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The twin paradox and time dilation have equivalents in polarization optics

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Abstract. The phenomena known as the twin-paradox and time dilation, which are familiar effects in the special theory of relativity, have analogous counterparts in polarization optics. To show that, we present the concept of proper irradiance for a partially polarized lightwave. The analogous effect of the twin paradox is that the proper irradiance of the incoherent addition of two partially polarized lightwaves exceeds the sum of the proper irradiances of each individual lightwave. This effect does not pose a paradox in optics. The analog of time dilation is the increase in irradiance experienced by a lightwave as it propagates through a pure dichroic device.

Keywords. Minkowski space, polarization optics, proper irradiance, quaternionic representation of polarized light, special theory of relativity, time dilation, twin paradox.

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OCIS codes:

1. Introduction

Both the special theory of relativity and polarization optics are based on the same geometry, specifically that of the Minkowski space and proper Lorentz rotations^{1,2}. As a result, certain relativistic phenomena have corresponding manifestations in polarization optics, as they stem from underlying geometrical properties. For instance, Lorentz boosts in relativity and pure dichroics in polarization optics represent physical expressions of hyperbolic rotations in the abstract Minkowski space, leading to analogous effects in both domains.

The triangle inequality, which exhibits a particular form in Minkowski space^{2,3}, gives rise to the well-known twin paradox in relativity^{2,4-6}. In the present article, we explore the implications of this inequality on the proper irradiance of a partially polarized lightwave, a concept equivalent to the proper time in relativity. Our findings reflect the twin paradox in relativity within the context of polarization optics. Those results align with established characteristics of partially polarized lightwaves and do not introduce any paradoxical elements in polarization optics.

Additionally, we interpret the increase in irradiance under a pure dichroic as analogous, in polarization optics, to time dilation in the realm of relativity.

2. Geometric analysis on Minkowski space

2.1. Minkowski space

We represent Minkowski's 4-vectors by minquats^{1,2,7}. (The term "minquat," due to Synge⁷, is an abbreviation for "Minkowskian quaternion.") If $\mathbf{x} = (x_0, x_1, x_2, x_3)$ is a

4-vector in Minkowski space, the x_μ 's ($\mu = 0, 1, 2, 3$) being real numbers, the associated minquat is the complex quaternion x such that

$$x = x_0e_0 + i(x_1e_1 + x_2e_2 + x_3e_3), \quad (1)$$

where e_0, e_1, e_2 and e_3 are unit quaternions, with

$$(e_1)^2 = (e_2)^2 = (e_3)^2 = -(e_0)^2 = -e_0, \quad (2)$$

$$e_0e_j = e_j = e_je_0, \quad j = 1, 2, 3, \quad (3)$$

$$e_1e_2 = e_3 = -e_2e_1. \quad (4)$$

The Minkowski space is (\mathbb{R}^4, Q) where Q denotes the Lorentz quadratic form, defined for every 4-vector \mathbf{x} by

$$Q(\mathbf{x}) = Q(x_0, x_1, x_2, x_3) = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2. \quad (5)$$

We denote as \mathbb{M} the (real) vector space of minquats, a basis of which is $\{e_0, ie_1, ie_2, ie_3\}$. It is endowed with the quaternionic norm N , such that for every complex quaternion $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$ (the q_μ 's are complex numbers)

$$N(q) = (q_0)^2 + (q_1)^2 + (q_2)^2 + (q_3)^2. \quad (6)$$

If x is a minquat, then

$$N(x) = N[x_0e_0 + i(x_1e_1 + x_2e_2 + x_3e_3)] = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2. \quad (7)$$

We conclude that the Minkowski space and the minquat space are quadratically isomorphic: $(\mathbb{M}, N) \simeq (\mathbb{R}^4, Q)$.

We say that

- x is a spacelike minquat if $N(x) < 0$;
- x is a null (or isotropic) minquat if $N(x) = 0$;
- x is a scalarlike (or timelike) minquat if $N(x) > 0$.

Correspondingly a 4-vector \mathbf{x} may be spacelike, null or timelike, if $Q(\mathbf{x})$ is strictly negative, zero or strictly positive.

Remark. “Scalarlike” should not be confused with “scalar”. A scalar quaternion takes the form $q = q_0 e_0$, whereas x is a scalarlike minquat if its scalar component $x_0 e_0$ is the dominant factor in $N(x)$, that is, if $(x_0)^2 > (x_1)^2 + (x_2)^2 + (x_3)^2$.

