Nonlocal Dynamic Homogenization of Fluid-Saturated Metamaterials

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Chapter 7
Nonlocal Dynamic Homogenization of Fluid-Saturated Metamaterials

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Abstract The electromagnetic analogy introduced in the previous chapter is used here to construct an original macroscopic theory of sound propagation, allowing for both temporal and spatial dispersion, in fluid-saturated homogeneous porous media having arbitrary microstructure—including "metamaterials". The theory can be formulated for stationary random materials, periodic materials, and using different conceptions of the averaging operation (ensemble-average, volume-average). For simplicity, we have assumed that the structure is rigid and motionless, and the propagation occurs along a symmetry axis. The theory will have to be generalized to account for anisotropy, finite dimensions and frame deformations. In Appendix, we show that the preceding macroscopic descriptions in use in literature, leave aside spatial dispersion: this is a warning that the asymptotic two-scale homogenization method, often used to infer them, cannot be fully consistent.

7.1 Sound Propagation in Fluid-Saturated Rigid-Framed Porous Media

In the previous Chap. 6, we have studied sound propagation in a viscothermal fluid, and shown that it can be put in the form of Maxwellian nonlocal equations. Moreover, we have suggested that this Maxwellian nonlocal form of the equations should apply quite generally to media that can be described in macroscopic averaged sense. "Maxwellian nonlocal" here, means that the corresponding nonlocal pattern of the equations is similar to that which describes, (with only symmetry and variances interchanges, in the most general version detailed in the Appendix), macroscopic electromagnetic wave propagation in a medium in the presence of temporal and spatial dispersion. We now fix our attention to the propagation of sound waves when the fluid is permeating a porous structure, as illustrated in Fig. 7.1. We will show that at a suitable macroscopic level it can again be put in the form of Maxwellian nonlocal

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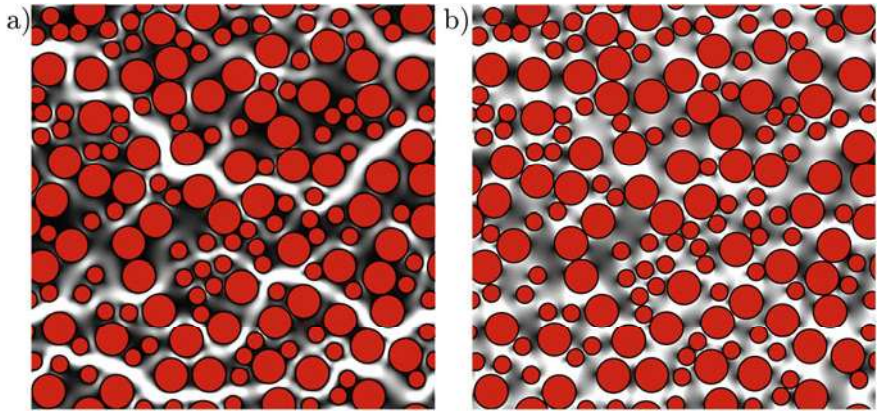

Fig. 7.1 Acoustic wave propagation in fluid-saturated rigid-framed porous media: **a** viscous-dominated low-frequency regime, **b** inertial-dominated high-frequency regime [Adapted with permission from calculations by N. Martys and E. J. Garboczi at the National Institute of Standards and Technology]

equations, and will indicate how to compute in principle the nonlocal operators. For the sake of simplicity here, we limit ourselves to materials with a, (connected or not), rigid structure, which do not move or deform during the propagation of sound in the saturating fluid. The geometry of the structure, which will determine that of the connected fluid domain, is left arbitrary except that it will appear homogeneous, (in the sense of ensemble- or volume-average), at some outer macroscopic level. In that case, and for the description of macroscopic compressional waves along a symmetry axis, we expect ending up with the same pattern of nonlocal macroscopic equations as seen in the text of Chap. 6, for the longitudinal waves in the fluid.[1]

What is well known in the literature [1], is the situation where the microstructure is simple enough to ensure that, because the macroscopic wavelengths are large relative to the dimensions of a representative volume, (scale separation), the fluid moves without divergence at the pore scale [2, 3]. This situation is generally described by applying the two-scale asymptotic method of homogenization [3, 4]. It leads to an asymptotic "local theory" description, very useful in practice. The type of geometries associated to this description is typically that represented above,[2] when the sizes of

[1] For materials with deformable structure, the expected pattern of nonlocal macroscopic equations will be the more general one seen in the Appendix of Chap. 6, but this is left for further studies.

[2] These figures are taken with permission from the electronic monograph https://concrete.nist.gov/monograph, Part III, General Random Porous Materials, Length scales relating the fluid permeability and electrical conductivity in random two-dimensional porous media, alinea: Comparison between electrical and fluid-flow problems. They represent the fluid flow, resp. in low-frequency (viscous Darcy) regime (see the velocity pattern $\boldsymbol{v}_0$ (7.189)–(7.192) in Appendix), and high-frequency (inviscid) regime (see the velocity pattern $\boldsymbol{v}_\infty$ (7.198)–(7.201) or $\boldsymbol{E}$ (7.202)–(7.205) in Appendix). The fluid flow is modelled as incompressible because long-wavelengths are assumed, *and* the microgeometry is "simple" (it does not involve widely different pore-scale lengths).

Fig. 7.2 Helmholtz resonators

grains and separations are very small compared to wavelengths. To be complete, and because of its interest in practice, we study this local theory in Appendix. We show that it is a simplification of the complete propagation problem, that makes abstraction of the spatial dispersion. For this reason, it fails, as soon as the geometry is complicated enough, (so-called metamaterials in which very different pore-scale lengths are simultaneously present), to permit that compressible movements can occur at the small-scale, in spite of scale separation. This is particularly the case when the solid includes Helmholtz resonators, as shown in Fig. 7.2, (necks and cavities have very different dimensions).

Indeed, as we will see, there is a direct link between resonances and spatial dispersion.

In what follows, by pursuing the ideas established in the Chap. 6, we develop explicitly the general principles of nonlocal theory, that will remain valid regardless of microgeometry. In comparison to the local description in Appendix, it is much more difficult to draw all the consequences from the nonlocal description. Much of this remains to be done: metamaterials will have much richer possible macroscopic behaviours than allowed by the conventional local description.

7.2 Statement of the Problem

To simplify the treatment, our medium is macroscopically homogeneous, thus boundless (see footnotes 2 and 23 in Chap. 6), and either, macroscopically isotropic, or the considered plane wave propagation occurs along a macroscopic symmetry axis x. To arrive at a precise definition of the macroscopic level, we assume that the geometric configuration is stationary random in some respects, however we will also consider the important but ambiguous case of periodic geometry. The material extends in the same stationary random way, or else periodic way, throughout all space, and is made of a solid part $\mathcal{V}_s$, shown in red colour in Fig. 7.1, (not necessarily connected), and a simply connected complementary fluid part $\mathcal{V}_f$ fully saturated with the viscothermal fluid, shown in bench of white and grey shades. The pore surface between solid and fluid is denoted $\partial\mathcal{V}$. The solid, mechanically and thermally inert, is supposed to remain perfectly still, either because it is heavy or rigid or both, or attached to external unmoving parts. Its thermal inertia also is assumed large, so that it remains at room temperature. Therefore, only the saturating fluid, which is compressible and can move, carries a disturbance. This disturbance is either due to longitudinal pressure waves coming from without, (no source in the medium), or directly created by a source-term of the type considered in the Chap. 6, (longitudinal

bulk force per unit fluid volume, $\boldsymbol{f} = -\partial\mathcal{P}$, $\mathcal{P} = \tilde{\mathcal{P}}e^{-i\omega t+ikx}$, where ω and k are independently chosen[3]), and acting on the fluid. At the solid-fluid contact surfaces $\partial\mathcal{V}$, the following boundary conditions apply: the mechanical inertia of the solid and the viscous nature of the fluid, (nonzero shear viscosity), result in a no-slip cancellation condition, $\boldsymbol{v} = \boldsymbol{0}$, for the velocity; likewise, the thermal inertia of the solid and the thermal nature of the fluid, (nonzero thermal conduction coefficient), result in the cancellation, $\tau = 0$, of the excess temperature. In general, in response to the field coming from without, or in response to the direct source action, it will appear in the fluid, complex and phase-lagged distributions of velocities, excess temperature, excess pressure, etc.

For small-amplitude motions, their governing equations will be those of the Navier–Stokes–Fourier model discussed in Chap. 6, completed by the mentioned boundary conditions, where we put in bracket the source term, as it may, or not, be present:

$$\partial\cdot\boldsymbol{v} + \frac{\partial b}{\partial t} = 0, \qquad \text{in } \mathcal{V}_f, \tag{7.1}$$

$$\rho_0\frac{\partial\boldsymbol{v}}{\partial t} = -\partial p + \eta\partial^2\boldsymbol{v} + \left(\frac{\eta}{3}+\zeta\right)\partial(\partial\cdot\boldsymbol{v}) + \left[\boldsymbol{f}\right], \qquad \text{in } \mathcal{V}_f, \tag{7.2}$$

$$\gamma\chi_0 p = b + \beta_0\tau, \qquad \text{in } \mathcal{V}_f, \tag{7.3}$$

$$\rho_0 c_P\frac{\partial\tau}{\partial t} = \beta_0 T_0\frac{\partial p}{\partial t} + \kappa\partial^2\tau, \qquad \text{in } \mathcal{V}_f, \tag{7.4}$$

and

$$\boldsymbol{v} = \boldsymbol{0}, \qquad \text{on } \partial\mathcal{V}, \tag{7.5}$$

$$\tau = 0, \qquad \text{on } \partial\mathcal{V}. \tag{7.6}$$

Note that in (7.2), while it is ultimately a matter of describing the propagation of macroscopic dilatation-compression waves in the material, we have kept the vortical viscous term $-\eta\partial\times\partial\times\boldsymbol{v}$. Indeed, whether the fluid motion is due to longitudinal pressure waves from the outside or created by a longitudinal bulk-force source term, viscous shearing movements must be taken into account, as they are automatically generated at the pore walls as a result of the application of boundary conditions (7.5).

In the Fig. 7.1, a low-frequency viscous-flow regime is represented on the left, (viscous term $\eta\partial^2\boldsymbol{v}$ much greater than inertial term $\rho_0\partial\boldsymbol{v}/\partial t$, i.e. viscous skin depth greater than pore sizes), and a high-frequency inertial-flow regime is on the right, (inertial term $\rho_0\partial\boldsymbol{v}/\partial t$ much greater than viscous term $\eta\partial^2\boldsymbol{v}$, i.e. viscous skin depth very small compared to pore sizes). In the represented domain $\mathcal{V}_f$, white colour corresponds to higher velocities, and grey colour to lower velocities. In general, the patterned, complex, and phase-lagged distributions of "high" and "low" values, will depend on both the time and the spatial variations of the macroscopic fields. The first dependence will be associated to temporal dispersion, the second to spatial

[3] Using this complex notation, we understand that $\mathcal{P} = \Re\left(\tilde{\mathcal{P}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}\right)$, omitting the real part symbol $\Re\,(\,)$.

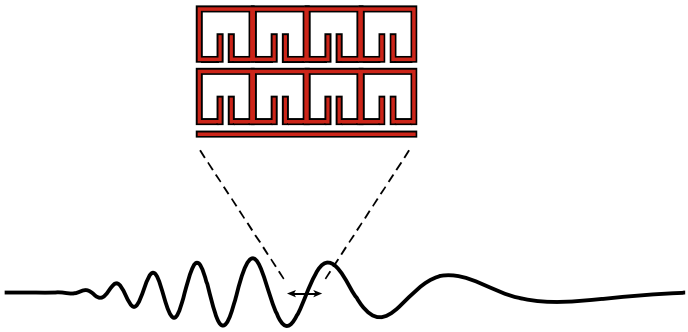

Fig. 7.3 Long-wavelength sound-field in a line of Helmholtz resonators

dispersion. Here on the two Figures 1, only the first dependence is present; the second, very feeble, is not even considered: the sound propagation is assumed to be described by the conventional approach, (local homogenization, see Appendix), which assumes *divergence-free fluid motion at the pore scale*. We mentioned that this local homogenization description falls in error in presence of Helmholtz resonators in the medium, and that this failure is linked to the failure to take spatial dispersion into account. As we saw in the Chap. 6, spatial dispersion means that the medium properties depend on the spatial variations of the external fields and it should be obvious that *this dependence cannot be described by making the simplification of locally incompressible fluid motion*.

The close relationship between the presence of resonances and the presence of spatial dispersion can be immediately understood graphically as follows. Consider a geometry with Helmholtz' resonators in succession and in which, we assume, waves having long wavelengths are established, as shown in Fig. 7.3. A Helmholtz resonance cannot occur, i.e. a significant flow cannot go to and fro an Helmholtz cavity, (red arrow in Fig. 7.4), without simultaneously being associated with significantly different flows in and out the unit cell, (black and blue arrows in Fig. 7.4), that is, without a concomitant spatial variation of the macroscopic fields. Therefore, as we cannot separate the occurrence of resonances from the occurrence of associated macroscopic spatial variations, a consistent complete treatment necessitates introducing the spatial dispersion.

The conventional description also falls in error when the wavelengths reduce and become comparable to the characteristic averaging lengths allowing to homogenize

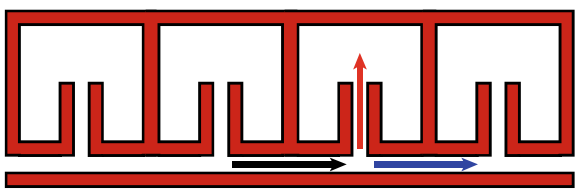

Fig. 7.4 Link between resonances and spatial dispersion

the medium. In this case also, the cornerstone simplification of the local theory, (incompressibility at the microstructure scale), is faulty, and taking spatial dispersion into account will have to be done to arrive at a consistent macroscopic theory capable to describe, in ensemble-averaged sense, the averaged fields, ("coherent fields" in multiple-scattering theory language).[4]

In short, we need a general nonlocal theory, as the local theory is insufficient at long wavelengths in resonant geometries, and in all geometries when wavelengths reduce. The generalization will have to be done outside the partly inconsistent framework of conventional two-scale homogenization. In what follows we detail how the Maxwellian nonlocal description we have introduced in the Chap. 6 in the fluid in absence of solid, and applied to longitudinal wave propagation, precisely furnishes the appropriate basis for such a generalization. This was anticipated in Appendix of Chap. 6. Here, we will explicitly show, for the case of compressional motions in fluid-saturated rigid framed materials, that the general macroscopic nonlocal theory can be put in the suggested Maxwellian nonlocal form. It will lead us to the formulation of a general Maxwellian description valid whatever microgeometries, frequencies, and wavelengths. First, in the next section and subsections, we must develop the concepts of macroscopic averages and establish their properties. While the theory is most clearly formulated for stationary random materials, in practice, however, the case of the periodical media is of interest. We will consider the two cases successively.

7.3 The Operations of Macroscopic Averaging

Remember that for the fluid we were not interested in the detailed behaviour of each molecule, but in an average macroscopic behaviour. Here, we must introduce a similar notion: we are not interested by what happens in the fluid at one particular pore-scale location, we are interested in the overall statistical macroscopic behaviour. To define it, we can use one of the two different conceptions of an average we have mentioned, Lorentz or Gibbs. The first is volume average when we are given one sample and there is scale separation between the outer scale of inhomogeneities in the medium, and the scale of considered wavelengths. The second is ensemble average when we are given an ensemble of samples, considered equivalent from a macroscopic point of view. In this case the wavelengths are not required to respect a scale separation condition, however, what is described is not what happens at a "macroscopic level" in a sample, but what happens on average in all samples, under the same conditions of excitation.

[4] It could be thought that when the wavelengths reduce, spatial dispersion is automatically modelled in simple microgeometries by means of the higher-order terms of the two-scale asymptotic homogenization method. But this expectation is illusory. Partially inconsequent, this method will not give meaningful higher-order terms.

Let us start with the case of materials which are in some respect stationary random and examine in more details the concept of calculating a Lorentz or Gibbs macroscopic mean.

7.3.1 The Well-Defined Case of Stationary-Random Media

The fields $a(t, \boldsymbol{x})$ we want to average concern what happens in the fluid phase; in the inert, immobile solid phase domain, these fields are extended to be zero. To automatically ensure it, we introduce the fluid domain indicator function $I(\boldsymbol{x})$ that takes the value 1 in the fluid and the value 0 in the solid. It will be an inherent part of all fields $a(t, \boldsymbol{x}) \equiv I(\boldsymbol{x})a(t, \boldsymbol{x})$ we consider.

7.3.1.1 The Definitions of Lorentz's and Gibbs' Averages

In Lorentz's averaging conception we are dealing with one sample; we view its indicator function I as a stationary random function of position. For I, we may think of some function devoid of a preferred origin in space, that may repeat itself but without true periodicity and with some inherent randomness. In Gibbs' averaging conception, we are given infinitely many such stationary random samples, or realizations ϖ of the medium, taken from a probability space Ω, the ensemble of which defines the homogeneous macroscopic medium in question. As an example, we can imagine that, viewing the Lorentz sample from infinitely many different origins in space, would produce a suitable Gibbs' ensemble. The fluid domain indicator function I is noted, respectively:

Lorentz	Gibbs
$I_0(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \mathcal{V}_f, \\ 0 & \boldsymbol{x} \in \mathcal{V}_s, \end{cases}$	$\forall \varpi \in \Omega,\ I(\boldsymbol{x}, \varpi) = \begin{cases} 1 & \boldsymbol{x} \in \mathcal{V}_f(\varpi), \\ 0 & \boldsymbol{x} \in \mathcal{V}_s(\varpi), \end{cases}$

(7.7)

(we put index 0 in Lorentz's case to insist that we have only one configuration). We first discuss Lorentz's average.

Lorentz's Average

The Lorentz volume-average $\langle a \rangle(t, \boldsymbol{x})$ of a given field $a(t, \boldsymbol{x})$ such as a fluid velocity component, the condensation, the pressure, etc., is best defined in Russakoff's manner [5] by convolution with a smooth and finite-width test function $w_{L_h}(\boldsymbol{x})$ of characteristic extent L_h, (an homogenization length, giving the size of a representative elementary volume or REV), such as e.g. $w_{L_h}(\boldsymbol{x}) = (\pi L_h^2)^{-3/2} e^{-\boldsymbol{x}^2/L_h^2}$, centred at the considered location, ($\boldsymbol{x} = \boldsymbol{0}$ in the given function), and normalized to one upon

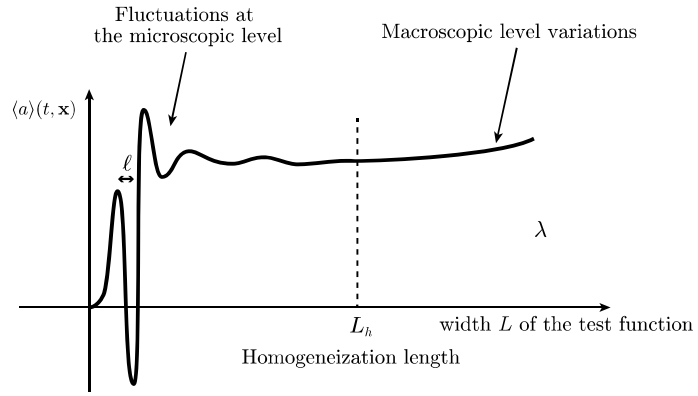

Fig. 7.5 Homogenization length for Lorentz's average

integrating over whole space. The shape of the test function is not essential, and a natural choice is that of a plateau function as discussed and plotted in Jackson [6].

The average is then defined as the following convolution product:

$$A(t, \boldsymbol{x}) = \langle a \rangle(t, \boldsymbol{x}) = \left(w_{L_h} * a\right)(t, \boldsymbol{x}) = \int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x} - \boldsymbol{x}') a(t, \boldsymbol{x}'),$$
$$\int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x} - \boldsymbol{x}') = 1. \tag{7.8}$$

Recall that the indicator function $I_0(\boldsymbol{x})$ is present in the field $a(t, \boldsymbol{x})$. When convenient, (see e.g. the calculation in (7.16)), it can be explicitly noted as an additional factor $I_0(\boldsymbol{x}')$, in the integral (7.8.1), with no change on the result of the integration.

If we take w_L with arbitrary L for the test function, the way the average varies in function of the size L of the averaging window, is schematized in Fig. 7.5.

The mean $\langle a \rangle(t, \boldsymbol{x})$ first presents fluctuations when L is on the order of typical small-scale distances ℓ over which the fields vary. It then smooths out and reaches a plateau when $L \sim L_h$, with $\ell \ll L_h \ll \lambda$, where λ is an estimate of macroscopic wavelengths. Finally, because of the macroscopic variations, it starts to again vary when L is further increased to become comparable to λ. The length L_h defines the suitable "homogenization" length or appropriate size of a REV. As it is assumed sufficiently large compared to ℓ, so that the average tends to a definite limit, and sufficiently small compared to typical wavelengths λ, so that the macroscopic variations of the fields are not polluting the result, the introduction of this homogenization length L_h supposes a wide scale separation $\ell \lll \lambda$.

