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# Chapter 6

## Acoustic Wave Propagation in Viscothermal Fluids



### An Electromagnetic Analogy

Denis Lafarge

**Abstract** First, we recall the Navier-Stokes-Fourier model linearized equations, which govern the propagation of small amplitude, long wavelength waves in viscothermal fluids; we specify how these equations are derived from several thermodynamic simplifications, and examine some of their solutions. Then, we analyze the general pattern of macroscopic nonlocal equations of propagation of small amplitude electromagnetic waves in effective homogeneous media, taking into account both the temporal and spatial dispersion. We argue that we lack a whole thermodynamics to fully precise all intervening quantities; proceeding by analogy, we then suggest that for the general acoustics of a homogeneous fluid, an analogous general pattern of nonlocal equations of propagation would arise, if we had sufficient thermodynamics. These ideas are finally implemented to obtain, within the available Navier-Stokes-Fourier's model, a nonlocal description of compressional waves.

### 6.1 Introduction

In the many works carried out on acoustic phononic crystals and metamaterials, viscous and thermal losses are usually neglected. This is often unrealistic. In this chapter, we first recall the classical theory of small amplitude wave propagation in a viscothermal fluid, and next reexamine it in a new light: that of a deep *electromagnetic analogy*. Indeed, we show that to some extent it can be put in the form of “nonlocal acoustic” equations allowing for both temporal and spatial dispersion. This is closely similar to the general nonlocal Maxwell equations that describe electromagnetic wave propagation in dispersive materials with temporal and spatial dispersion. Working within the usual (Navier-Stokes-Fourier) near-equilibrium thermodynamic framework, only a reduced version of this analogy, concerned with compressional motions only, will be accessible here. As outlined in the Appendix, we believe that the anal-

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ogy would undertake its more complete nondegenerate form, (including Frenkel's solid-type of shearing at short times, see Sect. 6.6.2), if the thermodynamic framework could be sufficiently extended. This extension, however, is at present time a distant perspective.

In the next chapter the reduced analogy describing compressional waves in the homogeneous fluid, then is generalized to describe macroscopic compressional waves along a symmetry axis in macroscopically homogeneous unbounded metamaterials constituted of the viscothermal fluid permeating a rigid<sup>1</sup> porous structure having arbitrary microgeometry. It expresses in a nonlocal dynamic homogenization attributing to the medium, from microstructure, an equivalent nonlocal density and bulk modulus, playing the role of nonlocal electric and (inverse) magnetic susceptibilities. The developed nonlocal homogenization description is new and, we believe, will predict the right metamaterial properties when generalized further to describe inhomogeneous materials, and in particular, materials having finite dimensions.<sup>2</sup>

Finally, in Appendix of next chapter we recall how the developed nonlocal description simplifies in a classic local one, in some simple microgeometries. It is more convenient not to derive this simplified local description from the general nonlocal one, but to calculate afresh the relevant quantities.

Usually, this limit is deduced from a long-wavelength condition by blindly applying the method of two-scale asymptotic homogenization<sup>3</sup>; in reality, this "deduction" is not only illusory, (in it, is tacitly introduced a prerequisite of the local theory, namely, that the microgeometry is "simple" i.e. does not involve very different characteristic pore sizes), it is also fundamentally faulty. Here, we deduce the same limit in a better physical way by explicitly granting *as a simplification of the real problem*, the total absence of spatial dispersion.<sup>4</sup> As this total absence is the exact, (but unphysical), expression of the leading order terms of the method of two-scale asymptotic homogenization, (considered here, in the case of rigid porous structure) it should be clear, a priori, that the latter method cannot be an entirely consistent method of homogenization, (despite its large literature, practical usefulness, and original introduction by mathematicians).<sup>5</sup>

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<sup>1</sup> If deformable, the macroscopic description is expected to manifest the full nonlocal pattern of equations expressed in Appendix: see the concluding considerations.

<sup>2</sup> In the context of a macroscopic nonlocal description any material having finite dimensions is to be viewed as an inhomogeneous material.

<sup>3</sup> And retaining only the leading order, see e.g. [3] (Appendix A) and [2] and [4] in next chapter.

<sup>4</sup> To take an image, this simplification transforms the full Kirchhoff's theory of the propagation of sound waves in cylindrical circular tubes, in Zwicker and Kosten's, classical but incomplete treatment, only slightly inexact at long wavelengths.

<sup>5</sup> In particular, we can anticipate that the addition of all higher order terms appearing in sequence in this abstract homogenization process, will always prove to yield results departing significantly from the exact ones.

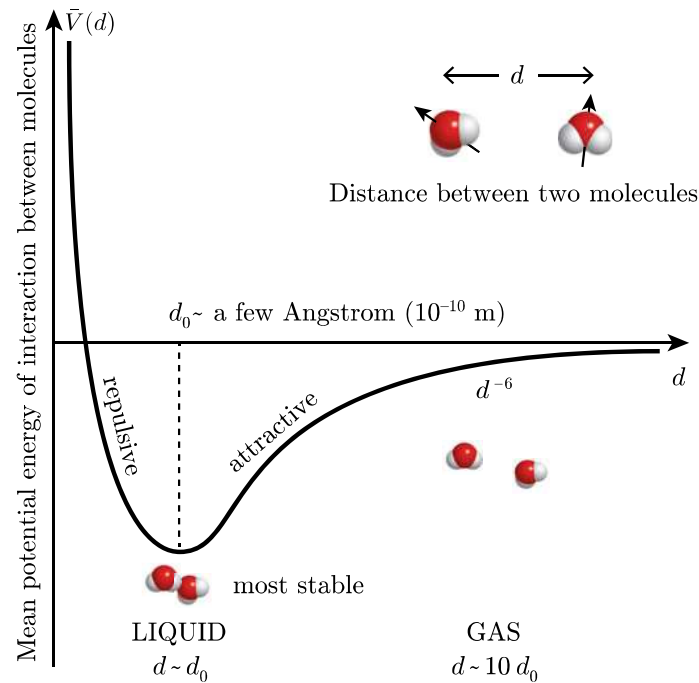


Fig. 6.1 Typical pair-interaction potential

## 6.2 Molecular Constitution of Fluids

We start by recalling basic facts about the molecular constitution of fluids. Virtually everything we are sensitive to, on earth and in the sky, is made up of countless molecules that respond to electromagnetic interactions. Molecules are usually small electrical dipoles, (either permanent or induced): a small separation of negative and positive charges is present, which produces a dipole moment equal to the amount of separated charges times the distance. The dipole averaged pair-interaction energy potentials  $\bar{V}(d)$ , have a simple Boscovich form, shown in Fig. 6.1, in function of the distance  $d$  between two of them.<sup>6</sup>

A fluid, liquid or gas, comprises an incredibly large number of molecules in any “physically infinitesimal” volume. A number so large (recall Avogadro’s number:  $N_A \cong 6.10^{23}$ ) that it corresponds to a so-called “thermodynamic limit”.

In a liquid (solid) such as water (ice), the molecules are bound at distances  $d$  on the order of a few Ångström. Typically,  $d_0 \sim 3.1 \cdot 10^{-10}$  m, for the distance where the pair-interaction potential  $\bar{V}(d)$  is close to its minimum. In a gas such as air,

<sup>6</sup> In the averaging over the relative orientations of two dipoles, a thermalization is performed, so that a configuration having pair-interaction energy  $V$  is considered to have a probability proportional to  $e^{-V/kT}$  to occur.

the molecules are widely distanced on the order of, say, mean distances  $\bar{d} \sim 10 d_0$ . The molecules then are mostly non-interacting, in uniform inertial motion except for binary collisions. The mean free path, (mean distance travelled by a molecule between two collisions), is on the order of the microscopic distance  $\ell = \eta/\rho_0 c_0$ , (as can be estimated by elementary kinetic theory considerations), where  $\eta$  is first viscosity,  $\rho_0$  ambient density, and  $c_0$  is (adiabatic) speed of sound. This distance is very small, ( $\sim 4.5 \cdot 10^{-8}$  m in air), only one order of magnitude higher than the mean distance  $\bar{d}$  between molecules, ( $\ell \sim 10\bar{d} \sim 100d_0$ ). In a liquid, as the molecules are bound, a mean free path distance is not well-defined, but can be considered on the order of the mean distance ( $\bar{d} \sim d_0$ ) between molecules, (in water we find  $5.9 \cdot 10^{-10}$  m for the above microscopic distance  $\ell$ , which is roughly two-times  $d_0$ ).

The thermal expansion coefficient  $\beta_0$  of a fluid, (liquid or gas), in thermodynamic equilibrium at temperature  $T$  and pressure  $P$ , is defined as

$$\beta_0 = \frac{1}{v} \left( \frac{\partial v}{\partial T} \right)_P = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_P, \quad (6.1)$$

where  $v = 1/\rho$  is the specific volume, (volume per unit of mass). The ideal gas is the limit where the molecules are considered point mass, possessing mass but no significant volume. Simple kinetic theory considerations lead to the ideal gas law

$$PV = NkT,$$

where  $V$  is the volume,  $N$  is the number of particles in it, and  $k$  is Boltzmann's constant. From this law and the definition (6.1), we see that the thermal expansion coefficient of an ideal gas is given by

$$\beta_0 = 1/T_0, \quad (6.2)$$

with  $T_0$  the ambient temperature. In a real gas, there are deviations from (6.2) to account for the nonzero tail of the actual potential at large distances; but as the potential goes to zero very rapidly, (see the exponent  $-6$  in Fig. 6.1), the above is only slightly modified:

$$\beta_0 \cong 1/T_0. \quad (6.3)$$

A physical expression of these molecular characteristics is that a gas is easily compressible and also significantly expands when heated.

In a liquid, because the molecules are bound, trapped at the minimum distance  $d_0$  allowed by intermolecular forces, but not occupying fixed mean positions, (on the contrary, undergoing permanent diffusive wandering from one location to the other), the coefficient of thermal expansion is drastically lower, in order of magnitude:

$$\beta_0 \lll 1/T_0. \quad (6.4)$$

Indeed, when some heat quantity is given to the fluid so that the equilibrium temperature increases, the molecules augment their characteristic diffusive velocities, wandering more rapidly from one location to the other. In so doing, however, they almost remain trapped at the minimum distance  $d_0$ . The thermal expansion coefficient is non zero,  $\beta_0 \neq 0$ , because there always subsists a small thermal expansion on the order of that found in solids.

In what follows, we limit ourselves to considering simple liquids and gases whose composition is invariable, i.e. so-called trivariate fluids whose thermodynamic state is specified by two independent thermodynamic variables, (such as pressure, temperature, density, etc.). A brief recap on the thermodynamics of trivariate fluids is given in Sect. 6.5. There is the following important thermodynamic identity, between fluid parameters, (see below (6.20.1)):

$$\gamma - 1 = \frac{T_0 \beta_0^2 c_0^2}{c_P}, \quad (6.5)$$

with  $\gamma$  and  $c_P$ , the heat capacity ratio (or adiabatic exponent) and specific heat coefficient at constant pressure. In view of this general identity, the above molecular characteristics of the liquid and gas phases, can be seen to have the following important consequences.

For a liquid, as  $\beta_0$  is very small, the deviation (6.5) is almost zero, (it is a quadratic effect on  $\beta_0$ ), and  $\gamma$  turns out to be nearly indistinguishable from 1. Therefore, the isothermal bulk modulus  $K_0$ , is nearly indistinguishable from the adiabatic bulk modulus<sup>7</sup>  $K_a = \gamma K_0$ , i.e.:  $K_a \cong K_0$ . At variance, in a gas,  $\beta_0$  is not to be considered a small quantity, and it turns out that the factor  $\gamma - 1$  is generally of order 1. Simple kinetic theory considerations give:  $\gamma - 1 = 2/n_d$ , where  $n_d$  is the number of excited degrees of freedom of a typical molecule. For example, for air, 99% composed of diatomic molecules ( $N_2$  and  $O_2$ ) whose vibrational movements are only slightly excited,  $n_d \cong 5$ , as there are 3 translational and only 2 rotational degrees of freedom to consider. This gives  $\gamma - 1 \cong 0.4$ , and in normal conditions the adiabatic bulk modulus of air,  $K_a = \gamma P_0$ , is 40% greater than the isothermal bulk modulus,  $K_0 = P_0$ , where  $P_0$  is atmospheric pressure.

To finish, we recall that a special notation  $\chi_0$  is used for the inverse of the adiabatic bulk modulus, the adiabatic compressibility:

$$1/K_a \equiv \chi_0. \quad (6.6)$$

For the case of propagation in the fluid with losses disregarded, no heat enters or leaves an element of fluid during its alternate cooling and warming due to the passage of sound waves, so that the pressure-density cycle is adiabatic, and the following relation occurs between the fluid ambient density  $\rho_0$ , the fluid adiabatic compressibility  $\chi_0$ , and the (adiabatic) speed of sound  $c_0$

<sup>7</sup> See below (6.19.1) and (6.19.3) for the general definition of the adiabatic and isothermal bulk moduli  $K_a$  and  $K_0$ .

$$\rho_0 \chi_0 c_0^2 = 1. \quad (6.7)$$

This relation reminds us that observed for the propagation of electromagnetic waves in a lossless non-dispersive medium equivalent to vacuum:

$$\epsilon_0 \mu_0 c^2 = 1. \quad (6.8)$$

In the presence of losses, we will see that the density and compressibility eventually become operators  $\hat{\rho}$  and  $\hat{\chi}$ . They will play the role assigned in electromagnetism to the electrical susceptibility  $\hat{\epsilon}$  and magnetic susceptibility  $\hat{\mu}$  operators, respectively.<sup>8</sup>

### 6.3 Wave Propagation in a Viscothermal Fluid: Use of Near-Equilibrium Thermodynamics Because of Scale Separation

Let us now precise notions such as, “fluid particle”, “macroscopic level” and “scale separation”. A “fluid particle” refers to the macroscopic collection of molecules contained in a “physically infinitesimal” volume. Despite its small size, it is a “macroscopic” object in the sense that it contains an enormous number – “thermodynamic” – of molecules. To have an order of magnitude in mind, we typically have  $10^9$  molecules in the smallest experimentally resolvable volumes [3]. A given volume  $\ell^3 = dV$  containing a “thermodynamic” number of molecules will be considered an infinitesimal macroscopic volume, or “microscopic” volume, when its size  $\ell$  is very much smaller than any macroscopic characteristic distance  $L$ , such as sample size, characteristic structure size, or wavelength  $\lambda$ .

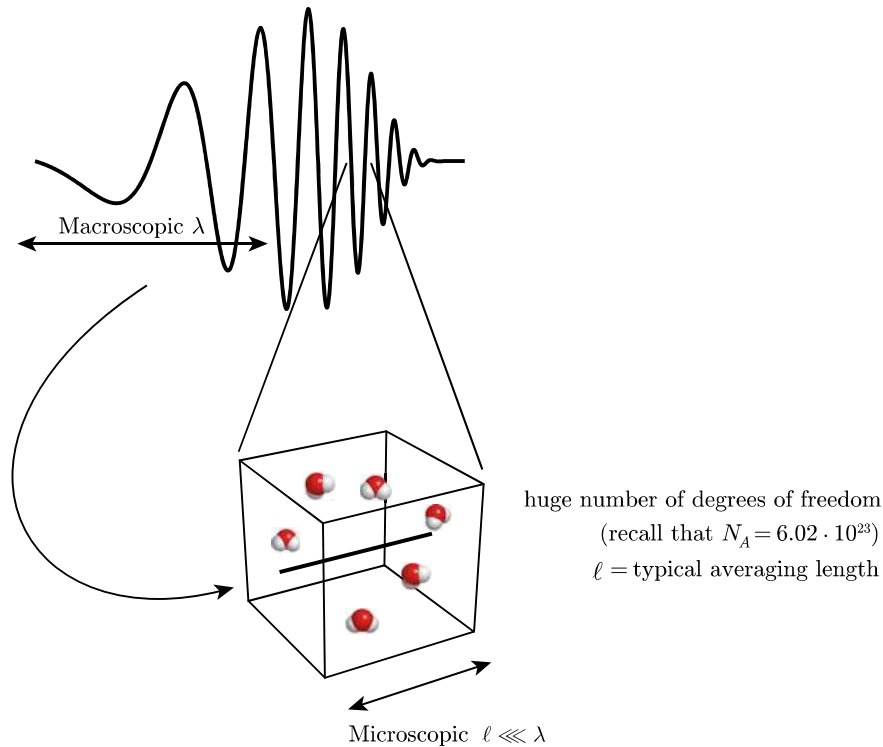
As an example of a microscopic size  $\ell$  in a homogeneous fluid, we may think of the mean free path,  $\ell \sim \eta / \rho_0 c_0$ . We do not aim at a description of the underlying motion of interacting molecules in such volumes. With numbers  $N > 10^9$  this would be not only impossible but also useless. Instead, we aim at a macroscopic description statistical in nature, feasible because of there being a “thermodynamic-limit”, very large number of particles. In fact, through the statistical Law of Large Numbers, the physical state of each infinitesimal “fluid particle” is specified by remarkably few macroscopic variables.

This is related to the very large “scale separation” between the typical macroscopic and microscopic scales, as shown in Fig. 6.2.

Because of the assumed “scale separation”, the fluid in the infinitesimal volume  $\ell^3 = dV$  is very nearly in a state of *local thermodynamic equilibrium*. Instead of

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<sup>8</sup> More precisely and as discussed in the text and in Appendix, this affinity will be fully expressed only when the thermodynamic framework is sufficiently extended, with density and compressibility kernels becoming second order and fourth order tensors. In the main text they will remain scalars because we will restrict to long-wavelengths compressional waves only described within classical near-equilibrium framework.



**Fig. 6.2** Scale separation

doing a microscopic description, we can make a macroscopic description in which the state of “macroscopic motion” of each fluid particle is summed up in terms of only five numbers: the mean or macroscopic velocity, and two equilibrium, (for a trivariate fluid), thermodynamic variables.

Indeed, our wide “scale separation” condition implies in particular that any typical relaxation time associated with irreversible processes in the fluid, will be very short compared to the typical periods of the acoustic vibrations. As a result, we will be justified in discarding the effects of the so-called molecular relaxation. Consider for example the case of air. It is principally made of polyatomic – diatomic – molecules. Their energy will be distributed between the translational (external) degrees of freedom and the rotational and vibrational (internal) degrees of freedom. Obviously the sound wave disturbs to some extent the equilibrium distribution of energy between external and internal degrees of freedom. The concomitant departures from local thermodynamic equilibrium have tendency to return to equilibrium after certain relaxation times. Here, assuming that the typical periods of motion are sufficiently large compared to the molecular relaxation times, we have that the distribution of energy between external and internal degrees of freedom always remains



that of thermodynamic equilibrium. In practice, this limit is well-satisfied for audible sounds.

Pressure and temperature is a possible convenient choice for the two thermodynamic variables. These thermodynamic variables – as is apparent from the fact that they become meaningless for the case of a system having only a few degrees of freedom – are macroscopic statistical variables having more than a purely “mechanical” meaning. Their averaged nature can be conceived using the conception of volume average, referred here to H.A. Lorentz who used it in his landmark program of the “theory of electrons” [4], (see Sect. 6.11.2), or, better, the conception of ensemble average, referred to J.W. Gibbs, who introduced it in his fundamental studies of the theory of gases and the statistical mechanics foundations of thermodynamics [1]: the average is performed over the ensemble of all systems that are “macroscopically identical”. This notion of macroscopically identical systems can be clarified in the case of thermodynamic equilibrium. In this case, we have to think as equiprobable, all configurations that a set of molecules can take in a given volume and which share a given total energy. We will imagine that the notion of Gibbs ensemble could also in principle be generalized in an appropriate way in the case of deviations from the thermodynamic equilibrium, which would enable us to describe in a generalized Gibbs sense the propagation of waves of arbitrarily short wavelengths or arbitrarily high frequencies,<sup>9</sup> not respecting any scale-separation condition.

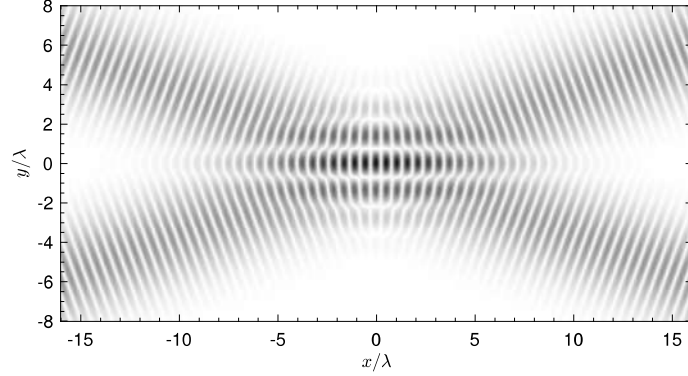
As for the macroscopic mean velocity  $\mathbf{v}$ , we have the same two conceptions. With volume averaging, we can define  $\mathbf{v}$  as the velocity of the center of mass of the collection of molecules present in the fluid particle volume. With ensemble averaging, we will assume possible to define a collection of different molecular realizations of the same fluid medium and wave propagation phenomenon, that are equivalent from a macroscopic point of view. We can define  $\mathbf{v}$  at a given position, as, say, the velocity of the closest molecule, averaged over all realizations. (Of course, these naive definitions would have to be refined as they are, both, not consistent quantum mechanically, but this is not our focus here). At long wavelengths, we will assume that ergodicity properties will ensure the equivalence of the two points of view.

## 6.4 Governing Navier-Stokes-Fourier Equations for Small Amplitude Wave Propagation

We are interested here in the first place in describing, at the macroscopic thermodynamic level, small amplitude long-wavelength wave propagation in a homogeneous fluid, (see e.g. the situation depicted on (Fig. 6.3) with  $\lambda \gg \ell$ ,  $\ell$  the mean free path). At each spatial position  $\mathbf{x}$  and instant of time  $t$ , the state of the fluid, nearly in equilibrium, is for the essential described by the velocity  $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ , pressure  $P(t, \mathbf{x})$  and temperature  $T(t, \mathbf{x})$ . But as the fluid is not, exactly, in a local state of pure thermo-

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<sup>9</sup> We will later need thinking in this strongly out-of-equilibrium, hypothetical Gibbs averaging sense, when discussing hypothetical Frenkel-type of motions in gases.



**Fig. 6.3** Wave propagation example: two intersecting Gaussian beams

dynamic equilibrium, this description is not complete. To account for the dissipative processes induced by the nonequilibrium, there are additional variables, such as the dissipative viscous stresses  $\sigma'_{vij}$  and heat flux  $\mathbf{q}$ , that will be necessary to introduce, and relate, by constitutive laws, to the variations of the former variables. All variables are “total variables”,  $I$ , sum of an equilibrium variable,  $I_0$ , representing the value of this variable in the ambient thermodynamic state, (zero for the velocity and additional variables), and a wave variable,  $I_w$ , representing an excess value associated to the wave:

$$I \left( \begin{array}{c} \text{total} \\ \text{variable} \end{array} \right) = I_0 \left( \begin{array}{c} \text{equilibrium} \\ \text{variable} \end{array} \right) + I_w \left( \begin{array}{c} \text{wave} \\ \text{variable} \end{array} \right). \quad (6.9)$$

We now recall how are derived the basic Navier-Stokes-Fourier viscothermal linearized model equations of motion, governing the small “wave variables”,  $\mathbf{v}$ ,  $p$ ,  $\tau$  and  $b$ , namely:  $\mathbf{v}$  = part “w” of velocity, its equilibrium part being zero,  $p$  = part “w” of pressure, or “excess pressure”, the equilibrium part being  $P_0$ ,  $\tau$  = part “w” of temperature, or “excess temperature”, the equilibrium part being  $T_0$ , and  $b = \rho'/\rho_0$ , the condensation, equal to excess density  $\rho'$  – part “w” of density – over ambient density  $\rho_0$ <sup>10</sup>

<sup>10</sup> This variable  $b$ , as we already have  $p$  and  $\tau$  in the set of variables, is redundant thermodynamically, see (6.12). It is introduced here mainly because it will later facilitate the discussion of the electromagnetic analogy. In the context of the electromagnetic analogy, however,  $b$  will in fact not be viewed as the quantity  $b = \rho'/\rho_0$ . It will either be viewed as the purely kinematic quantity directly defined with (6.10), (see Sect. 6.12.1 and the definitions (6.98.1) and (6.98.2), equivalent to (6.10)); more precisely, in the full analogy we introduce a symmetric tensor  $b_{ij}$  playing the role of antisymmetric magnetic tensor field, obeying  $\partial b_{ij}/\partial t = -(\partial_i v_j + \partial_j v_i)/2$ , and representing the opposite strain, with  $b \equiv \sum_i b_{ii}$ , (more details are in Appendix).

$$\frac{\partial b}{\partial t} + \partial \cdot \mathbf{v} = 0, \quad (6.10)$$

$$\begin{aligned} \rho_0 \frac{\partial \mathbf{v}}{\partial t} &= -\partial p - \eta \partial \times \partial \times \mathbf{v} + \left( \frac{4\eta}{3} + \zeta \right) \partial (\partial \cdot \mathbf{v}), \\ &= -\partial p + \eta \partial^2 \mathbf{v} - \left( \frac{\eta}{3} + \zeta \right) \partial \left( \frac{\partial b}{\partial t} \right), \end{aligned} \quad (6.11)$$

$$\gamma \chi_0 p = b + \beta_0 \tau, \quad (6.12)$$

$$\rho_0 c_P \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \partial^2 \tau. \quad (6.13)$$

The parameters  $\eta$  and  $\zeta$ , first and second viscosities, and  $\kappa$ , thermal conduction coefficient, are constitutive constants of the fluid. Their appearance is the expression of the mentioned additional irreversible-thermodynamic variables, ( $\sigma'_{vij}$  and  $\mathbf{q}$ ), related by constitutive laws to the former variables, (here  $\mathbf{v}$  and  $\tau$ , resp.). To arrive at these linearized model equations (6.10)–(6.13), first, we recall some basic facts and definitions about equilibrium thermodynamics of trivariate fluids, and next, we introduce the additional variables associated to near-equilibrium deviations, and state the conservation and constitutive laws.