2.2. Triangle inequality

The triangle inequality in Minkowski space takes a form reverse to its form in an Euclidean space^{2,3}. If \mathbf{x} and \mathbf{y} are two 4-vectors belonging to Minkowski space, independently null or timelike, then (triangle inequality)

$$\sqrt{Q(\mathbf{x} + \mathbf{y})} \geq \sqrt{Q(\mathbf{x})} + \sqrt{Q(\mathbf{y})}. \quad (8)$$

The equality holds if, and only if, 4-vectors \mathbf{x} and \mathbf{y} are collinear.

By isomorphism, the triangle inequality holds in the minquat space \mathbb{M} under the form

$$\sqrt{N(x + y)} \geq \sqrt{N(x)} + \sqrt{N(y)}, \quad (9)$$

where x and y are two independently null or scalarlike (timelike) minquats.

3. The twin paradox in relativity

3.1. Basic elements of relativity⁴⁻⁶

In the special theory of relativity an event, denoted as \mathcal{X} , is represented by a 4-vector in Minkowski space, say \mathbf{x} , with coordinates x_μ 's referring to an inertial (or Galilean) frame (the x_μ 's are real numbers). We set $x_0 = ct$, where c denotes the speed of light (in a vacuum) and t the time, in the considered frame. The x_j 's ($j = 1, 2, 3$) are spatial coordinates along three orthogonal directions (also denoted as x_j).

If \mathcal{R} and \mathcal{R}' denote two inertial frames in relative uniform translatory motion, the coordinates of an event in \mathcal{R} and in \mathcal{R}' are connected by a hyperbolic rotation^{1,2,7}, provided that the axis x_j is parallel to the axis x'_j , and that $x'_0 = 0$ if, and only if, $x_0 = 0$. The event \mathcal{X} is represented by the minquat x in \mathcal{R} and x' in \mathcal{R}' , with $N(x) = N(x')$.

A spacetime interval between two events \mathcal{X} and \mathcal{Y} is also a 4-vector, represented by the minquat $y - x$ in \mathcal{R} (coordinates $y_\mu - x_\mu$, $\mu = 0, 1, 2, 3$), and $y' - x'$ in \mathcal{R}' (coordinates $y'_\mu - x'_\mu$), with $N(y - x) = N(y' - x')$.

In the following we will denote a spacetime interval by $\Delta\mathcal{X}$, represented by the minquat Δx in \mathcal{R} (coordinates Δx_μ), and by $\Delta x'$ in \mathcal{R}' (coordinates $\Delta x'_\mu$). We have $N(\Delta x) = N(\Delta x')$.

We employ the notion of observer^{5,6}, attached to an inertial frame. The spatial coordinates x_j ($j = 1, 2, 3$) of an observer \mathcal{O} in an inertial frame, say \mathcal{R} , may be regarded as continuous functions of time t , or of parameter $x_0 = ct$. We write $\Delta x_j(x_0) = x_j(x_0) - x_j(0)$, where $x_0 = 0$ is the origin of time in \mathcal{R} . The observer \mathcal{O} is at rest in \mathcal{R} if $\Delta x_j(x_0) = 0$ ($j = 1, 2, 3$) for every x_0 (\mathcal{R} is thus the frame to which \mathcal{O} is attached). Therefore \mathcal{O} is represented in \mathcal{R} by the minquat $\Delta x = e_0 \Delta x_0$, and Δx_0 is called the proper time of \mathcal{O} . In another inertial frame, say \mathcal{R}' , the observer \mathcal{O} is represented by the minquat $\Delta x' = e_0 \Delta x'_0 + i(e_1 \Delta x'_1 + e_2 \Delta x'_2 + e_3 \Delta x'_3)$, and we have $N(\Delta x) = N(\Delta x')$, that is

$$(\Delta x_0)^2 = (\Delta x'_0)^2 - (\Delta x'_1)^2 - (\Delta x'_2)^2 - (\Delta x'_3)^2. \quad (10)$$

3.2. Twin paradox (timelike vectors)

An observer \mathcal{O} stays at point $(x_1, x_2, x_3) = (0, 0, 0)$, in an inertial frame \mathcal{R} , while another observer \mathcal{O}' travels from abscissa $x_1 = 0$ to abscissa $x_1 = b_1 \neq 0$, on the x_1 -axis, and then travels back to abscissa $x_1 = 0$. The travel diagram is shown on Fig. 1: the observer \mathcal{O}' travels from the point $A = (a_0, 0, 0, 0)$ to $B = (b_0, b_1, 0, 0)$ and then from B to $C = (c_0, 0, 0, 0)$. We have $c_0 > b_0 > a_0$. Both motions of \mathcal{O}' are accomplished in uniform translation with respect to \mathcal{R} . An inertial frame \mathcal{R}' is attached to \mathcal{O}' , when travelling from A to B , and an inertial frame \mathcal{R}'' , when travelling from B to C .