As we use the total volume normalization (7.8.2), the mean of the characteristic function I is just the porosity ϕ, (fluid volume per unit total—fluid plus solid—volume):

$$\langle I_0 \rangle = \int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x} - \boldsymbol{x}') I_0(\boldsymbol{x}') = \phi. \tag{7.9}$$

Because of this normalization, the definition (7.8) of the averaging symbol $\langle\,\rangle$ interprets as a "total volume" average: porosity times the mean value in the fluid phase. The mean value in the fluid phase will be denoted by an index f putted on the mean symbol, so that for example, we will have

$$\langle \boldsymbol{v} \rangle = \phi \langle \boldsymbol{v} \rangle_f. \tag{7.10}$$

The mean operation in the fluid phase $\langle\,\rangle_f$ can be defined by a convolution as noted in (7.8.1), but with the following different normalization of the test function:

$$\int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x} - \boldsymbol{x}') = 1/\phi, \tag{7.11}$$

i.e. an extra factor $1/\phi$ is put on the test function, so that as wanted

$$\langle I_0 \rangle_f = 1. \tag{7.12}$$

Gibbs' Average

Let us now consider the Gibbs-average $\langle a \rangle(t, \boldsymbol{x})$ of a given field $a(t, \boldsymbol{x}; \varpi)$ such as a fluid velocity component, the condensation, the pressure, etc., which is a function of $t, \boldsymbol{x}$ in each realization ϖ, and is extended to zero in $\mathcal{V}_s(\varpi)$. This average is simply the expectation value $\overline{a(t, \boldsymbol{x}; \varpi)}$ of the given field a, at time t, position $\boldsymbol{x}$, over all realizations ϖ in Ω:

$$A(t, \boldsymbol{x}) = \langle a \rangle(t, \boldsymbol{x}) = \overline{a(t, \boldsymbol{x}; \varpi)}. \tag{7.13}$$

Because the field $a(t, \boldsymbol{x}; \varpi)$ is extended to zero in the solid, this average over realizations (7.13) can be seen as a "total volume" average. As an example, looking at the values of the field $I(\boldsymbol{x}; \varpi)$ at a given position $\boldsymbol{x}$, and taking the realization ϖ at random, we have a probability ϕ to be in the fluid, and thus find the value 1, and a probability $1 - \phi$ to be in the solid, and thus find the value 0. Therefore

$$\langle I \rangle = \overline{I(\boldsymbol{x}; \varpi)} = \phi. \tag{7.14}$$

Also note that, to compute the Gibbs expectation values (7.13), only the fields $a(t, \boldsymbol{x}; \varpi)$ at a single (pointlike) position $\boldsymbol{x}$ in the different realizations ϖ are involved. When performing the average (7.13), we a priori deal with discontinuous functions as the functions $I(\boldsymbol{x}, \varpi)$ are discontinuous over $\boldsymbol{x}$, (for example the functions associated to the condensation or the excess pressure, extended to zero in the solid, are discontinuous at the pore-wall surfaces; velocity and excess temperature are not discontinuous because they vanish on $\partial\mathcal{V}(\varpi)$). This discontinuous nature of the fields brings unessential mathematical difficulties when willing

to see how their Gibbs' averages vary in space. To avoid this, it is convenient to replace single points by infinitesimal volumes, or more precisely, replace the possibly discontinuous set of values $a(t, \boldsymbol{x}; \varpi)$ by the set of continuous mean values, $\int d\boldsymbol{x}' w_{dL}(\boldsymbol{x} - \boldsymbol{x}') a(t, \boldsymbol{x}'; \varpi)$, performed in a "physically infinitesimal" neighbourhood of $\boldsymbol{x}$, and where $I(\boldsymbol{x}'; \varpi)$ is inherent part of the field $a(t, \boldsymbol{x}'; \varpi)$. We consider dL as a much smaller length than all the lengths involved in defining geometry such as the radii of curvatures associated with pore wall surface $\partial\mathcal{V}$, and use the normalization $\int d\boldsymbol{x}' w_{dL}(\boldsymbol{x} - \boldsymbol{x}') = 1$, (at $dL = 0$, w_{dL} will be the delta-function). Following this purely technical refinement, we redefine more conveniently the Gibbs expectation value $\langle a \rangle(t, \boldsymbol{x})$ (7.13) as:

$$A(t, \boldsymbol{x}) = \langle a \rangle(t, \boldsymbol{x}) = \overline{\int d\boldsymbol{x}' w_{dL}(\boldsymbol{x} - \boldsymbol{x}') a(t, \boldsymbol{x}'; \varpi)}. \tag{7.15}$$

Finally, also willing, with Gibbs' conception, to dispose of a mean operation $\langle . \rangle_f$ having "fluid volume" instead of "total volume" normalization, we define Gibbs' fluid volume average such that $\langle a \rangle_f(t, \boldsymbol{x})$ is given by (7.15), but with normalization $\int d\boldsymbol{x}' w_{dL}(\boldsymbol{x} - \boldsymbol{x}') = 1/\phi$ of the test function w_{dL}, (at $dL = 0$, it gives for the test function the delta-function $\times\ 1/\phi$).

7.3.1.2 Commutation Relations

We now precise how the above Lorentz's and Gibbs' averaging symbols $\langle \cdot \rangle(\boldsymbol{x})$, (7.8) and (7.13) refined in the form (7.15), behave with respect to the spatial derivative symbol $\partial = \partial/\partial\boldsymbol{x}$. We show that, in general, for fields nonzero on the pore walls, the averaging do not commute with the spatial derivative.

Lorentz's Average

With Lorentz's average, the spatial derivative symbol refers to the variation of the central position $\boldsymbol{x}$ of the test function. Previously in electromagnetic theory, because the fields extended all over space, there was direct commutation between the Lorentz-Russakoff average and the spatial derivative symbol, (see the Chap. 6, Sect. 11.2, (79)). Here, because the fields are set to zero in the solid, it is convenient to substitute them explicitly, in the definition (7.8), in the form of, $a(t, \boldsymbol{x}') = I_0(\boldsymbol{x}') a(t, \boldsymbol{x}')$. As detailed below an extra surface term then generally appears in the commutation relation between average and derivative symbol:

$$\begin{aligned}\partial A(t,\boldsymbol{x}) = \frac{\partial}{\partial \boldsymbol{x}}\langle a\rangle(t,\boldsymbol{x}) &= \int d\boldsymbol{x}' \frac{\partial}{\partial \boldsymbol{x}} w_{L_h}(\boldsymbol{x}-\boldsymbol{x}') I_0(\boldsymbol{x}') a(t,\boldsymbol{x}'),\\ &= -\int d\boldsymbol{x}' \frac{\partial}{\partial \boldsymbol{x}'}\left[w_{L_h}(\boldsymbol{x}-\boldsymbol{x}')\right] I_0(\boldsymbol{x}') a(t,\boldsymbol{x}'),\\ &= -\int d\boldsymbol{x}' \frac{\partial}{\partial \boldsymbol{x}'}\left[w_{L_h}(\boldsymbol{x}-\boldsymbol{x}') I_0(\boldsymbol{x}') a(t,\boldsymbol{x}')\right]\\ &\quad + \int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x}-\boldsymbol{x}') \frac{\partial}{\partial \boldsymbol{x}'}\left[I_0(\boldsymbol{x}')\right] a(t,\boldsymbol{x}')\\ &\quad + \int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x}-\boldsymbol{x}') I_0(\boldsymbol{x}') \frac{\partial}{\partial \boldsymbol{x}'} a(t,\boldsymbol{x}').\end{aligned} \tag{7.16}$$

The integral of the total derivative term vanishes because the quantity inside brackets contains the test function which quickly tends to zero at infinity. The gradient $\partial\left[I_0(\boldsymbol{x}')\right]/\partial\boldsymbol{x}'$ is a Dirac delta distribution, supported by the pore surface $\partial\mathcal{V}$, and directed along $-\hat{\boldsymbol{n}}(\boldsymbol{x}')$, where $\hat{\boldsymbol{n}}(\boldsymbol{x}')$ is the outward normal to the fluid region at position $\boldsymbol{x}'$ on this pore surface. The integral containing it, expresses as a pore-surface integral. The last integral is the definition of the Lorentz mean of the derived field, (which is extended to zero in the solid). Therefore we find:

$$\begin{aligned}\partial A(t,\boldsymbol{x}) =& \partial\langle a\rangle(t,\boldsymbol{x}) = \langle \partial a\rangle(t,\boldsymbol{x}) + \langle a\partial I_0\rangle(t,\boldsymbol{x}),\\ &= \int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x}-\boldsymbol{x}')\partial_{\boldsymbol{x}'} a(t,\boldsymbol{x}') - \int_{\partial\mathcal{V}} d\boldsymbol{x}' a(t,\boldsymbol{x}')\hat{\boldsymbol{n}}(\boldsymbol{x}') w_{L_h}(\boldsymbol{x}-\boldsymbol{x}').\end{aligned} \tag{7.17}$$

This relation is sometimes called the "averaging theorem". Note that, as the material is assumed macroscopically homogeneous, we have $\partial\phi = \mathbf{0}$, that is

$$\int_{\partial\mathcal{V}} d\boldsymbol{x}' \hat{\boldsymbol{n}}(\boldsymbol{x}') w_L(\boldsymbol{x}-\boldsymbol{x}') = \mathbf{0}. \tag{7.18}$$

This identity (7.18) will be useful later on. It can be obtained either by applying the averaging theorem (7.17) for $a = 1$ in the fluid, or else, by directly taking the derivative of (7.9):

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{x}}\phi &= \int d\boldsymbol{x}' \frac{\partial}{\partial \boldsymbol{x}} w_{L_h}(\boldsymbol{x}-\boldsymbol{x}') I_0(\boldsymbol{x}') = -\int d\boldsymbol{x}' \frac{\partial}{\partial \boldsymbol{x}'}\left[w_{L_h}(\boldsymbol{x}-\boldsymbol{x}')\right] I_0(\boldsymbol{x}'),\\ &= -\int d\boldsymbol{x}' \frac{\partial}{\partial \boldsymbol{x}'}\left[w_{L_h}(\boldsymbol{x}-\boldsymbol{x}') I_0(\boldsymbol{x}')\right] + \int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x}-\boldsymbol{x}') \frac{\partial}{\partial \boldsymbol{x}'} I_0(\boldsymbol{x}'),\\ &= -\int_{\partial\mathcal{V}} d\boldsymbol{x}' \hat{\boldsymbol{n}}(\boldsymbol{x}') w_L(\boldsymbol{x}-\boldsymbol{x}') = \mathbf{0}.\end{aligned}$$

Gibbs' Average

With Gibbs' average (7.13) refined in the form (7.15), the calculation of the spatial derivative of macroscopic averages is performed in same manner. Starting with

$$\partial A(t,\boldsymbol{x}) = \partial\langle a\rangle(t,\boldsymbol{x}) = \overline{\int d\boldsymbol{x}' \frac{\partial}{\partial \boldsymbol{x}} w_{dL}(\boldsymbol{x}-\boldsymbol{x}') I(\boldsymbol{x}';\varpi) a(t,\boldsymbol{x}';\varpi)},$$

(the I-function factor is made apparent for convenience in the calculation), using $\partial w_{dL}(\boldsymbol{x}-\boldsymbol{x}')/\partial \boldsymbol{x} = -\partial w_{dL}(\boldsymbol{x}-\boldsymbol{x}')/\partial \boldsymbol{x}'$, and integrating by parts, we obtain:

$$\begin{aligned} \partial A(t,\boldsymbol{x}) =& \partial\langle a\rangle(t,\boldsymbol{x}) = \langle \partial a\rangle(t,\boldsymbol{x}) + \langle a\partial I\rangle(t,\boldsymbol{x}), \\ =& \overline{\int d\boldsymbol{x}' w_{dL}(\boldsymbol{x}-\boldsymbol{x}')\partial a(t,\boldsymbol{x}';\varpi)} \\ & - \overline{\int_{\partial\mathcal{V}(\varpi)} d\boldsymbol{x}' a(t,\boldsymbol{x}';\varpi)\hat{\boldsymbol{n}}(\boldsymbol{x}';\varpi) w_{dL}(\boldsymbol{x}-\boldsymbol{x}')}. \end{aligned} \tag{7.19}$$

Discussion

As in Lorentz's or Gibbs' conceptions the symbols $\langle\cdot\rangle$ and ∂ obey the relations (7.17) or (7.19), they do not commute except if the surface integral terms vanish. To take different examples, the gradient of a mean field such as $B \equiv \langle b\rangle$ or $\langle p\rangle$, is, in general, not equal to the mean of the gradient, i.e.

$$\partial B \equiv \partial\langle b\rangle \neq \langle \partial b\rangle, \qquad \text{or} \qquad \partial\langle p\rangle \neq \langle \partial p\rangle. \tag{7.20}$$

Because, however, the velocity vanishes on the solid-fluid interface, (boundary condition (7.5)), the divergence of the mean velocity is always automatically the same as the mean of the velocity divergence:

$$\partial \cdot \langle \boldsymbol{v}\rangle = \langle \partial \cdot \boldsymbol{v}\rangle. \tag{7.21}$$

This remains true for an inviscid fluid: the normal component of the velocity vanishes on the fluid-solid interface as long as the solid is impenetrable, which ensures cancellation of the surface term.

In addition, there is a generic class of fields a, for which, as the surface terms happen to vanish, (while a is nonzero on the interface), the average operation commute with the derivative operation:

$$\langle \partial a\rangle = \partial\langle a\rangle. \tag{7.22}$$

This class is that of fields that would vary, (except for their systematic extension to zero in the solid), like "macroscopic fields" or "external fields". For example, the source-term field $\boldsymbol{f}$ in (7.2), or its associated potential $\mathcal{P}$, or any field a that is

the macroscopic mean, $a \equiv \langle b \rangle$, of some response field b in the pore space, (and is extended to zero in the solid), are this same type of fields, verifying (7.22). Let us show it, successively using Lorentz's or Gibbs' averaging conception.

With Lorentz's average, scale separation needs to be assumed. Therefore, the fields which, (apart from their extension to zero in the solid), vary only at the macroscopic scale, present almost linear variations in the fluid within an averaging volume. In first approximation, they write, (in the fluid): $a(t, \boldsymbol{x}') = a_0 + a_1 \hat{\boldsymbol{n}}_0 \cdot \boldsymbol{x}'$, with $\hat{\boldsymbol{n}}_0$ the unit direction of the gradient, and $a_{0,1}$ two constants. As the medium is stationary random, it can then be shown that such linear variations automatically cancel the surface term:

$$\int_{\partial \mathcal{V}} d\boldsymbol{x}' a(t, \boldsymbol{x}') \hat{\boldsymbol{n}}(\boldsymbol{x}') w_{L_h}(\boldsymbol{x} - \boldsymbol{x}') = \\ = \int_{\partial \mathcal{V}} d\boldsymbol{x}' \left[a_0 + a_1 \hat{\boldsymbol{n}}_0 \cdot \boldsymbol{x}' \right] \hat{\boldsymbol{n}}(\boldsymbol{x}') w_{L_h}(\boldsymbol{x} - \boldsymbol{x}') = \boldsymbol{0},$$

and thus, (7.22) is satisfied. Indeed by (7.18) the first integral identically vanishes and it remains to show the additional identity

$$\int_{\partial \mathcal{V}} d\boldsymbol{x}' \left[\hat{\boldsymbol{n}}_0 \cdot \boldsymbol{x}' \right] \hat{\boldsymbol{n}}(\boldsymbol{x}') w_{L_h}(\boldsymbol{x} - \boldsymbol{x}') = \boldsymbol{0}. \tag{7.23}$$

By symmetry reasons, the mean of a purely linear variation should be a linear variation, modified by the porosity factor to account for the total volume normalization:

$$\langle a_0 + a_1 \hat{\boldsymbol{n}}_0 \cdot \boldsymbol{x}' \rangle(\boldsymbol{x}) = \phi \left(a_0 + a_1 \hat{\boldsymbol{n}}_0 \cdot \boldsymbol{x} \right).$$

Taking the spatial derivative this gives

$$\partial \langle \hat{\boldsymbol{n}}_0 \cdot \boldsymbol{x}' \rangle(\boldsymbol{x}) = \phi \hat{\boldsymbol{n}}_0.$$

But we also evidently have, by averaging the identity $\partial_{\boldsymbol{x}'}(\boldsymbol{n}_0 \cdot \boldsymbol{x}') = \boldsymbol{n}_0$,

$$\langle \partial_{\boldsymbol{x}'}(\boldsymbol{n}_0 \cdot \boldsymbol{x}') \rangle(\boldsymbol{x}) = \langle \boldsymbol{n}_0 \rangle = \phi \boldsymbol{n}_0.$$

Therefore, the averaging theorem (7.17) applied to the field $a = \boldsymbol{n}_0 \cdot \boldsymbol{x}$, yields the identity (7.23). This completes the proof and justifies (7.22) in Lorentz's averaging conception, for an arbitrary "external" field a, i.e. a field having only "long-wavelength" variations, (except for its extension to zero in solid).

With Gibbs' average, saying that the field a varies like "macroscopic" fields, means that, $a(t, \boldsymbol{x}; \varpi) \equiv I(\boldsymbol{x}; \varpi) a(t, \boldsymbol{x})$. In all realizations this field in the fluid is given by a single $a(t, \boldsymbol{x})$, and the latter writes as a superposition of exponentials $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ with associated Fourier coefficients independent of ϖ. The Gibbs surface terms $\overline{\int_{\partial \mathcal{V}(\varpi)} d\boldsymbol{x}' a(t, \boldsymbol{x}'; \varpi) \hat{\boldsymbol{n}}(\boldsymbol{x}'; \varpi) w_{dL}(\boldsymbol{x} - \boldsymbol{x}')}$ will have contributions

$\int_{\partial\mathcal{V}(\varpi)} d\boldsymbol{x}' e^{i\boldsymbol{k}\cdot\boldsymbol{x}'} \hat{\boldsymbol{n}}(\boldsymbol{x}';\varpi) w_{dL}(\boldsymbol{x}-\boldsymbol{x}')$ in front of the Fourier coefficients and these vanish whatever the value of $\boldsymbol{k}$. To see this, we note that, realization after realization, the exponential variations $e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}$ automatically become slow variations in the limit $dL \to 0$. They thus factor out as the central value $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ in front of an integral $\int_{\partial\mathcal{V}(\varpi)} d\boldsymbol{x}' \hat{\boldsymbol{n}}(\boldsymbol{x}';\varpi) w_{dL}(\boldsymbol{x}-\boldsymbol{x}')$ which is identically zero. Indeed, at any given fixed value dL, it cancels out when making the ensemble averaging, because, as small as can be dL, the mean of the integral over realizations becomes, (when the number of realizations increases indefinitely), representative of the integral performed in the full medium, (that is, made in one sample with a value of L equal to homogenization length L_h); but this integral is the null integral (7.18), hence the cancellation of Gibbs' surface terms. It justifies (7.22) in Gibbs' conception, for an arbitrary "macroscopic" or "external" field a.

Now that the notions of Lorentz's and Gibbs' averages and some of their properties have been precised, let us return to the problem (7.1)–(7.6), alternatively stated, solved, and averaged, using Lorentz's and Gibbs' conceptions.

7.3.1.3 Ergodic Equivalences

We first argue that, at long wavelengths, the two conceptions can be used interchangeably, manifesting a property of ergodicity.

By the principle of superposition it suffices to consider and solve the problem of the response of the fluid to a longitudinal force, $\boldsymbol{f} = -\partial\mathcal{P}$, given in the form of a single plane wave variation, $\mathcal{P} = \tilde{\mathcal{P}} e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$, with arbitrary ω, and $\boldsymbol{k}$, respecting long-wavelength condition, $\Re(k_i L_h), \Im(k_i L_h) \ll 1$, $(i = 1, 2, 3)$, a prerequisite to the definition of Lorentz's average.

In the Lorentz formulation, we are given a single sample, stationary random. We call it the reference sample ϖ_0 and denote its indicator function $I(\boldsymbol{x};\varpi_0) = I_0(\boldsymbol{x})$; it determines the fluid domain $\mathcal{V}_f$, the solid-fluid interface $\partial\mathcal{V}$, and the response solution $\boldsymbol{v}(t,\boldsymbol{x};\varpi_0) = \boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x};\varpi_0)e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$, $\quad b(t,\boldsymbol{x};\varpi_0) = b(\omega,\boldsymbol{k},\boldsymbol{x};\varpi_0)e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$, etc., of the motion equations (7.1)–(7.6), taken with the above source term. An important point is that the amplitudes in front of the exponentials, proportional to the source amplitude constant $\tilde{\mathcal{P}}$, are uniquely determined, bounded functions of $\boldsymbol{x}$. These bounded functions are also stationary random functions.

In the Gibbs formulation, we are given an ensemble of stationary random samples $\varpi \in \Omega$, defined by their indicator functions $I(\boldsymbol{x};\varpi)$. These determine a collection of fluid domains $\mathcal{V}_f(\varpi)$, solid-fluid interfaces $\partial\mathcal{V}(\varpi)$, and response solutions $\boldsymbol{v}(t,\boldsymbol{x};\varpi) = \boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x};\varpi)e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$, $b(t,\boldsymbol{x};\varpi) = b(\omega,\boldsymbol{k},\boldsymbol{x};\varpi)e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$, etc., of the motion equations (7.1)–(7.6), with source term as above. The amplitudes in front of the exponentials, proportional to $\tilde{\mathcal{P}}$, are unique stationary random and bounded functions of $\boldsymbol{x}$.

In the macroscopic theory we are concerned with macroscopic averages of the fields or product of fields, such as $\langle\boldsymbol{v}\rangle$, $\langle b\rangle$ or $\langle p\boldsymbol{v}\rangle$. As an example of the general principle we consider the mean $\langle\boldsymbol{v}\rangle$.

First consider Lorentz's formulation. We have

$$\begin{aligned}\boldsymbol{V}(t,\boldsymbol{x}) = \langle \boldsymbol{v}\rangle(t,\boldsymbol{x}) &= \int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{v}(t,\boldsymbol{x}';\varpi_0),\\ &= \int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x}';\varpi_0)e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}'}.\end{aligned}$$

Now, as a scale separation (long-wavelength limit) is imposed, the exponential $e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}$, which varies slowly in the averaging $\boldsymbol{x}'$ region defined by the test function $w_{L_h}(\boldsymbol{x}-\boldsymbol{x}')$, can be nearly replaced by its central value $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ and extracted from the integral. Hence we will have, nearly

$$\begin{aligned}\boldsymbol{V}(t,\boldsymbol{x}) =& e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}\int d\boldsymbol{x}' w_{L_h}(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x}';\varpi_0),\\ =& e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}\langle \boldsymbol{v}\rangle_{Lor},\end{aligned}$$

where the index Lor on the average $\langle \boldsymbol{v}\rangle$ indicates that it is Lorentz's average.

Next consider Gibbs' formulation. The mean $\langle \boldsymbol{v}\rangle$ writes

$$\begin{aligned}\boldsymbol{V}(t,\boldsymbol{x}) = \langle \boldsymbol{v}\rangle(t,\boldsymbol{x}) =& \overline{\int d\boldsymbol{x}' w_{dL}(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{v}(t,\boldsymbol{x}';\varpi)},\\ =& \overline{\int d\boldsymbol{x}' w_{dL}(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x}';\varpi)e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}'}}.\end{aligned}$$

In the limit $dL\to 0$, the exponential automatically factors out as the central value $e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$ whatever $\boldsymbol{k}$ (long or short wavelengths) and we find, this time without approximation

$$\begin{aligned}\boldsymbol{V}(t,\boldsymbol{x}) &= e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}\overline{\int d\boldsymbol{x}' w_{dL}(\boldsymbol{x}-\boldsymbol{x}')\boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x}';\varpi)},\\ &= e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}\langle \boldsymbol{v}\rangle_{Gib},\end{aligned}$$

where the index Gib on the average $\langle \boldsymbol{v}\rangle$ reminds that it is Gibbs' average.

