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Duality for some models of epidemic spreading

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Abstract

We examine three non-conservative interacting particle systems in one dimension modeling epidemic spreading: the diffusive contact process (DCP), a model that we introduce and call generalized diffusive contact process (GDCP), and the susceptible-infectious-recovered (SIR) model. For each system, we establish duality relations through an analytical approach. As an application, we provide a new expression of the correlation functions for the steady state of the finite volume DCP and GDCP in contact with boundary reservoirs. We explicitly compute the one point correlation function of the steady state of a small sized DCP, and for arbitrary size in a particular setting of the GDCP. For the infinite SIR model, the duality relation obtained yields an explicit expression of the time evolution of some specific correlation functions, describing the time decay of the sizes of chains of susceptible individuals.

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1 Introduction

In the context of Markov processes, duality is a remarkable tool to analyze a model of interest using another model, its dual, via an observable for both models called duality function. For interacting particle systems, there are various ways of deriving a duality relation, when it is available: using analytical tools via computations on generators or intensity matrices, through pathwise methods via a graphical representation, or by an algebraic approach; in the latter case, a unified treatment of conservative systems has been derived. A non exhaustive list of books or surveys that deal with duality is [23, 22, 16, 29, 33, 6], see also references therein.

In this article, we study three different non-conservative interacting particle systems, namely *(i)* the diffusive contact process (see for instance [20]), which we augment with open boundaries that allow particle exchanges with external reservoirs, abbreviated below by DCP, *(ii)* a generalization thereof with a variable death rate, abbreviated below by GDCP, and *(iii)* the susceptible-infectious-recovered (SIR) model ([30]). All these particle systems, described informally below and defined formally in terms of their generators in Sections 2–4, are Markovian models for the spreading of infectious diseases, but with different microscopic dynamics. For each of these models we prove a duality relation using analytical tools. Nontrivial duality relations are, in general, rare and exceptional. They occur, for example, in conservative particle systems, in the presence of non-Abelian symmetries [26, 14], or when the time evolution of n -point correlations is fully determined by k -point correlations of lower or equal order $k \leq n$ ([27]). This is not the case in general for models such as ours, where the total number of particles is not conserved by the bulk dynamics, and the evolution of correlations involves higher order correlations. It should also be noted that in general it is not guaranteed that the dual process turns out to be simpler. As pointed out in Chapter II of [23], “for the contact process the dual Markov chain may be no easier than the original problems for the particle system; duality yields interesting and important connections between these problems which add insight into what is going on”. Therefore, whether duality is useful or not, depends not only on the dual process, but also on the type of question that one asks. Indeed, it will turn out that dualities for models of epidemic spreading of diseases may be of different nature and address different questions.

Dualities that frequently appear in interacting particle systems are *factorized* dualities, meaning that the global duality function is a product over local duality functions on each lattice site. Such a duality is well-known for the contact process ([22]). Here, we prove that the same duality function can be used both for the open DCP, and for its generalization GDCP, to obtain information about the invariant measure. In the case of the GDCP, it also yields information on the time-dependent one-point function for arbitrary initial states. For the SIR model, we prove a different kind of duality, that we call *cluster duality*, since the duality function relates the particle configuration in a cluster of neighboring sites, to the dual process which involves the boundaries of the cluster. This type of duality function, reminiscent of the so-called empty-interval method used to study correlations in reaction-diffusion processes ([1]), appears to be unexplored in the

context of interacting particle systems, and may have further applications to other models. We use it to study the time-dependence of correlations in the SIR model. Our results induce many open questions that we plan to address in the future. Also, note that although we treat each model separately, we aim, in future work, at a unified treatment of such models, via an algebraic approach.

We now briefly describe the three models studied in this work, that are one-dimensional interacting particle systems, either on \mathbb{Z} or on $\Lambda_N := \{1, \dots, N\}$ with $N \geq 1$ sites, and the results we obtain for them through duality. Let us first recall the definition of duality between two Markov processes.

Definition 1 (Definition 3.1 in [23], Chapter II). *Let $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ be two continuous time Markov processes with state spaces Ω and Ω^{dual} , respectively. We say that X and Y are dual with respect to a local duality function $D : \Omega \times \Omega^{\text{dual}} \rightarrow \mathbb{R}$ if*

$$\mathbb{E}_x D(X_t, y) = \mathbb{E}_y D(x, Y_t) \tag{1.1}$$

for all $(x, y) \in \Omega \times \Omega^{\text{dual}}$ and $t \geq 0$. In equation (1.1), \mathbb{E}_x (respectively \mathbb{E}_y) is the expectation w.r.t. the law of the process X initialized at x (respectively the process Y initialized at y).

In the diffusive contact process (DCP), each lattice site can be occupied by at most one particle. Its dynamics is the superposition of the (basic) contact process (CP) with birth rate $\lambda > 0$, with the symmetric simple exclusion process (SSEP), a particle exchange dynamics parameterized by a diffusion coefficient $\mathcal{D} \geq 0$. In the CP, a particle on site x on the lattice “dies” (i.e., disappears from the lattice) with rate 1 and “infects” (i.e. attempts to create another particle), with rate $\lambda > 0$, on a neighboring site, provided the latter is empty; otherwise the creation attempt fails. In the SSEP, particles attempt to jump to nearest neighbor sites, at rate $\mathcal{D} \geq 0$, provided the chosen target site is empty; otherwise the jump attempt fails, see [23, 22] for a precise definition of both the CP and the SSEP.

The dynamics of the DCP is irreducible and its state space finite, so it admits a unique invariant measure (or stationary state). For the closed DCP (that is, with no reservoirs), the empty configuration is an absorbing state, so the invariant measure is completely explicit and corresponds to the Dirac measure concentrated on the empty configuration. When adding reservoirs of particles, which can remove or inject particles into the system at random times, the empty configuration is no longer an absorbing state, as the reservoirs maintain the process “alive”.

The goal of using duality for the DCP, is to get a better understanding of its stationary state, when in contact with reservoirs. In Section 2, we establish in Theorem 1 a duality result between the DCP with an open boundary, and the DCP with a purely absorbing boundary. From this duality relation, we provide a new expression for the correlation functions of the invariant measure in terms of absorption probabilities, and compute them for small size systems in Subsection 2.3. Somewhat unexpectedly, the bulk duality function is the same as for the CP with closed boundaries. We also comment briefly on the fast stirring limit, in which the process effectively reduces to a birth-death chain on the integers $\{0, 1, \dots, N\}$. Note that fast stirring limits have been much investigated for infinite volume systems in \mathbb{Z}^d , to establish hydrodynamic limits (see e.g. [7, 9]); much less is known in our case with a finite diffusion coefficient \mathcal{D} . This is also why, in the appendices A and B, we investigate other direct approaches to compute, on small sized systems, the invariant measure, and its correlation functions.

The generalized diffusive contact process (GDGP) has the same structure as the DCP, namely the superposition of a SSEP with diffusion coefficient \mathcal{D} , and a generalization of the contact process, that is, a process with infection rate $\lambda > 0$, but with death rates depending on the

occupation variables on the neighboring sites, instead of being constant equal to 1. This model is a special case of general models introduced in [25, 27]. As for the DCP, the goal of using duality for the GDCP is to get a better understanding of its stationary state when in contact with reservoirs. In Section 3, we derive in Subsection 3.2 several duality relations for the GDCP, and focus on one of them that induces a dual process with no birth rate. As an application of the latter dual relation, we compute in Subsection 3.3 the one point correlation functions of the stationary state, for a general bulk of N sites. Moreover, we obtain a system of ordinary differential equations for the time-dependent one-point correlation functions, for arbitrary initial distributions. After some elementary transformations, this system of ODE's becomes identical to that of the SSEP with open boundaries, but without any nonconservative bulk processes.

The one-dimensional susceptible-infectious-recovered (SIR) model, that goes back to the partial differential-equation model for epidemics introduced in [19], describes the evolution on the infinite lattice \mathbb{Z} (hence without reservoirs), of three species of individuals: susceptible (in state S), infectious (I), recovered (R). An infectious individual infects a susceptible neighbor at rate $\beta > 0$, it recovers at rate $\gamma > 0$ and becomes immune (that is, it remains forever in state R).

As in the previous sections, our goal is to compute correlation functions. However for the SIR model, this is quite different to the DCP and the GDCP where we were interested in the invariant measure. Here, the state space is not irreducible and there are infinitely many invariant measures (Dirac measures on blocked states), so the asymptotic behavior of the system is highly dependent on the initial condition. We are rather interested in the time evolution of correlation functions according to the initial state of the system. In Section 4, we study the “ n -point cluster functions”, named as such in [30], which are quantities written in terms of higher order correlation functions. Thanks to these cluster functions, we derive in Subsection 4.1 a duality relation between the SIR model and a two dimensional random walk on two layers with a trap. From that, using a probabilistic approach, we provide in Subsection 4.2 an explicit expression of the average of these cluster functions at any time. This new treatment of the model allows us to extend the results of [30] for translation invariant initial distributions, to the case of non translation invariant initial distributions.

The paper is organised as follows. In Section 2, we study the DCP, adding in Appendices A and B computations that do not rely on duality. In Section 3 we study the GDCP, and in Section 4, we study the SIR model.

2 The diffusive contact process (DCP)

In this section, we define the DCP with open boundaries, and establish the duality relation with the DCP with purely absorbing boundaries. A duality relation which turns an open boundary into a purely absorbing one, is quite common in interacting particle systems, we refer to [31] for an early result, where absorbing sites in dual processes were discussed. This also appears in the SSEP on a discrete segment with two open boundaries, see [32, 5, 10]. However, the analysis of the effect of reservoirs on the stationary state of the DCP substantially differs from the case of conservative particle systems such as the SSEP, where the system is driven out of equilibrium due to a current of particles produced by the effect of reservoirs tuned differently, see [8, 28, 15]. In our non-conservative setting, the system is constantly out of equilibrium.

2.1 The model

For $N \geq 1$, denote by $\Lambda_N = \{1, \dots, N\}$ the one dimensional finite chain of size N , which we refer to as the bulk. A site of Λ_N is either empty (in state 0), or occupied (in state 1). Therefore,

the state of the system is described by an element $\eta = (\eta_1, \dots, \eta_N) \in \Omega_N := \{0, 1\}^{\Lambda_N}$ such that for $1 \leq x \leq N$,

$$\eta_x = \begin{cases} 1 & \text{if } x \text{ is occupied} \\ 0 & \text{if } x \text{ is empty.} \end{cases} \quad (2.1)$$

In the bulk, the dynamics considered is the superposition of a contact process with parameter $\lambda > 0$, with a symmetric simple exclusion process (SSEP). Both these dynamics are of nearest-neighbor type and are defined as follows. For $x, y \in \Lambda_N$, $x \sim y$ means that x and y are neighboring sites in Λ_N .

In the contact process with parameter $\lambda > 0$, a site $x \in \Lambda_N$ becomes occupied at rate $\lambda \sum_{y \sim x} \eta_y$ if it is empty, and becomes empty at rate 1 if it is occupied. The generator of this dynamics acts on $f : \Omega_N \rightarrow \mathbb{R}$ as follows, for $\eta \in \Omega_N$:

$$\begin{aligned} \mathcal{L}^{\text{CP}} f(\eta) &= \sum_{x=2}^{N-1} [\eta_x + \lambda(1 - \eta_x)(\eta_{x-1} + \eta_{x+1})] [f(\eta^x) - f(\eta)] \\ &+ [\eta_1 + \lambda(1 - \eta_1)\eta_2] [f(\eta^1) - f(\eta)] \\ &+ [\eta_N + \lambda(1 - \eta_N)\eta_{N-1}] [f(\eta^N) - f(\eta)]. \end{aligned} \quad (2.2)$$

where for $x, y \in \Lambda_N$,

$$\eta_y^x = \begin{cases} \eta_y & \text{if } y \neq x \\ 1 - \eta_x & \text{if } y = x \end{cases} \quad (2.3)$$

represents the flip of the occupation variable at site $x \in \Lambda_N$. Note that this generator can be re-written as follows:

$$\mathcal{L}^{\text{CP}} f(\eta) = \frac{\eta_1}{2} [f(\eta^1) - f(\eta)] + \sum_{x=1}^{N-1} \mathcal{L}_{x,x+1}^{\text{CP}} f(\eta) + \frac{\eta_N}{2} [f(\eta^N) - f(\eta)] \quad (2.4)$$

with the bond generator

$$\begin{aligned} \mathcal{L}_{x,x+1}^{\text{CP}} f(\eta) &= \frac{1}{2} \eta_x [f(\eta^x) - f(\eta)] + \lambda(1 - \eta_x) \eta_{x+1} [f(\eta^x) - f(\eta)] \\ &+ \frac{1}{2} \eta_{x+1} [f(\eta^{x+1}) - f(\eta)] + \lambda(1 - \eta_{x+1}) \eta_x [f(\eta^{x+1}) - f(\eta)]. \end{aligned} \quad (2.5)$$

Finally, one could also highlight the action on the site instead of the bond, i.e. by defining, for $1 \leq x \leq N - 1$,

$$\mathcal{L}_{x,+}^{\text{CP}} f(\eta) = \frac{1}{2} \eta_x [f(\eta^x) - f(\eta)] + \lambda(1 - \eta_x) \eta_{x+1} [f(\eta^x) - f(\eta)] \quad (2.6)$$

and

$$\mathcal{L}_{x+1,-}^{\text{CP}} f(\eta) = \frac{1}{2} \eta_{x+1} [f(\eta^{x+1}) - f(\eta)] + \lambda(1 - \eta_{x+1}) \eta_x [f(\eta^{x+1}) - f(\eta)], \quad (2.7)$$

so that the bond action is

$$\mathcal{L}_{x,x+1}^{\text{CP}} f(\eta) = \mathcal{L}_{x,+}^{\text{CP}} f(\eta) + \mathcal{L}_{x+1,-}^{\text{CP}} f(\eta).$$

In the SSEP, particles jump to one of their neighboring sites, under the exclusion rule, namely each site can accommodate at most one particle, and the direction of the jump is not biased to the left or right. The generator of this dynamics acts on $f : \Omega_N \rightarrow \mathbb{R}$ as follows, for $\eta \in \Omega_N$:

$$\mathcal{L}^{\text{SSEP}} f(\eta) = \sum_{x=1}^{N-1} \mathcal{L}_{x,x+1}^{\text{SSEP}} f(\eta), \quad \text{with } \mathcal{L}_{x,x+1}^{\text{SSEP}} f(\eta) = f(\eta^{x,x+1}) - f(\eta), \quad (2.8)$$

where for $x, y \in \{1, \dots, N\}$,

$$\eta_z^{x,y} = \begin{cases} \eta_z & \text{if } z \notin \{x, y\} \\ \eta_y & \text{if } z = x \\ \eta_x & \text{if } z = y. \end{cases} \quad (2.9)$$

The generator of the bulk dynamics resulting from the superposition of the contact process and the SSEP is therefore given by $\mathcal{L}^{\text{CP}} + \mathcal{L}^{\text{SSEP}}$.

We study the open boundary version of the DCP, namely we attach reservoirs of particles at each end point of the system, which introduce or remove a particle from sites 1 and N . These reservoirs are parameterized by four positive parameters $\alpha, \gamma, \delta, \beta$. A particle is introduced at site 1, resp. N , at rate α , resp. δ , provided the site is empty. A particle is removed from the system at site 1, resp. N , at rate γ , resp. β , provided the site is occupied (see Figure 2.1). The associated generators of these boundary dynamics act on $f : \Omega_N \rightarrow \mathbb{R}$ as follows, for $\eta \in \Omega_N$:

$$\mathcal{L}_- f(\eta) = \alpha(1 - \eta_1)[f(\eta^1) - f(\eta)] + \gamma\eta_1[f(\eta^1) - f(\eta)] \quad (2.10)$$

and

$$\mathcal{L}_+ f(\eta) = \delta(1 - \eta_N)[f(\eta^N) - f(\eta)] + \beta\eta_N[f(\eta^N) - f(\eta)]. \quad (2.11)$$

Note that we do not put an index ‘DCP’ in these generators since they are not specific to our dynamics.

Finally, the generator for the open DCP is given by

$$\mathcal{L}^{\text{DCP}} = \mathcal{L}_- + \mathcal{L}^{\text{CP}} + \mathcal{D}\mathcal{L}^{\text{SSEP}} + \mathcal{L}_+, \quad (2.12)$$

where $\mathcal{D} \geq 0$ is the diffusion parameter.

Since the DCP and the GDCP are particle systems on a finite lattice, and therefore with a finite state space, the duality function can be written as a matrix and the Markov generators of the processes can be written as intensity matrices [24]. Recall that an intensity matrix L is such that

$$L(x, y) \geq 0 \quad \text{for } x \neq y \quad \text{and} \quad \sum_y L(x, y) = 0.$$

For $x \neq y$, $L(x, y)$ represents the rate to go from state x to state y .

Remark 1 (Countable state space). *If both the original process $X = (X_t)_{t \geq 0}$ and the dual process $Y = (Y_t)_{t \geq 0}$ are Markov processes with a countable state space, the duality relation can be written in matrix notation as ([34])*

$$LD = D \left(L^{\text{dual}} \right)^T \quad (2.13)$$

where L is the intensity matrix corresponding to the Markov generator \mathcal{L} , L^{dual} is the intensity matrix corresponding to $\mathcal{L}^{\text{dual}}$ and the superscript T denotes matrix transposition. Equivalently one could choose to adopt the quantum Hamiltonian representation [25, 28] of the intensity matrix, namely $H = -L^T$, and in this case duality reads $H^T D = D H^{\text{dual}}$, see [26].

Therefore, as the state space of the DCP is finite, one can encode the dynamics in a finite sized matrix. To write these matrices in the canonical basis $(e_i)_{1 \leq i \leq 2^N}$ of \mathbb{R}^{2^N} , to each vector e_i , we associate the unique configuration $\eta^i = (\eta_1^i, \dots, \eta_N^i)$ such that

$$i = 1 + \sum_{k=1}^N 2^{N-k} \eta_k^i. \quad (2.14)$$

For example, for $N = 2$, the order is

$$\begin{aligned} e_1 &= (1, 0, 0, 0) \text{ associated to } (0, 0), & e_2 &= (0, 1, 0, 0) \text{ associated to } (0, 1), \\ e_3 &= (0, 0, 1, 0) \text{ associated to } (1, 0) & \text{and } e_4 &= (0, 0, 0, 1) \text{ associated to } (1, 1). \end{aligned}$$

With this ordering of configurations, the bulk dynamics matrix writes

$$L^{\text{CP}} + \mathcal{D}L^{\text{SSEP}} = \sum_{x=1}^{N-1} \ell_x + \tilde{\ell}_1 + \tilde{\ell}_N, \quad (2.15)$$

where for $x = 1, \dots, N-1$, denoting by \otimes the Kronecker product and $\mathbf{1}$ the two-dimensional unit matrix,

$$\ell_x := \mathbf{1}^{\otimes(x-1)} \otimes \ell \otimes \mathbf{1}^{\otimes(N-1-x)}, \quad (2.16)$$

with ℓ corresponding to $\mathcal{L}_{x,x+1}^{\text{CP}} + \mathcal{D}\mathcal{L}_{x,x+1}^{\text{SSEP}}$:

$$\ell = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/2 & -(1/2 + \mathcal{D} + \lambda) & \mathcal{D} & \lambda \\ 1/2 & \mathcal{D} & -(1/2 + \mathcal{D} + \lambda) & \lambda \\ 0 & 1/2 & 1/2 & -1 \end{pmatrix},$$

and, for $x = 1$ and $x = N$, $\tilde{\ell}_1$ and $\tilde{\ell}_N$ correspond to the first and last terms in the right hand side of (2.4) with,

$$\tilde{\ell}_1 := \tilde{\ell} \otimes \mathbf{1}^{\otimes(N-1)} \quad \text{and} \quad \tilde{\ell}_N := \mathbf{1}^{\otimes(N-1)} \otimes \tilde{\ell} \quad (2.17)$$

where

$$\tilde{\ell} = \begin{pmatrix} 0 & 0 \\ 1/2 & -1/2 \end{pmatrix}.$$

Regarding the boundary dynamics the matrices ℓ_1^- and ℓ_1^+ correspond respectively to \mathcal{L}_- in (2.10) and \mathcal{L}_+ in (2.11), with

$$\ell_1^- := \ell^- \otimes \mathbf{1}^{\otimes(N-1)}, \quad \text{and} \quad \ell_1^+ := \mathbf{1}^{\otimes(N-1)} \otimes \ell^+, \quad (2.18)$$

where

$$\ell^- = \begin{pmatrix} -\alpha & \alpha \\ \gamma & -\gamma \end{pmatrix}, \quad \ell^+ = \begin{pmatrix} -\delta & \delta \\ \beta & -\beta \end{pmatrix}.$$

In total, the intensity matrix of the DCP dynamics is given by

$$L^{\text{DCP}} = \sum_{x=1}^{N-1} \ell_x + \tilde{\ell}_1 + \tilde{\ell}_N + \ell_1^- + \ell_1^+. \quad (2.19)$$

The dynamics of the DCP is irreducible and as the state space Ω_N is finite, the process admits a unique invariant measure ν^{DCP} , which depends on $\alpha, \delta, \gamma, \beta, \mathcal{D}$ and λ . As soon as $\alpha \neq 0$ or $\delta \neq 0$, the empty configuration is no longer an absorbing state and ν^{DCP} cannot be found by direct computations when N is large.