The “travel” in time of \mathcal{O} from A to C is represented in \mathcal{R} by the minquat

$$\Delta x = (c_0 - a_0) e_0, \quad (11)$$

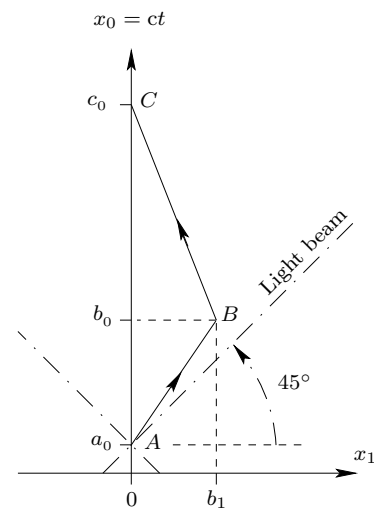


Fig. 1. Twin paradox in relativity. The proper time of the space-time interval AC is greater than the sum of the individual proper times of AB and BC . Dashed-dotted lines represent the trace of the light cone at A . The diagram is applicable in both Minkowski space and minquat space \mathbb{M} , with the latter involving basis vectors e_0 (x_0 -axis) and ie_1 (x_1 -axis).

and the corresponding proper time is

$$\Delta t_x = \frac{1}{c} \sqrt{N(\Delta x)} = \frac{1}{c} \sqrt{(c_0 - a_0)^2}. \quad (12)$$

The travel from A to B is represented in \mathcal{R} by the minquat

$$\Delta y = (b_0 - a_0) e_0 + i b_1 e_1, \quad (13)$$

and since \mathcal{O}' is at rest in \mathcal{R}' , the corresponding proper time is

$$\Delta t'_y = \frac{1}{c} \sqrt{N(\Delta y)} = \frac{1}{c} \sqrt{(b_0 - a_0)^2 - (b_1)^2}. \quad (14)$$

The travel from B to C is represented by

$$\Delta z = (c_0 - b_0) e_0 - i b_1 e_1, \quad (15)$$

and the corresponding proper time is

$$\Delta t''_z = \frac{1}{c} \sqrt{N(\Delta z)} = \frac{1}{c} \sqrt{(c_0 - b_0)^2 - (b_1)^2}. \quad (16)$$

Since $\Delta x = \Delta y + \Delta z$, and since Δy and Δz are not collinear (because $b_1 \neq 0$), the triangle inequality leads to

$$\sqrt{N(\Delta x)} > \sqrt{N(\Delta y)} + \sqrt{N(\Delta z)}, \quad (17)$$

that is

$$\Delta t_x > \Delta t'_y + \Delta t''_z, \quad (18)$$

which means that the travel time-length is measured greater by \mathcal{O} than by \mathcal{O}' .

The whole-travel time-length measured by \mathcal{O} (in \mathcal{R}) is longer than the travel-length measured by \mathcal{O}' (on his own clock). If the two observers are twins, \mathcal{O} is older than \mathcal{O}' when they meet again. That constitutes a paradox according to our common ways of conceiving time.

To compare with the equivalent effect in polarization optics (Sect. 4), we note that the speed of \mathcal{O}' when travelling from A to B is

$$v_y = c \frac{b_1}{b_0 - a_0}, \quad (19)$$

and the speed in the travel from B to C is

$$v_z = c \frac{-b_1}{c_0 - b_0}. \quad (20)$$

We assume $b_1 > 0$ and define $\beta_y = v_y/c$ and $\beta_z = -v_z/c$; we have $0 < \beta_y < 1$ and $0 < \beta_z < 1$. The travel from A to B may be represented by the minquat

$$\begin{aligned} Y &= \Delta y = (b_0 - a_0)(e_0 + i\beta_y e_1) \\ &= Y_0(e_0 + i\beta_y e_1), \end{aligned} \quad (21)$$

and the travel from B to C by

$$\begin{aligned} Z &= \Delta z = (c_0 - b_0)(e_0 - i\beta_z e_1) \\ &= Z_0(e_0 - i\beta_z e_1), \end{aligned} \quad (22)$$

with $Y_0\beta_y = Z_0\beta_z$ (Y_0 and Z_0 are strictly positive, because $c_0 > b_0 > a_0$).