## 6.5 Equilibrium Thermodynamics: *Brief Recap on Trivariate Fluids*

Classical equilibrium thermodynamics of a trivariate fluid asserts that all thermodynamic quantities – thermodynamic variables and thermodynamic constants – can be deduced when we know a single relationship that links three independent additive quantities: entropy, internal energy and volume. It is the so-called fundamental relation.<sup>11</sup> It can be written either in the original form of entropy  $S$  in function of internal energy  $E$  and volume  $V$ , or in the form of internal energy  $E$  in function of entropy  $S$  and volume  $V$ :

$$\begin{aligned} S &= S(E, V), \\ E &= E(S, V). \end{aligned} \quad (6.14)$$

Indeed, as the entropy  $S$  is postulated to be a single valued continuous and differentiable function of  $E$  and  $V$ , and moreover a monotonic increasing function of  $E$ , the fundamental relation (6.14.1) can be inverted with respect to energy, giving (6.14.2) with a function  $E$  that is also single valued, continuous and differentiable.

Knowing the fundamental relation, the pressure and temperature can be written by the formulae:

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<sup>11</sup> See [2, 3, 5]; nice discussions of several topics of interest here, e.g., thermodynamics, equations of motion in a fluid, and averaging operations in a material, are given by Marle [5].

$$P \equiv -\left(\frac{\partial E}{\partial V}\right)_S, \quad T \equiv \left(\frac{\partial E}{\partial S}\right)_V, \quad (6.15)$$

and likewise, all other quantities can be expressed as well in terms of  $S$ ,  $V$  and the partial derivatives of  $E$ .

Equation (6.15) derive immediately from the thermodynamic identity  $TdS = dE + PdV$ . And the latter identity is a consequence of the First Law,  $dQ = dE + PdV$ , stating that the internal energy can change (by  $dE$ ) only by receiving work ( $-PdV$ ) and/or heat ( $dQ$ ), joined to the additional condition that the latter is always the form  $dQ = TdS$ , (the notation  $dQ$  reminds that the infinitesimal heat quantity is not an exact differential). Note that, as all quantities  $S$ ,  $E$  and  $V$  in (6.14) are additive, the fundamental relation is homogeneous of order 1. Hence writing (6.14.2) for a *unit mass fluid particle* which can be in translational motion, we have the so-called caloric equation:

$$\epsilon = \epsilon(s, v). \quad (6.16)$$

It gives the specific internal energy  $\epsilon$  in the fluid, (i.e. internal energy per unit mass, which includes all forms of energy except the specific kinetic energy  $v^2/2$  associated with the global mass translation movement), as a function of the specific entropy  $s$ , (entropy per unit mass), and the specific volume  $v = 1/\rho$ . Pressure and temperature are then functions  $P = P(s, v)$  and  $T = T(s, v)$  given by:

$$P \equiv -\left(\frac{\partial \epsilon}{\partial v}\right)_s, \quad T \equiv \left(\frac{\partial \epsilon}{\partial s}\right)_v. \quad (6.17)$$

As all thermodynamic variables can be expressed in terms of  $s$ ,  $v$ , and the partial derivatives of  $\epsilon$ , there are only two independent thermodynamic variables. Any variable, e.g. temperature, can be expressed as a function of two any other independent variables, by means of equations of state characteristic of the fluid, e.g.  $T = T(P, v)$ ,  $T = T(P, \epsilon)$ , etc. Contrary to the fundamental relation, the equations of state do not constitute complete knowledge of the thermodynamic properties of the fluid. Knowledge of all equations of state, however, is complete, and equivalent to the knowledge of the fundamental relation or caloric equation.

The specific heat coefficients  $c_P$  and  $c_V$ , (amount of heat needed to rise a temperature of the unit of mass by 1 Kelvin under constant pressure, or under constant volume), adiabatic exponent, and thermal expansion coefficient, are:

$$c_P \equiv T \left(\frac{\partial s}{\partial T}\right)_P, \quad c_V \equiv T \left(\frac{\partial s}{\partial T}\right)_v, \quad \gamma \equiv \frac{c_P}{c_V}, \quad \beta \equiv \frac{1}{v} \left(\frac{\partial v}{\partial T}\right)_P. \quad (6.18)$$

For a trivariate fluid  $\beta$  is not completely negligible, ( $\beta \neq 0$ ), or  $\gamma$  is not strictly equal to one. There follows that the caloric (6.16) cannot be written in the form  $\epsilon = Y(s) + Z(v)$  and none of the equations of state degenerate into relations connecting only two thermodynamic variables.

Other useful thermodynamic definitions are those of the reference adiabatic and isothermal, bulk moduli  $K_a$  and  $K_i$  (or  $K_0$ ), and velocities  $c_a$  (or  $c_0$ ) and  $c_i$  (or  $c'_i$ , later on), in the fluid:

$$K_a \equiv \rho \left( \frac{\partial P}{\partial \rho} \right)_s, \quad c_a^2 \equiv \left( \frac{\partial P}{\partial \rho} \right)_s, \quad K_i \equiv \rho \left( \frac{\partial P}{\partial \rho} \right)_T, \quad c_i^2 \equiv \left( \frac{\partial P}{\partial \rho} \right)_T. \quad (6.19)$$

By purely thermodynamic arguments, the various constitutive constants introduced so far (6.18)–(6.19) are not independent. One can show the following general thermodynamic identities:

$$\gamma - 1 = \frac{T\beta^2 c_a^2}{c_P}, \quad c_a^2 = \gamma c_i^2. \quad (6.20)$$

These are consequences of the First Law and the fact that  $\delta Q = T dS$ . Indeed the fundamental thermodynamic requirement that  $\delta Q/T$  is actually an exact differential, meaning that the entropy is a function of state, imposes a restriction on the analytical form of the internal energy, also implying that the caloric equation  $\epsilon = \epsilon(s, v)$  is not independent of the thermal equation of state  $P = P(T, v)$ . In the proof of (6.20), will be involved:

- (i) Maxwell's thermodynamic relations such as  $-(\partial P/\partial s)_v = (\partial T/\partial v)_s$  expressing the fact that energy  $\epsilon$  or other potentials such as enthalpy  $H$  (whose differential is  $dH = T ds + v dP$ ) and the Helmholtz free energy  $F$  (whose differential is  $dF = -s dT - P dv$ ) are functions of state; the process of obtaining this kind of relations is as follows: let  $\epsilon = \epsilon(s, v)$ , then writing  $d\epsilon = T ds - P dv$  and  $\partial^2 \epsilon / \partial s \partial v = \partial^2 \epsilon / \partial v \partial s$ , we get,  $-(\partial P/\partial s)_v = (\partial T/\partial v)_s$ , and similarly for the other thermodynamic potentials, e.g. with the enthalpy one finds,  $(\partial v/\partial s)_P = (\partial T/\partial P)_s$ .
- (ii) General relations of the type  $(\partial v/\partial s)_P = -(\partial v/\partial P)_s (\partial P/\partial s)_v$  satisfied by any three quantities  $v, P, s$ , satisfying a single functional relationship, which again expresses the existence of equations of state; the process of obtaining these relations is as follows: let  $P = P(s, v)$ , then  $dP = (\partial P/\partial s)_v ds + (\partial P/\partial v)_s dv$ , and if  $dP = 0$ , then  $(\partial P/\partial v)_s (dv/ds) = -(\partial P/\partial s)_v$ , which also writes  $(\partial v/\partial s)_P = -(\partial v/\partial P)_s (\partial P/\partial s)_v$ .

The detailed proof of (6.20) is classically found in textbooks (a derivation is also available at <https://www.astro.princeton.edu/~gk/A403/termo.pdf>). We will always have to use the identity (6.20.1) for quantities taken in the ambient state, denoted by index 0. For simplicity, however, we do not put the index 0 on all the quantities involved. In what follows we write  $c_a^2 \equiv c_0^2$  and (6.20.1) in the form of (6.5).

## 6.6 Derivation of the Equations of Motion

Let us now turn to the derivation of the Navier-Stokes-Fourier model equations (6.10)–(6.13) that govern the small deviations from the equilibrium state. Consider the thermodynamic total variables pressure, temperature, entropy, specific volume, density, and specific internal energy. Removing for simplicity the index “w”, we write them in the form:

$$\begin{array}{ll}
 \text{Pressure:} & P = P_0 + p, \\
 \text{Temperature:} & T = T_0 + \tau, \\
 \text{Specific entropy:} & s = s_0 + s', \\
 \text{Specific volume:} & v = v_0 + v', \\
 \text{Density:} & \rho = \rho_0 + \rho', \\
 \text{Specific internal energy:} & \epsilon = \epsilon_0 + \epsilon'.
 \end{array}$$

Because of scale separation, the fluid in the infinitesimal volumes  $dV$  used to define these quantities is very nearly in equilibrium state. Therefore, both the ambient quantities and the total variable quantities, are related by equations of state. For example, writing the caloric equation for ambient state and actual state, we have,  $\epsilon_0 = \epsilon(s_0, v_0)$  and  $\epsilon_0 + \epsilon' = \epsilon(s_0 + s', v_0 + v) = \epsilon(s_0, v_0) + s' (\partial\epsilon/\partial s)(s_0, v_0) + v' (\partial\epsilon/\partial v)(s_0, v_0) + \dots$ , whence in the linear approximation,  $\epsilon' = s' (\partial\epsilon/\partial s)(s_0, v_0) + v' (\partial\epsilon/\partial v)(s_0, v_0)$ . In that way, starting with the different state equations, and within linear approximation valid for small amplitude perturbations, many linear relations can be obtained, relating any three of the above deviatoric (“w”) quantities, and making apparent some of the ambient quantities, and also, the fluid thermodynamic constants such as  $\beta_0$ ,  $c_P$  and  $c_V$ . Examples of such relationships will appear below.

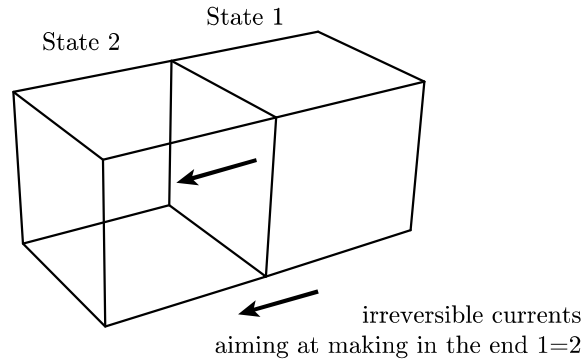
The general equations governing the spatiotemporal evolution of all total variables quantities – the velocity  $\mathbf{v}(t, \mathbf{x})$  and the thermodynamic fields  $P(t, \mathbf{x})$ ,  $T(t, \mathbf{x})$ ,  $\rho(t, \mathbf{x})$ , etc. – will be derived from:

- (i) General conservation laws,
- (ii) Equilibrium thermodynamics,
- (iii) Constitutive equations, outside the realm of equilibrium thermodynamics.

The reason why the evolution of the fields cannot be derived solely from (i) general conservation laws and (ii) equilibrium thermodynamics (despite the long-wavelength scale-separation condition  $\lambda \gg \ell$  ensuring the existence of local equilibrium states, where  $\ell$  is a microscopic length such as mean free path), is because *the fields and, thus, the thermodynamic state of the fluid vary to some extent in space and time*, meaning that the system is, slightly, out-of-equilibrium. As later discussed with the electromagnetic analogy, taking into account the effect of these variations will already consist in taking *spatial dispersion* and *temporal dispersion* effects into account.

Due to spatial and temporal variations, irreversible transient currents are generated, aimed at bringing the system back to equilibrium. Indeed, consider two neighbouring infinitesimal particles. At each instant of time they are in slightly different

**Fig. 6.4** Generation of irreversible currents



equilibrium thermodynamic states of motion (where by “thermodynamic state of motion” here we also include the information of the mean velocity), say state 1 and state 2, as sketched in Fig. 6.4.

During any physically infinitesimal period, many molecules (a thermodynamic number) pass through the boundary surface between particles 1 and 2. Particles passing in either direction carry information on the state 1 or 2, which are different; therefore, their passage is necessarily associated with an irreversible exchange of momentum and energy, aimed at suppressing the difference of state. The appearance of these transient irreversible currents is described in practice by the introduction of additional variables, related to the former by empirical constitutive laws. These laws lie outside the realm of equilibrium thermodynamics. Essentially, they must precise how much momentum and heat energy are irreversibly transferred, (resp. by so-called viscous effects and thermal conduction effects), per unit of time across the surface.

### 6.6.1 Conservation Laws and Equation of State

We first state the general conservation or balance laws.

#### **Mass Conservation**

It is classically found and derived in any textbook:

$$\frac{\partial \rho}{\partial t} + \partial \cdot (\rho \mathbf{v}) = 0.$$

After linearization, it reads

$$\frac{\partial b}{\partial t} + \partial \cdot \mathbf{v} = 0, \quad (6.21)$$

where  $b = \rho' / \rho_0$  is the condensation.

**Momentum Balance**

This balance law is best written using the convective derivative,  $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \partial$ :

$$\rho \frac{d}{dt}(v_i) = \partial_j \sigma_{ij},$$

where  $\sigma_{ij}$  is the excess stress tensor in the fluid, which incorporates the mentioned momentum irreversibly transferred. By integration over an infinitesimal volume  $dV$  in motion, it reads,  $dm \frac{d}{dt}(v_i) = \frac{d}{dt}(v_i dm) = \oint \sigma_{ij} \hat{n}_j dS$  ( $dm = \rho dV$ ). This is Newton's law, stating that the time rate of change of momentum is the result of the excess stress contact forces exerted at the particle bounding surface, (with  $\hat{\mathbf{n}}$  the unit outward normal).

If there were present an external bulk force acting at a distance, pulling the volume  $dV$  with force  $d\mathbf{f}$ , it would be necessary to rewrite the above, as,  $dm \frac{d}{dt}(v_i) = \frac{d}{dt}(v_i dm) = \oint \sigma_{ij} \hat{n}_j dS + df_i$ . It would correspond, writing the starting equation with a bulk-source term of force,  $[df_i/dV]$ , added in the right-hand side. Obviously,  $df_i/dV = \rho df_i/dm$ , and if we count the force per unit mass, we write:

$$\rho \frac{d}{dt}(v_i) = \partial_j \sigma_{ij} + [\rho f_i].$$

On using the expression of the convective derivative and inserting mass Conservation, it also reads

$$\frac{\partial}{\partial t}(\rho v_i) + \partial_j(\rho v_i v_j) = \partial_j \sigma_{ij} + [\rho f_i],$$

and after linearization

$$\rho_0 \frac{\partial}{\partial t} v_i = \partial_j \sigma_{ij} + [\rho_0 f_i]. \quad (6.22)$$

**Energy Balance**

This balance law is also stated in the most simple and explicit form using the convective derivative:

$$\rho \frac{d}{dt} \left( \frac{1}{2} v^2 + \epsilon \right) = -\partial_i (-\Sigma_{ij} v_j + q_i),$$

where  $\mathbf{q}$  is the heat flux density due to thermal conduction, that determines the above-mentioned heat energy irreversibly transferred, and  $\Sigma_{ij} = -P_0 \delta_{ij} + \sigma_{ij}$  is the total variables stress tensor. By integration over an infinitesimal volume  $dV$  in motion, (and such that  $\rho dV = dm$ ), it reads  $\frac{d}{dt} (dm \frac{1}{2} v^2 + dm \epsilon) = \oint (-\Sigma_{ij} v_j + q_i)(-\hat{n}_i) dS$ , where the surface integrals are over the bounding surface of element  $dV$ . This is the basic thermodynamic principle, stating that the time rate of change of the energy of a given fluid volume, (kinetic energy plus internal energy), is equal to the work done on it, (by the total stress contact forces exerted at the particle bounding surface), plus the heat it receives, (through this bounding surface), per unit of time.



If there were present a bulk-source of heat, directly acting in the fluid, and giving a quantity of heat  $dQ$  in the volume  $dV$  during time  $dt$ , it would be necessary to rewrite the above, as,  $\frac{d}{dt}(dm\frac{1}{2}v^2 + dm\epsilon) = \oint(-\Sigma_{ij}v_j)(-\hat{n}_i)dS + \oint q_i(-\hat{n}_i)dS + dQ/dt$ . It would correspond writing the above starting balance law with a bulk-source term [ $dQ/dVdt$ ] added in the right-hand side. But,  $dQ/dVdt = \rho dQ/dmdt$ , and choosing to count the heat per unit of mass, i.e. noting the present  $dQ/dm$  as  $dQ$ , we have

$$\rho \frac{d}{dt} \left( \frac{1}{2}v^2 + \epsilon \right) = -\partial_i (-\Sigma_{ij}v_j + q_i) + \left[ \rho \frac{dQ}{dt} \right].$$

After linearization the energy balance equation is, (omitting the prime on  $\epsilon$ )

$$\rho_0 \frac{\partial \epsilon}{\partial t} = -P_0 \partial_i v_i - \partial_i q_i + \left[ \rho_0 \frac{dQ}{dt} \right]. \quad (6.23)$$

We next introduce the equation of state, which expresses the condition of local thermodynamic equilibrium, to obtain another form of this energy balance equation.

Starting with the thermodynamic equilibrium relation,  $d\epsilon = Tds - PdV = Tds + Pd\rho/\rho^2$ , substituting in (6.23), and taking into account (6.21), we find

$$\rho_0 T_0 \frac{\partial s}{\partial t} = -\partial_i q_i + \left[ \rho_0 \frac{dQ}{dt} \right]. \quad (6.24)$$

Finally, to complete these equations and make apparent only the excess thermodynamic pressure and temperature, we write the equilibrium equations of state,  $\rho = \rho(P, T)$  and  $T = T(P, s)$ , whose linearized versions give, resp.

$$\rho' = \left( \frac{\partial \rho}{\partial P} \right)_T p + \left( \frac{\partial \rho}{\partial T} \right)_P \tau, \quad \tau = \left( \frac{\partial T}{\partial P} \right)_s p + \left( \frac{\partial T}{\partial s} \right)_P s, \quad (6.25)$$

that is, using (6.19.3) and (6.18.4) and noting that  $1/K_0 = \gamma\chi_0$ ,

$$b = \rho'/\rho_0 = \gamma\chi_0 p - \beta_0 \tau, \quad \text{and} \quad \tau = \frac{\beta_0 T_0}{\rho_0 c_P} p + \frac{T_0}{c_P} s. \quad (6.26)$$

In the last equation, (6.18.1) has been used to evaluate  $(\partial T/\partial s)_P = (T_0/c_P)$ , and the remaining coefficient has been expressed using the thermodynamic identity  $(\partial T/\partial P)_s = (\beta_0 T_0)/(\rho_0 c_P)$ , given in Pierce [18] Sect. (6.10). Substituting (6.26.2) in (6.24), we obtain finally the following useful form of the energy balance equation:

$$\rho_0 c_P \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} - \partial_i q_i + \left[ \rho_0 \frac{dQ}{dt} \right]. \quad (6.27)$$

To summarize, after consideration of the general conservation laws and equilibrium thermodynamic relations, we have obtained the following set, (6.28)–(6.31), of

linearized equations:

$$\text{Mass: } \frac{\partial b}{\partial t} + \partial \cdot \mathbf{v} = 0, \quad (6.28)$$

$$\text{Momentum: } \rho_0 \frac{\partial v_i}{\partial t} = \partial_j \sigma_{ij} + [\rho_0 f_i], \quad (6.29)$$

$$\text{Equation of state: } \gamma \chi_0 p = b + \beta_0 \tau, \quad (6.30)$$

$$\text{Energy: } \rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} - \partial_i q_i + \left[ \rho_0 \frac{dQ}{dt} \right], \quad (6.31)$$

with  $f_i$ , a possible external body force per unit mass, and  $dQ/dt$ , a possible external rate of heat supply per unit mass, acting on the fluid.

### 6.6.2 Constitutive Laws: Stokes and Fourier

Finally, we state irreversible-thermodynamic constitutive laws, for the evaluation of the heat flux  $q_i$ , and an additional lossy part  $\sigma'_{ij}$  present in the excess stresses  $\sigma_{ij}$ :

$$\sigma_{ij} = -p_a \delta_{ij} + \sigma'_{ij}, \quad p_a = \chi_0^{-1} b, \quad (6.32)$$

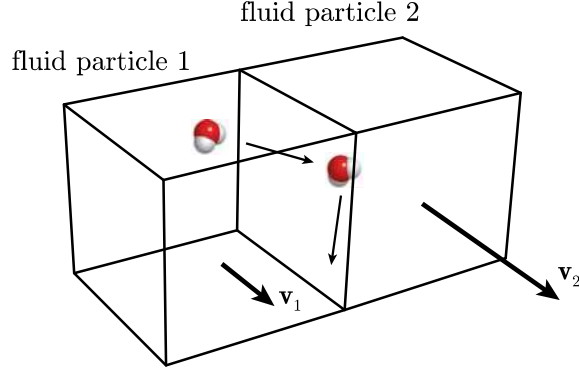
with  $p_a$ , the fictitious excess pressure in adiabatic relation with the condensation. The additional stress  $\sigma'_{ij}$  owes its presence to the losses processes generated in response to the temporal and spatial variations in the fields. Using terminology inspired by the electromagnetic analogy that will later be discussed, (Sects. 6.10–6.12), the first adiabatic term would give the right result in the absence of intrinsically irreversible “polarization processes” induced by temporal and spatial variations in the fields; the additional term  $\sigma'_{ij}$ , is there to take into account the presence of such processes. A priori it would decompose in two parts:

$$\sigma'_{ij} = \sigma'_{\tau ij} + \sigma'_{v ij}, \quad (6.33)$$

one,  $\sigma'_{\tau ij}$ , caused by the irreversible processes generated by the temperature variations, and the other,  $\sigma'_{v ij}$ , caused by the irreversible processes generated by the velocity variations.

Indeed, in an inhomogeneous temperature field such as found in a wavefield, the condensation-rarefaction do not occur in exact adiabatic manner due to the small temperature exchanges which appear mainly by thermal conduction between neighbouring particles, (having slightly different instantaneous temperatures). For this reason, the natural excess pressure variable will not be the adiabatic one,  $p_a = \chi_0^{-1} b$ . It will be the thermodynamic excess pressure,  $p(b, \tau)$ , by definition related to  $b$  and

**Fig. 6.5** Momentum transfer between two fluid particles 1 and 2 in contact



$\tau$  by the equation of state, (6.30).<sup>12</sup> This is because, as said, scale-separation here, means that the local thermodynamic equilibrium tends to be reached. Actually, as the effect of temperature spatial variations must express in  $\sigma'_{\tau ij}$ , we guess that this thermal stress term, once added to the adiabatic first term in (6.32), leads to the excess thermodynamic pressure, hydrostatic term:

$$-p_a \delta_{ij} + \sigma'_{\tau ij} = -\chi_0^{-1} b \delta_{ij} + \sigma'_{\tau ij} = -p \delta_{ij}. \quad (6.34)$$

Then the total excess stress tensor also simply writes:

$$\sigma_{ij} = -p \delta_{ij} + \sigma'_{vij}. \quad (6.35)$$

This last form (6.35) in place of (6.32), is the way the stresses are usually directly written in literature, without mentioning (6.32), (6.33), and (6.34), see e.g. Landau and Lifshitz [6]. We will soon see, in the analysis of the Fourier constitutive law which specifies the additional variable  $q_i$ , that the thermal stress tensor  $\sigma'_{\tau ij}$  which allows (6.32)–(6.34) to be satisfied is:

$$\sigma'_{\tau ij} = -\kappa \frac{\gamma - 1}{\beta_0 T_0} \int_{-\infty}^t dt' \partial'^2 \tau \delta_{ij}, \quad (6.36)$$

where  $\kappa$  is the coefficient of thermal conduction in Fourier's law.

The remaining  $\sigma'_{vij}$  in (6.33) or (6.35), is the viscous contribution, separately identified in Stokes's law. It must determine the rate of momentum transfer between two fluid particles 1 and 2 in contact, that do not have the same average velocity, as sketched in Fig. 6.5.

Indeed, over a physically infinitesimal time period, many molecules (a thermodynamic number) pass through the boundary surface between particles 1 and 2. But

<sup>12</sup> It will be shown that  $p$  is related to  $b$  only, by an operational nonlocal bulk modulus. The operator in the relation  $p = \hat{\chi} b$  will be derived in Sect. 6.12.3

as the mean velocity in 2 is, say, greater than that in 1, this exchange is not symmetrical and particle 1 receives more momentum than does particle 2. Therefore, an irreversible exchange of momentum between the two particles is generated across the surface. This exchange creates the viscous stress tensor  $\sigma'_{vij}$ . It has two indices as one is defined by the direction of the normal to the considered surface, and the other is the direction defining the considered momentum component. The simplest *ansatz* is that at a given time  $t$  and position, it depends only on the first derivatives,  $\partial v_i/\partial x_j$ , evaluated at the same time and position, with other dependencies, (higher order spatial derivatives and time-memory effects), leading to negligible corrections.<sup>13</sup> With the additional observation that it should vanish if the fluid is rotating uniformly as a whole, one deduces that only the symmetric combinations  $\partial v_i/\partial x_j + \partial v_j/\partial x_i$ , can be involved. As a result,  $\sigma'_{vij}$  must be constructed with these combinations and the unit tensor  $\delta_{ij}$ , the only other tensor at our disposal with isotropy assumed. There follows that the viscous irreversible current density of momentum is principally given in the form:

$$\sigma'_{vij} = \eta \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_l v_l \right) + \zeta \delta_{ij} \partial_l v_l, \quad (6.37)$$

with  $\eta$  and  $\zeta$ , two constitutive constants of the fluid, which are named the first and second viscosity. Equation (6.37) is Stokes's constitutive law.

