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Duality for some models of epidemic spreading

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Abstract

We examine the role of boundaries and the structure of nontrivial duality functions for three non-conservative interacting particle systems in one dimension that model epidemic spreading: (i) the diffusive contact process (DCP), (ii) a model that we introduce and call generalized diffusive contact process (GDCP), both in finite volume in contact with boundary reservoirs, i.e., with open boundaries, and (iii) the susceptible-infectious-recovered (SIR) model on \mathbb{Z} . We establish duality relations for each system through an analytical approach. It turns out that with open boundaries self-duality breaks down and qualitatively different properties compared to closed boundaries (i.e., finite volume without reservoirs) arise: Both the DCP and GDCP are ergodic but no longer absorbing, while the respective dual processes are absorbing but not ergodic. We provide expressions for the stationary correlation functions in terms of the dual absorption probabilities. We perform explicit computations for a small sized DCP, and for arbitrary size in a particular setting of the GDCP. The duality function is factorized for the DCP and GDCP, contrary to the SIR model for which the duality relation is nonlocal and yields an explicit expression of the time evolution of some specific correlation functions, describing the time decay of the sizes of clusters of susceptible individuals.

1 Introduction

In the context of Markov processes, duality is a remarkable tool to analyze a model of interest using another model, its dual, via an observable for both models called duality function. For interacting particle systems, there are various ways of deriving a duality relation : using analytical tools via computations on generators or intensity matrices, through pathwise methods via a graphical representation, or by an algebraic approach. A nonexhaustive list of books or surveys that deal with duality is [30, 29, 20, 39, 44, 7], see also references therein.

In this article, we study three different non-conservative interacting particle systems defined on a one-dimensional lattice, namely (i) the diffusive contact process (see for instance [25]) on a finite lattice, abbreviated below by DCP, (ii) a generalization thereof with a variable death rate, abbreviated below by GDCP which like the DCP is a special case of a broad class of reaction diffusion systems surveyed with a view on duality in [37, 46, 45], and (iii) the susceptible-infectious-recovered (SIR) model on \mathbb{Z} (as in [40]). All these particle systems, described informally below and defined formally in terms of their generators in Sections 2–4,

are Markovian models for the spreading of infectious diseases, but with different microscopic dynamics. For each of these models we prove a duality relation using analytical tools.

Informative duality relations are, in general, rare and exceptional. They occur, for example, in conservative particle systems in the presence of non-Abelian symmetries of the generator, as first pointed out in [36] and further developed in [18], or when the time evolution of n -point correlations is fully determined by k -point correlations of lower or equal order $k \leq n$ [37, 16]. This is not the case in general for models such as ours, where the total number of particles is not conserved by the bulk dynamics, and the evolution of correlations involves higher order correlations.

The main novelty of our discussion of duality in the DCP and the GDCP are open boundaries that allow for particle exchanges with external reservoirs at the boundary sites of the one-dimensional finite lattice, much like in the paradigmatic open simple exclusion process [43, 41, 10, 6, 13]. In the papers [37, 46] such reservoirs are not considered and in [45] they are actually explicitly ruled out as they include spontaneous creation of particles which invalidates the results obtained in that paper. Reservoirs have been considered in [42], but at generic lattice sites and for the purpose of establishing ergodic theorems, while we focus on boundary reservoirs and study their effect on correlations.

Indeed, it turns out that open boundaries lead to qualitatively different properties: For closed boundaries the empty configuration is an absorbing state, so the invariant measure is unique and completely explicit. It is the trivial Dirac measure concentrated on the empty configuration. Moreover, both the closed DCP and closed GDCP are self-dual [25, 45]. In contrast, when adding reservoirs of particles, which can remove or insert particles into the system at random times, three new features arise: (i) The empty configuration is no longer an absorbing state, as the reservoirs maintain the process “alive”. (ii) Self-duality is lost. The dual process has absorbing sinks which are additional sites at the edges of the one-dimensional lattice which absorb particles without ever releasing them. (iii) The dual process is not ergodic. As time tends to infinity, the limiting measure depends strongly on the initial configuration.

Despite these differences some features of duality persist in the presence of reservoirs. This is the factorization of the duality function, meaning that the global duality function is a product over local duality functions on each lattice site. Such a duality is long-known for the contact process [29] both on \mathbb{Z} and for closed boundaries and appears also in the closed DCP and GDCP [25, 45]. Here we prove – somewhat unexpectedly – that apart from extra boundary terms the same factorized duality function can be used also for the open DCP and open GDCP and yields information about the invariant measure in terms of absorption probabilities of the dual process. In a special case of the GDCP, it also yields explicit results for the time-dependent one-point function for arbitrary initial states.

Duality functions that factorize into local duality functions for each lattice site appear also in the general discussions of duality for reaction-diffusion systems of [37, 46, 16, 45] and thus factorization appears to be a natural feature of informative dualities [34]. For the SIR model on \mathbb{Z} , the main novelty is the appearance of a non-factorized kind of duality, that we call *cluster duality*, since the duality function relates the particle configuration in a cluster of neighboring sites to the dual process which involves the boundaries of the cluster. This type of non-local duality function, reminiscent of the so-called empty-interval method used to study correlations in reaction-diffusion processes [5, 1], appears to be unexplored in the context of duality in interacting particle systems and may have further applications to other models. Also, note that although we treat each model separately, we aim, in future work, at a unified treatment of such models, via an algebraic approach.

We now briefly describe the three models studied in this work, that are one-dimensional interacting particle systems, either on \mathbb{Z} or on $\Lambda_N := \{1, \dots, N\}$ with $N \geq 1$ sites, and the results

we obtain for them through duality. Let us first recall the definition of duality between two Markov processes.

Definition 1 (Definition 3.1 in Chapter II of [30]). *Let $X = (X_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ be two continuous time Markov processes with state spaces Ω and Ω^{dual} , respectively. We say that X and Y are dual with respect to a local duality function $D : \Omega \times \Omega^{\text{dual}} \rightarrow \mathbb{R}$ if*

$$\mathbb{E}_x D(X_t, y) = \mathbb{E}_y D(x, Y_t) \quad (1.1)$$

for all $(x, y) \in \Omega \times \Omega^{\text{dual}}$ and $t \geq 0$. In equation (1.1), \mathbb{E}_x (respectively \mathbb{E}_y) is the expectation w.r.t. the law of the process X initialized at x (respectively the process Y initialized at y).

In the diffusive contact process on Λ_N (DCP), each lattice site can be occupied by at most one particle. Its dynamics is the superposition of the (basic) contact process (CP) with birth rate $\lambda > 0$, with the symmetric simple exclusion process (SSEP), a particle exchange dynamics parameterized by a diffusion coefficient $\mathcal{D} \geq 0$. In the CP, a particle on site x on the lattice “dies” (i.e., disappears from the lattice) with rate 1 and “infects” (i.e. attempts to create another particle), with rate $\lambda > 0$, on a neighboring site, provided the latter is empty; otherwise the creation attempt fails. In the SSEP, particles attempt to jump to nearest neighbor sites, at rate $\mathcal{D} \geq 0$, provided the chosen target site is empty; otherwise the jump attempt fails, see [30, 29] for a precise definition of both the CP and the SSEP.

The goal of using duality for the DCP, is to get a better understanding of its stationary state, when in contact with boundary reservoirs. In Section 2, we establish in Theorem 1 a duality result between the DCP with an open boundary, and the DCP with a purely absorbing boundary. From this duality relation, we provide a new expression for the correlation functions of the invariant measure in terms of absorption probabilities, and compute them for small sized systems in appendix A.2. We also comment briefly on the fast stirring limit, in which the process effectively reduces to a birth-death chain on the integers $\{0, 1, \dots, N\}$. Note that fast stirring limits have been much investigated for infinite volume systems in \mathbb{Z}^d , to establish bounds on the critical value of the contact process, see e.g. [26, 3, 28, 33] and also with a rescaling of time to study hydrodynamic limits, see e.g. [8, 11].

The generalized diffusive contact process on Λ_N (GDCP) has the same structure as the DCP, namely the superposition of a SSEP with diffusion coefficient \mathcal{D} , and a generalization of the contact process, that is, a process with infection rate $\lambda > 0$, but with death rates depending on the occupation variables on the neighboring sites, instead of being constant equal to 1. In Section 3, we derive in Subsection 3.2 a duality relation for the GDCP, and focus on a particular case that induces a dual process with no birth rate. As an application of the latter dual relation, we compute in Subsection 3.3 the one-point function in the invariant measure for a general lattice of N sites. Moreover, we obtain a system of ordinary differential equations for the time-dependent one-point function, for arbitrary initial distributions. After some elementary transformations, this system of ODE’s becomes – again somewhat surprisingly – identical to that of the SSEP with open boundaries, which has no nonconservative bulk processes.

The one-dimensional susceptible-infectious-recovered (SIR) model, that goes back to the partial differential-equation model for epidemics introduced in [24], describes the evolution on the infinite lattice \mathbb{Z} (hence without reservoirs), of three species of individuals: susceptible (in state S), infectious (I), recovered (R). An infectious individual infects a susceptible neighbor at rate $\beta > 0$, it recovers and becomes immune (that is, it remains forever in state R) at rate $\gamma > 0$. This process has an interesting feature in common with the classical problem of random sequential adsorption [12] which is a discrete version of Rényi’s random space filling problem [35]: an ongoing fragmentation of the state space into independent ergodic components occurs

as the process evolves in time. This is accompanied by a strong dependence of the limiting measure on the initial configuration.

As in the previous sections, our goal is to compute correlation functions which, like in random sequential absorption, remains a non-trivial task despite the fragmentation of the state space. However for the SIR model, this is quite different to the DCP and the GDCP where we were interested in the invariant measure. Here, the state space is not irreducible and there are infinitely many invariant measures (that are Dirac measures on blocked states), so the asymptotic behavior of the system is highly dependent on the initial condition. We are rather interested in the time evolution of correlation functions according to the initial state of the system. In Section 4, we study the “ n -point cluster functions”, named as such in [40], which are quantities written in terms of higher order correlation functions. Thanks to these cluster functions, we derive in Subsection 4.1 a duality relation between the SIR model and a two dimensional random walk on two layers with a trap. From that, using a probabilistic approach, we provide in Subsection 4.2 an explicit expression of the average of these cluster functions at any time. This new treatment of the model allows us to extend the results for translation invariant initial distributions derived in [40], to the case of non translation invariant initial distributions.

The paper is organised as follows. In Section 2, we study the DCP, adding in Appendix A.1 computations that do not rely on duality. In Section 3 we study the GDCP, and in Section 4, we study the SIR model. The proofs for all results are provided at the end of their respective sections.

2 The diffusive contact process (DCP)

In physical interacting particle systems that at their edges are in contact with external particle reservoirs, the particle exchange with such reservoirs is commonly modelled by so-called open boundaries. This means that particles at the edge sites of the lattice on which the particle system is defined are created and annihilated at certain rates. The reservoirs themselves do not enter the mathematical description of the process. In this section, we define the DCP with open boundaries, and establish the duality relation with the DCP with purely absorbing boundaries. These are edge sites from which particles jump into auxiliary sites that correspond to sinks from which no particle can escape. In contrast to the reservoirs envisaged in the open boundary setting, these sinks become part of the dual process by enlarging the dual state space (compared to the state space of the DCP with open boundaries) to keep track of the number of absorbed particles.

A duality relation which turns reservoirs into sinks is quite common in interacting particle systems. We refer to [42] for an early result where absorbing sites in dual processes are discussed with a view on establishing ergodic theorems. Such a duality also appears in the SSEP on a discrete segment with two open boundaries, see [43, 6, 13]. However, the analysis of the effect of reservoirs on the stationary state of the DCP substantially differs from the case of conservative particle systems such as the SSEP, where the system is driven out of equilibrium due to a current of particles produced by the effect of reservoirs tuned differently, see [9, 38, 19]. In our non-conservative setting, the system is constantly out of equilibrium.

2.1 The model

For $N \geq 1$, denote by $\Lambda_N = \{1, \dots, N\}$ the one dimensional finite chain of size N , which we refer to as the bulk. A site of Λ_N is either empty (in state 0), or occupied (in state 1). Therefore, the state of the system is described by an element $\eta = (\eta_1, \dots, \eta_N) \in \Omega_N := \{0, 1\}^{\Lambda_N}$ such that

for $x \in \Lambda_N$,

$$\eta_x = \begin{cases} 1 & \text{if } x \text{ is occupied} \\ 0 & \text{if } x \text{ is empty.} \end{cases} \quad (2.1)$$

In the bulk, the dynamics considered is the superposition of a contact process with parameter $\lambda > 0$ (CP), with a symmetric simple exclusion process (SSEP). Both these dynamics are of nearest-neighbor type and are defined as follows. For $x, y \in \Lambda_N$, $x \sim y$ means that x and y are neighboring sites in Λ_N .