We have $X = e_0 \Delta t_x = Y + Z = (Y_0 + Z_0) e_0$, and eventually Eq. (18) also writes

$$Y_0 + Z_0 > Y_0 \sqrt{1 - \beta_y^2} + Z_0 \sqrt{1 - \beta_z^2}. \quad (23)$$

4. Equivalence in polarization optics

4.1. Norm and degree of polarization

The polarization state of a partially polarized lightwave is represented by a minquat of the form

$$X = X_0 e_0 + i(X_1 e_1 + X_2 e_2 + X_3 e_3), \quad (24)$$

where the X_μ 's are the Stokes parameters of the lightwave (they are homogeneous to irradiances and proportional to the lightwave power; moreover $X_0 > 0$)^{2,8,9}.

The minquat X can be written $X = X_0(e_0 + i\rho_x e_n)$, where ρ_x is the degree of polarization^{2,8} of the lightwave ($0 \leq \rho_x \leq 1$), and where e_n is a real unit pure quaternion ($e_n = n_1 e_1 + n_2 e_2 + n_3 e_3$, with $\rho_x n_j = X_j/X_0$, $j = 1, 2, 3$, if $\rho_x \neq 0$, so that $(n_1)^2 + (n_2)^2 + (n_3)^2 = 1$). The unpolarized component of the lightwave is $(1 - \rho_x)X_0 e_0$; its polarized component is $X_0 \rho_x (e_0 + i e_n)$, and e_n represents its polarization state on the Poincaré sphere^{2,9}.

The norm of X is

$$N(X) = (X_0)^2 (1 - \rho_x^2). \quad (25)$$

The minquat X represents a physical state of polarization if, and only if, $N(X) \geq 0$. If X is a null (or isotropic) minquat, i.e. if $N(X) = 0$, the corresponding lightwave is completely polarized. If X is a scalarlike (or timelike) minquat, i.e. if $N(X) > 0$, the lightwave is partially polarized.

The minquat $X = e_0 + i e_n$ represents a unitary completely polarized state of polarization and $X_\perp = e_0 - i e_n$ the unitary orthogonal state. By abuse we say ‘‘polarization state X ,’’ and ‘‘orthogonal polarization state X_\perp .’’

4.2. Pure dichroics

The polarization of a lightwave may be altered when the lightwave propagates through certain media or reflects off smooth surfaces. Among the physical devices that act on light polarization, we mention birefringent media and pure dichroic ones² (a pure dichroic has an isotropic absorption equal to 1, whereas a physical passive dichroic has an isotropic absorption factor smaller than 1). Birefringents and pure dichroics generate a group, the group of pure ‘‘dephasers’’ (or pure phase-shifters), which can be shown to be isomorphic to the group $SO_+(1, 3)$ of proper Lorentz rotations on Minkowski space².

On the other hand, proper Lorentz rotations can be represented by quaternions of the form $u = \exp e_u \psi$, where ψ is a complex number and e_u a complex pure quaternion (e_u may be a unit quaternion or a null quaternion²). The proper Lorentz rotation represented by u operates on \mathbb{M} according to $X \mapsto u X \bar{u}^*$, where \bar{u}^* denotes the Hamilton conjugate of the complex conjugate of u . The transformation preserves the norm of minquats: $N(u X \bar{u}^*) = N(X)$. If ψ is a purely imaginary number and if e_u is a real pure unit quaternion (then denoted as e_n), the previous Lorentz rotation is a hyperbolic rotation.

If polarization states are represented by minquats (see Sect. 4.1), a pure dichroic operates on \mathbb{M} as a hyperbolic rotation² and is represented by a unit quaternion of the form $u = \exp(i e_n \delta/2)$, where δ is the dichroism (a real number) and e_n is a real unit pure quaternion, called the axis of the dichroic (by reference to the Poincaré sphere representation of polarization states). The image of the polarization state X through the dichroic is X' given by

$$X' = u X u = \exp\left(i e_n \frac{\delta}{2}\right) X \exp\left(i e_n \frac{\delta}{2}\right), \quad (26)$$

because $u = \bar{u}^*$ for a dichroic.

For actual derivations we use

$$\exp\frac{i e_n \delta}{2} = e_0 \cosh \frac{\delta}{2} + i e_n \sinh \frac{\delta}{2}. \quad (27)$$

For example, if $e_n = e_1$, Eq. (26) gives

$$X'_0 = X_0 \cosh \delta + X_1 \sinh \delta, \quad (28)$$

$$X'_1 = X_0 \sinh \delta + X_1 \cosh \delta, \quad (29)$$

$$X'_2 = X_2, \quad (30)$$

$$X'_3 = X_3. \quad (31)$$

If $u = \exp(i e_n \delta/2)$, the minquats $X^+ = e_0 + i e_n$ and $X^- = e_0 - i e_n$ represent two orthogonal unit eigenstates of polarization. They are such that $u X^+ u = e^\delta X^+$ and $u X^- u = e^{-\delta} X^-$, which means that for $\delta > 0$, the polarization state X^+ is amplified, whereas the state X^- is mitigated.