2.2 Duality for the DCP

Introduce the dual space

$$\Omega_N^{\text{dual}} = \mathbb{N}_0 \times \{0, 1\}^{\Lambda_N} \times \mathbb{N}_0,$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ is the set of non-negative integers.

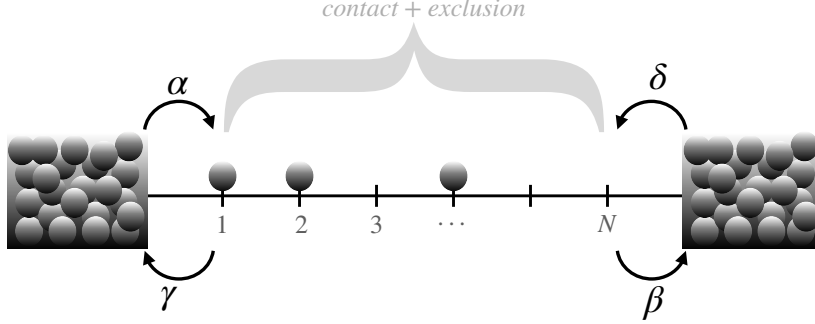


Figure 1: Contact + exclusion process on Λ_N with reservoirs.

Theorem 1. *The open DCP $(\eta_t)_{t \geq 0}$ with generator (2.12) is dual to a purely absorbing contact process $(\xi_t)_{t \geq 0}$ with duality function $D : \Omega_N \times \Omega_N^{\text{dual}} \rightarrow \mathbb{R}$, given by*

$$D(\eta, \xi) = \left(\frac{\gamma}{\alpha + \gamma} \right)^{\xi_0} H(\eta, \xi) \left(\frac{\delta}{\beta + \delta} \right)^{\xi_{N+1}}, \quad (2.20)$$

where the bulk duality function is:

$$H(\eta, \xi) = \prod_{x \in A(\xi)} (1 - \eta_x) = \prod_{x \in \Lambda_N} (1 - \eta_x)^{\xi_x}, \quad (2.21)$$

with $A(\xi) = \{y \in \{1, \dots, N\}, \xi_y = 1\}$. The dual generator is given by

$$\mathcal{L}^{\text{DCP, Dual}} = \mathcal{L}_-^{\text{Dual}} + \mathcal{L}^{\text{CP}} + \mathcal{D}\mathcal{L}^{\text{SSEP}} + \mathcal{L}_+^{\text{Dual}}, \quad (2.22)$$

where \mathcal{L}^{CP} is the generator of the contact process defined in (2.2), $\mathcal{L}^{\text{SSEP}}$ is the generator of the exclusion process defined in (2.8) and the action at the boundary on function $f : \Omega_N^{\text{dual}} \rightarrow \mathbb{R}$ is

$$\mathcal{L}_-^{\text{Dual}} f(\xi) = (\alpha + \gamma) \xi_1 [f(\xi^{1,0}) - f(\xi)] \quad (2.23)$$

$$\mathcal{L}_+^{\text{Dual}} f(\xi) = (\delta + \beta) \xi_N [f(\xi^{N,N+1}) - f(\xi)]. \quad (2.24)$$

We denote the reservoir densities by

$$c_- := \frac{\gamma}{\alpha + \gamma}, \quad \text{and} \quad c_+ := \frac{\delta}{\beta + \delta}. \quad (2.25)$$

From now on, in this section, the letter η will be relative to a DCP and ξ to the dual process introduced in Theorem 1.

First, notice that in our case Definition 1 is equivalent to the following relation involving the corresponding Markov generators of the processes:

Proposition 1. ([24, Theorem 3.42]) *Let \mathcal{L} the generator of the process X and $\mathcal{L}^{\text{dual}}$ the generator of the process Y . Then X and Y are dual with local duality function D if*

$$(\mathcal{L}D(\cdot, x))(y) = (\mathcal{L}^{\text{dual}}D(y, \cdot))(x). \quad (2.26)$$

for all $x \in \Omega$ and $y \in \Omega^{\text{dual}}$.

It is known in the literature that both the contact and the exclusion processes with closed boundary conditions are self-dual with duality function (2.21), see in particular [23, Chapter 3, Section 4] and [20, Chapter 5]. This leads to a self-duality relation for the bulk of the DCP, namely we have that for any $\eta, \xi \in \Omega_N$,

$$(\mathcal{L}^{\text{CP}} + \mathcal{L}^{\text{SSEP}})H(\eta, \cdot)(\xi) = (\mathcal{L}^{\text{CP}} + \mathcal{L}^{\text{SSEP}})H(\cdot, \xi)(\eta). \quad (2.27)$$

Remark 2. As noticed in [29], for the system with either closed or periodic boundary conditions, given $a, b, c, d \in \mathbb{R}$, the function

$$\tilde{H}(\eta, \xi) = \prod_{x \in \Lambda_N} [a + b\eta_x]^{c+d\xi_x}$$

is always a self-duality function for the exclusion process. The case $a = 1, b = -1, c = 0$ and $d = 1$ yields $\tilde{H} = H$, the self-duality function for the contact process.

Proof of Theorem 1. Taking into account Proposition 1, and by equation (2.27), we are left to show that

$$(\mathcal{L}_- + \mathcal{L}_+)D(\cdot, \xi)(\eta) = (\mathcal{L}_-^{\text{Dual}} + \mathcal{L}_+^{\text{Dual}})D(\eta, \cdot)(\xi),$$

where we recall that $\mathcal{L}_-^{\text{Dual}}$ is defined in (2.23), $\mathcal{L}_+^{\text{Dual}}$ in (2.24), \mathcal{L}_- in (2.10) and \mathcal{L}_+ in (2.11). Without loss of generality, we prove the result for the left reservoir, i.e.

$$\mathcal{L}_-D(\cdot, \xi)(\eta) = \mathcal{L}_-^{\text{Dual}}D(\eta, \cdot)(\xi),$$

and the proof that $\mathcal{L}_+D(\cdot, \xi)(\eta) = \mathcal{L}_+^{\text{Dual}}D(\eta, \cdot)(\xi)$ follows the same steps. We have

$$\mathcal{L}_-D(\cdot, \xi)(\eta) = \alpha(1 - \eta_1)[D(\eta^1, \xi) - D(\eta, \xi)] + \gamma\eta_1[D(\eta^1, \xi) - D(\eta, \xi)].$$

Now,

$$\begin{aligned} D(\eta^1, \xi) &= \left(\frac{\gamma}{\alpha + \gamma}\right)^{\xi_0} \prod_{x \in A(\xi)} (1 - \eta_x^1) \left(\frac{\delta}{\beta + \delta}\right)^{\xi_{N+1}} \\ &= \left(\frac{\gamma}{\alpha + \gamma}\right)^{\xi_0} \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \left(\frac{\delta}{\beta + \delta}\right)^{\xi_{N+1}} \eta_1 \mathbf{1}_{1 \in A(\xi)} + D(\eta, \xi) \mathbf{1}_{1 \notin A(\xi)}. \end{aligned}$$

Therefore,

$$\begin{aligned} D(\eta^1, \xi) - D(\eta, \xi) &= D(\eta^1, \xi) - D(\eta, \xi) \mathbf{1}_{1 \in A(\xi)} - D(\eta, \xi) \mathbf{1}_{1 \notin A(\xi)} \\ &= \left(\frac{\gamma}{\alpha + \gamma}\right)^{\xi_0} \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \left(\frac{\delta}{\beta + \delta}\right)^{\xi_{N+1}} \eta_1 \mathbf{1}_{1 \in A(\xi)} - D(\eta, \xi) \mathbf{1}_{1 \in A(\xi)} \\ &= \left(\frac{\gamma}{\alpha + \gamma}\right)^{\xi_0} (2\eta_1 - 1) \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \left(\frac{\delta}{\beta + \delta}\right)^{\xi_{N+1}} \mathbf{1}_{1 \in A(\xi)}. \end{aligned}$$

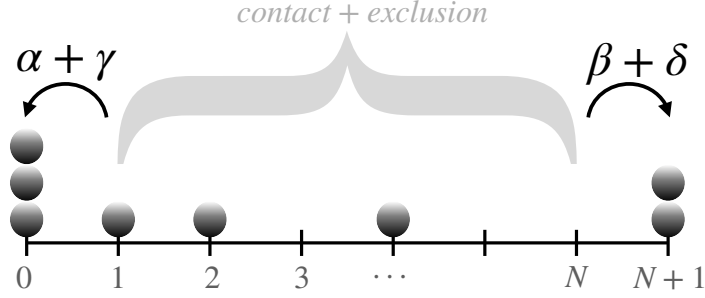


Figure 2: Diffusive contact process with absorbing reservoirs.

Finally, we are left with

$$\begin{aligned}
\mathcal{L}_- D(\cdot, \xi)(\eta) &= \alpha(1 - \eta_1) \left(\frac{\gamma}{\alpha + \gamma} \right)^{\xi_0} (2\eta_1 - 1) \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \left(\frac{\delta}{\beta + \delta} \right)^{\xi_{N+1}} \mathbf{1}_{1 \in A(\xi)} \\
&+ \gamma \eta_1 \left(\frac{\gamma}{\alpha + \gamma} \right)^{\xi_0} (2\eta_1 - 1) \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \left(\frac{\delta}{\beta + \delta} \right)^{\xi_{N+1}} \mathbf{1}_{1 \in A(\xi)} \\
&= -\alpha D(\eta, \xi) \mathbf{1}_{1 \in A(\xi)} + \gamma \left(\frac{\gamma}{\alpha + \gamma} \right)^{\xi_0} \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \left(\frac{\delta}{\beta + \delta} \right)^{\xi_{N+1}} \eta_1 \mathbf{1}_{1 \in A(\xi)} \\
&= -(\alpha + \gamma) D(\eta, \xi) \mathbf{1}_{1 \in A(\xi)} + (\alpha + \gamma) \left(\frac{\gamma}{\alpha + \gamma} \right)^{\xi_0 + 1} \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \left(\frac{\delta}{\beta + \delta} \right)^{\xi_{N+1}} \mathbf{1}_{1 \in A(\xi)} \\
&= (\alpha + \gamma) \xi_1 \left[D(\eta, \xi^{1,0}) - D(\eta, \xi) \right],
\end{aligned}$$

which is exactly the dual absorbing generator of the left boundary. \square

2.3 Correlation functions via duality

In order to provide a better understanding of the invariant measure ν^{DCP} of the DCP with absorbing boundaries, we study its correlation functions thanks to the duality relation established in Theorem 1. The dual process is seemingly simpler, as it almost surely becomes extinct. We are thus able to express the correlation functions of the original process in terms of absorption probabilities of the dual process. This allows to get an upper bound for the one point correlation function, see subsection 2.3.1. However, the explicit computation of these correlation functions relies on solving large systems of linear equations, see subsection 2.3.2, and it is not clear whether they are simpler to solve than the ones obtained in the appendices A and B, through a more standard approach based on solving Fokker-Planck equations.

In the following, for $1 \leq \ell \leq N$, the ℓ -point correlation function of ν^{DCP} is the function ρ_ℓ^{DCP} defined by

$$\forall x_1 < \dots < x_\ell \in \Lambda_N, \rho_\ell^{\text{DCP}}(x_1, \dots, x_\ell) := \mathbb{E}_{\nu^{\text{DCP}}} [\eta_{x_1} \dots \eta_{x_\ell}] \quad (2.28)$$

with the convention $\rho_0^{\text{DCP}} = 1$. Introduce τ_N the extinction time of the DCP $(\xi(t))_{t \geq 0}$ with purely absorbing boundaries:

$$\tau_N := \inf \{t \geq 0, \forall x \in \{1, \dots, N\}, \xi_x(t) = 0\}. \quad (2.29)$$

Also, introduce the total number of particles absorbed on the left and right hand side boundary:

$$\xi_0(\infty) := \lim_{t \rightarrow \infty} \xi_0(t), \quad \text{and} \quad \xi_{N+1}(\infty) := \lim_{t \rightarrow \infty} \xi_{N+1}(t), \quad (2.30)$$

where the limits are almost sure. As the process almost surely becomes extinct, τ_N , $\xi_0(\infty)$ and $\xi_{N+1}(\infty)$ are almost surely finite and $\xi_0(\infty) = \xi_0(\tau_N)$, and $\xi_{N+1}(\infty) = \xi_{N+1}(\tau_N)$ almost surely.

2.3.1 One point correlation functions

For $y \in \Lambda_N$, denote by δ_y the element of Ω_N^{dual} where there is a particle in site $y \in \Lambda_N$ and all other sites $x \neq y$ are empty.

Proposition 2. *The one point correlation function of ν^{DCP} is given by*

$$\forall y \in \Lambda_N, \rho_1^{\text{DCP}}(y) = \mathbb{E}_{\nu^{\text{DCP}}}[\eta_y] = 1 - \mathbb{E}_{\delta_y}[c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)}] = 1 - \mathbb{E}_{\delta_y}[c_-^{\xi_0(\tau_N)} c_+^{\xi_{N+1}(\tau_N)}]. \quad (2.31)$$

Proof. By definition (2.20) of the duality function D given in Theorem 1, if we choose as dual process $\xi(t) = \delta_y(t)$ we have

$$D(\eta, \delta_y) = 1 - \eta_y.$$

The duality relation (2.26) then yields that for any $t \geq 0$,

$$\int \mathbb{E}_{\eta}[D(\eta(t), \delta_y)] \nu^{\text{DCP}}(d\eta) = \int \mathbb{E}_{\delta_y}[D(\eta, \xi(t))] \nu^{\text{DCP}}(d\eta). \quad (2.32)$$

By invariance of ν^{DCP} , the left hand side in (2.32) does not depend on t and equals

$$\int \mathbb{E}_{\eta}[D(\eta, \delta_y)] \nu^{\text{DCP}}(d\eta) = 1 - \mathbb{E}_{\nu^{\text{DCP}}}[\eta_y], \quad (2.33)$$

while the right hand equals

$$\int \mathbb{E}_{\delta_y}[D(\eta, \xi(t))] \nu^{\text{DCP}}(d\eta) \xrightarrow{t \rightarrow \infty} \mathbb{E}_{\delta_y}[c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)}], \quad (2.34)$$

since for $t \rightarrow \infty$ all dual particles will eventually be absorbed at the boundary. Collecting (2.33) and (2.34) yields the result. \square

Although the dual process is seemingly simpler than the original one, it remains a non-conservative process and the total number of particles absorbed by the reservoirs can reach arbitrary values. Therefore, we are not able to provide an explicit expression of the right hand side term in (2.31), contrary to the case of conservative particle systems where exact formulas are available, we refer to section 2.3 of [12] and Theorem 2.2 of [11], where a closed formula for the n points correlation is found in the non-equilibrium steady state of two models with open boundary and the same purely absorbing dual.

However, note that the dual process $(\xi(t))_{t \geq 0}$ either dies out before any particle lying in the bulk has had time to reach the boundary, or, it dies out and at least one particle has been absorbed by the boundary reservoirs. Thanks to this observation, the following upper bound holds:

Lemma 1. For $y \in \Lambda_N$, denote by $A_y(\infty)$ the following event:

$$A_y(\infty) = \left(\xi(0) = \delta_y, c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)} < 1 \right),$$

that is, starting from a single dual particle at site y , at least one particle is absorbed by the boundary of the system. Then,

$$\rho_1^{\text{DCP}}(y) \leq \mathbb{P}_{\delta_y}[A_y(\infty)]. \quad (2.35)$$

Proof. We have

$$\begin{aligned} \mathbb{E}_{\delta_y}[c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)}] &= \mathbb{P}_{\delta_y}[A_y(\infty)^c] + \mathbb{E}_{\delta_y}[c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)} \mathbb{1}_{A_y(\infty)}] \\ &= 1 - \mathbb{P}_{\delta_y}[A_y(\infty)] + \mathbb{E}_{\delta_y}[c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)} \mathbb{1}_{A_y(\infty)}], \end{aligned}$$

therefore,

$$\rho_1^{\text{DCP}}(y) = 1 - \mathbb{E}_{\delta_y}[c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)}] \leq \mathbb{P}_{\delta_y}[A_y(\infty)].$$

□

Remark 3. This inequality is of interest if the upper bound is small, that is, when the probability that starting from a particle at y , the process reaches the boundary before it becomes extinct, is small. Heuristically, if the time of extinction τ_N of the process is small, then the particle should not have time to reach the boundary. The extinction time is small when λ is small. Precisely, an estimate of τ_N is known in the case of a subcritical contact process on Λ_N (without exclusion), and we refer to [22, Part 1, Theorem 3.3].

2.3.2 Higher order correlation functions

The idea of relating the ℓ points correlations using the dual absorption probabilities goes back to [13] for a model of stochastic diffusion of energy. In the same spirit as in Proposition 2 in [13] we can show the following results for the DCP model. Given $1 \leq \ell \leq N$ and sites $1 \leq x_1 < \dots < x_\ell \leq N$, denote by $\delta_{x_1, x_2, \dots, x_\ell}$, the element of Ω_N^{dual} where there is a particle at sites x_1, \dots, x_ℓ , and none elsewhere.

Proposition 3. For any $1 \leq \ell \leq N$ and any $1 \leq x_1 < \dots < x_\ell \leq N$,

$$\begin{aligned} \mathbb{E}_{\nu^{\text{DCP}}}\left[\prod_{j=1}^{\ell} (1 - \eta_{x_j})\right] &= \sum_{k=0}^{\ell} (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq \ell} \rho_k^{\text{DCP}}(x_{i_1}, \dots, x_{i_k}) \\ &= \sum_{m, n \geq 0} \mathbb{P}_{\delta_{x_1, x_2, \dots, x_\ell}}[\xi_0(\infty) = m, \xi_N(\infty) = n] c_-^m c_+^n. \end{aligned} \quad (2.36)$$

Proof. For simplicity, we consider that only the left reservoir interacts with the bulk, that is, $\delta = \beta = 0$, but the reasoning remains the same if that is not the case. Our goal is therefore to prove that for any $1 \leq x_1 < \dots < x_\ell \leq N$,

$$\mathbb{E}_{\nu^{\text{DCP}}}\left[\prod_{j=1}^{\ell} (1 - \eta_{x_j})\right] = \sum_{k \geq 0} \mathbb{P}_{\delta_{x_1, x_2, \dots, x_\ell}}[\xi_0(\infty) = k] c_-^k. \quad (2.37)$$

In this case, it follows that the duality relation (1.1) holds with duality function

$$D(\eta, \xi) = c_-^{\xi_0} \prod_{x \in \Lambda_N} (1 - \eta_x)^{\xi_x}. \quad (2.38)$$

In other words, the initial process is dual to a DCP with an absorbing reservoir with rate $\alpha + \gamma$. Then, proceeding as in the proof of Proposition 2.31, we consider the duality relation (1.1) under the invariant measure for the original process and, taking $\xi = \delta_{x_1, x_2, \dots, x_\ell}$, we have:

$$\mathbb{E}_{\nu^{\text{DCP}}} \left[(1 - \eta_{x_1}) (1 - \eta_{x_2}) \dots (1 - \eta_{x_\ell}) \right] = \mathbb{E}_{\delta_{x_1, x_2, \dots, x_\ell}} \left[c_-^{\xi_0(\infty)} \right]. \quad (2.39)$$

Expanding both sides of (2.39), we are left with

$$\sum_{k=0}^{\ell} (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq \ell} \rho_k^{\text{DCP}}(x_{i_1}, \dots, x_{i_k}) = \sum_{k \geq 0} \mathbb{P}_{\delta_{x_1, x_2, \dots, x_\ell}} \left[\xi_0(\infty) = k \right] c_-^k.$$

□

Below, we show that, if we consider a small bulk, we can explicitly compute the absorption probabilities. The idea is to perform a conditioning on the first step of the dual dynamics, and to get an explicit expression for the absorption probabilities of the dual process.