In the CP, a site $x \in \Lambda_N$ becomes occupied at rate $\lambda \sum_{y \sim x} \eta_y$ if it is empty, and becomes empty at rate 1 if it is occupied. The generator of this dynamics acts on $f : \Omega_N \rightarrow \mathbb{R}$ as follows, for $\eta \in \Omega_N$ [29]:

$$\begin{aligned} \mathcal{L}^{\text{CP}} f(\eta) &= \sum_{x=2}^{N-1} [\eta_x + \lambda(1 - \eta_x)(\eta_{x-1} + \eta_{x+1})] [f(\eta^x) - f(\eta)] \\ &+ [\eta_1 + \lambda(1 - \eta_1)\eta_2] [f(\eta^1) - f(\eta)] \\ &+ [\eta_N + \lambda(1 - \eta_N)\eta_{N-1}] [f(\eta^N) - f(\eta)]. \end{aligned} \quad (2.2)$$

where for $x, y \in \Lambda_N$,

$$\eta_y^x = \begin{cases} \eta_y & \text{if } y \neq x \\ 1 - \eta_x & \text{if } y = x \end{cases} \quad (2.3)$$

represents the flip of the occupation variable at site $x \in \Lambda_N$. Notice that this generator can be re-written as follows:

$$\mathcal{L}^{\text{CP}} f(\eta) = \frac{\eta_1}{2} [f(\eta^1) - f(\eta)] + \sum_{x=1}^{N-1} \mathcal{L}_{x,x+1}^{\text{CP}} f(\eta) + \frac{\eta_N}{2} [f(\eta^N) - f(\eta)] \quad (2.4)$$

with the bond generator

$$\begin{aligned} \mathcal{L}_{x,x+1}^{\text{CP}} f(\eta) &= \frac{1}{2} \eta_x [f(\eta^x) - f(\eta)] + \lambda(1 - \eta_x) \eta_{x+1} [f(\eta^x) - f(\eta)] \\ &+ \frac{1}{2} \eta_{x+1} [f(\eta^{x+1}) - f(\eta)] + \lambda(1 - \eta_{x+1}) \eta_x [f(\eta^{x+1}) - f(\eta)]. \end{aligned} \quad (2.5)$$

In the SSEP, particles jump to one of their neighboring sites, under the exclusion rule, namely each site can accommodate at most one particle, and the direction of the jump is not biased to the left or right. The generator of this dynamics acts on $f : \Omega_N \rightarrow \mathbb{R}$ as follows, for $\eta \in \Omega_N$:

$$\mathcal{L}^{\text{SSEP}} f(\eta) = \sum_{x=1}^{N-1} \mathcal{L}_{x,x+1}^{\text{SSEP}} f(\eta), \quad (2.6)$$

with

$$\mathcal{L}_{x,x+1}^{\text{SSEP}} f(\eta) = \eta_x(1 - \eta_{x+1})(f(\eta^{x,x+1}) - f(\eta)) + \eta_{x+1}(1 - \eta_x)(f(\eta^{x+1,x}) - f(\eta)), \quad (2.7)$$

where for $x, y \in \{1, \dots, N\}$, if $\eta_x(1 - \eta_y) = 1$,

$$\eta_z^{x,y} = \begin{cases} \eta_z & \text{if } z \notin \{x, y\} \\ \eta_x - 1 & \text{if } z = x \\ \eta_y + 1 & \text{if } z = y. \end{cases} \quad (2.8)$$

and $\eta_z^{x,y} = \eta$ otherwise.

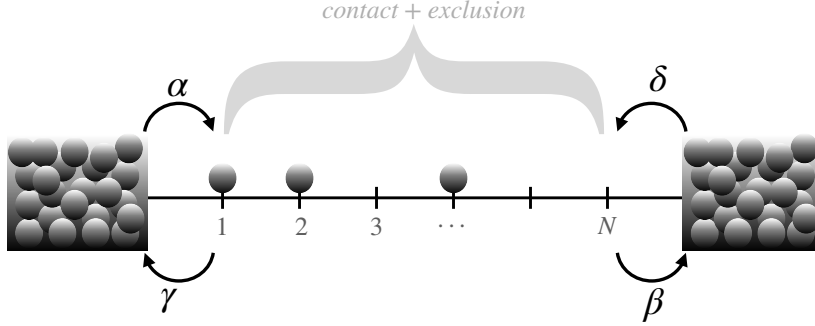


Figure 1: Contact + exclusion process on Λ_N with reservoirs.

We study the open boundary version of the DCP, namely we imagine reservoirs of particles at each edge of the system, which insert or remove a particle on sites 1 and N . These reservoirs are parameterized by four positive parameters $\alpha, \gamma, \beta, \delta$. A particle is inserted at site 1, resp. N , at rate α , resp. δ , provided the site is empty. A particle is removed from the system at site 1, resp. N , at rate γ , resp. β , provided the site is occupied (see Figure 2.1). The associated generators of these boundary dynamics act on $f : \Omega_N \rightarrow \mathbb{R}$ as follows, for $\eta \in \Omega_N$:

$$\mathcal{L}_- f(\eta) = \alpha(1 - \eta_1)[f(\eta^1) - f(\eta)] + \gamma\eta_1[f(\eta^1) - f(\eta)] \quad (2.9)$$

$$\text{and } \mathcal{L}_+ f(\eta) = \delta(1 - \eta_N)[f(\eta^N) - f(\eta)] + \beta\eta_N[f(\eta^N) - f(\eta)]. \quad (2.10)$$

Notice that we do not put an index ‘DCP’ in these generators since they are not specific to our dynamics.

Finally, the generator for the open DCP is given by

$$\mathcal{L}^{\text{DCP}} = \mathcal{L}_- + \mathcal{L}^{\text{CP}} + \mathcal{D}\mathcal{L}^{\text{SSEP}} + \mathcal{L}_+, \quad (2.11)$$

where $\mathcal{D} \geq 0$ is the diffusion parameter. The dynamics of the DCP is irreducible and as the state space Ω_N is finite, the process admits a unique invariant measure ν^{DCP} , which depends on $\alpha, \delta, \gamma, \beta, \mathcal{D}$ and λ . When $\alpha = \gamma = \beta = \delta = 0$ we speak of closed boundaries. In this case the empty configuration is an absorbing state which means that the invariant measure is the trivial Dirac measure concentrated on the empty configuration. As soon as $\alpha \neq 0$ or $\delta \neq 0$, the empty configuration is no longer an absorbing state and ν^{DCP} cannot be found by direct computations when N is large.

Matrix formulation

Since the DCP and the GDCP are particle systems on a finite lattice, and therefore with a finite state space, the duality function can be written as a matrix and the Markov generators of the processes can be written as intensity matrices [31]. This matrix approach with the choice of basis (2.13) below is often a convenient tool to treat interacting particle systems with countable state space [21, 32, 38, 18] and will be used below alongside with the more traditional generator treatment used above.

Recall that an intensity matrix L is such that

$$L(x, y) \geq 0 \quad \text{for} \quad x \neq y \quad \text{and} \quad \sum_y L(x, y) = 0.$$

For $x \neq y$, $L(x, y)$ represents the rate to go from state x to state y .

Remark 1 (Countable state space). *If in Definition 1 both the original process $X = (X_t)_{t \geq 0}$ and the dual process $Y = (Y_t)_{t \geq 0}$ are Markov processes with a countable state space, the duality relation can be written in matrix notation as [46]*

$$LD = D \left(L^{\text{dual}} \right)^T \quad (2.12)$$

where L is the intensity matrix corresponding to the Markov generator \mathcal{L} of X , L^{dual} is the intensity matrix corresponding to $\mathcal{L}^{\text{dual}}$ and the superscript T denotes matrix transposition. Equivalently one could choose to adopt the quantum Hamiltonian representation [32, 38] of the intensity matrix, namely $H = -L^T$, and in this case duality reads $H^T D = D H^{\text{dual}}$, see [36].

Therefore, as the state space of the DCP is finite, one can encode the dynamics in a finite sized matrix. To write these matrices in the canonical basis $(e_i)_{1 \leq i \leq 2^N}$ of \mathbb{R}^{2^N} , to each vector e_i , we associate the unique configuration $\eta^i = (\eta_1^i, \dots, \eta_N^i)$ such that

$$i = 1 + \sum_{k=1}^N 2^{N-k} \eta_k^i. \quad (2.13)$$

For example, for $N = 2$, the order is

$$\begin{aligned} e_1 &= (1, 0, 0, 0) \text{ associated to } (0, 0), & e_2 &= (0, 1, 0, 0) \text{ associated to } (0, 1), \\ e_3 &= (0, 0, 1, 0) \text{ associated to } (1, 0) & \text{and } e_4 &= (0, 0, 0, 1) \text{ associated to } (1, 1). \end{aligned}$$

With this ordering of configurations, the bulk dynamics matrix writes

$$L^{\text{CP}} + \mathcal{D}L^{\text{SSEP}} = \sum_{x=1}^{N-1} \ell_x + \tilde{\ell}_1 + \tilde{\ell}_N, \quad (2.14)$$

where for $x = 1, \dots, N-1$, denoting by \otimes the Kronecker product and $\mathbf{1}$ the two-dimensional unit matrix,

$$\ell_x := \mathbf{1}^{\otimes(x-1)} \otimes \ell \otimes \mathbf{1}^{\otimes(N-1-x)}, \quad (2.15)$$

with ℓ corresponding to $\mathcal{L}_{x,x+1}^{\text{CP}} + \mathcal{D}\mathcal{L}_{x,x+1}^{\text{SSEP}}$:

$$\ell = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1/2 & -(1/2 + \mathcal{D} + \lambda) & \mathcal{D} & \lambda \\ 1/2 & \mathcal{D} & -(1/2 + \mathcal{D} + \lambda) & \lambda \\ 0 & 1/2 & 1/2 & -1 \end{pmatrix},$$

and, for $x = 1$ and $x = N$, $\tilde{\ell}_1$ and $\tilde{\ell}_N$ correspond to the first and last terms in the right hand side of (2.4) with,

$$\tilde{\ell}_1 := \tilde{\ell} \otimes \mathbf{1}^{\otimes(N-1)} \quad \text{and} \quad \tilde{\ell}_N := \mathbf{1}^{\otimes(N-1)} \otimes \tilde{\ell} \quad (2.16)$$

where

$$\tilde{\ell} = \begin{pmatrix} 0 & 0 \\ 1/2 & -1/2 \end{pmatrix}.$$

Regarding the boundary dynamics the matrices ℓ_1^- and ℓ_1^+ correspond respectively to \mathcal{L}_- in (2.9) and \mathcal{L}_+ in (2.10), with

$$\ell_1^- := \ell^- \otimes \mathbf{1}^{\otimes(N-1)}, \quad \text{and} \quad \ell_1^+ := \mathbf{1}^{\otimes(N-1)} \otimes \ell^+, \quad (2.17)$$

where

$$\ell^- = \begin{pmatrix} -\alpha & \alpha \\ \gamma & -\gamma \end{pmatrix}, \quad \ell^+ = \begin{pmatrix} -\delta & \delta \\ \beta & -\beta \end{pmatrix}.$$

In total, the intensity matrix of the DCP dynamics is given by

$$L^{\text{DCP}} = \sum_{x=1}^{N-1} \ell_x + \tilde{\ell}_1 + \tilde{\ell}_N + \ell_1^- + \ell_N^+. \quad (2.18)$$

2.2 Duality for the DCP

The open DCP has a diffusive and contact bulk dynamics with spontaneous creation and annihilation of particles at sites 1 and N due to the presence of two reservoirs in ghost sites 0 and $N + 1$ which, as discussed above, do not appear in the process. Theorem 1 shows that the dual DCP preserves the same dynamics in the bulk while the boundary mechanism is quite different. The dual process is defined on an extended lattice which includes sites 0 and $N + 1$ where particles are permanently absorbed, thus describing particle sinks. Because of this the dual state space is different,

$$\Omega_N^{\text{dual}} = \mathbb{N}_0 \times \{0, 1\}^{\Lambda_N} \times \mathbb{N}_0, \quad (2.19)$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ is the set of non-negative integers, while $\mathbb{N} = \{1, 2, \dots\}$.

Theorem 1. *The open DCP $(\eta_t)_{t \geq 0}$ with generator (2.11) is dual to a purely absorbing contact process $(\xi_t)_{t \geq 0}$ with duality function $D : \Omega_N \times \Omega_N^{\text{dual}} \rightarrow \mathbb{R}$, given by*

$$D(\eta, \xi) = \left(\frac{\gamma}{\alpha + \gamma} \right)^{\xi_0} H(\eta, \xi) \left(\frac{\beta}{\beta + \delta} \right)^{\xi_{N+1}}, \quad (2.20)$$

where the bulk duality function is:

$$H(\eta, \xi) = \prod_{x \in A(\xi)} (1 - \eta_x) = \prod_{x \in \Lambda_N} (1 - \eta_x)^{\xi_x}, \quad (2.21)$$

with $A(\xi) = \{y \in \{1, \dots, N\}, \xi_y = 1\}$. The dual generator is given by

$$\mathcal{L}^{\text{DCP, Dual}} = \mathcal{L}_-^{\text{Dual}} + \mathcal{L}^{\text{CP}} + \mathcal{D}\mathcal{L}^{\text{SSEP}} + \mathcal{L}_+^{\text{Dual}}, \quad (2.22)$$

where \mathcal{L}^{CP} is the generator of the contact process defined in (2.2), $\mathcal{L}^{\text{SSEP}}$ is the generator of the exclusion process defined in (2.6) and the action at the boundary on function $f : \Omega_N^{\text{dual}} \rightarrow \mathbb{R}$ is

$$\mathcal{L}_-^{\text{Dual}} f(\xi) = (\alpha + \gamma) \xi_1 [f(\xi^{1,0}) - f(\xi)] \quad (2.23)$$

$$\mathcal{L}_+^{\text{Dual}} f(\xi) = (\delta + \beta) \xi_N [f(\xi^{N,N+1}) - f(\xi)]. \quad (2.24)$$

As pointed out in the introduction, both the contact and the exclusion processes with *closed* boundary conditions are self-dual with duality function (2.21), see in particular [30, Chapter 3, Section 4] and [25, Chapter 5]. This leads to a self-duality relation for the bulk of the DCP [45], namely we have that for any $\eta, \xi \in \Omega_N$,

$$(\mathcal{L}^{\text{CP}} + \mathcal{L}^{\text{SSEP}})H(\eta, \cdot)(\xi) = (\mathcal{L}^{\text{CP}} + \mathcal{L}^{\text{SSEP}})H(\cdot, \xi)(\eta). \quad (2.25)$$

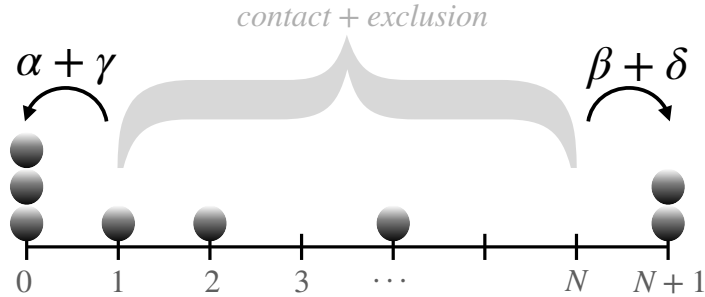


Figure 2: Dual diffusive contact process with sinks at sites 0 and $N + 1$.

Remark 2. (a) It was noticed in [39] that for the symmetric simple exclusion process with either closed or periodic boundary conditions, the function

$$\tilde{H}(\eta, \xi) = \prod_{x \in \Lambda_N} [a_1 + a_2 \eta_x]^{a_3 + a_4 \xi_x}, \quad a_i \in \mathbb{R}, i \in \{1, 2, 3, 4\} \quad (2.26)$$

is a self-duality function for any choice of the parameters a_i . The case $a_1 = 1$, $a_2 = -1$, $a_3 = 0$ and $a_4 = 1$ yields $\tilde{H} = H$, the self-duality function for the contact process.