Now, invoking the stationary random character of the geometries, it should be the same thing to perform the ensemble-average $\langle \boldsymbol{v}\rangle_{Gib}$ or to perform the volume-average $\langle \boldsymbol{v}\rangle_{Lor}$, because, as small as can be dL, the mean of the integral over realizations becomes representative of the full-medium Lorentz integral, when the number of realizations increases indefinitely. It means that, at long-wavelengths, the Gibbs ensemble average will be equivalent to the Lorentz average in one realization; the two will be used interchangeably.

We can go a step further. While Lorentz's method of averaging loses its physical meaning when the wavelengths reduce sufficiently, the Gibbs method remains feasible. As mentioned, in this case the macroscopic theory will not describe what happens in one sample, but what happens on average in the ensemble of realizations.

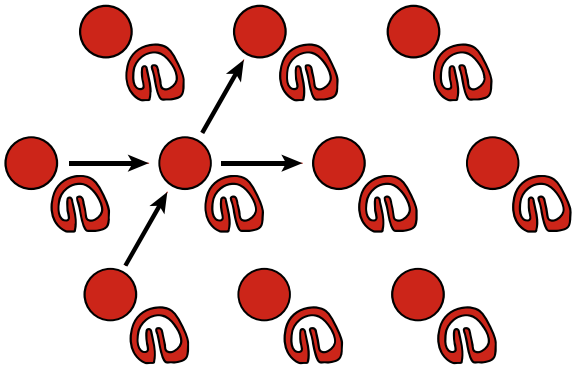

Fig. 7.6 Periodic sample ϖ_0

Now, whatever $\boldsymbol{k}$ is long- or short-wavelengths, for the fields created by the potential $\mathcal{P} = \tilde{\mathcal{P}} e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$, we always have the right to write Gibbs' average as

$$\boldsymbol{V}(t,\boldsymbol{x}) = \langle \boldsymbol{v} \rangle(t,\boldsymbol{x}) = e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \boldsymbol{v} \rangle_{Gib} = e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \boldsymbol{v} \rangle_{Lor}. \tag{7.24}$$

because the ergodicity property $\langle \boldsymbol{v} \rangle_{Gib} = \langle \boldsymbol{v} \rangle_{Lor}$ seen above is not linked to the size of wavelengths but to the stationary random nature of the geometry. Therefore, in general, when making Gibbs' average, we can always choose to use Lorentz's average to perform the average of the tilde part of the fields.[5]

7.3.2 *The Ambiguous Case of Periodic Media*

We now consider the case of periodic materials as well. Obviously, the idea of a periodicity clashes with the preceding idea of a macroscopic homogeneity obtained through underlying uniform randomness. In this context, similar averaging operations and properties as mentioned in the previous section, can now only be obtained *cum grano salis*. Because of the periodicity, ambiguities will appear in the definitions, which will complicate the presentation (Fig. 7.6).[6]

[5] By "tilde part", we mean, to take the example of the velocity field which writes $\boldsymbol{v}(t,\boldsymbol{x};\varpi) = \boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x};\varpi)e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$, the amplitude in the right-hand side, in front of the exponential; usually it is denoted with a tilde which we suppress here for the simplicity of the notation. In the equality $\langle \boldsymbol{v} \rangle_{Gib} = \langle \boldsymbol{v} \rangle_{Lor}$, the $\boldsymbol{v}$ are the tilde parts. At short wavelengths $\langle \boldsymbol{v} \rangle_{Gib}(t,\boldsymbol{x})$ and $\langle \boldsymbol{v} \rangle_{Lor}(t,\boldsymbol{x})$, with $\boldsymbol{v}$ the original space velocities, are not the same; the last average $\langle \boldsymbol{v} \rangle_{Lor}(t,\boldsymbol{x})$ in general will have no precise utility or significance.

[6] Ultimately, however, the theory will have to deal with media having finite dimensions. In this case, true periodicity will be lost, which will hopefully help mitigate the mentioned ambiguities.

7.3.2.1 The Definitions of Lorentz's and Gibbs' Averages

In Lorentz's averaging conception, we are, (as we make abstraction of finite dimensions), dealing with one indefinite sample ϖ_0 of the medium, which is now assumed periodic. Thus its indicator function $I(\boldsymbol{x}, \varpi_0) = I_0(\boldsymbol{x})$ verifies:

$$I_0\left(\boldsymbol{x} + \sum_{i=1}^{3} m_i \boldsymbol{a}_i\right) = I_0(\boldsymbol{x}), \quad \forall \boldsymbol{x}, \quad \forall m_{1,2,3} \in \mathbb{Z},$$

with $\boldsymbol{a}_{1,2,3}$, one of the possible equivalent choices of three primitive translation vectors, leaving unchanged the sample. Also willing to introduce a Gibbs' averaging conception, we would like to have infinitely many periodic samples, or realizations ϖ of the "same macroscopic medium", taken from a probability space Ω, the ensemble of which defines the homogeneous macroscopic medium in question. A natural idea that comes to mind is that at the macroscopic level we will not pay attention to the underlying positioning in space of the crystal, so that the "different realizations ϖ of the same macroscopic medium" can be defined here as produced by the infinitely many random translations of the original realization ϖ_0. To characterize them we introduce a random translation vector $\boldsymbol{X}$

$$\boldsymbol{X} = \sum_{i=1}^{3} \xi_i \boldsymbol{a}_i,$$

with each of the ξ_i, a random variable uniformly distributed in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Accordingly, the fluid domain periodic indicator function will be:

Lorentz	Gibbs	
		(7.25)
$I_0(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \mathcal{V}_f \\ 0 & \boldsymbol{x} \in \mathcal{V}_s \end{cases}$	$\forall \varpi \in \Omega, I(\boldsymbol{x}, \varpi) = \begin{cases} 1 & \boldsymbol{x} \in \mathcal{V}_f(\varpi) \\ 0 & \boldsymbol{x} \in \mathcal{V}_s(\varpi) \end{cases}$	(7.26)
$I_0(\boldsymbol{x}) = I_0(\boldsymbol{x} + \sum_{i=1}^{3} m_i \boldsymbol{a}_i)$	$I(\boldsymbol{x}, \varpi) = I_0(\boldsymbol{x} - \boldsymbol{X}), \quad \boldsymbol{X} = \sum_{i=1}^{3} \xi_i \boldsymbol{a}_i$	(7.27)
$m_{1,2,3} \in \mathbb{Z}$	$\xi_{1,2,3} \in \left[-\frac{1}{2}, \frac{1}{2}\right]$	(7.28)

Lorentz's Average

Because the underlying geometry is periodic, we can define it by giving $I_0(\boldsymbol{x})$ in restricted regions, and next, complete the rest by duplication: we can give $I_0(\boldsymbol{x})$ in the irreducible region

$$\Delta_{111} = \left\{ \boldsymbol{x} \mid \boldsymbol{x} = \sum_{i=1}^{3} \xi_i \boldsymbol{a}_i, \xi_{1,2,3} \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}, \tag{7.29}$$

and complete the rest by requiring the periodicity, $I_0(\boldsymbol{x}) = I_0(\boldsymbol{x} + \sum_{i=1}^{3} m_i \boldsymbol{a}_i)$, $m_{1,2,3} \in \mathbb{Z}$. We can also define it by giving $I_0(\boldsymbol{x})$ in the more extended non-irreducible region

$$\Delta_{M_1 M_2 M_3} = \left\{ \boldsymbol{x} \mid \boldsymbol{x} = \sum_{i=1}^{3} \xi_i M_i \boldsymbol{a}_i, \xi_i \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}, \quad M_{1,2,3} \in \mathbb{N}_{\neq 0}, \tag{7.30}$$

and complete the rest by requiring the lower periodicity, $I_0(\boldsymbol{x}) = I_0(\boldsymbol{x} + \sum_{i=1}^{3} m_i M_i \boldsymbol{a}_i)$, $m_{1,2,3} \in \mathbb{Z}$. This is a first instance of the ambiguities that will appear below.

Proceeding as before, it is convenient to perform the average of a given field, $a(t, \boldsymbol{x}) \equiv I_0(\boldsymbol{x}) a(t, \boldsymbol{x})$, by convolution with a finite-width test function. To play the role of the test function, the periodic nature of the geometry suggests taking a slot function $w_{M_1 M_2 M_3}(\boldsymbol{x})$ centred at $\boldsymbol{x} = \boldsymbol{0}$, equal to the inverse cell volume $1/V_{M_1 M_2 M_3} = 1/M_1 \boldsymbol{a}_1 \cdot (M_2 \boldsymbol{a}_2 \times M_3 \boldsymbol{a}_3)$, if $\boldsymbol{x} \in \Delta_{M_1 M_2 M_3}$, and equal to zero, if $\boldsymbol{x} \notin \Delta_{M_1 M_2 M_3}$, as shown in Fig. 7.7.

With this test function, Lorentz's average is thus defined as:

$$\begin{aligned} A(t, \boldsymbol{x}) = \langle a \rangle (t, \boldsymbol{x}) &= \int d\boldsymbol{x}' w_{M_1 M_2 M_3}(\boldsymbol{x} - \boldsymbol{x}') a(t, \boldsymbol{x}'), \\ \int d\boldsymbol{x}' w_{M_1 M_2 M_3}(\boldsymbol{x} - \boldsymbol{x}') &= 1. \end{aligned} \tag{7.31}$$

where as before we can, if we wish, make apparent the function $I_0(\boldsymbol{x}')$ in the integrand (7.31.1). We dispense from noting the choice of $M_1 M_2 M_3$ on A and $\langle . \rangle$ to lighten the notation.

We later clarify the reason for multivocity, (i.e. why we might be interested in taking $M_1 M_2 M_3 \neq 111$). We note that, as we use the total volume normalization (7.31.2), the mean of the characteristic function I is the porosity:

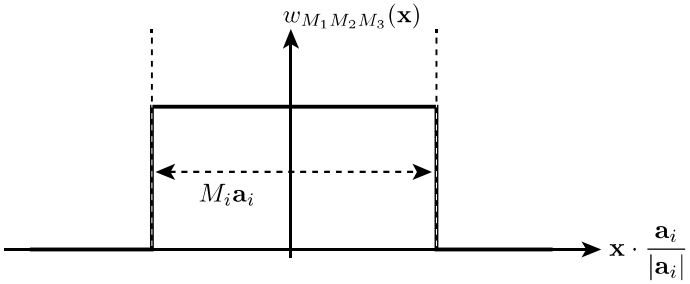

Fig. 7.7 Test function

$$\langle I_0 \rangle = \int d\boldsymbol{x}' w_{M_1 M_2 M_3}(\boldsymbol{x} - \boldsymbol{x}') I_0(\boldsymbol{x}') = \phi. \tag{7.32}$$

Because of this normalization, the averaging symbol $\langle \cdot \rangle$ in (7.31) interprets as one ("$M_1 M_2 M_3$") "total volume" average, i.e., porosity times the ("$M_1 M_2 M_3$") mean value in the fluid phase: $\langle \cdot \rangle = \phi \langle \cdot \rangle_f$.

Gibbs' Average

Just as the definition of the Lorentz mean given above is multiple, so is the Gibbs mean that we are defining now. At first, not considering multivocity, we would define Gibbs' average as:

$$A(t, \boldsymbol{x}) = \langle a \rangle (t, \boldsymbol{x}) = \overline{a(t, \boldsymbol{x}; \varpi)},$$

where we extend the field to zero in the solid and the overline is average over the random realizations ϖ. These are defined by applying the random translations $\boldsymbol{X} = \sum_{i=1}^{3} \xi_i \boldsymbol{a}_i$ to the reference configuration ϖ_0, so that $I(\boldsymbol{x}; \varpi) = I(\boldsymbol{x} - \boldsymbol{X}; \varpi_0) = I_0(\boldsymbol{x} - \boldsymbol{X})$. Introducing multivocity, however, we consider instead writing the random translations in the form of, $\boldsymbol{X}_{M_1 M_2 M_3} = \sum_{i=1}^{3} \xi_i M_i \boldsymbol{a}_i$, with some choice for integers M_i. It means that the realization ϖ will have for indicator function, $I(\boldsymbol{x}, \varpi) = I_0(\boldsymbol{x} - \boldsymbol{X}_{M_1 M_2 M_3})$. To remember this and make appear the choice of integers M_i in the definition, (we again dispense from noting it on A and $\langle . \rangle$), we denote finally the corresponding Gibbs average:

$$A(t, \boldsymbol{x}) = \langle a \rangle (t, \boldsymbol{x}) = \overline{a(t, \boldsymbol{x}; \varpi)}_{M_1 M_2 M_3}. \tag{7.33}$$

As before, it is convenient to rewrite the definition in a form that allows easy expression of the spatial derivative of the mean. To effect this refinement we introduce Δ, a region centred at $\boldsymbol{x} = \boldsymbol{0}$:

$$\Delta = \left\{ \boldsymbol{x} \mid \boldsymbol{x} = \sum_{i=1}^{3} \xi_i \epsilon \boldsymbol{a}_i, \quad \xi_i \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}. \tag{7.34}$$

and that is considered in the limit $\epsilon \to 0$ where it is vanishingly small.

We introduce also the corresponding test function w_ϵ, which equals the inverse volume $1/V_\Delta = 1/\epsilon \boldsymbol{a}_1 \cdot (\epsilon \boldsymbol{a}_2 \times \epsilon \boldsymbol{a}_3)$ of this region if $\boldsymbol{x} \in \Delta$, and is zero if $\boldsymbol{x} \notin \Delta$. Then using the form of (7.15) we have the following definition of Gibbs' average:

$$A(t, \boldsymbol{x}) = \langle a \rangle (t, \boldsymbol{x}) = \overline{\int d\boldsymbol{x}' w_\epsilon(\boldsymbol{x} - \boldsymbol{x}') a(t, \boldsymbol{x}'; \varpi)}_{M_1 M_2 M_3}. \tag{7.35}$$

It has total volume normalization because of $\int d\boldsymbol{x}' w_\epsilon(\boldsymbol{x} - \boldsymbol{x}') = 1$.

In case we need an average $\langle a \rangle_f$ with fluid volume normalization, we just multiply by $1/\phi$ the test functions, ($w_{M_1M_2M_3}$ for Lorentz, w_ϵ for Gibbs), so that, again, $\langle . \rangle = \phi \langle . \rangle_f$. We now clarify the reason of the multivocity.

Reason of the Multivocity

The averaging operations are destined to be used to perform averages of fields, (or product of fields), solutions to one problem of type (7.1)–(7.6) stated for one periodic sample and at long wavelengths (Lorentz), or solutions to an ensemble of problems of type (7.1)–(7.6) for the collection of translated samples and at arbitrary wavelengths (Gibbs). For the present discussion we will assume that an impressed source term, $\boldsymbol{f} = -\partial(\tilde{\mathcal{P}} e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}})$, is present.[7] In response to the source term, fields varying like $\boldsymbol{v}(t,\boldsymbol{x}) = \boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x})e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$, $b(t,\boldsymbol{x}) = b(\omega,\boldsymbol{k},\boldsymbol{x})e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$, etc., will appear in the fluid, (omitting the realization argument ϖ_0 or ϖ). In the stationary random case, the response-amplitudes in front of the exponentials were uniquely fixed by the condition to be bounded fields. Now in the periodic case, this condition becomes insufficient to uniquely fix them: *we can have different choices for the periodicities*. Along the direction of vector $\boldsymbol{a}_1$ for example, we can require that the solutions amplitudes verify $\boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x}+\boldsymbol{a}_1) = \boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x})$, etc. We can instead require that they do not verify this, but verify $\boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x}+2\boldsymbol{a}_1) = \boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x})$, etc., and so on, with increasing value of periodicity. Thus, to unambiguously fix the response solutions, we have to precise what are the minimal periodicities of the (tilde-)amplitudes we are selecting among the different possible ones, i.e. specify the minimal integers, $M_1M_2M_3 \in (\mathbb{N}_{\neq 0})^3$, that will be such that, $\forall \boldsymbol{x}$, $\boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x}+M_i\boldsymbol{a}_i) = \boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x})$, for $i = 1, 2, 3$, and as soon as $N_i < M_i$, $\exists \boldsymbol{x}$, $\boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x}+N_i\boldsymbol{a}_i) \neq \boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x})$. Evidently, to perform macroscopic averages on a solution determined by such a choice, the corresponding "$M_1M_2M_3$" Lorentz or Gibbs average will have to be employed. This is the reason of the previous multivoked definitions. It is intrinsically connected with the unbounded nature of the periodic geometries considered. Having made this point, the discussion of commutation relations and ergodic equivalences, follows without difficulty.

7.3.2.2 Commutation Relations

Using the definitions of Lorentz's and Gibbs' averages, (7.31) and (7.33)–(7.35), it is easy to rewrite what has been stated in the stationary random case.

[7] In its absence, the same problematics also arises for the specification of the possible normal waves, with however additional technicalities as $\boldsymbol{k}$ then interprets as a Bloch wave-vector, and is thus determined only up to the addition of a reciprocal lattice vector, $\boldsymbol{K} = \sum_{i=1}^{3} m_i \boldsymbol{b}_i$, (with $\boldsymbol{b}_i \cdot \boldsymbol{a}_j = 2\pi\delta_{ij}$), which induces a corresponding ambiguity in the definition of (tilde-)amplitudes.

Lorentz's Average

Rewriting (7.17) we have the following commutation relation or averaging theorem:

$$\begin{aligned}\partial A(t,\boldsymbol{x}) &= \partial\langle a\rangle(t,\boldsymbol{x}) = \langle \partial a\rangle(t,\boldsymbol{x}) + \langle a\partial I_0\rangle(t,\boldsymbol{x}),\\ &= \int d\boldsymbol{x}' w_{M_1M_2M_3}(\boldsymbol{x}-\boldsymbol{x}')\partial a(t,\boldsymbol{x}') - \int_{\partial\mathcal{V}} d\boldsymbol{x}' a(t,\boldsymbol{x}')\hat{\boldsymbol{n}}(\boldsymbol{x}')w_{M_1M_2M_3}(\boldsymbol{x}-\boldsymbol{x}').\end{aligned} \tag{7.36}$$

The relation (7.18), that expressed macroscopic homogeneity, can be written here

$$\int_{\partial\mathcal{V}} d\boldsymbol{x}'\hat{\boldsymbol{n}}(\boldsymbol{x}')w_{M_1M_2M_3}(\boldsymbol{x}-\boldsymbol{x}') = \boldsymbol{0}. \tag{7.37}$$

Gibbs' Average

For Gibbs' average the previous commutation relation (7.19) now writes

$$\begin{aligned}\partial A(t,\boldsymbol{x}) =& \partial\langle a\rangle(t,\boldsymbol{x}) = \langle \partial a\rangle(t,\boldsymbol{x}) + \langle a\partial I\rangle(t,\boldsymbol{x}),\\ =& \overline{\int d\boldsymbol{x}' w_\epsilon(\boldsymbol{x}-\boldsymbol{x}')\partial a(t,\boldsymbol{x}';\varpi)}^{M_1M_2M_3}\\ &- \overline{\int_{\partial\mathcal{V}(\varpi)} d\boldsymbol{x}' a(t,\boldsymbol{x}';\varpi)\hat{\boldsymbol{n}}(\boldsymbol{x}';\varpi)w_\epsilon(\boldsymbol{x}-\boldsymbol{x}')}^{M_1M_2M_3}.\end{aligned} \tag{7.38}$$

Discussion

The discussion to be done here is the same as before, *mutatis mutandis*. Therefore we will not repeat it.

7.3.2.3 Ergodic Equivalences

Here we can explicitly check the ergodic equivalences, ensuring that the Gibbs ensemble average can always be done in terms of a Lorentz volume average.

The periodic Lorentz medium is defined by the indicator periodic function $I_0(\boldsymbol{x})$ of reference configuration ϖ_0, *and*, a choice of the integers M_1, M_2, M_3. When subjected to the action of the external force $\boldsymbol{f} = -\partial\mathcal{P}$ specified by a potential, $\mathcal{P} = \tilde{\mathcal{P}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$, the response fields, $\boldsymbol{v} = \boldsymbol{v}(\omega,\boldsymbol{k},\boldsymbol{x};\varpi_0)e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$, $b = b(\omega,\boldsymbol{k},\boldsymbol{x};\varpi_0)e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}$, etc., are set, uniquely, by the condition that the tilde amplitudes[8] are proportional to the source tilde amplitude $\tilde{\mathcal{P}}$, and are periodic functions,

[8] Meaning the amplitudes in front of the exponentials; as always we omit the tilde symbol to lighten the writing.

$f(\boldsymbol{x} + M_i \boldsymbol{a}_i) = f(\boldsymbol{x})$, $\forall \boldsymbol{x}$, $i = 1, 2, 3$, with the minimal periodicities specified by the integers M_1, M_2, M_3.

An associated Gibbs medium will be the ensemble of randomly translated samples $\varpi \in \Omega$, each defined by the indicator periodic function

$$I_0(\boldsymbol{x} - \boldsymbol{X}_{M_1 M_2 M_3}), \text{ with } \boldsymbol{X}_{M_1 M_2 M_3} = \sum_{i=1}^{3} \xi_i M_i \boldsymbol{a}_i, \text{ and } \xi_i \in [-1/2, 1/2]\,.$$

As fixed by $M_1 M_2 M_3$, the random translations are over several periods because they will serve to average fields having tilde-amplitudes varying with corresponding periodicities. Indeed, when subjected to the action of the same external force as above, the response fields we consider, $\boldsymbol{v} = \boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x}; \varpi) e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$, $b = b(\omega, \boldsymbol{k}, \boldsymbol{x}; \varpi)$ $e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$, etc., will be set as before, uniquely, by the condition that the tilde amplitudes are proportional to source tilde amplitude $\tilde{\mathcal{P}}$, and are periodic functions, $f(\boldsymbol{x} + M_i \boldsymbol{a}_i) = f(\boldsymbol{x})$, having the minimal periodicities specified by M_1, M_2, M_3.