$N = 1$ site

The behavior of the bulk is conservative as the contact dynamics has no effect here (no possible birth). This provides a finite sum in the expression of the correlation function (2.37) obtained through duality, allowing us to compute it conveniently. By (2.31), we have

$$\mathbb{E}_{\nu^{\text{DCP}}} [1 - \eta_1] = 1 - \rho_1^{\text{DCP}}(1) = \mathbb{P}_{\delta_1}(\xi_0(\infty) = 0) c_-^0 + \mathbb{P}_{\delta_1}(\xi_0(\infty) = 1) c_-.$$

To compute the absorption probabilities $\mathbb{P}_{\delta_1}(\xi_0(\infty) = 0)$ and $\mathbb{P}_{\delta_1}(\xi_0(\infty) = 1)$, we perform a conditioning on the first possible outcome of the process, starting from a particle at site 1:

$$\begin{aligned} \mathbb{P}_{\delta_1}(\xi_0(\infty) = 0) &= \mathbb{P}_{\delta_1}(\xi_0(\infty) = 0 \mid \text{left jump}) \mathbb{P}_{\delta_1}(\text{left jump}) \\ &\quad + \mathbb{P}_{\delta_1}(\xi_0(\infty) = 0 \mid \text{it dies}) \mathbb{P}_{\delta_1}(\text{it dies}) \\ &= 0 \cdot \frac{\alpha + \gamma}{\alpha + \gamma + 1} + 1 \cdot \frac{1}{\alpha + \gamma + 1} \\ &= \frac{1}{\alpha + \gamma + 1} \end{aligned}$$

and,

$$\begin{aligned} \mathbb{P}_{\delta_1}(\xi_0(\infty) = 1) &= \mathbb{P}_{\delta_1}(\xi_0(\infty) = 1 \mid \text{left jump}) \mathbb{P}_{\delta_1}(\text{left jump}) \\ &\quad + \mathbb{P}_{\delta_1}(\xi_0(\infty) = 1 \mid \text{it dies}) \mathbb{P}_{\delta_1}(\text{it dies}) \\ &= 1 \cdot \frac{\alpha + \gamma}{\alpha + \gamma + 1} + 0 \cdot \frac{1}{\alpha + \gamma + 1} \\ &= \frac{\alpha + \gamma}{\alpha + \gamma + 1} \end{aligned}$$

if there is one particle absorbed. Therefore,

$$\rho_1^{\text{DCP}}(1) = \frac{\alpha}{\alpha + \gamma + 1},$$

which matches what we get in Appendix B, for $N = 1$, through the Fokker-Planck equations.

$N = 2$ sites

This is the first non trivial case. In addition to the dynamics for one bulk site, the effect of the exclusion process (with rate \mathcal{D}) as well as the effect of the contact process (with rate λ) are present, therefore we have an infinite sum in equation (2.37), as expressed below in (2.40). Recall the expressions of the correlation functions (2.28) provided by the duality relations (2.31), and (2.37):

$$\begin{cases} \rho_1^{\text{DCP}}(1) &= 1 - \sum_{k \geq 0} \mathbb{P}_{\delta_1}(\xi_0(\infty) = k) c_-^k \\ \rho_2^{\text{DCP}}(1, 2) &= 1 - \sum_{k \geq 0} \mathbb{P}_{\delta_{1,2}}(\xi_0(\infty) = k) c_-^k \\ \rho_1^{\text{DCP}}(2) &= 1 - \sum_{k \geq 0} \mathbb{P}_{\delta_2}(\xi_0(\infty) = k) c_-^k. \end{cases} \quad (2.40)$$

To compute the absorption probabilities appearing in the infinite sums, as in the case $N = 1$, we condition on the first possible outcome of the process, starting from a particle at site 1, 2 or at both sites. For $k \geq 0$, we define

$$x_1^k := \mathbb{P}_{\delta_1}[\xi_0(\infty) = k], \quad x_2^k := \mathbb{P}_{\delta_{1,2}}[\xi_0(\infty) = k], \quad \text{and} \quad x_3^k := \mathbb{P}_{\delta_2}[\xi_0(\infty) = k],$$

the probabilities that $k \geq 0$ dual particles are absorbed in the left reservoir starting from a particle at site 1, 2 or at both sites. To alleviate notation we do not write the dependence on the model parameters α, γ, λ and \mathcal{D} on these absorption probabilities. For $k \geq 0$, define the column vector

$$C_k = \begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \end{pmatrix}. \quad (2.41)$$

- For $k = 0$, the conditioning on the first step leads to the following system:

$$\begin{cases} x_1^0 - \frac{\lambda}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_2^0 - \frac{\mathcal{D}}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_3^0 = \frac{1}{\alpha + \gamma + \mathcal{D} + \lambda + 1} \\ \frac{-1}{\alpha + \gamma + 2} x_1^0 + x_2^0 - \frac{1}{\alpha + \gamma + 2} x_3^0 = 0 \\ \frac{-\mathcal{D}}{\mathcal{D} + \lambda + 1} x_1^0 - \frac{\lambda}{\mathcal{D} + \lambda + 1} x_2^0 + x_3^0 = \frac{1}{\mathcal{D} + \lambda + 1}, \end{cases} \quad (2.42)$$

that is,

$$M_2 C_0 = \begin{pmatrix} \frac{1}{\alpha + \gamma + \mathcal{D} + \lambda + 1} \\ 0 \\ \frac{1}{\mathcal{D} + \lambda + 1} \end{pmatrix}, \quad (2.43)$$

where

$$M_2 := \begin{pmatrix} 1 & \frac{-\lambda}{\alpha + \gamma + \mathcal{D} + \lambda + 1} & \frac{-\mathcal{D}}{\alpha + \gamma + \mathcal{D} + \lambda + 1} \\ \frac{-1}{\alpha + \gamma + 2} & 1 & \frac{-1}{\alpha + \gamma + 2} \\ \frac{-\mathcal{D}}{\mathcal{D} + \lambda + 1} & \frac{-\lambda}{\mathcal{D} + \lambda + 1} & 1 \end{pmatrix}, \quad (2.44)$$

and where 2 in M_2 refers to the fact that $N = 2$. We have

$$d := \det M_2 = \frac{\tilde{d}}{(\mathcal{D} + \lambda + 1)(\alpha + \gamma + 2)(\mathcal{D} + \alpha + \gamma + \lambda + 1)}, \quad (2.45)$$

where

$$\tilde{d} := \mathcal{D}\alpha + (\alpha + \gamma)[(\alpha + \gamma + \lambda + 2)(\lambda + 1) + 1] + 2(\lambda + 1), \quad (2.46)$$

with

$$A := (\alpha + \gamma + \lambda + 2)^2 - \lambda(\lambda + 4) = (\alpha + \gamma + 2)^2 + 2\lambda(\alpha + \gamma) \quad (2.47)$$

So $\det M_2 \neq 0$, and solving (2.43) by inverting M_2 yields

$$\begin{cases} x_1^0 = \tilde{d}^{-1}(2\mathcal{D} + \lambda + 1)(\alpha + \gamma + 2) \\ x_2^0 = \tilde{d}^{-1}[4\mathcal{D} + \alpha + \gamma + 2\lambda + 2] \\ x_3^0 = \tilde{d}^{-1}(2\mathcal{D} + \alpha + \gamma + \lambda + 1)(\alpha + \gamma + 2). \end{cases} \quad (2.48)$$

- For $k = 1$, the same conditioning leads to

$$\begin{cases} x_1^1 - \frac{\lambda}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_2^1 - \frac{\mathcal{D}}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_3^1 = \frac{\alpha + \gamma}{\alpha + \gamma + \mathcal{D} + \lambda + 1} \\ \frac{-1}{\alpha + \gamma + 2} x_1^1 + x_2^1 - \frac{1}{\alpha + \gamma + 2} x_3^1 = \frac{\alpha + \gamma}{\alpha + \gamma + 2} x_3^0 \\ \frac{-\mathcal{D}}{\mathcal{D} + \lambda + 1} x_1^1 - \frac{\lambda}{\mathcal{D} + \lambda + 1} x_2^1 + x_3^1 = 0, \end{cases} \quad (2.49)$$

that is,

$$M_2 C_1 = \begin{pmatrix} \frac{\alpha + \gamma}{\alpha + \gamma + \mathcal{D} + \lambda + 1} \\ \frac{\alpha + \gamma}{\alpha + \gamma + 2} x_3^0 \\ 0 \end{pmatrix} \quad (2.50)$$

Again, as $\det M_2 \neq 0$, we can invert this and, to lighten the formulas of the solution, we introduce the notation

$$\varphi := \mathcal{D}B + \lambda^2 + \lambda(\alpha + \gamma + 2) + \alpha + \gamma + 1, \quad (2.51)$$

where

$$B := (\alpha + \gamma + 2\lambda + 2). \quad (2.52)$$

We get:

$$\begin{cases} x_1^1 = \tilde{d}^{-1}(\alpha + \gamma)(\alpha + \gamma + 2) \left[\mathcal{D} + 1 + \frac{\lambda(\alpha + \gamma + 1)}{\alpha + \gamma + 2} \right. \\ \quad \left. + \lambda \tilde{d}^{-1}(2\mathcal{D} + \lambda + 1)(2\mathcal{D} + \alpha + \gamma + \lambda + 1) \right] \\ x_2^1 = \tilde{d}^{-1}(\alpha + \gamma)(\alpha + \gamma + 2) \left[\frac{2\mathcal{D} + \lambda + 1}{\alpha + \gamma + 2} + \tilde{d}^{-1}\varphi(2\mathcal{D} + \alpha + \gamma + 1 + \lambda) \right] \\ x_3^1 = \tilde{d}^{-1}(\alpha + \gamma)(\alpha + \gamma + 2) \left[\mathcal{D} + \frac{\lambda}{\alpha + \gamma + 2} + \tilde{d}^{-1}\lambda(2\mathcal{D} + \lambda + \alpha + \gamma + 1)^2 \right]. \end{cases} \quad (2.53)$$

- For $k = 2$, the same conditioning leads to

$$\begin{cases} x_1^k - \frac{\lambda}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_2^k - \frac{\mathcal{D}}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_3^k = 0 \\ \frac{-1}{\alpha + \gamma + 2} x_1^k + x_2^k - \frac{1}{\alpha + \gamma + 2} x_3^k = \frac{\alpha + \gamma}{\alpha + \gamma + 2} x_3^{k-1} \\ \frac{-\mathcal{D}}{\mathcal{D} + \lambda + 1} x_1^k - \frac{\lambda}{\mathcal{D} + \lambda + 1} x_2^k + x_3^k = 0 \end{cases} \quad (2.54)$$

that is,

$$M_2 C_k = \begin{pmatrix} 0 \\ \frac{\alpha+\gamma}{\alpha+\gamma+2} x_3^{k-1} \\ 0 \end{pmatrix} \quad (2.55)$$

Again, inverting M_2 , using the definition of φ given in (2.51), and introducing

$$\psi := \mathcal{D}(\alpha + \gamma + 2) + \lambda + \tilde{d}^{-1} \lambda (\alpha + \gamma + 2) (2\mathcal{D} + \alpha + \gamma + \lambda + 1)^2 \quad (2.56)$$

we get:

$$\begin{cases} x_1^2 = \tilde{d}^{-2} \lambda (\alpha + \gamma)^2 (2\mathcal{D} + \lambda + 1) \psi \\ x_2^2 = \tilde{d}^{-2} (\alpha + \gamma)^2 \varphi \psi \\ x_3^2 = \tilde{d}^{-2} \lambda (\alpha + \gamma)^2 (2\mathcal{D} + \alpha + \gamma + \lambda + 1) \psi. \end{cases} \quad (2.57)$$

- For $k = 3$, the same reasoning as for the cases above yields

$$\begin{cases} x_1^3 = \tilde{d}^{-3} \lambda^2 (\alpha + \gamma)^3 (2\mathcal{D} + \lambda + 1) (2\mathcal{D} + \alpha + \gamma + \lambda + 1) \psi \\ x_2^3 = \tilde{d}^{-3} \lambda (\alpha + \gamma)^3 (2\mathcal{D} + \alpha + \gamma + \lambda + 1) \varphi \psi \\ x_3^3 = \tilde{d}^{-3} \lambda^2 (\alpha + \gamma)^3 (2\mathcal{D} + \alpha + \gamma + \lambda + 1)^2 \psi. \end{cases} \quad (2.58)$$

- For $k > 3$, using the same conditioning and reasoning recursively, we find that

$$\begin{cases} x_1^k = \tilde{d}^{-k} \lambda^{(k-2)} (\alpha + \gamma)^k (2\mathcal{D} + \lambda + 1) (2\mathcal{D} + \alpha + \gamma + \lambda + 1)^{(k-3)} \varphi \psi \\ x_2^k = \tilde{d}^{-k} \lambda^{(k-3)} (\alpha + \gamma)^k (2\mathcal{D} + \alpha + \gamma + \lambda + 1)^{(k-3)} \varphi^2 \psi \\ x_3^k = \tilde{d}^{-k} \lambda^{(k-2)} (\alpha + \gamma)^k (2\mathcal{D} + \alpha + \gamma + \lambda + 1)^{(k-2)} \varphi \psi. \end{cases} \quad (2.59)$$

Finally, we are left to plug in the terms x_1^k , x_2^k and x_3^k in (2.40). We refer to Appendix B where an alternative explicit expression of $\rho_1(1)$, $\rho_1(2)$ and $\rho_2(1, 2)$ is computed using the Fokker-Planck equations.

N sites

Generalizing the above strategy to N sites and showing that the corresponding matrix M_N is invertible would allow us to completely determine the correlation functions. To this end, we stress that the order of the unknown variable is chosen in an arbitrary way. One possibility to systematically ordering the variables is the following, defined recursively for N sites. There are

$\sum_{k=1}^N \binom{N}{k} = 2^N - 1$ total number of configurations to order: the first 2^{N-1} unknown variables

have a particle at site 1 and the rest is completed by a configuration from the $N - 1$ th ordering. The $2^{N-1} - 1$ coordinates left are the ones such that site 1 is empty and the rest is completed by a configuration from the $N - 1$ th ordering. More explicitly, for $N = 2$ sites we have $\xi^1 = \delta_1$, $\xi^2 = \delta_{1,2}$, $\xi^3 = \delta_2$ which is the order we used (See Figure 3). For $N = 3$, this rule would give us the ordering $\xi^1 = \delta_1$, $\xi^2 = \delta_{1,2}$, $\xi^3 = \delta_{1,2,3}$, $\xi^4 = \delta_{1,3}$, $\xi^5 = \delta_2$, $\xi^6 = \delta_{2,3}$, $\xi^7 = \delta_3$ and so on.

For N fixed and for $k \geq 0$, denote $x_i^k = \mathbb{P}_{\eta^i}[\xi_0(\infty) = k]$. Also, for $\eta \in \Lambda_N$ introduce $\mathcal{A}(\eta)$ the set of configurations containing at least one particle and resulting from η after the jump, birth or death of a particle in η (but not an absorption in the left side reservoir). Denote by $(p(\eta^i, \eta^j))_{1 \leq i, j \leq n}$ the transition probabilities between configurations. Then for any $1 \leq i \leq 2^N - 1$ and for any $k \geq 2$, the following recurrence relation holds:

$$x_i^k = \sum_{\substack{j=1 \\ \eta^j \in \mathcal{A}(\eta^i)}}^{2^N-1} p(\eta^i, \eta^j) x_j^k + \sum_{\substack{j=1 \\ \eta^j \in \Lambda_N \setminus \mathcal{A}(\eta^i)}}^{2^N-1} p(\eta^i, \eta^j) x_j^{k-1}.$$

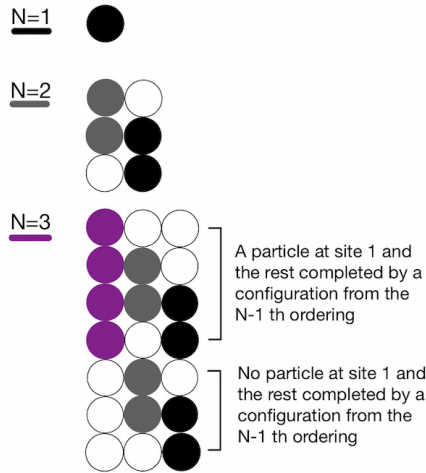


Figure 3: Ordering of configurations

For $k = 0, 1$ the recurrence relation remains the same modulo some changes on the second sum in the right hand side. Therefore, denoting C_k the column vector $(x_i^k)_{1 \leq i \leq 2^N - 1}$, for any $k \geq 0$, there is a matrix of size $2^N - 1$ of the form $M = I_{2^N - 1} - P$ such that

$$MC_k = R_k,$$

where R_k is a columns vector involving some constant terms for $k = 0, 1$ and terms from C_{k-1} for $k \geq 2$, and where

$$P = (p(\eta^i, \eta^j) \mathbb{1}_{\eta^j \in \mathcal{A}(\eta^i)})_{1 \leq i, j \leq 2^N - 1}.$$

By the Perron-Frobenius Theorem, as P is the transition matrix of an irreducible dynamics, one has that M is invertible and

$$M^{-1} = \sum_{k \geq 0} P^k.$$

2.3.3 Fast stirring limit

It is interesting to consider the fast stirring limit, that is, $\mathcal{D} \rightarrow \infty$ for the DCP model. Unlike in hydrodynamics, here, we discuss the setting where time is not rescaled when performing this limit.

When $\mathcal{D} \rightarrow \infty$, the DCP essentially reduces to a birth-death chain on the lattice $\{0, 1, \dots, N\}$. This can be seen by considering a configuration $\eta(t)$ with $n(t)$ particles, where $0 \leq n(t) \leq N$, and where the process $n(t)$ is defined in a way that is analogous to the derivation of the reaction term in the fast stirring limit of a reaction-diffusion processes, as discussed in [7, 9]. More precisely, between any attempt to create or annihilate a particle in the bulk or at the boundaries, the process is just a SSEP with infinite rate, which has the uniform measure as its stationary distribution. This means that in between any creation/annihilation event of a particle (death, infection, and reservoir events), any configuration $\eta(t)$ with $n(t)$ particles, moves uniformly to another configuration $\eta'(t)$ with $n'(t)$ particles, so that on the time scale of the creation

and annihilation processes, the following transitions occur: any given configuration with $n(t)$ particles will turn uniformly into another one with $n(t) \pm 1$ particles. The invariant measure of the process in this limit thus becomes a convex combination of uniform measures on configurations with n particles, where the weights are given by the stationary weights of the birth-death chain that describes the transitions between configurations of n and $n \pm 1$ particles.

Specifically, the rate at which a particle dies in the closed DCP at a given site (transition $n(t) \rightarrow n(t) - 1$), is n/N (which is the probability of finding an occupied site in the uniform distribution), and since there are N sites where particles can die, the rate of transition is n . In the open DCP, particles can also die at the boundary sites with rates γ , and β respectively. Hence the transition rate from a configuration with $n(t)$ particles to a configuration with $n(t) - 1$ particles is $n + (\beta + \gamma)n/N$. Likewise, the rate at which a particle is created in the DCP by infection at a bond, is given by the probability of finding an occupied site with a vacant neighbor in the uniform distribution (which is $n(N - n)/(N(N - 1))$) times the number of bonds (which is $N - 1$) times 2λ (since at each bond the infection can occur with rate λ from a particle on the right or on the left site of the bond). In the open DCP particles can also be created at the boundary sites with rates α , and δ respectively. Hence the transition rate from a configuration with $n(t)$ particles to a configuration with $n(t) + 1$ particles is $2\lambda n(1 - n/N) + (\alpha + \delta)(1 - n/N)$. Thus, the DCP degenerates into a birth-death chain $n(t)$ with the following transition rates:

$$n \rightarrow n - 1, \text{ at rate } n[1 + (\beta + \gamma)/N], \quad n \rightarrow n + 1, \text{ at rate } [2\lambda n + (\alpha + \delta)](1 - n/N). \quad (2.60)$$

Without reservoirs, $n = 0$ is an absorbing state, but not when $\alpha \neq 0$ or $\delta \neq 0$.