It remains to state Fourier's constitutive law, indicating how much thermal energy is irreversibly transferred by thermal conduction. To state this law, we assume that the fluid particles 1 and 2 in contact do not have the same temperature. As before, particles 1 and 2 exchange a thermodynamic number of molecules, across their common boundary surface, over a physically infinitesimal period of time. But as the mean temperature in 2 is, say, greater than that in 1, particle 1 receives molecules having higher thermal speed than does particle 2. Therefore, a heat exchange is generated across the surface, that is described in terms of a thermal-conduction vector heat flux  $q_i$ . The simplest *ansatz* is that at a given time  $t$  and position, this vector depends only on the first derivatives  $\partial \tau/\partial x_i$  evaluated at the same time and position, with other dependencies leading to negligible corrections.<sup>14</sup> With isotropy assumed there follows that the irreversible current density of heat transfer is principally:

$$q_i = -\kappa \partial_i \tau, \quad (6.38)$$

with  $\kappa$  a constitutive constant of the fluid, which is named the thermal conduction coefficient. Equation (6.38) is Fourier's constitutive law. Note that with these laws the density of entropy production can be shown to be

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<sup>13</sup> In fact, these corrections should imperatively be taken into account, (which raises deep irreversible-thermodynamic questions), in such a way that the description, once reviewed in the light of the nonlocal electromagnetic analogy, remains physical and mathematically well conditioned: see end of Sect. 6.12.3.

<sup>14</sup> Again, the electromagnetic analogy suggests that these corrections should also be taken into account to ensure that the description remains entirely physical: see end of Sect. 6.12.3.

$$\sigma_s = -T_0^{-2} q_i \partial_i \tau + T_0^{-1} \sigma'_{vij} \partial_i v_j, \quad (6.39)$$

and by the second principle of thermodynamics which imposes  $\sigma_s > 0$ , we have that the introduced viscosities and thermal conduction coefficients are all positive.

Substituting Stokes's and Fourier's constitutive laws in the set (6.28)–(6.31) completed by (6.35), we finally obtain the governing linearized equations stated in (6.10)–(6.13).

We can now check our previous identifications (6.34) and (6.36). The combination of (6.12) and (6.13) gives,  $\rho_0 c_p \partial_t (\gamma \chi_0 p - b) / \beta_0 = \beta_0 T_0 \partial_t p + \kappa \partial^2 \tau$ , and using the first thermodynamic identity (6.20), we derive after straightforward calculation:

$$p = \chi_0^{-1} b + \kappa \frac{\gamma - 1}{\beta_0 T_0} \int_{-\infty}^t dt' \partial^2 \tau. \quad (6.40)$$

This expression of the thermodynamic excess pressure is coherent with (6.34), if only, we have the expression (6.36) of the thermal stresses.

We note that, in the same way as it is logical to write the momentum equation (6.29) as,  $\rho_0 \partial_t v_i = \partial_j (-\chi_0^{-1} b \delta_{ij} + \sigma'_{ij})$ , (using (6.32)), but it is, actually, more convenient to use the thermodynamic pressure and write it,  $\rho_0 \partial_t v_i = \partial_j (-p \delta_{ij} + \sigma'_{vij})$ , (using instead (6.35)), it would be logical to write, for the energy equation,  $\rho_0 c_p \partial_t \tau = \beta_0 T_0 \chi_0^{-1} \partial_i b \delta_i q'_i$ , with some irreversible  $q'_i$  flux, but it is, again, more convenient to make appear the thermodynamic pressure, and use (6.31). The irreversible flux  $q'_i$  would be:  $q'_i = -\gamma \kappa \partial_i \tau = \gamma q_i$ , and it is preferable to deal with  $q_i$ , which is the thermal energy flux.

There is still one important point to be made here. The laws we have considered are special in that they have explicitly assumed scale separation, or sufficiently slow times variations. In reality, at very short times or very short wavelengths, the used near-equilibrium thermodynamic framework becomes unadapted and the appropriate description is expected to become much more subtle. In particular, solid-like behaviours should systematically appear in any fluid, to some extent, at very short times. This hypothesis was put forward long ago, in 1925, by the self-taught Russian theoretical physicist Yakov Il'ich Frenkel [7].

Frenkel's ideas which have long been overlooked, have recently been substantiated in liquids [8]. This suggests that, if the thermodynamic framework in which we work were broad enough, we would have to write, instead of (6.32):

$$\sigma_{ij} = -(\chi_0^{-1})_{ijkl} b_{kl} + \sigma'_{ij}, \quad b_{ij} \equiv -\frac{1}{2} (\partial_i a_j + \partial_j a_i), \quad \mathbf{v} \equiv \partial \mathbf{a} / \partial t, \quad (6.41)$$

$$(\chi_0^{-1})_{ijkl} = \chi_0^{-1} \delta_{ij} \delta_{kl} + \mu_0 \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right). \quad (6.42)$$

In the last equation,  $\mu_0$  would be a new constant, which we could denote the ‘‘Frenkel-Lamé’’ adiabatic shear modulus of the fluid. Its nonzero value would allow for the propagation of undamped shear waves in a very-short-times limit (times so short that the dissipative reactions expressed in the lossy  $\sigma'_{ij}$ , have no time to be built).

In Appendix we see how the electromagnetic-acoustic analogy is automatically, by construction, coherent with these physical views.

In a gas, the large separation between the molecules will make the modulus  $\mu_0$  a priori extraordinarily small; nevertheless, from the strict point of view of questions of principle, it should be considered nonzero, which would imply the presence of additional terms in the dissipative part  $\sigma'_{ij}$  of the stresses, in turn determining at least in Gibbs' sense,<sup>15</sup> the dynamics of propagation and attenuation of Frenkel's "elastic" waves, of extraordinarily low amplitudes. Of course, well before considering these effects in a thermodynamic extension, it would have been necessary, in the first place, to account for the molecular relaxation, and go beyond the too simplistic Stokes and Fourier's laws. A complete thermodynamic extension, we believe, would reveal the full nonlocal pattern of equations and quantities conjectured in the Appendix, inspired by the electromagnetic analogy.

### 6.6.3 *Inherent Thermodynamic Simplifications in Navier-Stokes-Fourier's Framework*

Let us now summarize and comment the above development, which has justified the Navier-Stokes-Fourier linearized-motion model equations (6.10)–(6.13). It rests on simplifications well-verified in practice: the assumption of local equilibrium allowing to use classical equilibrium thermodynamic relations and to neglect molecular relaxation phenomena, the Stokes and Fourier constitutive laws which are the simplest ones to be considered in this framework, and finally, the neglect of Frenkel's shear motions. Nevertheless, as we have just suggested, these are important simplifications from a general physical point of view. As such they can have some unwanted consequences. The motion equations obtained have an elliptic nature through the absence of Frenkel's terms and the Stokes law in the momentum equations, and a parabolic nature through Fourier's law in the energy equation. Their solutions then manifest unphysical features such as the totally diffusive nature of shear waves mode solutions,<sup>16</sup> and instant propagation of an initial temperature disturbance.<sup>17</sup>

In short, when these model equations (6.10)–(6.13) will be reexamined in the light of a general nonlocal electromagnetic analogy, (see Sects. 6.10–6.12 and Appendix), the simplifications they convey will cause defects, particularly visible in the degenerate description they will give of the shearing movements. These degeneracies could

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<sup>15</sup> As in this context the wavelengths could be reduced to the point of becoming comparable to the mean free path, the separation of scale would not be ensured and macroscopic means would have to be taken in the Gibbs sense.

<sup>16</sup> See (6.66) below, this unphysical feature being related to the absence, in purely viscous shearing motions, of associated nonzero "acoustic stress field  $H_{ij}$ " – see in Appendix, after (6.205).

<sup>17</sup> See (6.145) below, this unphysical feature being related to the nonvanishing of the kernel  $\chi_b(t, x)$ , whatever the values, however large, of  $|x|$ , contradicting the finite velocity of propagation of physical influences.

only be totally overcome by working within an extended framework of the thermodynamics of irreversible processes, if necessary with the Gibbs mean, abandoning local equilibrium, going beyond Stokes and Fourier's laws, and introducing Frenkel's shearing motions. Nevertheless, by limiting our considerations to longitudinal movements as described by current laws, (see Sects. 6.10 and 6.12), we will deliberately avoid entering into this difficult subject, which belongs to the unclarified field of Extended Irreversible Thermodynamics far from equilibrium.

## 6.7 The Different Normal Modes at Fixed Real Wavevector

We detail here the different type of solutions of the model equations of motion (6.10)–(6.13). Eliminating the condensation, we have a system of five first-order-in-time differential equations for the five fields  $\mathbf{v}(t, \mathbf{x})$ ,  $p(t, \mathbf{x})$  and  $\tau(t, \mathbf{x})$ . Therefore, given a fixed wavevector  $\mathbf{k}$ , the solutions can be analyzed in terms of five “hydrodynamic modes” or “normal modes” solutions varying like<sup>18</sup>:

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{v}_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}, \quad p(t, \mathbf{x}) = p_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}, \quad \tau(t, \mathbf{x}) = \tau_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}.$$

Consider first the case where the wavevector  $\mathbf{k} = k_x \hat{\mathbf{x}} + k_y \hat{\mathbf{y}} + k_z \hat{\mathbf{z}}$  has real components. It is the form  $\mathbf{k} = k \hat{\mathbf{n}}$ , specified by a unit vector direction  $\hat{\mathbf{n}}$  and modulus  $k$ . We must find five hydrodynamic modes solutions having this real wavevector, and complex frequencies, (to account for the damping brought by irreversible processes). To clarify these five solutions, we introduce three independent polarization directions, the longitudinal (1) direction ( $\parallel \hat{\mathbf{n}}$ ), and transversal (2, 3) directions ( $\perp \hat{\mathbf{n}}$ ), in which the velocity amplitude vector  $\mathbf{v}_0$  can point, as shown in Fig. 6.6.

We write the complex frequencies as,  $\omega = \omega' - i\omega''$ . Hence the exponential factor will have the form  $e^{-\omega'' t} e^{-i\omega' t + i\mathbf{k}\hat{\mathbf{n}}\cdot\mathbf{x}}$ . Physically, here, as the imaginary parts owe their existence to the losses terms, they are to be taken negative, i.e.  $\omega''$  positive, to ensure that the solutions fade away over time. The real parts will be either zero, corresponding to diffusive purely damped solutions, or nonzero, with the two signs possible, corresponding to right-going or left-going waves.

The five modes solutions, therefore, are as follows<sup>19</sup>:

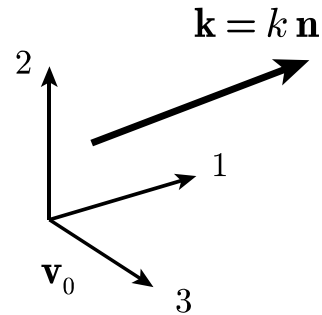
### Polarization 1:

$$\omega' = c_0 k - O(k^3), \quad \omega'' = \Gamma k^2 + O(k^4),$$

<sup>18</sup> When using this complex notation, we understand that  $\mathbf{v}(t, \mathbf{x}) = \Re(\mathbf{v}_0 e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}})$ , etc., systematically omitting here and elsewhere the real part symbol  $\Re(\ )$ .

<sup>19</sup> The results for polarization 1 are the complex solutions,  $\omega = \omega' - i\omega''$ , of Kirchhoff-Langevin's dispersion equation (6.52). The results for polarization 2, 3, are the complex solutions of the vortical diffusion equation, first (6.61).

Fig. 6.6 Polarization



damped longitudinal wave propagating in the direction of vector  $\mathbf{k} = k\hat{\mathbf{n}}$  (a so-called “acoustic wave”; it has pressure and temperature variations but almost no entropy variations),

**Polarization 1:**

$$\omega' = -c_0 k + O(k^3), \quad \omega'' = \Gamma k^2 + O(k^4),$$

damped longitudinal wave of the same type (acoustic wave) propagating in the reverse direction,

**Polarization 1:**

$$\omega' = 0, \quad \omega'' = \frac{\kappa}{\rho_0 c_P} k^2 - O(k^4),$$

purely damped longitudinal wave (a so-called “entropic wave”; it has entropy and temperature variations but almost no pressure variation),

**Polarization 2:**

$$\omega' = 0, \quad \omega'' = \frac{\eta}{\rho_0} k^2,$$

purely damped shear wave (transversal, also called vortical; it has no pressure, temperature, or entropy variations),

**Polarization 3:**

$$\omega' = 0, \quad \omega'' = \frac{\eta}{\rho_0} k^2,$$

purely damped shear wave of the same type but other polarization,

and where,



$$\Gamma \equiv \frac{1}{2\rho_0} \left[ \frac{4\eta}{3} + \zeta + \kappa \left( \frac{1}{c_V} - \frac{1}{c_P} \right) \right], \quad (6.43)$$

is a small constitutive constant called the classical sound damping constant. Its order of magnitude is that of the kinematic viscosity

$$\nu \equiv \frac{\eta}{\rho_0}, \quad (6.44)$$

respectively  $10^{-5} \text{ m}^2\text{s}^{-1}$  and  $10^{-6} \text{ m}^2\text{s}^{-1}$  for air and water.

Consider next the case with real frequency and complex wavevectors. The above three kinds of waves, shear, “entropic” and “acoustic”, will correspond to three different sorts of complex wavenumbers. In the following, we show how the equations of motion (6.10)–(6.13), are used to derive the characteristics of the “acoustic” and “entropic” harmonic modes; this calculation is classical and can be found in Rayleigh [9].

## 6.8 Some Explicit Calculations for the Longitudinal Modes

Insert (6.12) in (6.13) and use the first thermodynamic identity (6.20) and third definition (6.18) to obtain an alternative form of (6.13):

$$\partial_t \tau = \frac{\gamma - 1}{\beta_0} \partial_t b + \frac{\kappa}{\rho_0 c_V} \partial^2 \tau.$$

It simplifies the equations to introduce  $\tau' = \beta_0 \tau / (\gamma - 1)$ . Using this temperature variable and denoting  $c_0$  and  $c'_0$  the adiabatic and isothermal reference sound velocities ( $c_a$  and  $c_i$  see (6.19)),<sup>20</sup> the (6.12) and (6.13) take the following form:

$$\frac{P}{\rho_0} = c_0^2 b + (c_0^2 - c_0'^2) \tau', \quad (6.45)$$

and

$$\partial_t \tau' = \partial_t b + \frac{\kappa}{\rho_0 c_V} \partial^2 \tau'. \quad (6.46)$$

Let us work in harmonic regime  $e^{-i\omega t}$ . Substituting (6.45) inside the last form of (6.11) gives an equation having the form,  $\partial_t \mathbf{v} - \nu \partial^2 \mathbf{v} = -\partial X$ , with

$$X = \left[ c_0^2 - i\omega \frac{\zeta + \frac{\eta}{3}}{\rho_0} \right] b + (c_0^2 - c_0'^2) \tau',$$

<sup>20</sup> Which verify  $\rho_0 \chi_0 c_0^2 = 1$ ,  $\rho_0 \gamma \chi_0 c_0'^2 = 1$ ,  $c_0^2 = \gamma c_0'^2$ ,  $\gamma - 1 = \frac{c_0^2 - c_0'^2}{c_0'^2}$  and  $\frac{\gamma - 1}{\gamma} = \frac{c_0^2 - c_0'^2}{c_0^2}$ .

Equation (6.46) gives

$$b = \tau' + \frac{\kappa}{\rho_0 c_V i \omega} \partial^2 \tau'. \quad (6.47)$$

Finally, with (6.10) giving  $\partial \cdot \mathbf{v} - i\omega b = 0$ , we obtain the following velocity-temperature equations:

$$-i\omega \mathbf{v} - \nu \partial^2 \mathbf{v} = -\partial X, \quad X = \left[ c_0'^2 - i\omega \frac{\zeta + \frac{\eta}{3}}{\rho_0} \right] b + (c_0^2 - c_0'^2) \tau', \quad (6.48)$$

$$\partial \cdot \mathbf{v} - i\omega \tau' - \frac{\kappa}{\rho_0 c_V} \partial^2 \tau' = 0. \quad (6.49)$$

By taking the divergence of (6.48.1), and eliminating the velocity-divergence terms using (6.49), there results the following equation on temperature:

$$\begin{aligned} -\omega^2 \tau' - \left[ c_0^2 - i\omega \left( \frac{\kappa}{\rho_0 c_V} + \nu + \frac{\zeta + \frac{\eta}{3}}{\rho_0} \right) \right] \partial^2 \tau' + \dots \\ - \frac{\kappa}{\rho_0 c_V i \omega} \left[ c_0'^2 - i\omega \left( \nu + \frac{\zeta + \frac{\eta}{3}}{\rho_0} \right) \right] \partial^4 \tau' = 0. \end{aligned} \quad (6.50)$$

Looking for a solution having the form  $e^{-i\omega t + ik \cdot \mathbf{x}}$  and setting by definition

$$-\mathbf{k}^2 \equiv \lambda, \quad (6.51)$$

the constant  $\lambda$  must be the solution of the following Kirchhoff-Langevin's<sup>21</sup> characteristic equation:

$$\begin{aligned} -\omega^2 - \left[ c_0^2 - i\omega \left( \frac{\kappa}{\rho_0 c_V} + \nu + \frac{\zeta + \frac{\eta}{3}}{\rho_0} \right) \right] \lambda + \dots \\ - \frac{\kappa}{\rho_0 c_V i \omega} \left[ c_0'^2 - i\omega \left( \nu + \frac{\zeta + \frac{\eta}{3}}{\rho_0} \right) \right] \lambda^2 = 0. \end{aligned} \quad (6.52)$$

For small real  $k$  ( $k(\ell \sim \nu/c_0) \ll 1$ , long wavelengths), the complex frequencies  $\omega$ , solutions to this equation, expand in powers of  $k$ . Writing such expansions and substituting them in (6.52), along with  $\lambda = -k^2$ , one obtains the three "polarization 1" solutions, (two acoustic, one entropic), given in Sect. 6.7. For real frequency, the dispersion equation (6.52) has two complex roots  $\lambda_{1,2}$ :

$$-\lambda_1 = k_1^2 = \frac{1}{2A} [-B + \sqrt{B^2 - 4AC}], \quad -\lambda_2 = k_2^2 = \frac{1}{2A} [-B - \sqrt{B^2 - 4AC}], \quad (6.53)$$

with

<sup>21</sup> See [2] for an explanation of this denomination.

$$\begin{aligned}
A &= -\frac{\kappa}{\rho_0 c_V i \omega} \left[ c_0'^2 - i \omega \left( \nu + \frac{\zeta + \frac{\eta}{3}}{\rho_0} \right) \right], \quad C = -\omega^2, \\
B &= c_0^2 - i \omega \left( \frac{\kappa}{\rho_0 c_V} + \nu + \frac{\zeta + \frac{\eta}{3}}{\rho_0} \right).
\end{aligned} \tag{6.54}$$

Using by convention, square-root determinations  $\Re(\sqrt{\phantom{x}}) > 0$ ,  $\lambda_1$  represents the smallest solution, mainly real, that describes acoustic waves having very small absorption, and  $\lambda_2$  represents the other much larger solution,<sup>22</sup> mainly purely imaginary, that describes the highly damped diffusive entropic waves. The associated acoustic and entropic wavenumbers, such that the dependencies  $e^{-i\omega t + ik_{1,2}x}$  represent acoustic and entropic waves propagating in the direction  $+x$ , are

$$k_1 = \sqrt{\frac{1}{2A}[-B + \sqrt{B^2 - 4AC}]}, \quad k_2 = \sqrt{\frac{1}{2A}[-B - \sqrt{B^2 - 4AC}]}, \tag{6.55}$$

(with this choice,  $\Im(k_{1,2}) > 0$ , so that the waves attenuate along  $+x$ ).

After straightforward calculations we find expansions having the form:

$$-\lambda_1 = \frac{\omega^2}{c_0^2} \left[ 1 + i \omega \frac{2\Gamma}{c_0^2} + \dots \right], \quad -\lambda_2 = i \frac{\rho_0 c_P \omega}{\kappa} \left[ 1 + i \omega \frac{2\Gamma'}{c_0^2} + \dots \right], \tag{6.56}$$

$$k_1 = \frac{\omega}{c_0} \left[ 1 + i \omega \frac{\Gamma}{c_0^2} + \dots \right], \quad k_2 = (1 + i) \sqrt{\frac{\rho_0 c_P \omega}{2\kappa}} \left[ 1 + i \omega \frac{\Gamma'}{c_0^2} + \dots \right], \tag{6.57}$$

with  $\Gamma$  the expression previously given in (6.43) and  $\Gamma'$  the expression

$$\Gamma' = \frac{\gamma - 1}{2\rho_0} \left( \frac{4\eta}{3} + \zeta - \frac{\kappa}{c_P} \right). \tag{6.58}$$

The solution for the wavenumber  $k_1$  describes a pressure wave propagating with a soundspeed velocity always slightly less than the reference adiabatic value  $c_0$ , and which is slightly attenuated because of the small factor  $\Gamma$ . The solution for the wavenumber  $k_2$  describes a mainly purely diffusive ( $\Re(k) = \Im(k)$ ), highly attenuating entropic (or thermal) wave. The characteristic penetration length of this wave is, as seen on the exponential  $\exp(ik_2x) \cong \exp(ix\sqrt{\rho_0 c_P \omega / 2\kappa}) \exp(-x\sqrt{\rho_0 c_P \omega / 2\kappa})$ , determined by the characteristic entropic, or thermal boundary layer, length  $\delta_\tau = (2\kappa / \rho_0 c_P \omega)^{1/2}$ .

It can be checked that, for these two types of waves (respectively  $T = 1, 2$ ) there are the following relations:

<sup>22</sup> But still small in the sense  $|k_2 \ell| \lll 1$  (long wavelengths), ensuring the scale separation discussed in Sect. 6.3.

$$v_i = \frac{\omega}{k_T^2} k_{Ti} b, \quad (i = x, y, z), \quad p = \frac{1}{\gamma \chi_0} \left( 1 + \frac{\gamma - 1}{1 + \gamma \frac{\kappa}{\rho_0 c_P} \frac{1}{-i\omega} k_T^2} \right) b, \quad (6.59)$$

$$\tau = \frac{1}{\beta_0} \frac{\gamma - 1}{1 + \gamma \frac{\kappa}{\rho_0 c_P} \frac{1}{-i\omega} k_T^2} b, \quad s = \frac{\gamma - 1}{\gamma} \frac{c_P}{\beta_0 T_0} \left( -1 + \frac{1}{1 + \gamma \frac{\kappa}{\rho_0 c_P} \frac{1}{-i\omega} k_T^2} \right) b$$

For the type 1, the “acoustic wave”, the entropy variations are almost zero,  $s \cong 0$ . In effect,  $(\gamma \kappa / -i\omega \rho_0 c_P) k_1^2$  is usually close to zero because of the small  $\kappa / \rho_0 c_P$  value (as long as frequencies are not too high, or wavelengths, not too small). Hence the parenthesis in the expression of  $s$  almost vanishes. In the expression of  $p$ , the parenthesis almost takes the value  $1 + (\gamma - 1)/1 = \gamma$ , leading to pressure-density relation  $p \cong \chi_0^{-1} b$ , close to the adiabatic. The temperature also, is almost in adiabatic relation to the condensation,  $\tau \cong b(\gamma - 1)/\beta_0$ .

For the type 2, the “entropic wave”, on the contrary,  $(\gamma \kappa / -i\omega \rho_0 c_P) k_2^2 \cong -\gamma$ . Then the entropy variations are significant:

$$\left( -1 + \frac{1}{1 + \gamma \frac{\kappa}{\rho_0 c_P} \frac{1}{-i\omega} k_2^2} \right) \cong \left( -1 + \frac{1}{1 - \gamma} \right) = \frac{\gamma}{1 - \gamma} \Rightarrow s \cong -\frac{c_P}{\beta_0 T_0} b.$$

The temperature variations are also non-negligible  $\tau \cong b(\gamma - 1)/\beta_0(1 - \gamma) \cong -b/\beta_0$ . The pressure variations are practically zero,

$$\left( 1 + \frac{\gamma - 1}{1 + \gamma \frac{\kappa}{\rho_0 c_P} \frac{1}{-i\omega} k_2^2} \right) \cong 1 + \frac{\gamma - 1}{1 - \gamma} = 0 \Rightarrow p \cong 0.$$

Indeed, using (6.57.2) we find, for this entropic pressure,  $p \cong \frac{1}{\chi_0} \frac{2i\omega \Gamma'}{(\gamma - 1)c_0^2} b$ , which is very small compared to the adiabatic pressure  $b/\chi_0$ . This is why the first and second type of longitudinal waves are often termed pressure waves and entropy waves, in spite of the fact that pressure and entropy variations are always simultaneously present in these two waves.