(b) The bulk duality functions (2.21) and (2.26) have a locally factorized form, namely

$$G(\eta, \xi) = \prod_{x \in \Lambda_N} g(\eta_x, \xi_x)$$

which corresponds to a duality matrix of the form $G^{\otimes N}$, for a local duality matrix

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (2.27)$$

where for the symmetric simple exclusion process the matrix elements are arbitrary, while for the contact process we have $a = b = c = 1$, $d = 0$.

From now on the letter $\eta \in \Omega_N$ will be relative to a DCP and $\xi \in \Omega_N^{\text{dual}}$ to the dual process introduced in Theorem 1.

2.3 Correlation functions via duality

In order to provide a better understanding of the invariant measure ν^{DCP} of the DCP with absorbing boundaries, we study its correlation functions thanks to the duality relation established in Theorem 1. The dual process is seemingly simpler, as it almost surely becomes extinct. We are thus able to express the correlation functions of the original process in terms of absorption probabilities of the dual process. This allows to get an upper bound for the one-point function, see subsection 2.3.1. However, the explicit computation of correlation functions relies on solving large systems of linear equations, see subsection 2.3.2, and it is not clear whether they are simpler to solve than the ones obtained in Appendix A.1 through a more standard approach.

In the following, the one-point function is the expectation $\rho_1^{\text{DCP}}(y) := \mathbb{E}_{\nu^{\text{DCP}}}[\eta_y]$ of the occupation number η_y w.r.t. the invariant measure of the DCP. For $2 \leq \ell \leq N$, the ℓ -point correlation function of ν^{DCP} is the function ρ_ℓ^{DCP} defined by

$$\forall x_1 < \dots < x_\ell \in \Lambda_N, \rho_\ell^{\text{DCP}}(x_1, \dots, x_\ell) := \mathbb{E}_{\nu^{\text{DCP}}}[\eta_{x_1} \dots \eta_{x_\ell}]. \quad (2.28)$$

Introduce the extinction time τ_N of the DCP $(\xi(t))_{t \geq 0}$ with purely absorbing boundaries:

$$\tau_N := \inf \{t \geq 0, \forall x \in \{1, \dots, N\}, \xi_x(t) = 0\}. \quad (2.29)$$

Also, introduce the total number of particles absorbed on the left and right hand side boundary:

$$\xi_0(\infty) := \lim_{t \rightarrow \infty} \xi_0(t), \quad \text{and} \quad \xi_{N+1}(\infty) := \lim_{t \rightarrow \infty} \xi_{N+1}(t), \quad (2.30)$$

where the limits are almost sure. As the process almost surely becomes extinct, τ_N , $\xi_0(\infty)$ and $\xi_{N+1}(\infty)$ are almost surely finite and $\xi_0(\infty) = \xi_0(\tau_N)$, $\xi_{N+1}(\infty) = \xi_{N+1}(\tau_N)$ almost surely. To state and prove the results it is convenient to denote

$$c_- := \frac{\gamma}{\alpha + \gamma} \quad \text{and} \quad c_+ := \frac{\beta}{\beta + \delta}, \quad (2.31)$$

which are related to the left (resp. right) reservoir densities $1 - c_-$ (resp. $1 - c_+$).

2.3.1 One-point function

For $y \in \Lambda_N$, denote by δ_y the element of Ω_N^{dual} where there is a particle in site $y \in \Lambda_N$ and all other sites $x \neq y$ are empty.

Proposition 1. *The one-point function of ν^{DCP} is given by*

$$\forall y \in \Lambda_N, \rho_1^{\text{DCP}}(y) = 1 - \mathbb{E}_{\delta_y}[c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)}] = 1 - \mathbb{E}_{\delta_y}[c_-^{\xi_0(\tau_N)} c_+^{\xi_{N+1}(\tau_N)}]. \quad (2.32)$$

Although the dual process is seemingly simpler than the original one, it remains a non-conservative process and the total number of particles absorbed by the reservoirs can reach arbitrary values. Therefore, we are not able to provide an explicit expression of the right hand side term in (2.32), contrary to the case of conservative particle systems where exact formulas are available. We refer to section 2.3 of [15] and Theorem 2.2 of [14], where a closed formula for the n points correlation is found in the non-equilibrium steady state of two models with open boundary and the same purely absorbing dual.

However, note that the dual process $(\xi(t))_{t \geq 0}$ either dies out before any particle lying in the bulk has had time to reach the boundary, or, it dies out and at least one particle has been absorbed by the boundary reservoirs. Thanks to this observation, the following upper bound holds:

Lemma 1. *For $y \in \Lambda_N$, denote by $A_y(\infty)$ the following event:*

$$A_y(\infty) = \left(\xi(0) = \delta_y, c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)} < 1 \right),$$

that is, starting from a single dual particle at site y , at least one particle is absorbed by the boundary of the system. Then,

$$\rho_1^{\text{DCP}}(y) \leq \mathbb{P}_{\delta_y}[A_y(\infty)]. \quad (2.33)$$

Remark 3. *This inequality is of interest if the upper bound is small, that is, that starting from a particle at y , it is unlikely that the process reaches the boundary before it becomes extinct. Heuristically, if the time of extinction τ_N of the process is small, corresponding to a small birth rate λ , then, the particle should not have time to reach the boundary. Precisely, an estimate of τ_N is known in the case of a subcritical contact process on Λ_N (without exclusion), and we refer to [29, Part 1, Theorem 3.3].*

2.3.2 Higher order correlation functions

The idea of relating the ℓ -point correlations using the dual absorption probabilities goes back to [17] for a model of stochastic diffusion of energy. In the same spirit as in Proposition 2 in [17] we can show the following results for the DCP. Given $2 \leq \ell \leq N$ and sites $1 \leq x_1 < \dots < x_\ell \leq N$, denote by $\delta_{x_1, x_2, \dots, x_\ell}$, the element of Ω_N^{dual} where there is a particle at sites x_1, \dots, x_ℓ , and none elsewhere.

Proposition 2. *For any $1 \leq \ell \leq N$ and any $1 \leq x_1 < \dots < x_\ell \leq N$,*

$$\begin{aligned} \mathbb{E}_{\nu^{\text{DCP}}} \left[\prod_{j=1}^{\ell} (1 - \eta_{x_j}) \right] &= \sum_{k=0}^{\ell} (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq \ell} \rho_k^{\text{DCP}}(x_{i_1}, \dots, x_{i_k}) \\ &= \sum_{m, n \geq 0} \mathbb{P}_{\delta_{x_1, x_2, \dots, x_\ell}} \left[\xi_0(\infty) = m, \xi_N(\infty) = n \right] c_-^m c_+^n. \end{aligned} \quad (2.34)$$

If we consider a small bulk, we can explicitly compute the above absorption probabilities. The idea is to perform a conditioning on the first step of the dual dynamics. We leave to Appendix A.2 the explicit computations for $N = 1$ and $N = 2$.

For N sites notice that there are

$$\sum_{k=1}^N \binom{N}{k} = 2^N - 1$$

nontrivial initial dual configurations. We define M_N to be the $2^N - 1$ square matrix given by the condition on the first steps starting from the dual initial configuration. Then our goal is to show that M_N is invertible so that all the absorption probabilities – and so the corresponding correlation functions – are determined as solutions of a linear system. For $k \geq 0$, denote by $x_i^k = \mathbb{P}_{\xi^i}[\xi_0(\infty) = k]$ the absorption probabilities of k dual particles starting from the initial configuration ξ^i , $1 \leq i \leq 2^N - 1$ that we want to find. The order of these unknown variables is chosen in an arbitrary way and one option to systematically ordering them is the following, defined recursively for N sites. There are $2^N - 1$ total number of non trivial configurations: the first 2^{N-1} unknown variables have a particle at site 1 and the rest is completed by a configuration from the $N - 1$ th ordering. The $2^{N-1} - 1$ coordinates left are the ones such that site 1 is empty and the rest is completed by a configuration from the $N - 1$ th ordering.

More explicitly, for $N = 2$ sites we have $\xi^1 = \delta_1$, $\xi^2 = \delta_{1,2}$, $\xi^3 = \delta_2$ which is the order we used, see Fig. 3. For $N = 3$, this rule would give us the ordering $\xi^1 = \delta_1$, $\xi^2 = \delta_{1,2}$, $\xi^3 = \delta_{1,2,3}$, $\xi^4 = \delta_{1,3}$, $\xi^5 = \delta_2$, $\xi^6 = \delta_{2,3}$, $\xi^7 = \delta_3$ and so on.

Also, for $\xi \in \Omega_N^{\text{dual}}$, introduce $\mathcal{A}(\xi)$ the set of configurations containing at least one particle and resulting from ξ after the jump, birth or death of a particle in η (but not an absorption in the left side reservoir). Denote by $(p(\xi^i, \xi^j))_{1 \leq i, j \leq n}$ the transition probabilities between configurations. Then for any $1 \leq i \leq 2^N - 1$ and for any $k \geq 2$, the following recurrence relation holds:

$$x_i^k = \sum_{\substack{j=1 \\ \xi^j \in \mathcal{A}(\xi^i)}}^{2^N - 1} p(\xi^i, \xi^j) x_j^k + \sum_{\substack{j=1 \\ \xi^j \in \Lambda_N \setminus \mathcal{A}(\xi^i)}}^{2^N - 1} p(\xi^i, \xi^j) x_j^{k-1}.$$

For $k = 0, 1$ the recurrence relation remains the same modulo some changes on the second sum in the right hand side. Therefore, denoting C_k the column vector $(x_i^k)_{1 \leq i \leq 2^N - 1}$, for any $k \geq 0$, there is a matrix of size $2^N - 1$ of the form $M_N = I_{2^N - 1} - P$ such that

$$M_N C_k = R_k,$$

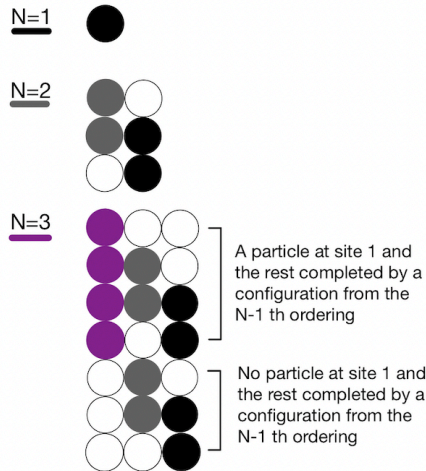


Figure 3: Ordering of configurations

where R_k is a column vector involving some constant terms for $k = 0, 1$ and terms from C_{k-1} for $k \geq 2$, and where

$$P = (p(\xi^i, \xi^j) \mathbb{1}_{\xi^j \in \mathcal{A}(\xi^i)})_{1 \leq i, j \leq 2^{N-1}}.$$

By the Perron-Frobenius Theorem, as P is the transition matrix of an irreducible dynamics, one has that M_N is invertible and

$$M_N^{-1} = \sum_{k \geq 0} P^k.$$

2.3.3 Fast stirring limit

It is interesting to consider the fast stirring limit, that is, $\mathcal{D} \rightarrow \infty$ for the DCP model. Unlike in hydrodynamics, here, we discuss the setting where time is not rescaled when performing this limit.

When $\mathcal{D} \rightarrow \infty$, the DCP reduces to a birth-death chain on the lattice $\{0, 1, \dots, N\}$. This can be seen by considering a configuration $\eta(t)$ with $n(t)$ particles, where $0 \leq n(t) \leq N$, and where the process $n(t)$ is defined in a way that is analogous to the derivation of the reaction term in the fast stirring limit of a reaction-diffusion processes, as discussed in [8, 11]. More precisely, between any attempt to create or annihilate a particle in the bulk or at the boundaries, the process is just a SSEP with infinite rate, which has the uniform measure as its stationary distribution. This means that in between any creation/annihilation event of a particle (death, infection, and reservoir events), any configuration $\eta(t)$ with $n(t)$ particles, moves uniformly to another configuration $\eta'(t)$ with $n'(t)$ particles, so that on the time scale of the creation and annihilation processes, the following transitions occur: any given configuration with $n(t)$ particles will turn uniformly into another one with $n(t) \pm 1$ particles. The invariant measure of the process in this limit thus becomes a convex combination of uniform measures on configurations with n particles, where the weights are given by the stationary weights of the birth-death chain that describes the transitions between configurations of n and $n \pm 1$ particles.

Specifically, the rate at which a particle dies in the closed DCP at a given site (transition $n(t) \rightarrow n(t) - 1$), is n/N (which is the probability of finding an occupied site in the uniform distribution), and since there are N sites where particles can die, the rate of transition is n . In the open DCP, particles can also die at the boundary sites with rates γ , and β respectively. Hence the transition rate from a configuration with $n(t)$ particles to a configuration with $n(t) - 1$ particles is $n + (\beta + \gamma)n/N$. Likewise, the rate at which a particle is created in the DCP by infection at a bond, is given by the probability of finding an occupied site with a vacant neighbor in the uniform distribution (which is $n(N - n)/(N(N - 1))$) times the number of bonds (which is $N - 1$) times λ (since at each bond the infection can occur with rate $\lambda/2$ from a particle on the right or on the left site of the bond). In the open DCP particles can also be created at the boundary sites with rates α , and δ respectively. Hence the transition rate from a configuration with $n(t)$ particles to a configuration with $n(t) + 1$ particles is $\lambda n(1 - n/N) + (\alpha + \delta)(1 - n/N)$. Thus, the DCP degenerates into a birth-death chain $n(t)$ with the following transition rates:

$$n \rightarrow n - 1, \text{ at rate } n[1 + (\beta + \gamma)/N], \quad n \rightarrow n + 1, \text{ at rate } [\lambda n + (\alpha + \delta)](1 - n/N). \quad (2.35)$$

Without reservoirs, $n = 0$ is an absorbing state, but not when $\alpha \neq 0$ or $\delta \neq 0$.

Since the arguments that lead to these rates are analogous to those of [8, 11] we refrain from providing a formal proof. For the stationary probabilities and general properties of this birth-death chain we refer to [23]. Again, we point to Appendix A.2 where we compute explicitly the absorption probabilities for $N = 2$ under the fast stirring limit.

2.4 Proofs for Section 2

2.4.1 Proofs for Section 2.2

First, notice that in our case Definition 1 is equivalent to the following relation involving the corresponding Markov generators of the processes:

Proposition 3. ([31, Theorem 3.42]) *Given X a process with generator \mathcal{L} and Y , with generator $\mathcal{L}^{\text{dual}}$, then X and Y are dual with local duality function D if*

$$(\mathcal{L}D(\cdot, x))(y) = \left(\mathcal{L}^{\text{dual}}D(y, \cdot) \right)(x). \quad (2.36)$$

for all $x \in \Omega$ and $y \in \Omega^{\text{dual}}$.