Since the arguments that lead to these rates are analogous to those of [7, 9] we refrain from providing a formal proof. For the stationary probabilities and general properties of this birth-death chain we refer to [18].

Here, we illustrate the role of the limit $\mathcal{D} \rightarrow \infty$ in the computations, done in subsection 2.3.2, of the absorption probability for $N = 2$ sites, and $\beta = \delta = 0$. For all $k \geq 0$ and $i = 1, 2, 3$ we denote

$$x_i^k(\infty) := \lim_{\mathcal{D} \rightarrow \infty} x_i^k$$

The solutions obtained in (2.48), (2.53), (2.57), (2.58), (2.59) become (recall the definitions (2.47) of A and (2.52) of B):

- For $k = 0$,

$$\begin{cases} x_1^0(\infty) = 2A^{-1}(\alpha + \gamma + 2) \\ x_2^0(\infty) = 4A^{-1} \\ x_3^0(\infty) = 2A^{-1}(\alpha + \gamma + 2). \end{cases} \quad (2.61)$$

- For $k = 1$, we get:

$$\begin{cases} x_1^1(\infty) = A^{-1}(\alpha + \gamma)(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_2^1(\infty) = A^{-1}2(\alpha + \gamma) [1 + A^{-1}B(\alpha + \gamma + 2)] \\ x_3^1(\infty) = A^{-1}(\alpha + \gamma)(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \end{cases} \quad (2.62)$$

- For $k = 2$, we get:

$$\begin{cases} x_1^2(\infty) = 2A^{-2}\lambda(\alpha + \gamma)^2(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_2^2(\infty) = A^{-2}B(\alpha + \gamma)^2(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_3^2(\infty) = 2A^{-2}\lambda(\alpha + \gamma)^2(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda]. \end{cases} \quad (2.63)$$

- For $k = 3$, the same reasoning as for the cases above yields

$$\begin{cases} x_1^3(\infty) = 4A^{-3}\lambda^2(\alpha + \gamma)^3(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_2^3(\infty) = 2A^{-3}B\lambda(\alpha + \gamma)^3(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_3^3(\infty) = 4A^{-3}\lambda^2(\alpha + \gamma)^3(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda]. \end{cases} \quad (2.64)$$

- For $k > 3$, we find that

$$\begin{cases} x_1^k(\infty) = A^{-k}(2\lambda)^{k-2}B(\alpha + \gamma)^k(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_2^k(\infty) = A^{-k}(2\lambda)^{k-3}B^2(\alpha + \gamma)^k(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_3^k(\infty) = A^{-k}(2\lambda)^{k-2}B(\alpha + \gamma)^k(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda]. \end{cases} \quad (2.65)$$

These formulas induce simple expressions for the values (2.40) when $\mathcal{D} \rightarrow \infty$, where in particular for all $k \geq 0$, $x_1^k(\infty) = x_3^k(\infty)$, that is, the absorption probability to the left reservoirs of one particle at site 1 is the same as the one of a particle at site 2.

3 Generalized diffusive contact process (GDGP)

In this section, we introduce a generalized diffusive contact (GDGP) which also exhibits a factorized duality property, similar to the one for the DCP. For a specific choice of the parameters of the GDGP on a finite one dimensional lattice with reservoirs, one can extract an explicit expression of the one point correlation function for its invariant measure. This is in contrast with the DCP, where we were not able to extract explicit expressions of the correlation functions from the duality result provided by Theorem 1. Duality also leads to a system of first-order ordinary differential equations with constant coefficients.

3.1 The model

Here, as for the DCP, particles evolve on the open one dimensional finite lattice Λ_N . The reservoir dynamics is the same as in Section 2, and given by the generators \mathcal{L}_- and \mathcal{L}_+ , defined in (2.10) and (2.11). The exclusion dynamics is the same as for the DCP, and given by the generator $\mathcal{L}^{\text{SSEP}}$, as defined in (2.8).

What differs is the contact dynamics which is here more general. The birth rate is $\lambda > 0$, while the death rates, instead of being 1, will depend on the occupation variables in the nearest neighbor sites: a particle dies with rate $2\mu_1$ when both neighbors are empty, with rate $\mu_1 + \mu_2$ when one of the two neighbors is empty and the other is occupied, and with rate $2\mu_2$ when both neighbors are occupied. The generator of the generalized diffusive contact processes is given by

$$\mathcal{L}^{\text{GDGP}} = \mathcal{L}_- + \mathcal{L}^{\text{GCP}} + \mathcal{D}\mathcal{L}^{\text{SSEP}} + \mathcal{L}_+. \quad (3.1)$$

Above, \mathcal{L}_- , resp. \mathcal{L}_+ , is the left, resp. right hand side reservoir generator, and is given by (2.10), resp. (2.11), with the boundary rates $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ for the GDGP, instead of $\alpha, \beta, \gamma, \delta$ for the DCP. As before, $\mathcal{D} \geq 0$ is the diffusion parameter which tunes the exclusion dynamics described by $\mathcal{L}^{\text{SSEP}}$ in equation (2.8). While \mathcal{L}^{GCP} is the generator of the generalized contact process, whose action on a given function $f : \Omega_N \rightarrow \mathbb{R}$, is, for $\eta \in \Omega_N$

$$\mathcal{L}^{\text{GCP}} f(\eta) = \sum_{x=1}^{N-1} \mathcal{L}_{x,x+1}^{\text{GCP}} f(\eta), \quad (3.2)$$

with the bond generator

$$\begin{aligned}\mathcal{L}_{x,x+1}^{\text{GCP}}f(\eta) &= \eta_x [\mu_2\eta_{x+1} + \mu_1(1 - \eta_{x+1})] [f(\eta^x) - f(\eta)] + \lambda(1 - \eta_x)\eta_{x+1} [f(\eta^x) - f(\eta)] \\ &\quad + \eta_{x+1} [\mu_2\eta_x + \mu_1(1 - \eta_x)] [f(\eta^{x+1}) - f(\eta)] + \lambda(1 - \eta_{x+1})\eta_x [f(\eta^{x+1}) - f(\eta)].\end{aligned}\quad (3.3)$$

Notice that we can rewrite this bond generator as follows. Define

$$\mathcal{L}_{x,+}^{\text{GCP}}f(\eta) = \eta_x [\mu_2\eta_{x+1} + \mu_1(1 - \eta_{x+1})] [f(\eta^x) - f(\eta)] + \lambda(1 - \eta_x)\eta_{x+1} [f(\eta^x) - f(\eta)], \quad (3.4)$$

and

$$\mathcal{L}_{x+1,-}^{\text{GCP}}f(\eta) = \eta_{x+1} [\mu_2\eta_x + \mu_1(1 - \eta_x)] [f(\eta^{x+1}) - f(\eta)] + \lambda(1 - \eta_{x+1})\eta_x [f(\eta^{x+1}) - f(\eta)], \quad (3.5)$$

so that

$$\mathcal{L}_{x,x+1}^{\text{GCP}}f(\eta) = \mathcal{L}_{x,+}^{\text{GCP}}f(\eta) + \mathcal{L}_{x+1,-}^{\text{GCP}}f(\eta).$$

Then,

$$\begin{aligned}\mathcal{L}^{\text{GCP}}f(\eta) &= \sum_{x=1}^{N-1} (\mathcal{L}_{x,+}^{\text{GCP}}f(\eta) + \mathcal{L}_{x+1,-}^{\text{GCP}}f(\eta)) \\ &= \sum_{x=1}^{N-1} \mathcal{L}_{x,+}^{\text{GCP}}f(\eta) + \sum_{x=2}^N \mathcal{L}_{x,-}^{\text{GCP}}f(\eta) \\ &= \mathcal{L}_{1,+}^{\text{GCP}}f(\eta) + \sum_{x=2}^{N-1} (\mathcal{L}_{x,+}^{\text{GCP}}f(\eta) + \mathcal{L}_{x,-}^{\text{GCP}}f(\eta)) + \mathcal{L}_{N,-}^{\text{GCP}}f(\eta),\end{aligned}\quad (3.6)$$

that is,

$$\begin{aligned}\mathcal{L}^{\text{GCP}}f(\eta) &= \left[\eta_1(\mu_1 + (\mu_2 - \mu_1)\eta_2) + \lambda(1 - \eta_1)\eta_2 \right] [f(\eta^1) - f(\eta)] \\ &\quad + \sum_{x=2}^{N-1} \left[\eta_x(2\mu_1 + (\mu_2 - \mu_1)(\eta_{x-1} + \eta_{x+1})) + \lambda(1 - \eta_x)(\eta_{x-1} + \eta_{x+1}) \right] [f(\eta^x) - f(\eta)] \\ &\quad + \left[\eta_N(\mu_1 + (\mu_2 - \mu_1)\eta_{N-1}) + \lambda(1 - \eta_N)\eta_{N-1} \right] [f(\eta^N) - f(\eta)].\end{aligned}\quad (3.7)$$

Remark 4. (1) One can recover an open DCP with boundary parameters $\alpha, \beta, \gamma, \delta$, from an open GDCP with boundary parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, and death parameters μ_1, μ_2 , by taking

$$\mu_1 = \mu_2 = \frac{1}{2}, \quad \tilde{\alpha} = \alpha, \quad \tilde{\gamma} = \gamma + \frac{1}{2}, \quad \tilde{\delta} = \delta, \quad \text{and} \quad \tilde{\beta} = \beta + \frac{1}{2}. \quad (3.8)$$

(2) For $\mu_2 = 0$, the GDCP reduces to a biased voter model with stirring by symmetric simple exclusion, and when $\mu_2 = 0$ and $\mu_1 = \lambda$ this is the usual voter model with stirring, studied in [3] for asymmetric simple exclusion with step initial condition on \mathbb{Z} , and [21] for totally asymmetric simple exclusion on Λ_N and open boundaries with $\tilde{\gamma} = \tilde{\delta} = 0$.

As for the DCP, since the state space is finite, one can encode the dynamics in a finite sized matrix. In this setting, the intensity matrix of the GDCP in the bulk writes as the following 2^N -sized matrix

$$L^{\text{GCP}} + \mathcal{D}L^{\text{SSEP}} = \sum_{x=1}^{N-1} \ell_x, \quad (3.9)$$

where for $x = 1, \dots, N - 1$

$$\ell_x := \mathbb{1}^{\otimes(x-1)} \otimes \ell \otimes \mathbb{1}^{\otimes(N-1-x)}, \quad (3.10)$$

with the two-dimensional unit matrix $\mathbb{1}$, and the local intensity matrix

$$\ell = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mu_1 & -(\mu_1 + \mathcal{D} + \lambda) & \mathcal{D} & \lambda \\ \mu_1 & \mathcal{D} & -(\mu_1 + \mathcal{D} + \lambda) & \lambda \\ 0 & \mu_2 & \mu_2 & -2\mu_2 \end{pmatrix}. \quad (3.11)$$

The intensity matrices of the boundary dynamics are the same as for the DCP dynamics, see (1) in Remark 4.

The bond transition rates for neighboring sites $(x, x + 1)$ can be schematically represented in tabular form as

<i>Initial</i>	<i>Final</i>	<i>Rate</i>
01	→ 00	μ_1
01	→ 10	\mathcal{D}
01	→ 11	λ
10	→ 00	μ_1
10	→ 01	\mathcal{D}
10	→ 11	λ
11	→ 01	μ_2
11	→ 10	μ_2

(3.12)

3.2 Duality results for the GDCP

In this section we show several duality relations for the GDCP. One, in particular, will be of interest for applications, as it allows to find a dual process with no birth rate. This means that the sum in the correlations equation (2.37) is finite, and can be explicitly computed for a general bulk of N sites. We start by considering the closed GDCP, namely we set $\tilde{\alpha} = \tilde{\delta} = \tilde{\gamma} = \tilde{\beta} = 0$. In order to prove a duality relation we have to show that (2.13) holds with the intensity matrix corresponding to the initial process being L^{GDCP} . Again, our goal is to obtain a bulk duality function in factorized form, namely

$$G(\eta, \xi) = \prod_{x \in \Lambda_N} g(\eta_x, \xi_x)$$

which corresponds to a duality matrix of the form $G^{\otimes N}$, with a local duality matrix

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (3.13)$$

3.2.1 Main results

Below we state the three main duality results for the GDCP, whose corresponding proofs can be found in the next subsection.

Theorem 2. *Assume $\mathcal{D} + \mu_1 - \mu_2 \geq 0$ and $\lambda + \mu_2 - \mu_1 \geq 0$. When $d = 0$ and $a = b = c = 1$, the open GDCP with generator (3.1) is dual with a purely absorbing GDCP with generator*

$$\mathcal{L}^{\text{GDCP, Dual}} = \mathcal{L}_-^{\text{Dual}} + \mathcal{L}^{\text{GCP, Dual}} + \tilde{\mathcal{D}}\mathcal{L}^{\text{SSEP}} + \mathcal{L}_+^{\text{Dual}}, \quad (3.14)$$

with duality function the one obtained for the open diffusive contact process in equation (2.20), with rates $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, in place of $\alpha, \beta, \gamma, \delta$.

Above, $\tilde{\mathcal{D}} = \mathcal{D} + \mu_1 - \mu_2$ is the diffusion parameter of the exclusion dynamics, $\mathcal{L}_-^{\text{Dual}}$ and $\mathcal{L}_+^{\text{Dual}}$ are defined in (2.23) and (2.24). Finally, $\mathcal{L}^{\text{GCP, Dual}}$ is the generator (3.7) of the generalized contact process with birth rate $\tilde{\lambda} = \lambda + \mu_2 - \mu_1$ and death rates exchanged, that is,

$$\begin{aligned} \mathcal{L}^{\text{GCP, Dual}} f(\xi) = & \\ & \sum_{x=2}^{N-1} \left[\xi_x (2\mu_2 + (\mu_1 - \mu_2) (\xi_{x-1} + \xi_{x+1})) + (\lambda + \mu_2 - \mu_1) (1 - \xi_x) (\xi_{x-1} + \xi_{x+1}) \right] \left[f(\xi^x) - f(\xi) \right] \\ & + \left[\xi_1 (\mu_2 + (\mu_1 - \mu_2) \xi_2) + (\lambda + \mu_2 - \mu_1) (1 - \xi_1) \xi_2 \right] \left[f(\xi^1) - f(\xi) \right] \\ & + \left[\xi_N (\mu_2 + (\mu_1 - \mu_2) \xi_{N-1}) + (\lambda + \mu_2 - \mu_1) (1 - \xi_N) \xi_{N-1} \right] \left[f(\xi^N) - f(\xi) \right]. \end{aligned} \quad (3.15)$$

A special case of the above setting is elucidated in the Corollary below.

Corollary 1 (Annihilating dual process). *Under the same hypothesis as before, setting $\mu_1 = \lambda + \mu_2$, the dual GDCP has birth rate $\lambda = 0$, i.e. no particle is ever created. The intensity matrix associated to the local dual generator is given by*

$$\ell^{\text{Dual}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mu_2 & -(\mathcal{D} + \lambda + \mu_2) & \mathcal{D} + \lambda & 0 \\ \mu_2 & \mathcal{D} + \lambda & -(\mathcal{D} + \lambda + \mu_2) & 0 \\ 0 & \lambda + \mu_2 & \lambda + \mu_2 & -2(\lambda + \mu_2) \end{pmatrix}. \quad (3.16)$$

We are now left to study the setting where $d \neq 0$. In this setting, we establish a self-duality result (Theorem 3), as well as another duality relation where both processes involved have no birth mechanism. (Theorem 4).

Theorem 3. *When $d \neq 0$ and $a = b = c = 1$, the closed GDCP with generator $\mathcal{L}^{\text{GCP}} + \mathcal{D}\mathcal{L}^{\text{SSEP}}$ is self-dual with duality matrix*

$$G = \begin{pmatrix} 1 & 1 \\ 1 & \frac{\mu_1 - \mu_2}{\lambda} \end{pmatrix}. \quad (3.17)$$

Theorem 4. *When $a = b = c = 0$ and $d = 1$ the closed GDCP process with generator $\mathcal{L}^{\text{GCP}} + \mathcal{D}\mathcal{L}^{\text{SSEP}}$ and no birth, i.e. $\lambda = 0$, is dual with duality matrix*

$$G = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.18)$$

to a GDCP with no birth, i.e. $\tilde{\lambda} = 0$, arbitrary diffusive rate $\tilde{\mathcal{D}}$, arbitrary death rate $\tilde{\mu}_1$ and $\tilde{\mu}_2 = \mu_2$.

Note that in the case where $a = b = c = d = 0$, we have a trivial duality relation with duality function equal to zero.

3.2.2 Proofs for Subsection 3.2.1

Proof of Theorem 2. In order to show the matrix duality relation of equation (2.13), as the generator acts on two sites, it is enough to show that

$$\ell G^{\otimes 2} = G^{\otimes 2} \left(\ell^{\text{Dual}} \right)^T. \quad (3.19)$$

For G, ℓ defined respectively in (3.13) and (3.11); we have

$$G^{\otimes 2} = \begin{pmatrix} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{pmatrix}. \quad (3.20)$$

For any choice of the parameters a, b, c, d , this defines a self-duality function for the SSEP (see Remark 2), and we only have to take into consideration the action of the GCP. The left hand side of equation (3.19) becomes

$$\ell G^{\otimes 2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x[a\mu_1 - b\lambda] & ay\mu_1 - dx\lambda - \Delta\mathcal{D} & cx\mu_1 - by\lambda + \Delta\mathcal{D} & y[c\mu_1 - d\lambda] \\ x[a\mu_1 - b\lambda] & cx\mu_1 - by\lambda + \Delta\mathcal{D} & ay\mu_1 - dx\lambda - \Delta\mathcal{D} & y[c\mu_1 - d\lambda] \\ 2bx\mu_2 & (dx + by)\mu_2 & (dx + by)\mu_2 & 2d(c - d)\mu_2 \end{pmatrix}, \quad (3.21)$$

where $\Delta := ad - bc$, $x := a - b$ and $y := c - d$. For the right hand side of (3.19), we first consider a local generator ℓ^{Dual} which describes a GDCP with possibly different rates: $\tilde{\mathcal{D}}$ is the diffusion coefficient, $\tilde{\lambda}$ the birth rate while $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are the death rates. This gives the following dual local intensity matrix

$$\ell^{\text{Dual}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \tilde{\mu}_1 & -(\tilde{\mathcal{D}} + \tilde{\lambda} + \tilde{\mu}_1) & \tilde{\mathcal{D}} & \tilde{\lambda} \\ \tilde{\mu}_1 & \tilde{\mathcal{D}} & -(\tilde{\mathcal{D}} + \tilde{\lambda} + \tilde{\mu}_1) & \tilde{\lambda} \\ 0 & \tilde{\mu}_2 & \tilde{\mu}_2 & -2\tilde{\mu}_2 \end{pmatrix}. \quad (3.22)$$

Computing the right hand side of (3.19), we get

$$G^{\otimes 2} (\ell^{\text{Dual}})^T = \begin{pmatrix} 0 & (a - c)(a\tilde{\mu}_1 - c\tilde{\lambda}) & (a - c)(a\tilde{\mu}_1 - c\tilde{\lambda}) & 2(a - c)c\tilde{\mu}_2 \\ 0 & a(b - d)\tilde{\mu}_1 - d(a - c)\tilde{\lambda} - \Delta\tilde{\mathcal{D}} & b(a - c)\tilde{\mu}_1 - c(b - d)\tilde{\lambda} + \Delta\tilde{\mathcal{D}} & (ad + bc - 2cd)\tilde{\mu}_2 \\ 0 & b(a - c)\tilde{\mu}_1 - c(b - d)\tilde{\lambda} + \Delta\tilde{\mathcal{D}} & a(b - d)\tilde{\mu}_1 - d(a - c)\tilde{\lambda} - \Delta\tilde{\mathcal{D}} & (ad + bc - 2cd)\tilde{\mu}_2 \\ 0 & (b - d)[b\tilde{\mu}_1 - d\tilde{\lambda}] & (b - d)[b\tilde{\mu}_1 - d\tilde{\lambda}] & 2(b - d)d\tilde{\mu}_2 \end{pmatrix}. \quad (3.23)$$