Consider then Gibbs' average

$$\boldsymbol{V}(t, \boldsymbol{x}) = \langle \boldsymbol{v} \rangle(t, \boldsymbol{x}) = e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \overline{\boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x}; \varpi)}_{M_1 M_2 M_3}. \tag{7.39}$$

In evaluating it, there is the following relation that can be used between $\boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x}; \varpi)$ and $\boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x}; \varpi_0)$:

$$\boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x}; \varpi) = \boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x} - \boldsymbol{X}_{M_1 M_2 M_3}; \varpi_0). \tag{7.40}$$

To see it, let us apply the translation $\boldsymbol{X}_{M_1 M_2 M_3}$ to the reference configuration ϖ_0, then obtaining the translated configuration ϖ whose indicator function is $I(\boldsymbol{x}, \varpi) = I_0(\boldsymbol{x} - \boldsymbol{X}_{M_1 M_2 M_3})$. In translated coordinate axes $\boldsymbol{y}$ related to the $\boldsymbol{x}$ by $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{X}_{M_1 M_2 M_3}$, this translated configuration ϖ is the same as ϖ_0 in the axes $\boldsymbol{x}$, that is, $I(\boldsymbol{y} + \boldsymbol{X}_{M_1 M_2 M_3}; \varpi) = I_0(\boldsymbol{y}; \varpi_0)$. The response fields would then be the same, that is, $\boldsymbol{v}(t, \boldsymbol{x}; \varpi) = \boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{y}; \varpi_0) e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{y}}$, if the source potential $\tilde{\mathcal{P}} e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$ had also been "displaced", so as to write $\tilde{\mathcal{P}} e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{y}}$ and have amplitude $\tilde{\mathcal{P}}$ at $\boldsymbol{y} = \boldsymbol{0}$, (new position, in the present Gibbs realization ϖ, of the material that was in $\boldsymbol{x} = \boldsymbol{0}$ in the reference configuration ϖ_0). But the source term, independent of realization, is not displaced. It possesses now a multiplicative factor $e^{i\boldsymbol{k}\cdot\boldsymbol{X}_{M_1 M_2 M_3}}$, giving it the value $\tilde{\mathcal{P}} e^{i\boldsymbol{k}\cdot\boldsymbol{X}_{M_1 M_2 M_3}}$ at the new origin $\boldsymbol{y} = \boldsymbol{0}$. Hence correcting for this factor, we conclude that in the translated configuration, with source unchanged, we have $\boldsymbol{v}(t, \boldsymbol{x}; \varpi) = \boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{y}; \varpi_0) e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{y}} e^{i\boldsymbol{k}\cdot\boldsymbol{X}_{M_1 M_2 M_3}}$, that is, $\boldsymbol{v}(t, \boldsymbol{x}; \varpi) = \boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x} - \boldsymbol{X}_{M_1 M_2 M_3}; \varpi_0) e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$, which is the indicated relation (7.40).

Inserting (7.40) in (7.39) we then have

$$\boldsymbol{V}(t, \boldsymbol{x}) = \langle \boldsymbol{v} \rangle(t, \boldsymbol{x}) = e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \overline{\boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x} - \boldsymbol{X}; \varpi_0)}_{M_1 M_2 M_3}. \tag{7.41}$$

The overline, average over realizations, amounts here in distributing the random displacement $\boldsymbol{X}_{M_1 M_2 M_3}$ uniformly in a "unit cell" region $\Delta_{M_1 M_2 M_3}$ given by

$$\Delta_{M_1M_2M_3} = \left\{ \boldsymbol{x} \mid \boldsymbol{x} = \sum_{i=1}^{3} \xi_i M_i \boldsymbol{a}_i, \quad \xi_i \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}. \tag{7.42}$$

As the field is extended to zero in the solid, we have

$$\overline{\boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x} - \boldsymbol{X}_{M_1M_2M_3}; \varpi_0)} = \frac{1}{V_{\Delta_{M_1M_2M_3}}} \int_{\Delta_{M_1M_2M_3}} d\boldsymbol{X} \times \cdots \cdots \boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x} - \boldsymbol{X}; \varpi_0). \tag{7.43}$$

By inspection, we can see that this is the same as the Lorentz mean

$$\langle \boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x}; \varpi_0) \rangle_{Lor} = \int d\boldsymbol{x}' w_{M_1M_2M_3}(\boldsymbol{x} - \boldsymbol{x}') \boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x}'; \varpi_0), \tag{7.44}$$

with $w_{M_1M_2M_3}(\boldsymbol{x})$ the slot function represented in Fig. 7.7. Finally, it shows us that the Gibbs mean, is, whatever the chosen $\boldsymbol{k}$:

$$\begin{aligned} \boldsymbol{V}(t, \boldsymbol{x}) =& \langle \boldsymbol{v} \rangle(t, \boldsymbol{x}) = e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \overline{\boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x}; \varpi)}, \\ =& e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \overline{\boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x} - \boldsymbol{X}; \varpi_0)} = e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \langle \boldsymbol{v}(\omega, \boldsymbol{k}, \boldsymbol{x}; \varpi_0) \rangle_{Lor}, \end{aligned}$$

where we suppressed mention of the retained choice $M_1M_2M_3$. Therefore, apart from an undesirable multiplicity, we have demonstrated in the periodic case, for our macroscopic averaging operations, the same properties as we asserted before in the stationary random case. Whatever the wavelengths, the Gibbs mean can always be performed in terms of Lorentz mean directly made on the tilde amplitudes. At long wavelengths, both averages are interchangeable.

7.4 Macroscopic Equations and Definition of the Acoustic *H*-Field from Electromagnetic Analogy

We now address the question of formulating the general macroscopic equations describing the compressional-dilatational wave propagation in a macroscopically homogeneous rigid-framed porous medium saturated with a viscothermal fluid.

In Lorentz's conception we are given one sample and the theory we are to formulate is intended to describe long-wavelength sound propagation in this sample. In the Gibbs' conception we are given an ensemble of samples and the theory is intended to describe ensemble-averaged fields, whatever the frequencies and wavelengths.

In what follows, to simplify the discussion, we limit ourselves to considering wave propagation or excitation along a single axis x, which is also assumed to be a macroscopic symmetry axis. Precisely, because the external actions $\boldsymbol{f}$ in (7.2) are directed along x, a symmetry axis, we can assume that :

$$\boldsymbol{f} = -\partial\mathcal{P}, \qquad \mathcal{P} = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\tilde{\mathcal{P}}(\omega,k)e^{-i\omega t+ikx}, \qquad \langle \boldsymbol{v}\rangle \parallel \hat{\boldsymbol{x}}, \qquad \langle p\boldsymbol{v}\rangle \parallel \hat{\boldsymbol{x}}. \quad (7.45)$$

The most general fields we consider here will have a "forced" part coming in response to the impressed excitation $\boldsymbol{f}$, and a "free" part that superposes, (solution to the homogeneous equations (7.1)–(7.6) with $\boldsymbol{f}$ suppressed), and that can be viewed as the result of the presence of an "external" incident field, coming from without, in the directions $\pm x$.

We recall first, the results we have obtained in Chap. 6. We recall in Sect. (7.4.1), the pattern of nonlocal equations in the homogeneous viscothermal fluid itself, without solid, and written for compressional motions along axis x to comply with the above-mentioned restrictions; with no ambiguity, we do not indicate the indice x on v, d, and f. We recall next, in Sect. (7.4.2), the pattern of the equations in macroscopic nonlocal electromagnetics, written for propagation/excitation along a principal axis x of a general homogeneous structured material. Finally, in Sect. (7.4.3), we pass to the pattern of the macroscopic equations we would like to write, by analogy, for the propagation/excitation of compressional waves along macroscopic axis x in our homogeneous, structured fluid/solid medium.

7.4.1 Unbounded Fluid (Longitudinal Motions)

In the unbounded fluid, the corresponding pattern of nonlocal acoustic equations found in Chap. 6, was as follows:

Field equations

$$\frac{\partial b}{\partial t} + \partial_x v = 0, \qquad \frac{\partial d}{\partial t} = \partial_x h + [f], \quad (7.46)$$

where

$$f = -\partial_x\mathcal{P}, \qquad \mathcal{P} = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\tilde{\mathcal{P}}(\omega,k)e^{-i\omega t+ikx}. \quad (7.47)$$

Constitutive relations

$$d(t,x) = \hat{\rho}v(t,x) = \int \rho(t-t',x-x')v(t',x')dt'dx', \quad (7.48)$$

$$h(t,x) = -\hat{\chi}^{-1}b(t,x) = -\int \chi^{-1}(t-t',x-x')b(t',x')dt'dx'. \quad (7.49)$$

Definition of the h-field

$$h = -p. \quad (7.50)$$

The last "acoustic Heaviside-Poynting" identification is obtained as a result of setting, at the same time

$$s = -vh, \tag{7.51}$$

$$s = vp. \tag{7.52}$$

The first equation, where s is the "energy current density transported in acoustic form", is to be viewed as satisfied by principle, by definition of what will be the h-field. The second, which specifies s, is obtained by using the irreversible thermodynamics of the Navier–Stokes–Fourier model.

7.4.2 Macroscopic Electromagnetics

In the structured homogeneous electromagnetic medium, the corresponding pattern of nonlocal electromagnetic equations found in Chap. 6, was as follows:

Field equations

$$\frac{\partial \boldsymbol{B}}{\partial t} + \partial \times \boldsymbol{E} = \boldsymbol{0}, \qquad \frac{\partial \boldsymbol{D}}{\partial t} = \partial \times \boldsymbol{H} - [\boldsymbol{J}], \tag{7.53}$$

where

$$\boldsymbol{E} = \langle \boldsymbol{e} \rangle, \qquad \boldsymbol{B} = \langle \boldsymbol{b} \rangle, \tag{7.54}$$

$$\boldsymbol{J} = J\hat{\boldsymbol{x}}, \qquad J = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\tilde{\mathcal{J}}(\omega, k)e^{-i\omega t + ikx}. \tag{7.55}$$

Constitutive relations

$$\boldsymbol{D}(t,x) = \hat{\epsilon}\boldsymbol{E}(t,x) = \int \epsilon(t-t', x-x')\boldsymbol{E}(t',x')dt'dx', \tag{7.56}$$

$$\boldsymbol{H}(t,x) = \hat{\mu}^{-1}\boldsymbol{B}(t,x) = \int \mu^{-1}(t-t', x-x')\boldsymbol{B}(t',x')dt'dx'. \tag{7.57}$$

Definition of the $\boldsymbol{H}$-field

$$\boldsymbol{H} = \text{thermodynamic field } \boldsymbol{\mathcal{P}}. \tag{7.58}$$

The last would result from setting, at the same time:

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}, \tag{7.59}$$

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{\mathcal{P}}. \tag{7.60}$$

The first equation, where $\boldsymbol{S}$ is the "energy current density transported in electromagnetic form", is to be viewed as satisfied by principle, by definition of what will be the $\boldsymbol{H}$-field. To be written, the second, which specifies $\boldsymbol{S}$, would require an irreversible thermodynamic description of the behaviour of charges and currents in material

media. Currently, we cannot properly define the field $\boldsymbol{H}$, because, we do not have this irreversible thermodynamic description.

7.4.3 Macroscopic Acoustics (Fluid-Saturated Rigid-Framed Porous Medium)

By analogy and extension, we now write the pattern of required equations for the propagation of macroscopic compressional waves in our structured fluid/solid medium, along a macroscopic symmetry axis. Since it is only in the fluid part that the movement is not null, new technicalities intervene in the presentation where one can for example choose to make ϕ factors appear or not. We will therefore start with some preliminary points regarding our definitions.

There is assumed to be an external longitudinal bulk force, $\boldsymbol{f}(\boldsymbol{x}) = f(\boldsymbol{x})\hat{\boldsymbol{x}}$, acting in the fluid, such that, $\boldsymbol{f}(\boldsymbol{x}) = -\partial_{\boldsymbol{x}}\mathcal{P}(\boldsymbol{x})$, $f(\boldsymbol{x}) = -\partial_x\mathcal{P}(\boldsymbol{x})$, $\mathcal{P}(\boldsymbol{x}) = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}$ $\tilde{\mathcal{P}}(\omega, k)e^{-i\omega t+ikx}$, if $\boldsymbol{x}$ is in the fluid, and $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$, $\mathcal{P}(\boldsymbol{x}) = 0$, if $\boldsymbol{x}$ is in the solid. We note that the fields $\boldsymbol{f}(\boldsymbol{x})$, $f(\boldsymbol{x})$ and $\mathcal{P}(\boldsymbol{x})$, have the nature of the "macroscopic impressed fields" or "external fields" discussed in Sect. 7.3.1: they vary independently of the geometry except for their conventional extension to zero in the solid. Therefore in particular we will have, using the commutation relation (7.22), $\langle f\rangle = \langle -\partial_x\mathcal{P}\rangle = -\partial_x\langle\mathcal{P}\rangle$. Now, for later convenience, we introduce different notations for the macroscopic means $\langle f\rangle$ and $\langle\mathcal{P}\rangle$. For the first, we chose to denote it $\langle f\rangle \equiv \phi F$, then adopting for $F = \langle f\rangle_f$ a fluid-volume-average normalization. For $\langle\mathcal{P}\rangle$ we chose to write, by language abuse

$$\langle\mathcal{P}\rangle = \phi\mathcal{P}, \tag{7.61}$$

denoting with *same* letter $\mathcal{P}$ the field, macroscopic proper, given by $\mathcal{P}(\boldsymbol{x}) = \mathcal{P}(x) = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\tilde{\mathcal{P}}(\omega, k)e^{-i\omega t+ikx}$, (with same coefficients $\tilde{\mathcal{P}}$ as above), *whatever the values* $\boldsymbol{x}$, x, (*including* $\boldsymbol{x}$ *in the solid*). In fact, this language abuse is made possible by the "external field" nature of the $\mathcal{P}$ field. Owing to this nature of the $\mathcal{P}$ field, this interpretation of the letter $\mathcal{P}$ in the right side of (7.61) is evident to see with Gibbs' averaging conception. The same writing (7.61) can also easily be obtained with Lorentz's averaging, with reasonings of the type previously seen in Sect. 7.3.1.2 to justify the cancellation (7.23). We do not use it for $\langle f\rangle = \phi F$, (that could in same manner be denoted $\langle f\rangle = \phi f$), just to have a capital letter for the external force in our macroscpic equations. In this way we have the following notation, whose form is simple:

$$F \equiv \frac{\langle f\rangle}{\phi} = \frac{\langle -\partial_x\mathcal{P}\rangle}{\phi} = \frac{-\partial_x\langle\mathcal{P}\rangle}{\phi} = -\partial_x\mathcal{P}. \tag{7.62}$$

With these preliminaries made, the juxtaposition of the two sets of equations, acoustic (7.46)–(7.51) in the unbounded fluid, and electromagnetic (7.53)–(7.60) in

the material, suggests that for compressional macroscopic motions along symmetry axis x in the fluid-saturated porous medium, we should have macroscopic nonlocal equations taking the following form:

Field equations

$$\frac{\partial B}{\partial t} + \partial_x V = 0, \qquad \frac{\partial D}{\partial t} = \partial_x H + [F], \tag{7.63}$$

where

$$V = \langle v \rangle, \qquad B = \langle b \rangle, \tag{7.64}$$

$$F = -\partial_x \mathcal{P}, \qquad \mathcal{P} = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\tilde{\mathcal{P}}(\omega, k)e^{-i\omega t + ikx}. \tag{7.65}$$

Constitutive relations

$$D(t,x) = \hat{\rho} V(t,x) = \int \rho(t-t', x-x')V(t',x')dt'dx', \tag{7.66}$$

$$H(t,x) = -\hat{\chi}^{-1}B(t,x) = -\int \chi^{-1}(t-t', x-x')B(t',x')dt'dx'. \tag{7.67}$$

Definition of the H-field

$$-H\langle v \rangle = \langle pv \rangle. \tag{7.68}$$

The last is obtained as a result of setting, at the same time, for the "macroscopic acoustic part of the energy current density":

$$S = -\langle v \rangle H, \tag{7.69}$$

$$S = \langle s \rangle = \langle vp \rangle. \tag{7.70}$$

Equation (7.70) is an interpretation, made possible by the existence of thermodynamic concepts, of what is the macroscopic acoustic part of the energy current density $\boldsymbol{S} = S\hat{\boldsymbol{x}}$: it is the mean of pore-scale acoustic part of the energy current density, $\boldsymbol{S} = \langle \boldsymbol{s} \rangle$, and this gives a usable definition, because we have a thermodynamic background to identify $\boldsymbol{s}$: $\boldsymbol{s} = p\boldsymbol{v}$, where p is the thermodynamic pressure. Equation (7.69), written as an "acoustic Heaviside-Poynting" relation that must be satisfied by definition of what will be the acoustic H-field, then gives the identification (7.68) which is a usable definition of this acoustic H-field. It is in fact nothing but the customary definition of lumped acoustic pressure discussed in Pierce [7], Sect. 7.2 Lumped-Parameters Models. In Sect. 7.7, after entering in the detail of the micro-macro passage, we will see how the above definition (7.68) of macroscopic H-field, is a powerful statement capable to fix the macroscopic properties of the medium, i.e. the operators $\hat{\rho}$ and $\hat{\chi}^{-1}$. Before this, however, we want to examine what other definitions we could make, if we were to proceed like in conventional electromagnetics, or if we were to use, as is done in local theory, (see Appendix), a macroscopic pressure defined by direct volume averaging.

7.5 Macroscopic Equations: Other Points of View

First recall that the customary point of view in electromagnetics of spatially dispersive media, found e.g. in Landau and Lifshitz [8], Agranovich and Ginzburg [9], Melrose and McPhedran [10], and which we view as a stopgap measure, consists in setting $\boldsymbol{M} = \boldsymbol{0}$, in the presence of spatial dispersion, i.e. write by definition:

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0}, \tag{7.71}$$

and put all effects in an effective electric susceptibility $\hat{\epsilon}$. It leads to write the nonlocal equations in a form seen in Chap. 6 and recalled in next subsection.

7.5.1 Macroscopic Electromagnetics—Customary Point of View

Following the customary point of view, the pattern of nonlocal electromagnetic equations is set as follows:

Field equations

$$\frac{\partial \boldsymbol{B}}{\partial t} + \partial \times \boldsymbol{E} = \boldsymbol{0}, \qquad \frac{\partial \boldsymbol{D}}{\partial t} = \frac{1}{\mu_0}\partial \times \boldsymbol{B} - [\boldsymbol{J}], \tag{7.72}$$

where

$$\boldsymbol{E} = \langle \boldsymbol{e} \rangle, \qquad \boldsymbol{B} = \langle \boldsymbol{b} \rangle, \tag{7.73}$$

$$\boldsymbol{J} = J\hat{\boldsymbol{x}}, \qquad J = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\tilde{\mathcal{J}}(\omega, k)e^{-i\omega t + ikx}. \tag{7.74}$$

Constitutive relation

$$\boldsymbol{D}(t, x) = \hat{\epsilon}\boldsymbol{E}(t, x) = \int \epsilon(t - t', x - x')\boldsymbol{E}(t', x')dt'dx'. \tag{7.75}$$

In a similar manner, it would perfectly be possible here, to formulate the acoustic nonlocal equations by introducing a conventional H-field, different from that of (7.68). Let us take two examples.

7.5.2 Acoustics Translation of the Customary Point of View in Electromagnetics

The formulation that would be the exact counterpart of the above customary electromagnetic point of view, would consist in setting $M = 0$, i.e. write by definition, (the occurrence of porosity factor here, is as in (7.62), see also Sect. 7.6):

$$H = -\frac{B}{\chi_0 \phi}. \tag{7.76}$$

This would lead to write the nonlocal acoustic equations in the form:

Field equations

$$\frac{\partial B}{\partial t} + \partial_x V = 0, \qquad \frac{\partial D}{\partial t} = -\frac{1}{\chi_0 \phi} \partial_x B + [F], \tag{7.77}$$

where

$$V = \langle v \rangle, \qquad B = \langle b \rangle, \tag{7.78}$$

$$F = -\partial_x \mathcal{P}, \qquad \mathcal{P} = \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} \tilde{\mathcal{P}}(\omega, k) e^{-i\omega t + ikx}. \tag{7.79}$$

Constitutive relation

$$D(t, x) = \hat{\rho} V(t, x) = \int \rho(t - t', x - x') V(t', x') dt' dx'. \tag{7.80}$$

This point of view sets the bulk modulus operator $\hat{\chi}^{-1}$ to a constant $1/\phi\chi_0$, determined by the adiabatic value. It is clearly an artificial replacement.

7.5.3 Acoustics Formulation in Terms of Volume-Averaged Pressure

Still another definition, inspired this time by what is done within the local homogenization, would be to use in the macroscopic equations the ordinary macroscopic pressure, $P = \langle p \rangle_f$, defined by a fluid-volume-average. Using this variable as the opposite H-field, we would have the following formulation:

Field equations

$$\frac{\partial B}{\partial t} + \partial_x V = 0, \qquad \frac{\partial D}{\partial t} = -\partial_x P + [F], \tag{7.81}$$

where

$$V = \langle v \rangle, \qquad B = \langle b \rangle, \tag{7.82}$$

$$F = \partial_x \mathcal{P}, \qquad \mathcal{P} = \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\tilde{\mathcal{P}}(\omega,k)e^{-i\omega t+ikx}. \tag{7.83}$$

Constitutive relations

$$D(t,x) = \hat{\rho}V(t,x) = \int \rho(t-t',x-x')V(t',x')dt'dx', \tag{7.84}$$

$$P(t,x) = \hat{\chi}^{-1}B(t,x) = \int \chi^{-1}(t-t',x-x')B(t',x')dt'dx'. \tag{7.85}$$

Definition of the macroscopic stress field H and pressure field P

$$H = -P = -\langle p\rangle/\phi = -\langle p\rangle_f. \tag{7.86}$$

This point of view, however, sets undue importance to the direct volume-average of the pressure $\langle p\rangle_f$. As soon as the pressure is significantly distributed at the pore-scale, this average is not a very meaningful macroscopic variable. A distributed pore-scale pressure occurs principally in two instances: long wavelengths with complex geometry producing local resonances, and short wavelengths, (described within a Gibbs conception). In further work, the interest of using, preferentially the definition (7.68) in these cases should be studied, in particular, when the finite dimensions of the media are taken into account. The conception (7.68) would have to generalize smoothly and we should see that it is advantageous to use, compared to others.

Indeed, consider a slab of material and compare how the H-variables of the different conceptions will behave at a boundary, $x = Cst$, between the material and the fluid, and in its vicinity.