Proof of Theorem 1. Taking into account Proposition 3, and by equation (2.25), we are left to show that

$$(\mathcal{L}_- + \mathcal{L}_+)D(\cdot, \xi)(\eta) = (\mathcal{L}_-^{\text{Dual}} + \mathcal{L}_+^{\text{Dual}})D(\eta, \cdot)(\xi),$$

where we recall that $\mathcal{L}_-^{\text{Dual}}$ is defined in (2.23), $\mathcal{L}_+^{\text{Dual}}$ in (2.24), \mathcal{L}_- in (2.9) and \mathcal{L}_+ in (2.10). Without loss of generality, we prove the result for the left reservoir, i.e.

$$\mathcal{L}_-D(\cdot, \xi)(\eta) = \mathcal{L}_-^{\text{Dual}}D(\eta, \cdot)(\xi),$$

and the proof that $\mathcal{L}_+D(\cdot, \xi)(\eta) = \mathcal{L}_+^{\text{Dual}}D(\eta, \cdot)(\xi)$ follows the same steps. We have

$$\mathcal{L}_-D(\cdot, \xi)(\eta) = \alpha(1 - \eta_1)[D(\eta^1, \xi) - D(\eta, \xi)] + \gamma\eta_1[D(\eta^1, \xi) - D(\eta, \xi)].$$

Now,

$$\begin{aligned} D(\eta^1, \xi) &= c_-^{\xi_0} c_+^{\xi_{N+1}} \prod_{x \in A(\xi)} (1 - \eta_x^1) \\ &= c_-^{\xi_0} c_+^{\xi_{N+1}} \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \eta_1 \mathbb{1}_{1 \in A(\xi)} + D(\eta, \xi) \mathbb{1}_{1 \notin A(\xi)}. \end{aligned}$$

Therefore,

$$\begin{aligned}
D(\eta^1, \xi) - D(\eta, \xi) &= D(\eta^1, \xi) - D(\eta, \xi) \mathbb{1}_{1 \in A(\xi)} - D(\eta, \xi) \mathbb{1}_{1 \notin A(\xi)} \\
&= c_-^{\xi_0} c_+^{\xi_{N+1}} \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \eta_1 \mathbb{1}_{1 \in A(\xi)} - D(\eta, \xi) \mathbb{1}_{1 \in A(\xi)} \\
&= c_-^{\xi_0} c_+^{\xi_{N+1}} (2\eta_1 - 1) \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \mathbb{1}_{1 \in A(\xi)}.
\end{aligned}$$

Finally, we are left with

$$\begin{aligned}
\mathcal{L}_- D(\cdot, \xi)(\eta) &= \alpha(1 - \eta_1)(2\eta_1 - 1) c_-^{\xi_0} c_+^{\xi_{N+1}} \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \mathbb{1}_{1 \in A(\xi)} \\
&\quad + \gamma \eta_1 (2\eta_1 - 1) c_-^{\xi_0} c_+^{\xi_{N+1}} \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \mathbb{1}_{1 \in A(\xi)} \\
&= -\alpha D(\eta, \xi) \mathbb{1}_{1 \in A(\xi)} + \gamma c_-^{\xi_0} c_+^{\xi_{N+1}} \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \eta_1 \mathbb{1}_{1 \in A(\xi)} \\
&= -(\alpha + \gamma) D(\eta, \xi) \mathbb{1}_{1 \in A(\xi)} + (\alpha + \gamma) c_-^{\xi_0 + 1} c_+^{\xi_{N+1}} \prod_{x \in A(\xi) \setminus \{1\}} (1 - \eta_x) \mathbb{1}_{1 \in A(\xi)} \\
&= (\alpha + \gamma) \xi_1 \left[D(\eta, \xi^{1,0}) - D(\eta, \xi) \right],
\end{aligned}$$

which is exactly the dual absorbing generator of the left boundary. \square

2.4.2 Proofs for Section 2.3

Proof of Proposition 1. By definition (2.20) of the duality function D given in Theorem 1, if we choose as dual process $\xi(t) = \delta_y(t)$, we have

$$D(\eta, \delta_y) = 1 - \eta_y.$$

The duality relation (2.36) then yields that for any $t \geq 0$,

$$\int \mathbb{E}_\eta [D(\eta(t), \delta_y)] \nu^{\text{DCP}}(d\eta) = \int \mathbb{E}_{\delta_y} [D(\eta, \xi(t))] \nu^{\text{DCP}}(d\eta). \quad (2.37)$$

By invariance of ν^{DCP} , the left hand side in (2.37) does not depend on t and equals

$$\int \mathbb{E}_\eta [D(\eta, \delta_y)] \nu^{\text{DCP}}(d\eta) = 1 - \mathbb{E}_{\nu^{\text{DCP}}} [\eta_y], \quad (2.38)$$

while the right hand equals

$$\int \mathbb{E}_{\delta_y} [D(\eta, \xi(t))] \nu^{\text{DCP}}(d\eta) \xrightarrow[t \rightarrow \infty]{} \mathbb{E}_{\delta_y} [c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)}], \quad (2.39)$$

since for $t \rightarrow \infty$ all dual particles will eventually be absorbed at the boundary. Collecting (2.38) and (2.39) yields the result. \square

Proof of Lemma 1. We have

$$\begin{aligned}
\mathbb{E}_{\delta_y} [c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)}] &= \mathbb{P}_{\delta_y} [A_y(\infty)^c] + \mathbb{E}_{\delta_y} [c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)} \mathbb{1}_{A_y(\infty)}] \\
&= 1 - \mathbb{P}_{\delta_y} [A_y(\infty)] + \mathbb{E}_{\delta_y} [c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)} \mathbb{1}_{A_y(\infty)}],
\end{aligned}$$

therefore,

$$\rho_1^{\text{DCP}}(y) = 1 - \mathbb{E}_{\delta_y} [c_-^{\xi_0(\infty)} c_+^{\xi_{N+1}(\infty)}] \leq \mathbb{P}_{\delta_y} [A_y(\infty)].$$

\square

Proof of Proposition 2. For simplicity, we consider that only the left reservoir interacts with the bulk, that is, $\delta = \beta = 0$, but the reasoning remains the same if that is not the case. Our goal is therefore to prove that for any $1 \leq x_1 < \dots < x_\ell \leq N$,

$$\mathbb{E}_{\nu^{\text{DCP}}} \left[\prod_{j=1}^{\ell} (1 - \eta_{x_j}) \right] = \sum_{k \geq 0} \mathbb{P}_{\delta_{x_1, x_2, \dots, x_\ell}} \left[\xi_0(\infty) = k \right] c_-^k. \quad (2.40)$$

In this case, it follows that the duality relation (1.1) holds with duality function

$$D(\eta, \xi) = c_-^{\xi_0} \prod_{x \in \Lambda_N} (1 - \eta_x)^{\xi_x}. \quad (2.41)$$

In other words, the initial process is dual to a DCP with an absorbing sink with rate $\alpha + \gamma$. Then, proceeding as in the proof of Proposition 1, we consider the duality relation (1.1) under the invariant measure for the original process and, taking $\xi = \delta_{x_1, x_2, \dots, x_\ell}$, we have:

$$\mathbb{E}_{\nu^{\text{DCP}}} \left[(1 - \eta_{x_1}) (1 - \eta_{x_2}) \dots (1 - \eta_{x_\ell}) \right] = \mathbb{E}_{\delta_{x_1, x_2, \dots, x_\ell}} \left[c_-^{\xi_0(\infty)} \right]. \quad (2.42)$$

Expanding both sides of (2.42), we are left with

$$\sum_{k=0}^{\ell} (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq \ell} \rho_k^{\text{DCP}}(x_{i_1}, \dots, x_{i_k}) = \sum_{k \geq 0} \mathbb{P}_{\delta_{x_1, x_2, \dots, x_\ell}} \left[\xi_0(\infty) = k \right] c_-^k,$$

where $\mathbb{P}_{\delta_{x_1, x_2, \dots, x_\ell}} \left[\xi_0(\infty) = k \right]$ is the probability that k dual particles are absorbed in the sink when the dual process is initialized with ℓ particles in sites $0 < x_1 < \dots < x_\ell$. \square

3 Generalized diffusive contact process (GDCP)

In this section, we introduce a generalized diffusive contact (GDCP) on Λ_N which also exhibits a factorized duality property, similar to the one for the DCP. For a specific choice of the parameters of the GDCP on a finite one-dimensional lattice with reservoirs, one can extract an explicit expression of the one-point function for its invariant measure. This is in contrast with the DCP, where we were not able to extract explicit expressions of the correlation functions from the duality result provided by Theorem 1. For time-dependent correlations, duality leads to a closed system of first-order ordinary differential equations with constant coefficients. This is demonstrated in explicit form for the one-point function.

3.1 The model

Here, as for the DCP, particles evolve on the open one-dimensional finite lattice Λ_N . The reservoir dynamics is the same as in Section 2, given by the generators \mathcal{L}_- and \mathcal{L}_+ defined in (2.9) and (2.10). The exclusion dynamics is the same as for the DCP, given by the generator $\mathcal{L}^{\text{SSEP}}$ defined in (2.6).

What differs is the contact dynamics which is here more general. The birth rate is $\lambda > 0$, while the death rates, instead of being 1, depend on the occupation variables in the nearest neighbor sites: a particle dies with rate $2\mu_1$ when both neighbors are empty, with rate $\mu_1 + \mu_2$ when one of the two neighbors is empty and the other is occupied, and with rate $2\mu_2$ when both neighbors are occupied. We refer (3.9) for a description of the rates in a tabular form. The generator of the generalized diffusive contact processes is given by

$$\mathcal{L}^{\text{GDCP}} = \mathcal{L}_- + \mathcal{L}^{\text{GCP}} + \mathcal{D}\mathcal{L}^{\text{SSEP}} + \mathcal{L}_+. \quad (3.1)$$

Above, \mathcal{L}_- , resp. \mathcal{L}_+ , is the left, resp. right hand side reservoir generator, and is given by (2.9), resp. (2.10), with the boundary rates $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ for the GDCP, instead of $\alpha, \beta, \gamma, \delta$ for the DCP. As before, $\mathcal{D} \geq 0$ is the diffusion parameter which tunes the exclusion dynamics described by $\mathcal{L}^{\text{SSEP}}$ in equation (2.6). While \mathcal{L}^{GCP} is the generator of the generalized contact process, whose action on a given function $f : \Omega_N \rightarrow \mathbb{R}$, is, for $\eta \in \Omega_N$

$$\mathcal{L}^{\text{GCP}} f(\eta) = \sum_{x=1}^{N-1} \mathcal{L}_{x,x+1}^{\text{GCP}} f(\eta), \quad (3.2)$$

with the bond generator

$$\begin{aligned} \mathcal{L}_{x,x+1}^{\text{GCP}} f(\eta) &= \eta_x [\mu_2 \eta_{x+1} + \mu_1 (1 - \eta_{x+1})] [f(\eta^x) - f(\eta)] + \lambda (1 - \eta_x) \eta_{x+1} [f(\eta^x) - f(\eta)] \\ &\quad + \eta_{x+1} [\mu_2 \eta_x + \mu_1 (1 - \eta_x)] [f(\eta^{x+1}) - f(\eta)] + \lambda (1 - \eta_{x+1}) \eta_x [f(\eta^{x+1}) - f(\eta)]. \end{aligned} \quad (3.3)$$

That is,

$$\begin{aligned} \mathcal{L}^{\text{GCP}} f(\eta) &= \left[\eta_1 (\mu_1 + (\mu_2 - \mu_1) \eta_2) + \lambda (1 - \eta_1) \eta_2 \right] [f(\eta^1) - f(\eta)] \\ &\quad + \sum_{x=2}^{N-1} \left[\eta_x (2\mu_1 + (\mu_2 - \mu_1) (\eta_{x-1} + \eta_{x+1})) + \lambda (1 - \eta_x) (\eta_{x-1} + \eta_{x+1}) \right] [f(\eta^x) - f(\eta)] \\ &\quad + \left[\eta_N (\mu_1 + (\mu_2 - \mu_1) \eta_{N-1}) + \lambda (1 - \eta_N) \eta_{N-1} \right] [f(\eta^N) - f(\eta)]. \end{aligned} \quad (3.4)$$

Remark 4. (1) One can recover an open DCP with boundary parameters $\alpha, \beta, \gamma, \delta$, from an open GDCP with boundary parameters $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$, and death parameters μ_1, μ_2 , by taking

$$\mu_1 = \mu_2 = \frac{1}{2}, \quad \tilde{\alpha} = \alpha, \quad \tilde{\gamma} = \gamma + \frac{1}{2}, \quad \tilde{\delta} = \delta, \quad \text{and} \quad \tilde{\beta} = \beta + \frac{1}{2}. \quad (3.5)$$

(2) For $\mu_2 = 0$, the GDCP reduces to a biased voter model with stirring by symmetric simple exclusion, and when $\mu_2 = 0$ and $\mu_1 = \lambda$, this is the usual voter model with stirring, studied in [2] for asymmetric simple exclusion with step initial condition on \mathbb{Z} , and [27] for totally asymmetric simple exclusion on Λ_N and open boundaries with $\tilde{\gamma} = \tilde{\delta} = 0$.

As for the DCP, since the state space is finite, one can encode the dynamics in a finite sized matrix. In this setting, the intensity matrix of the GDCP in the bulk writes as the following 2^N -sized matrix

$$L^{\text{GCP}} + \mathcal{D}L^{\text{SSEP}} = \sum_{x=1}^{N-1} \ell_x, \quad (3.6)$$

where for $x = 1, \dots, N-1$

$$\ell_x := \mathbf{1}^{\otimes(x-1)} \otimes \ell \otimes \mathbf{1}^{\otimes(N-1-x)}, \quad (3.7)$$

with the two-dimensional unit matrix $\mathbf{1}$, and the local intensity matrix

$$\ell = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mu_1 & -(\mu_1 + \mathcal{D} + \lambda) & \mathcal{D} & \lambda \\ \mu_1 & \mathcal{D} & -(\mu_1 + \mathcal{D} + \lambda) & \lambda \\ 0 & \mu_2 & \mu_2 & -2\mu_2 \end{pmatrix}. \quad (3.8)$$

The intensity matrices of the boundary dynamics are the same as for the DCP dynamics, see (1) in Remark 4.