Since we are aiming to match expressions (3.21) and (3.23), we first notice that equality in the first column requires $x = 0$, i.e. $a = b$, while equality in the first row requires $a = c$. Thus $a = b = c$ yields $\Delta = -a(a - d)$, and we are left with

$$\ell G^{\otimes 2} = (a - d) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a\mu_1 + a\mathcal{D} & -a\lambda - a\mathcal{D} & a\mu_1 - d\lambda \\ 0 & -a\lambda - a\mathcal{D} & a\mu_1 + a\mathcal{D} & a\mu_1 - d\lambda \\ 0 & a\mu_2 & a\mu_2 & 2d\mu_2 \end{pmatrix} \quad (3.24)$$

for the left hand side, and

$$G^{\otimes 2} (\ell^{\text{Dual}})^T = (a - d) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a\tilde{\mu}_1 + a\tilde{\mathcal{D}} & -a\tilde{\lambda} - a\tilde{\mathcal{D}} & a\tilde{\mu}_2 \\ 0 & -a\tilde{\lambda} - a\tilde{\mathcal{D}} & a\tilde{\mu}_1 + a\tilde{\mathcal{D}} & a\tilde{\mu}_2 \\ 0 & a\tilde{\mu}_1 - d\tilde{\lambda} & a\tilde{\mu}_1 - d\tilde{\lambda} & 2d\tilde{\mu}_2 \end{pmatrix} \quad (3.25)$$

for the right hand side. Consider $d = 0$ in both equations (3.24) and (3.25). Then, the identity holds if we set $\tilde{\mu}_2 = \mu_1$ and $\tilde{\mu}_1 = \mu_2$, $\tilde{\mathcal{D}} = \mathcal{D} + \mu_1 - \mu_2$ and $\tilde{\lambda} = \lambda + \mu_2 - \mu_1$. Furthermore, without loss of generality, choosing $a = 1$, we are left with the duality matrix

$$G = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.26)$$

which corresponds to the duality function

$$G(\eta, \xi) = \prod_{x \in \Lambda_N} (1 - \eta(x))^{\xi(x)}. \quad (3.27)$$

Note that G is the same duality function as in (2.21). Since the bulk duality function is the same as the one for the diffusive contact process, we can extend the result to the GDCP with the corresponding boundary parameters. Hence, the duality function has the same form as for the diffusive contact process, that is: $D : \Omega_N \times \Omega_N^{\text{dual}} \rightarrow \mathbb{R}$ given by

$$D(\eta, \xi) = \left(\frac{\tilde{\gamma}}{\tilde{\alpha} + \tilde{\gamma}} \right)^{\xi_0} \prod_{x \in \Lambda_N} (1 - \eta_x)^{\xi_x} \left(\frac{\tilde{\delta}}{\tilde{\beta} + \tilde{\delta}} \right)^{\xi_{N+1}}. \quad (3.28)$$

Notice that for $\mu_1 = \mu_2 = \frac{1}{2}$, we recover Theorem 1. □

The proof of Corollary 1 follows from Theorem 2.

Proof of Theorem 3. If $d \neq 0$ and $a \neq 0$ in both (3.24) and (3.25), it has to be verified that $\tilde{\mu}_2 = \mu_2$, $a\tilde{\mu}_1 = a\mu_2 + d\tilde{\lambda}$, and the constraint $d = a\frac{\mu_1 - \mu_2}{\lambda}$ where, again, without loss of generality we can choose $a = 1$. Again, as the death rates do not depend on the number of neighbors, i.e. $\mu_1 = \mu_2$, we get the same function as in (3.26). □

Proof of Theorem 4. Setting $a = 0$ in equations (3.24) and (3.25) leads to

$$\ell G^{\otimes 2} = -d \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d\lambda \\ 0 & 0 & 0 & -d\lambda \\ 0 & 0 & 0 & 2d\mu_2 \end{pmatrix}, \quad (3.29)$$

and

$$G^{\otimes 2} (\ell^{\text{Dual}})^T = -d \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -d\tilde{\lambda} & -d\tilde{\lambda} & 2d\tilde{\mu}_2 \end{pmatrix}. \quad (3.30)$$

Since $d \neq 0$, we require that $\tilde{\mu}_2 = \mu_2$ and that there is no birth in both the initial and the dual process, namely $\lambda = \tilde{\lambda} = 0$. □

3.3 Application of duality: computing correlations

As for the DCP, the dynamics of the GDCP is irreducible and the state space is finite so there is a unique invariant measure ν^{GDCP} for the GDCP, which depends on the birth, death, exchange and boundary rates. For a particular choice of these rates, the one point correlation function of ν^{GDCP} can be explicitly computed, see Proposition 5. Furthermore, some information on the time-dependent one-point function for arbitrary initial distributions can be obtained.

3.3.1 Invariant measure

Here, for $1 \leq \ell \leq N$, the ℓ -point correlation function ρ^{GDCP} of ν^{GDCP} is the function ρ_ℓ^{GDCP} defined by

$$\forall x_1 < \dots < x_\ell \in \Lambda_N, \quad \rho_\ell^{\text{GDCP}}(x_1, \dots, x_\ell) = \mathbb{E}_{\nu^{\text{GDCP}}}[\eta_{x_1} \dots \eta_{x_\ell}]. \quad (3.31)$$

From now on, we denote by

$$\tilde{c}_- := \frac{\tilde{\gamma}}{\tilde{\alpha} + \tilde{\gamma}}, \quad \text{and} \quad \tilde{c}_+ := \frac{\tilde{\delta}}{\tilde{\beta} + \tilde{\delta}}. \quad (3.32)$$

As for the DCP, the dual $(\xi(t))_{t \geq 0}$ of the GDCP (we keep the same notation ξ to refer to the dual process) becomes extinct almost surely and, recall that the total number of particles absorbed on the left, resp. right hand side boundary is:

$$\xi_0(\infty) := \lim_{t \rightarrow \infty} \xi_0(t), \quad \text{resp.} \quad \xi_{N+1}(\infty) := \lim_{t \rightarrow \infty} \xi_{N+1}(t), \quad (3.33)$$

where the limits are almost sure.

Following the same lines as the proof of Proposition 3, one can prove the following general formula for the correlation functions of the GDCP:

Proposition 4. *For any $1 \leq \ell \leq N$, and any $1 \leq x_1 < \dots < x_\ell \leq N$,*

$$\begin{aligned} \mathbb{E}_{\nu^{\text{GDCP}}} \left[\prod_{j=1}^{\ell} (1 - \eta_{x_j}) \right] &= \sum_{k=0}^{\ell} (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq \ell} \rho_k^{\text{GDCP}}(x_{i_1}, \dots, x_{i_k}) \\ &= \sum_{m, n \geq 0} \mathbb{P}_{\delta_{x_1, x_2, \dots, x_\ell}} \left[\xi_0(\infty) = m, \xi_N(\infty) = n \right] \tilde{c}_-^m \tilde{c}_+^n, \end{aligned} \quad (3.34)$$

where $\delta_{x_1, x_2, \dots, x_\ell}$ denotes the configuration with a particle in x_1, \dots, x_ℓ , and none elsewhere.

In the particular setting of Corollary 1, the one point correlation function can be computed explicitly.

Proposition 5. *Consider a GDCP with birth rate $\lambda > 0$, diffusion coefficient \mathcal{D} , boundary rates $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ and death rates μ_1 and μ_2 , with $\mu_1 = \mu_2 + \lambda$. Then,*

$$\rho_1^{\text{GDCP}}(x) = u_x(1 - \tilde{c}_-) + v_x(1 - \tilde{c}_+),$$

where with the constants

$$\begin{aligned} \tilde{a} &:= \frac{\tilde{\alpha} + \tilde{\gamma}}{\tilde{\alpha} + \tilde{\gamma} + \mathcal{D} + \lambda + \mu_2}, & \tilde{b} &:= \frac{\mathcal{D} + \lambda}{\tilde{\alpha} + \tilde{\gamma} + \mathcal{D} + \lambda + \mu_2}, \\ \tilde{c} &:= \frac{\tilde{\beta} + \tilde{\delta}}{\tilde{\beta} + \tilde{\delta} + \mathcal{D} + \lambda + \mu_2}, & \tilde{d} &:= \frac{\mathcal{D} + \lambda}{\tilde{\beta} + \tilde{\delta} + \mathcal{D} + \lambda + \mu_2} \end{aligned} \quad (3.35)$$

and

$$A := \frac{\mathcal{D} + \lambda}{\mathcal{D} + \lambda + \mu_2}, \quad r_\pm := \frac{1 \pm \sqrt{1 - A^2}}{A}, \quad (3.36)$$

$$B_N := r_-(1 - \tilde{b}r_-)(1 - \tilde{d}r_+^{-1}) + r_-^N r_+^{1-N} (\tilde{b}r_+ - 1)(1 - \tilde{d}r_-^{-1}), \quad (3.37)$$

$$B'_N := \tilde{a}(\tilde{c}N + 1 - \tilde{c}) + (1 - 2\tilde{a})\tilde{c}, \quad (3.38)$$

the terms u_x and v_x are given by, for all $x \in \{1, \dots, N\}$,

$$u_x = \begin{cases} \frac{\tilde{a}}{B_N} [(1 - \tilde{d}r_+^{-1})r_-^x + (\tilde{d}r_-^{-1} - 1)r_-^N r_+^{x-N}] & \mu_2 > 0 \\ \frac{\tilde{a}}{B'_N} [1 - \tilde{c} + \tilde{c}(N - x)] & \mu_2 = 0, \end{cases} \quad (3.39)$$

$$v_x = \begin{cases} \frac{\tilde{c}}{B_N} [(\tilde{b}r_+ - 1)r_+^{1-N} r_-^x + r_- (1 - \tilde{b}r_-)r_+^{x-N}] & \mu_2 > 0 \\ \frac{\tilde{c}}{B'_N} [1 - \tilde{a} + \tilde{a}(x - 1)] & \mu_2 = 0 \end{cases} \quad (3.40)$$

Remark 5. • For $\mu_2 > 0$ one finds that the stationary bulk density

$$\rho_{bulk}^{\text{GDGP}}(s) := \lim_{N \rightarrow \infty} \rho_1([sN]) \quad (3.41)$$

vanishes for all $s \in (0, 1)$. A non-trivial exponentially decaying density profile appears near the boundaries.

• In contrast, for $\mu_2 = 0$ a linear bulk density profile emerges,

$$\rho_{bulk}^{\text{GDGP}}(s) = \frac{\tilde{\alpha}}{\tilde{\alpha} + \tilde{\gamma}}(1 - s) + \frac{\tilde{\beta}}{\tilde{\beta} + \tilde{\delta}}s = \frac{\tilde{\alpha}}{\tilde{\alpha} + \tilde{\gamma}} - \frac{\tilde{\alpha}\tilde{\delta} - \tilde{\beta}\tilde{\gamma}}{(\tilde{\alpha} + \tilde{\gamma})(\tilde{\beta} + \tilde{\delta})}s \quad (3.42)$$

This phenomenon is reminiscent of a superposition of shock measures appearing in the voter model with totally asymmetric stirring (rather than symmetric stirring as in the present case) on Λ_N and open boundaries with $\tilde{\gamma} = \tilde{\delta} = 0$ ([21]). The invariant measure of that model is a convex combination of shock product measures with marginals $\mu(\eta_x) = 1$ for $1 \leq x < x_s$ and $\mu(\eta_x) = 0$ for $x_s \leq x \leq N$ and uniform distribution of the microscopic shock position $x_s \in \Lambda_N$ which leads to a linear stationary density profile and a simple form of the two-point correlation function computed in [17]. For the present more complicated case of symmetric stirring and arbitrary boundary parameters we leave a detailed investigation of the two-point correlation function for future work.

Proof of Proposition 4. In this setting, by Corollary 1, the dual process of the GDGP is a GDGP with no birth rate, diffusion coefficient $\mathcal{D} + \lambda$, boundary rates $0, \tilde{\alpha} + \tilde{\gamma}, 0, \tilde{\beta} + \tilde{\delta}$ (purely absorbing boundaries), and death rates μ_2 and $\lambda + \mu_2$. Thus, the generator of the dual process is given by

$$\mathcal{L}^{\text{GDGP, Dual}} = \mathcal{L}_-^{\text{Dual}} + \mathcal{L}^{\text{GCP, Dual}} + (\mathcal{D} + \lambda)\mathcal{L}^{\text{SSEP}} + \mathcal{L}_+^{\text{Dual}},$$

where we recall that $\mathcal{L}_-^{\text{Dual}}$ and $\mathcal{L}_+^{\text{Dual}}$ are defined in (2.23) (with $\tilde{\alpha} + \tilde{\gamma}$ instead of $\alpha + \gamma$ and $\tilde{\beta} + \tilde{\delta}$ instead of $\beta + \delta$) and,

$$\begin{aligned} \mathcal{L}^{\text{GDGP, Dual}} f(\xi) &= \sum_{x=2}^{N-1} \left[\eta_x (2\mu_2 + \lambda(\xi_{x-1} + \xi_{x+1})) \right] [f(\xi^x) - f(\xi)] \\ &+ \left[\xi_1 (\mu_2 + \lambda\xi_2) \right] [f(\xi^1) - f(\xi)] \\ &+ \left[\xi_N (\mu_2 + \lambda\xi_{N-1}) \right] [f(\xi^N) - f(\xi)]. \end{aligned}$$

By Theorem 2 the duality function $D : \Omega_N \times \Omega_N^{\text{dual}} \rightarrow \mathbb{R}$ is the same as for the DCP, that is given by (3.28). The same reasoning as in the proof of Proposition 2 yields that for $x \in \{1, \dots, N\}$,

$$\begin{aligned} \mathbb{E}_{\nu, \text{GDPC}} [1 - \eta_x] &= \mathbb{P}_{\delta_x} [\xi_0(\infty) = 1 \cap \xi_{N+1}(\infty) = 0] \tilde{c}_- \\ &\quad + \mathbb{P}_{\delta_x} [\xi_0(\infty) = 0 \cap \xi_{N+1}(\infty) = 1] \tilde{c}_+ \\ &\quad + \left(1 - \mathbb{P}_{\delta_x} [\xi_0(\infty) = 1 \cap \xi_{N+1}(\infty) = 0] - \mathbb{P}_{\delta_x} [\xi_0(\infty) = 0 \cap \xi_{N+1}(\infty) = 1] \right) \\ &=: u_x \tilde{c}_- + v_x \tilde{c}_+ + (1 - u_x - v_x). \end{aligned}$$

Therefore,

$$\rho_1^{\text{GDPC}}(x) = u_x(1 - \tilde{c}_-) + v_x(1 - \tilde{c}_+).$$

In this case the sum (3.34) is finite due to the fact that the rate of birth is zero, so $\xi_0(\infty)$ and $\xi_{N+1}(\infty)$ can be at most equal to one. \square

Proof of Proposition 5. To prove Proposition 5, we are now left to compute $(u_x)_{1 \leq x \leq N}$ and $(v_x)_{1 \leq x \leq N}$. By conditioning on the first possible event (jump, birth, death, or absorption by a reservoir) of the process starting from a particle at site x , we have the following discrete equations

- For $(u_x)_{1 \leq x \leq N}$:

$$\begin{aligned} u_1 &= \tilde{a} + \tilde{b}u_2 \\ u_x &= \frac{A}{2}(u_{x-1} + u_{x+1}) \quad \forall x \in \{2, \dots, N-1\} \\ u_N &= \tilde{d}u_{N-1}, \end{aligned}$$

with the constants $\tilde{a}, \tilde{b}, \tilde{d}$, and A , defined in (3.35) and (3.36).

- For $(v_x)_{1 \leq x \leq N}$:

$$\begin{aligned} v_1 &= \tilde{b}v_2 \\ v_x &= \frac{A}{2}(v_{x-1} + v_{x+1}) \quad \forall x \in \{2, \dots, N-1\} \\ v_N &= \tilde{c} + \tilde{d}v_{N-1}, \end{aligned}$$

with the constants $\tilde{b}, \tilde{c}, \tilde{d}$, and A , defined in (3.35) and (3.36).

Both recurrence relations are of the form

$$\begin{aligned} w_1 &= a + bw_2 \\ w_x &= \frac{A}{2}(w_{x-1} + w_{x+1}) \quad \forall x \in \{2, \dots, N-1\} \\ w_N &= c + dw_{N-1} \end{aligned} \tag{3.43}$$

with $a = \tilde{a}$, $b = \tilde{b}$, $c = 0$, $d = \tilde{d}$, for $(u_x)_{1 \leq x \leq N}$ and $a = 0$, $b = \tilde{b}$, $c = \tilde{c}$, $d = \tilde{d}$, for $(v_x)_{1 \leq x \leq N}$. The properties of the solution of this recursion depend on μ_2 as follows.

Case 1: $\mu_2 > 0$

To solve the general recurrence relation, notice that the constants defined in (3.36) satisfy $A < 1$, $r_- < 1$ and $r_+ > 1$. Then, there are $p, q \in \mathbb{R}$ such that for $2 \leq x \leq N-1$,

$$w_x = pr_-^x + qr_+^x, \tag{3.44}$$

which follows from the bulk part of the recurrence relation (3.43). Furthermore, the recurrence relation involving sites 1 and 2 yields $pr_- + qr_+ = a + b[pr_-^2 + qr_+^2]$, therefore $pr_-(1 - br_-) = a + qr_+(bqr_+ - 1)$. Since $0 < r_- < 1$ and $b < 1$, for both u_x and v_x , one obtains

$$p = \frac{a + qr_+(br_+ - 1)}{r_-(1 - br_-)}.$$

In a similar fashion, the recurrence relation involving sites N and $N - 1$ yields $pr_-^N + qr_+^N = c + d[pr_-^{N-1} + qr_+^{N-1}]$, therefore $pr_-^N(1 - dr_-^{-1}) = c + qr_+^N(dr_+^{-1} - 1)$. Since $r_+ > 1$ and $d < 1$, for both u_x and v_x , one obtains

$$q = \frac{c - pr_-^N(1 - dr_-^{-1})}{r_+^N(1 - dr_+^{-1})}.$$

Finally, introducing

$$B_N := r_-(1 - br_-)(1 - dr_+^{-1}) + r_-^N r_+^{1-N}(br_+ - 1)(1 - dr_-^{-1}), \quad (3.45)$$

one finds

$$q = \frac{1}{B_N} [cr_-(1 - br_-) + ar_-^N(dr_-^{-1} - 1)]r_+^{-N} \quad (3.46)$$

$$p = \frac{1}{B_N} [a(1 - dr_+^{-1}) + cr_+^{1-N}(br_+ - 1)]. \quad (3.47)$$

This result yields (3.39) and (3.40) for $\mu_2 > 0$.