Consider first the artificial definition $H = -B/\chi_0\phi$. There is no reason that $\chi_0\phi$ be a meaningful compressibility in the material, especially in presence of resonances. Therefore, we expect that this variable H can quickly vary in the vicinity of the boundary, and be discontinuous at it. Likewise, consider the definition $H = -\langle p\rangle_f$. In presence of local Helmholtz resonances in the material, the pressure will be rapidly variable on the small scale, and there is no reason that its direct mean shouldn't exhibit rapid variation at the boundary, and be discontinuous at it. This is at variance with the indirect mean or "lumped" variable H, in the "acoustic Heaviside-Poynting" definition (7.68), $S = \langle pv\rangle = -H\langle v\rangle$. Indeed, as we have continuity of normal flow $\langle v\rangle$, and we can also expect continuity of normal acoustic power flow S, this definition (7.68) generates a H-field continuous at the material boundary, and that will not vary rapidly in its vicinity. That is why it should lead to natural determination of the density and compressibility operators, when the finite dimensions are taken into account. In particular, not only appropriate to describe characteristic wavenumbers, it would give also appropriate characteristic impedances (see Sect. 7.9).

With these remarks in mind, the deficiencies of the acoustic conceptions, $H = -B/\chi_0\phi$, and $H = -\langle p\rangle_f$, cast a harsh light on the present day definitions of electromagnetic Maxwell fields $\boldsymbol{H}$ and $\boldsymbol{D}$, generally effected in presence of spatial dispersion through setting $\boldsymbol{H} = \boldsymbol{B}/\mu_0$.

7.6 Derivation of the Macroscopic Equations by a Micro-Macro Passage

In preceding Sects. 7.4 and 7.5, by following an electromagnetic analogy, we proposed different specific patterns for the macroscopic acoustic equations. Referring to conventional ideas in electromagnetics we were led to the pattern (7.76)–(7.80); referring to other conventional ideas in acoustics we were led to the pattern (7.81)–(7.86). Finally, by deepening the analogy itself, with new perspectives that we defended and that concerned as much electromagnetics, irreversible-thermodynamics, and acoustics, we were brought to the pattern (7.63)–(7.68), that we consider is the preferable one, physically.

We now scrutinize how microscopic equations (7.1)–(7.6) applied for longitudinal motions along x, supposed to be a macroscopic symmetry axis, indeed lead, after averaging, to macroscopic equations that can be put in the given various patterns (7.63)–(7.68) or (7.76)–(7.80) or (7.81)–(7.86). It will also lead us clarify, next, how the respective operators can be in principle derived, from microstructure.

The "micro-macro" passage, from microscopic to macroscopic equations is as follows.[9]

Using Lorentz's conception, we write the equations (7.1)–(7.6) in one single realization of our stationary-random or periodic medium, say ϖ_0, we solve them, and then we make a "micro-macro" transition by taking a Lorentz volume average. Using Gibbs' conception, we write the equations in an arbitrary realization ϖ, we solve them, and make a "micro-macro" transition by taking the Gibbs ensemble average over realizations.

In this process, when we take the average of the (7.1) and apply the general commutation relation (7.21), we directly obtain the first macroscopic field equation:

$$0 = \frac{\partial\langle b\rangle}{\partial t} + \langle\partial\cdot\boldsymbol{v}\rangle = \frac{\partial\langle b\rangle}{\partial t} + \partial\cdot\langle\boldsymbol{v}\rangle \;\rightarrow\; \frac{\partial B}{\partial t} + \partial\cdot\boldsymbol{V} = 0.$$

[9] "Micro" do not refer here to any level comparable to that of molecules and electrons in electromagnetics. It refers to the inner macroscopic level where equations of fluid mechanics apply and the material appears inhomogeneous, by opposition to the outer macroscopic level, defined in Lorentz's or Gibbs' sense. Note also that, in the general reasonings made here, as well as in the two preceding sections, we work in the stationary-random case to make the discussion definite and avoid the ambiguities encountered in the periodic case.

Here, as we restrict to considering free wave propagation along symmetry axis x, or else, source-driven problem with external density of force directed along this axis, the velocity has only nonzero component x noted V:

$$\frac{\partial B}{\partial t} + \partial_x V = 0. \tag{7.87}$$

The second macroscopic field equation, which will be either (7.63.2) with (7.68), or, (7.77.2), or, (7.81.2) with (7.86), is not at all obtained in the same direct manner, by averaging. It involves an additional crucial step, playing the role of the Lorentz splitting seen in electromagnetics, and which is related to the choice of definition of the H-field. Let us detail it now.

Based on the rewrite in the unbounded fluid, (6.99), Sect. 6.12.1 of the Chap. 6, we begin here by rewriting (7.2) as follows

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\phi \chi_0} \partial \langle b \rangle + \boldsymbol{f}_{pol} - [\partial \mathcal{P}], \tag{7.88}$$

with

$$\boldsymbol{f}_{pol} = \frac{1}{\phi \chi_0} \partial \langle b \rangle - \partial p + \eta \partial^2 \boldsymbol{v} + \left(\frac{\eta}{3} + \zeta \right) \partial \left(\partial \cdot \boldsymbol{v} \right). \tag{7.89}$$

This induced force is the expression of processes provoked by the presence of the macroscopic perturbation, the existence of viscous and thermal losses, and the solid interface on which the boundary conditions (7.5)–(7.6) apply. Its precise value $\boldsymbol{f}_{pol}$ is fixed by the expression (7.89), and the considered solution of the complete system of equations (7.1)–(7.6).[10] In the right-hand side of (7.88), the first term is chosen so that, after averaging, ($\langle \frac{1}{\phi \chi_0} \partial \langle b \rangle \rangle = \frac{1}{\chi_0} \partial \langle b \rangle$), this Equation compares well with the electromagnetic equation obtained by combining the (6.80.4) and (6.82) of Sect. 6.11.2 of the Chap. 6:

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{\mu_0} \partial \times \boldsymbol{B} - \boldsymbol{J}_{pol} - [\boldsymbol{J}]. \tag{7.90}$$

Indeed, by taking the macroscopic mean (Lorentz or Gibbs) of (7.88), and noting that the direct switching relationship (7.22) applies to the fields $a = \langle b \rangle$ and $a = \partial \mathcal{P}$, which only vary at the macroscopic scale, we find

$$\rho_0 \frac{\partial \langle \boldsymbol{v} \rangle}{\partial t} = -\frac{1}{\phi \chi_0} \partial \langle \langle b \rangle \rangle + \langle \boldsymbol{f}_{pol} \rangle - [\partial \langle \mathcal{P} \rangle].$$

As we extend the fields to zero in the solid, we have $\langle \langle b \rangle \rangle = \phi \langle b \rangle$, and $\langle \mathcal{P} \rangle = \phi \mathcal{P}$, and thus

[10] This solution is not unique because it is made of a unique forced response proportional to the excitation, and a nonunique source-free solution which depends on a possible incident field, coming from without: see comment after (7.45).

$$\rho_0 \frac{\partial \langle \boldsymbol{v} \rangle}{\partial t} = -\frac{1}{\chi_0} \partial \langle b \rangle + \langle \boldsymbol{f}_{pol} \rangle - [\phi \partial \mathcal{P}] \,.$$

Finally, by setting

$$\phi \boldsymbol{F}_{pol} \equiv \langle \boldsymbol{f}_{pol} \rangle = \frac{1}{\chi_0} \partial \langle b \rangle - \langle \partial p \rangle + \langle \eta \partial^2 \boldsymbol{v} + \left(\frac{\eta}{3} + \zeta \right) \partial \left(\partial \cdot \boldsymbol{v} \right) \rangle, \tag{7.91}$$

the averaged equation reads

$$\rho_0 \frac{\partial \langle \boldsymbol{v} \rangle}{\partial t} = -\frac{1}{\chi_0} \partial \langle b \rangle + \phi \boldsymbol{F}_{pol} - [\phi \partial \mathcal{P}] \,. \tag{7.92}$$

Now we proceed by considering that this (7.92) is a meaningful acoustic counterpart of the electromagnetic equation (7.90). In electromagnetics we have formally decomposed, in the Lorentz splitting—see the Chap. 6, Sect. 6.11.2, (6.83), the macroscopic polarization current density $\boldsymbol{J}_{pol}$ in (7.90) in two macroscopic terms, one given by temporal derivatives, the other by spatial derivatives. Here, as we have previously done with success in the unbounded fluid—see the Chap. 6, Sect. 6.12.1, (6.101), we consider that the polarization force density $\phi \boldsymbol{F}_{pol}$ in (7.92), given by (7.91), is formally decomposed in two macroscopic terms, one given by temporal derivatives, the other by spatial derivatives[11]:

$$\phi \boldsymbol{F}_{pol} = -\phi \frac{\partial \boldsymbol{P}}{\partial t} + \phi \partial M, \tag{7.93}$$

what we call an acoustic Lorentz splitting. As we restrict to a macroscopic motion along x, we have $\boldsymbol{F}_{pol} = F_{pol} \hat{\boldsymbol{x}}$, $\boldsymbol{P} = P \hat{\boldsymbol{x}}$, (with "polarization" P not to be confused with the mean pressure), and the above decomposition writes

$$F_{pol} = -\frac{\partial P}{\partial t} + \partial_x M. \tag{7.94}$$

There are nonlocal operators $\hat{\chi}_V$ and $\hat{\chi}_B$, or kernels $\chi_V(t, x)$ and $\chi_B(t, x)$, establishing how the "polarization fields P and M" are determined by the velocity and condensation macroscopic fields[12]:

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[11] Note that if we were not to assume isotropy or propagation along a symmetry axis, we would define symmetric tensors H-field and M-field, through writing $\left(F_{pol}\right)_i = \frac{-\partial P_i}{\partial t} + \partial_j M_{ji}$, and $S_i = -H_{ij} V_j = \langle p v_i \rangle$; in the electromagnetic case we have $S_i = -H_{ji} E_j$ for the Poynting vector (not paying attention to the variances), and, $\left(J_{pol}\right)_i = \frac{\partial P_i}{\partial t} - \partial_j M_{ji}$, which looks the same, with however the important difference that H_{ij} and M_{ij} are antisymmetric instead of symmetric tensors.

[12] Recall that we limit ourselves here for simplicity to macroscopically homogeneous media (unbounded media), so that we have difference-kernels, i.e. kernels which depend on the (Cartesian-coordinates) difference $x - x'$.

$$P(t,x) = \int dt'dx' \chi_V(t-t',x-x')V(t',x'), \tag{7.95}$$

$$M(t,x) = \int dt'dx' \chi_B(t-t',x-x')B(t',x'). \tag{7.96}$$

In (7.95) there is no need to add a similar term with the condensation field, because, as V and B are related by (7.87), such a term could also be rewritten in the integral form (7.95). Likewise, in (7.96), because of the complete form of the dispersion, there is no need to add a similar term with the velocity field. Substituting (7.94) in (7.92) we get

$$\rho_0 \frac{\partial \langle v \rangle}{\partial t} + \phi \frac{\partial P}{\partial t} = -\frac{1}{\chi_0} \partial_x \langle b \rangle + \phi \partial_x M - [\phi \partial_x \mathcal{P}], \tag{7.97}$$

which, by setting

$$D = \frac{\rho_0}{\phi} \langle v \rangle + P = \frac{\rho_0}{\phi} V + P, \qquad H = -\frac{1}{\chi_0 \phi} \langle b \rangle + M = -\frac{1}{\chi_0 \phi} B + M, \tag{7.98}$$

takes the form

$$\begin{aligned} \frac{\partial D}{\partial t} &= \partial_x H + [F], \\ F &= -\partial_x \mathcal{P} \quad \mathcal{P} = \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} \tilde{\mathcal{P}}(\omega,k) e^{-i\omega t + ikx}. \end{aligned} \tag{7.99}$$

And finally, saying that we necessarily have nonlocal relations of type (7.95) and (7.96), is the same as saying that we have nonlocal relations of type:

$$D(t,x) = \hat{\rho} V(t,x) = \int \rho(t-t',x-x')V(t',x')dt'dx', \tag{7.100}$$

$$H(t,x) = -\hat{\chi}^{-1} B(t,x) = -\int \chi^{-1}(t-t',x-x')B(t',x')dt'dx', \tag{7.101}$$

with the connection

$$\rho(t-t',x-x') = \frac{\rho_0}{\phi} \delta(t-t')\delta(x-x') + \chi_V(t-t',x-x'), \tag{7.102}$$

$$\chi^{-1}(t-t',x-x') = \frac{1}{\chi_0 \phi} \delta(t-t')\delta(x-x') - \chi_B(t-t',x-x'). \tag{7.103}$$

In this way, the formal pattern (7.63)–(7.67) of the macroscopic acoustics equations is now evidenced.

Nevertheless, as the decomposition (7.93) is not unique, (we can add an arbitrary term $\partial\Phi$ to the polarization $\boldsymbol{P}$, if we simultaneously add a term $\frac{\partial \Phi}{\partial t}$ to polarization M), the description is not unique. It remains to be shown that by using an additional

condition of definition of the H-field, which will be either (7.68), or (7.76), or else (7.86), to take up the various possibilities successively considered previously, we arrive at unique definitions of the relationships between the "Maxwell" fields M or H and the "Lorentz" field B, on one hand, and the "Maxwell" fields P or D and the "Lorentz" field V, on the other hand, so that the associated nonlocal operators, respectively, $\hat{\chi}_B$ or $\hat{\chi}^{-1}$, and $\hat{\chi}_V$ or $\hat{\rho}$, are determined in a unique way from the microstructure.

Of the various determinations, the one obtained with the "acoustic Heaviside-Poynting" identification (7.68), we believe, will be the most advantageous. We start with it.

7.7 Action-Response Problem to Determine the Nonlocal Dynamic Operators from Microstructure

We show here how the definition (7.68) uniquely determines in principle the operators, $(\hat{\chi}_B, \hat{\chi}^{-1})$, and $(\hat{\chi}_V, \hat{\rho})$, in a direct generalization of the previous "action-response problem" 6.12.4 in the unbounded fluid.

We consider the saturating fluid response to an harmonic excitation density of force along x, $f = -\partial_x \mathcal{P}, \mathcal{P} = \tilde{\mathcal{P}} e^{-i\omega t + ikx}$, in the fluid, paying our attention, either directly at the macroscopic level, or initially at the microscopic level, and then at macroscopic level after taking an average. For consistency, both considerations will have to lead to the same macroscopic response. This, in conjunction with the definition (7.68), will fix the Fourier coefficients of the kernels, $\rho(\omega, k)$ and $\chi^{-1}(\omega, k)$, in a unique way.

In the first, directly macroscopic consideration, we write by principle the following macroscopic equations:

$$\frac{\partial B}{\partial t} + \partial_x V = 0, \qquad \frac{\partial D}{\partial t} = \partial_x H - \partial_x \mathcal{P}, \tag{7.104}$$

$$D(t,x) = \hat{\rho} V(t,x) = \int \rho(t-t', x-x') V(t',x') dt' dx', \tag{7.105}$$

$$H(t,x) = -\hat{\chi}^{-1} B(t,x) = -\int \chi^{-1}(t-t', x-x') B(t',x') dt' dx', \tag{7.106}$$

where the H-field is set so that the "Heaviside-Poynting" definition (7.68)

$$\langle pv \rangle (t,x) = -\langle v \rangle (t,x) H(t,x), \tag{7.107}$$

will be satisfied.

The kernels can be represented in Fourier-transforms as

$$\begin{aligned}\rho(t-t',x-x') &= \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\rho(\omega,k)e^{-i\omega(t-t')+ik(x-x')},\\ \chi^{-1}(t-t',x-x') &= \int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\chi^{-1}(\omega,k)e^{-i\omega(t-t')+ik(x-x')}.\end{aligned} \tag{7.108}$$

As the excitation source term is here taken as a single Fourier-component

$$F = -\partial_x \mathcal{P} = -ik\tilde{\mathcal{P}}e^{-i\omega t+ikx}, \tag{7.109}$$

the macroscopic response-fields are sought in the form, (as usual, by language abuse, we dispense noting the tilde on the fields)

$$V = Ve^{-i\omega t+ikx}, \quad \text{etc.} \tag{7.110}$$

The above equations then directly express as

$$-i\omega B = -ikV, \qquad -i\omega D = ikH - ik\tilde{\mathcal{P}}, \tag{7.111}$$

$$D = \rho(\omega,k)V, \qquad H = -\chi^{-1}(\omega,k)B, \tag{7.112}$$

and we find that

$$\rho(\omega,k) = \frac{ik\left(H-\tilde{\mathcal{P}}\right)}{-i\omega V}, \quad \chi^{-1}(\omega,k) = -\frac{H}{B}. \tag{7.113}$$

In the second consideration, starting at the microscopic level, we write the microscopic action-response problem to be solved, (7.1)–(7.6), with the excitation $\boldsymbol{f} = -\partial_x \mathcal{P}\hat{\boldsymbol{x}}, -\partial_x \mathcal{P}$ given by (7.109) in the fluid, and we look for its solution with fields varying like

$$\boldsymbol{v} = \boldsymbol{v}(\omega,k,\boldsymbol{x})e^{-i\omega t+ikx}, \quad \text{etc.} \tag{7.114}$$

If the geometry is stationary random, the solution amplitudes $\boldsymbol{v}(\omega,k,\boldsymbol{x})$, etc., are uniquely fixed by the condition to be stationary random, (and proportional to the excitation amplitude); if it is periodic, they are fixed by the condition to be periodic, (and proportional to the excitation amplitude), with an arbitrary choice to be made, on the minimal periodicities of the tilde-fields.[13]

If we are to use Lorentz's averaging conception we solve the above in one single realization; if we are to use Gibbs' average, we solve it in the different realizations, and for short, above and in what follows, we do not mention the presence of the realization argument ϖ in the amplitudes: $\boldsymbol{v}(\omega,k,\boldsymbol{x}) \to \boldsymbol{v}(\omega,k,\boldsymbol{x};\varpi)$, etc., and in the fluid and pore-surface domains: $\mathcal{V}_f, \partial\mathcal{V} \to \mathcal{V}_f(\varpi), \partial\mathcal{V}(\varpi)$.

The macroscopic velocity $V = Ve^{-i\omega t+ikx}$, and macroscopic condensation, $B = Be^{-i\omega t+ikx}$, in the first consideration, are the macroscopic averages of the (x com-

[13] In that case, here, we only have to precise the integer M_1, fixing the retained period along x.

ponent of) velocity and condensation in the second consideration, i.e.

$$\begin{aligned} V = Ve^{-i\omega t+ikx} &= \langle \boldsymbol{v}(\omega,k,\boldsymbol{x})e^{-i\omega t+ikx}\rangle \cdot \hat{\boldsymbol{x}} \\ &= \langle \boldsymbol{v}(\omega,k,\boldsymbol{x})\rangle \cdot \hat{\boldsymbol{x}} e^{-i\omega t+ikx} = \langle v(\omega,k,\boldsymbol{x})\rangle e^{-i\omega t+ikx}, \end{aligned} \tag{7.115}$$

$$B = Be^{-i\omega t+ikx} = \langle b(\omega,k,\boldsymbol{x})e^{-i\omega t+ikx}\rangle = \langle b(\omega,k,\boldsymbol{x})\rangle e^{-i\omega t+ikx}, \tag{7.116}$$

(these equalities are exactly obtained with Gibbs' averaging, with no restriction on k; with Lorentz's averaging, it is assumed that the macroscopic wavelength $\lambda = 2\pi/k$ is large compared to the homogenization length). Applying the definition (7.107) in which we substitute, (the notation $+c.c.$ adds the complex conjugate)

$$\begin{aligned} p &= \frac{1}{2}\left(p(\omega,k,\boldsymbol{x})e^{-i\omega t+ikx} + c.c.\right), \\ v &= \frac{1}{2}\left(v(\omega,k,\boldsymbol{x})e^{-i\omega t+ikx} + c.c.\right), \\ H &= \frac{1}{2}\left(H(\omega,k)e^{-i\omega t+ikx} + c.c.\right), \end{aligned} \tag{7.117}$$

and noting that the exponentials can be extracted from the averaging symbols in (7.107), (for Lorentz's average, long-wavelength regime is assumed in extracting the exponentials; for Gibbs' average, the extraction is general), we derive, after identification of the terms having same exponentials:

$$\langle p(\omega,k,\boldsymbol{x})v(\omega,k,\boldsymbol{x})\rangle = -\langle v(\omega,k,\boldsymbol{x})\rangle H(\omega,k). \tag{7.118}$$

Requiring then the compatibility of the above two considerations—directly macroscopic, and microscopically averaged—there follows that the kernels operators, by definition, can be computed after solving the microscopic action-response problem (7.1)–(7.6), averaging, and plugging in the definitions (7.113) the following values:

$$V = \langle v(\omega,k,\boldsymbol{x})\rangle, \qquad B = \langle b(\omega,k,\boldsymbol{x})\rangle, \qquad H = -\frac{\langle p(\omega,k,\boldsymbol{x})v(\omega,k,\boldsymbol{x})\rangle}{\langle v(\omega,k,\boldsymbol{x})\rangle}. \tag{7.119}$$

In brief, an "action-response problem" to determine the nonlocal $\rho(\omega,k)$ and $\chi^{-1}(\omega,k)$ is obtained, that can be summarized as follows, (we do not mention the presence of the realization argument, fixed argument ϖ_0 for Lorentz's conception, variable argument ϖ for Gibbs' conception, in the amplitudes and in the fluid and pore-surface domains).

Action-response problem to determine $\rho(\omega,k)$ ***and*** $\chi^{-1}(\omega,k)$:

(i) Let us subject the permeating fluid to the action of a longitudinal bulk force $\boldsymbol{f}$ per unit fluid volume, deriving from a potential $\mathcal{P}$, and varying as follows in the fluid:

$$\boldsymbol{f} = f\hat{\boldsymbol{x}} = -\partial_x \mathcal{P}\hat{\boldsymbol{x}}, \qquad \mathcal{P} = \tilde{\mathcal{P}} e^{-i\omega t + ikx}, \qquad \tilde{\mathcal{P}} = Cst. \tag{7.120}$$

(ii) Let us find the response of the permeating fluid to this action.
We seek fields $\boldsymbol{v} = \boldsymbol{v}(\omega,k,\boldsymbol{x})e^{-i\omega t+ikx}$, $b = b(\omega,k,\boldsymbol{x})e^{-i\omega t+ikx}$, $p = p(\omega,k,\boldsymbol{x})e^{-i\omega t+ikx}$, $\tau = \tau(\omega,k,\boldsymbol{x})e^{-i\omega t+ikx}$, satisfying the motion equations

$$\partial\cdot\boldsymbol{v} + \frac{\partial b}{\partial t} = 0, \qquad \text{in } \mathcal{V}_f, \tag{7.121}$$

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\partial p + \eta\partial^2\boldsymbol{v} + \left(\frac{\eta}{3}+\zeta\right)\partial(\partial\cdot\boldsymbol{v}) + \boldsymbol{f}, \qquad \text{in } \mathcal{V}_f, \tag{7.122}$$

$$\gamma\chi_0 p = b + \beta_0\tau, \qquad \text{in } \mathcal{V}_f, \tag{7.123}$$

$$\rho_0 c_P \frac{\partial\tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa\partial^2\tau, \qquad \text{in } \mathcal{V}_f, \tag{7.124}$$

and

$$\boldsymbol{v} = \boldsymbol{0}, \qquad \text{on } \partial\mathcal{V}, \tag{7.125}$$

$$\tau = 0, \qquad \text{on } \partial\mathcal{V}, \tag{7.126}$$

and whose tilde-amplitudes in $(\omega,k,\boldsymbol{x})$ are proportional to $\tilde{\mathcal{P}}$.