In general, dichroics are “passive” devices, they cannot amplify the incident-lightwave power. To represent them, we introduce an isotropic absorption factor κ ($\kappa > 0$), such that $\kappa \exp \delta < 1$ (for $\delta > 0$), so that a dichroic is represented by a quaternion of the form $\sqrt{\kappa} \exp(i e_n \delta/2)$. It should be clear that a passive dichroic does not preserve the norm of minquats as pure dichroics do.

A pure dichroic ($\kappa = 1$) can be obtained with an active medium: an eigenstate is amplified, whereas the orthogonal state is mitigated. For instance, this can be achieved in a laser cavity.

4.3. Proper irradiance

Let us consider the state $X = X_0(e_0 + i \rho_x e_n)$ and a pure dichroic of axis e_n and dichroism δ . The image of X under the dichroic is

$$\begin{aligned} X' &= \exp\left(i e_n \frac{\delta}{2}\right) X \exp\left(i e_n \frac{\delta}{2}\right) \\ &= X'_0(e_0 + i \rho_{x'} e_n), \end{aligned} \quad (32)$$

where

$$X'_0 = X_0(\cosh \delta + \rho_x \sinh \delta), \quad (33)$$

and

$$\rho_{x'} = \frac{\sinh \delta + \rho_x \cosh \delta}{\cosh \delta + \rho_x \sinh \delta}. \quad (34)$$

We assume that X does not represent a completely polarized lightwave: $0 \leq \rho_x < 1$ (i.e. the lightwave is strictly partially polarized and X is a scalarlike minquat). Let us denote $\tilde{\delta}$ the value of δ such that $\tanh \tilde{\delta} = -\rho_x$, for which X' becomes \tilde{X} . Then $\rho_{\tilde{x}} = 0$, and \tilde{X} is a scalar quaternion: $\tilde{X} = \tilde{X}_0 e_0$; it represents an unpolarized lightwave.

We remark that $\tilde{\delta} < 0$, which means that the state $e_0 + i e_n$ is the mitigated eigenstate of the dichroic (with eigenvalue $\exp \tilde{\delta} < 1$), whereas the state $e_0 - i e_n$ is the amplified eigenstate (eigenvalue $\exp(-\tilde{\delta}) > 1$).

From $\tanh \tilde{\delta} = -\rho_x$, we deduce

$$\tilde{X} = \tilde{X}_0 e_0 = \frac{X_0}{\cosh \tilde{\delta}} e_0, \quad (35)$$

and

$$\begin{aligned} (\tilde{X}_0)^2 &= \frac{(X_0)^2}{\cosh^2 \tilde{\delta}} = N(\tilde{X}) = N(X) \\ &= (X_0)^2(1 - \rho_x^2). \end{aligned} \quad (36)$$

(We have $N(\tilde{X}) = N(X)$, because X is transformed into \tilde{X} under a pure dichroic.)

We call \tilde{X}_0 the proper irradiance of the lightwave represented by X and we state the following proposition:

Proposition 4.1 *For every partially polarized lightwave whose degree of polarization is strictly smaller than 1, there exists a pure dichroic that transforms it into an unpolarized lightwave. The irradiance of the unpolarized lightwave is called the proper irradiance of the initial lightwave.*

The proper irradiance \tilde{X}_0 of a lightwave represented by the scalarlike minquat X has the following property. Let us consider an arbitrary pure dichroic that transforms X into X'' , with $X'' = X''_0(e_0 + i \rho_{x''} e_m)$. Then

$$\begin{aligned} (X''_0)^2(1 - \rho_{x''}^2) &= N(X'') = N(X) \\ &= N(\tilde{X}) = (\tilde{X}_0)^2, \end{aligned} \quad (37)$$

so that

$$\tilde{X}_0 \leq X''_0. \quad (38)$$

Let \mathfrak{D}_X be the set of all minquats X'' that are images of X under all pure dichroics. Such a minquat X'' is given by $X'' = \exp(i e_\ell \delta/2) X \exp(i e_\ell \delta/2)$, where δ is a real number and where e_ℓ is a real unit pure quaternion. It can also be written $X'' = X''_0(e_0 + i \rho_{x''} e_m)$ and then, according to Eq. (38)

$$\tilde{X}_0 = \min_{X'' \in \mathfrak{D}_X} X''_0. \quad (39)$$

Equation (39) delineates a characteristic property of the proper irradiance of a partially polarized lightwave: it represents the minimum achievable irradiance when the lightwave traverses a pure dichroic.