Case 2, $\mu_2 = 0$

In this case, $A = 1$ and the bulk part of the recurrence relation (3.43) can be written $\Delta w_x = 0$, where Δ is the discrete one-dimensional Laplacian. The general solution is the linear function

$$w_x = p' + q'x. \quad (3.48)$$

The boundary condition at site 1 yields $p' + q' = a + b(p' + 2q')$ and therefore $p'(1 - b) = a + (2b - 1)q'$. Likewise, the boundary condition at site N yields $p' + q'N = c + d(p' + q'(N - 1))$ and therefore $p'(1 - d) = c - q'[(1 - d)N - d]$. We then discuss the following cases:

- (i) $\tilde{b} = \tilde{d} = 1$. In this case, we have closed boundary conditions, and the empty lattice is the absorbing state, implying without further computation $u_x = v_x = \rho_1^{\text{GDGP}}(x) = 0$ for all $x \in \{1, \dots, N\}$.
- (ii) $\tilde{b} = 1, \tilde{d} < 1$. This corresponds to a closed left boundary, with $\tilde{a} = 0$, but with open right boundary, leaving the invariant measure non-trivial. The recurrence for w_x yields $q' = 0$ and one obtains $u_x = 0, v_x = 1$ for all $x \in \{1, \dots, N\}$.
- (iii) $\tilde{d} = 1, \tilde{b} < 1$. This corresponds to a closed right boundary, with $\tilde{c} = 0$, but with open left boundary, leaving the invariant measure non-trivial. The recurrence for w_x yields $q' = 0$ and one obtains $u_x = 1, v_x = 0$ for all $x \in \{1, \dots, N\}$.
- (iv) $\tilde{d} < 1, \tilde{b} < 1$. The recurrence yields

$$q' = \frac{c(1 - b) - a(1 - d)}{B'_N} \quad (3.49)$$

$$p' = \frac{a[(1 - d)N + d] + c(2b - 1)}{B'_N}, \quad (3.50)$$

with

$$B'_N := (1 - b)[(1 - d)N + d] + (2b - 1)(1 - d).$$

Hence, observing that for $\mu_2 = 0$, one has $1 = \tilde{a} + \tilde{b} = \tilde{c} + \tilde{d}$, we arrive at (3.39) and (3.40) for $\mu_2 = 0$, which covers all four cases (i)-(iv). \square

Remark 6. *Computing higher order correlations is in general a complicated task. For these models, the difficulty lies in the fact that if there are two or more dual particles, the death rates depend on the configuration. We leave this for future work.*

3.3.2 Time-dependence of the one-point function

For the choice $\mu_1 = \lambda + \mu_2$ the duality function of the GDCP yields the finite inhomogeneous system of ordinary differential equations (ODEs)

$$\begin{aligned}\frac{d}{dt}\langle \eta_x \rangle &= (\mathcal{D} + \lambda)(\langle \eta_{x+1} \rangle + \langle \eta_{x-1} \rangle - 2\langle \eta_x \rangle) - 2\mu_2\langle \eta_x \rangle, \quad 1 < x < N \\ \frac{d}{dt}\langle \eta_1 \rangle &= (\mathcal{D} + \lambda)(\langle \eta_2 \rangle - \langle \eta_1 \rangle) - (\mu_2 + \tilde{\alpha} + \tilde{\gamma})\langle \eta_1 \rangle + \tilde{\alpha}, \\ \frac{d}{dt}\langle \eta_N \rangle &= (\mathcal{D} + \lambda)(\langle \eta_{N-1} \rangle - \langle \eta_N \rangle) - (\mu_2 + \tilde{\beta} + \tilde{\delta})\langle \eta_N \rangle + \tilde{\delta},\end{aligned}\tag{3.51}$$

where $\langle \cdot \rangle$ means expectation at time t for an arbitrary initial measure ν . The inhomogeneity arising from the constants $\tilde{\alpha}$ and $\tilde{\delta}$ can be removed by considering the centered variables $\bar{\eta}_x := \eta_x - \rho_1(x)$, and by noting that invariance of the measure yields $\tilde{\alpha} = (\mathcal{D} + \lambda + \mu_2 + \tilde{\alpha} + \tilde{\gamma})\rho_1(1) - (\mathcal{D} + \lambda)\rho_1(2)$ and $\tilde{\delta} = (\mathcal{D} + \lambda + \mu_2 + \tilde{\beta} + \tilde{\delta})\rho_1(N) - (\mathcal{D} + \lambda)\rho_1(N-1)$. With the recursion (3.43), which by linearity is also valid for $\rho_1(x)$, we conclude that

$$\begin{aligned}\frac{d}{dt}\langle \bar{\eta}_x \rangle &= (\mathcal{D} + \lambda)(\langle \bar{\eta}_{x+1} \rangle + \langle \bar{\eta}_{x-1} \rangle - 2\langle \bar{\eta}_x \rangle) - 2\mu_2\langle \bar{\eta}_x \rangle, \quad 1 < x < N, \\ \frac{d}{dt}\langle \bar{\eta}_1 \rangle &= (\mathcal{D} + \lambda)(\langle \bar{\eta}_2 \rangle - \langle \bar{\eta}_1 \rangle) - (\mu_2 + \tilde{\alpha} + \tilde{\gamma})\langle \bar{\eta}_1 \rangle, \\ \frac{d}{dt}\langle \bar{\eta}_N \rangle &= (\mathcal{D} + \lambda)(\langle \bar{\eta}_{N-1} \rangle - \langle \bar{\eta}_N \rangle) - (\mu_2 + \tilde{\beta} + \tilde{\delta})\langle \bar{\eta}_N \rangle,\end{aligned}\tag{3.52}$$

which is a *homogeneous* system of first order ordinary differential equations. Denoting by

$$g(x, t) := e^{2\mu_2 t} \langle \bar{\eta}_x \rangle,\tag{3.53}$$

one then finds

$$\begin{aligned}\frac{d}{dt}g(x, t) &= (\mathcal{D} + \lambda)(g(x+1, t) + g(x-1, t) - 2g(x, t)), \quad 1 < x < N, \\ \frac{d}{dt}g(1, t) &= (\mathcal{D} + \lambda)(g(2, t) - g(1, t)) - (\tilde{\alpha} + \tilde{\gamma} - \mu_2)g(1, t), \\ \frac{d}{dt}g(N, t) &= (\mathcal{D} + \lambda)(g(N-1, t) - g(N, t)) - (\tilde{\beta} + \tilde{\delta} - \mu_2)g(N, t).\end{aligned}\tag{3.54}$$

In this system, we recognize the ODE's for the time-dependent centered one-point correlation function of the open SSEP with diffusion coefficient $\hat{D} = \mathcal{D} + \lambda$, and reservoir rates $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$, satisfying

$$\hat{\alpha} + \hat{\gamma} = \tilde{\alpha} + \tilde{\gamma} - \mu_2, \quad \text{and} \quad \hat{\beta} + \hat{\delta} = \tilde{\beta} + \tilde{\delta} - \mu_2.\tag{3.55}$$

Following [28], this system can be solved in a closed form by a discrete Fourier transformation with a reflected wave, in the special cases obtained from the four combinations of reservoir parameters given by $\tilde{\alpha} + \tilde{\gamma} \in \{\mu_2, \mu_2 + \mathcal{D} + \lambda\}$ and $\tilde{\beta} + \tilde{\delta} \in \{\mu_2, \mu_2 + \mathcal{D} + \lambda\}$. We find surprising that the system of equations (3.54) matches the one for the space time empirical profile of an

open SSEP, see equation (19) and below in [15]. In particular, the choice $\tilde{\alpha} + \tilde{\gamma} = \tilde{\beta} + \tilde{\delta} = \mu_2$ corresponds to $\hat{\alpha} = \hat{\beta} = \hat{\gamma} = \hat{\delta} = 0$ for the boundary parameters of the SSEP (i.e. $\kappa = 0$ in [15]) and leads to the same system of equations, both for the SSEP with reflecting boundary which conserves the total number of particles and for our GCDP which does not conserve the total number of particles due to positive birth and death rates. We believe it would be interesting to prove the hydrodynamic limit for this non-conservative process with open boundary, in the same spirit as for the open finite volume SSEP.

4 The susceptible-infectious-recovered (SIR) model

We now consider another non conservative model, originally introduced in [19], in terms of a nonlinear system of differential equations for the sizes of populations of three species of individuals which are subject to an infection/recovery mechanism. The susceptible-infectious-recovered (SIR) model describes propagation of infections in the following sense: a susceptible individual (characterized by its state S) can become an infectious individual (characterized by its state I), according to an infection rate β , if it is in contact with an infectious individual. An infectious individual recovers (then it is characterized by its state R) with recovery rate γ ; once an individual has recovered it stays immune, that is, it remains in state R forever.

Despite the simplicity of the original mean-field type model, it is known to capture important features of the temporal dynamics of an infection. However, only limited results are available if fluctuations (which inevitably occur in a real system) are taken into account. This question was addressed in the SIR model of [30] where particles evolve on a one dimensional space of sites, as in the diffusive contact process. However, contrary to the diffusive contact process, the model is defined on the infinite translation invariant lattice \mathbb{Z} , so that in particular, there are no reservoirs. Notice that the parameters β and γ for the SIR model have nothing to do with the reservoir parameters in the previous sections. The reason for which we keep this notation is that it is canonically used in the literature on the SIR model.

The stochastic evolution of the collection of particles in the system is governed by a Markov process denoted by $\{\eta(t), t \geq 0\}$, with state space $\tilde{\mathcal{S}} =: \{S, I, R\}^{\mathbb{Z}}$, so that at each time $t \geq 0$, $\eta(t) = (\eta_1(t), \dots, \eta_N(t)) \in \tilde{\mathcal{S}}$. For $x \in \mathbb{Z}$, $a \in \{S, I, R\}$ and a configuration $\eta \in \tilde{\mathcal{S}}$, $\eta_x = a$ means that x is in state a .

For $\eta \in \tilde{\mathcal{S}}$, denote by (η^S, η^I, η^R) the element of $\mathcal{S} := \{0, 1\}^{\mathbb{Z}} \times \{0, 1\}^{\mathbb{Z}} \times \{0, 1\}^{\mathbb{Z}}$ such that:

$$\forall x \in \mathbb{Z}, \forall a \in \{S, I, R\}, \quad \eta_x^a = \mathbb{1}_{\eta_x = a}. \quad (4.1)$$

The correspondence (4.1) between $\tilde{\mathcal{S}}$ and \mathcal{S} allows us to use indifferently the representation of a configuration as an element of $\tilde{\mathcal{S}}$ or of \mathcal{S} . Note that for $x \in \mathbb{Z}$, since $\eta_x \in \{S, I, R\}$,

$$\eta_x^S + \eta_x^I + \eta_x^R = 1. \quad (4.2)$$

The generator of the SIR dynamics acts on local functions $f : \mathcal{S} \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} \mathcal{L}^{\text{SIR}} f(\eta) = & \sum_{x \in \mathbb{Z}} \left\{ \beta \eta_x^I \eta_{x+1}^S [f((T_{x+1}\eta)^S) - f(\eta)] \right. \\ & + \beta \eta_x^I \eta_{x-1}^S [f((T_{x-1}\eta)^S) - f(\eta)] \\ & \left. + \gamma \eta_x^I [f((\hat{T}_x\eta)^I) - f(\eta)] \right\}, \end{aligned} \quad (4.3)$$

where the operators T and \widehat{T} are defined, for $y \in \mathbb{Z}$, a configuration η , and a state $a \in \{S, I, R\}$, by: when $\eta_y^S = 1$,

$$(T_y \eta)_z^a = \begin{cases} \eta_z^a & \text{if } z \neq y \\ \eta_z^I & \text{if } z = y, \end{cases} \quad (4.4)$$

while, when $\eta_y^I = 1$,

$$(\widehat{T}_y \eta)_z^a = \begin{cases} \eta_z^a & \text{if } z \neq y \\ \eta_z^R & \text{if } z = y. \end{cases} \quad (4.5)$$

In other words, recovering the notation in [30], the (translation invariant) transitions between nearest neighboring sites are

<i>Initial</i>	<i>Final</i>	<i>Rate</i>	
<i>IS</i>	\rightarrow <i>II</i>	β	(4.6)
<i>SI</i>	\rightarrow <i>II</i>	β	
<i>I</i>	\rightarrow <i>R</i>	γ	

Note that this dynamics is not attractive (one cannot check the conditions required in [4]), so we cannot rely on monotonicity and need another technique to study it.

Our first goal is to find a duality relation for the SIR model with generator (4.3).

4.1 Clusters as duality function

This section is devoted to showing a duality relation for the SIR model, for the purpose of studying the expected population size, and correlations as a function of time for arbitrary initial distributions, rather than only for translation invariant initial distributions, as studied in [30]. This is achieved using a duality relation between cluster functions (defined below) with a bi-layered two-dimensional random walk on two copies of the semi-infinite lattice $\mathbb{N} \times \mathbb{Z}$ that we shall label below by G and J respectively. The random walk is asymmetric on each lattice and lattice G is absorbing, i.e., once the random walker has left lattice H it cannot return to it. In addition, there is a further (single) absorbing cemetery state which the random walker can reach from lattice G . On both lattices the random walk is asymmetric.

To make this qualitative picture of the dual process precise and obtain information about the original process, we use the notation $\langle \cdot \rangle_\nu := \mathbb{E}_\nu[\cdot]$ to denote the expectation with respect to some initial distribution ν , e.g. $\langle \eta_x^a(t) \rangle_\nu$ is the expected state at site x and time t with respect to ν . As already noticed in [30] quantities of interest are written in terms of high order correlation functions, called n -point cluster functions, for $n \in \mathbb{N}$, the set of strictly positive integers:

$$G^\nu(r, n, t) := \langle \eta_{r-1}^I(t) \left(\prod_{j=0}^{n-1} \eta_{r+j}^S(t) \right) \eta_{r+n}^I(t) \rangle_\nu \quad (4.7)$$

$$H^\nu(r, n, t) := \left\langle \left(\prod_{j=0}^{n-1} \eta_{r+j}^S(t) \right) \eta_{r+n}^I(t) \right\rangle_\nu, \quad (4.8)$$

where in what follows, to lighten the notation we do not write the dependence in time explicitly.

Instead of the cluster H , it will be more convenient to define a different cluster which is of the same size as G . Namely,

$$J^\nu(r, n, t) := \langle \eta_{r-1}^R(t) \left(\prod_{j=0}^{n-1} \eta_{r+j}^S(t) \right) \eta_{r+n}^I(t) \rangle_\nu. \quad (4.9)$$

Due to (4.2), one has

$$J^\nu(r, n, t) = H^\nu(r, n, t) - H^\nu(r-1, n+1, t) - G^\nu(r, n, t), \quad (4.10)$$

so that given $J^\nu(r, n, t)$ and $G^\nu(r, n, t)$, the cluster function $H^\nu(r, n, t)$ can be computed recursively.

In particular, one can gather information about the correlations of the SIR model by choosing short clusters, see also Remark 8 below. We are now ready to give the duality result for the SIR model. To this end, denote by

$$\mathcal{S}^{\text{dual}} := (\mathbb{Z} \times \mathbb{N} \times \{G, J\}) \cup \{\partial\}, \quad (4.11)$$

where ∂ will be a trap for the dual evolution.

Theorem 5 (Duality relation for SIR). *The SIR model $(\eta_t)_{t \geq 0}$ with generator (4.3) is dual to a two dimensional biased random walk on two layers (see Figure 4) with duality function $d : \mathcal{S} \times \mathcal{S}^{\text{dual}} \rightarrow \mathbb{R}$ given by: for $\eta \in \mathcal{S}$ and $\xi \in \mathcal{S}^{\text{dual}}$, if $\xi = (r, n, i) \in \mathbb{Z} \times \mathbb{N} \times \{G, H\}$,*

$$\begin{aligned} d(\eta, (r, n, i)) &:= \eta_{r-1}^I \eta_r^S \cdots \eta_{r+n-1}^S \eta_{r+n}^I \mathbb{1}_{i=G} + \eta_{r-1}^R \eta_r^S \cdots \eta_{r+n-1}^S \eta_{r+n}^I \mathbb{1}_{i=J} \\ &= [\eta_{r-1}^I \mathbb{1}_{i=G} + \eta_{r-1}^R \mathbb{1}_{i=J}] \left(\prod_{j=0}^{n-1} \eta_{r+j}^S(t) \right) \eta_{r+n}^I, \end{aligned} \quad (4.12)$$

and

$$d(\eta, \partial) = 0. \quad (4.13)$$

The dual generator acts on local functions $f : \mathcal{S}^{\text{dual}} \rightarrow \mathbb{R}$ as follows: for $\xi \in \mathcal{S}^{\text{dual}}$, if $\xi = \partial$,

$$\mathcal{L}^{\text{dual}} f(\xi) = 0, \quad (4.14)$$

while if $\xi = (r, n, i) \in \mathbb{Z} \times \mathbb{N} \times \{G, H\}$,

$$\begin{aligned} \mathcal{L}^{\text{dual}} f(r, n, i) &= \beta \mathbb{1}_{i=G} [f(r-1, n+1, i) - f(r, n, i)] \\ &\quad + \beta [f(r, n+1, i) - f(r, n, i)] \\ &\quad + \gamma \mathbb{1}_{i=J} [f(r, n, \phi(i)) - f(r, n, i)] \\ &\quad + 2\gamma \mathbb{1}_{i=G} [f(\partial) - f(r, n, i)], \end{aligned} \quad (4.15)$$

where we define the flip operator as

$$\phi : \{G, J\} \rightarrow \{G, J\}, \text{ such that } \phi(G) = J \text{ and } \phi(J) = G. \quad (4.16)$$

Remark 7. By (4.14), ∂ is indeed a trap: once the process reaches ∂ , it remains there forever. Furthermore, the generator of the dual dynamics defines the following transitions on the dual state space, illustrated in Figure 4:

$$(r, n, i) \rightarrow (r, n+1, i) \text{ at rate } \beta \text{ for both layers: } i \in \{G, J\} \quad (4.17)$$

$$(r, n, G) \rightarrow (r-1, n+1, G) \text{ at rate } \beta \text{ only for layer } G. \quad (4.18)$$

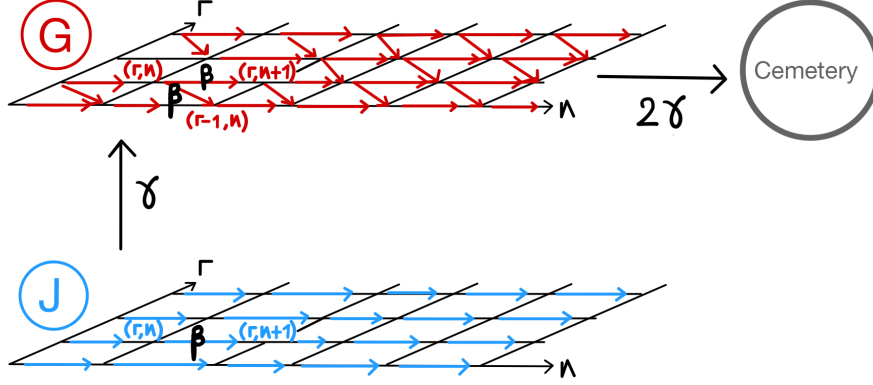


Figure 4: Dual dynamics for the SIR model

It is only possible to go from layer J to layer G , (but not the other way around):

$$(r, n, J) \rightarrow (r, n, G) \text{ at rate } \gamma, \quad (4.19)$$

and absorption in the trap is only possible if the walker is in layer G :

$$(r, n, G) \rightarrow \partial \text{ at rate } 2\gamma. \quad (4.20)$$

Notice the non-translation invariance nature of the dynamics in the second transition (4.18) above.

Proof. First, note that due to the definitions and conventions (4.11), (4.13), (4.14), (4.15), we have, for any configuration η ,

$$\mathcal{L}^{\text{SIR}} d(\cdot, \partial)(\eta) = \mathcal{L}^{\text{dual}} d(\eta, \cdot)(\partial). \quad (4.21)$$

Then, in order to show the following duality relation for all configuration η and $(r, n, i) \in \mathbb{Z} \times \mathbb{N} \times \{G, H\}$

$$\mathcal{L}^{\text{SIR}} d(\cdot, (r, n, i))(\eta) = \mathcal{L}^{\text{dual}} d(\eta, \cdot)(r, n, i), \quad (4.22)$$

it is convenient to split the SIR generator into three different actions, namely

$$\mathcal{L}^{\text{SIR}} d(\cdot, (r, n, i))(\eta) = \sum_{y \in \mathbb{Z}} [A_y + B_y + C_y], \quad (4.23)$$

where

$$\begin{aligned} A_y &:= \beta \eta_y^I \eta_{y+1}^S \left[d(T_{y+1} \eta, (r, n, i)) - d(\eta, (r, n, i)) \right] \\ B_y &:= \beta \eta_y^I \eta_{y-1}^S \left[d(T_{y-1} \eta, (r, n, i)) - d(\eta, (r, n, i)) \right] \\ C_y &:= \gamma \eta_y^I \left[d(\widehat{T}_y \eta, (r, n, i)) - d(\eta, (r, n, i)) \right], \end{aligned} \quad (4.24)$$

and where the operators T and \widehat{T} were defined in (4.4), (4.5). We now proceed by direct computation of each of the three operators.