(iii) There is unique solution $\boldsymbol{v}(\omega,k,\boldsymbol{x})$, $b(\omega,k,\boldsymbol{x})$, $p(\omega,k,\boldsymbol{x})$ and $\tau(\omega,k,\boldsymbol{x})$, to this action-response problem. We denote $v(\omega,k,\boldsymbol{x}) = \boldsymbol{v}(\omega,k,\boldsymbol{x})\cdot\hat{\boldsymbol{x}}$. Then according to (7.113), the effective density $\rho(\omega,k)$ and effective bulk modulus $\chi^{-1}(\omega,k)$ are obtained through the definitions

$$\rho(\omega,k) = \frac{ik\left(H-\tilde{\mathcal{P}}\right)}{-i\omega V}, \quad \chi^{-1}(\omega,k) = -\frac{H}{B}, \tag{7.127}$$

where we plug the values (7.119) of V, B, and H:

$$V = \langle v(\omega,k,\boldsymbol{x})\rangle, \ B = \langle b(\omega,k,\boldsymbol{x})\rangle, \ H = -\frac{\langle p(\omega,k,\boldsymbol{x})v(\omega,k,\boldsymbol{x})\rangle}{\langle v(\omega,k,\boldsymbol{x})\rangle}. \tag{7.128}$$

According to (7.102) and (7.103), the operators $\hat{\chi}_V$, $\hat{\rho}$, and $\hat{\chi}_B$, $\hat{\chi^{-1}}$, have Fourier kernels which verify the relations

$$\rho(\omega,k) = \frac{\rho_0}{\phi} + \chi_V(\omega,k), \tag{7.129}$$

and

$$\chi^{-1}(\omega, k) = \frac{1}{\chi_0 \phi} - \chi_B(\omega, k). \tag{7.130}$$

Therefore, the above determination of $\hat{\rho}$ and $\hat{\chi}^{-1}$ is also a determination of $\hat{\chi}_V$ and $\hat{\chi}_B$.

Finally, we note that the physical content of the above procedure is just equivalent to stating that the acoustic equations can be put in the form (7.63)–(7.67) which expresses the electromagnetic-acoustic analogy, with in addition, the H-field, taken according to the "acoustic Heaviside-Poynting" identification, (7.68). We view this identification as a final deepening of the electromagnetic analogy, even if, at present, in macroscopic electromagnetics, we have no clue on the corresponding notion of "energy current density carried out in electromagnetic form".

7.8 Other Points of View

If we were to use the other points of view, sketched in Sect. 7.5, we would write the following.

7.8.1 *Acoustics Translation of the Customary Point of View in Electromagnetics*

Considering that the (7.76)–(7.80) obtained by setting $M \equiv 0$ would apply, we would write, instead of (7.111)–(7.112):

$$-i\omega B = -ikV, \qquad -i\omega D = ikH - ik\tilde{\mathcal{P}}, \tag{7.131}$$

$$D = \rho(\omega, k)V, \qquad H = -\chi^{-1}(\omega, k)B = -\frac{B}{\chi_0 \phi}, \tag{7.132}$$

and it would give

$$\rho(\omega, k) = \frac{-ik\left(\dfrac{B}{\chi_0 \phi} + \tilde{\mathcal{P}}\right)}{-i\omega V}, \quad \chi^{-1}(\omega, k) = \frac{1}{\chi_0 \phi}. \tag{7.133}$$

The action-response problem to determine $\rho(\omega, k)$ would be as follows.

Action-response problem to determine $\rho(\omega, k)$:

(i) Let us subject the permeating fluid to the action of a longitudinal bulk force $\boldsymbol{f}$ per unit fluid volume, deriving from a potential $\mathcal{P}$, and varying as follows in the fluid:

$$\boldsymbol{f} = f\hat{\boldsymbol{x}} = -\partial_x \mathcal{P}\hat{\boldsymbol{x}}, \qquad \mathcal{P} = \tilde{\mathcal{P}} e^{-i\omega t + ikx}, \qquad \tilde{\mathcal{P}} = Cst. \tag{7.134}$$

(ii) Let us find the response of the permeating fluid to this action.
We seek fields $\boldsymbol{v} = \boldsymbol{v}(\omega, k, \boldsymbol{x}) e^{-i\omega t + ikx}$, $b = b(\omega, k, \boldsymbol{x}) e^{-i\omega t + ikx}$, $p = p(\omega, k, \boldsymbol{x}) e^{-i\omega t + ikx}$, $\tau = \tau(\omega, k, \boldsymbol{x}) e^{-i\omega t + ikx}$, satisfying the motion equations

$$\partial \cdot \boldsymbol{v} + \frac{\partial b}{\partial t} = 0, \qquad \text{in } \mathcal{V}_f, \tag{7.135}$$

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\partial p + \eta \partial^2 \boldsymbol{v} + \left(\frac{\eta}{3} + \zeta\right) \partial(\partial \cdot \boldsymbol{v}) + \boldsymbol{f}, \qquad \text{in } \mathcal{V}_f, \tag{7.136}$$

$$\gamma \chi_0 p = b + \beta_0 \tau, \qquad \text{in } \mathcal{V}_f, \tag{7.137}$$

$$\rho_0 c_P \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \partial^2 \tau, \qquad \text{in } \mathcal{V}_f, \tag{7.138}$$

and

$$\boldsymbol{v} = \boldsymbol{0}, \qquad \text{on } \partial\mathcal{V}, \tag{7.139}$$

$$\tau = 0, \qquad \text{on } \partial\mathcal{V}, \tag{7.140}$$

and whose amplitudes in $(\omega, k, \boldsymbol{x})$ are proportional to $\tilde{\mathcal{P}}$.

(iii) There is unique solution $\boldsymbol{v}(\omega, k, \boldsymbol{x})$, $b(\omega, k, \boldsymbol{x})$, $p(\omega, k, \boldsymbol{x})$ and $\tau(\omega, k, \boldsymbol{x})$, to this action-response problem. We denote $v(\omega, k, \boldsymbol{x}) = \boldsymbol{v}(\omega, k, \boldsymbol{x}) \cdot \hat{\boldsymbol{x}}$. Then according to (7.133.1), the effective density $\rho(\omega, k)$ is obtained through the definition

$$\rho(\omega, k) = \frac{-ik\left(\dfrac{B}{\chi_0 \phi} + \tilde{\mathcal{P}}\right)}{-i\omega V}, \tag{7.141}$$

where we plug the following values of V and B

$$V = \langle v(\omega, k, \boldsymbol{x}) \rangle, \quad B = \langle b(\omega, k, \boldsymbol{x}) \rangle. \tag{7.142}$$

This formulation—whose electromagnetic counterpart is just that ordinarily used in literature on spatial dispersion [8–10]—fixes the bulk modulus of the material to an artificial constant value (7.133.2) determined by the adiabatic bulk modulus of the fluid, independently of ω and k. But this constant does not appear quite meaningful, in the present acoustic context, where the pressure can be distributed at the pore scale, (with its direct mean, not necessarily being the meaningful macroscopic variable, see below), and thermal exchanges can occur. It suggests that in electromagnetics, the counterpart setting $\boldsymbol{H} = \boldsymbol{B}/\mu_0$, common in the presence of spatial dispersion, will also be found to be an unappropriate definition, in some respect and some cases.

Finally, let us conclude with the case where, as is done in local theory, we identify the H-field with the volume-averaged opposite pressure (7.86).

7.8.2 Acoustics Formulation in Terms of Volume-Averaged Pressure

Considering that the (7.81)–(7.86) would apply, we would have no change in (7.111)–(7.112), and obtain as before the relations (7.113). The difference would be that, for H, we would have to use the artificial definition, $H = -\langle p \rangle_f$. Therefore, the action-response problem to determine $\rho(\omega, k)$ and $\chi^{-1}(\omega, k)$ would be as follows.

Action-response problem to determine $\rho(\omega, k)$ ***and*** $\chi^{-1}(\omega, k)$:

(i) Let us subject the permeating fluid to the action of a longitudinal bulk force $\boldsymbol{f}$ per unit fluid volume, deriving from a potential $\mathcal{P}$, and varying as follows in the fluid:

$$\boldsymbol{f} = f\hat{\boldsymbol{x}} = -\partial\mathcal{P}, \qquad \mathcal{P} = \tilde{\mathcal{P}}e^{-i\omega t+ikx}, \qquad \tilde{\mathcal{P}} = Cst. \tag{7.143}$$

(ii) Let us find the response of the permeating fluid to this action.
We seek fields $\boldsymbol{v} = \boldsymbol{v}(\omega, k, \boldsymbol{x})e^{-i\omega t+ikx}$, $b = b(\omega, k, \boldsymbol{x})e^{-i\omega t+ikx}$, $p = p(\omega, k, \boldsymbol{x})e^{-i\omega t+ikx}$, $\tau = \tau(\omega, k, \boldsymbol{x})e^{-i\omega t+ikx}$, satisfying the motion equations

$$\partial \cdot \boldsymbol{v} + \frac{\partial b}{\partial t} = 0, \qquad \text{in } \mathcal{V}_f, \tag{7.144}$$

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\partial p + \eta\partial^2\boldsymbol{v} + \left(\frac{\eta}{3} + \zeta\right)\partial(\partial \cdot \boldsymbol{v}) + \boldsymbol{f}, \qquad \text{in } \mathcal{V}_f, \tag{7.145}$$

$$\gamma\chi_0 p = b + \beta_0\tau, \qquad \text{in } \mathcal{V}_f, \tag{7.146}$$

$$\rho_0 c_P \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa\partial^2\tau, \qquad \text{in } \mathcal{V}_f, \tag{7.147}$$

and

$$\boldsymbol{v} = \boldsymbol{0}, \qquad \text{on } \partial\mathcal{V}, \tag{7.148}$$

$$\tau = 0, \qquad \text{on } \partial\mathcal{V}, \tag{7.149}$$

and whose amplitudes in $(\omega, k, \boldsymbol{x})$ are proportional to $\tilde{\mathcal{P}}$.

(iii) There is unique solution $\boldsymbol{v}(\omega, k, \boldsymbol{x})$, $b(\omega, k, \boldsymbol{x})$, $p(\omega, k, \boldsymbol{x})$ and $\tau(\omega, k, \boldsymbol{x})$, to this action-response problem. We denote $v(\omega, k, \boldsymbol{x}) = \boldsymbol{v}(\omega, k, \boldsymbol{x}) \cdot \hat{\boldsymbol{x}}$. Then, the effective density $\rho(\omega, k)$ and effective bulk modulus $\chi^{-1}(\omega, k)$ are obtained through the definitions

$$\rho(\omega,k)=\frac{ik\left(H-\tilde{\mathcal{P}}\right)}{-i\omega V},\quad \chi^{-1}(\omega,k)=-\frac{H}{B}, \tag{7.150}$$

where we plug the following values of V, B, and H

$$V=\langle v(\omega,k,\boldsymbol{x})\rangle,\quad B=\langle b(\omega,k,\boldsymbol{x})\rangle,\quad H=-\frac{\langle p(\omega,k,\boldsymbol{x})\rangle}{\phi}. \tag{7.151}$$

This determination has no counterpart in electromagnetics in the absence of the hypothesized thermodynamic laws associated to the behaviour of polarization charges and currents in matter. When the pressure is distributed at the pore scale because of long-wavelengths local resonances, or short wavelengths, there is no reason that its direct volume average would be the meaningful variable to define an effective bulk modulus, (see again end of Sect. 7.5.3). To repeat ourselves, we expect that the two preceding formulations, contrary to the first (7.104)–(7.107), will not lead to natural generalization when the finite dimensions of materials will be taken into account.

7.9 Characteristic Wavenumbers and Impedances

A characteristic feature of a nonlocal effective medium theory is that it allows for the propagation of several normal waves at a given angular frequency ω. Here, as we focus on the macroscopic propagation along a symmetry axis x, we can have a series of normal waves varying like $e^{-i\omega t+ikx}$, with ω and k related by the dispersion equation:

$$\rho(\omega,k)\chi(\omega,k)\omega^2=k^2. \tag{7.152}$$

At given real ω, as our medium is lossy, the imaginary parts of the wavenumbers $k(\omega)$ solutions to (7.152), will be positive, $\Im\left[k(\omega)\right]>0$, for the waves propagating in the direction $+x$, (that can be created by a source in the direction $-x$). This ensures that these waves are damped. Depending on the case, the waves propagating in the direction $+x$ can have positive or negative sign of $\Re\left[k(\omega)\right]$, corresponding to positive or negative phase velocity, $c_\varphi(\omega)=\omega/\Re\left[k(\omega)\right]$.

In the well-defined stationary-random case, assuming that these solutions $k(\omega)$ are nondegenerate, they can be labelled

$$k=k_n(\omega), \tag{7.153}$$

with a discrete label $n=1,2,\ldots$, which orders them from the least-attenuated mode, to the highly attenuated ones, giving, $0<\Im\left[k_1(\omega)\right]<\Im\left[k_2(\omega)\right]<\cdots$.

The normal-mode fields will have the form, (we do not indicate the realization argument, fixed argument ϖ_0 for Lorentz's conception, variable argument ϖ for

Gibbs' conception), $\boldsymbol{v}(t,\boldsymbol{x})=\tilde{\boldsymbol{v}}_n(\omega,\boldsymbol{x})e^{-i\omega t+ik_n(\omega)x}$, $b(t,\boldsymbol{x})=\tilde{b}_n(\omega,\boldsymbol{x})e^{-i\omega t+ik_n(\omega)x}$, $p(t,\boldsymbol{x})=\tilde{p}_n(\omega,\boldsymbol{x})e^{-i\omega t+ik_n(\omega)x}$, etc., with unique stationary-random amplitude functions, $\tilde{\boldsymbol{v}}_n(\omega,\boldsymbol{x})$, $\tilde{b}_n(\omega,\boldsymbol{x})$, $\tilde{p}_n(\omega,\boldsymbol{x})$, etc., presenting when n increases, more and more rapid variations at the small scale.

To the nth normal-mode solution is associated an H-field varying like $H(t,\boldsymbol{x})=\tilde{H}_n(\omega)e^{-i\omega t+ik_n(\omega)x}$, and having, resp., characteristic amplitudes, $\tilde{H}_n(\omega)=-\langle\tilde{p}_n(\omega,\boldsymbol{x})\tilde{\boldsymbol{v}}_n(\omega,\boldsymbol{x})\rangle\cdot\hat{\boldsymbol{x}}/\langle\tilde{\boldsymbol{v}}_n(\omega,\boldsymbol{x})\rangle\cdot\hat{\boldsymbol{x}}$, in the acoustic Heaviside-Poynting conception (7.63)–(7.68), or $\tilde{H}_n(\omega)=-\chi_0^{-1}\langle\tilde{b}_n(\omega,\boldsymbol{x})\rangle$, in the acoustic conception (7.76)–(7.80) counterpart of conventional electromagnetics, or $\tilde{H}_n(\omega)=-\langle\tilde{p}_n(\omega,\boldsymbol{x})\rangle$, in the acoustic conception (7.81)–(7.86) inspired by the local-theory usual definition of macroscopic pressure as a volumic mean. These conceptions, in turn, lead to define frequency-dependent, characteristic macroscopic modal impedances $(Z_c)_n(\omega)$, all given by, in the different cases

$$(Z_c)_n(\omega)=\frac{-\tilde{H}_n(\omega)}{\langle\tilde{\boldsymbol{v}}_n(\omega,\boldsymbol{x})\rangle\cdot\hat{\boldsymbol{x}}}, \tag{7.154}$$

as well as associated frequency-dependent characteristic modal density and bulk-modulus functions, all given by

$$\rho_n(\omega)=\frac{k_n(\omega)}{\omega}(Z_c)_n(\omega),\qquad \chi_n^{-1}(\omega)=\frac{\omega}{k_n(\omega)}(Z_c)_n(\omega). \tag{7.155}$$

In the presence of resonances, these modal functions may have much more general and complicated behaviours than the simple relaxational-ones, described in the Appendix for the $\rho(\omega)$ and $\chi^{-1}(\omega)$ functions of local theory.

The different formulations we have given of the nonlocal equations and operators, namely based on (7.63)–(7.68) or (7.76)–(7.80) or (7.81)–(7.86), lead to the same wavenumbers but different impedances, densities and compressibilities. Subsequent work, we believe, will show, (in particular when considering inhomogeneous materials), that the latter quantities are best defined in the formulation (7.63)–(7.68) using Heaviside-Poynting's identification (7.68).

In the periodic case we will obtain comparable results, with however ambiguities. There will be direct relation between the normal modes in the considered macroscopic homogeneous medium, and the so-called Bloch modes in one periodic realization. General properties will have to be closely examined in further work. We anticipate that the normal mode wavenumbers defined by the macroscopic Gibbs nonlocal homogeneous medium will allow attributing unambiguously and successively, definite Brillouin zones to a given Bloch-wave, when frequency increases.

7.10 Conclusions

By following an electromagnetic analogy introduced in the Chap. 6, we have shown that the general equations governing sound propagation in a direction x of symmetry,

in macroscopically homogeneous rigid-framed fluid-saturated porous metamaterials, assume the nonlocal Maxwellian pattern of (7.63)–(7.67).

In these equations, V and B are the mean (volume- or ensemble-averaged) x-velocity and condensation, and D and H, related to the former fields by nonlocal constitutive operator relations, can be defined in different manners, leading to different definitions of the operators.

We suggested that there is a physically preferred choice, which will be to choose the H-field according to the "acoustic Heaviside-Poynting" identification (7.68). In future work, this statement will have to be supported by explicitly showing the advantages of this identification (7.68).

In this connection we should mention an imprecision made in preliminary works on the present nonlocal theory [11–14]. The fact that the electromagnetic analogy led directly to the simultaneous definition of the two operators density $\hat{\rho}$ and bulk-modulus $\hat{\chi}^{-1}$, by solving a single action-response problem, (7.120)–(7.128), where the medium is subjected to an external longitudinal force, was missed in these works. Instead of being deduced from the analogy, the action-response procedures for calculating density and bulk-modulus operators were independently postulated, based on what was observed to be true in the unbounded fluid. For density, it led to the correct action-response procedure. But for bulk-modulus, it turns out that the proposed procedure, [11], (71), is a slightly faulty one, which is working correctly only when k is a characteristic wavenumber of the medium. This error could not be detected in the verifications that were made in subsequent work [12–14], because the latter were restricted to considering only the Bloch modes, for which k was, by force, one of the characteristic wavenumbers. When the forcing is made at ω and k equal to a characteristic wavenumber solution $k_n(\omega)$ to the dispersion equation (7.152), a resonance occurs, meaning that a finite response is produced by a vanishingly small forcing. In that case, the faulty procedure [11], (71), gives back the present (7.127.2), because the forcing amplitudes disapppear.

In future work, we will have to clarify if and how the same operators can also be obtained by giving heat instead of doing work.

Finally, we recall that, much remains to be done to generalize the description in the case where the medium has finite dimensions, is anisotropic and poroelastic, and to express all the consequences of nonlocalities.

Appendix: Local Dynamic Homogenization of Rigid-Framed Fluid-Saturated Porous Materials

In some geometries and at long wavelengths, rather than trying to solve the very complex system of coupled (7.1)–(7.6), we can break it down and solve it into independent and simplified pieces, encapsulating the main effects. In the density operator will be put inertial and viscous effects, in the compressibility operator, elastic and thermal effects, evaluated in a corresponding special local limit. At the

macroscopic level where one writes, for macroscopic motion along principal axis x[14]:

$$\frac{\partial \langle b \rangle}{\partial t} + \partial_x \langle v \rangle = 0, \quad \frac{\partial D}{\partial t} = \partial_x H + [F], \tag{7.156}$$

this will mean using a simplified conception of the fields D and H, that will correspond to introducing the approximations:

$$-H\langle v \rangle = \langle pv \rangle \cong \langle p \rangle_f \langle v \rangle, \tag{7.157}$$

(note that, as the wavelengths are large, the mean operation $\langle\,\rangle$ can be conceived in Lorentz's manner), and simultaneously, discarding spatial dispersion:

$$D(t,x) = \hat{\rho}\langle v \rangle(t,x) \cong \int_{-\infty}^{t} \rho(t-t')\langle v \rangle(t',x)dt', \tag{7.158}$$

$$H(t,x) = -\hat{\chi}^{-1}\langle b \rangle(t,x) \cong -\int_{-\infty}^{t} \chi^{-1}(t-t')\langle b \rangle(t',x)dt'. \tag{7.159}$$

This simplified conception will be justified when the geometries are simple, characterized by one typical pore size, meaning that the fluid motion is practically divergence-free at the pore scale. Usually, it is obtained at first order by applying the two-scale asymptotic homogenization method [3, 4], however, the corresponding process does not appear to be truly consistent. Here we will get the same results "with our hands", from the simplification that the spatial dispersion phenomena are absent, and its corollary here, the divergence-free nature of fluid motion at the pore scale.

Dynamic Viscous and Thermal Tortuosities and Permeabilities, and Definition of the Local Density $\rho(\omega)$ and Compressibility $\chi(\omega)$

Here, we reason for convenience with external, long-wavelength force present. Analogous considerations will hold without it, still assuming long-wavelengths.