## 6.9 General Representation of the Fields in Harmonic Regime Without Source

Introducing acoustic and entropic potentials,  $\phi_1$  and  $\phi_2$ , such as

$$\partial^2 \phi_{1,2} = \lambda_{1,2} \phi_{1,2}, \quad (6.60)$$

the general field  $\tau'$ , solution to the temperature equation (6.50), will be

$$\tau' = A_1 \phi_1 + A_2 \phi_2,$$

with  $A_{1,2}$  two constants. The general velocity field  $\mathbf{v}$  will be

$$\mathbf{v} = \mathbf{v}' + B_1 \partial \phi_1 + B_2 \partial \phi_2,$$

with the vortical part  $\mathbf{v}'$  obeying the equations

$$\partial^2 \mathbf{v}' = \frac{-i\omega}{\nu} \mathbf{v}', \quad \partial \cdot \mathbf{v}' = 0. \quad (6.61)$$

One expresses the coefficients  $B$  from the coefficients  $A$  by using the fact that the (6.49) is independently satisfied by terms 1 and 2. This yields

$$B_{1,2} = \left( \frac{\kappa}{\rho_0 c_V} - \frac{-i\omega}{\lambda_{1,2}} \right) A_{1,2}.$$

From the temperature field, the condensation field can be written using (6.47). The pressure field then derives using (6.45). In summary, the general form of the fields in harmonic regime and without source, is:

$$\mathbf{v} = \mathbf{v}' + \left( \frac{\kappa}{\rho_0 c_V} + \frac{i\omega}{\lambda_1} \right) A_1 \partial \phi_1 + \left( \frac{\kappa}{\rho_0 c_V} + \frac{i\omega}{\lambda_2} \right) A_2 \partial \phi_2, \quad (6.62)$$

$$\tau' = A_1 \phi_1 + A_2 \phi_2, \quad (6.63)$$

$$b = \left( 1 + \frac{\kappa}{\rho_0 c_V i \omega} \lambda_1 \right) A_1 \phi_1 + \left( 1 + \frac{\kappa}{\rho_0 c_V i \omega} \lambda_2 \right) A_2 \phi_2, \quad (6.64)$$

$$\frac{p}{\rho_0} = \left( c_0^2 + c_0'^2 \frac{\kappa}{\rho_0 c_V i \omega} \lambda_1 \right) A_1 \phi_1 + \left( c_0^2 + c_0'^2 \frac{\kappa}{\rho_0 c_V i \omega} \lambda_2 \right) A_2 \phi_2. \quad (6.65)$$

The wavenumber for the vortical motions (shear motions described in the part  $\mathbf{v}'$ ), is purely diffusive, highly attenuated. Indeed, putting the form,  $\mathbf{v}' = \mathbf{v}'_0 e^{-i\omega t + i k_v \cdot \mathbf{x}}$ , in (6.61.1), gives,  $k_v^2 = i\omega/\nu$ , or:

$$k_v = \frac{1+i}{\sqrt{2}} \left( \frac{\omega}{\nu} \right)^{1/2}. \quad (6.66)$$

The characteristic penetration length of this wave is seen on the exponential  $\exp(i k_v \cdot \mathbf{x}) = \exp(i x / \delta_v) \exp(-x / \delta_v)$ . It is the characteristic shear, or viscous boundary layer, length  $\delta_v = (2\nu/\omega)^{1/2}$ . Because the Prandtl number,  $\text{Pr} \equiv \eta c_P / \kappa$ , is generally of order one, (e.g. 0.71 for air), the viscous and thermal penetration lengths,

$\delta_\tau = \delta_v / \text{Pr}^{1/2}$ , are of a comparable order. This order is very small: for example for air at audible frequencies, typical viscous and thermal attenuation lengths are in the order of  $10^{-4}$  m, (still very “long-wavelength” compared to the mean free path  $\ell \sim \eta / \rho_0 c_0 = 4.5 \cdot 10^{-8}$  m).

## 6.10 Revisiting the Longitudinal Acoustic Equations Using an Electromagnetic Analogy

We now want to make clear, that the propagation of waves in the fluid, in the presence of viscous and thermal losses, can be considered in exactly the same way as the nonlocal macroscopic propagation of electromagnetic waves in a medium. We believe that this electromagnetic analogy would assume its full beautiful form, sketched in Appendix, if it were possible to extend sufficiently the thermodynamic framework, so as to model also Frenkel’s shearing movements; if we work within Navier-Stokes-Fourier model, however, the absence of Frenkel’s shearing, among other shortcomings, makes the description degenerate. We mentioned previously in Sect. 6.6.3 these insufficiencies of the description. As the completion of the analogy is until now a distant perspective, in what follows, to avoid degeneration, we limit ourselves to considering the Navier-Stokes-Fourier model when shearing movements are not excited.

Recall that the momentum equation can be written,  $\rho_0 \partial_t \mathbf{v} = -\partial p - \eta \partial \times (\partial \times \mathbf{v}) + (4\eta/3 + \zeta) \partial(\partial \cdot \mathbf{v})$ . Focusing on longitudinal motions only, the term  $-\eta \partial \times (\partial \times \mathbf{v})$  is not excited, and the set of equations to be revisited is

$$\partial_t b + \partial \cdot \mathbf{v} = 0, \quad (6.67)$$

$$\rho_0 \partial_t \mathbf{v} = -\partial p - \left( \frac{4\eta}{3} + \zeta \right) \partial \partial_t b + [\mathbf{f} = -\partial \mathcal{P}], \quad (6.68)$$

$$\gamma \chi_0 p = b + \beta_0 \tau, \quad (6.69)$$

$$\rho_0 c_F \partial_t \tau = \beta_0 T_0 \partial_t p + \kappa \partial^2 \tau. \quad (6.70)$$

In (6.68) we have included the possibility that an external density of longitudinal bulk force  $\mathbf{f}$ , that is thus given as a gradient, is acting on the fluid. We will see that these (6.67)–(6.70) can be transformed into a general pattern inspired by that of macroscopic electromagnetic equations in a homogeneous material medium, with a source  $[\mathbf{J}]$ , density of external electrical current, playing the role of source  $[\mathbf{f}]$ , density of external longitudinal body force. To prepare this way of looking at the acoustic equations of motion, we now go to macroscopic electromagnetic theory and study its formal pattern.

## 6.11 Macroscopic Electromagnetics

Due to the lack, to date, of a precise relationship between macroscopic electromagnetics and thermodynamics in the broad sense, (out of equilibrium), it is still not possible to perform a precise specification of some of the macroscopic quantities involved in the macroscopic equations that describe the propagation of electromagnetic waves in, say, a neutral material medium given in an ambient rest state. This point is generally not sufficiently well recognized in textbooks and literature. For example in Jackson [10, Chap. 6], there is written, about the macroscopic electromagnetic equations: “Although these equations are familiar and totally acceptable, we have yet to present a serious derivation of them from a microscopic starting point”; there follows a “microscopic derivation” of the stated equations, which is apparently intended to clarify the question, but which, in fact, remains superficial and does not address a number of difficult and embarrassing questions. This is not surprising: one could hardly expect deriving results belonging to the field of macroscopic physics proper, from scratch microscopics, (and even less so, when the microscopic level is that of semiclassical models). We try here to bring out some of the irritating pending questions, in a formal discussion. The subject will not come out of it clarified, quite the contrary, but the simple recognition of the presence of difficulties of a macroscopic thermodynamic nature in the broad sense, will be useful to us later in acoustics. Moreover, the forthcoming acoustic implementation of our discussion – in this chapter and the next – will particularly retrospectively highlight the thermodynamic nature of these difficulties, which will be circumvented in acoustics, only because of an existing (even if imperfect) thermodynamic framework, that of the Navier-Stokes-Fourier model.

### 6.11.1 *Nonlocal Maxwellian Pattern of the Macroscopic Equations*

Here, we are interested in the description of time variable and space variable phenomena such as occurring in macroscopic wave propagation, either corresponding to free motion (no external source), or forced motion (with external source). Although, as we have said, no complete derivations of the required macroscopic electromagnetic wave equations have ever been made from the microstructure, we will argue that, what is known at the microscopic level, suggests that, at the macroscopic level can be written a system of two field equations and two nonlocal constitutive equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -\partial \times \mathbf{E}, \quad \frac{\partial \mathbf{D}}{\partial t} = \partial \times \mathbf{H} - [\mathbf{J}], \quad \mathbf{D} = \hat{\epsilon} \mathbf{E}, \quad \mathbf{H} = \hat{\mu}^{-1} \mathbf{B}. \quad (6.71)$$

The impressed term  $[\mathbf{J}]$  is a current density source term which we put inside brackets, as it may, or may not be applied. The quantities  $\hat{\epsilon}$  and  $\hat{\mu}$  are operators intrinsic to the

medium, establishing action-response relations between fields, and having the same following *general nonlocal form*, whatever the source term is present, or not<sup>23</sup>:

$$\mathbf{D}(t, \mathbf{x}) = \int \epsilon(t - t', \mathbf{x}, \mathbf{x}') \mathbf{E}(t', \mathbf{x}') dt' d\mathbf{x}', \quad (6.72)$$

$$\mathbf{H}(t, \mathbf{x}) = \int \mu^{-1}(t - t', \mathbf{x}, \mathbf{x}') \mathbf{B}(t', \mathbf{x}') dt' d\mathbf{x}'. \quad (6.73)$$

Of the four electromagnetic macroscopic fields  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{D}$ ,  $\mathbf{H}$ , two of them,  $\mathbf{E}$  and  $\mathbf{B}$ , will be called Lorentz's fields and directly interpreted in the next subsection as the macroscopic means of the microscopic electric and magnetic fields  $\mathbf{e}$  and  $\mathbf{b}$ :  $\mathbf{E} \equiv \langle \mathbf{e} \rangle$  and  $\mathbf{B} \equiv \langle \mathbf{b} \rangle$ . The other two,  $\mathbf{D}$  and  $\mathbf{H}$ , will be called Maxwell's fields and seen to reflect the response of the medium to the preceding fields, and thus, the nature of the microstructures. They are not to be viewed as the macroscopic averages of corresponding microscopic fields. They are, in fact, *problematic to define in principle, in a material environment*. By the way, not apparent in the above Gibbs-Heaviside's standard vector notations, Lorentz's and Maxwell's fields have different tensor nature, (see Appendix<sup>24</sup>). Whereas  $\mathbf{E}$  and  $\mathbf{B}$  express in terms of true tensors,  $\mathbf{D}$  and  $\mathbf{H}$  express in terms of tensor densities of weight  $-1$ . The impressed densities of charges and current  $\rho$  and  $\mathbf{J}$ <sup>25</sup> also are tensor densities of weight  $-1$ . Like the Maxwell fields they are problematic to define in principle, in a material environment, and this is also generally obscured in literature.

The nonlocal form of the relationships (6.72) and (6.73) is essential to assure that the description is general; as it is rarely considered in the literature, it is worthwhile opening a parenthesis to comment on it in some detail.

The integration over time  $t'$ , i.e. temporal nonlocality, gives rise to what is called "temporal dispersion"; the integration over space  $\mathbf{x}'$ , i.e. spatial nonlocality, gives rise to what is called "spatial dispersion". Because both integrations are accounted for, we are directly assured of the general validity of the relation (6.72) for a time-invariant medium: it would be useless to add in (6.72) an integral term built on the values  $\mathbf{B}(t', \mathbf{x}')$ . Indeed, as analyzed in Landau and Lifshitz [13], because the fields  $\mathbf{E}$  and  $\mathbf{B}$  are not totally independent but related by (6.71.1), it would be possible to view this term as an effect of spatial dispersion, already described in the first term. Concerning

<sup>23</sup> The dependencies are over  $t - t'$ ,  $\mathbf{x}$  and  $\mathbf{x}'$ , because the medium is assumed time-invariant but not translational invariant. When considering an *homogeneous unbounded medium*, the spatial dependency will be over differences  $\mathbf{x} - \mathbf{x}'$ . Note that the indefinite integrals become definite when considering that the kernels factors must vanish at some point – if only when the event  $(t', \mathbf{x}')$  lies outside the relativistic cone of causality whose tip is the event  $(t, \mathbf{x})$ :  $\int dt' \int d\mathbf{x}' \leftrightarrow \int_{-\infty}^t dt' \int_{|\mathbf{x}-\mathbf{x}'| < c(t-t')} d\mathbf{x}'$ .

<sup>24</sup> These aspects are often overlooked in electromagnetic literature – see however Fournet [11]. See Weinberg [12] for general notions on Gauss coordinates, tensors, tensor densities, covariance, contravariance.

<sup>25</sup>  $\rho$  appears in another general equation,  $\partial \cdot \mathbf{D} = \rho$ , which we need not explicitly consider here, because, in the time variable, space variable regimes, it automatically follows from taking the divergence of (6.71.2), and using the conservation equation,  $\partial \rho / \partial t + \partial \cdot \mathbf{J} = 0$ .



the other nonlocal relationship (6.73), a similar justification of its generality, can be made, based on the relation established between  $\mathbf{D}$  and  $\mathbf{H}$  through (6.71.2), and the generality of (6.72). In brief, because they allow for spatial dispersion in addition to temporal dispersion, the relationships (6.72) and (6.73) between Lorentz's and Maxwell's fields are the most general possible ones, (including, in particular, the so-called bianisotropic constitutive relations). Often, in literature, by a convention, the temporal and spatial dispersion is introduced only in (6.72), not in (6.73), which is replaced by a direct definition-relation,  $\mathbf{H} = \mu_0^{-1} \mathbf{B}$ . We will comment in more detail on this convention in Sect. 6.11.3, and a similar problematic, in the acoustic analogy, will later shed useful light on it, illustrating its simple nature as an expedient.

As long as we use Cartesian coordinates and work with Gibbs-Heaviside's notation, the kernel functions  $\epsilon(t - t', \mathbf{x}, \mathbf{x}')$  and  $\mu^{-1}(t - t', \mathbf{x}, \mathbf{x}')$  retain the nature of second-rank tensor quantities.<sup>26</sup> As a rule, they are independent of the fields and only determined by the microstructure. Nevertheless, different distributions of the Lorentz fields must produce different resulting distributions of Maxwell fields  $\mathbf{D}$  and  $\mathbf{H}$ . Precisely because the created  $\mathbf{D}$  and  $\mathbf{H}$  will depend on the variations of  $\mathbf{E}$  and  $\mathbf{B}$ , all effective properties meaningful for wave propagation, such as impedances and wavenumbers, will reflect associated temporal and spatial variations of Lorentz fields.

Therefore in this sense, one may think of "temporal dispersion" as something that tells us that the effective properties of the medium depend on the time variations of the macroscopic fields; likewise, "spatial dispersion" is something that tells us that the effective properties of the medium depend on the spatial variations of the macroscopic fields.

In wave propagation problems, there are both time and space variations, therefore, temporal dispersion and spatial dispersion, should always be simultaneously present. Nevertheless, often, spatial dispersion effects are very weak: they are neglected in almost all textbooks; or else, once introduced, they are assumed to be small corrections meaningful to consider only when the wavelengths reduce sufficiently [13]. It is only very recently understood [14], that, spatial dispersion effects can be very strong effects at long wavelengths. This occurs in presence of localized resonating structures.

The same generalities will apply to acoustics. Often, spatial dispersion effects are necessary to consider only as corrections to the conventional (local) homogenization, (see next chapter), when the wavelengths reduce sufficiently to become commensurable with the microstructure characteristic lengths. But in some other instances, this time completely failing the conventional views, they can be strong effects at long wavelengths. This occurs in presence of resonances. People often speak of "local resonances" as these resonances are generally those of finite structures, localized in space. But this wording is, in-part, unfortunate: when "local resonances" are present,

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<sup>26</sup> In general Gauss coordinates,  $\epsilon$  remains second-rank but  $\mu^{-1}$  becomes fourth-rank tensor; the first and last (one or two) indices, transform respectively with coefficients associated to the different points  $\mathbf{x}$  and  $\mathbf{x}'$ .

the wave propagation physics becomes strongly nonlocal (spatially).<sup>27</sup> A nonlocal homogenization description will be developed in next chapter, for rigid-framed fluid-saturated materials. In this chapter, as we consider a simple fluid with no structure, the spatial dispersion effects will be weak, (but essential to allow for the propagation of other waves than the so-called acoustic one).

We close our parenthesis here. We now want to examine why it is not so evident to fix the meaning of the Maxwell fields  $\mathbf{D}$ ,  $\mathbf{H}$ , and impressed source term  $[\mathbf{J}]$ . To do this, we try to sketch in the next section how these fields appear, in a material environment, and lead to the formal pattern of the above equations.

### 6.11.2 Passage from Microscopics to Macroscopics

To evoke the difficult nature of the questions raised by the specification of the fields  $\mathbf{D}$ ,  $\mathbf{H}$ , and  $[\mathbf{J}]$ , let's start with the equations that hold true at the microscopic level. In a semi-classic description these are the well-known electromagnetic equations first written at the turn of the 20th century by Lorentz in his celebrated series of papers, and monograph, on "The theory of electrons". They comprise first, the so-called "microscopic Maxwell-Lorentz equations":

$$\partial \cdot \mathbf{b} = 0, \quad \frac{\partial \mathbf{b}}{\partial t} + \partial \times \mathbf{e} = \mathbf{0}, \quad \partial \cdot \mathbf{e} = \frac{\rho}{\epsilon_0}, \quad \epsilon_0 \frac{\partial \mathbf{e}}{\partial t} = \frac{1}{\mu_0} \partial \times \mathbf{b} - \mathbf{j}. \quad (6.74)$$

We can be assured of their profound physical significance because there is a route, from Quantum Electro-Dynamics describing the photonic field, to these equations, describing electric and magnetic fields. They comprise next, the following inherently semi-classic expressions for the microscopic densities of charge  $\rho$  and current  $\mathbf{j}$ :

$$\rho = \sum_{\alpha} q_{\alpha} \delta[\mathbf{x} - \mathbf{x}_{\alpha}(t)], \quad \mathbf{j} = \sum_{\alpha} q_{\alpha} \mathbf{v}_{\alpha}(t) \delta[\mathbf{x} - \mathbf{x}_{\alpha}(t)], \quad (\mathbf{v}_{\alpha} \equiv d\mathbf{x}_{\alpha}/dt), \quad (6.75)$$

where  $\delta$  is the Dirac delta, and the symbolic summation<sup>28</sup>  $\sum_{\alpha}$ ,  $\alpha = 1, 2, \dots$ , is over electrons and nuclei. They comprise finally, the following, also inherently semi-classic and symbolic equations of motion, separately obeyed by the charges moving under the action of the microscopic electric and magnetic fields:

$$\frac{d}{dt} \left( m_{0\alpha} \mathbf{v}_{\alpha} / \sqrt{1 - \frac{v_{\alpha}^2}{c^2}} \right) = q_{\alpha} (\mathbf{e} + \mathbf{v}_{\alpha} \times \mathbf{b}), \quad \alpha = 1, 2, \dots \quad (6.76)$$

<sup>27</sup> In this connection, see the discussion around Fig. 7.4, next Chap. 7, Sect. 7.2.

<sup>28</sup> The labelling  $\alpha$  of all charged particles is here symbolic as it makes abstraction of counting subtleties related to quantum indiscernibility of identical particles.

The justification for using delta functions in (6.75) is that electrons have no perceptible dimensions, while those of nuclei are still astonishingly small,  $\sim 10^{-14}$  m. Interested in what happens at scales much larger than this, everything is supposed to happen as if charged particles were pointlike, and Maxwell-Lorentz's microscopic equations (6.74) applied everywhere in space. We take with reservation this semi-classical expression of things, which is not a complete reflection of the microscopic quantum world, and among other shortcomings, does not consider the collective aspect of electron behaviour.<sup>29</sup> We note that the conservation relation

$$\frac{\partial \rho}{\partial t} + \partial \cdot \mathbf{j} = 0, \quad (6.77)$$

can be obtained from the conservation of individual charges,  $q_\alpha = Cst$ , and the semi-classic picture, but more deeply, it is also a consequence of Maxwell-Lorentz's microscopic equations themselves.

Now, we observe that we are not directly concerned by microscopic equations and quantities. The fields will have rapid variations at the Ångström scale of molecular dimensions and less ( $\text{Å} = 10^{-10}$  m), and we have to fix our attention not on all these irregularities but only on certain mean values that vary at a much larger ("macroscopic") scale. As Gibbs' conception of ensemble average is delicate to be made explicit here, in literature these mean values  $\langle \cdot \rangle$  are generally conceived using Lorentz's conception of volume average. The idea is to smooth-out irregularities by integrating in averaging spheres or "homogenization" volumes, neither too small nor too large. The sphere radius must be sufficient to be representative of the structure of the material. Since the purpose is to get rid of the irregularities, the sphere must contain a very large number of particles. The averaging on the other hand must not obliterate the changes from point to point that can really be observed. Molecular dimensions ( $\sim \text{Å}$ ) are so much smaller than ordinary electromagnetic wavelengths that both conditions can be met at the same time. In homogeneous substances, a suitable macroscopic homogenization length  $L_h$ , will be on the order of  $100\text{Å}$ , typically. In a Lorentz averaging sphere having this size, there is still on the order of  $10^6$  nuclei and electrons. The average can be best performed using Russakoff's signal-theory-type refinement of a convolution with an isotropic test function  $w(\mathbf{x})$ , of characteristic extent  $L_h$ , and normalized to unity,  $\int d\mathbf{x} w(\mathbf{x}) = 1$ , (see [15] and the discussion in Jackson [10], Chap. 6). The macroscopic volume-averaged electric field  $\mathbf{E}$ , for example, will be

$$\mathbf{E}(t, \mathbf{x}) = \langle \mathbf{e}(t, \mathbf{x}) \rangle = \langle \mathbf{e} \rangle(t, \mathbf{x}) = \int d\mathbf{x}' w(\mathbf{x}') \mathbf{e}(t, \mathbf{x} - \mathbf{x}'). \quad (6.78)$$

If we put a derivative symbol in front of the above integral, we can move it inside:  $\partial_{x_i} \int d\mathbf{x}' w(\mathbf{x}') \mathbf{e}(t, \mathbf{x} - \mathbf{x}') = \int d\mathbf{x}' w(\mathbf{x}') \partial_{x_i} \mathbf{e}(t, \mathbf{x} - \mathbf{x}')$ . Therefore the operations of space differentiation and averaging commute by construction:

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<sup>29</sup> In this respect, see in next chapter, Appendix, the discussion about Curie-von Schweidler's law.

$$\partial_{x_i} \mathbf{E}(t, \mathbf{x}) = \partial_{x_i} \langle \mathbf{e}(t, \mathbf{x}) \rangle = \langle \partial_{x_i} \mathbf{e} \rangle(t, \mathbf{x}). \quad (6.79)$$

The commutation with time differentiation is also evident. Thus, by averaging, the microscopic Maxwell-Lorentz equations (6.74) become the following macroscopic ones

$$\partial \cdot \mathbf{B} = 0, \quad \frac{\partial \mathbf{B}}{\partial t} + \partial \times \mathbf{E} = \mathbf{0}, \quad \partial \cdot \mathbf{E} = \frac{\langle \rho \rangle}{\epsilon_0}, \quad \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\mu_0} \partial \times \mathbf{B} - \langle \mathbf{j} \rangle. \quad (6.80)$$

and the conservation relation (6.77) becomes

$$\frac{\partial \langle \rho \rangle}{\partial t} + \partial \cdot \langle \mathbf{j} \rangle = 0. \quad (6.81)$$

Equation (6.80.1) is, for (time-variable) wave phenomena, a consequence of (6.80.2), whereas, (6.80.3) is a consequence of (6.80.4) and the conservation of charges (6.81). Thus, in what follows, we limit our attention to (6.80.2) and (6.80.4). Comparison of these two equations with (6.71.1) and (6.71.2), suggests that Maxwell's fields  $\mathbf{D}$  and  $\mathbf{H}$ , are introduced by the extraction from  $\langle \mathbf{j} \rangle$  of certain contributions, that can be identified either, with “intrinsic polarization current” determined by bulk properties of the medium, or with “extrinsic” to the medium, external contribution imposed from without. That is,  $\langle \mathbf{j} \rangle$  will have to be decomposed in a (intrinsic) polarization response current  $\mathbf{J}_{pol}$ , expressing the rearrangements of all charges (bound or free) and internal currents (circulation currents responsible for magnetism) in the medium, in relation to the presence of the macroscopic Lorentz fields,<sup>30</sup> and an additional (extrinsic) impressed charge current  $[\mathbf{J}]$ , imposed from the outside:

$$\langle \mathbf{j} \rangle = \mathbf{J}_{pol} + [\mathbf{J}]. \quad (6.82)$$

Moreover the (intrinsic) polarization current  $\mathbf{J}_{pol}$ , characteristic response of the medium to the presence of macroscopic fields  $\mathbf{E}$  and  $\mathbf{B}$ , is supposed to decompose in two parts electric and magnetic:

$$\mathbf{J}_{pol} = \frac{\partial \mathbf{P}}{\partial t} + \partial \times \mathbf{M}, \quad (6.83)$$

what we call Lorentz-splitting,<sup>31</sup> with nonlocal operators  $\hat{\chi}_E$  and  $\hat{\chi}_B$ , or kernels  $\chi_E(t, \mathbf{x}, \mathbf{x}')$ ,  $\chi_B(t, \mathbf{x}, \mathbf{x}')$ , establishing how the “electric” and “magnetic” polarization fields  $\mathbf{P}$  and  $\mathbf{M}$  are determined by the electric and magnetic macroscopic fields:

<sup>30</sup> Say, whatever this will precisely mean, some “electric” and “magnetic” polarization rearrangements of charges and currents.