Let us compute A_y :

- If $y + 1 \notin \{r - 1, r, r + 1, \dots, r + n - 1, r + n\}$, $A_y = 0$;
- If $y + 1 \in \{r, \dots, r + n\}$, then $\eta_y^I \eta_{y+1}^S d(T_{y+1}\eta, (r, n, i)) = 0$;
- If $y \in \{r, \dots, r + n\}$, then $\eta_y^I \eta_{y+1}^S d(\eta, (r, n, i)) = 0$;
- If $y + 1 = r - 1$, then

$$\eta_y^I \eta_{y+1}^S d(T_{y+1}\eta, (r, n, i)) = \eta_{r-2}^I \eta_{r-1}^S \left(\prod_{j=0}^{n-1} \eta_{r+j}^S \right) \eta_{r+n}^I \mathbb{1}_{i=G}.$$

Therefore,

$$A_{r-2} = \beta \mathbb{1}_{i=G} d(\eta, (r - 1, n + 1, i)). \quad (4.25)$$

- If $y + 1 = r$, then

$$\eta_y^I \eta_{y+1}^S d(\eta, (r, n, i)) = \eta_{r-1}^I \left(\prod_{j=0}^{n-1} \eta_{r+j}^S \right) \eta_{r+n}^I \mathbb{1}_{i=G}.$$

Therefore,

$$A_{r-1} = -\beta \mathbb{1}_{i=G} d(\eta, (r, n, i)). \quad (4.26)$$

Collecting (4.26) and (4.25), we are left with

$$\sum_{y \in \mathbb{Z}} A_y = \beta \mathbb{1}_{i=G} [d(\eta, (r - 1, n + 1, i)) - d(\eta, (r, n, i))]. \quad (4.27)$$

Now, we similarly compute B_y :

- For $y - 1 \notin \{r - 1, \dots, r + n\}$, $B_y = 0$.
- If $y - 1 \in \{r - 1, \dots, r + n - 1\}$, then $\eta_y^I \eta_{y-1}^S d(T_{y-1}\eta, (r, n, i)) = 0$.
- If $y \in \{r, \dots, r + n - 1\}$, then $\eta_y^I \eta_{y-1}^S d(\eta, (r, n, i)) = 0$.
- For $y - 1 = r + n - 1$, then

$$\eta_y^I \eta_{y-1}^S d(\eta, (r, n, i)) = \eta_{r+n}^I \eta_{r+n-1}^S [\eta_{r-1}^I \mathbb{1}_{i=G} + \eta_{r-1}^R \mathbb{1}_{i=J}] \left(\prod_{j=0}^{n-1} \eta_{r+j}^S \right) \eta_{r+n}^I.$$

Therefore,

$$B_{r+n} = -\beta d(\eta, (r, n, i)). \quad (4.28)$$

- For $y - 1 = r + n$, then

$$\eta_y^I \eta_{y-1}^S d(T_{y-1}\eta, (r, n, i)) = \eta_{r+n+1}^I \eta_{r+n}^S [\eta_{r-1}^I \mathbb{1}_{i=G} + \eta_{r-1}^R \mathbb{1}_{i=J}] \left(\prod_{j=0}^{n-1} \eta_{r+j}^S \right).$$

Therefore,

$$B_{r+n+1} = \beta d(\eta, (r, n + 1, i)). \quad (4.29)$$

Collecting (4.28) and (4.29), we are left with

$$\sum_{y \in \mathbb{Z}} B_y = \beta [d(\eta, (r, n+1, i)) - d(\eta, (r, n, i))]. \quad (4.30)$$

Finally, let us compute C_y :

- If $y \notin \{r-1, \dots, r+n\}$, $C_y = 0$.
- If $y \in \{r, \dots, r+n\}$, then $\eta_y^I d(\widehat{T}_y \eta, (r, n, i)) = 0$.
- If $y \in \{r, \dots, r+n-1\}$, then $\eta_y^I d(\eta, (r, n, i)) = 0$.
- If $y = r-1$,

$$\begin{aligned} \eta_y^I d(\widehat{T}_y \eta, (r, n, i)) &= \eta_{r-1}^I \left(\prod_{j=0}^{n-1} \eta_{r+j}^S \right) \eta_{r+n}^I \mathbb{1}_{i=J} \\ \eta_y^I d(\eta, (r, n, i)) &= \eta_{r-1}^I \mathbb{1}_{i=G} \left(\prod_{j=0}^{n-1} \eta_{r+j}^S \right) \eta_{r+n}^I. \end{aligned}$$

Therefore, recalling the definition (4.16) of the layer-flip operator, we have

$$C_{r-1} = \gamma \mathbb{1}_{i=J} d(\eta, (r, n, \phi(i))) - \gamma \mathbb{1}_{i=G} d(\eta, (r, n, i)). \quad (4.31)$$

- If $y = r+n$,

$$\eta_y^I d(\eta, (r, n, i)) = \eta_{r+n}^I [-\eta_{r-1}^I \mathbb{1}_{i=G} - \eta_{r-1}^R \mathbb{1}_{i=J}] \left(\prod_{j=0}^{n-1} \eta_{r+j}^S \right).$$

Therefore,

$$C_{r+n} = -\gamma d(\eta, (r, n, i)). \quad (4.32)$$

Collecting (4.31) and (4.32) we are left with

$$\sum_{y \in \mathbb{Z}} C_y = \gamma \mathbb{1}_{i=J} [d(\eta, (r, n, \phi(i))) - d(\eta, (r, n, i))] - 2\gamma \mathbb{1}_{i=G} d(\eta, (r, n, i)). \quad (4.33)$$

Finally, collecting (4.27), (4.30) and (4.33), we get

$$\begin{aligned} \mathcal{L}^{\text{SIR}} d(\cdot, (r, n, i))(\eta) &= \beta \mathbb{1}_{i=G} [d(\eta, (r-1, n+1, i)) - d(\eta, (r, n, i))] \\ &\quad + \beta [d(\eta, (r, n+1, i)) - d(\eta, (r, n, i))] \\ &\quad + \gamma \mathbb{1}_{i=J} [d(\eta, (r, n, \phi(i))) - d(\eta, (r, n, i))] \\ &\quad + 2\gamma \mathbb{1}_{i=G} [d(\eta, \partial) - d(\eta, (r, n, i))] = \mathcal{L}^{\text{dual}} d(\eta, \cdot)(r, n, i), \end{aligned} \quad (4.34)$$

where we used that $d(\eta, \partial) = 0$, so that we have recovered the generator $\mathcal{L}^{\text{dual}}$ of equation (4.15). \square

We remark that the trap state is in the same spirit as in [2], where the dual operator is interpreted as a particle system (an asymmetric exclusion process) with a cemetery state ∂ and a bigger state space, see equation (2.15) in [2].

4.2 Applications of the duality relation

Consider a measure ν on $\{S, I, R\}^{\mathbb{Z}}$ and recall the n -point cluster functions $G^\nu(r, n, t)$ and $J^\nu(r, n, t)$ defined respectively in (4.7) and (4.9). Note that $J^\nu(r, n, t)$ is not the same as $H(n)$ in [30], because of the term $\mathbb{1}_{\eta_{r-1}(t)=R}$ but we still obtain closed equations for G and J type chains.

Remark 8 (SIR correlations for small clusters). *Choosing $n = 1$ the duality functions can give information on the three point correlations given in terms of the clusters G or J , which can then be used to get two point correlations via the cluster H .*

4.2.1 Coupled equations for the cluster functions

In [30] (see equation (20)), two first order differential equations are provided and explicitly solved for $G^\nu(r, n, t)$ and $H^\nu(r, n, t)$, provided that the initial measure ν is translation invariant. Therefore, the initial measure and the dynamics (we are on \mathbb{Z}), being both translation invariant, one can drop the dependence on r and write $G^\nu(r, n, t) = G^\nu(n, t)$ and $H^\nu(r, n, t) = H^\nu(n, t)$. Below, we show that one can recover these equations for $G^\nu(n, t)$ and $J^\nu(n, t)$, by only using the duality relation of Theorem 5. Fix $n \in \mathbb{N}$, then

$$\begin{aligned} \frac{d}{dt}G^\nu(n, t) &= \frac{d}{dt}\mathbb{E}_\nu \left[d(\cdot, (r, n, G))(\eta(t)) \right] = \mathbb{E}_\nu \left[\mathcal{L}^{\text{SIR}} d(\cdot, (r, n, G))(\eta(t)) \right] \\ &= \mathbb{E}_\nu \left[\mathcal{L}^{\text{Dual}} d(\eta(t), \cdot)(r, n, G) \right] \\ &= \beta \left[\mathbb{E}_\nu \left[d(\cdot, (r-1, n+1, G))(\eta(t)) \right] - \mathbb{E}_\nu \left[d(\cdot, (r, n, G))(\eta(t)) \right] \right] \\ &+ \beta \left[\mathbb{E}_\nu \left[d(\cdot, (r, n+1, G))(\eta(t)) \right] - \mathbb{E}_\nu \left[d(\cdot, (r, n, G))(\eta(t)) \right] \right] \\ &- 2\gamma \mathbb{E}_\nu \left[d(\cdot, (r, n, G))(\eta(t)) \right] \\ &= 2\beta [G^\nu(n+1, t) - G^\nu(n, t)] - 2\gamma G^\nu(n, t), \end{aligned}$$

where we used the duality relation in the first line and the fact that G^ν does not depend on r , by translation invariance, in the last line. In the same way, we get:

$$\begin{aligned} \frac{d}{dt}J^\nu(n, t) &= \frac{d}{dt}\mathbb{E}_\nu \left[d(\cdot, (r, n, H))(\eta(t)) \right] = \mathbb{E}_\nu \left[\mathcal{L}^{\text{SIR}} d(\cdot, (r, n, H))(\eta(t)) \right] \\ &= \mathbb{E}_\nu \left[\mathcal{L}^{\text{Dual}} d(\eta(t), \cdot)(r, n, H) \right] \\ &= \beta \left[\mathbb{E}_\nu \left[d(\cdot, (r, n+1, H))(\eta(t)) \right] - \mathbb{E}_\nu \left[d(\cdot, (r, n, H))(\eta(t)) \right] \right] \\ &+ \gamma \left[\mathbb{E}_\nu \left[d(\cdot, (r, n, G))(\eta(t)) \right] - \mathbb{E}_\nu \left[d(\cdot, (r, n, H))(\eta(t)) \right] \right] \\ &= \beta [J^\nu(n+1, t) - J^\nu(n, t)] + \gamma [G^\nu(n, t) - J^\nu(n, t)]. \end{aligned}$$

that is:

$$\begin{cases} \frac{dG^\nu(n, t)}{dt} = -2(\gamma + \beta)G^\nu(n, t) + 2\beta G^\nu(n+1, t) \\ \frac{dJ^\nu(n, t)}{dt} = -(\gamma + \beta)J^\nu(n, t) + \beta J^\nu(n+1, t) + \gamma G^\nu(n, t). \end{cases}$$

Note that indeed these are the analogous of equation (20) in [30]. The first equation is exactly the same one as for G in [30] and the solution is given by:

$$G^\nu(n, t) = e^{-2(\gamma+\beta)t} \sum_{\ell \geq 0} \frac{(2\beta t)^\ell}{\ell!} G^\nu(\ell + n, 0). \quad (4.35)$$

The second equation can be solved similarly in a recursive way, by treating the terms $J^\nu(n+1, t)$ and $G^\nu(n, t)$ as inhomogeneities. The solution is given by

$$J^\nu(n, t) = e^{-(\gamma+\beta)t} \sum_{\ell \geq 0} \frac{(\beta t)^\ell}{\ell!} J^\nu(\ell+n, 0) + \gamma \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{\ell \geq 0} \frac{(\beta(t-s))^\ell}{\ell!} G^\nu(\ell+n, s) ds. \quad (4.36)$$

Then, using that $\eta_{r-1}^S + \eta_{r-1}^I + \eta_{r-1}^R = 1$, and noticing that

$$H^\nu(n, t) = H^\nu(n+1, t) + J^\nu(n, t) + G^\nu(n, t),$$

the equation for H^ν , which is the same as the one in [30], is

$$\frac{dH^\nu(n, t)}{dt} = -(\gamma + \beta)H^\nu(n, t) + \beta(H^\nu(n+1, t) - G^\nu(n, t)),$$

with solution

$$H^\nu(n, t) = e^{-(\gamma+\beta)t} \sum_{\ell \geq 0} \frac{(\beta t)^\ell}{\ell!} H(\ell+n, 0) - \beta \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{\ell \geq 0} \frac{(\beta(t-s))^\ell}{\ell!} G(\ell+n, s) ds.$$

Therefore, we can write the solution J^ν in terms of the H^ν cluster as

$$J^\nu(n, t) = H^\nu(n, t) - H^\nu(n+1, t) - G^\nu(n, t). \quad (4.37)$$

4.2.2 Non-translation invariant case

Recall that in section 4.1, we have proved that the SIR model is dual with a process defined on the state space $\Omega_{\text{Dual}} := (\mathbb{Z} \times \mathbb{N} \times \{G, J\}) \cup \{\partial\}$, where ∂ is a trap state. Recall the dynamics of the dual process given in Remark 7. Denote by ξ an element of Ω_{Dual} . For any SIR configuration η , by abuse of notation, we write

$$G^\eta(r, n, t) =: G^{\delta_\eta}(r, n, t) \text{ and } J^\eta(r, n, t) =: J^{\delta_\eta}(r, n, t),$$

the cluster correlation functions when the system is initially in state η . Note that the measure δ_η is no longer a translation invariant measure for the SIR process (unless it is a constant configuration, in which case the system remains unchanged).

In [30], these cluster correlation functions have not been solved in the case where ν is not translation invariant, because the equations obtained in G and J or H are not closed. Our goal now is to get an explicit expression of such cluster functions in the non translation invariant case, by using the dual process.

Theorem 6. *The $G^\eta(r, n, t)$ cluster is given by:*

$$G^\eta(r, n, t) = e^{-2(\gamma+\beta)t} \sum_{\substack{(a,b) \in (\mathbb{N})^2 \\ a \leq b}} \frac{\binom{b}{a} (2\beta t)^b}{2^b b!} G^\eta(r-a, n+b, 0). \quad (4.38)$$

Theorem 7. *The $J^\eta(r, n, t)$ cluster is given by:*

$$\begin{aligned} J^\nu(r, n, t) &= e^{-(\gamma+\beta)t} \sum_{k \geq 0} \frac{(\beta t)^k}{k!} J^\nu(r, n+k, 0) \\ &+ \gamma e^{-(\gamma+\beta)t} \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a,b) \in \mathbb{N}^2}} \frac{(\beta s)^k}{k!} \frac{\binom{b}{a} (2\beta(t-s))^b}{2^b b!} G^\nu(r-a, n+k+b, 0) ds. \end{aligned} \quad (4.39)$$

For the proofs of Theorems 6 and 7, we introduce the dynamics on $\mathbb{Z} \times \mathbb{N} \times \{G\}$ given by the following transition rates

$$(r, n, G) \rightarrow (r - 1, n + 1, G) : \text{ at rate } \beta, \quad \text{and} \quad (r, n, G) \rightarrow (r, n + 1, G) : \text{ at rate } \beta. \quad (4.40)$$

Denote by $\mathbb{P}_{(r,n,G)}^G$, resp. $\mathbb{E}_{(r,n,G)}^{\text{Dual},G}$ the probability measure, resp. expectation, under this dynamics, when starting from (r, n, G) .

Also, consider the dynamics on $\mathbb{Z} \times \mathbb{N} \times \{J\}$ given by the following transition rates

$$(r, n, J) \rightarrow (r, n + 1) : \text{ at rate } \beta. \quad (4.41)$$

Denote by $\mathbb{P}_{(r,n,J)}^J$, resp. $\mathbb{E}_{(r,n,J)}^J$ the probability measure, resp. expectation, under this dynamics when starting from (r, n, G) .

Proof of Theorem 6. Fix $(r, n) \in \mathbb{Z}^2$:

$$\begin{aligned} G^\eta(r, n, t) &= \mathbb{E}_\eta[d(\eta(t), (r, n, G))] = \mathbb{E}_{(r,n,G)}^{\text{Dual}}[d(\eta, \xi(t))] = \mathbb{E}_{(r,n,G)}^{\text{Dual}}[d(\eta, \xi(t)) \cap (\xi(t) \neq \partial)] \\ &= \mathbb{E}_{(r,n,G)}^{\text{Dual}}[d(\eta, \xi(t)) \cap (\forall 0 \leq s \leq t, \xi_s \neq \partial)] \\ &= \mathbb{E}_{(r,n,G)}^{\text{Dual}}[\mathbb{1}_{\{\forall 0 \leq s \leq t, \xi_s \neq \partial\}} d(\eta, \xi(t))] \\ &= \mathbb{P}_{(r,n,G)}[\forall 0 \leq s \leq t, \xi_s \neq \partial] \mathbb{E}_{(r,n,G)}^{\text{Dual}}[d(\eta, \xi(t)) | \forall 0 \leq s \leq t, \xi_s \neq \partial] \\ &= \mathbb{P}_{(r,n,G)}[\forall 0 \leq s \leq t, \xi_s \neq \partial] \times \mathbb{E}_{(r,n,G)}^G[d(\eta, (r(t), n(t), G))] \\ &= e^{-2\gamma t} \mathbb{E}_{(r,n,G)}^G[d(\eta, (r(t), n(t), G))]. \end{aligned}$$

The fourth line comes from the strong Markov property. The fifth line is due to the fact that when conditioned on not having reached the trap ∂ up to time t , the dual process starting from (r, n, G) has the same dynamics as the one defined in (4.40). The last line comes from the fact that the time it takes to reach the trap ∂ from any (r, n, ∂) , is given by an exponential clock of parameter 2γ . Expanding the expectation in the last line, we are left with:

$$\begin{aligned} G^\eta(r, n, t) &= e^{-2\gamma t} \sum_{(\tilde{r}, \tilde{n}) \in \mathbb{Z} \times \mathbb{N}} d(\eta, (\tilde{r}, \tilde{n})) \mathbb{P}_{(r,n,G)}^G[(r(t), n(t), G) = (\tilde{r}, \tilde{n}, G)] \\ &= e^{-2\gamma t} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b}} d(\eta, (r - a, n + b)) \mathbb{P}_{(r,n,G)}^G[(r(t), n(t), G) = (r - a, n + b, G)] \\ &= e^{-2\gamma t} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b}} d(\eta, (r - a, n + b)) e^{-2\beta t} \frac{\binom{b}{a}}{2^b} \frac{(2\beta t)^b}{b!} \\ &= e^{-2(\gamma+\beta)t} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b}} \frac{\binom{b}{a}}{2^b} \frac{(2\beta t)^b}{b!} G^\eta(r - a, n + b, 0). \end{aligned}$$

The third line comes from the fact that in order to reach $(r - a, n + b)$ starting from (r, n) and with the dynamics (4.40), one has to perform $\binom{b}{a}$ steps, where r decreases by 1 and n increases by 1 and, the rest of the steps where it is only n that increases by 1. The time between two such jumps is an exponential clock with parameter 2β . \square

The other cluster can be found in a similar fashion.