For the external force F, it suffices to consider a single exponential form, $\boldsymbol{F} = F\hat{\boldsymbol{x}} = -\partial_x \mathcal{P}\hat{\boldsymbol{x}}$, with $\mathcal{P} = \frac{1}{2}(\tilde{\mathcal{P}}e^{-i\omega t+ikx} + c.c.)$, and $\tilde{\mathcal{P}}$ a complex constant. Whenever convenient and without notice, we work in complex representation, e.g. $\mathcal{P} = \tilde{\mathcal{P}}e^{-i\omega t+ikx}$, omitting the real part symbol $\Re\,(\,)$. A long-wavelength limit will be considered, i.e. $kL_h \to 0$, where L_h is a homogenization length.[15]

[14] Generalization to anisotropic materials presents no difficulty.

[15] k is set as $2\pi/\lambda$, with λ taken to be on the same order as the macroscopic wavelength in the medium, (which is unique within local theory).

We will first observe that, as there are no local resonances in the simple geometries considered, the pressure in the fluid must be everywhere, very close to its average value, $\langle p\rangle_f$, where $\langle\,\rangle_f$ is the Lorentz average in the fluid at the given point. Let us indeed write the pressure in terms of its mean and deviatoric part, at the given position:

$$p = \langle p\rangle_f + \delta p, \qquad\qquad \langle \delta p\rangle_f = 0. \tag{7.160}$$

Considering that the gradient of the two terms are comparable, and since $\langle p\rangle_f \sim e^{ikx}$ varies over macroscopic distances $\lambda = 2\pi/k$, whereas δp varies over small-scale distances ℓ, (an estimate of the pore size[16]), we write $\langle p\rangle_f/\lambda \sim \delta p/\ell$. Consequently, the order of magnitude of the deviatoric part δp is that of the mean part $\langle p\rangle_f$, times the very small ratio ℓ/λ. In porous materials used for noise control at audible frequencies, this ratio is typically very small, on order of 10^{-4}, coherent with considering a limit $\ell/\lambda \to 0$. Hence, because of the huge scale separation, we have that

$$p \cong \langle p\rangle_f, \quad \langle pv\rangle \cong \langle p\rangle_f\langle v\rangle, \quad H \cong -\langle p\rangle_f. \tag{7.161}$$

In particular, for the distribution of excess temperature in a representative elementary volume (REV) around a given position x_0, we can consider, instead of the profile of the exact solution of the (7.1)–(7.6), the approximate profile, generated in the following simplified problem, where the pressure field is, in the REV, assimilated to its average part or $(-H)$-part $\cong \langle p\rangle_f$, calculated at the central position:

$$\rho_0 c_P \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial \langle p\rangle_f}{\partial t} + \kappa \partial^2 \tau, \qquad \text{in } \mathcal{V}_f, \tag{7.162}$$

$$\langle p\rangle_f = \text{spatial constant}, \qquad \text{in } \mathcal{V}_f, \tag{7.163}$$

$$\tau = 0, \qquad \text{on } \partial\mathcal{V}. \tag{7.164}$$

With $\langle p\rangle_f = -[\tilde{H}e^{ikx_0}]e^{-i\omega t}$, the field τ in the REV is uniquely fixed by the constant $-[\tilde{H}e^{ikx_0}]$ and the frequency. The field τ solution to the above problem, can be used to define a response function $\alpha'(\omega)$, known as "dynamic thermal tortuosity" [15], determined by the microgeometry, and such that, by definition

$$\rho_0 c_P \alpha'(\omega) \frac{\partial \langle \tau\rangle_f}{\partial t} = \beta_0 T_0 \frac{\partial \langle p\rangle_f}{\partial t}. \tag{7.165}$$

Equivalently, one defines a "dynamic thermal permeability" $k'(\omega)$ by setting [3]

$$\phi \langle \tau\rangle_f = \frac{k'(\omega)}{\kappa} \beta_0 T_0 \frac{\partial \langle p\rangle_f}{\partial t}. \tag{7.166}$$

[16] Note that the idealization that there is mainly "one" pore-size, excludes the presence of Helmholtz resonators: resonators involve widely different sizes in their necks and cavities.

The denominations of "tortuosity" and "permeability" are not quite appropriate here, but they come from an analogy with corresponding viscous functions, seen below. Obviously, both functions are related by

$$k'(\omega)\alpha'(\omega) = \frac{\nu'\phi}{-i\omega}, \tag{7.167}$$

where

$$\nu' \equiv \frac{\kappa}{\rho_0 c_P} = \nu/\,\mathrm{Pr}, \quad \nu \equiv \frac{\eta}{\rho_0}. \tag{7.168}$$

Taking the fluid-average of the equation of state, and its time derivative, we write

$$\gamma\chi_0 \frac{\partial\langle p\rangle_f}{\partial t} = \frac{\partial\langle b\rangle_f}{\partial t} + \beta_0 \frac{\partial\langle\tau\rangle_f}{\partial t}. \tag{7.169}$$

Inserting (7.165) in (7.169) and using the thermodynamic identity seen in the Chap. 6, Sect. 6.2, (6.5), this gives the relation, (after removal of the time derivatives)

$$\chi_0\left[\gamma - \frac{\gamma-1}{\alpha'(\omega)}\right]\langle p\rangle_f = \langle b\rangle_f. \tag{7.170}$$

Therefore in accordance with (7.161) and (7.159), we find a relation having the form, in harmonic regime

$$H = -\chi^{-1}(\omega)\langle b\rangle, \tag{7.171}$$

with

$$\chi(\omega) = \phi\chi_0\left[\gamma - \frac{\gamma-1}{\alpha'(\omega)}\right] = \phi\chi_0\left[\gamma - (\gamma-1)\frac{-i\omega}{\nu'\phi}k'(\omega)\right]. \tag{7.172}$$

This gives the Fourier coefficients of the kernel function $\chi(t)$ in (7.159). It represents a dynamic compressibility, function of frequency because of the thermal exchanges between fluid and solid. This is often written as a relation

$$\chi_0\beta(\omega)\frac{\partial\langle p\rangle_f}{\partial t} = -\partial\cdot\langle\boldsymbol{v}\rangle_f, \tag{7.173}$$

with the function $\beta(\omega)$

$$\beta(\omega) = \frac{\chi(\omega)}{\phi\chi_0} = \gamma - \frac{\gamma-1}{\alpha'(\omega)} = \gamma - (\gamma-1)\frac{-i\omega}{\boldsymbol{v}'(\phi)}\boldsymbol{k}'(\omega) \tag{7.174}$$

a normalized dynamic compressibility, modifying the adiabatic value [3, 15].

In defining the above excess temperature pattern τ, whose average served us to compute the wanted response functions, we made abstraction of the spatial variations of $\langle p\rangle_f$ in the representative volume. Exactly the same average $\langle\tau\rangle_f$ and hence

response functions, would have been obtained, however, if we had also taken into account, to first order, the variation of $\langle p \rangle_f$ in the representative volume: to first order the quantity possesses a linear variation, which doesn't contribute, by symmetry, to the mean excess temperature.

To obtain the Fourier coefficients of the kernel function $\rho(t)$, we proceed in similar manner. To compute the velocity pattern in a REV around a given position x_0, we consider that the fluid is subjected to a spatially-constant macroscopic force

$$\partial H + \boldsymbol{F} \cong -\partial \langle p \rangle_f + \boldsymbol{F} = \left[-ik \left(\tilde{\mathcal{P}} + \langle \tilde{p} \rangle_f \right) e^{ikx_0} \right] e^{-i\omega t} \hat{\boldsymbol{x}}, \tag{7.175}$$

equal to the sum of external force $\boldsymbol{F}$ and macroscopic pressure-gradient force. That is, instead of the exact solution of (7.1)–(7.6), we consider the approximated velocity profile generated in the following simplified problem where $\partial H + \boldsymbol{F}$ is introduced as the constant (7.175):

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\partial(\delta p) + \eta \partial^2 \boldsymbol{v} + \partial H + \boldsymbol{F}, \qquad \text{in } \mathcal{V}_f, \tag{7.176}$$

$$\delta p = \text{bounded, stationary random field}, \qquad \text{in } \mathcal{V}_f, \tag{7.177}$$

$$\partial \cdot \boldsymbol{v} = 0, \qquad \text{in } \mathcal{V}_f, \tag{7.178}$$

$$\boldsymbol{v} = \boldsymbol{0}, \qquad \text{on } \partial\mathcal{V}. \tag{7.179}$$

The condition that δp is a stationary random field, applies to stationary random geometries; in periodic geometries it is to be replaced by the condition that δp is a periodic field. Unambiguously here, because of the incompressibility condition and related spatial constancy of the excitation, the periodicity can be taken as the smallest possible. The solution field $\boldsymbol{v}$ is uniquely fixed by the amplitude in brackets $[\cdot]$ in (7.175), the condition set on δp (bounded field), the frequency, and the macroscopic direction $\hat{\boldsymbol{x}}$. There is however an arbitrary constant in the field δp, and it can be noted that it is fixed to the same value by requiring either $\langle \delta p \rangle = 0$ or $\langle \boldsymbol{v} \delta p \rangle = \boldsymbol{0}$.[17] The solution field $\boldsymbol{v}$ to the above problem, can be used to define a response function $\alpha(\omega)$, known as "dynamic viscous tortuosity", determined by the microgeometry, and such that, by definition [2]

$$\rho_0 \alpha(\omega) \frac{\partial \langle \boldsymbol{v} \rangle_f}{\partial t} = -\partial \langle p \rangle_f + F \hat{\boldsymbol{x}}. \tag{7.180}$$

Equivalently, one defines a "dynamic viscous permeability" $k(\omega)$ by setting

$$\phi \langle \boldsymbol{v} \rangle_f = \frac{k(\omega)}{\eta} \left[-\partial \langle p \rangle_f + F \hat{\boldsymbol{x}} \right]. \tag{7.181}$$

Both functions are related by

[17] Actually it means that within local theory, and as also expressed in (7.161), we do not see the difference between the two definitions $H = -\langle p \rangle_f$ and $H = -\langle pv \rangle / \langle v \rangle$ of the H-field.

$$k(\omega)\alpha(\omega) = \frac{\nu\phi}{-i\omega}. \tag{7.182}$$

This can now be compared with the equation obtained by combining (7.156.2) and (7.158):

$$\rho(\omega)\frac{\partial\langle v\rangle}{\partial t} = \partial_x H + F. \tag{7.183}$$

Recalling (7.10) and (7.161) we see that

$$\rho(\omega) = \frac{\rho_0\alpha(\omega)}{\phi} = \frac{\eta}{-i\omega k(\omega)}. \tag{7.184}$$

It represents a dynamic density, function of frequency because of the viscous effects. For later use, we observe that the dynamic tortuosities $\alpha(\omega)$ and $\alpha'(\omega)$ are related as follows, to the velocity and excess temperature patterns $\boldsymbol{v}$ and τ, (where the star denotes complex conjugate):

$$\frac{\alpha(\omega)}{\phi} = \frac{\langle \boldsymbol{v}\cdot\boldsymbol{v}^*\rangle}{\langle \boldsymbol{v}\rangle\cdot\langle \boldsymbol{v}^*\rangle} + \frac{\nu}{-i\omega}\frac{\langle -\boldsymbol{v}\cdot\partial^2\boldsymbol{v}^*\rangle}{\langle \boldsymbol{v}\rangle\cdot\langle \boldsymbol{v}^*\rangle}, \tag{7.185}$$

and

$$\frac{\alpha'(\omega)}{\phi} = \frac{\langle \tau\tau^*\rangle}{\langle \tau\rangle\langle \tau^*\rangle} + \frac{\nu'}{-i\omega}\frac{\langle -\tau\partial^2\tau^*\rangle}{\langle \tau\rangle\langle \tau^*\rangle}. \tag{7.186}$$

To see this, we take the dot product of (7.176.1) with $\boldsymbol{v}$, and perform volume average:

$$-i\omega\rho_0\langle \boldsymbol{v}\cdot\boldsymbol{v}\rangle = -\langle \boldsymbol{v}\cdot\partial(\delta p)\rangle + \eta\langle \boldsymbol{v}\cdot\partial^2\boldsymbol{v}\rangle + \langle \boldsymbol{v}\cdot(\partial_x H + F)\hat{\boldsymbol{x}}\rangle.$$

The term $\langle \boldsymbol{v}\cdot\partial(\delta p)\rangle$ vanishes after integration by part, owing to incompressibility, no-slip condition, and the stationary random nature of the fields (periodic nature, in periodic geometries). The term $\langle \boldsymbol{v}\cdot(\partial_x H + F)\hat{\boldsymbol{x}}\rangle$ factorizes as $(\partial_x H + F)\langle \boldsymbol{v}\rangle\cdot\hat{\boldsymbol{x}}$ since $(\partial H + F)$ is treated as a spatial constant, (in so doing, spatial dispersion effects are discarded). This gives, dividing by $\langle \boldsymbol{v}\rangle\cdot\hat{\boldsymbol{x}}$,

$$-i\omega\rho_0\frac{\langle \boldsymbol{v}\cdot\boldsymbol{v}\rangle}{\langle \boldsymbol{v}\rangle\cdot\hat{\boldsymbol{x}}} = \frac{\eta\langle \boldsymbol{v}\cdot\partial^2\boldsymbol{v}\rangle}{\langle \boldsymbol{v}\rangle\cdot\hat{\boldsymbol{x}}} + (\partial_x H + F),$$

from which we obtain the representation:

$$\frac{\alpha(\omega)}{\phi} = \frac{\langle \boldsymbol{v}\cdot\boldsymbol{v}\rangle}{\langle \boldsymbol{v}\rangle^2} + \frac{\nu}{-i\omega}\frac{\langle -\boldsymbol{v}\cdot\partial^2\boldsymbol{v}\rangle}{\langle \boldsymbol{v}\rangle^2}. \tag{7.187}$$

The given form (7.185) is then obtained with the same reasoning, if we work with the complex conjugate of the starting equations, and account for the fact that the fields are stationary. The calculation leading to (7.186) is made in analogous manner, by

multiplying the complex conjugate of (7.162) by τ, averaging, and taking into account the stationary random nature of the field.

Finally, knowing $\alpha(\omega)$ or $k(\omega)$, and $\alpha'(\omega)$ or $k'(\omega)$, the effective density and compressibility, are given by (7.184) and (7.172).

Low-Frequency Parameters: k_0, α_0, k'_0, α'_0

In the low-frequency limit, $\omega \to 0$, where viscous effects are "relaxed", viscous shearing motions have time to fully develop in the fluid. It corresponds to a limit of large viscous skin depths $\delta_v = (2\eta/\rho_0\omega)^{1/2}$ with respect to characteristic pore sizes ℓ, i.e. $\omega \ll \ell^2\rho_0/2\eta$. The dynamic permeability tends to Darcy's permeability k_0, and the dynamic tortuosity is mainly purely positive imaginary:

$$\alpha(\omega) \to \frac{\nu\phi}{-i\omega k_0} + \alpha_0, \tag{7.188}$$

with a correction α_0 which is Norris' inertial factor T_0 [16]. We can denote $\boldsymbol{v}_0$ the fluid velocity pattern which appears in d.c. permanent flow ($\omega = 0$):

$$\mathbf{0} = -\partial\delta p_0 + \eta\partial^2\boldsymbol{v}_0 + \left[-\partial_x\langle p\rangle_f + F\right]\hat{\boldsymbol{x}}, \qquad \text{in } \mathcal{V}_f \tag{7.189}$$

$$\partial\cdot\boldsymbol{v}_0 = 0, \qquad \text{in } \mathcal{V}_f \tag{7.190}$$

$$p_0 = \text{stationary random}, \qquad \text{in } \mathcal{V}_f \tag{7.191}$$

$$\boldsymbol{v}_0 = \mathbf{0}, \qquad \text{on } \partial\mathcal{V} \tag{7.192}$$

A comparison of (7.188) with (7.185) taken in the limit $\omega \to 0$ shows that

$$\frac{\phi}{k_0} = \frac{\langle -\boldsymbol{v}_0\cdot\partial^2\boldsymbol{v}_0\rangle_f}{\langle\boldsymbol{v}_0\rangle_f^2}, \quad \alpha_0 = \frac{\langle\boldsymbol{v}_0^2\rangle_f}{\langle\boldsymbol{v}_0\rangle_f^2}, \tag{7.193}$$

where we have suppressed complex conjugates as in this "Poiseuille" limit the velocities at different positions in the pores are all in phase. Parameters k_0 and α_0, resp. dimension of surface and dimensionless, are named static or d.c. permeability and tortuosity.

Likewise, in the low-frequency limit, $\omega \to 0$, where thermal effects are "relaxed", thermal exchanges between solid and fluid have time to occur completely. It corresponds to a limit of large thermal skin depths $\delta_\tau = (2\eta/\rho_0\,\mathrm{Pr}\,\omega)^{1/2}$ with respect to characteristic pore sizes ℓ, i.e. $\omega \ll \ell^2\rho_0\,\mathrm{Pr}\,/2\eta$. The solid, which generally has large thermal inertia, imposes its steady ambient temperature to the fluid. The condensation-rarefactions occur in isothermal regime and the fluid "effective bulk modulus" is equal to the isothermal modulus K_0, equal to P_0 the ambient pressure, meaning that $\beta(\omega) \to \gamma$, the adiabatic constant of the saturating fluid. In this limit, corrections to the leading isothermal behaviour are easily obtained by using (7.174)

and by noting that the dynamic thermal permeability $k'(\omega)$ tends to Lafarge's d.c. thermal permeability k'_0, (equal to the inverse trapping constant, see [3, 17]), and the dynamic thermal tortuosity is mainly purely positive imaginary:

$$\alpha'(\omega) \to \frac{\nu'\phi}{-i\omega k'_0} + \alpha'_0. \tag{7.194}$$

We can denote τ_0 the fluid excess temperature pattern which appears in d.c. regime, (where d.c. is here only in the sense $\omega \to 0$, and not $\omega = 0$):

$$0 = \kappa\partial^2\tau_0 + \beta_0 T_0 \partial\langle p\rangle_f/\partial t, \qquad \text{in } \mathcal{V}_f, \tag{7.195}$$

$$\tau_0 = 0, \qquad \text{on } \partial\mathcal{V}. \tag{7.196}$$

A comparison of (7.194) with (7.186) shows that

$$\frac{\phi}{k'_0} = \frac{\langle -\tau_0\partial^2\tau_0\rangle_f}{\langle\tau_0\rangle_f^2}, \quad \alpha'_0 = \frac{\langle\tau_0^2\rangle_f}{\langle\tau_0\rangle_f^2}, \tag{7.197}$$

where we have as before suppressed complex conjugates as the temperature variations in the pores are all in phase in this d.c. limit.

High-Frequency Parameters: α_∞, Λ, Λ'

In the opposite high-frequency limit, $\omega \to \infty$, where the viscous effects are "frozen", viscous shearing motions only have time to develop in an immediate vicinity of the pore walls. It corresponds to a limit of small viscous skin depths, $\omega \gg \ell^2\rho_0/2\eta$. The fluid velocity tends to the ideal-fluid flow pattern $\boldsymbol{v}_\infty$, such that:

$$\rho_0\frac{\partial\boldsymbol{v}_\infty}{\partial t} = -\partial\delta p_\infty + \left[-\partial_x\langle p\rangle_f + F\right]\hat{\boldsymbol{x}}, \qquad \text{in } \mathcal{V}_f, \tag{7.198}$$

$$\partial\cdot\boldsymbol{v}_\infty = 0, \qquad \text{in } \mathcal{V}_f, \tag{7.199}$$

$$\delta p_\infty = \text{stationary random}, \qquad \text{in } \mathcal{V}_f, \tag{7.200}$$

$$\boldsymbol{v}_\infty\cdot\hat{\boldsymbol{n}} = 0, \qquad \text{in } \partial\mathcal{V}. \tag{7.201}$$

excepted for a vanishingly small viscous boundary layer region at the pore walls, and for an additional potential-flow perturbation in the bulk, of comparatively vanishingly small amplitude.

The above leading order ideal-fluid flow pattern can be described in terms of the scaled electric field $\boldsymbol{E}$ that appears in the pores, when the fluid is conducting, the solid is insulating, and a unit macroscopic electric field $1\hat{\boldsymbol{x}}$ is imposed, (see [17]):

$$\boldsymbol{E} = -\partial\Phi + \hat{\boldsymbol{x}}, \qquad \text{in } \mathcal{V}_f, \tag{7.202}$$

$$\partial \cdot \boldsymbol{E} = 0, \qquad \text{in } \mathcal{V}_f, \tag{7.203}$$

$$\Phi = \text{stationary random}, \qquad \text{in } \mathcal{V}_f, \tag{7.204}$$

$$\boldsymbol{E} \cdot \hat{\boldsymbol{n}} = 0, \qquad \text{on } \partial\mathcal{V}. \tag{7.205}$$

Obviously,

$$\boldsymbol{v}_\infty = \frac{\left[-\partial_x \langle p \rangle_f + F\right]}{-i\omega\rho_0}\boldsymbol{E}, \quad \delta p_\infty = \left[-\partial_x \langle p \rangle_f + F\right]\Phi. \tag{7.206}$$

The actual fluid velocity in the high-frequency limit, will be

$$\boldsymbol{v} = \boldsymbol{v}_\infty + \delta\boldsymbol{v}, \tag{7.207}$$

with $\boldsymbol{v}_\infty \propto \boldsymbol{E}$ the above ideal-fluid profile, and $\delta\boldsymbol{v} = \delta\boldsymbol{v}_p + \delta\boldsymbol{v}_v$ made of two parts: the vortical boundary-layer part, $\delta\boldsymbol{v}_v$, such that $\eta\partial^2\boldsymbol{v} = \eta\partial^2\delta\boldsymbol{v}_v \neq \boldsymbol{0}$, rapidly decreasing away from the pore-walls, and the additional small-amplitude bulk potential-flow perturbation, $\delta\boldsymbol{v}_p$, $\eta\partial^2\boldsymbol{v}_p = \boldsymbol{0}$, with no contribution to the Laplacian viscous term $\eta\partial^2\boldsymbol{v}$.

As regards the fluid excess-temperature pattern in the high-frequency limit, $\omega \to \infty$, where thermal exchanges are "frozen", (small thermal skin depths, $\omega \gg \ell^2\rho_0 c_P/2\kappa$), and again because we make the important simplification that the fluid motion is that of an incompressible fluid at the pore scale, it tends to the adiabatic constant profile τ_∞, such that:

$$\rho_0 c_P \frac{\partial\tau_\infty}{\partial t} = \beta_0 T_0 \frac{\partial\langle p \rangle_f}{\partial t}, \qquad \text{in } \mathcal{V}_f, \tag{7.208}$$

or

$$\tau_\infty = \frac{\beta_0 T_0}{\rho_0 c_P}\langle p \rangle_f, \qquad \text{in } \mathcal{V}_f, \tag{7.209}$$

excepted for a vanishingly small thermal boundary layer region at the pore walls. The actual excess-temperature in the high-frequency limit, will be:

$$\tau = \tau_\infty + \delta\tau, \tag{7.210}$$

with $\delta\tau$, a diffusive part near the pore-walls, associated with $\kappa\partial^2\tau = \kappa\partial^2\delta\tau \neq 0$, and rapidly decreasing away from the pore-walls.