4.4. The equivalent of twin paradox

Let $Y = Y_0(e_0 + i\rho_Y e_n)$ and $Z = Z_0(e_0 - i\rho_Z e_n)$ represent two strictly partially polarized lightwaves \mathcal{Y} and \mathcal{Z} with $Y_0\rho_Y = Z_0\rho_Z \neq 0$. Lightwaves \mathcal{Y} and \mathcal{Z} have orthogonal polarized components. Let \mathcal{X} be the incoherent addition⁹ (or superposition) of \mathcal{Y} and \mathcal{Z} . The polarization state of \mathcal{X} is then represented by the minquat

$$X = Y + Z = (Y_0 + Z_0)e_0, \quad (40)$$

and corresponds to an unpolarized lightwave, whose degree of polarization is 0. Since X , Y and Z are scalarlike minquats, not collinear, the triangle inequality gives

$$\sqrt{N(X)} > \sqrt{N(Y)} + \sqrt{N(Z)}, \quad (41)$$

that is

$$X_0 = Y_0 + Z_0 > Y_0\sqrt{1 - \rho_Y^2} + Z_0\sqrt{1 - \rho_Z^2}. \quad (42)$$

Equation (42) is no more than Eq. (23) in which β_y and β_z are replaced with ρ_Y and ρ_Z . Moreover, since X represents an unpolarized lightwave, we have $\tilde{X}_0 = X_0$, and Eq. (36), applied to Y and Z , leads us to write Inequality (42) in the form

$$\tilde{X}_0 > \tilde{Y}_0 + \tilde{Z}_0, \quad (43)$$

which means that the proper irradiance of X is greater than the sum of the individual proper irradiances of Y and Z . Equation (43) is equivalent, in polarization optics, to Eq. (18) in relativity.

4.5. Graphical equivalence

If the minquat X represents a partially or completely polarized lightwave, then $X_0 \geq 0$ and $N(X) \geq 0$. If $Y = \alpha X$, with $\alpha > 0$, then $Y_0 = \alpha X_0 \geq 0$ and $N(Y) = \alpha^2 N(X) \geq 0$, so that Y also represents a

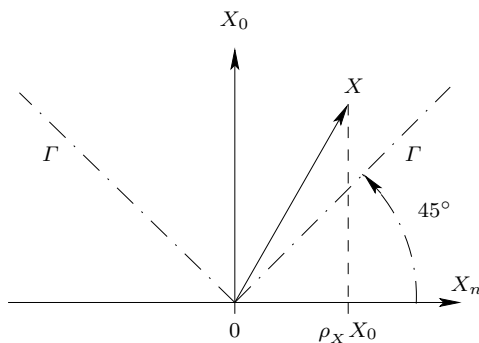


Fig. 2. Graphical representation of a polarization state in \mathbb{M} . The minquat $X = X_0(e_0 + i\rho_X e_n) = X_0 e_0 + iX_n e_n$ lies in the plane spanned by e_0 (X_0 -axis) and $i e_n$ (X_n -axis). The dashed-dotted lines are the trace of the null-minquat half-cone Γ , whose surface represents the set of completely-polarized states. On the diagram, the minquat X belongs to the interior of Γ : it is then a scalarlike minquat and it represents a partially-polarized state.

partially or completely polarized lightwave. We conclude that minquats that represent lightwave polarization states form a subset of \mathbb{M} which is a half-cone Γ and its interior. The half-cone equation is

$$N(X) = 0, \quad X_0 \geq 0. \quad (44)$$

The symmetry-axis of the cone is the straightline of equation $X_1 = X_2 = X_3 = 0$.

In polarization optics the half-cone Γ is the equivalent of the future light-cone in relativity (Fig. 1).

The minquat $X = X_0(e_0 + i\rho_X e_n) = X_0 + iX_n e_n$ represents a polarization state and lies in the plane $(e_0, i e_n)$ (included in \mathbb{M}), as shown in Fig. 2.

The incoherent addition of states Y and Z as in Sect. 4.4 is represented in Fig. 3. The comparison with Fig. 1 emphasizes the analogy with the twin paradox.