The bond transition rates for neighboring sites $(x, x + 1)$ can be schematically represented in tabular form as

<i>Initial</i>	<i>Final</i>	<i>Rate</i>
01	→ 00	μ_1
01	→ 10	\mathcal{D}
01	→ 11	λ
10	→ 00	μ_1
10	→ 01	\mathcal{D}
10	→ 11	λ
11	→ 01	μ_2
11	→ 10	μ_2

(3.9)

3.2 Duality results for the GDCP

In this section we establish in Theorem 2 a duality relation for the GDCP. A special case, derived in Corollary 1 will be of interest for applications, as it allows to find a dual process with no birth rate. This means that the sum in the equation (2.40) for correlations is finite, and can be explicitly computed for a general bulk of N sites.

Theorem 2. *Assume $\mathcal{D} + \mu_1 - \mu_2 \geq 0$ and $\lambda + \mu_2 - \mu_1 \geq 0$. The open GDCP with generator (3.1) is dual with a purely absorbing GDCP with generator*

$$\mathcal{L}^{\text{GDCP,Dual}} = \mathcal{L}_-^{\text{Dual}} + \mathcal{L}^{\text{GCP,Dual}} + \widehat{\mathcal{D}}\mathcal{L}^{\text{SSEP}} + \mathcal{L}_+^{\text{Dual}}, \quad (3.10)$$

w.r.t. the duality function (2.20) obtained for the open diffusive contact process. We have that $\mathcal{L}^{\text{GCP,Dual}}$ is the generator (3.4) of the generalized contact process with birth rate $\widehat{\lambda} = \lambda + \mu_2 - \mu_1$ and death rates exchanged, that is,

$$\begin{aligned} \mathcal{L}^{\text{GCP,Dual}} f(\xi) = & \\ & \sum_{x=2}^{N-1} \left[\xi_x (2\mu_2 + (\mu_1 - \mu_2) (\xi_{x-1} + \xi_{x+1})) + (\lambda + \mu_2 - \mu_1) (1 - \xi_x) (\xi_{x-1} + \xi_{x+1}) \right] \left[f(\xi^x) - f(\xi) \right] \\ & + \left[\xi_1 (\mu_2 + (\mu_1 - \mu_2) \xi_2) + (\lambda + \mu_2 - \mu_1) (1 - \xi_1) \xi_2 \right] \left[f(\xi^1) - f(\xi) \right] \\ & + \left[\xi_N (\mu_2 + (\mu_1 - \mu_2) \xi_{N-1}) + (\lambda + \mu_2 - \mu_1) (1 - \xi_N) \xi_{N-1} \right] \left[f(\xi^N) - f(\xi) \right], \end{aligned} \quad (3.11)$$

$\widehat{\mathcal{D}} = \mathcal{D} + \mu_1 - \mu_2$ is the diffusion parameter of the exclusion dynamics, and $\mathcal{L}_-^{\text{Dual}}$, $\mathcal{L}_+^{\text{Dual}}$ are defined in (2.23), (2.24) with rates $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}$, in place of $\alpha, \beta, \gamma, \delta$.

The proof of Theorem 2 is given in subsection 3.4. A special case of the above setting is elucidated in the Corollary below.

Corollary 1 (Annihilating dual process). *Under the same hypothesis as before, setting $\mu_1 = \lambda + \mu_2$, the dual GDCP has birth rate $\widehat{\lambda} = 0$, i.e. no particle is ever created. The intensity matrix associated to the local dual generator is given by*

$$\ell^{\text{Dual}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mu_2 & -(\mathcal{D} + \lambda + \mu_2) & \mathcal{D} + \lambda & 0 \\ \mu_2 & \mathcal{D} + \lambda & -(\mathcal{D} + \lambda + \mu_2) & 0 \\ 0 & \lambda + \mu_2 & \lambda + \mu_2 & -2(\lambda + \mu_2) \end{pmatrix}. \quad (3.12)$$

3.3 Application of duality: computing correlations

As for the DCP, the dynamics of the GDCP is irreducible and the state space is finite so there is a unique invariant measure ν^{GDCP} for the GDCP, which depends on the birth, death, exchange and boundary rates. For a particular choice of these rates, the one-point function of ν^{GDCP} can be explicitly computed, see Proposition 5. Furthermore, some information on the time-dependent one-point function for arbitrary initial distributions can be obtained using duality. To this end, consider $\xi = \delta_{x_1, \dots, x_K}$ namely the dual configuration with K particles respectively on sites $0 < x_1 < \dots < x_K < N + 1$ and none elsewhere, then the duality function reads

$$D(\eta, \delta_{x_1, \dots, x_K}) = \prod_{x \in A(\xi)} (1 - \eta_x) = \prod_{i=1}^K (1 - \eta_{x_i}), \quad (3.13)$$

where $A(\xi) = \{x_1, \dots, x_K\}$. Then one can write the time-dependent K -points correlations via duality as

$$\begin{aligned} \mathbb{E}_\eta [D(\eta(t), \delta_{x_1, \dots, x_K})] &= \mathbb{E}_\eta \left[\prod_{x \in A(\xi)} (1 - \eta_x(t)) \right] = \mathbb{E}_{\delta_{x_1, \dots, x_K}} \left[c_-^{\xi_0(t)} c_+^{\xi_{N+1}(t)} \prod_{x \in A(\xi(t))} (1 - \eta_x) \right] \\ &= \sum_{n_- = 0}^{\infty} \sum_{n_+ = 0}^{\infty} \sum_{\zeta \in \Omega_N} c_-^{n_-} c_+^{n_+} \prod_{i=1}^M (1 - \zeta_{y_i}) p_t((0; x_1, \dots, x_K; 0), (n_-; y_1, \dots, y_M; n_+)) \end{aligned}$$

where $p_t((0; x_1, \dots, x_K; 0), (n_-; y_1, \dots, y_M; n_+))$ is the probability that, starting with a dual configuration δ_{x_1, \dots, x_K} , at time t there are n_- dual particles absorbed in the left sink in site 0, n_+ dual particles absorbed in the right sink in site $N + 1$, and a dual particle in sites y_1, \dots, y_M .

3.3.1 Invariant measure

For $1 \leq \ell \leq N$, the ℓ -point correlation function ρ^{GDCP} of ν^{GDCP} is the function ρ_ℓ^{GDCP} defined by

$$\forall x_1 < \dots < x_\ell \in \Lambda_N, \rho_\ell^{\text{GDCP}}(x_1, \dots, x_\ell) = \mathbb{E}_{\nu^{\text{GDCP}}} [\eta_{x_1} \dots \eta_{x_\ell}]. \quad (3.14)$$

From now on, we denote by

$$\tilde{c}_- := \frac{\tilde{\gamma}}{\tilde{\alpha} + \tilde{\gamma}}, \quad \text{and} \quad \tilde{c}_+ := \frac{\tilde{\beta}}{\tilde{\beta} + \tilde{\delta}}. \quad (3.15)$$

As for the DCP, the dual $(\xi(t))_{t \geq 0}$ of the GDCP (we keep the same notation ξ to refer to the dual process) becomes extinct almost surely and, recall that the total number of particles absorbed on the left, resp. right hand side boundary is:

$$\xi_0(\infty) := \lim_{t \rightarrow \infty} \xi_0(t), \quad \text{resp.} \quad \xi_{N+1}(\infty) := \lim_{t \rightarrow \infty} \xi_{N+1}(t), \quad (3.16)$$

where the limits are almost sure.

Following the same lines as for the proof of Proposition 2, one can prove the following general formula for the correlation functions of the GDCP:

Proposition 4. *For any $1 \leq \ell \leq N$, and any $1 \leq x_1 < \dots < x_\ell \leq N$,*

$$\begin{aligned} \mathbb{E}_{\nu^{\text{GDCP}}} \left[\prod_{j=1}^{\ell} (1 - \eta_{x_j}) \right] &= \sum_{k=0}^{\ell} (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq \ell} \rho_k^{\text{GDCP}}(x_{i_1}, \dots, x_{i_k}) \\ &= \sum_{m, n \geq 0} \mathbb{P}_{\delta_{x_1, x_2, \dots, x_\ell}} \left[\xi_0(\infty) = m, \xi_N(\infty) = n \right] \tilde{c}_-^m \tilde{c}_+^n, \end{aligned} \quad (3.17)$$

In the particular setting of Corollary 1, the one-point function can be computed explicitly.

Proposition 5. Consider a GDCP with birth rate $\lambda > 0$, diffusion coefficient $\mathcal{D} > 0$, boundary rates $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ and death rates μ_1 and μ_2 , with $\mu_1 = \mu_2 + \lambda$. Then,

$$\rho_1^{\text{GDCP}}(x) = u_x(1 - \tilde{c}_-) + v_x(1 - \tilde{c}_+),$$

where with the constants

$$\begin{aligned} \tilde{a} &:= \frac{\tilde{\alpha} + \tilde{\gamma}}{\tilde{\alpha} + \tilde{\gamma} + \mathcal{D} + \lambda + \mu_2}, & \tilde{b} &:= \frac{\mathcal{D} + \lambda}{\tilde{\alpha} + \tilde{\gamma} + \mathcal{D} + \lambda + \mu_2}, \\ \tilde{c} &:= \frac{\tilde{\beta} + \tilde{\delta}}{\tilde{\beta} + \tilde{\delta} + \mathcal{D} + \lambda + \mu_2}, & \tilde{d} &:= \frac{\mathcal{D} + \lambda}{\tilde{\beta} + \tilde{\delta} + \mathcal{D} + \lambda + \mu_2} \end{aligned} \quad (3.18)$$

and

$$A := \frac{\mathcal{D} + \lambda}{\mathcal{D} + \lambda + \mu_2}, \quad r_{\pm} := \frac{1 \pm \sqrt{1 - A^2}}{A}, \quad (3.19)$$

$$B_N := r_-(1 - \tilde{b}r_-)(1 - \tilde{d}r_+^{-1}) + r_-^N r_+^{1-N} (\tilde{b}r_+ - 1)(1 - \tilde{d}r_-^{-1}), \quad (3.20)$$

$$B'_N := \tilde{a}(\tilde{c}N + 1 - \tilde{c}) + (1 - 2\tilde{a})\tilde{c}, \quad (3.21)$$

the terms u_x and v_x are given by, for all $x \in \{1, \dots, N\}$,

$$u_x = \begin{cases} \frac{\tilde{a}}{B_N} [(1 - \tilde{d}r_+^{-1})r_-^x + (\tilde{d}r_-^{-1} - 1)r_-^N r_+^{x-N}] & \mu_2 > 0 \\ \frac{\tilde{a}}{B'_N} [1 - \tilde{c} + \tilde{c}(N - x)] & \mu_2 = 0, \end{cases} \quad (3.22)$$

$$v_x = \begin{cases} \frac{\tilde{c}}{B_N} [(\tilde{b}r_+ - 1)r_+^{1-N} r_-^x + r_-(1 - \tilde{b}r_-)r_+^{x-N}] & \mu_2 > 0 \\ \frac{\tilde{c}}{B'_N} [1 - \tilde{a} + \tilde{a}(x - 1)] & \mu_2 = 0. \end{cases} \quad (3.23)$$

Remark 5. (a) For $\mu_2 > 0$ one finds that the stationary bulk density

$$\rho_{\text{bulk}}^{\text{GDCP}}(s) := \lim_{N \rightarrow \infty} \rho_1([sN]) \quad (3.24)$$

vanishes for all $s \in (0, 1)$. A non-trivial exponentially decaying density profile appears near the boundaries.

(b) In contrast, for $\mu_2 = 0$ a linear bulk density profile emerges,

$$\rho_{\text{bulk}}^{\text{GDCP}}(s) = \frac{\tilde{\alpha}}{\tilde{\alpha} + \tilde{\gamma}}(1 - s) + \frac{\tilde{\beta}}{\tilde{\beta} + \tilde{\delta}}s = \frac{\tilde{\alpha}}{\tilde{\alpha} + \tilde{\gamma}} - \frac{\tilde{\alpha}\tilde{\delta} - \tilde{\beta}\tilde{\gamma}}{(\tilde{\alpha} + \tilde{\gamma})(\tilde{\beta} + \tilde{\delta})}s \quad (3.25)$$

This phenomenon is reminiscent of a superposition of shock measures appearing in the voter model with totally asymmetric stirring (rather than symmetric stirring as in the present case) on Λ_N and open boundaries with $\tilde{\gamma} = \tilde{\delta} = 0$ [27]. The invariant measure of that model is a convex combination of shock product measures with marginals $\mu(\eta_x) = 1$ for $1 \leq x < x_s$ and $\mu(\eta_x) = 0$ for $x_s \leq x \leq N$ and uniform distribution of the microscopic shock position $x_s \in \Lambda_N$ which leads to a linear stationary density profile and a simple form of the two-point correlation function computed in [22]. For the present more complicated case of symmetric stirring and arbitrary boundary parameters we leave a detailed investigation of the two-point correlation function for future work.

(c) Moreover, computing higher order correlations is in general a complicated task. For these models, the difficulty lies in the fact that if there are two or more dual particles, the death rates depend on the configuration. We leave this for future work.

3.3.2 Time-dependence of the one-point function

For the choice $\mu_1 = \lambda + \mu_2$ the absence of particle birth or insertion in the dual process guarantees that the duality function yields a finite inhomogeneous system of ordinary differential equations (ODE's) for the one-point function.