As shown below, in the present simplification a consistent boundary-layer calculation gives

$$\alpha(\omega) = \alpha_\infty\left[1 + \left(\frac{\nu}{-i\omega}\right)^{1/2}\frac{2}{\Lambda} + \cdots\right], \; \alpha'(\omega) = \alpha'_\infty\left[1 + \left(\frac{\nu'}{-i\omega}\right)^{1/2}\frac{2}{\Lambda'} + \cdots\right], \tag{7.211}$$

when the pore-walls $\partial\mathcal{V}$ are assumed locally plane, where:

$$\alpha_\infty = \frac{\langle \boldsymbol{v}_\infty^2 \rangle_f}{\langle \boldsymbol{v}_\infty \rangle_f^2}, \qquad \alpha'_\infty = 1, \tag{7.212}$$

$$\frac{2}{\Lambda} = \frac{\int_{\partial\mathcal{V}} \boldsymbol{v}_\infty^2 dS}{\int_{\mathcal{V}_f} \boldsymbol{v}_\infty^2 dV}, \quad \frac{2}{\Lambda'} = \frac{\int_{\partial\mathcal{V}} dS}{\int_{\mathcal{V}_f} dV}, \tag{7.213}$$

where $\int_{\partial\mathcal{V}} dS$ denotes integral on the pore-walls, and $\int_{\mathcal{V}_f} dV$ integral in the fluid volume. Here, α_∞ is the ideal-fluid or electrical tortuosity ($\langle \boldsymbol{E} \rangle_f \cdot \boldsymbol{e} = 1/\alpha_\infty$), Λ is Johnson's viscous characteristic length [2] and Λ' is Allard's thermal characteristic length [18].

To show (7.211)–(7.213), we rectify, in the presence of thermal effects, a reasoning given in [2], (see [19] for a different, more involved but direct calculation). It consists in requiring the compatibility of two sound attenuation calculations, one directly macroscopic and the other starting at the microscopic level. Looking for plane-wave solutions varying as $e^{-i\omega t + iq(\omega)x}$, of the macroscopic motion equations (7.156)–(7.159), (without the source term), we find a dispersion equation

$$\rho_0 \alpha(\omega) \chi_0 \beta(\omega) \omega^2 = q(\omega)^2. \tag{7.214}$$

On a macroscopic scale, the intensity will decay like $e^{-2q''x}$, with $q'' = \Im q(\omega)$. With asymptotics (7.211), the attenuation constant q'' is found to be:

$$q'' = \frac{\omega}{c_0}\sqrt{\alpha_\infty}\left[\frac{1}{\Lambda}\sqrt{\frac{\nu}{2\omega}} + (\gamma - 1)\frac{1}{\Lambda'}\sqrt{\frac{\nu'}{2\omega}}\right] + \cdots \tag{7.215}$$

On the other hand, starting at the microscopic scale, with a classical reasoning given in Landau and Lifshitz [20], p. 299, q'' can be related to the velocity and excess-temperature fields $\boldsymbol{v}(\boldsymbol{x})$ and $\tau(\boldsymbol{x})$:

$$q'' = \frac{\overline{|\dot{E}_{mech}|}}{2\overline{S_0}}, \tag{7.216}$$

where $\overline{S_0}$ is the time-averaged acoustic energy flux calculated by making abstraction of the effect of losses,[18] and $\overline{\dot{E}_{mech}}$ is the time-averaged rate of energy dissipation per unit total volume $V = V_f/\phi$. The mean acoustic energy flux $\overline{S_0}$ is estimated as $c\overline{E}$ where c is the speed of sound without losses, $c = c_0/\sqrt{\alpha_\infty}$, and $\overline{E}$ is the mean acoustic energy per unit total volume, equal to twice the mean kinetic or potential energy per unit total volume:

[18] Equation (7.216) is a first order calculation, which will give the leading term in (7.215). The losses determine the numerator so they need not be taken into account in the denominator.

$$\overline{S_0} = \frac{c_0}{\sqrt{\alpha_\infty}} 2 \frac{1}{V} \int_{V_f} \frac{1}{2} \rho_0 \overline{\boldsymbol{v}_\infty^2(t,\boldsymbol{x})} dV = \frac{c_0}{\sqrt{\alpha_\infty}} \frac{1}{V} \int_{V_f} \frac{1}{2} \rho_0 |\boldsymbol{v}_\infty(\boldsymbol{x})|^2 dV$$
$$= \frac{c_0}{\sqrt{\alpha_\infty}} 2 \frac{1}{V} \int_{V_f} \frac{1}{2} \chi_0 \overline{\langle p_\infty \rangle_f^2(t,\boldsymbol{x})} dV = \frac{c_0}{\sqrt{\alpha_\infty}} \frac{1}{V} \int_{V_f} \frac{1}{2} \chi_0 |\langle p_\infty(\boldsymbol{x}) \rangle_f|^2 dV, \tag{7.217}$$

where in the right are complex amplitudes, and we have added index ∞ to recall high-frequency asymptotic limit.

The rate of energy dissipation per unit volume $\dot{E}_{mech}$ is [20]:

$$\dot{E}_{mech} = -\frac{\kappa}{T_0} \frac{1}{V} \int (\partial \tau)^2 \, dV + \cdots$$
$$- \frac{1}{2} \eta \frac{1}{V} \int \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial v_l} \right)^2 dV - \zeta \frac{1}{V} \int (\partial \cdot \boldsymbol{v})^2 \, dV,$$

(this is the volume integral of the dissipation rate $-\mathcal{D}$, see the Chap. 6, (6.111)), and after average over a cycle and replacement of the real quantities by their complex amplitudes, it reads

$$\overline{\dot{E}_{mech}} = -\frac{\kappa}{T_0} \frac{1}{2} \frac{1}{V} \int |\partial \tau|^2 dV - \frac{1}{4} \eta \frac{1}{V} \int |\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}|^2 dV, \tag{7.218}$$

where the incompressibility of the fluid on the size scale of the pores has been accounted for. By integrations by parts, we have:

$$\int |\partial \tau|^2 \, dV = \int \frac{\partial}{\partial x_k} \left(\tau \frac{\partial \tau^*}{\partial x_k} \right) dV - \int \tau \partial^2 \tau^* dV,$$
$$\int \left| \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right|^2 dV = 2 \int \frac{\partial}{\partial x_k} \left(v_i \frac{\partial v_i^*}{\partial x_k} \right) dV + 2 \int \frac{\partial}{\partial x_k} \left(v_i \frac{\partial v_k^*}{\partial x_i} \right) dV + \cdots$$
$$- 2 \int \left(v_i \frac{\partial}{\partial x_i} \frac{\partial v_k^*}{\partial x_k} \right) dV - 2 \int v_i \partial^2 v_i^* dV.$$

In the two expressions, only the last integrals subsist: for the total derivative integrals, their vanishing comes from the stationary nature of the fields and the vanishing of velocity and excess temperature on $\partial \mathcal{V}$; for the above third integral, its vanishing comes from incompressibility. The remaining two integrals come from the viscous and thermal boundary layers where $\partial^2 v_i^*$ and $\partial^2 \tau^*$, nonzero, are rapidly decreasing. To evaluate these integrals, a high-frequency limit is considered, in which the pore-surface interface $\partial \mathcal{V}$ is assumed to be locally plane at the scale of the relevant boundary-layer thicknesses, resp. $\delta = (2\nu/\omega)^{1/2}$ for the velocity field, and $\delta' = \left(2\nu'/\omega\right)^{1/2}$ for the excess temperature field. In this case, we find by a classic

calculation, (boundary-layer profile near a flat surface [20], p. 91), that the excess-temperature and velocity fields have the following form, where $\boldsymbol{x}_w$ denotes a position on $\partial\mathcal{V}$, and ξ is a coordinate along the normal, ($-\hat{\boldsymbol{n}}\xi = \boldsymbol{x} - \boldsymbol{x}_w$, with $\hat{\boldsymbol{n}}$, the outward normal to the fluid region at $\boldsymbol{x}_w$):

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}_\infty(\boldsymbol{x}_w)\left[1 - e^{ik_v\xi}\right], \quad \tau(\boldsymbol{x}) = \tau_\infty(\boldsymbol{x}_w)\left[1 - e^{ik_\tau\xi}\right], \tag{7.219}$$

and $k_v = (1+i)/\delta$ and $k_\tau = (1+i)/\delta'$ are the shear and entropic wavenumbers, (6.66) and (6.57.2), of the Chap. 6. Substituting (7.219) in the integrals and making the integrations $\int dV$ in the form of $\int_{\partial\mathcal{V}} dS \int_0^\infty d\xi$, we find

$$\begin{aligned}\overline{\dot{E}_{mech}} = &-\frac{1}{T_0 2V}\left(\frac{1}{2}\omega\kappa\rho_0 c_P\right)^{1/2}\int_{\partial\mathcal{V}}|\tau_\infty|^2 dS + \cdots \\ &-\frac{1}{2V}\left(\frac{1}{2}\omega\eta\rho_0\right)^{1/2}\int_{\partial\mathcal{V}}|\boldsymbol{v}_\infty(\boldsymbol{x}_w)|^2 dS.\end{aligned} \tag{7.220}$$

Then dividing the two terms by the appropriate (kinetic-energy or potential-energy) form of (7.217), we get, by (7.216), the following expression of q'':

$$\begin{aligned}q'' = &\frac{1}{T_0 2V}\left(\frac{1}{2}\omega\kappa\rho_0 c_P\right)^{1/2}\frac{\int_{\partial\mathcal{V}}|\tau_\infty(\boldsymbol{x}_w)|^2 dS}{2\dfrac{c_0}{\sqrt{\alpha_\infty}}\dfrac{1}{V}\int\dfrac{1}{2}\chi_0|\langle p_\infty(\boldsymbol{x})\rangle_f|^2 dV} + \cdots \\ &+\frac{1}{2V}\left(\frac{1}{2}\omega\eta\rho_0\right)^{1/2}\frac{\int_{\partial\mathcal{V}}|\boldsymbol{v}_\infty(\boldsymbol{x}_w)|^2 dS}{2\dfrac{c_0}{\sqrt{\alpha_\infty}}\dfrac{1}{V}\int\dfrac{1}{2}\rho_0|\boldsymbol{v}_\infty(\boldsymbol{x})|^2 dV} + \cdots\end{aligned} \tag{7.221}$$

According to (7.209), $|\langle p_\infty\rangle_f|^2$ can be replaced by $|\tau_\infty|^2\,(\beta_0 T_0/\rho_0 c_P)^2$, so that, after using the general thermodynamic relation Equation (5) of the Chap. 6, we obtain

$$q'' = \frac{\omega}{c_0}\sqrt{\alpha_\infty}\left[\frac{\int_{\partial\mathcal{V}}|\boldsymbol{v}_\infty|^2 dS}{2\int_{\mathcal{V}_f}|\boldsymbol{v}_\infty|^2 dV}\sqrt{\frac{\nu}{2\omega}} + (\gamma-1)\frac{\int_{\partial\mathcal{V}}|\tau_\infty|^2 dS}{2\int_{\mathcal{V}_f}|\tau_\infty|^2 dV}\sqrt{\frac{\nu'}{2\omega}}\right] + \cdots \tag{7.222}$$

Comparison with (7.215) shows that this expression of q'' justifies the asymptotics (7.211), with expressions (7.212) and (7.213) of the parameters. The $|.|$ are useless and can be removed, as the different velocities and temperatures are in phase in a REV. The factor of τ_∞^2 do not appear in the characteristic thermal length, because, as it is a pore-scale constant, it can be removed from numerator and denominator.[19]

[19] In nonlocal theory, there will be a generalization to be made, and when the pressure is distributed at the pore scale, the factors $|\tau_\infty|^2$ should be maintained since they are not constant.

Intermediate Frequencies: Constraints, and Simple Model-Functions

At intermediate frequencies, the functions $\alpha(\omega)$ and $\alpha'(\omega)$ will interpolate smoothly between the low-frequencies and high-frequencies behaviours (7.188), (7.194) and (7.211). Indeed, if we write these functions in the form:

$$\alpha(\omega) = \frac{\eta\phi}{-i\omega\rho_0 k_{app}(\omega)} + \alpha_{app}(\omega),\ \alpha'(\omega) = \frac{\kappa\phi}{-i\omega\rho_0 c_P k'_{app}(\omega)} + \alpha'_{app}(\omega), \tag{7.223}$$

the introduced new functions $k_{app}(\omega)$, $\alpha_{app}(\omega)$, $k'_{app}(\omega)$, $\alpha'_{app}(\omega)$, are apparent tortuosities and permeabilities in intermediate regime, given by (see (7.185) and (7.186)):

$$\frac{\phi}{k_{app}} = \frac{\langle -\boldsymbol{v}\cdot\partial^2\boldsymbol{v}^\star\rangle_f}{\langle\boldsymbol{v}\rangle_f\cdot\langle\boldsymbol{v}^\star\rangle_f}, \qquad \alpha_{app} = \frac{\langle\boldsymbol{v}\cdot\boldsymbol{v}^\star\rangle_f}{\langle\boldsymbol{v}\rangle_f\cdot\langle\boldsymbol{v}^\star\rangle_f}, \tag{7.224}$$

$$\frac{\phi}{k'_{app}} = \frac{\langle -\tau\partial^2\tau^\star\rangle_f}{\langle\tau\rangle_f\langle\tau^\star\rangle_f}, \qquad \alpha'_{app} = \frac{\langle\tau\tau^\star\rangle_f}{\langle\tau\rangle_f\langle\tau^\star\rangle_f}, \tag{7.225}$$

where $\boldsymbol{v}$ and τ are the complex patterns of the velocity and excess temperature fields appearing at frequency ω, *and we can show that, because of the divergence-free nature of the pore-scale motion, these new functions are monotonic strictly decreasing functions of frequency* [21]:

$$\frac{dk_{app}}{d\omega} < 0, \quad \frac{d\alpha_{app}}{d\omega} < 0, \quad \frac{dk'_{app}}{d\omega} < 0, \quad \frac{d\alpha'_{app}}{d\omega} < 0. \tag{7.226}$$

These behaviours are related to the condition, (expressing the divergence-free motion), that the singularities, poles, and zeros, of the functions $\alpha(\omega)$, $k(\omega)$, and $\alpha'(\omega)$, $k'(\omega)$, of complex ω, are located on the negative imaginary ω axis, (see [2, 3, 17]). This is a much more severe condition than that imposed solely by causality. It excludes resonant behaviours because the effect of spatial nonlocalities is not accounted for, in the modeling. There follows that there are relatively simple analytical models of the frequency dependence of functions $\alpha(\omega)$ and $\alpha'(\omega)$, which allow them to be represented at any real frequency, with reasonable precision, in terms of the physical constants of the fluid and the preceding low- and high-frequency geometric parameters.

To express the characteristic relaxation these functions present, it is appropriate to consider their inverses $1/\alpha(\omega)$ and $1/\alpha'(\omega)$. These inverses pass from the values $[\alpha^{-1}]_0 = [\alpha'^{-1}]_0 = 0$ at $\omega = 0$, to the values $[\alpha^{-1}]_\infty = 1/\alpha_\infty$ and $[\alpha'^{-1}]_\infty = 1$ at $\omega = \infty$. Thus we write them in terms of relaxation functions $\chi(\omega)$ and $\chi'(\omega)$, relaxing from 1, at $\omega = 0$, in relaxed state, to 0, at $\omega = \infty$, in frozen state:

$$\begin{aligned}\frac{1}{\alpha(\omega)} &= [\alpha^{-1}]_\infty + \left([\alpha^{-1}]_0 - [\alpha^{-1}]_\infty\right)\chi(\omega) = \frac{1}{\alpha_\infty}\left(1-\chi(\omega)\right),\\ \frac{1}{\alpha'(\omega)} &= [\alpha'^{-1}]_\infty + \left([\alpha'^{-1}]_0 - [\alpha'^{-1}]_\infty\right)\chi'(\omega) = 1-\chi'(\omega).\end{aligned} \tag{7.227}$$

Based on the property that the poles and zeros are on the negative imaginary axis, and the limiting behaviours (7.188), (7.194), and (7.211), must be satisfied, it can be anticipated that the following simple expressions of the relaxation functions $\chi(\omega)$ and $\chi'(\omega)$, will provide a very satisfactory full-frequency description, (with discrepancies in the transition region, less than a few per cent):

$$\frac{1}{\chi(\omega)} = 1 - \frac{ix}{S(x)}, \qquad \frac{1}{\chi'(\omega)} = 1 - \frac{ix'}{S'(x')}, \tag{7.228}$$

$$S(x) = 1 - \mathtt{P} + \mathtt{P}\sqrt{1 - \frac{ixM}{2\mathtt{P}^2}}, \qquad S'(x') = 1 - \mathtt{P}' + \mathtt{P}'\sqrt{1 - \frac{ix'M'}{2\mathtt{P}'^2}}, \tag{7.229}$$

$$x = \frac{\omega}{\omega_v}, \qquad x' = \frac{\omega}{\omega_\tau}, \tag{7.230}$$

$$\omega_v = \frac{\nu}{Fk_0}, \quad \left(F = \frac{\alpha_\infty}{\phi}\right), \qquad \omega_\tau = \frac{\nu'}{F'k_0'}, \quad \left(F' = \frac{1}{\phi}\right), \tag{7.231}$$

$$M = \frac{8Fk_0}{\Lambda^2}, \qquad M' = \frac{8F'k_0'}{\Lambda'^2}, \tag{7.232}$$

$$\mathtt{P} = \frac{M}{4\left(\dfrac{\alpha_0}{\alpha_\infty} - 1\right)}, \qquad \mathtt{P}' = \frac{M'}{4\left(\alpha_0 - 1\right)}. \tag{7.233}$$

The quantities ω_v and ω_τ, are characteristic viscous and thermal, pulsation rollover frequencies, between low and high frequency regimes. The purely geometrical quantities M, $\mathtt{P}$, M' and $\mathtt{P}'$, are dimensionless, viscous and thermal, form factors of order one, that will determine the precise shape of the viscous and thermal relaxation. The notation M is borrowed from [2]. The notation $\mathtt{P}$ ($\mathtt{P}$ for Pride), refers to the paper [22] rectified in [15]. When $\mathtt{P}$ is set to the value 1, one obtains the model of $\alpha(\omega)$ proposed by Johnson et al. [2]. When M' and $\mathtt{P}'$ are set to the value 1, one obtains the model of $\beta(\omega)$ proposed by Allard and Champoux [18]; when only $\mathtt{P}' = 1$, the model proposed by Lafarge et al. [3].

The behaviours described by these functions closely remind behaviours observed in electromagnetics of dielectric and ionic conductors. With forms factors taken around 1, the above relaxation forms of the functions $\chi(\omega)$ and $\chi'(\omega)$, best represented using Cole-Cole (also named Argand) plots ($\Im\chi(\omega)$ versus $\Re\chi(\omega)$), are roughly comparable to a Davidson-Cole relaxation with exponent 1/2: $\chi, \chi' \approx 1/(1-i\omega\Theta)^{1/2}, 1/(1-i\omega\Theta')^{1/2}$. When the M M' are small, the relaxation tends to Debye's relaxation $\chi, \chi' \approx 1/(1-i\omega\Theta), 1/(1-i\omega\Theta')$, excepted at sufficiently high frequencies. Notably, also, it can be observed that the characteristic high-frequency limits (7.211) imply, for the relaxation functions, the limiting behaviours,

$\omega \to \infty, \chi, \chi' \approx (M\omega_v/-2i\omega\mathbb{P}^2)^{1/2}, (M'\omega_\tau/-2i\omega\mathbb{P}'^2)^{1/2}$. For the associated time-domain functions $\chi(t)$ and $\chi'(t)$, this give laws of the empirical "Curie-von Schweidler" power-law type [23, 24]: $\chi(t)$ and $\chi'(t) \propto t^{-1/2}$, for the typical decrease of any field, at small times, just after an excitation. This fractional power-law type of behaviour has caused interrogations in electromagnetics; it has been interpreted as indicative of a many-body problem, revealing collective behaviour of electrons. Here, it is interpreted in terms of the existence of lossy boundary layers; following [17] we could show that it expresses in terms of fractal accumulation of viscous or thermal relaxation times, near zero, with dimension $1/2$ (see [17], Appendix C). With pore-walls taken cusped with some fractality, instead of being locally plane, we would expect obtaining a similar relaxational behaviour, but with different values for the exponent (because of the different fractal dimension of the accumulation), leading in particular to the more general form of the Curie-von Schweidler law: $\chi(t)$, $\chi'(t) \propto t^{-n}, t^{-n'}, 0 < n \text{ and } n' < 1$.

Finally, let us recall the situation in electromagnetics of non-ferromagnetic materials, when one considers that the dispersion effects on the magnetic susceptibility $\mu(\omega)$ are small relativistic effects, of second order on the small parameter $\beta = v/c$, (with v an estimate of electronic velocities in molecules, and c the speed of light). In this case, because of the smallness of β and quadratic (β^2) nature of relativistic effects, the low-frequency and high-frequency values of $\mu(\omega)$ coincide, $\mu(\omega) \cong \mu_0$, and no dispersion intervenes. There is a similar situation here when the permeating fluid is a liquid. In a simple fluid there is the general thermodynamic identity (Chap. 6, (6.5)):

$$\frac{\beta_0^2 T_0 c_0^2}{c_P} = \gamma - 1, \tag{7.234}$$

where β_0 is the thermal expansion coefficient of the fluid, T_0 is ambient temperature, c_0 is the adiabatic velocity of sound and c_P is the heat coefficient at constant pressure. It turns out that in a liquid, $\gamma \cong 1$, because the deviation $\gamma - 1$ is a quadratic effect on the thermal expansion coefficient β_0, which is very small. Therefore, for a liquid-saturated material *described by the present local theory*, (i.e. when the material has a sufficiently *simple microstructure*), the low-frequency and high-frequency bulk-moduli values coincide, $\beta(\omega) \cong 1$, and no dispersion intervenes on the compressibility.

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