<sup>31</sup> The idea can be attributed to Lorentz, though we have not found a clear expression of it in original papers. Through it, the polarization current is formally decomposed in time derivative and spatial derivative terms; the part  $\partial \times \mathbf{M}$  actually represents a contracted expression  $M_{;j}^{ij}$ , see Appendix, with  $M^{ij}$  an antisymmetric tensor density.

$$\mathbf{P}(t, \mathbf{x}) = \int dt' d\mathbf{x}' \chi_E(t - t', \mathbf{x}, \mathbf{x}') \mathbf{E}(t', \mathbf{x}'), \quad (6.84)$$

$$\mathbf{M}(t, \mathbf{x}) = \int dt' d\mathbf{x}' \chi_B(t - t', \mathbf{x}, \mathbf{x}') \mathbf{B}(t', \mathbf{x}'). \quad (6.85)$$

Indeed, in (6.84) it is useless to add a similar term with the magnetic field, because, as  $\mathbf{E}$  and  $\mathbf{B}$  are related by (6.80.2), such a term could also be rewritten in the nonlocal integral form (6.84). Likewise, in (6.85), because of the complete nonlocal form of the relation, there is no need to add a similar term with the electric field. Therefore, when (6.82)–(6.83) are substituted in (6.80.4), the addition of the electric polarization current term  $\partial \mathbf{P} / \partial t$  to  $\epsilon_0 \partial \mathbf{E} / \partial t$  will produce the term  $\partial \mathbf{D} / \partial t$  in (6.71.2), and the subtraction of the magnetic polarization current term  $\partial \times \mathbf{M}$  to  $\partial \times \mathbf{B} / \mu_0$  will produce the term  $\partial \times \mathbf{H}$ , provided  $\mathbf{D}$  and  $\mathbf{H}$  are interpreted as:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}. \quad (6.86)$$

Then, (6.72) and (6.73) will apply, with

$$\epsilon(t - t', \mathbf{x}, \mathbf{x}') = \epsilon_0 \delta(t - t') \delta(\mathbf{x} - \mathbf{x}') + \chi_E(t - t', \mathbf{x}, \mathbf{x}'), \quad (6.87)$$

$$\mu^{-1}(t - t', \mathbf{x}, \mathbf{x}') = \mu_0^{-1} \delta(t - t') \delta(\mathbf{x} - \mathbf{x}') - \chi_B(t - t', \mathbf{x}, \mathbf{x}'). \quad (6.88)$$

In this way, the formal pattern of the electromagnetic macroscopic equations set out in Sect. 6.11.1 has been reconstructed and justified; it appears to be entirely general as ~~soon~~ as a macroscopic description is possible.

*soon*

### 6.11.3 Ambiguities and a Suggested Way to Resolve Them

In this “formal derivation” of (6.71)–(6.73) we have left open two difficult questions. We have not explained, how are to be performed the successive separations (6.82) and (6.83). The first separation is clearly elusive: how do we precisely distinguish between the “intrinsic”  $\mathbf{J}_{pol}$  and “extrinsic”  $[\mathbf{J}]$  currents? The second separation is in itself ambiguous: the rotational  $\partial \times \Phi$  of an arbitrary vector potential  $\Phi$ , can always be added to  $\mathbf{P}$ , provided simultaneously, a corresponding term  $\partial \Phi / \partial t$ , be subtracted to  $\mathbf{M}$ . This is not like Helmholtz’s decomposition,  $\mathbf{F} = -\partial \Phi + \partial \times \mathbf{A}$ , of a vector field  $\mathbf{F}$ , into a gradient of a scalar potential and a rotational of a vector potential, which is unambiguous when requiring the vanishing of the fields at infinity and the divergence-free nature of the vector potential.

Due to these shortcomings, we are not able to explain unambiguously, what are the electric and magnetic polarizations  $\mathbf{P}$  and  $\mathbf{M}$ , the Maxwell fields  $\mathbf{D}$  and  $\mathbf{H}$ , the polarization current  $\mathbf{J}_{pol}$ , the impressed current  $[\mathbf{J}]$ , and how to compute them in principle.

We will argue by comparison with a forthcoming acoustic implementation of a similar problematic, that we can have no answers to the above questions, in absence of thermodynamic considerations and laws for the motion of, say, the “fluid” of electronic charges in the material lattice of nuclei, considerations and laws which are however so far absent in the present state of macroscopic electromagnetics and thermodynamics.

Here we say that, if we disposed of the suitable thermodynamic knowledge, a solution to the indetermination in (6.83) would be to require that the Heaviside-Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \quad (6.89)$$

possesses the thermodynamic meaning of the *current density of energy transported in electromagnetic form*. We will denote this, the “Heaviside-Poynting”, thermodynamic identification. “In electromagnetic form” means the part of the energy current density, origin of the subsequent propagation and attenuation of the electromagnetic disturbance, and not the part, “lost” for the propagation, which is right-on converted in thermal degraded form.

If we knew what this current  $\mathbf{S}$  is, in the sense that we would have a thermodynamic framework (so far missing) to express it, we could, by working within this framework, define  $\mathbf{H}$  by applying (6.89), then  $\mathbf{M}$  by applying (6.86.2). Assuming that the missing knowledge would also determine what the “impressed part”  $[\mathbf{J}]$  is, we could then use (6.82)–(6.83) to define  $\mathbf{P}$ , and finally, by (6.86.1), define  $\mathbf{D}$ . These specifications being performed whatever the chosen source term,  $[\mathbf{J}]$ , all nonlocal operators would then be uniquely defined.

Basically, this is what we will try to do further on, in acoustics, with substituted quantities and corresponding changes. But the concrete exploitation of these ideas in electromagnetics is not feasible now, as long as a whole slice of thermodynamics, concerned with the response of electrons in matter, is missing. For the motion of a viscothermal fluid permeating a solid structure (see next chapter), or not (no solid structure, see this chapter), we dispose of the thermodynamic Navier-Stokes-Fourier equations we have stated, and this will suffice to us to progress significantly in the direction of a satisfactory nonlocal description.<sup>32</sup> But for the response of an “electronic fluid” (partially bounded and free) in the material nuclei lattice, we have so far nothing comparable. It is therefore not surprising that in the electromagnetic literature, the problem is not defined in the above general terms, which remain without content for the moment. Shortcuts and simplifications, which we now try to summarize, are used instead.

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<sup>32</sup> Note that in expressing ourselves in this manner we assume that the properties of the fluid will be unaffected by its proximity to the walls of the solid; for the “electronic fluid”, on the contrary, one can expect such complications to occur, and the development of an appropriate description seems to be a distant prospect.

### 6.11.4 Ambiguities and the Customary Way

A priori, there is in the polarization current (6.83),  $\mathbf{J}_{pol} = \mathbf{J}_{bound} + \mathbf{J}_{cond}$ , two different parts: a part  $\mathbf{J}_{bound}$ , coming from the motion of bound electrons, and a part  $\mathbf{J}_{cond}$ , coming from the motion of free electrons. The first  $\mathbf{J}_{bound}$ , includes a first contribution related to the global displacement of bound charges, to be incorporated in the term  $\partial\mathbf{P}/\partial t$ , and a second contribution not related to this but to the modifications of underlying magnetic circulation currents (e.g. in the molecules or other greater structures), to be incorporated in the  $\partial \times \mathbf{M}$  term in (6.83). The remaining,  $\mathbf{J}_{cond}$ , will be an induced conduction current, coming from the motion of electrons not bounded to one individual molecule, and that will complete the above first contribution to produce the term  $\partial\mathbf{P}/\partial t$  in (6.83), but also, for some structured materials, the  $\partial \times \mathbf{M}$  term itself. Thus writing (6.82), we have

$$\langle \mathbf{j} \rangle = \mathbf{J}_{bound} + \mathbf{J}_{cond} + [\mathbf{J}]. \quad (6.90)$$

In the literature, often, by a tacite redefinition,  $\mathbf{J}_{cond} + [\mathbf{J}] = \mathbf{J}$ , of the current  $\mathbf{J}$  which appears in the macroscopic equations, one writes (6.90) as

$$\langle \mathbf{j} \rangle = \mathbf{J}_{bound} + \mathbf{J}. \quad (6.91)$$

Finally, in presence of spatial dispersion and as explained in Landau and Lifshitz [13] or Agranovich and Ginzburg [16] or Melrose and McPhedran [17], one decides to put all different effects coming from the bound charges, in an abstract new  $\mathbf{P}$  obeying the definition:

$$\mathbf{J}_{bound} = \frac{\partial \mathbf{P}}{\partial t}. \quad (6.92)$$

Then, macroscopic equations are obtained, which have the form:

$$\frac{\partial \mathbf{B}}{\partial t} = -\partial \times \mathbf{E}, \quad \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{\mu_0} \partial \times \mathbf{B} - \mathbf{J}, \quad \mathbf{D} = \hat{\epsilon} \mathbf{E}, \quad (6.93)$$

with  $\mathbf{J}$ , a current, that is either taken equal to  $\mathbf{J}_{cond}$  and obeying Ohm's law in absence of source,<sup>33</sup> or taken as an imposed term in presence of source,<sup>34</sup> and  $\hat{\epsilon}$  an operator establishing a relation having the same following *general nonlocal form* whatever the  $\mathbf{J}$  term is present or not:

$$\mathbf{D}(t, \mathbf{x}) = \int \epsilon(t - t', \mathbf{x}, \mathbf{x}') \mathbf{E}(t', \mathbf{x}') dt' d\mathbf{x}'. \quad (6.94)$$

<sup>33</sup> In the most general case this will be a nonlocal Ohm's law  $\mathbf{J}(t, \mathbf{x}) = \int dt' d\mathbf{x}' \sigma(t - t', \mathbf{x}, \mathbf{x}') \mathbf{E}(t', \mathbf{x}')$ .

<sup>34</sup> It also obeys the above Ohm's law, with  $\mathbf{E}$  containing two parts, respectively associated to the forced and free motion.

In this conception, the magnetization  $\mathbf{M}$  is, by force, defined to be zero, and the  $\mathbf{H}$  field, is, simply, viewed as:

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B}. \quad (6.95)$$

Contrary to the interpretation made of the term  $\mathbf{J}$  in the set of (6.71)–(6.73), here, in the above (6.93)–(6.95), because of the presence of  $\mathbf{J}_{cond}$  in it, the term  $\mathbf{J}$  is not truly an external source-term. It becomes so, however, if we reincorporate the induced conduction current  $\mathbf{J}_{cond}$  in the definition of  $\mathbf{D}$ , i.e. decide to write  $\mathbf{J}_{pol} = \partial \mathbf{P} / \partial t$ , instead of (6.83) or (6.92), now artificially putting all effects coming from bound and free charges, in the abstract  $\mathbf{P}$ . Anyway, whatever ones uses or not the redefinition  $\mathbf{J}_{cond} + [\mathbf{J}] = \mathbf{J}$ , certainly by setting (6.95), some artificiality is introduced in the definitions. We confirm this later in the acoustic case, where in this respect, similar problematics will arise.

### 6.11.5 Discussion of Our Proposal

Our idea is that there should exist a dynamical field  $\mathcal{P}$ , having thermodynamic status, and such that  $\mathbf{S} = \mathbf{E} \times \mathcal{P}$  has the interpretation of current density of energy transported in electromagnetic form. Then setting  $\mathbf{H} = \mathcal{P}$  would allow identifying the field  $\mathbf{H}$ , removing the ambiguity in the Lorentz decomposition, and fixing the meaning of all fields and operators.

To precise this, we observe that, as discussed in textbooks, the polarization  $\mathbf{P}$ , verifies,  $\partial \cdot \mathbf{P} = \rho_{pol}$ , where  $\partial \rho_{pol} / \partial t + \partial \cdot \mathbf{J}_{pol} = 0$ . If we were to add the equation,  $\partial \times \mathbf{P} = \mathbf{0}$ , no ambiguity would arise for  $\mathbf{P}$ , once given the  $\rho_{pol}$ . But we are guessing here, that, in general, what should be called the electric polarization will have,  $\partial \times \mathbf{P} \neq \mathbf{0}$ . Let us thus denote  $\mathbf{P}'$ , the field satisfying,  $\partial \cdot \mathbf{P}' = \rho_{pol}$ , and,  $\partial \times \mathbf{P}' = \mathbf{0}$ . The actual  $\mathbf{P}$ , will be,  $\mathbf{P} = \mathbf{P}' + \partial \times \Phi$ , (with some  $\Phi$ ), with the additional term, compensated by a corresponding additional one,  $-\partial \Phi / \partial t$ , in  $\mathbf{M}$ . We are guessing that the requirement,  $\mathbf{E} \times (\mathbf{B} / \mu_0 - \mathbf{M}) = \mathbf{E} \times \mathcal{P}$ , would then be capable to fix the appropriate term,  $-\partial \Phi / \partial t$ , resolving the ambiguity of the Lorentz splitting.

For later reference, we name this process of hypothetical identification of what should be the Maxwell field  $\mathbf{H}$ , based on (6.89), Heaviside-Poynting's thermodynamic identification of the field  $\mathbf{H}$ , or in short, Heaviside-Poynting's conception of the field  $\mathbf{H}$ .

We hope that these comments will help clarify the difference between how Maxwell's fields are generally defined in the electromagnetic literature, and how they should be better defined, following (6.89), but unfortunately, in a way that is not feasible at this time, due to the absence of irreversible thermodynamic descriptions of the movement of an "electronic-fluid", in part bound and free, in material media. We note that many recent theoretical studies and experimental observations are now pointing in this direction. Electrons flowing like liquid have been reported



in graphene, and notions resembling viscosity have appeared, to describe newly observed “fluidic” aspects of their collective behaviour.

In the next section we observe that a similar nonlocal pattern of relations applies to the case of the propagation of acoustic small-amplitude perturbations in our viscothermal homogeneous and unbounded fluid. In this case, however, we will not have to resort to stopgap measures of the type (6.95) expounded just above. Because in acoustics, we will have an existing irreversible thermodynamic framework – that of Navier-Stokes-Fourier equations – it will be possible to make definite use of a similarly imposed energetic condition of the type (6.89), allowing us to identify an “acoustic  $\mathbf{H}$ –field” and then to determine also the other acoustic quantities. For the reasons mentioned above (Sect. 6.6.3, and comments above (6.67) in Sect. 6.10), however, we will limit ourselves to considering the case of longitudinal movements only, (corresponding, without source, to the “acoustic” and “entropic” modes in the fluid).

Finally, we insist that the acoustic example will help us, perceive the fundamental physical insufficiencies of present-day definitions of electromagnetic Maxwell’s fields  $\mathbf{H} = \mathbf{B}/\mu_0$  and  $\mathbf{D}$ . It is to be hoped that in the future, a thermodynamics of the behaviour of electrons in material media will be developed, that will be capable to precise the notions of electric and magnetic polarizations  $\mathbf{P}$  and  $\mathbf{M}$ , and polarization- and impressed-currents  $\mathbf{J}_{pol}$  and  $[\mathbf{J}]$ , but this, again, seems to be a distant perspective.

## 6.12 Nonlocal Maxwellian Pattern of Longitudinal Acoustics

We now can come back to the “acoustic” longitudinal equations in a viscothermal fluid (6.67)–(6.70), and explain the formal pattern, that makes them comparable to the macroscopic electromagnetic equations discussed in the previous Sects. 6.11.1 and 6.11.2. Consider, on one hand, the two longitudinal acoustic equations (6.67)–(6.68), written with a source term  $\mathbf{f} = -\partial\mathcal{P}$ , density of external longitudinal body force:

$$\partial_t b + \partial \cdot \mathbf{v} = 0, \quad \rho_0 \partial_t \mathbf{v} = -\partial p - \left( \frac{4\eta}{3} + \zeta \right) \partial \partial_t b + [\mathbf{f} = -\partial\mathcal{P}], \quad (6.96)$$

and completed by the additional thermodynamic ones (6.69)–(6.70), that tell us what the variable  $p$  is. Consider, on the other hand, the two macroscopic electromagnetic equations (6.80.2) and (6.80.4) with (6.82) inserted:

$$\frac{\partial \mathbf{B}}{\partial t} + \partial \times \mathbf{E} = \mathbf{0}, \quad \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\mu_0} \partial \times \mathbf{B} - \mathbf{J}_{pol} - [\mathbf{J}], \quad (6.97)$$

and supposedly completed by additional ones, that should tell us, in the end, what the polarization current  $\mathbf{J}_{pol}$  is. The brackets [ ] recall that the source terms may not be present. We say that the two sets can quite be seen in the same way. To discuss this affinity we have to avoid entering in too much details: (6.96) are directly macroscopic<sup>35</sup> equations involving thermodynamic results, while (6.97) are also macroscopic equations, but here we completely lack the necessary thermodynamics to understand the currents. Indeed, already because it would need giving a prohibitively large amount of information to specify the system, we cannot start at the microscopic non-thermodynamic quantum level, give the configuration, state in full the equations and find and average their solution to define  $\mathbf{J}$  and  $\mathbf{J}_{pol}$ , (see also footnote 45 in Appendix); and we cannot, either, work directly at a thermodynamic level as it would be desirable and as is done in the acoustic case, because so far, we completely lack any irreversible-thermodynamic description of the motion of “electronic fluid” flowing in matter. Nevertheless, with this in mind, we can make appear as follows the strong affinity between the two wave-propagation problems.

### 6.12.1 Electromagnetic Analogy

First, to see that the philosophy of (6.96.1) can be made comparable to that of (6.97.1), we note that the macroscopic basic fields involved,  $\mathbf{v}$ ,  $b$  (acoustic) and  $\mathbf{E}$ ,  $\mathbf{B}$  (electromagnetic), are “derived” fields, obtained in comparable manner from an underlying macroscopic three-vector “potential” field, which we denote  $\mathbf{a}$  in acoustics, and  $\mathbf{A}$  in electromagnetics.<sup>36</sup> In acoustics, working in the reference-frame in which the fluid is globally at rest,  $\mathbf{a}(t, \mathbf{x})$  is the macroscopic displacement of fluid particles. In electromagnetics, working in the reference-frame in which the medium is globally at rest,  $\mathbf{A}(t, \mathbf{x})$  is the macroscopic mean potential in Weyl’s temporal gauge.<sup>37</sup>

The relations between the acoustic (displacement) and electromagnetic (potential) fields  $\mathbf{a}$  and  $\mathbf{A}$  and the derived acoustic and electromagnetic basic fields  $\mathbf{v}$ ,  $b$  and  $\mathbf{E}$ ,  $\mathbf{B}$  are as follows:

$$\begin{array}{ll} \text{Acoustics:} & \text{Electromagnetics:} \\ b \equiv -\partial \cdot \mathbf{a}, \quad \mathbf{v} \equiv \frac{\partial \mathbf{a}}{\partial t}, & \mathbf{B} \equiv \partial \times \mathbf{A}, \quad \mathbf{E} \equiv -\frac{\partial \mathbf{A}}{\partial t}. \end{array} \quad (6.98)$$

<sup>35</sup> The acoustic variables are from the beginning macroscopic variables, see Sect. 6.3.

<sup>36</sup> We have  $\mathbf{A} = \langle \mathbf{a} \rangle$  with  $\mathbf{a}$  the microscopic potential associated to the fields  $\mathbf{e}$ ,  $\mathbf{b}$ . The acoustic  $\mathbf{a}$ , in spite of his lowercase notation, is macroscopic and the correspondent of  $\mathbf{A}$ . There’s no point in looking for an acoustic match for the electromagnetic  $\mathbf{a}$ .

<sup>37</sup> The electromagnetic potential is a relativistic object  $A_\mu = \langle a_\mu \rangle$  with four temporal and spatial covariant indices  $\mu = 0, 1, 2, 3$ ; general gauge invariance of electromagnetic equations describing fields in interaction with charges, expresses in the fact that the covariant vector  $A_\mu$  is determined only up to a four-gradient  $\varphi_{;\mu}$ ; then working in the fixed reference-frame where the medium is at rest, it is possible to choose the four-gradient in such a way that the temporal component  $A_0$  vanishes. This is Weyl’s temporal gauge, in which  $A_\mu = (A_0 \equiv 0, \mathbf{A})$ . The electromagnetic fields  $\mathbf{E}$  and  $\mathbf{B}$ , then derive in this gauge from a three-vector  $\mathbf{A}$  with the written relations (6.98.3), (6.98.4).

The relations (6.96.1) and (6.97.1) are then seen to be, both, the tautological consequences of the way the fields  $b$ ,  $\mathbf{v}$ , or  $\mathbf{B}$ ,  $\mathbf{E}$ , are derived from  $\mathbf{a}$  or  $\mathbf{A}$  in (6.98).

The properties of the medium are not expressed in these first field equations (6.96.1) and (6.97.1); they come from the other relations, which are to be written as second field equations, constitutive relations, and other energetic (Heaviside-Poynting's) condition. If we want to see the acoustic relation (6.96.2) as a counterpart to the electromagnetic relation (6.97.2), we have an interest in putting it in the form of

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\chi_0} \partial b + \mathbf{f}_{pol} + [\mathbf{f}]. \quad (6.99)$$

The term  $-\partial b/\chi_0$  compares to the term  $\partial \times \mathbf{B}/\mu_0$ . It comes from the diagonal adiabatic part  $-p_a \delta_{ij}$  in the stresses (6.32), that is, from a medium response part making abstraction of losses, and the rest, denoted  $\mathbf{f}_{pol}$  here to comply with electromagnetic notations in (6.97.2), describes an “induced” viscous and thermal “polarization” force density which is the expression of the previously mentioned, unavoidable and accompanying, irreversible exchanges of momentum and energy. Here, as the motion is assumed longitudinal, this induced “polarization” force density is simply given by, (compare (6.99) and (6.68))

$$\mathbf{f}_{pol} = \frac{1}{\chi_0} \partial b - \partial p - \left( \frac{4\eta}{3} + \zeta \right) \partial \frac{\partial b}{\partial t}. \quad (6.100)$$

The interpretation of the acoustic (6.99) and electromagnetic (6.97.2) equations is now made along similar physical lines of reasoning. In absence of “polarization” reactions of the media, the set of equations would be,  $\partial b/\partial t = -\partial \cdot \mathbf{v}$ ,  $\rho_0 \partial \mathbf{v}/\partial t = -\chi_0^{-1} \partial b + [\mathbf{f}]$ , in acoustics, and  $\partial \mathbf{B}/\partial t = -\partial \times \mathbf{E}$ ,  $\epsilon_0 \partial \mathbf{E}/\partial t = \mu_0^{-1} \partial \times \mathbf{B} - [\mathbf{J}]$ , in macroscopic electromagnetics, leading to nondispersive wave propagation with soundspeed  $c_0$  and lightspeed  $c$ , such that,  $\rho_0 \chi_0 c_0^2 = 1$ ,  $\epsilon_0 \mu_0 c^2 = 1$ . But as soon as the wavefields  $b$ ,  $\mathbf{v}$ , or  $\mathbf{B}$ ,  $\mathbf{E}$ , are present and variable in time and space, progressive “polarization” processes inevitably arise in response, resulting in an induced polarization force  $\mathbf{f}_{pol}$ , or an induced polarization current  $\mathbf{J}_{pol}$ , affecting in turn the variations of the fields and causing the wave dispersion.

This analogy then suggests introducing, for the polarization force  $\mathbf{f}_{pol}$ , a general representation in terms of a time-derived and space-derived term:

$$\mathbf{f}_{pol} = -\frac{\partial \mathbf{p}}{\partial t} + \partial m. \quad (6.101)$$

This will be the acoustic counterpart of the Lorentz splitting (6.83) of the macroscopic polarization current, (the “acoustic polarization” fields  $m$  and  $\mathbf{p}$  are written in lowercase here but they have macroscopic nature). As in electromagnetics, there is an ambiguity in writing (6.101). Given some  $\mathbf{p}$  and  $m$  satisfying (6.101), new  $\mathbf{p}$  and  $m$  given by  $\mathbf{p}' = \mathbf{p} + \partial \varphi$  and  $m' = m + \partial \varphi/\partial t$  also satisfy (6.101), where  $\varphi$  is a scalar field. Nevertheless, if some criterium can be found to fix the scalar unknown

$\varphi$ , so that unique polarizations  $\mathbf{p}$  and  $m$  exist, we have the following consequences. Substituting (6.101) in (6.99) we arrive at

$$\frac{\partial \mathbf{d}}{\partial t} = \partial h + [\mathbf{f}], \quad (6.102)$$

with

$$\mathbf{d} = \rho_0 \mathbf{v} + \mathbf{p}, \quad h = -\frac{1}{\chi_0} b + m. \quad (6.103)$$

Finally, in the same way as it has been suggested previously that macroscopic electromagnetic equations should ultimately be written in the form of (6.71)–(6.73), with the Maxwell field  $\mathbf{H}$  to be fixed according to the Heaviside-Poynting condition that (6.89) should represent “the electromagnetic part” of the energy current density, here, it is suggested that the acoustic equations for longitudinal motions in the fluid, should ultimately be written in the following general form<sup>38</sup>

$$\frac{\partial b}{\partial t} + \partial \cdot \mathbf{v} = 0, \quad \frac{\partial \mathbf{d}}{\partial t} = \partial h + [\mathbf{f} = -\partial \mathcal{P}], \quad (6.104)$$

$$\mathbf{d}(t, \mathbf{x}) = \hat{\rho} \mathbf{v}(t, \mathbf{x}) = \int \rho(t - t', \mathbf{x} - \mathbf{x}') \mathbf{v}(t', \mathbf{x}') dt' d\mathbf{x}', \quad (6.105)$$

$$\mathbf{h}(t, \mathbf{x}) = -\hat{\chi}^{-1} \mathbf{b}(t, \mathbf{x}) = -\int \chi^{-1}(t - t', \mathbf{x} - \mathbf{x}') \mathbf{b}(t', \mathbf{x}') dt' d\mathbf{x}', \quad (6.106)$$

with the “Maxwell field  $h$ ” to be fixed according to the condition that

$$s = -vh \quad (6.107)$$

should represent “the acoustic part” of the energy current density, or else, the *current density of energy transported in acoustic form*. Like before in the electromagnetic case, we mean by it the part of the energy current density, origin of the subsequent propagation and attenuation of a disturbance, and not the part, “lost” for the propagation, which is right-on converted in thermal degraded form. We will denote this, the “acoustic Heaviside-Poynting” thermodynamic identification of the  $h$ -field. With it, the  $h$ -field will represent an acoustic stress field, and the  $\mathbf{d}$ -field an acoustic momentum field, different from the ideal fluid expressions,  $h = -\chi_0^{-1} b$ , and  $\mathbf{d} = \rho_0 \mathbf{v}$ , because of the “acoustic polarization” processes.