Proof of Theorem 7. Similarly to the previous computation we have

$$\begin{aligned}
J^\eta(r, n, t) &= \mathbb{E}_\eta[d(\eta_t, (r, n, J))] = \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, \xi(t))] = \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, \xi(t)) \cap (\xi(t) \neq \partial)] \\
&= \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), i(t))) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)] \\
&= \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), G)) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)] \\
&+ \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), J)) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)],
\end{aligned} \tag{4.42}$$

where in the last line, we partitioned according to having or not a jump from J to G before time t . Defining:

$$\mathcal{A}_s^t := (\forall 0 \leq u \leq s^-, \xi_3(u) = J) \cap (\forall s^+ \leq u \leq t, \xi_3(u) = G) \cap (\forall 0 \leq u \leq t, \xi(u) \neq \partial),$$

the event that before time t , the walk does not reach the trap ∂ and at time $0 \leq s \leq t$, the walk jumps from lane J to G , we have:

$$\begin{aligned}
\mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), G)) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)] &= \int_0^t \mathbb{E}_{(r, n, J)}^{\text{Dual}}[\mathcal{A}_s^t] ds \\
&= \int_0^t d\mathbb{P}_{(r, n, J)}^{\text{Dual}}[\mathcal{A}_s^t] \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), i(t))) | \mathcal{A}_s^t].
\end{aligned} \tag{4.43}$$

Using that the law of the jump from state J to G is an exponential law of parameter γ , and that the rate of jump from state G to the trap ∂ is 2γ , we have

$$\begin{aligned}
d\mathbb{P}_{(r, n, J)}^{\text{Dual}}[\mathcal{A}_s^t] &= \gamma e^{-\gamma s} (1 - \int_0^{t-s} 2\gamma e^{-2\gamma u} du) ds \\
&= \gamma e^{-\gamma s} e^{-2\gamma(t-s)} ds = \gamma e^{-2\gamma t} e^{\gamma s} ds.
\end{aligned}$$

Now, we expand (4.43) by partitioning on the number of increases in n up to time s for the dual process conditioned on being of type J between times 0 and s , for a certain $s \in [0, t]$:

$$\begin{aligned}
&\mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), i(t))) | \mathcal{A}_s^t] \\
&= \sum_{k \geq 0} \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), i(t))) \cap (r(s), n(s), i(s)) = (r, n+k, J) | \mathcal{A}_s^t] \\
&= \sum_{k \geq 0} \mathbb{P}_{(r, n, J)}^J[(r(s), n(s), J) = (r, n+k, J)] \mathbb{E}_{(r, n+k, G)}^G[d(\eta, (r(t-s), n(t-s), G))] \\
&= \sum_{k \geq 0} e^{-\beta s} \frac{(\beta s)^k}{k!} \sum_{\substack{a \leq b \\ (a, b) \in \mathbb{N}^2}} d(\eta, (r-a, n+k+b, G)) \\
&\times \mathbb{P}_{(r, n+k, G)}^G[(r(t-s), n(t-s), G) = (r-a, r+k+b, G)] \\
&= \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a, b) \in \mathbb{N}^2}} e^{-\beta s} \frac{(\beta s)^k}{k!} e^{-2\beta(t-s)} \frac{\binom{b}{a}}{2^b} \frac{(2\beta(t-s))^b}{b!} G^\eta(r-a, n+k+b, 0),
\end{aligned}$$

where we used the strong Markov property in the second line, and expanded the expectation under the process (4.40) in the third line. Therefore,

$$\begin{aligned}
&\mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), G)) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)] = \\
&\gamma e^{-2\gamma t} \int_0^t e^{\gamma s} \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a, b) \in \mathbb{N}^2}} e^{-\beta s} \frac{(\beta s)^k}{k!} e^{-2\beta(t-s)} \frac{\binom{b}{a}}{2^b} \frac{(2\beta(t-s))^b}{b!} G^\eta(r-a, n+k+b, 0) ds.
\end{aligned} \tag{4.44}$$

Finally, the second expectation in the last line in (4.42) is given by:

$$e^{-\gamma t} \sum_{k \geq 0} e^{-\beta t} \frac{(\beta t)^k}{k!} J^\eta(r, n+k, 0). \quad (4.45)$$

Collecting (4.44) and (4.45) yields:

$$\begin{aligned} J^\eta(r, n, t) &= e^{-(\gamma+\beta)t} \sum_{k \geq 0} \frac{(\beta t)^k}{k!} J^\eta(r, n+k, 0) \\ &+ \gamma e^{-2\gamma t} \int_0^t e^{\gamma s} \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a,b) \in \mathbb{Z}^2}} e^{-\beta s} \frac{(\beta s)^k}{k!} e^{-2\beta(t-s)} \frac{\binom{b}{a}}{2^b} \frac{(2\beta(t-s))^b}{b!} G^\eta(r-a, n+k+b, 0) ds \\ &= e^{-(\gamma+\beta)t} \sum_{k \geq 0} \frac{(\beta t)^k}{k!} J^\eta(r, n+k, 0) \\ &+ \gamma e^{-(\gamma+\beta)t} \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a,b) \in \mathbb{Z}^2}} \frac{(\beta s)^k}{k!} \frac{\binom{b}{a}}{2^b} \frac{(2\beta(t-s))^b}{b!} G^\eta(r-a, n+k+b, 0) ds. \end{aligned}$$

□

Theorems 6 and 7 hold for any initial measure ν which is not necessarily translation invariant. We conclude this section showing that under the extra assumption of translation invariance we can recover the expressions (21) and (22) in [30].

Special case: translation invariance

Let us recover the formulas obtained in [30] in the translation invariant case using ODE's (here we have not used such ODE's), from (4.38) and (4.39). If ν is invariant under translation, $G^\nu(r, n, 0)$ and $J^\nu(r, n, 0)$ are independent of r , so

$$\begin{aligned} G^\nu(n, t) &:= G^\nu(r, n, t) = e^{-2(\gamma+\beta)t} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b}} \frac{\binom{b}{a}}{2^b} \frac{(2\beta t)^b}{b!} G^\nu(r-a, n+b, 0) \\ &= e^{-2(\gamma+\beta)t} \sum_{b \geq 0} \sum_{a=0}^b \frac{\binom{b}{a}}{2^b} \frac{(2\beta t)^b}{b!} G^\nu(n+b, 0). \end{aligned} \quad (4.46)$$

Therefore, $G^\nu(n, t) = e^{-2(\gamma+\beta)t} \sum_{b \geq 0} \frac{(2\beta t)^b}{b!} G^\nu(n+b, 0)$,

which matches (21) in [30]. Now,

$$\begin{aligned}
J^\nu(n, t) &:= J^\nu(r, n, t) = e^{-(\gamma+\beta)t} \sum_{k \geq 0} \frac{(\beta t)^k}{k!} J^\nu(n+k, 0) \\
&+ \gamma e^{-(\gamma+\beta)t} \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a,b) \in \mathbb{N}^2}} \frac{(\beta s)^k}{k!} \frac{\binom{b}{a}}{2^b} \frac{(2\beta(t-s))^b}{b!} G^\nu(n+k+b, 0) ds \\
&= e^{-(\gamma+\beta)t} \sum_{k \geq 0} \frac{(\beta t)^k}{k!} J^\nu(n+k, 0) \\
&+ \gamma e^{-(\gamma+\beta)t} \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{k \geq 0} \sum_{b \geq 0} \frac{(\beta s)^k}{k!} \frac{(2\beta(t-s))^b}{b!} G^\nu(n+k+b, 0) ds.
\end{aligned} \tag{4.47}$$

Performing the change of variable $u = t - s$ in the second integral, we get

$$\begin{aligned}
&\int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{k \geq 0} \sum_{b \geq 0} \frac{(\beta s)^k}{k!} \frac{(2\beta(t-s))^b}{b!} G^\nu(n+k+b, 0) ds \\
&= \int_0^t \sum_{k \geq 0} \frac{(\beta(t-u))^k}{k!} e^{-(\gamma+\beta)u} \sum_{b \geq 0} \frac{(2\beta u)^b}{b!} G^\nu(n+k+b, 0) ds \\
&= \int_0^t \sum_{k \geq 0} \frac{(\beta(t-u))^k}{k!} G^\nu(n+k, s) ds,
\end{aligned} \tag{4.48}$$

where we used the expression for $G^\nu(n, t)$ given by (4.46). Finally, we get

$$J^\nu(n, t) = e^{-(\gamma+\beta)t} \sum_{\ell \geq 0} \frac{(\beta t)^\ell}{\ell!} J^\nu(\ell+n, 0) + \gamma \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{\ell \geq 0} \frac{(\beta(t-s))^\ell}{\ell!} G^\nu(\ell+n, s) ds, \tag{4.49}$$

which is exactly the solution given in (4.36), found by solving the ODE for J , in the same spirit as in [30].

A Invariant measure of the diffusive contact process on small finite sets

Here, in the same spirit as [20], we compute the stationary distribution of the diffusive contact process for small N , where we recall that the dynamics is given by the generator \mathcal{L}^{DCP} , defined in (2.12), here we stress the dependence on the lattice size with the subscript N . This approach is based on the definition of the invariant measure as the probability distribution ν^{DCP} satisfying $\nu^{\text{DCP}} \mathcal{L}^{\text{DCP}} = 0$. Here, configurations are represented in their vectorial form, for instance, for $N = 1$, (0) is the empty configuration and (1) the full one.

We notice that in our case the presence of boundary reservoirs allows to see the effect of the diffusive coefficient \mathcal{D} already when the bulk size is $N = 2$. This is in contrast to the scenario studied in Section 6.2 of [20] with boundary conditions $\eta_0 = \eta_{N+1} = 1$ when the results are equivalent to the case of $\mathcal{D} = 0$.

$N = 1$ site

Let us solve the stationary condition $\nu^{\text{DCP}} \mathcal{L}^{\text{DCP}} = 0$. For $N = 1$, the corresponding intensity matrix is

$$L^{\text{DCP}} = \begin{pmatrix} -(\alpha + \delta) & (\alpha + \delta) \\ (\gamma + \beta + 1) & -(\gamma + \beta + 1) \end{pmatrix}$$

and the stationary condition yields

$$\begin{cases} \nu^{\text{DCP}}((0)) &= \frac{\gamma + \beta + 1}{1 + \beta + \gamma + \delta + \alpha} \\ \nu^{\text{DCP}}((1)) &= \frac{\alpha + \delta}{1 + \beta + \gamma + \delta + \alpha}. \end{cases}$$

Notice that for just one bulk site there is no diffusive effect.

$N = 2$ sites

This is the first non trivial case where the diffusion parameter \mathcal{D} plays a role. In this case the intensity matrix is

$$L^{\text{DCP}} = \begin{pmatrix} -(\alpha + \delta) & \delta & \alpha & 0 \\ 1 + \beta & -(1 + \beta + \mathcal{D} + \lambda + \alpha) & \mathcal{D} & \lambda + \alpha \\ 1 + \gamma & \mathcal{D} & -(1 + \gamma + \mathcal{D} + \lambda + \delta) & \lambda + \delta \\ 0 & 1 + \gamma & 1 + \beta & -(2 + \beta + \gamma) \end{pmatrix}$$

and the stationary condition yields

$$\begin{cases} c\nu^{\text{DCP}}((0,0)) &= \mathcal{D}(\beta + \gamma + 2)^2 + (\alpha + \beta + \gamma + \delta + 2 + 2\lambda)(1 + \beta + \gamma + \beta\gamma) \\ c\nu^{\text{DCP}}((1,0)) &= \mathcal{D}(2 + \beta + \gamma)(\alpha + \delta) + (\gamma + 1)[\lambda(\alpha + \delta) + \delta(\alpha + \beta + \gamma + \delta + 2)] \\ c\nu^{\text{DCP}}((0,1)) &= \mathcal{D}(2 + \beta + \gamma)(\alpha + \delta) + (\beta + 1)[\lambda(\alpha + \delta) + \alpha(\alpha + \beta + \gamma + \delta + 2)] \\ c\nu^{\text{DCP}}((1,1)) &= \mathcal{D}(\alpha^2 + \delta^2 + 2\alpha + 2\lambda(\alpha + \delta)) + \lambda(\alpha + \delta)(\alpha + \delta + \lambda + 1) + \lambda(\alpha\beta + \gamma\delta), \end{cases} \quad (\text{A.1})$$

where

$$c = \mathcal{D} [(\alpha + \beta + \gamma + \delta + 2)^2 + 2\lambda(\alpha + \delta)] + (\alpha + \beta + \gamma + \delta)^2 + 3(\alpha + \beta + \gamma + \delta) + (\alpha + \gamma)(\beta + \delta)(\alpha + \beta + \gamma + \delta + 2) + \lambda(\alpha + \delta)(\alpha + \delta + \lambda + 3) + 2\lambda(\beta + \gamma + 1)$$

In both cases, setting $\alpha = \delta = \lambda$ and $\gamma = \beta = 0$, our results correspond to the ones in [20], as expected. This strategy explicitly characterizes the stationary measure but only works when the size of the system is small.

Taking the limit $\mathcal{D} \rightarrow \infty$, the solutions in (A.1) become

$$\begin{cases} \nu^{\text{DCP}}((0,0))(\infty) &= \frac{(\beta + \gamma + 2)^2}{(\alpha + \beta + \gamma + \delta + 2)^2 + 2\lambda(\alpha + \delta)} \\ \nu^{\text{DCP}}((1,0))(\infty) &= \frac{(2 + \beta + \gamma)(\alpha + \delta)}{(\alpha + \beta + \gamma + \delta + 2)^2 + 2\lambda(\alpha + \delta)} \\ \nu^{\text{DCP}}((0,1))(\infty) &= \frac{(2 + \beta + \gamma)(\alpha + \delta)}{(\alpha + \beta + \gamma + \delta + 2)^2 + 2\lambda(\alpha + \delta)} \\ \nu^{\text{DCP}}((1,1))(\infty) &= \frac{(\alpha^2 + \delta^2 + 2\alpha + 2\lambda(\alpha + \delta))}{(\alpha + \beta + \gamma + \delta + 2)^2 + 2\lambda(\alpha + \delta)} \end{cases} \quad (\text{A.2})$$

Due to the infinite diffusion, $\nu^{\text{DCP}}((1,0))(\infty) = \nu^{\text{DCP}}((0,1))(\infty)$, so this quantity only depends on the total number of particles, regardless of their position.

B Fokker-Planck equations

Recall that given \mathcal{L} , the generator of a dynamics on a state space Ω , we have the following characterization of an invariant measure

Theorem 8. (in [23, Chapter 1, Proposition 1.8]) A measure μ is invariant for \mathcal{L} if and only if, the following Fokker-Planck equation is satisfied:

$$\int_{\Omega} \mathcal{L}f d\mu = 0, \quad \forall f \in C(\Omega). \quad (\text{B.1})$$

Consider \mathcal{L}^{DCP} , the generator of the open diffusive contact process, defined in (2.12) with state space Ω_N , and where in the boundary generators \mathcal{L}_- and \mathcal{L}_+ , we take $\alpha, \gamma \neq 0$ and $\delta = \beta = 0$, in other words, the right hand side boundary is closed. Recall that we denoted by ν^{DCP} , the unique invariant measure of the process. Applying (B.1) to the functions $f(\eta) = \eta_{x_1} \dots \eta_{x_\ell}$ allows to compute the correlation functions.

- For $N = 1$, take $f(\eta) = \eta_1$. Then,

$$\mathcal{L}_- \eta_1 = \alpha(1 - \eta_1) - \gamma \eta_1, \quad \mathcal{L}^{\text{CP}} \eta_1 = -\eta_1,$$

so (B.1) yields

$$\alpha(1 - \rho_1(1)) - \gamma \rho_1(1) - \rho_1(1) = 0,$$

that is,

$$\rho_1(1) = \frac{\alpha}{1 + \gamma + \alpha}.$$

- For $N = 2$, we have

$$\begin{aligned} \mathcal{L}_- \eta_1 &= \alpha(1 - \eta_1) - \gamma \eta_1 = \alpha - (\alpha + \gamma) \eta_1, & \mathcal{L}_- \eta_2 &= 0, \\ \mathcal{L}_- \eta_1 \eta_2 &= \eta_2 \mathcal{L}_- \eta_1 = \alpha \eta_2 - (\alpha + \gamma) \eta_1 \eta_2, \\ \mathcal{L}^{\text{CP}} \eta_1 &= -\eta_1 + \lambda(1 - \eta_1) \eta_2, & \mathcal{L}^{\text{CP}} \eta_2 &= -\eta_2 + \lambda(1 - \eta_2) \eta_1, \\ \mathcal{L}^{\text{CP}} \eta_1 \eta_2 &= \eta_1 \mathcal{L}^{\text{CP}} \eta_2 + \eta_2 \mathcal{L}^{\text{CP}} \eta_1 = -2(\lambda + 1) \eta_1 \eta_2 + \lambda \eta_1 + \lambda \eta_2, \\ \mathcal{D} \mathcal{L}^{\text{SSEP}} \eta_1 &= D(\eta_2 - \eta_1), & \mathcal{D} \mathcal{L}^{\text{SSEP}} \eta_2 &= D(\eta_1 - \eta_2), & \mathcal{D} \mathcal{L}^{\text{SSEP}} \eta_1 \eta_2 &= 0. \end{aligned}$$

Now we have, recalling the notation $\rho_1^{\text{DCP}}(1) = \mathbb{E}_{\nu^{\text{DCP}}}[\eta_1] =: x$, $\rho_1^{\text{DCP}}(2) = \mathbb{E}_{\nu^{\text{DCP}}}[\eta_2] =: y$ and $\rho_2^{\text{DCP}}(1, 2) = \mathbb{E}_{\nu^{\text{DCP}}}[\eta_1 \eta_2] =: z$:

$$\begin{aligned} & \int (\mathcal{L}_- \eta_1 + \mathcal{L}^{\text{CP}} \eta_1 + \mathcal{D} \mathcal{L}^{\text{SSEP}} \eta_1) d\nu^{\text{DCP}}(\eta) = 0 \\ \Leftrightarrow & \alpha - (\alpha + \gamma)x - x + \lambda y - \lambda z + \mathcal{D}y - Dx = 0 \end{aligned} \quad (\text{B.2})$$

then

$$\begin{aligned} & \int (\mathcal{L}_- \eta_2 + \mathcal{L}^{\text{CP}} \eta_2 + \mathcal{D} \mathcal{L}^{\text{SSEP}} \eta_2) d\nu^{\text{DCP}}(\eta) = 0 \\ \Leftrightarrow & -y + \lambda x - \lambda z + Dx - Dy = 0, \end{aligned} \quad (\text{B.3})$$

and

$$\begin{aligned} & \int (\mathcal{L}_- \eta_1 \eta_2 + \mathcal{L}^{\text{CP}} \eta_1 \eta_2 + \mathcal{D} \mathcal{L}^{\text{SSEP}} \eta_1 \eta_2) d\nu^{\text{DCP}}(\eta) = 0 \\ \Leftrightarrow & \alpha y - (\alpha + \gamma)z - 2(\lambda + 1)z + \lambda x + \lambda y = 0, \end{aligned} \quad (\text{B.4})$$

Gathering (B.2), (B.3), (B.4), we are left to solve

$$\begin{cases} (\alpha + \gamma + \mathcal{D} + 1)x - (\lambda + \mathcal{D})y + \lambda z = \alpha \\ (\lambda + \mathcal{D})x - (\mathcal{D} + 1)y - \lambda z = 0 \\ \lambda x + (\alpha + \lambda)y - (\alpha + \gamma + 2\lambda + 2)z = 0. \end{cases} \quad (\text{B.5})$$

The solution of this system is given by:

$$\begin{cases} x = \widehat{d}(\mathcal{D})^{-1}\alpha [(\mathcal{D} + 1)(\alpha + \gamma + 2\lambda + 2) + \alpha\lambda + \lambda^2], \\ y = \widehat{d}(\mathcal{D})^{-1}\alpha [\mathcal{D}(\alpha + \gamma + 2\lambda + 2) + \alpha\lambda + \gamma\lambda + \lambda^2 + 2\lambda], \\ z = \widehat{d}(\mathcal{D})^{-1}\alpha [\mathcal{D}(\alpha + 2\lambda) + \alpha\lambda + \lambda^2 + \lambda]. \end{cases} \quad (\text{B.6})$$

where

$$\widehat{d}(\mathcal{D}) = \mathcal{D} [(\alpha + \gamma + 2)^2 + 2\alpha\lambda] + (\alpha + \gamma + \lambda + 1)(\alpha + \gamma + \lambda\alpha + 2) + \lambda(\alpha + \gamma).$$

Taking the limit $\mathcal{D} \rightarrow \infty$, the solutions in (B.6) simplifies to

$$\begin{cases} x(\infty) = \frac{\alpha(\alpha + \gamma + 2\lambda + 2)}{(\alpha + \gamma + 2)^2 + 2\alpha\lambda} \\ y(\infty) = \frac{\alpha(\alpha + \gamma + 2\lambda + 2)}{(\alpha + \gamma + 2)^2 + 2\alpha\lambda} \\ z(\infty) = \frac{\alpha(\alpha + 2\lambda)}{(\alpha + \gamma + 2)^2 + 2\alpha\lambda}. \end{cases} \quad (\text{B.7})$$

Notice that, as remarked in Section 2.3.3 for the absorption probabilities of one dual particle either in site 1 or site 2, also the corresponding correlations $\rho_1^{\text{DCP}}(1)$ and $\rho_1^{\text{DCP}}(2)$ are the same.

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