Proposition 6. *We assume that $\mu_1 = \lambda + \mu_2$. Denoting by $\bar{\eta}_x := \eta_x - \rho_1(x)$ centered variables, and defining the function*

$$g(x, t) := e^{2\mu_2 t} \langle \bar{\eta}_x \rangle, \quad (3.26)$$

where $\langle \cdot \rangle$ means expectation at time t for an arbitrary initial measure ν , we have

$$\begin{aligned} \frac{d}{dt}g(x, t) &= (\mathcal{D} + \lambda)(g(x+1, t) + g(x-1, t) - 2g(x, t)), \quad 1 < x < N, \\ \frac{d}{dt}g(1, t) &= (\mathcal{D} + \lambda)(g(2, t) - g(1, t)) - (\tilde{\alpha} + \tilde{\gamma} - \mu_2)g(1, t), \\ \frac{d}{dt}g(N, t) &= (\mathcal{D} + \lambda)(g(N-1, t) - g(N, t)) - (\tilde{\beta} + \tilde{\delta} - \mu_2)g(N, t). \end{aligned} \quad (3.27)$$

In the system (3.27), we recognize the ODE's for the time-dependent centered one-point function of the open SSEP with diffusion coefficient $\hat{D} = \mathcal{D} + \lambda$, and reservoir rates $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$, satisfying

$$\hat{\alpha} + \hat{\gamma} = \tilde{\alpha} + \tilde{\gamma} - \mu_2, \quad \text{and} \quad \hat{\beta} + \hat{\delta} = \tilde{\beta} + \tilde{\delta} - \mu_2. \quad (3.28)$$

Following [38], this system can be solved in a closed form by a discrete Fourier transformation with a reflected wave, in the special cases obtained from the four combinations of reservoir parameters given by $\tilde{\alpha} + \tilde{\gamma} \in \{\mu_2, \mu_2 + \mathcal{D} + \lambda\}$ and $\tilde{\beta} + \tilde{\delta} \in \{\mu_2, \mu_2 + \mathcal{D} + \lambda\}$. We find surprising that the system of equations (3.27) matches the one for the space time empirical profile of an open SSEP, see equation (19) and below in [19]. In particular, the choice $\tilde{\alpha} + \tilde{\gamma} = \tilde{\beta} + \tilde{\delta} = \mu_2$ corresponds to $\hat{\alpha} = \hat{\beta} = \hat{\gamma} = \hat{\delta} = 0$ for the boundary parameters of the SSEP (i.e. $\kappa = 0$ in [19]) and leads to the same system of equations, both for the SSEP with reflecting boundary which conserves the total number of particles and for our GCDP which does not conserve the total number of particles due to positive birth and death rates. We believe it would be interesting to prove the hydrodynamic limit for this non-conservative process with open boundary, in the same spirit as for the open finite volume SSEP.

3.4 Proofs for Section 3

3.4.1 Proofs for Subsection 3.2

Proof of Theorem 2. In order to show the matrix duality relation of equation (2.12), as the generator acts on two sites, it is enough to show that

$$\ell G^{\otimes 2} = G^{\otimes 2} \left(\ell^{\text{Dual}} \right)^T, \quad (3.29)$$

for G, ℓ defined respectively in (2.27) and (3.8); we have

$$G^{\otimes 2} = \begin{pmatrix} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{pmatrix}. \quad (3.30)$$

For any choice of the parameters a, b, c, d , this defines a self-duality function for the SSEP (see Remark 2), and we only have to take into consideration the action of the GCP. The left hand side of equation (3.29) becomes

$$\ell G^{\otimes 2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x[a\mu_1 - c\lambda] & ay\mu_1 - dx\lambda - \Delta\mathcal{D} & bx\mu_1 - cy\lambda + \Delta\mathcal{D} & y[b\mu_1 - d\lambda] \\ x[a\mu_1 - c\lambda] & bx\mu_1 - cy\lambda + \Delta\mathcal{D} & ay\mu_1 - dx\lambda - \Delta\mathcal{D} & y[b\mu_1 - d\lambda] \\ 2cx\mu_2 & (dx + cy)\mu_2 & (dx + cy)\mu_2 & 2dy\mu_2 \end{pmatrix}, \quad (3.31)$$

where $\Delta := ad - bc$, $x := a - c$ and $y := b - d$. For the right hand side of (3.29), we first consider a local generator ℓ^{Dual} which describes a GDGP with possibly different rates: $\tilde{\mathcal{D}}$ is the diffusion coefficient, $\tilde{\lambda}$ the birth rate while $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are the death rates. This gives the following dual local intensity matrix

$$\ell^{\text{Dual}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \tilde{\mu}_1 & -(\tilde{\mathcal{D}} + \tilde{\lambda} + \tilde{\mu}_1) & \tilde{\mathcal{D}} & \tilde{\lambda} \\ \tilde{\mu}_1 & \tilde{\mathcal{D}} & -(\tilde{\mathcal{D}} + \tilde{\lambda} + \tilde{\mu}_1) & \tilde{\lambda} \\ 0 & \tilde{\mu}_2 & \tilde{\mu}_2 & -2\tilde{\mu}_2 \end{pmatrix}. \quad (3.32)$$

Computing the right hand side of (3.29), we get

$$G^{\otimes 2} (\ell^{\text{Dual}})^T = \begin{pmatrix} 0 & z(a\tilde{\mu}_1 - b\tilde{\lambda}) & z(a\tilde{\mu}_1 - b\tilde{\lambda}) & 2\tilde{\mu}_2bz \\ 0 & aw\tilde{\mu}_1 - dz\tilde{\lambda} - \Delta\tilde{\mathcal{D}} & cz\tilde{\mu}_1 - bw\tilde{\lambda} + \Delta\tilde{\mathcal{D}} & (bw + dz)\tilde{\mu}_2 \\ 0 & cz\tilde{\mu}_1 - bw\tilde{\lambda} + \Delta\tilde{\mathcal{D}} & aw\tilde{\mu}_1 - dz\tilde{\lambda} - \Delta\tilde{\mathcal{D}} & (bw + dz)\tilde{\mu}_2 \\ 0 & w[c\tilde{\mu}_1 - d\tilde{\lambda}] & w[c\tilde{\mu}_1 - d\tilde{\lambda}] & 2dw\tilde{\mu}_2 \end{pmatrix}, \quad (3.33)$$

where $z = a - b$ and $w = c - d$. Since we are aiming to match expressions (3.31) and (3.33), we first notice that equality in the first column requires $x = 0$, i.e. $a = c$, while equality in the first row requires $z = 0$, i.e. $a = b$. Thus $a = b = c$ yields $\Delta = -a(a - d)$, and we are left with

$$\ell G^{\otimes 2} = (a - d) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a\mu_1 + a\mathcal{D} & -a\lambda - a\mathcal{D} & a\mu_1 - d\lambda \\ 0 & -a\lambda - a\mathcal{D} & a\mu_1 + a\mathcal{D} & a\mu_1 - d\lambda \\ 0 & a\mu_2 & a\mu_2 & 2d\mu_2 \end{pmatrix} \quad (3.34)$$

for the left hand side, and

$$G^{\otimes 2} (\ell^{\text{Dual}})^T = (a - d) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a\tilde{\mu}_1 + a\tilde{\mathcal{D}} & -a\tilde{\lambda} - a\tilde{\mathcal{D}} & a\tilde{\mu}_2 \\ 0 & -a\tilde{\lambda} - a\tilde{\mathcal{D}} & a\tilde{\mu}_1 + a\tilde{\mathcal{D}} & a\tilde{\mu}_2 \\ 0 & a\tilde{\mu}_1 - d\tilde{\lambda} & a\tilde{\mu}_1 - d\tilde{\lambda} & 2d\tilde{\mu}_2 \end{pmatrix} \quad (3.35)$$

for the right hand side.

Consider $d = 0$ in both equations (3.34) and (3.35). Then, the identity holds if we set $\tilde{\mu}_2 = \mu_1$ and $\tilde{\mu}_1 = \mu_2$, $\tilde{\mathcal{D}} = \mathcal{D} + \mu_1 - \mu_2$ and $\tilde{\lambda} = \lambda + \mu_2 - \mu_1$. Furthermore, without loss of generality, choosing $a = 1$, we are left with the duality matrix

$$G = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.36)$$

which corresponds to the duality function

$$G(\eta, \xi) = \prod_{x \in \Lambda_N} (1 - \eta_x)^{\xi_x}. \quad (3.37)$$

Note that G is the same duality function as in (2.21) (see Remark 2). Since the bulk duality function is the same as the one for the diffusive contact process, we can extend the result to the GDCP with the corresponding boundary parameters. Hence, the duality function has the same form as for the diffusive contact process, that is: $D : \Omega_N \times \Omega_N^{\text{dual}} \rightarrow \mathbb{R}$ given by

$$D(\eta, \xi) = \left(\frac{\tilde{\gamma}}{\tilde{\alpha} + \tilde{\gamma}} \right)^{\xi_0} \prod_{x \in \Lambda_N} (1 - \eta_x)^{\xi_x} \left(\frac{\tilde{\beta}}{\tilde{\beta} + \tilde{\delta}} \right)^{\xi_{N+1}}. \quad (3.38)$$

Notice that for $\mu_1 = \mu_2 = \frac{1}{2}$, we recover Theorem 1. \square

The proof of Corollary 1 follows from Theorem 2.

3.4.2 Proofs for subsection 3.3

Proof of Proposition 4. In this setting, by Corollary 1, the dual process of the GDCP is a GDCP with no birth rate, diffusion coefficient $\mathcal{D} + \lambda$, boundary rates $0, \tilde{\alpha} + \tilde{\gamma}, 0, \tilde{\beta} + \tilde{\delta}$ (purely absorbing boundaries), and death rates μ_2 and $\lambda + \mu_2$. Thus, the generator of the dual process is given by

$$\mathcal{L}^{\text{GDCP, Dual}} = \mathcal{L}_-^{\text{Dual}} + \mathcal{L}^{\text{GCP, Dual}} + (\mathcal{D} + \lambda) \mathcal{L}^{\text{SSEP}} + \mathcal{L}_+^{\text{Dual}},$$

where we recall that $\mathcal{L}_-^{\text{Dual}}$ and $\mathcal{L}_+^{\text{Dual}}$ are defined in (2.23) (with $\tilde{\alpha} + \tilde{\gamma}$ instead of $\alpha + \gamma$ and $\tilde{\beta} + \tilde{\delta}$ instead of $\beta + \delta$) and,

$$\begin{aligned} \mathcal{L}^{\text{GDCP, Dual}} f(\xi) &= \sum_{x=2}^{N-1} \left[\eta_x (2\mu_2 + \lambda (\xi_{x-1} + \xi_{x+1})) \right] [f(\xi^x) - f(\xi)] \\ &\quad + \left[\xi_1 (\mu_2 + \lambda \xi_2) \right] [f(\xi^1) - f(\xi)] \\ &\quad + \left[\xi_N (\mu_2 + \lambda \xi_{N-1}) \right] [f(\xi^N) - f(\xi)]. \end{aligned}$$

By Theorem 2 the duality function $D : \Omega_N \times \Omega_N^{\text{dual}} \rightarrow \mathbb{R}$ is the same as for the DCP, that is, given by (3.38). The same reasoning as in the proof of Proposition 1 yields that for $x \in \{1, \dots, N\}$,

$$\begin{aligned} \mathbb{E}_{\nu^{\text{GDCP}}} [1 - \eta_x] &= \mathbb{P}_{\delta_x} [\xi_0(\infty) = 1 \cap \xi_{N+1}(\infty) = 0] \tilde{c}_- \\ &\quad + \mathbb{P}_{\delta_x} [\xi_0(\infty) = 0 \cap \xi_{N+1}(\infty) = 1] \tilde{c}_+ \\ &\quad + \left(1 - \mathbb{P}_{\delta_x} [\xi_0(\infty) = 1 \cap \xi_{N+1}(\infty) = 0] - \mathbb{P}_{\delta_x} [\xi_0(\infty) = 0 \cap \xi_{N+1}(\infty) = 1] \right) \\ &=: u_x \tilde{c}_- + v_x \tilde{c}_+ + (1 - u_x - v_x). \end{aligned}$$

Therefore,

$$\rho_1^{\text{GDCP}}(x) = u_x (1 - \tilde{c}_-) + v_x (1 - \tilde{c}_+).$$

In this case the sum (3.17) is finite due to the fact that the rate of birth is zero, so $\xi_0(\infty)$ and $\xi_{N+1}(\infty)$ can be at most equal to one. \square

Proof of Proposition 5. To prove Proposition 5, we are now left to compute $(u_x)_{1 \leq x \leq N}$ and $(v_x)_{1 \leq x \leq N}$. By conditioning on the first possible event (jump, birth, death, or absorption by a reservoir) of the process starting from a particle at site x , we have the following discrete equations

- For $(u_x)_{1 \leq x \leq N}$:

$$\begin{aligned} u_1 &= \tilde{a} + \tilde{b}u_2 \\ u_x &= \frac{A}{2}(u_{x-1} + u_{x+1}) \quad \forall x \in \{2, \dots, N-1\} \\ u_N &= \tilde{d}u_{N-1}, \end{aligned}$$

with the constants $\tilde{a}, \tilde{b}, \tilde{d}$, and A defined in (3.18) and (3.19).

- For $(v_x)_{1 \leq x \leq N}$:

$$\begin{aligned} v_1 &= \tilde{b}v_2 \\ v_x &= \frac{A}{2}(v_{x-1} + v_{x+1}) \quad \forall x \in \{2, \dots, N-1\} \\ v_N &= \tilde{c} + \tilde{d}v_{N-1}, \end{aligned}$$

with the constants $\tilde{b}, \tilde{c}, \tilde{d}$, and A defined in (3.18) and (3.19).

Both recurrence relations are of the form

$$\begin{aligned} w_1 &= a + \tilde{b}w_2 \\ w_x &= \frac{A}{2}(w_{x-1} + w_{x+1}) \quad \forall x \in \{2, \dots, N-1\} \\ w_N &= c + \tilde{d}w_{N-1} \end{aligned} \tag{3.39}$$

with $a = \tilde{a}$, $c = 0$, for $(u_x)_{1 \leq x \leq N}$ and $a = 0$, $c = \tilde{c}$, for $(v_x)_{1 \leq x \leq N}$. The properties of the solution of this recursion depend on μ_2 as follows.

Case 1: $\mu_2 > 0$

To solve the general recurrence relation, notice that the constants defined in (3.19) satisfy $A < 1$, $r_- < 1$ and $r_+ > 1$. Then, there are $p, q \in \mathbb{R}$ such that for $2 \leq x \leq N-1$,

$$w_x = pr_-^x + qr_+^x, \tag{3.40}$$

which follows from the bulk part of the recurrence relation (3.39). Furthermore, the recurrence relation involving sites 1 and 2 yields $pr_- + qr_+ = a + \tilde{b}[pr_-^2 + qr_+^2]$, therefore $pr_-(1 - \tilde{b}r_-) = a + qr_+(\tilde{b}qr_+ - 1)$. Since $0 < r_- < 1$ and $\tilde{b} < 1$, for both u_x and v_x , one obtains

$$p = \frac{a + qr_+(\tilde{b}r_+ - 1)}{r_-(1 - \tilde{b}r_-)}.$$

In a similar fashion, the recurrence relation involving sites N and $N-1$ yields $pr_-^N + qr_+^N = c + \tilde{d}[pr_-^{N-1} + qr_+^{N-1}]$, therefore $pr_-^N(1 - \tilde{d}r_-^{-1}) = c + qr_+^N(\tilde{d}r_+^{-1} - 1)$. Since $r_+ > 1$ and $\tilde{d} < 1$, for both u_x and v_x , one obtains

$$q = \frac{c - pr_-^N(1 - \tilde{d}r_-^{-1})}{r_+^N(1 - \tilde{d}r_+^{-1})}.$$

Finally, for B_N defined in (3.20) one finds

$$q = \frac{1}{B_N}[cr_-(1 - \tilde{b}r_-) + ar_-^N(\tilde{d}r_-^{-1} - 1)]r_+^{-N} \tag{3.41}$$

$$p = \frac{1}{B_N}[a(1 - \tilde{d}r_+^{-1}) + cr_+^{1-N}(\tilde{b}r_+ - 1)]. \tag{3.42}$$

This result yields (3.22) and (3.23) for $\mu_2 > 0$.