Although the hypothesized Heaviside-Poynting thermodynamic identification (6.89) could not be performed, because of its current disconnection from available thermodynamics, its acoustic version (6.107) will prove to be feasible, and to remove the ambiguity inherent in Lorentz’s acoustic splitting (6.101). Nevertheless, in the same way as, in electromagnetics in presence of spatial dispersion it is customary to set,  $\mathbf{M} \equiv \mathbf{0}$ , or, (6.95), by definition, here in acoustics it could be considered that,

<sup>38</sup> In (6.105), the kernels are written to depend on the difference  $\mathbf{x} - \mathbf{x}'$ , because we work with Cartesian coordinates and the fluid, unbounded, defines an homogeneous medium.

$m \equiv 0$ , or,  $h \equiv -\chi_0^{-1}b$ , by definition. In what follows, after considering making the acoustic Heaviside-Poynting's thermodynamic identification (6.107), we will also have a look, for edification, at the different acoustic description resulting from this last definition, counterpart of the electromagnetic customary one.

### 6.12.2 Acoustic Maxwell Fields

We now consider specifying the Maxwell acoustic field  $h$ , the acoustic polarizations  $m$  and  $p$ , the other Maxwell field  $d$ , and thus the acoustic nonlocal operators  $\hat{\rho}$  and  $\hat{\chi}^{-1}$ , by effecting the acoustic Heaviside-Poynting fundamental thermodynamic identification (6.107).

First, we write down an Energy Conservation-Dissipation Corollary [18], which applies to the Navier-Stokes-Fourier linearized equations:

$$\frac{\partial w}{\partial t} + \partial \cdot \mathbf{I} = -\mathcal{D}, \quad (6.108)$$

where

$$w = \frac{1}{2}\rho_0 v^2 + \frac{1}{2}\frac{p^2}{\rho_0 c_0^2} + \frac{1}{2}\frac{\rho_0 T_0}{c_P} s^2, \quad (6.109)$$

$$I_i = p v_i - \sigma'_{vij} v_j - \frac{\kappa}{T_0} \tau \partial_i \tau, \quad (6.110)$$

$$\mathcal{D} = \sigma'_{vij} \partial_j v_i + \frac{\kappa}{T_0} (\partial \tau)^2. \quad (6.111)$$

Here  $w$  represents the density of disturbance energy, that is, the first term in (6.108), integrated in a control volume, represents the time rate of change of disturbance energy, whether this energy is accumulated in acoustic organised form, or other, degraded from. Vector  $\mathbf{I}$  represents the energy-flux vector, whether the energy is transported in usable acoustic form or other degraded form. Its surface integral on the boundary of the control volume represents the rate at which disturbance energy is flowing out, in acoustic or other form. Finally,  $\mathcal{D}$  represents the density of energy dissipated per unit volume and time, or in other terms, its integral in the control volume is the energy “unaccountably” lost in this volume and transformed in heat energy per unit of time. We see that  $\mathcal{D} = T_0 \sigma_s$ , where  $\sigma_s$  is the density of entropy production (6.39), in accordance with the general thermodynamic relation,  $dQ = T dS$ .

In covariant notations, (see Appendix),  $\mathbf{I}$  is a contravariant vector  $I^i$ . From its Cartesian-coordinates expression (6.110) we see that  $I^i$  doesn't write in the “Heaviside-Poynting” form  $-H_j^i(t, x)v^j(t, x)$ . Indeed, the last thermal term in  $I^i$  will be proportional to  $-\tau(t, x)\tau^{;i}(t, x)$ , and this cannot be put in the form of a

tensor at  $(t, x)$  contracted with the velocity at  $(t, x)$ . It means that we cannot count the thermal term in (6.110) in what we call the energy-flux vector “in acoustic form”  $\mathbf{s}$ : this term will correspond to an energy transported in a “degraded form”. As regards the viscous term in  $\mathbf{I}$ , it has the right form of a tensor at  $(t, x)$  contracted with the velocity at  $(t, x)$ , however we cannot count it in  $\mathbf{s}$  because it again corresponds to energy transported in degraded form. Indeed, we observe that these two thermal and viscous contributions to the energy flux  $\mathbf{I}$ , are transformed in similar manner,  $(-\frac{\kappa}{T_0}\tau\partial_i\tau \rightarrow \frac{\kappa}{T_0}\partial_i\tau\partial_i\tau, -\sigma'_{vij}v_j \rightarrow \sigma'_{vij}\partial_iv_j)$ , in the two thermal conduction and viscosity contributions to the dissipated energy density  $\mathcal{D}$ . If the thermal term  $-\frac{\kappa}{T_0}\tau\partial_i\tau$  carries only degraded energy, then it is in order also to assume that the viscous term  $-\sigma'_{vij}v_j$  corresponds to the transport of degraded energy only. The remaining term  $p\mathbf{v}$ , to which corresponds no contribution in  $\mathcal{D}$ , must be that associated to the energy transported in usable, acoustic form. Thus, finally, we set

$$\mathbf{s} = p\mathbf{v}. \quad (6.112)$$

Given the two expressions (6.107) and (6.112) for the same  $\mathbf{s}$ , it follows that we must identify the  $h$ -field with the (opposite) thermodynamic excess pressure:

$$h = -p. \quad (6.113)$$

This, turns out to be the translation of acoustic Heaviside-Poynting’s identification of the  $h$ -field. It completely determines all quantities: it sets the  $m$ -field through using (6.103.2) and the operator  $\hat{\chi}^{-1}$  through (6.106); after  $m$  is determined, it sets the  $\mathbf{p}$ -field through using (6.100), (6.101), then the  $\mathbf{d}$ -field through using (6.103.1), and then  $\hat{\rho}$  through using (6.105). Let us now proceed explicitly with these determinations. Putting the acoustic Heaviside-Poynting identification (6.113) in (6.103.2) yields the following expression of the “polarization  $m$ ”

$$m = -p + \frac{1}{\chi_0}b. \quad (6.114)$$

Eliminating  $b$  from using the state equation (6.69) we get

$$m = -\frac{\beta_0}{\chi_0} \left[ \tau - \frac{(\gamma - 1)\chi_0}{\beta_0} p \right]. \quad (6.115)$$

Inserting the thermodynamic identity (6.5), taking the time derivative, and applying the (6.70), this is rewritten  $\partial m / \partial t = -[(\gamma - 1) / \beta_0 T_0] \kappa \partial^2 \tau$ , or

$$m = -\frac{\gamma - 1}{\beta_0 T_0} \kappa \int_{-\infty}^t dt' \partial^2 \tau. \quad (6.116)$$

This equation joined to (6.114) allows us to see “polarization  $m$ ” as a thermal conduction part of the  $h$ -field or opposite pressure, which builds up due to the thermal

conduction processes created in spatially variable (divergent) temperature gradients. The diagonal thermal part,  $\sigma'_{rij}$ , in the additional lossy stress tensor (6.33), is now interpreted as being,  $m\delta_{ij}$ , and the  $h$ -field as being

$$h = -\frac{1}{\chi_0}b - \frac{\gamma - 1}{\beta_0 T_0}\kappa \int_{-\infty}^t dt' \partial^2 \tau. \quad (6.117)$$

We find back in this way (6.40), seen to express the fact that “acoustic” pressure  $p$  is not exactly in adiabatic relation with condensation, by the appearance of “polarization” processes related to thermal conduction. Putting the expression (6.114) in (6.100), (6.101) we next obtain that

$$\mathbf{p} = \left(\frac{4\eta}{3} + \zeta\right) \partial b = -\left(\frac{4\eta}{3} + \zeta\right) \partial(\partial \cdot \mathbf{a}), \quad (6.118)$$

or

$$\mathbf{p} = -\left(\frac{4\eta}{3} + \zeta\right) \int_{-\infty}^t dt' \partial(\partial \cdot \mathbf{v}). \quad (6.119)$$

This last equation compares with (6.116) and allows us to see “polarization  $\mathbf{p}$ ” as an additional viscous momentum, which builds up in response to the spatial variations of the velocity divergence. The sign before the  $\partial \mathbf{p} / \partial t$  term in (6.101) was chosen so that (6.119), for  $\mathbf{p}$ , has the same sign convention as (6.116), for  $m$ . Equation (6.119) also expresses as,  $-\partial p_i / \partial t = \partial_j \sigma'_{vij}$ , where  $\sigma'_{vij}$  is the viscous part in the stress tensor (6.32).<sup>39</sup> And  $\mathbf{d}$ , which finally is identified as

$$\mathbf{d} = \rho_0 \mathbf{v} - \left(\frac{4\eta}{3} + \zeta\right) \int_{-\infty}^t dt' \partial(\partial \cdot \mathbf{v}), \quad (6.120)$$

appears as an “acoustic momentum” that is augmented by the viscous polarization part  $\mathbf{p}$ . It represents only the part of the momentum that is transferred in reversible manner. Thus we see that, interestingly, *Heaviside-Poynting's thermodynamic identification of the  $h$ -field, leads to a natural separation of viscous effects and thermal conduction effects, the former being expressed in “viscous polarization  $\mathbf{p}$ ” (6.119), and the latter in “thermal polarization  $m$ ” (6.116).*

The laws and identifications obtained above have a somewhat unsatisfactory physical character, however, when viewed from the general nonlocal point of view. For example, looking at (6.116) or (6.119), it is obviously not reasonable in nonlocal physics, to treat on an equal footing the values of  $\partial^2 \tau$  or  $\partial(\partial \cdot \mathbf{v})$  at different instants. As we see in next subsection, these identifications do not lead to an existing density

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<sup>39</sup> For the case of longitudinal motions  $\mathbf{v} = \partial \phi$  considered here, we have  $\partial_i \partial_j v_j = \partial^2 v_i = \partial_i (\partial \cdot \mathbf{v})$ , so that  $\partial_j \sigma'_{vij}$  coincides with  $\left(\frac{4\eta}{3} + \zeta\right) \partial_i (\partial \cdot \mathbf{v})$ .

kernel in original space<sup>40</sup>; but in the Fourier space, they do correspond to definite kernels, whose expressions we will give. In Appendix we have given the more complete and general pattern of macroscopic quantities and equations, expected to hold true in presence of shear motions and when going far beyond the near-equilibrium simplifications made here. But corresponding quantities will await in the best case scenario an attempt to construct considerable advances in nonequilibrium thermodynamics.

### 6.12.3 Acoustic Nonlocal Operators

The acoustic kernels can be written in a Fourier decomposition as:

$$\begin{aligned}\rho(t-t', \mathbf{x}-\mathbf{x}') &= \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^3} \rho(\omega, \mathbf{k}) e^{-i\omega(t-t') + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}, \\ \chi^{-1}(t-t', \mathbf{x}-\mathbf{x}') &= \int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^3} \chi^{-1}(\omega, \mathbf{k}) e^{-i\omega(t-t') + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}. \end{aligned} \quad (6.121)$$

In what follows we show how to compute the Fourier kernels coefficients  $\rho(\omega, \mathbf{k})$  and  $\chi^{-1}(\omega, \mathbf{k})$ , even if the original space density kernel function  $\rho(t, \mathbf{x})$  turns out to be not well-defined mathematically by the Stokes law which interestingly, is not the case for the inverse operator  $\chi^{-1}$ . In accordance with the electromagnetic analogy, the obtained expressions will be the same with and without source.

Let us first consider a calculation without source. Passing over to the Fourier transform of the equations describing longitudinal motion (6.67)–(6.70), without source ( $\mathbf{f} = \mathbf{0}$ ), we get, on using the acoustic Heaviside-Poynting's identification (6.113)

$$-i\omega b(\omega, \mathbf{k}) = -i\mathbf{k} \cdot \mathbf{v}(\omega, \mathbf{k}), \quad (6.122)$$

$$-\rho_0 i\omega \mathbf{v}(\omega, \mathbf{k}) = i\mathbf{k}h(\omega, \mathbf{k}) - (4\eta/3 + \zeta)\omega \mathbf{k}b(\omega, \mathbf{k}), \quad (6.123)$$

$$-\gamma\chi_0 h(\omega, \mathbf{k}) = b(\omega, \mathbf{k}) + \beta_0 \tau(\omega, \mathbf{k}), \quad (6.124)$$

$$-\rho_0 c_P i\omega \tau(\omega, \mathbf{k}) = \beta_0 T_0 i\omega h(\omega, \mathbf{k}) - \kappa \mathbf{k}^2 \tau(\omega, \mathbf{k}). \quad (6.125)$$

As shown by (6.123), the velocity is collinear to  $\mathbf{k}$ , and we may write  $\mathbf{v}(\omega, \mathbf{k}) = v(\omega, \mathbf{k})\hat{\mathbf{n}}$ ,  $\mathbf{k} = k\hat{\mathbf{n}}$ . Equation (6.122) then is rewritten,  $-i\omega b(\omega, \mathbf{k}) = -ikv(\omega, \mathbf{k})$ , and, inserting this in (6.123), we get

$$\left[ -\rho_0 i\omega + \left( \frac{4\eta}{3} + \zeta \right) k^2 \right] v(\omega, \mathbf{k}) = i\mathbf{k}h(\omega, \mathbf{k}).$$

<sup>40</sup> A non-convergent integral is obtained. We do not see the mathematical difficulties encountered as an indication that our nonlocal recasting of equations is not physical, but as an expression of the fact that the thermodynamic framework used is unsatisfactory, (see Sect. 6.6.3), and should be extended.

$\rho^{-1}$



This compares directly to the equation,  $-i\omega\rho(\omega, \mathbf{k})\mathbf{v}(\omega, \mathbf{k}) = i\mathbf{k}h(\omega, \mathbf{k})$ , obtained by combining,  $-i\omega\mathbf{d}(\omega, \mathbf{k}) = i\mathbf{k}h(\omega, \mathbf{k})$ , with the function  $\mathbf{d}(\omega, \mathbf{k}) = \rho(\omega, \mathbf{k})\mathbf{v}(\omega, \mathbf{k})$ , which are respectively, the Fourier transform of (6.104.2) without source, and that of (6.105.1). The comparison yields the following expression of the density Fourier kernel:

$$\rho(\omega, \mathbf{k}) = \rho_0 + \left(\frac{4\eta}{3} + \zeta\right) \frac{k^2}{-i\omega}. \quad (6.126)$$

Next eliminating the excess temperature between (6.124) and (6.125), we find

$$\begin{aligned} -\rho_0 c_p i\omega \left( -\frac{\gamma\chi_0}{\beta_0} h(\omega, \mathbf{k}) - \frac{1}{\beta_0} b(\omega, \mathbf{k}) \right) = \dots \\ + \beta_0 T_0 i\omega h(\omega, \mathbf{k}) - \kappa k^2 \left( -\frac{\gamma\chi_0}{\beta_0} h(\omega, \mathbf{k}) - \frac{1}{\beta_0} b(\omega, \mathbf{k}) \right), \end{aligned}$$

and after straightforward calculation using the thermodynamic identity (6.5) and definition (6.18.3) we get

$$h(\omega, \mathbf{k}) = -\chi_0^{-1} \left[ 1 - \frac{\gamma-1}{\gamma} \left( 1 + \frac{i\omega}{-i\omega + \frac{\kappa}{\rho_0 c_V} k^2} \right) \right] b(\omega, \mathbf{k}).$$

This compares directly to the Fourier transform of (6.106), which is given by  $h(\omega, \mathbf{k}) = -\chi^{-1}(\omega, \mathbf{k})b(\omega, \mathbf{k})$ , showing that, for the bulk-modulus Fourier kernel:

$$\chi^{-1}(\omega, \mathbf{k}) = \chi_0^{-1} \left[ 1 - \frac{\gamma-1}{\gamma} \frac{\frac{\kappa}{\rho_0 c_V} k^2}{-i\omega + \frac{\kappa}{\rho_0 c_V} k^2} \right]. \quad (6.127)$$

From (6.119) and (6.114), the operators in the relations

$$\mathbf{p}(t, \mathbf{x}) = \int \chi_v(t-t', \mathbf{x}-\mathbf{x}') \mathbf{v}(t', \mathbf{x}') dt' d\mathbf{x}', \quad (6.128)$$

$$m(t, \mathbf{x}) = \int \chi_b(t-t', \mathbf{x}-\mathbf{x}') b(t', \mathbf{x}') dt' d\mathbf{x}', \quad (6.129)$$

inspired by (6.84) and (6.85), are found to be such that:

$$\chi_v(\omega, \mathbf{k}) = \left(\frac{4\eta}{3} + \zeta\right) \frac{k^2}{-i\omega}, \quad \chi_b(\omega, \mathbf{k}) = \chi_0^{-1} \frac{\gamma-1}{\gamma} \frac{\frac{\kappa}{\rho_0 c_V} k^2}{-i\omega + \frac{\kappa}{\rho_0 c_V} k^2}. \quad (6.130)$$

These expressions are simple and illustrate the mentioned separation of viscous and thermal “polarization” effects, automatically performed by using the acoustic Heaviside-Poynting identification (6.113).

The consistency of the obtained expressions (6.126) and (6.127) of the kernels  $\rho(\omega, \mathbf{k})$  and  $\chi^{-1}(\omega, \mathbf{k})$ , can be checked as follows. Passing over to the Fourier transform of (6.104)–(6.106) without source, we have

$$-i\omega b = -ikv, \quad -i\omega d = ikh, \quad (6.131)$$

$$d = \rho(\omega, \mathbf{k})v, \quad h = -\chi^{-1}(\omega, \mathbf{k})b, \quad (6.132)$$

from which we derive the dispersion relation

$$\rho(\omega, \mathbf{k})\chi(\omega, \mathbf{k})\omega^2 = k^2. \quad (6.133)$$

Putting in (6.133) the expressions (6.126) and (6.127), we can check that *the resulting dispersion relation is nothing but Kirchhoff-Langevin’s characteristic equation (6.52), rewritten in a nice factorized form.*

In the above Fourier-transform calculations no source term was introduced. Now, if our electromagnetic analogy is fruitful, we should find that the same expressions (6.126) and (6.127) of the density and bulk-modulus Fourier kernels, are obtained in presence of an arbitrary external density of bulk longitudinal force. Basically, what we have to check is that, in presence of an external density of bulk longitudinal force  $\mathbf{f}$  varying like  $e^{-i\omega t + ik \cdot \mathbf{x}}$ , with  $\omega$  and the components of  $\mathbf{k}$  all independently chosen with values taken in the complex plane, the response of the fluid is still described in terms of the functions  $\rho(\omega, \mathbf{k})$  and  $\chi^{-1}(\omega, \mathbf{k})$  given by (6.126) and (6.127). Since  $\omega$  and the components of  $\mathbf{k}$  are now arbitrarily chosen in the complex plane, these functions actually are the analytic continuation of the preceding.

To check this, let us apply the external longitudinal body force  $\mathbf{f} = -\partial\mathcal{P}$ ,  $\mathcal{P} = \tilde{\mathcal{P}}e^{-i\omega t + ik \cdot \mathbf{x}}$ , with  $\omega$  and  $\mathbf{k}$ , independent and possibly complex ( $\omega$  the upper complex plane to ensure physical excitation). With this source inserted in the equations (6.67)–(6.70), the response fields have the form  $f(t, \mathbf{x}) = f(\omega, \mathbf{k})e^{-i\omega t + ik \cdot \mathbf{x}}$ , and we find

$$-i\omega b(\omega, \mathbf{k}) = -i\mathbf{k} \cdot \mathbf{v}(\omega, \mathbf{k}), \quad (6.134)$$

$$-\rho_0 i\omega \mathbf{v}(\omega, \mathbf{k}) = i\mathbf{k}h(\omega, \mathbf{k}) - (4\eta/3 + \zeta)\omega \mathbf{k}b(\omega, \mathbf{k}) - i\mathbf{k}\tilde{\mathcal{P}}, \quad (6.135)$$

$$-\gamma\chi_0 h(\omega, \mathbf{k}) = b(\omega, \mathbf{k}) + \beta_0\tau(\omega, \mathbf{k}), \quad (6.136)$$

$$-\rho_0 c_p i\omega \tau(\omega, \mathbf{k}) = \beta_0 T_0 i\omega h(\omega, \mathbf{k}) - \kappa k^2 \tau(\omega, \mathbf{k}), \quad (6.137)$$

which leads, after straightforward calculations, to the equations:

$$-i\omega \left[ \rho_0 + \left( \frac{4\eta}{3} + \zeta \right) \frac{k^2}{-i\omega} \right] \mathbf{v}(\omega, \mathbf{k}) = i\mathbf{k}h(\omega, \mathbf{k}) - i\mathbf{k}\tilde{\mathcal{P}}, \quad (6.138)$$

and

$$h(\omega, \mathbf{k}) = -\chi_0^{-1} \left[ 1 - \frac{\gamma - 1}{\gamma} \frac{\frac{\kappa}{\rho_0 c_V} k^2}{-i\omega + \frac{\kappa}{\rho_0 c_V} k^2} \right] b(\omega, \mathbf{k}). \quad (6.139)$$

At the same time, setting in (6.104), (6.105),  $\mathbf{f} = -\partial\{\mathcal{P} = \tilde{\mathcal{P}}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\}$ , and taking fields having the form,  $a(t, \mathbf{x}) = a(\omega, \mathbf{k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ , we find:

$$-i\omega b(\omega, \mathbf{k}) = -i\mathbf{k} \cdot \mathbf{v}(\omega, \mathbf{k}), \quad -i\omega \mathbf{d}(\omega, \mathbf{k}) = i\mathbf{k}h(\omega, \mathbf{k}) - i\mathbf{k}\tilde{\mathcal{P}}, \quad (6.140)$$

$$\mathbf{d}(\omega, \mathbf{k}) = \rho(\omega, \mathbf{k})\mathbf{v}(\omega, \mathbf{k}), \quad h(\omega, \mathbf{k}) = -\chi^{-1}(\omega, \mathbf{k})b(\omega, \mathbf{k}), \quad (6.141)$$

where we have set, (in fact, because of convergence problem to define  $\rho(t, \mathbf{x})$  this calculation for the density should be done with the inverse operator and kernels)

$$\begin{aligned} \rho(\omega, \mathbf{k}) &= \int \rho(t, \mathbf{x})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} dt d\mathbf{x}, \\ \chi^{-1}(\omega, \mathbf{k}) &= \int \chi^{-1}(t, \mathbf{x})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} dt d\mathbf{x}. \end{aligned} \quad (6.142)$$

By comparison of (6.140.2, 6.141.1) with (6.138), and (6.141.2) with (6.139), we see that, as intended, the expressions of the kernels (6.126) and (6.127) obtained without source still apply in the same way, in the presence of source. This justifies the analogue way of reasoning which made us directly postulate the nonlocal pattern (6.104)–(6.106) of longitudinal acoustic equations, from the nonlocal pattern of macroscopic electromagnetic equations.