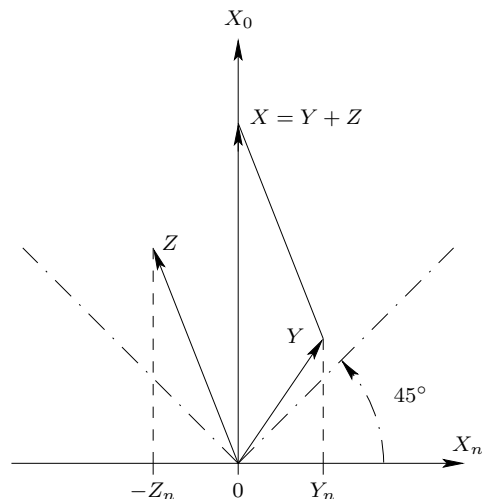


Fig. 3. Graphical equivalent representation of the twin paradox in polarization optics. Minquats Y and Z are such that $Y_n = Y_0\rho_Y = Z_0\rho_Z = Z_n$ (Sect. 4.4). The minquat X represents the incoherent addition of the polarization states represented by Y and Z . The proper irradiance of X is greater than the sum of the proper irradiances of Y and Z . To be compared with Fig. 1.

4.6. Conclusion (twin paradox for scalarlike minquats)

We summarize the preceding findings into the following three equivalent statements, applicable to scalarlike vectors or minquats:

- *Minkowski-space geometry (triangle inequality)*. The square root of the norm of the sum of two scalarlike (timelike) minquats is greater than the sum of the square roots of the individual norms.
- *Special Relativity (twin paradox)*. The proper time of the sum of two timelike 4-vectors in Minkowski spacetime is greater than the sum of the individual proper times.

- *Polarization optics.* The proper irradiance of the incoherent addition of two strictly partially polarized lightwaves is greater than the sum of the individual proper irradiances.

4.7. Particular case. Expression with the degree of polarization

We assume $Y = Y_0(e_0 + i\rho e_m)$ and $Z = Z_0(e_0 + i\rho e_n)$, with $e_m \neq e_n$ (and $\rho_Y = \rho_Z = \rho$). The incoherent superposition of Y and Z is

$$X = Y + Z = (Y_0 + Z_0)e_0 + i\rho(Y_0 e_m + Z_0 e_n), \quad (45)$$

which writes

$$X = (Y_0 + Z_0)(e_0 + i\rho_x e_\ell), \quad (46)$$

where e_ℓ is a real unit pure quaternion and $0 \leq \rho_x < 1$.

Since $e_m \neq e_n$ (which means that Y and Z are not collinear), the triangle inequality gives

$$1 - \rho_x^2 > 1 - \rho^2, \quad (47)$$

which is a special case of Eq. (43), according to Eq. (36). Equation (47) leads to

$$\rho_x < \rho, \quad (48)$$

that is, the degree of polarization of the incoherent superposition of Y and Z is smaller than the degree of polarization of Y and Z . Equation (48) may be regarded as a consequence of Eq. (43), expressing the equivalent of twin paradox in optics in terms of the degree of polarization. (Similar examples are given in a recent publication².)

More generally, the degree of polarization of the incoherent addition of two partially polarized lightwaves is smaller than—or at most equal to—the degree of polarization of the individual lightwaves. That property of partially polarized lightwaves, analogous to the relativistic twin-paradox, is well known and does not constitute any paradox in polarization optics.

5. Time dilation and its equivalent in polarization optics

5.1. Time dilation in relativity

Let us consider a spacetime interval represented by the minquat $\Delta x \neq 0$, in the inertial frame \mathcal{R} , and by $\Delta x' \neq 0$, in \mathcal{R}' . We assume \mathcal{R}' in translatory motion with velocity $v e_1$ with respect to \mathcal{R} ($|v| < c$). Minquats Δx and $\Delta x'$ are linked by a hyperbolic rotation according to

$$\Delta x' = \exp\left(i e_1 \frac{\delta}{2}\right) \Delta x \exp\left(i e_1 \frac{\delta}{2}\right), \quad (49)$$

where δ ($\delta \neq 0$) is the rapidity of the motion, such that

$$\cosh \delta = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (50)$$

$$\sinh \delta = -\gamma\beta = -\gamma \frac{v}{c}. \quad (51)$$

Equation (49) is equivalent to the following equations

$$\Delta x'_0 = \Delta x_0 \cosh \delta + \Delta x_1 \sinh \delta, \quad (52)$$

$$\Delta x'_1 = \Delta x_0 \sinh \delta + \Delta x_1 \cosh \delta, \quad (53)$$

$$\Delta x'_2 = \Delta x_2, \quad (54)$$

$$\Delta x'_3 = \Delta x_3. \quad (55)$$

Now, let us assume the spacetime interval such that $\Delta x'_j = 0$ ($j = 1, 2, 3$), which means that it is an interval between two events that take place at the same “spatial” position in \mathcal{R}' . Its proper time is equal to $\Delta x'_0$. For instance, the considered interval is a time interval given by a clock at rest in \mathcal{R}' . From $\Delta x'_1 = 0$, we deduce $\Delta x_1 = -\Delta x_0 \tanh \delta$ and then, by Eq. (52)

$$\Delta x_0 = \Delta x'_0 \cosh \delta > \Delta x'_0. \quad (56)$$

The time interval $\Delta x'_0$, which is the proper time of a clock at rest in \mathcal{R}' , is measured greater in \mathcal{R} . That is time dilation.

