on GIT theory.

Let us start with X being affine. Then, the algebra of G -invariant functions on X is finitely generated and the associated affine scheme $X//G$ comes equipped with a natural map $X \rightarrow X//G$ whose fibers are the closure of the orbits. This good quotient is geometric if and only all orbits are closed. The stable points, that is the points with closed orbits and finite stabilizers, form a Zariski open subset X^s of X and the restriction of $X \rightarrow X//G$ to X^s is a geometric quotient. Note however that X^s may be empty. Note also that there may exist an open subset of X bigger than X^s such that the restriction of $X \rightarrow X//G$ to it is a geometric quotient.

Proposition 13.16. *Let X be an affine scheme and G a reductive group acting rationally on it. Assume that $\mathcal{T}(M)$ is isomorphic to the quotient stack $[X/G]$. Then $X//G$ is not homeomorphic to the orbit space X/G , hence to the geometric quotient of $\mathcal{T}(M)$ if and only if one the following equivalent conditions are satisfied*

- i) *There exist a double point X_0 and a jumping family based at X_0 .*
- ii) *There exists an injective morphism from the quotient stack $[\mathbb{C}/\mathbb{C}^*]$ (with \mathbb{C}^* acting multiplicatively on \mathbb{C}) to $\mathcal{T}(M)$.*

Moreover, the subset X^s is included in the subset of points X_0 of $\mathcal{T}(M)$ with finite $\text{Aut}^1(X_0)$.

Proof. Since $\mathcal{T}(M)$ is isomorphic to $[X/G]$, then X is an atlas of $\mathcal{T}(M)$, hence comes equipped with a family of reduced M -deformations $\mathcal{X} \rightarrow X$ which is complete at any point. Now, $X//G$ is homeomorphic to the orbit space X/G if and only if all G -orbits are closed. If this is not the case, then the main Theorem of affine GIT asserts that there exists $x \in X$ and \mathbb{C}^* in G such that the limit of $g \cdot x$ when $g \in \mathbb{C}^*$ tends to zero is a point y not belonging to the x -orbit. In other words, this gives a holomorphic mapping from \mathbb{C} to X such that \mathbb{C}^* lands in the x -orbit and 0 in the distinct y -orbit. Pulling-back $\mathcal{X} \rightarrow X$ through this morphism gives a jumping family. Its base point 0 is a double point of $\mathcal{T}(M)$. At the same time, this morphism descends as an injective morphism from $[\mathbb{C}/\mathbb{C}^*]$ to $\mathcal{T}(M)$. Conversely, if there exists a jumping family, we may assume that its base is the unit disk \mathbb{D} and that it is obtained by pull-back from $\mathcal{X} \rightarrow X$ along a non-constant map $\mathbb{D} \rightarrow X$ with \mathbb{D}^* landing in a single G -orbit and 0 in a distinct one, proving the existence of a non-closed G -orbit. Applying what we just proved, this shows the existence of an injective morphism from $[\mathbb{C}/\mathbb{C}^*]$ to $\mathcal{T}(M)$. If

we assume the existence of such a morphism, arguing as above, we obtain a jumping family.

Finally, since $\mathcal{T}(M)$ is isomorphic to $[X/G]$, then the stabilizer of a point is the automorphism group $\text{Aut}^1(X_0)$ of the corresponding compact complex manifold. \square

Assume from now on that $X \subset \mathbb{P}^n$ is projective and choose a linearization of the action, that is a lift of the G -action to \mathbb{C}^{n+1} . The points $[z] \in X$ such that $G \cdot z$ does not accumulate onto 0 in \mathbb{C}^{n+1} form a Zariski open subset X^{ss} of X called the set of semistable points. Then, the algebra of G -invariant functions on the affine cone over X is finitely generated and the projective scheme $X//G$ associated to its projectivization comes equipped with a natural map $X^{ss} \rightarrow X//G$ whose fibers are the closure of the orbits. This good quotient is geometric if and only all orbits are closed in X^{ss} . The stable points, that is the points with closed orbits in X^{ss} and finite stabilizers, form a Zariski open subset X^s of X^{ss} and the restriction of $X^{ss} \rightarrow X//G$ to X^s is a geometric quotient. As in the affine case, X^s may be empty and there may exist an open subset of X bigger than X^s such that the restriction of $X^{ss} \rightarrow X//G$ to it is a geometric quotient.

The main difference with the affine case is the fact that unstable orbits must be thrown away before taking the GIT quotient. In particular, $X//G$ cannot be homeomorphic to the geometric quotient of $\mathcal{T}(M)$ in presence of unstable points. Now, the notion of unstable/semistable points depends on the choice of a linearization and is strongly related to the projectivity of the quotient. Indeed, considering the affine cone $\tilde{X} \subset \mathbb{C}^{n+1}$ above $X \subset \mathbb{P}^n$, then the GIT quotient $X//G$ is the projectivization of the affine GIT quotient $\tilde{X}//G$ minus zero. This is the projectivization of \tilde{X}^{ss}/G where z belongs to \tilde{X}^{ss} if $G \cdot z$ does not accumulate onto 0 in \mathbb{C}^{n+1} . For if z does not belong to \tilde{X}^{ss} , its G -orbit accumulates onto zero and it is sent to 0 in \tilde{X}/G , preventing from projectivizing the whole quotient \tilde{X}/G .

As a consequence, unstable points are not intrinsic and have no clear geometric meaning in the Teichmüller stack. Hence, we modify our setting and assume from now on that $\mathcal{T}(M)$ is isomorphic to the quotient stack $[X^{ss}/G]$. In this new setting, Proposition 13.16 can be easily adapted.

Proposition 13.17. *Let X be a projective scheme and G a reductive group acting rationally on it. Choose a linearization and assume that $\mathcal{T}(M)$ is isomorphic to the quotient stack $[X^{ss}/G]$. Then $X//G$ is not homeomorphic to the orbit space X^{ss}/G , hence to the geometric quotient of $\mathcal{T}(M)$ if and only if one the following equivalent conditions are satisfied*

- i) *There exist a double point X_0 and a jumping family based at X_0 .*
- ii) *There exists an injective morphism from the quotient stack $[\mathbb{C}/\mathbb{C}^*]$ (with \mathbb{C}^* acting multiplicatively on \mathbb{C}) to $\mathcal{T}(M)$.*

Moreover, the subset X^s is included in the subset of points X_0 of $\mathcal{T}(M)$ with finite $\text{Aut}^1(X_0)$.

Proof. Apply Proposition 13.16 to $\tilde{X} \rightarrow \tilde{X}/G$ restricted to \tilde{X}^{ss} . \square

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LAURENT MEERSSEMAN, UNIV ANGERS, CNRS, LAREMA, SFR MATHSTIC, F-49000 ANGERS, FRANCE, LAURENT.MEERSSEMAN@UNIV-ANGERS.FR