To conclude, it is interesting to complete our discussion in relation to the physical-mathematical deficiencies of our thermodynamic framework that have been left in the background. We can first explicitly show that the result (6.130.2) determines a definite, but physically unsatisfactory, original-space kernel function  $\chi_b(t, \mathbf{x})$ . For the purpose of simplifying the discussion, we will make the calculation for the 1D case where the propagation is along a single direction  $x$ . In that case the dependencies over  $\mathbf{k}$  and  $\mathbf{x}$  are replaced by dependencies over  $k$  and  $x$ , and in (6.121), the integral  $\int d\mathbf{k}/(2\pi)^3$  and variations  $e^{i\mathbf{k}\cdot\mathbf{x}}$  are replaced by  $\int dk/(2\pi)$  and  $e^{ikx}$ . In original-space the bulk modulus kernel is written as

$$\chi^{-1}(t, x) = \chi_0^{-1}\delta(t)\delta(x) - \chi_b(t, x), \quad (6.143)$$

where, according to (6.130.2)

$$\chi_b(t, x) = \chi_0^{-1}\alpha \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \frac{\beta k^2}{-i\omega + \beta k^2} e^{-i\omega t + ikx}, \quad (6.144)$$

with,  $\alpha = (\gamma - 1)/\gamma$ , and  $\beta = \kappa/(\rho_0 c_V)$ . Because of the exponentials, the double integral converges. Simple use of online integral calculator WolframAlpha yields the following result in closed form:

$$\chi_b(t, x) = \chi_0^{-1} \frac{\alpha}{8\sqrt{\pi}\beta^{3/2}t^{5/2}} (2\beta t - x^2) \exp\left[-\frac{x^2}{4\beta t}\right]. \quad (6.145)$$

It looks nice but is not completely physical: with finite propagation speeds, a truly physical kernel in original space would have to vanish exactly at sufficiently large values of  $x$ , (here,  $|x| > c_0 t$ ); the above kernel doesn't respect this condition in a way characteristic of diffusive, not propagative, processes. This reveals the unsatisfactory nature of our local-thermodynamic-equilibrium framework, just extended in the most basic manner, out of equilibrium, to include Stokes' and Fourier's constitutive laws. In the same framework, if we now try to compute in similar manner the density kernel,  $\rho(t, x) = \rho_0 \delta(t) \delta(x) + \chi_v(t, x)$ , where according to (6.130.1)

$$\chi_v(t, x) = \left(\frac{4\eta}{3} + \zeta\right) \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \frac{k^2}{-i\omega} e^{-i\omega t + ikx}, \quad (6.146)$$

we have an even more severe problem: the integral over  $k$  is not convergent. That over  $\omega$  is defined and convergent, but only in the sense of Cauchy's Principal Value. Therefore, while Fourier's law, through (6.130.2), leads to defined but partially unsatisfactory original-space kernels, Stokes' law, through (6.130.1), is unable to produce an existing original-space density kernel. (As mentioned before this problem does not occur for the inverse density operator). Direct comparison between (6.146) and (6.144), however, shows that the integral over  $k$  would become convergent, and that over  $\omega$  would become simply defined, if we extend the Stokes law in such a way to substitute, in place of the denominator  $-i\omega$ , the denominator  $-i\omega + \beta' k^2$ , with  $\beta'$  a new and supposedly small physical constant of same dimensions as  $\beta$ , i.e.  $\beta' = \epsilon \left(\frac{4\eta}{3} + \zeta\right) / \rho_0$ , with  $\epsilon \ll 1$  a small numerical constant. By inspection it can be seen that this substitution would correspond replacing, in the motion equation, the term  $\left(\frac{4\eta}{3} + \zeta\right) \partial_x^2 v$  leading to  $-\left(\frac{4\eta}{3} + \zeta\right) k^2 v$ , by a term  $-\left(\frac{4\eta}{3} + \zeta\right) k^2 v / \left(1 + \frac{\beta' k^2}{-i\omega}\right)$ , i.e. by an expansion

$$-\left(\frac{4\eta}{3} + \zeta\right) k^2 v \left(1 + \frac{-\beta' k^2}{-i\omega} + \left(\frac{-\beta' k^2}{-i\omega}\right)^2 + \left(\frac{-\beta' k^2}{-i\omega}\right)^3 + \dots\right).$$

That is, in Stokes law it would correspond replacing the term  $\left(\frac{4\eta}{3} + \zeta\right) \partial_x v$  by a more complex term with "more space and time memory"

$$\left(\frac{4\eta}{3} + \zeta\right) \left(\partial_x v + \beta' \partial_x^3 \int_{-\infty}^t dt' v(t', x) + \beta'^2 \partial_x^5 \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' v(t'', x) + \dots\right).$$

This modification of Stokes law would give a definite kernel in original space

$$\chi_v(t, x) = \frac{\rho_0}{\epsilon} \frac{1}{8\sqrt{\pi}\beta^{3/2}t^{5/2}} (2\beta't - x^2) \exp\left[-\frac{x^2}{4\beta't}\right],$$

but of course it is not intended to be an accurate modification. It only indicates the direction of the revisions to be made: extend the description by introducing more spatial and temporal memory effects in the constitutive laws.

#### 6.12.4 Summary: Action-Response Problem to Determine the Acoustic Kernels $\rho(\omega, \mathbf{k})$ and $\chi^{-1}(\omega, \mathbf{k})$

For later use and generalization in the next chapter, it is useful to reformulate here, what we have seen in the calculations with source in the preceding subsection. We have seen that, for arbitrary values of  $\omega$  and  $\mathbf{k}$  and when fixed by the Acoustic Heaviside-Poynting's condition (6.113), the effective nonlocal density  $\rho(\omega, \mathbf{k})$ , and the effective nonlocal bulk modulus  $\chi^{-1}(\omega, \mathbf{k})$ , can be determined by means of a simple action-response problem, in which work is performed on the system, in the form of external longitudinal bulk force  $\mathbf{f}$  acting on it:

*Action-response problem to determine  $\rho(\omega, \mathbf{k})$  and  $\chi^{-1}(\omega, \mathbf{k})$ :*

- (i) Let us subject the fluid to the action of a longitudinal body force  $\mathbf{f} = -\partial\mathcal{P}$  per unit fluid volume, deriving from a potential  $\mathcal{P}$  varying as:

$$\mathcal{P} = \tilde{\mathcal{P}}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}, \quad \tilde{\mathcal{P}} = Cst. \quad (6.147)$$

- (ii) Let us find the response of the fluid to this action. We seek fields in the form  $\mathbf{v} = v(\omega, \mathbf{k})\frac{\mathbf{k}}{k}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ ,  $b = b(\omega, \mathbf{k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ ,  $p = p(\omega, \mathbf{k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ ,  $\tau = \tau(\omega, \mathbf{k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ , satisfying the longitudinal motion equations

$$\partial_t b + \partial \cdot \mathbf{v} = 0, \quad (6.148)$$

$$\rho_0 \partial_t \mathbf{v} = -\partial p - (4\eta/3 + \zeta) \partial \partial_t b + \mathbf{f}, \quad (6.149)$$

$$\gamma \chi_0 p = b + \beta_0 \tau, \quad (6.150)$$

$$\rho_0 c_P \partial_t \tau = \beta_0 T_0 \partial_t p + \kappa \partial^2 \tau, \quad (6.151)$$

and whose amplitudes in  $(\omega, \mathbf{k})$  are proportional to  $\tilde{\mathcal{P}}$ .

- (iii) There is unique solution  $v(\omega, \mathbf{k})$ ,  $b(\omega, \mathbf{k})$ ,  $p(\omega, \mathbf{k})$  and  $\tau(\omega, \mathbf{k})$ , to this action-response problem, and we can easily write it by straightforward calculations. Then, the effective density  $\rho(\omega, \mathbf{k})$  of (6.126) is obtained through the definition

$$-i\omega \rho(\omega, \mathbf{k}) \mathbf{v}(\omega, \mathbf{k}) = -i\mathbf{k} \left( p(\omega, \mathbf{k}) + \tilde{\mathcal{P}} \right), \quad (6.152)$$

and the effective compressibility  $\chi(\omega, \mathbf{k})$  of (6.127) is obtained through the definition

$$p(\omega, \mathbf{k}) = \chi^{-1}(\omega, \mathbf{k})b(\omega, \mathbf{k}). \quad (6.153)$$

Note that we do not put  $\mathbf{f}$  into bracket in (6.149), as it is here necessarily present as the forcing term creating the observed response motion. The physical content of the above procedure is just equivalent to stating that the acoustic equations can be put in the form (6.104)–(6.105), which expresses the electromagnetic analogy, with in addition, the  $h$ –field taken according to the acoustic Heaviside-Poynting’s thermodynamic identification (6.113).

Finally, we note that the use we made here, in acoustics, of the Heaviside-Poynting thermodynamic identification, can be seen as an indirect confirmation of our analysis of macroscopic electromagnetic equations, even if we do not currently have the thermodynamics necessary to specify the corresponding notion of “energy current density carried out in electromagnetic form”. Nevertheless, to finish and conclude we now examine what would have been obtained if we had transposed here, the expedients traditionally used in electromagnetic literature, (Sect. 6.11.3).

### 6.12.5 Acoustics Translation of the Customary Viewpoint

Using the same approach as used in conventional nonlocal electromagnetics, we would have considered, that, by definition,  $m = 0$ :

$$\frac{\partial b}{\partial t} + \partial \cdot \mathbf{v} = 0, \quad \frac{\partial \mathbf{d}}{\partial t} = \partial h + [\mathbf{f}], \quad (6.154)$$

$$\mathbf{d}(t, \mathbf{x}) = \hat{\rho} \mathbf{v}(t, \mathbf{x}) = \int \rho(t - t', \mathbf{x} - \mathbf{x}') \mathbf{v}(t', \mathbf{x}') dt' d\mathbf{x}', \quad (6.155)$$

$$h(t, \mathbf{x}) = -\chi_0^{-1} b(t, \mathbf{x}). \quad (6.156)$$

The set of these formal equations (6.154)–(6.156), joined to the detailed longitudinal equations (6.67)–(6.70), allow determining the density operator in unique manner. Straightforward calculations show that the kernel coefficients

$$\rho(\omega, \mathbf{k}) = \int \rho(t, \mathbf{x}) e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}} dt d\mathbf{x},$$

are given by

$$\rho(\omega, \mathbf{k}) = \rho(\omega, \mathbf{k}) + \frac{k^2}{\omega^2} [\chi_0^{-1} - \chi^{-1}(\omega, \mathbf{k})], \quad (6.157)$$

where in the right, the quantities  $\rho(\omega, \mathbf{k})$  and  $\chi^{-1}(\omega, \mathbf{k})$  are those previously determined, given by (6.126) and (6.127). As before, the calculation can be done directly

by passing over to the Fourier transform, without source; it can also be done with source by means of the following action-response problem.

**Action-response problem to determine  $\rho(\omega, \mathbf{k})$ :**

- (i) Let us subject the fluid to the action of a longitudinal bulk force  $\mathbf{f} = -\partial\mathcal{P}$  per unit fluid volume, deriving from a potential  $\mathcal{P}$  varying as:

$$\mathcal{P} = \tilde{\mathcal{P}}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}, \quad \tilde{\mathcal{P}} = Cst. \quad (6.158)$$

- (ii) Let us find the response of the fluid to this action. We seek fields  $\mathbf{v} = v(\omega, \mathbf{k})\frac{\mathbf{k}}{k}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ ,  $b = b(\omega, \mathbf{k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ ,  $p = p(\omega, \mathbf{k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ ,  $\tau = \tau(\omega, \mathbf{k})e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ , satisfying the longitudinal motion equations

$$\partial_t b + \partial \cdot \mathbf{v} = 0, \quad (6.159)$$

$$\rho_0 \partial_t \mathbf{v} = -\partial p - (4\eta/3 + \zeta) \partial \partial_t b + \mathbf{f}, \quad (6.160)$$

$$\gamma \chi_0 p = b + \beta_0 \tau, \quad (6.161)$$

$$\rho_0 c_P \partial_t \tau = \beta_0 T_0 \partial_t p + \kappa \partial^2 \tau, \quad (6.162)$$

and whose amplitudes in  $(\omega, \mathbf{k})$  are proportional to  $\tilde{\mathcal{P}}$ .

- (iii) There is unique solution  $v(\omega, \mathbf{k})$ ,  $b(\omega, \mathbf{k})$ ,  $p(\omega, \mathbf{k})$  and  $\tau(\omega, \mathbf{k})$ , to this action-response problem, and we can easily write it by straightforward calculations. Then, the effective density  $\rho(\omega, \mathbf{k})$  in (6.154)–(6.156), is obtained through the definition

$$-i\omega \rho(\omega, \mathbf{k}) \mathbf{v}(\omega, \mathbf{k}) = -i\mathbf{k} \left( \frac{1}{\chi_0} b(\omega, \mathbf{k}) + \tilde{\mathcal{P}} \right). \quad (6.163)$$

The reader can easily check that this gives (6.157). The physical content of the above procedure is just equivalent to stating that the acoustic equations can be put in the form (6.104)–(6.106), which expresses the electromagnetic-acoustic analogy, with in addition, the  $h$ -field taken according to the definition

$$h = -b/\chi_0, \quad (6.164)$$

that is in fact artificial in presence of thermal conduction. Obviously, the dispersion equation that is obtained in this conception

$$\left[ \rho(\omega, \mathbf{k}) + \frac{k^2}{\omega^2} (\chi_0^{-1} - \chi^{-1}(\omega, \mathbf{k})) \right] \chi_0 \omega^2 = k^2, \quad (6.165)$$

is yet another way to rewrite the same Kirchhoff-Langevin's characteristic equation (6.133).

Therefore, this conception leads to exactly the same characteristic wavenumbers, but not the same characteristic impedances. Indeed this view, that mixes the different effects, (viscous, inertial, thermal, and elastic), into one abstract nonlocal density  $\rho(\omega, \mathbf{k})$ , with adiabatic compressibility, is not physical like the previous one, that places separately, inertial and viscous effects in the effective nonlocal density  $\rho(\omega, \mathbf{k})$ , and elastic and thermal effects in the effective nonlocal bulk modulus  $\chi^{-1}(\omega, \mathbf{k})$ . It provides a deep thermo-acoustic illustration of the unsatisfactory character of the current definitions in macroscopic nonlocal electromagnetics. We believe that, when generalized in acoustics to materials having finite dimensions, (as done in the next chapter as a first step, for rigid-framed and unbounded materials), the Heaviside-Poynting conception will allow us to overcome the so-called problem of “additional boundary conditions” in nonlocal media, encountered in electromagnetics [16].

## Appendix: Electromagnetic-Acoustic Analogy

We present here the general form of our electromagnetic-acoustic analogy.

In electromagnetics we consider a distribution of electrons and nuclei, which can be structured at different levels, but appears homogeneous at some outer macroscopic level. Regarding the macroscopic averaging operation, we assume that we can conceive it in Lorentz’s form [4], operating a spatial convolution of the considered field with a test function of suitable finite spatial extent [10, 15], or, with some advantages, in the Gibbs form of expectation values, obtained using a suitable ensemble of realizations of the environment: we imagine that the configuration of the microscopic charges, currents, as well as the structuring up to the outer macroscopic level, are appropriately varied to define a relevant average, over all the realizations.

In acoustics, we consider here (this is the subject of this chapter) a homogeneous fluid.<sup>41</sup> For the Lorentz conception we imagine that we have one sample, defined by one configuration of molecules, and that macroscopic averages are performed using a test function of spatial extent sufficiently large compared to mean distances between molecules and small compared to wavelengths; for the Gibbs conception we imagine that we have a suitable ensemble of configuration of molecules, and look at the expectations values over the configurations.

We consider, wave propagation of infinitesimal-amplitude electromagnetic or acoustic perturbations, either in the situation of free motion (no external source), or forced motion (an external current density or force density, is imposed, with same forcing in the different realizations when using Gibbs’ conception). We work in the inertial, rest, reference-frame attached to the ambient medium. Moreover, we work here with *arbitrary 3-space Gauss coordinates*  $x^i$ ,  $i = 1, 2, 3$ . Working with Gauss coordinates, and expressing the different equations and quantities in an invariant

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<sup>41</sup> As just mentioned, however, the analogy extends to a medium that can be structured at different levels and becomes homogeneous at some outer macroscopic level; the next chapter (fluid permeating a rigid solid) will illustrate this.



form under a general change of these coordinates, (general covariant notations), is a valuable heuristic tool, which will help us to detect both the affinities and differences appearing between the electromagnetic and acoustic equations and quantities.

For the acoustics of a viscothermal fluid, the electromagnetic-acoustic analogy will highlight thermodynamic deficiencies in our current acoustic laws, which prevent quantities and equations from appearing entirely in a well-formed manner; for macroscopic electromagnetics, it will highlight the absence of thermodynamic foundation of the notions of electric and magnetic polarizations and Maxwell's  $\mathbf{H}$  and  $\mathbf{D}$  fields, also preventing the concrete expression of the complete form of quantities and equations. These insufficient aspects of the current descriptions can only be overcome when substantial progress has been made in the corresponding fields of irreversible thermodynamics. Nevertheless, in acoustics we expect that the formal pattern of quantities and equations hypothesized here for a simple fluid, will find concrete application when considering structured fluid-solid materials, that become homogeneous at an outer macroscopic scale, with nonlocalities coming principally from the structuration. For such structured materials the missing thermodynamics should not impede making, in forthcoming work, successful illustrations of the full pattern of quantities and equations postulated here, (see discussion around (6.210)).

### *Macroscopic Electromagnetics in Covariant Notations*

Because we work in the rest reference-frame attached to the ambient medium, we are not concerned with the fact that, according to relativity theory, the electromagnetic potential is a four-vector. By using Weyl's gauge, (see footnote 37), we can set to zero its time component and thus associate, to the microscopic electromagnetic field, a covariant 3-vector potential field  $a_i(t, x)$ . From this 3-vector, the electric and magnetic fields,  $e_i(t, x)$  and  $b_{ij}(t, x)$ , are derived as follows, where the semicolon ; denotes the covariant derivative in the Gauss coordinates:

$$e_i = -\frac{\partial a_i}{\partial t}, \quad b_{ij} = \frac{1}{2} (a_{j;i} - a_{i;j}). \quad (6.166)$$

The usual pseudovector,  $\mathbf{b} = \partial \times \mathbf{a}$ , of the standard Gibbs-Heaviside vector notations, is obtained by working with Cartesian direct-system coordinates and setting,  $b_k = 2b_{ij}$ , where  $ijk$  is a circular permutation of 123; the factor of two in (6.166.2) and previous relation, is introduced for later convenience. By averaging, using a Gibbs' or Lorentz's conception, we will define corresponding macroscopic fields,  $A_i = \langle a_i \rangle$ ,  $E_i = \langle e_i \rangle$ ,  $B_{ij} = \langle b_{ij} \rangle$ . They will automatically satisfy

$$E_i = \langle e_i \rangle = -\frac{\partial A_i}{\partial t}, \quad B_{ij} = \langle b_{ij} \rangle = \frac{1}{2} (A_{j;i} - A_{i;j}), \quad (6.167)$$

as there is direct switching (commutation) between the averaging operation  $\langle \cdot \rangle$  and the time or spatial derivative  $\partial/\partial t$  or  $;$   $i$ :

$$\partial/\partial t \langle \cdot \rangle = \langle \partial/\partial t \cdot \rangle, \quad \langle \cdot \rangle ; i = \langle \cdot ; i \rangle. \quad (6.168)$$

The first switching is obvious. The last is also evident with Gibbs' conception: the covariant derivative of the expectation values of the field and the expectation values of its covariant derivative are involved at the same spatial point, thus, the order of the operations can be interchanged, (with Lorentz's conception the justification is a little longer).

To obtain the macroscopic electromagnetic equations appropriate to describe time-variable space-variable phenomena, we first start with the following microscopic equations, where  $g_{ij}$  is the metric tensor of coordinates  $x^i$ ,  $g = |g_{ij}|$  is the determinant of the metric tensor,  $g^{ij}$  is the inverse matrix of the metric tensor, ( $g_{ij}g^{jk} = \delta_i^k$ ,  $\delta_i^k$  is the Kronecker symbol, which is a true mixed tensor), and  $\epsilon_0$  and  $\mu_0$ , two scalar constants, are the electric and magnetic permittivities, (here and in the following we use Einstein's summation convention on repeated indices – one contravariant one covariant):

$$\frac{\partial b_{ij}}{\partial t} = \frac{1}{2} (e_{i;j} - e_{j;i}), \quad \frac{\partial d^i}{\partial t} = h^i_{;j} - j^i, \quad (6.169)$$

$$d^i = \epsilon_0^{ij} e_j, \quad h^{ij} = \mu_0^{-1} ijkl b_{kl}, \quad (6.170)$$

$$\epsilon_0^{ij} = \epsilon_0 g^{1/2} g^{ij}, \quad \mu_0^{-1} ijkl = \mu_0^{-1} g^{1/2} 2 (g^{ik} g^{jl} - g^{il} g^{jk}). \quad (6.171)$$

The usual pseudovector,  $\mathbf{h} = \mathbf{b}/\mu_0$ , is obtained by working in Cartesian coordinates and defining, this time,  $2h_k = h_{ij}$ , in order that (6.169.2) assumes the known form,  $\partial \mathbf{d}/\partial t = \partial \times \mathbf{h} - \mathbf{j}$ , in Gibbs-Heaviside notations. The factor of two in the relation between  $h_k$  and  $h_{ij}$  is chosen so that (6.169.2), like the above known relation, looks simple. We note that, in order for (6.169.2) to be consistent with the additional known field equation,  $d^i_{;i} = \rho$ , and the conservation of charge,  $\partial \rho/\partial t + j^i_{;i} = 0$ , the introduced field  $h^{ij}$  must be taken antisymmetric:  $h^{ij} = -h^{ji}$ . We next complete these first equations by giving additional equations, specifying how the electromagnetic current  $j^i$  is related to the motion of punctual charges  $q_\alpha$  (electrons and nuclei) located at  $x_\alpha$  ( $\delta$  is the Dirac delta function):

$$j^i = \sum_\alpha q_\alpha \frac{dx_\alpha^i}{dt} \delta [x - x_\alpha(t)], \quad (6.172)$$

and how their dynamics is related to the EM fields  $e$  and  $b$ :

$$\frac{d}{dt} \left[ m_{\alpha ij} \left[ \frac{dx^j}{dt} \right]_{\alpha} \right] = q_{\alpha} \left[ e_i + b_{ik} \left[ \frac{dx^k}{dt} \right]_{\alpha} \right], m_{\alpha ij} = \frac{g_{ij} m_{0\alpha}}{\sqrt{1 - \frac{g_{kl}}{c^2} \left[ \frac{dx^k}{dt} \frac{dx^l}{dt} \right]_{\alpha}}}. \quad (6.173)$$

where  $m_{0\alpha}$  are the rest masses of the particles labelled by the index  $\alpha = 1, 2, \dots$ , and  $c$  is the lightspeed constant. Finally, given the above microscopic equations, we have to average them up to the final outer macroscopic level.

Before proceeding we note that the additional equations (6.172)–(6.173) carry the non-quantum notion of “particles trajectories”; also, the summation over  $\alpha$  is symbolic as the electrons are indistinguishable in the quantum vision. These equations are therefore written within a semi-classic, incomplete vision, which do not describe, in particular, spin and the quantum collective behaviour of electrons and nuclei. Here they only superficially play the role of additional fundamental quantum equations, which should be written to complete (6.169)–(6.171).

Aside from its flaws, the expression (6.172) allows us to see that the variance of  $j^i$  is that of a contravariant vector density of weight  $-1$ . Its contravariant variance comes from the differentials  $dx^i_{\alpha}$ ; its weight comes from the  $\delta$  and the true scalar nature of charges  $q_{\alpha}$ .<sup>42</sup> From (6.169.2) and (6.170) it then follows that  $d^i$ ,  $h^{ij}$ ,  $\epsilon_0^{ij}$ ,  $\mu_0^{-1ijkl}$ , are contravariant tensor densities of weight  $-1$ .

From then on, to write the (6.170)–(6.171), we only have to assume that the density fields,  $d^i(t, x)$  and antisymmetric  $h^{ij}(t, x)$ , are, resp., directly related to the true tensor fields,  $e_i(t, x)$  and  $b_{ij}(t, x)$ , *expressed at the same time and space position*, i.e. there is no temporal or spatial dispersion in void space. The only available tensors to construct the contravariant tensor densities of weight  $-1$ ,  $\epsilon_0^{ij}$  and  $\mu_0^{-1ijkl}$ , in the relations (6.170), are introduced in the problem by the arbitrary choice of coordinates. This immediately gives the expressions (6.171),<sup>43</sup> since we must have antisymmetry of the coefficients  $\mu_0^{-1ijkl}$ , not only over  $kl$  but also over  $ij$ . We note that the resulting  $\epsilon_0^{ij}$  is automatically symmetric,  $\epsilon_0^{ij} = \epsilon_0^{ji}$ , and likewise, the resulting  $\mu_0^{-1ijkl}$  is automatically symmetric under the exchange of the first and second pair of indices,  $\mu_0^{-1ijkl} = \mu_0^{-1klji}$ .

We now consider taking the Gibbs’ or Lorentz’s averages  $\langle \rangle$  of the microscopic quantities and equations. For simplicity, it is advantageous to work with Gibbs’ averages, however, the results will also be valid (but at long wavelengths only) and look the same, with Lorentz’s averaging.

Because of the switching (6.168) it is immediate to average the equations (6.169)–(6.170) to write<sup>44</sup>:

<sup>42</sup> Since  $\int \delta [x - x_{\alpha}] d^3x = 1$ , and the differential element of volume  $d^3x$  is a scalar density of weight 1, the  $\delta$  has the nature of a scalar density of weight  $-1$ .

<sup>43</sup> The numerical constants in (6.171) are chosen so that these relations reduce, in Cartesian coordinates and in Gibbs-Heaviside’s notation, to proportionality relations,  $\mathbf{d} = \epsilon_0 \mathbf{e}$ , and  $\mathbf{h} = \mathbf{b}/\mu_0$ .