Case 2, $\mu_2 = 0$

In this case, $A = 1$ and the bulk part of the recurrence relation (3.39) can be written $\Delta w_x = 0$, where Δ is the discrete one-dimensional Laplacian. The general solution is the linear function

$$w_x = p' + q'x. \quad (3.43)$$

The boundary condition at site 1 yields $p' + q' = a + \tilde{b}(p' + 2q')$ and therefore $p'(1 - \tilde{b}) = a + (2\tilde{b} - 1)q'$. Likewise, the boundary condition at site N yields $p' + q'N = c + \tilde{d}(p' + q'(N - 1))$ and therefore $p'(1 - \tilde{d}) = c - q'[(1 - \tilde{d})N - \tilde{d}]$. We then discuss the following cases:

- (i) $\tilde{b} = \tilde{d} = 1$. In this case, we have closed boundary conditions, and the empty lattice is the absorbing state, implying without further computation $u_x = v_x = \rho_1^{\text{GDGP}}(x) = 0$ for all $x \in \{1, \dots, N\}$.
- (ii) $\tilde{b} = 1, \tilde{d} < 1$. This corresponds to a closed left boundary, with $\tilde{a} = 0$, but with open right boundary, leaving the invariant measure non-trivial. The recurrence for w_x yields $q' = 0$ and one obtains $u_x = 0, v_x = 1$ for all $x \in \{1, \dots, N\}$.
- (iii) $\tilde{d} = 1, \tilde{b} < 1$. This corresponds to a closed right boundary, with $\tilde{c} = 0$, but with open left boundary, leaving the invariant measure non-trivial. The recurrence for w_x yields $q' = 0$ and one obtains $u_x = 1, v_x = 0$ for all $x \in \{1, \dots, N\}$.
- (iv) $\tilde{d} < 1, \tilde{b} < 1$. The recurrence yields

$$q' = \frac{c(1 - \tilde{b}) - a(1 - \tilde{d})}{B'_N} \quad (3.44)$$

$$p' = \frac{a[(1 - \tilde{d})N + \tilde{d}] + c(2\tilde{b} - 1)}{B'_N}, \quad (3.45)$$

for B'_N defined in (3.21).

Hence, observing that for $\mu_2 = 0$, one has $1 = \tilde{a} + \tilde{b} = \tilde{c} + \tilde{d}$, we arrive at (3.22) and (3.23) for $\mu_2 = 0$, which covers all four cases (i)-(iv). \square

Proof of Proposition 6. Assuming $\mu_1 = \lambda + \mu_2$ the duality function of the GDGP yields the finite inhomogeneous system of ordinary differential equations (ODE's)

$$\begin{aligned} \frac{d}{dt} \langle \eta_x \rangle &= (\mathcal{D} + \lambda)(\langle \eta_{x+1} \rangle + \langle \eta_{x-1} \rangle - 2\langle \eta_x \rangle) - 2\mu_2 \langle \eta_x \rangle, \quad 1 < x < N \\ \frac{d}{dt} \langle \eta_1 \rangle &= (\mathcal{D} + \lambda)(\langle \eta_2 \rangle - \langle \eta_1 \rangle) - (\mu_2 + \tilde{\alpha} + \tilde{\gamma}) \langle \eta_1 \rangle + \tilde{\alpha}, \\ \frac{d}{dt} \langle \eta_N \rangle &= (\mathcal{D} + \lambda)(\langle \eta_{N-1} \rangle - \langle \eta_N \rangle) - (\mu_2 + \tilde{\beta} + \tilde{\delta}) \langle \eta_N \rangle + \tilde{\delta}. \end{aligned} \quad (3.46)$$

The inhomogeneity arising from the constants $\tilde{\alpha}$ and $\tilde{\delta}$ can be removed by considering the centered variables $\bar{\eta}_x := \eta_x - \rho_1(x)$, and by noting that invariance of the measure yields $\tilde{\alpha} = (\mathcal{D} + \lambda + \mu_2 + \tilde{\alpha} + \tilde{\gamma})\rho_1(1) - (\mathcal{D} + \lambda)\rho_1(2)$ and $\tilde{\delta} = (\mathcal{D} + \lambda + \mu_2 + \tilde{\beta} + \tilde{\delta})\rho_1(N) - (\mathcal{D} + \lambda)\rho_1(N - 1)$. With the recursion (3.39), which by linearity is also valid for $\rho_1(x)$, we conclude that

$$\begin{aligned} \frac{d}{dt} \langle \bar{\eta}_x \rangle &= (\mathcal{D} + \lambda)(\langle \bar{\eta}_{x+1} \rangle + \langle \bar{\eta}_{x-1} \rangle - 2\langle \bar{\eta}_x \rangle) - 2\mu_2 \langle \bar{\eta}_x \rangle, \quad 1 < x < N, \\ \frac{d}{dt} \langle \bar{\eta}_1 \rangle &= (\mathcal{D} + \lambda)(\langle \bar{\eta}_2 \rangle - \langle \bar{\eta}_1 \rangle) - (\mu_2 + \tilde{\alpha} + \tilde{\gamma}) \langle \bar{\eta}_1 \rangle, \\ \frac{d}{dt} \langle \bar{\eta}_N \rangle &= (\mathcal{D} + \lambda)(\langle \bar{\eta}_{N-1} \rangle - \langle \bar{\eta}_N \rangle) - (\mu_2 + \tilde{\beta} + \tilde{\delta}) \langle \bar{\eta}_N \rangle, \end{aligned} \quad (3.47)$$

that is (3.27), which is a *homogeneous* system of first order ordinary differential equations. \square

4 The susceptible-infectious-recovered (SIR) model

We now consider another non conservative model, originally introduced in [24], in terms of a nonlinear system of differential equations for the sizes of populations of three species of individuals which are subject to an infection/recovery mechanism. The susceptible-infectious-recovered (SIR) model describes propagation of infections in the following sense: a susceptible individual (characterized by its state S) can become an infectious individual (characterized by its state I), according to an infection rate β , if it is in contact with an infectious individual. An infectious individual recovers (then it is characterized by its state R) with recovery rate γ ; once an individual has recovered it stays immune, that is, it remains in state R forever.

Despite the simplicity of the original mean-field type model, it is known to capture important features of the temporal dynamics of an infection. However, only limited results are available if fluctuations (which inevitably occur in a real system) are taken into account. This question was addressed in the SIR model of [40] where particles evolve on a one dimensional space of sites, as in the diffusive contact process. However, contrary to the diffusive contact process studied in Section 2), the model is defined on the infinite translation invariant lattice \mathbb{Z} , so that in particular, there are no reservoirs. Notice that the parameters β and γ for the SIR model have nothing to do with the reservoir parameters in the previous sections. The reason for which we keep this notation is that it is canonically used in the literature on the SIR model.

The stochastic evolution of the collection of particles in the system is governed by a Markov process denoted by $\{\eta_x(t), x \in \mathbb{Z}, t \geq 0\}$, with state space

$$\mathcal{S} := \{S, I, R\}^{\mathbb{Z}}, \quad (4.1)$$

so that for $x \in \mathbb{Z}$, $a \in \{S, I, R\}$, $\eta_x = a$ means that x is in state a . Using the notation in [40], the (translation invariant) transition rates between nearest neighboring sites are given by

<i>Initial</i>	\rightarrow	<i>Final</i>	<i>Rate</i>
IS	\rightarrow	II	β
SI	\rightarrow	II	β
I	\rightarrow	R	γ

(4.2)

Note that this dynamics is not attractive (the necessary and sufficient conditions required for attractiveness in [4] are not satisfied), so we cannot rely on monotonicity for this model. In view of the duality result for the SIR model, see Theorem 3, we give an alternative definition of the dynamics in terms of its generator \mathcal{L}^{SIR} . For $f : \mathcal{S} \rightarrow \mathbb{R}$,

$$\begin{aligned} \mathcal{L}^{\text{SIR}} f(\eta) = & \sum_{x \in \mathbb{Z}} \{ \beta \eta_x^I \eta_{x+1}^S [f(T_{x+1}^I \eta) - f(\eta)] + \beta \eta_x^I \eta_{x-1}^S [f(T_{x-1}^I \eta) - f(\eta)] \\ & + \gamma \eta_x^I [f(T_x^R \eta) - f(\eta)] \} , \end{aligned} \quad (4.3)$$

where for $x, y \in \mathbb{Z}$ and $a \in \{S, I, R\}$

$$\eta_x^a = \mathbb{1}_{\{\eta_x = a\}}, \quad (4.4)$$

and T_y^a is the operator acting on elements of \mathcal{S} which flips the state of y into a , that is, for $z \in \mathbb{Z}$,

$$(T_y^a \eta)_z = \begin{cases} \eta_z & \text{if } z \neq y \\ a & \text{if } z = y. \end{cases} \quad (4.5)$$

Our first goal is to find a duality relation for the SIR model with generator (4.3).

4.1 Clusters as duality function

This section is devoted to showing duality for the SIR model, for the purpose of studying the expected population size, and correlations as a function of time for arbitrary initial distributions, rather than only for translation invariant ones, as studied in [40]. This is achieved using a duality relation between *cluster functions* (defined below) with a bi-layered two-dimensional random walk on two copies of the semi-infinite lattice $\mathbb{N} \times \mathbb{Z}$ that we shall label by G and J respectively. The random walk is asymmetric on each lattice and lattice G is absorbing, i.e., once the random walker has left lattice H it cannot return to it. In addition, there is a further (single) absorbing cemetery state that the random walker can reach from lattice G .

To make this qualitative picture of the dual process precise and obtain information about the original process, we use the notation $\langle \cdot \rangle_\nu := \mathbb{E}_\nu[\cdot]$ to denote the expectation with respect to some initial distribution ν , e.g. $\langle \eta_x^a(t) \rangle_\nu$ is the expected state at site x and time t with respect to ν . As already noticed in [40] quantities of interest are written in terms of high order correlation functions, called n -point cluster functions, for $n \in \mathbb{N}$, the set of strictly positive integers:

$$G^\nu(r, n, t) := \left\langle \eta_{r-1}^I(t) \left(\prod_{j=0}^{n-1} \eta_{r+j}^S(t) \right) \eta_{r+n}^I(t) \right\rangle_\nu \quad (4.6)$$

$$H^\nu(r, n, t) := \left\langle \left(\prod_{j=0}^{n-1} \eta_{r+j}^S(t) \right) \eta_{r+n}^I(t) \right\rangle_\nu, \quad (4.7)$$

where in what follows, to lighten the notation we do not write the dependence in time explicitly.

Instead of the cluster H , it will be more convenient to define a different cluster which is of the same size as G . Namely,

$$J^\nu(r, n, t) := \left\langle \eta_{r-1}^R(t) \left(\prod_{j=0}^{n-1} \eta_{r+j}^S(t) \right) \eta_{r+n}^I(t) \right\rangle_\nu. \quad (4.8)$$

Using that for any $x \in \mathbb{Z}$, $\eta_x^S + \eta_x^R + \eta_x^I = 1$, the relation between these clusters is

$$J^\nu(r, n, t) = H^\nu(r, n, t) - H^\nu(r-1, n+1, t) - G^\nu(r, n, t), \quad (4.9)$$

so that given $J^\nu(r, n, t)$ and $G^\nu(r, n, t)$, the cluster function $H^\nu(r, n, t)$ can be computed recursively.

In particular, one can gather information about the correlations of the SIR model by choosing short clusters, see also Remark 7 below. We are now ready to give the duality result for the SIR model. To this end, denote by

$$\mathcal{S}^{\text{dual}} := (\mathbb{Z} \times \mathbb{N} \times \{G, J\}) \cup \{\partial\}, \quad (4.10)$$

where ∂ will be a trap for the dual evolution.

Theorem 3 (Duality relation for SIR). *The SIR model $(\eta_t)_{t \geq 0}$ with generator (4.3) is dual to a two dimensional biased random walk on two layers (see Figure 4) with duality function $d : \mathcal{S} \times \mathcal{S}^{\text{dual}} \rightarrow \mathbb{R}$ given by: for $\eta \in \mathcal{S}$ and $\xi \in \mathcal{S}^{\text{dual}}$, if $\xi = (r, n, i) \in \mathbb{Z} \times \mathbb{N} \times \{G, J\}$,*

$$\begin{aligned} d(\eta, (r, n, i)) &:= \eta_{r-1}^I \eta_r^S \cdots \eta_{r+n-1}^S \eta_{r+n}^I \mathbb{1}_{i=G} + \eta_{r-1}^R \eta_r^S \cdots \eta_{r+n-1}^S \eta_{r+n}^I \mathbb{1}_{i=J} \\ &= [\eta_{r-1}^I \mathbb{1}_{i=G} + \eta_{r-1}^R \mathbb{1}_{i=J}] \left(\prod_{j=0}^{n-1} \eta_{r+j}^S \right) \eta_{r+n}^I, \end{aligned} \quad (4.11)$$

and

$$d(\eta, \partial) := 0. \quad (4.12)$$

The dual generator acts on local functions $f : \mathcal{S}^{\text{dual}} \rightarrow \mathbb{R}$ as follows: for $\xi \in \mathcal{S}^{\text{dual}}$, if $\xi = \partial$,

$$\mathcal{L}^{\text{dual}} f(\partial) = 0, \quad (4.13)$$

while if $\xi = (r, n, i) \in \mathbb{Z} \times \mathbb{N} \times \{G, J\}$,

$$\begin{aligned} \mathcal{L}^{\text{dual}} f(r, n, i) &= \beta \mathbb{1}_{i=G} [f(r-1, n+1, i) - f(r, n, i)] + \beta [f(r, n+1, i) - f(r, n, i)] \\ &\quad + \gamma \mathbb{1}_{i=J} [f(r, n, \phi(i)) - f(r, n, i)] + 2\gamma \mathbb{1}_{i=G} [f(\partial) - f(r, n, i)], \end{aligned} \quad (4.14)$$

where we define the flip operator ϕ as

$$\phi : \{G, J\} \rightarrow \{G, J\}, \text{ such that } \phi(G) = J \text{ and } \phi(J) = G. \quad (4.15)$$

Remark 6. By (4.13), ∂ is indeed a trap: once the process reaches ∂ , it remains there forever. Furthermore, the generator of the dual dynamics defines a bilayer random walk with the following transitions on the dual state space, illustrated in Fig. 4:

$$(r, n, i) \rightarrow (r, n+1, i) \text{ at rate } \beta \text{ for both layers: } i \in \{G, J\} \quad (4.16)$$

$$(r, n, G) \rightarrow (r-1, n+1, G) \text{ at rate } \beta \text{ only for layer } G. \quad (4.17)$$

It is only possible to go from layer J to layer G , (but not the other way around):

$$(r, n, J) \rightarrow (r, n, G) \text{ at rate } \gamma, \quad (4.18)$$

and absorption in the trap is only possible if the walker is in layer G :

$$(r, n, G) \rightarrow \partial \text{ at rate } 2\gamma. \quad (4.19)$$

Notice the non-translation invariance nature of the dynamics in the second transition (4.17).