Time dilation is reciprocal. Let us assume, indeed, that $\Delta x_j = 0$ ($j = 1, 2, 3$). Thus Δx_0 is the proper time given by a clock at rest in \mathcal{R} . From Eq. (52) we obtain

$$\Delta x'_0 = \Delta x_0 \cosh \delta > \Delta x_0, \quad (57)$$

with δ as above. The proper time given by the clock at rest in \mathcal{R} is measured greater in \mathcal{R}' .

5.2. Time-dilation equivalent in polarization optics

The equivalent of time dilation results from the representation of a pure dichroic by a hyperbolic rotation (see Sect. 4.2). If X represents the state of polarization of a lightwave incident on a dichroic medium of dichroism δ ($\delta > 0$) and axis e_1 on the Poincaré sphere, the emerging state of polarization is represented by the minquat X' , given by Eq. (26) for $e_n = e_1$, that is

$$X' = \exp\left(i e_1 \frac{\delta}{2}\right) X \exp\left(i e_1 \frac{\delta}{2}\right). \quad (58)$$

Equation (58) is no more than Eq. (49) where the rapidity in replaced with the dichroism and spacetime intervals with polarization-state representative minquats.

Let $X = X_0(e_0 + i\rho_x e_1)$ represent a strictly partially polarized lightwave and let $\exp(i e_1 \tilde{\delta}/2)$ represent the pure dichroic that transforms X into an unpolarized lightwave \tilde{X} (see Sect. 4.3). The proper irradiance of the lightwave is \tilde{X}_0 , and according to Eq. (35)

$$X_0 = \tilde{X}_0 \cos \tilde{\delta} > \tilde{X}_0, \quad (59)$$

which is Eq. (56) where the rapidity δ is replaced with the dichroism $\tilde{\delta}$, the proper time $\Delta x'_0$ with the proper irradiance \tilde{X}_0 , and Δx_0 with the irradiance X_0 .

For the reciprocal effect, let us assume that X takes the form $X = X_0 e_0$, and thus corresponds to an unpolarized lightwave. Then X_0 is also the proper irradiance of X , that is, $\tilde{X}_0 = X_0$.

According to Eq. (58), the minquat X' takes the form

$$\begin{aligned} X' &= X_0 \cosh \delta + iX_0 e_1 \sinh \delta \\ &= X_0 \cosh \delta (e_0 + i\rho_{X'} e_1) \\ &= X'_0 (e_0 + i\rho_{X'} e_1), \end{aligned} \quad (60)$$

where $\rho_{X'}$ denotes the corresponding degree of polarization, and

$$X'_0 = X_0 \cosh \delta = \tilde{X}_0 \cosh \delta > \tilde{X}_0. \quad (61)$$

For the analogy with time dilation, we note that Eq. (61) is Eq. (57) in which $\Delta x'_0$ (time interval) is replaced with X'_0 (irradiance), the proper time Δx_0 is replaced with the proper irradiance \tilde{X}_0 , and the rapidity with the dichroism.

Equation (61) means that the irradiance of the emerging lightwave is greater than the proper irradiance of X . Moreover, a dichroic is also a partial polarizer: X' is partially polarized, whereas X is unpolarized. Indeed $\rho_{X'} > 0$, because $\delta > 0$: the degree of polarization of X' is greater than that of X (which is zero). A part of the unpolarized component of the input state X is transformed into the polarized component of the output state X' . To obtain $N(X') = N(X)$, the irradiance X'_0 has to be greater than \tilde{X}_0 .

Eventually we point out that Lorentz boosts in relativity and pure dichroics in polarization optics are physical manifestations of hyperbolic rotations in the abstract Minkowski space, resulting in comparable physical consequences, including time dilation and its equivalent in optics.

6. Conclusion

Although widely accepted by the scientific community, certain relativistic effects like time dilation or twin paradox are perceived counterintuitive, sparking debates over their interpretations. Actually these effects stem from the geometry of Minkowski space; this one, however, is not the underlying vector space of the sole special theory of relativity: it also constitutes a basic structure of polarization optics. The aforementioned relativistic effects have then their counterparts in polarization optics

and this area of physics affords to illustrate them, as we have shown in this article. Polarization optics also offers a means to experimentally test geometrical properties of Minkowski space, while avoiding the paradoxes, whether real or perceived, linked to their interpretations in relativity.

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