<sup>44</sup> We obviously have  $\langle \epsilon_0^{ij} e_j \rangle = \epsilon_0^{ij} \langle e_j \rangle$ , etc., as we use identical Gauss coordinates in the different realizations.

$$\frac{\partial B_{ij}}{\partial t} = \frac{1}{2} (E_{i;j} - E_{j;i}), \quad \frac{\partial D_0^i}{\partial t} = H_{0;j}^{ij} - \langle j^i \rangle, \quad (6.174)$$

$$D_0^i = \epsilon_0^{ij} E_j, \quad H_0^{ij} = \mu_0^{-1 ijkl} B_{kl}, \quad (6.175)$$

where we have set

$$E_i = \langle e_i \rangle, \quad B_{ij} = \langle b_{ij} \rangle, \quad D_0^i = \langle d^i \rangle, \quad H_0^{ij} = \langle h^{ij} \rangle. \quad (6.176)$$

In (6.175), the index 0 over  $D$  and  $H$ , is put to reserve notations  $D$  and  $H$  for the final macroscopic fields that will appear below, (and that are not the direct averages of underlying  $d$  and  $h$  fields). For the macroscopic current  $\langle j^i \rangle$ , we view it as the sum of an external impressed macroscopic current denoted  $[J^i]$ , (by definition the same in the different realizations), and an intrinsic macroscopic polarization current denoted  $J_{pol}^i$ :

$$\langle j^i \rangle = J_{pol}^i + [J^i]. \quad (6.177)$$

This macroscopic polarization current reflects the initiation of irreversible thermodynamic processes, triggered by the presence of the disturbance macroscopic fields  $E$  and  $B$ , zero in the equilibrium state, and nonzero in the disturbed state. It is present even if the forcing  $[J^i]$  is absent. We certainly cannot truly express it from the semi-classical simplistic vision of the additional equations (6.172)–(6.173) that we have posed. We can also hardly express it by direct averaging from a complete microscopic quantum vision, because the laws that would link the macroscopic mean magnitudes together, being thermodynamic in nature, (in the sense of a macroscopic out-of-equilibrium statistics), would be difficult to obtain from the basic laws of QED.<sup>45</sup> Finally, as the corresponding thermodynamics of the motion of an “electronic fluid” in the lattice of nuclei, remains to be developed, a thermodynamic analysis and expression of  $J_{pol}^i$  is at present not feasible. Nevertheless, according to a division that we call Lorentz-splitting, we can always try to write this current as a spatial derivative, and a temporal derivative term:

$$J_{pol}^i = \frac{\partial P^i}{\partial t} + M_{;j}^{ij}, \quad (6.178)$$

with  $P^i$ , and antisymmetric  $M^{ij}$ , two unknown fields conventionally named the electric and magnetic polarization fields. The antisymmetry of  $M^{ij}$  is guessed on the basis of the characteristic antisymmetry observed in electromagnetic quantities and equations. Inserting (6.177)–(6.178) in (6.174.2) will provide a macroscopic equation having the form

$$\frac{\partial D^i}{\partial t} = H_{;j}^{ij} - [J^i], \quad (6.179)$$

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<sup>45</sup> By the way, fully explaining how the transition from a quantum microscopic level to a classical macroscopic level takes place, should be a difficult task, already because of the special physical position of quantum physics which requires the classical limit for its own formulation [19].

with

$$D^i = \epsilon_0^{ij} E_j + P^i, \quad H^{ij} = \mu_0^{-1ijkl} B_{kl} - M^{ij}. \quad (6.180)$$

The antisymmetry of  $M$  implies antisymmetry of the introduced  $H$  field. Because we consider infinitesimal perturbations, the macroscopic polarizations  $P$  and  $M$  appear as linear responses to the macroscopic electromagnetic field represented by  $E$  and  $B$ . There must be as a consequence, through (6.180), constitutive thermodynamic relations having, in the most general manner, the following linear form, (for small perturbations)

$$D^i(t, x) = \int_{-\infty}^t dt' \int d^3x' \epsilon^{ij}(t-t', x, x') E_j(t', x') + \dots \\ \int_{-\infty}^t dt' \int d^3x' \kappa^{ijkl}(t-t', x, x') B_{jk}(t', x'), \quad (6.181)$$

$$H^{ij}(t, x) = \int_{-\infty}^t dt' \int d^3x' \mu^{-1ijkl}(t-t', x, x') B_{kl}(t', x') + \dots \\ \int_{-\infty}^t dt' \int d^3x' \lambda^{ijk}(t-t', x, x') E_k(t', x'). \quad (6.182)$$

The integrations over time will describe temporal dispersion effects, and those over space, spatial dispersion effects. But under the first field equation (6.174.1), which in harmonic regime implies  $-i\omega B_{jk} = (E_{j;k} - E_{k;j})/2$ , it is always possible, by virtue of the presence of spatial dispersion, to incorporate in (6.181), the effect of the second integral term into the first, by appropriately redefining the nonlocal kernel function  $\epsilon^{ij}(t, x, x')$ . We can therefore generally simply rewrite

$$D^i(t, x) = \hat{\epsilon}^{ij} E_j(t, x) = \int_{-\infty}^t dt' \int d^3x' \epsilon^{ij}(t-t', x, x') E_j(t', x'). \quad (6.183)$$

Concerning (6.182), the second integral term is similarly useless, so that

$$H^{ij}(t, x) = \hat{\mu}^{-1ijkl} B_{kl}(t, x) = \int_{-\infty}^t dt' \int d^3x' \mu^{-1ijkl}(t-t', x, x') B_{kl}(t', x'). \quad (6.184)$$

It corresponds to expressing  $P_i$  as a nonlocal linear response to the electric field, and  $M_{ij}$  as a nonlocal linear response to the magnetic field; it justifies the denomination “electric” for  $P$  and “magnetic” for  $M$ .

There is still to be observed that the introduction of polarizations  $P$  and  $M$ , through the Lorentz-splitting (6.178), carries an essential ambiguity: an arbitrary response term  $\Phi_{;j}^{ij}$ , with  $\Phi^{ij}$  an antisymmetric tensor of weight  $-1$ , can always be added to  $P^i$ , if, simultaneously, a term  $\partial\Phi^{ij}/\partial t$  be subtracted to  $M^{ij}$ . Because of this indetermination, the fields  $D$  and  $H$ , and thus also the operators  $\hat{\epsilon}$  and  $\hat{\mu}^{-1}$ , are ambiguous and still need to be resolved. I assert that this inherent ambiguity would

be removed, i.e. all quantities above would become uniquely defined in principle, if one were able to give a precise thermodynamic content to the notion of the *current density of energy transported in usable electromagnetic form*, (by opposition to energy transported in degraded form). Given this notation, one then could require that, by definition of  $H^{ij}$ , this current density of energy be given by a Poynting-Heaviside vector  $S$ , such that

$$S^i(t, x) = -H^{ji}(t, x)E_j(t, x) = H^{ij}(t, x)E_j(t, x). \quad (6.185)$$

In the context of general nonlocal response of the media, the form of (6.185) is remarkable, in that, the current density of energy transported in electromagnetic form, *at a given time and position*, is expressed as the contraction of two fields evaluated *at the same time and position*. In fact, the corresponding remarkable choice of the field  $H^{ij}$  will determine, how is to be made the precise separation of the global effects, into the actual “time dispersion effects” per se, and “spatial dispersion effects” per se.

In electromagnetics of material media, we do not dispose of the necessary thermodynamic laws of motion of the system, comprising the electrons and solid nuclei structured lattice, to precise these notions and arrive at a definition of the electric polarization, magnetic polarization, field  $H^{ij}$ , field  $D^i$ , and operators  $\hat{\epsilon}$  and  $\hat{\mu}^{-1}$ . By thermodynamic arguments of a general nature, however, it can be shown that  $\hat{\epsilon}$  and  $\hat{\mu}^{-1}$  are “generalized susceptibilities” that satisfy the general symmetry relations [13, 20]:

$$\epsilon^{ij}(t, x, x') = \epsilon^{ji}(t, x', x), \quad \mu^{-1ijkl}(t, x, x') = \mu^{-1klij}(t, x', x). \quad (6.186)$$

Finally, in presence of spatial dispersion, it is customary in electromagnetic literature to consider that,  $M^{ij} = 0$ , i.e.  $H^{ij} = \mu_0^{-1ijkl} B_{kl}$ . This leads to the definition of conventional  $D$  field and  $\epsilon$  kernel, certainly slightly different from the true physical ones.

We believe that, future developments crossing the boundaries of present thermodynamics, could allow the introduction of the necessary thermodynamic variables, associated to the motion of “electronic fluid” interacting with the nuclei structured lattice, from which, in particular, the electromagnetic current density of energy,  $S^i$ , could be explicitly constructed. <sup>In</sup> ~~It is~~ a sequence of successive determinations of the type <sup>in acoustics</sup> illustrated in Sect (6.122.3) it would then imply, precise, and unambiguous, definition of  $H^{ij}$  field,  $M^{ij}$  field,  $D^i$  field,  $P^i$  field, operator  $\hat{\epsilon}$ , and operator  $\hat{\mu}^{-1}$ .

We summarize our discussion. Macroscopic electromagnetic equations have the following covariant (invariant) pattern in all coordinate systems fixed to the ambient medium:

$$\frac{\partial B_{ij}}{\partial t} = \frac{1}{2}(E_{i;j} - E_{j;i}), \quad \frac{\partial D^i}{\partial t} = H^i_j - [J^i], \quad (6.187)$$

$$D^i(t, x) = \hat{\epsilon}^{ij} E_j(t, x) = \int_{-\infty}^t dt' \int d^3x' \epsilon^{ij}(t-t', x, x') E_j(t', x'), \quad (6.188)$$

$$H^{ij}(t, x) = \hat{\mu}^{-1ijkl} B_{kl}(t, x) = \int_{-\infty}^t dt' \int d^3x' \mu^{-1ijkl}(t-t', x, x') B_{kl}(t', x'), \quad (6.189)$$

$$S^i(t, x) = H^{ij}(t, x) E_j(t, x) = -H^{ji}(t, x) E_j(t, x), \quad (6.190)$$

where a thermodynamic knowledge is assumed, (though currently missing), that could be used to construct the Heaviside-Poynting vector  $S^i$ , specifying the current density of energy transported in electromagnetic form. We name “Lorentz fields” the true tensor fields  $B_{ij} = \langle b_{ij} \rangle$  and  $E_i = \langle e_i \rangle$ , and “Maxwell fields” the tensor density fields  $H^{ij}$  and  $D^i$ . The first are the direct averages of the corresponding microscopic fields. The second are not. They include an additional polarization part.

Let us now consider the acoustic wave propagation in our simple homogeneous fluid, (directly a macroscopic wave propagation phenomenon), and let us try to write, by analogy, the most general pattern of macroscopic equations, that would manifest both temporal dispersion and spatial dispersion.

### *Acoustics in Covariant Notations*

In electromagnetics, working in the rest frame of the medium, we have introduced a 3-vector covariant macroscopic field  $A_i(t, x) = \langle a_i \rangle(t, x)$ , allowing to build the macroscopic electric and magnetic fields,  $E$ ,  $B$ , according to (6.167). Here in acoustics, by analogy, we want to test the idea that, working in the rest frame of the fluid, we also have, at the macroscopic level, at least in Cartesian coordinates, an obvious 3-vector field existing at any point and time and associated with an acoustic disturbance: the macroscopic displacement  $\mathbf{a}$  of the fluid particles.

An immediate objection to the possibility of such an analogy is that a finite displacement is not a vector when using Gauss’ general coordinates. However, as we restrict here to considering infinitesimal perturbations, we can consider that the associated displacement fields  $a^i$  will behave as coordinates differentials  $dx^i$ , that is, as true contravariant vectors. We can thus try to pursue a deep electromagnetic-acoustic analogy, where the macroscopic electromagnetic *covariant* potential field  $A_i$ , will be replaced by a *contravariant* displacement field  $a^i$ . From this basic acoustic contravariant 3-vector  $a^i$ , we derive the velocity and (opposite and linearized) strain fields<sup>46</sup>  $v^i$  and  $b^{ij}$ :

$$v^i = \frac{\partial a^i}{\partial t}, \quad b^{ij} = -\frac{1}{2} (a^{i;j} + a^{j;i}). \quad (6.191)$$

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<sup>46</sup> In spite of the macroscopic nature of our quantities  $v$  and  $b$  here, we use small letters for them, because in the next chapter the medium will be structured, (with solid parts), and these quantities will have to be averaged up to some outer macroscopic level, then becoming the outer macroscopic quantities  $V = \langle v \rangle$  and  $B = \langle b \rangle$ .

In these definitions, the characteristic electromagnetic antisymmetry,  $B_{ij} = -B_{ji}$ , becomes a characteristic acoustic symmetry,  $b^{ij} = b^{ji}$ : therefore, there is no sense to try to compare the choice of overall signs in (6.167) and (6.191). Here, we have introduced for convenience positive sign in (6.191.1) and negative sign in (6.191.2) in order that  $b_i^i = b$ , the condensation.<sup>47</sup>

We see by construction that, from (6.191), the following equation is satisfied

$$\frac{\partial b^{ij}}{\partial t} = -\frac{1}{2} (v^{i;j} + v^{j;i}). \quad (6.192)$$

It will play the role of the first electromagnetic equation (6.187).

Now, by analogy with the electromagnetic macroscopic equations (6.174), (6.175), and (6.177), we can guess that, when the fluid is forced by an arbitrary external force density  $[F^i](t, x)$ , its motion will be governed by equations having the form:

$$\frac{\partial b^{ij}}{\partial t} = -\frac{1}{2} (v^{i;j} + v^{j;i}), \quad \frac{\partial D_0^i}{\partial t} = H_{0;j}^{ij} + F^i, \quad (6.193)$$

$$D_{0i} = \rho_{0ij} v^j, \quad H_{0ij} = -\chi_{0ijkl}^{-1} b^{kl}, \quad (6.194)$$

$$F^i = F_{pol}^i + [F^i], \quad (6.195)$$

with, this time, we assume, a *symmetric*  $H_0^{ij}$ . Again, because of the interchange of symmetry there is no sense to compare signs in electromagnetic and acoustic equations. The signs appearing are chosen for later convenience.  $[F^i]$  is the external force density (force per unit volume) impressed on the fluid, and  $F_{pol}^i$  is an induced polarization force density, which reflects the initiation of irreversible thermodynamic processes of any kind, coming in response to the presence of the disturbance macroscopic fields  $v$  and  $b$ , zero in the equilibrium state, and nonzero in the disturbed state.

With regard to force densities  $[F^i]$ ,  $F_{pol}^i$ , we view them as contravariant vectors with a weight density of  $-1$ , (identical to that of electromagnetic current densities), and this variance comes, for example, from a comparison with a Newton's law term,  $\rho \partial v^i / \partial t$ , where  $\rho$  is an expression of type,  $\rho = \langle \sum_{\alpha} m_{\alpha} \delta [x - x_{\alpha}(t)] \rangle$ , the sum is on the molecules, and the masses  $m_{\alpha}$  are true scalars.

Therefore,  $D_{0i}$ ,  $H_{0ij}$ ,  $\rho_{0ij}$  and  $\chi_{0ijkl}^{-1}$ , are also tensor densities of weight  $-1$ . The equations (6.194) state that the "Maxwell" densities  $D_{0i}$  and  $H_{0ij}$  are, respectively, directly related to the "Loventz" true fields  $v^i$  and  $b^{ij}$ , *expressed at the same time and space position*. This is because all temporal and spatial dispersion effects, (all irreversible thermodynamic processes coming in response to the presence of the disturbance macroscopic fields), express, by definition, via the induced polarization force density  $F_{pol}^i$ . As the fluid is time invariant, homogeneous, isotropic, and invari-

<sup>47</sup> We recall that the lowering or raising of indices is done by contraction with the metric tensor or its inverse, (here,  $b_i^i = g_{ij} b^{ij} = g^{ij} b_{ij}$ ).



ant in spatial reflections, the only available tensors to construct the coefficients  $\rho_{0ij}$  and  $\chi_{0ijkl}^{-1}$ , are the metric tensor  $g_{ij}$  or its inverse  $g^{ij}$ , and the associated density of weight  $-1$ ,  $g^{1/2}$ . Moreover, as  $b^{kl}$  is symmetric, and we assume that  $H_{0ij}$  is also symmetric,  $\mu_{0ijkl}^{-1}$  must be symmetric, not antisymmetric, on indices  $ij$  and  $kl$ . With this exchange of symmetry, the only possible expressions are:

$$\rho_{0ij} = \rho_0 g^{1/2} g_{ij}, \quad \chi_{0ijkl}^{-1} = \chi_0^{-1} g^{1/2} g_{ij} g_{kl} + \mu_0 g^{1/2} \left( g_{ik} g_{jl} + g_{il} g_{jk} - \frac{2}{3} g_{ij} g_{kl} \right). \quad (6.196)$$

Note that we could have written

$$D_0^i = \rho_0^i v^j, \quad H_0^{ij} = -\chi_0^{-1 ij} b^{kl},$$

in place of (6.194), and

$$\rho_{0j}^i = \rho_0 g^{1/2} \delta_j^i, \quad \chi_{0kl}^{-1 ij} = \chi_0^{-1} g^{1/2} g^{ij} g_{kl} + \mu_0 g^{1/2} \left( \delta_i^k \delta_j^l + \delta_i^l \delta_j^k - \frac{2}{3} g^{ij} g_{kl} \right),$$

in place of (6.196). In Cartesian coordinates the above expressions of the  $\chi_0^{-1}$  tensor yield Frenkel's conjectured tensor (6.42).

The true scalar constants,  $\rho_0$ ,  $\chi_0^{-1}$  and  $\mu_0$ , interpret as the ambient density, the adiabatic bulk modulus, and the adiabatic Frenkel-Lamé shear coefficient: remarkably, the addition of the latter coefficient is a direct result of the transformation of antisymmetry to symmetry, when we pass from electromagnetic equations to acoustic equations.

Next, in the same manner as, in electromagnetics, we assumed Lorentz's splitting of the induced polarization current, here, we assume a similar splitting of the induced polarization force:

$$F_{pol}^i = -\frac{\partial P^i}{\partial t} + M_{;j}^{ij}, \quad (6.197)$$

with  $P^i$ , and symmetric  $M^{ij}$ , some "acoustic polarization" fields, (again, because of the interchange of symmetry there is no sense to try to compare signs in (6.178) and (6.197), and the sign  $-$  is introduced as inspired by (6.101)). Inserting (6.195, 6.197) in (6.193.2) will provide an equation

$$\frac{\partial D^i}{\partial t} = H_{;j}^{ij} + [F^i], \quad (6.198)$$

where

$$D_i = \rho_{0ij} v^j + P_i, \quad H_{ij} = -\chi_{0ijkl}^{-1} b^{kl} + M_{ij}. \quad (6.199)$$

The field  $D^i$  will be an “effective acoustic momentum” field, and the field  $H^{ij}$  an “effective acoustic stress” field. As the polarizations appear in linear response to the infinitesimal acoustic field represented by  $v$  and  $b$ , there must be constitutive thermodynamic linear relations having, in the most general manner, the form

$$D_i(t, x) = \int_{-\infty}^t dt' \int d^3 x' \rho_{ij}(t-t', x, x') v^j(t', x') + \dots \quad (6.200)$$

$$\int_{-\infty}^t dt' \int d^3 x' \kappa_{ijk}(t-t', x, x') b^{jk}(t', x'),$$

$$H_{ij}(t, x) = - \int_{-\infty}^t dt' \int d^3 x' \chi_{ijkl}^{-1}(t-t', x, x') b^{kl}(t', x') + \dots \quad (6.201)$$

$$\int_{-\infty}^t dt' \int d^3 x' \lambda_{ijk}(t-t', x, x') v^k(t', x').$$

But under the first field equation (6.193.1), which in harmonic regime implies  $i\omega b^{jk} = (v^{j;k} + v^{k;j})/2$ , it is always possible, by virtue of the presence of spatial dispersion, to incorporate in (6.200) the effect of the second integral into the first, by appropriately redefining the kernel  $\rho_{ij}$ . We can therefore generally simply rewrite

$$D_i(t, x) = \hat{\rho}_{ij} v^j(t, x) = \int_{-\infty}^t dt' \int d^3 x' \rho_{ij}(t-t', x, x') v^j(t', x'). \quad (6.202)$$

Concerning (6.201), the second integral is similarly useless, so that we can also simply rewrite

$$H_{ij}(t, x) = -\hat{\chi}_{ijkl}^{-1} b^{kl}(t, x) = - \int_{-\infty}^t dt' \int d^3 x' \chi_{ijkl}^{-1}(t-t', x, x') b^{kl}(t', x'). \quad (6.203)$$

It amounts to say that  $P_i$  is nonlocal response to the velocity field, and  $M_{ij}$  is nonlocal response to the strain field.

Finally and again, the introduction of polarizations  $P^i$  and  $M^{ij}$  through the splitting (6.197), carries an essential ambiguity: an arbitrary response term  $\Phi_{;j}^{ij}$ , with  $\Phi^{ij}$  a symmetric tensor of weight  $-1$ , can always be added to  $P^i$ , if, simultaneously, a term  $\partial\Phi^{ij}/\partial t$  be added to  $M^{ij}$ . Because of this indetermination, the fields  $D$  and  $H$ , and thus also the operators  $\hat{\rho}$  and  $\hat{\chi}^{-1}$ , are ambiguous and still need to be resolved. I assert that the ambiguity would be removed, i.e. all quantities above would become uniquely defined in principle, if one were able to give a precise thermodynamic content to the notion of the current density of energy transported in usable acoustic form, (by opposition to energy transported in degraded form). Given this notion, one then could require that, by definition of  $H^{ij}$ , this current density of energy be given by a “Poynting-Heaviside”, or here, “Umov” vector, such that

$$S^i(t, x) = -H_j^i(t, x)v^j(t, x), \quad (H_j^i = g_{jk}H^{ik} = g^{ik}H_{kj}). \quad (6.204)$$

At present time, we do not dispose of the necessary thermodynamic laws of motion, to fully precise this notion and arrive at a wholly satisfactory construction of the acoustic polarizations, acoustic fields  $H^{ij}$  and  $D^i$ , and acoustic kernels  $\rho_{ij}(t, x, x')$  and  $\chi_{ijkl}^{-1}(t, x, x')$ . By thermodynamic arguments of a general nature, however, we know that  $\hat{\rho}$  and  $\hat{\chi}^{-1}$  are “generalized susceptibilities” that satisfy the general symmetry relations [13, 20]:

$$\rho_{ij}(t, x, x') = \rho_{ji}(t, x', x), \quad \chi_{ijkl}^{-1}(t, x, x') = \chi_{klij}^{-1}(t, x', x). \quad (6.205)$$

Within the usual near-equilibrium Navier-Stokes-Fourier thermodynamic framework, what is obtained is that we must identify the effective acoustic stress field  $H_{ij}$  with  $-pg^{1/2}g_{ij}$ , (to respect Heaviside-Poynting’s identification (6.112), and the covariant writing of the (6.113), namely,  $S^i = g^{1/2}pv^i$ , where  $p$  interprets here as a scalar equal to the thermodynamic pressure), but this leads to degeneracies particularly apparent in the description of shear motions: for these motions, no pressure arises and the corresponding Maxwell stress field  $H_{ij}$  is then identically equal to zero. It highlights clearly the thermodynamic insufficiencies of this framework.

We summarize our discussion. Acoustic equations have the following covariant pattern in all coordinate systems fixed to the ambient medium:

$$\frac{\partial b^{ij}}{\partial t} = -\frac{1}{2}(v^{i;j} + v^{j;i}), \quad \frac{\partial D^i}{\partial t} = H_{;j}^{ij} + [F^i], \quad (6.206)$$

$$D_i(t, x) = \hat{\rho}_{ij}v^j(t, x) = \int_{-\infty}^t dt' \int d^3x' \rho_{ij}(t-t', x, x')v^j(t', x'), \quad (6.207)$$

$$H_{ij}(t, x) = -\hat{\chi}_{ijkl}^{-1}b^{kl}(t, x) = -\int_{-\infty}^t dt' \int d^3x' \chi_{ijkl}^{-1}(t-t', x, x')b^{kl}(t', x'), \quad (6.208)$$

$$S^i(t, x) = -H_j^i(t, x)v^j(t, x), \quad (6.209)$$

where a thermodynamic knowledge is assumed, (though currently missing), that could be used to construct the “Heaviside-Poynting” or else “Umov” vector  $S^i$ .

When the medium is structured, the analogy suggests to replace  $v$  and  $b$  by averages  $V = \langle v \rangle$  and  $B = \langle b \rangle$  that will be named the acoustic Lorentz fields. New acoustic Maxwell fields  $D$  and  $H$ , “Umov” vector  $S$ , and operators  $\hat{\rho}$  and  $\hat{\chi}^{-1}$ , will appear, respecting the same equation pattern as above. The mentioned deficiencies of the thermodynamic framework will not hinder the construction of a viable non-local description. Indeed, the nonlocal effects resulting from the structuring of the environment will be the ones that will be important to consider first. They will be sufficiently well described in the available imperfect thermodynamic framework. An highlighting, reduced exemplification of this, will be considered in the next chapter.

The full (nonreduced) nonlocal pattern of quantities and equations will arise for structured media, e.g. media having connected fluid and solid phases. For example, consider the case where the stress-strain relations in the solid are idealized as the following lossless relations,  $\sigma_{ij} = -\chi_{sijkl}^{-1} b^{kl}$ , with  $\chi_{sijkl}^{-1}$  given by (6.196), and  $\chi_0^{-1}$  and  $\mu_0$  replaced by  $\chi_s^{-1}$  and  $\mu_s$ , (the constants in the solid phase). Then, the full above nonlocal pattern of equations will appear, replacing  $v^i$  and  $b^{ij}$  by  $V^i = \langle v^i \rangle$  and  $B^{ij} = \langle b^{ij} \rangle$ , and defining the  $H_{ij}$  field through the generalized energetic “Umov” or “Heaviside-Poynting” relation, (making it a continuous field):

$$S^i(t, x) = \langle -\sigma_j^i v^j \rangle(t, x) = -H_j^i(t, x) V^j(t, x), \quad (6.210)$$

with  $\sigma_{ij}$ , equal to  $-pg^{1/2}g_{ij}$  in the fluid, where  $p$  is the thermodynamic excess pressure. In short, it should be possible to describe any fluid-saturated poroelastic material as a suitable nonlocal elastic solid.<sup>48</sup> We expect that this principled position of the problems will be verified in future detailed studies.

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<sup>48</sup> Biot’s theory [21, 22] will be a special, simplified case of this, in which at the small scale the fluid will move as an incompressible fluid and the solid will move as a whole. This limit will have the same physical position as the local-theory limit presented in Appendix of next chapter, for the case of rigid-framed materials.

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