4.2 Applications of the duality relation

Consider a measure ν on $\{S, I, R\}^{\mathbb{Z}}$ and recall the n -point cluster functions $G^\nu(r, n, t)$ and $J^\nu(r, n, t)$ defined respectively in (4.6) and (4.8). Note that $J^\nu(r, n, t)$ is not the same as $H(n)$ in [40], because of the term $\mathbb{1}_{\eta_{r-1}(t)=R}$ but we still obtain closed equations for G and J type chains.

Remark 7 (SIR correlations for small clusters). *Choosing $n = 1$ the duality functions can give information on the three point correlation functions given in terms of the clusters G or J , which can then be used to get two point correlation functions via the cluster H .*

4.2.1 Coupled equations for the cluster functions

In [40] (see equation (20)), two first order differential equations are provided and explicitly solved for $G^\nu(r, n, t)$ and $H^\nu(r, n, t)$, provided that the initial measure ν is translation invariant. Therefore, the initial measure and the dynamics (we are on \mathbb{Z}), being both translation invariant, one can drop the dependence on r and write $G^\nu(r, n, t) = G^\nu(n, t)$ and $H^\nu(r, n, t) = H^\nu(n, t)$. We first show that one can recover these equations for $G^\nu(n, t)$ and $J^\nu(n, t)$, by only using the duality relation of Theorem 3.

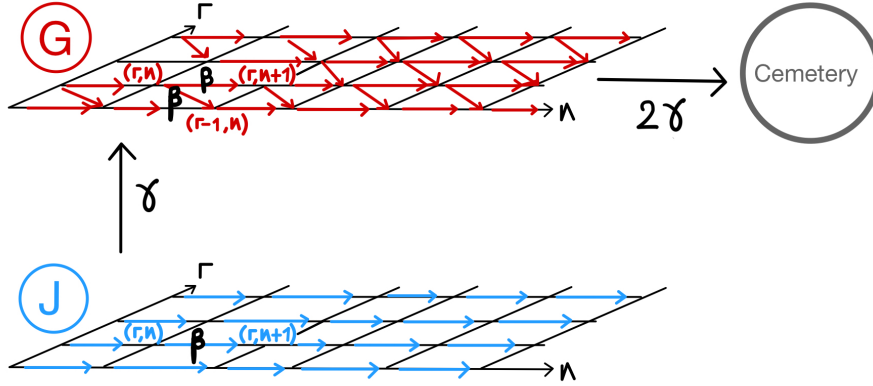


Figure 4: Dual dynamics for the SIR model

Proposition 7. *Assume that the initial distribution ν is translation invariant. Fix $n \in \mathbb{N}$, then we have*

$$\begin{cases} \frac{dG^\nu(n, t)}{dt} = -2(\gamma + \beta)G^\nu(n, t) + 2\beta G^\nu(n + 1, t) \\ \frac{dJ^\nu(n, t)}{dt} = -(\gamma + \beta)J^\nu(n, t) + \beta J^\nu(n + 1, t) + \gamma G^\nu(n, t) . \end{cases} \quad (4.20)$$

The solutions are given by

$$G^\nu(n, t) = e^{-2(\gamma+\beta)t} \sum_{\ell \geq 0} \frac{(2\beta t)^\ell}{\ell!} G^\nu(\ell + n, 0) \quad (4.21)$$

$$\begin{aligned} J^\nu(n, t) &= e^{-(\gamma+\beta)t} \sum_{\ell \geq 0} \frac{(\beta t)^\ell}{\ell!} J^\nu(\ell + n, 0) \\ &\quad + \gamma \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{\ell \geq 0} \frac{(\beta(t-s))^\ell}{\ell!} G^\nu(\ell + n, s) ds. \end{aligned} \quad (4.22)$$

Moreover

$$J^\nu(n, t) = H^\nu(n, t) - H^\nu(n + 1, t) - G^\nu(n, t) . \quad (4.23)$$

4.2.2 Non-translation invariant case

In [40], the cluster correlation functions have not been solved in the case where ν is not translation invariant, because the equations obtained in G and J or H are not closed. Our goal now is to get an explicit expression of such cluster functions in the non translation invariant case, by using the dual process.

Denote by ξ an element of $\mathcal{S}^{\text{dual}}$. For any SIR configuration η , by abuse of notation, we write

$$G^\eta(r, n, t) =: G^{\delta_\eta}(r, n, t) \text{ and } J^\eta(r, n, t) =: J^{\delta_\eta}(r, n, t),$$

the cluster correlation functions when the system is initially in state η . Note that the measure δ_η is no longer a translation invariant measure for the SIR process (unless it is a constant configuration, in which case the system remains unchanged).

Theorem 4. *The $G^\eta(r, n, t)$ cluster is given by:*

$$G^\eta(r, n, t) = e^{-2(\gamma+\beta)t} \sum_{\substack{(a,b) \in (\mathbb{N})^2 \\ a \leq b}} \frac{\binom{b}{a} (2\beta t)^b}{2^b b!} G^\eta(r-a, n+b, 0). \quad (4.24)$$

Theorem 5. *The $J^\eta(r, n, t)$ cluster is given by:*

$$\begin{aligned} J^\eta(r, n, t) &= e^{-(\gamma+\beta)t} \sum_{k \geq 0} \frac{(\beta t)^k}{k!} J^\eta(r, n+k, 0) \\ &+ \gamma e^{-(\gamma+\beta)t} \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a,b) \in \mathbb{N}^2}} \frac{(\beta s)^k \binom{b}{a} (2\beta(t-s))^b}{k! 2^b b!} G^\nu(r-a, n+k+b, 0) ds. \end{aligned} \quad (4.25)$$

Theorems 4 and 5 hold for any initial measure ν which is not necessarily translation invariant. We conclude this section showing that under the extra assumption of translation invariance we can recover the expressions (21) and (22) in [40].

Special case: translation invariance

Let us recover the formulas obtained in [40] in the translation invariant case using ODE's (here we have not used such ODE's), from (4.24) and (4.25). If ν is translation invariant, $G^\nu(r, n, 0)$ and $J^\nu(r, n, 0)$ are independent of r , so

$$\begin{aligned} G^\nu(n, t) &:= G^\nu(r, n, t) = e^{-2(\gamma+\beta)t} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b}} \frac{\binom{b}{a} (2\beta t)^b}{2^b b!} G^\nu(r-a, n+b, 0) \\ &= e^{-2(\gamma+\beta)t} \sum_{b \geq 0} \sum_{a=0}^b \frac{\binom{b}{a} (2\beta t)^b}{2^b b!} G^\nu(n+b, 0). \end{aligned} \quad (4.26)$$

$$\text{Therefore, } G^\nu(n, t) = e^{-2(\gamma+\beta)t} \sum_{b \geq 0} \frac{(2\beta t)^b}{b!} G^\nu(n+b, 0),$$

which matches (21) in [40]. Now,

$$\begin{aligned} J^\nu(n, t) &:= J^\nu(r, n, t) = e^{-(\gamma+\beta)t} \sum_{k \geq 0} \frac{(\beta t)^k}{k!} J^\nu(n+k, 0) \\ &+ \gamma e^{-(\gamma+\beta)t} \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a,b) \in \mathbb{N}^2}} \frac{(\beta s)^k \binom{b}{a} (2\beta(t-s))^b}{k! 2^b b!} G^\nu(n+k+b, 0) ds \\ &= e^{-(\gamma+\beta)t} \sum_{k \geq 0} \frac{(\beta t)^k}{k!} J^\nu(n+k, 0) \\ &+ \gamma e^{-(\gamma+\beta)t} \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{k \geq 0} \sum_{b \geq 0} \frac{(\beta s)^k (2\beta(t-s))^b}{k! b!} G^\nu(n+k+b, 0) ds. \end{aligned} \quad (4.27)$$

Performing the change of variable $u = t - s$, the second integral becomes

$$\begin{aligned} & \int_0^t \sum_{k \geq 0} \frac{(\beta(t-u))^k}{k!} e^{-(\gamma+\beta)u} \sum_{b \geq 0} \frac{(2\beta u)^b}{b!} G^\nu(n+k+b, 0) ds \\ &= \int_0^t \sum_{k \geq 0} \frac{(\beta(t-u))^k}{k!} G^\nu(n+k, s) ds, \end{aligned} \quad (4.28)$$

where we used the expression for $G^\nu(n, t)$ given by (4.26). Finally, we get

$$J^\nu(n, t) = e^{-(\gamma+\beta)t} \sum_{\ell \geq 0} \frac{(\beta t)^\ell}{\ell!} J^\nu(\ell+n, 0) + \gamma \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{\ell \geq 0} \frac{(\beta(t-s))^\ell}{\ell!} G^\nu(\ell+n, s) ds, \quad (4.29)$$

which is exactly the solution given in (4.22), found by solving the ODE for J , in the same spirit as in [40].

4.3 Proofs for Section 4

4.3.1 Proofs for Subsection 4.1

Proof of Theorem 3. First, note that due to the definitions and conventions (4.10), (4.12), (4.13), (4.14), we have, for any configuration η ,

$$\mathcal{L}^{\text{SIR}} d(\cdot, \partial)(\eta) = \mathcal{L}^{\text{dual}} d(\eta, \cdot)(\partial). \quad (4.30)$$

Then, in order to show the following duality relation for all configuration η and $(r, n, i) \in \mathbb{Z} \times \mathbb{N} \times \{G, J\}$

$$\mathcal{L}^{\text{SIR}} d(\cdot, (r, n, i))(\eta) = \mathcal{L}^{\text{dual}} d(\eta, \cdot)(r, n, i), \quad (4.31)$$

it is convenient to split the SIR generator into three different actions, namely

$$\mathcal{L}^{\text{SIR}} d(\cdot, (r, n, i))(\eta) = \sum_{y \in \mathbb{Z}} [A_y + B_y + C_y] d(\cdot, (r, n, i))(\eta), \quad (4.32)$$

where

$$\begin{aligned} A_y d(\cdot, (r, n, i))(\eta) &:= \beta \eta_y^I \eta_{y+1}^S \left[d(T_{y+1}^I \eta, (r, n, i)) - d(\eta, (r, n, i)) \right] \\ B_y d(\cdot, (r, n, i))(\eta) &:= \beta \eta_y^I \eta_{y-1}^S \left[d(T_{y-1}^I \eta, (r, n, i)) - d(\eta, (r, n, i)) \right] \\ C_y d(\cdot, (r, n, i))(\eta) &:= \gamma \eta_y^I \left[d(T_y^R \eta, (r, n, i)) - d(\eta, (r, n, i)) \right], \end{aligned} \quad (4.33)$$

for the operators T_y^a defined in (4.5). To conclude the proof we proceed by direct computations of each of the three items above:

Lemma 2. [Actions of A_y, B_y, C_y] The actions of the three operators A_y, B_y and C_y whose sum over $y \in \mathbb{Z}$ defines the SIR generator (4.3) are

$$\sum_{y \in \mathbb{Z}} A_y d(\eta, (r, n, i)) = \beta \mathbf{1}_{i=G} \left[d(\eta, (r-1, n+1, i)) - d(\eta, (r, n, i)) \right] \quad (4.34)$$

$$\sum_{y \in \mathbb{Z}} B_y d(\eta, (r, n, i)) = \beta \left[d(\eta, (r, n+1, i)) - d(\eta, (r, n, i)) \right] \quad (4.35)$$

$$\sum_{y \in \mathbb{Z}} C_y d(\eta, (r, n, i)) = \gamma \mathbf{1}_{i=J} \left[d(\eta, (r, n, \phi(i))) - d(\eta, (r, n, i)) \right] - 2\gamma \mathbf{1}_{i=G} d(\eta, (r, n, i)) \quad (4.36)$$

Collecting (4.34), (4.35), (4.36) and using that $d(\eta, \partial) = 0$ we get

$$\begin{aligned}
\mathcal{L}^{\text{SIR}} d(\cdot, (r, n, i))(\eta) &= \beta \mathbb{1}_{i=G} \left[d(\eta, (r-1, n+1, i)) - d(\eta, (r, n, i)) \right] \\
&\quad + \beta \left[d(\eta, (r, n+1, i)) - d(\eta, (r, n, i)) \right] \\
&\quad + \gamma \mathbb{1}_{i=J} \left[d(\eta, (r, n, \phi(i))) - d(\eta, (r, n, i)) \right] \\
&\quad + 2\gamma \mathbb{1}_{i=G} \left[d(\eta, \partial) - d(\eta, (r, n, i)) \right] = \mathcal{L}^{\text{dual}} d(\eta, \cdot)(r, n, i),
\end{aligned} \tag{4.37}$$

namely, we have recovered the generator $\mathcal{L}^{\text{dual}}$ of equation (4.14). \square

Proof of Lemma 2. We proceed by direct computations. Recall the definition of the duality function in (4.11).

(i) Action of A_y If $y+1 \notin \{r-1, \dots, r+n\}$ there is no action. Moreover,

$$\begin{aligned}
\eta_y^I \eta_{y+1}^S d(T_{y+1}^I \eta, (r, n, i)) &= \begin{cases} 0 & \text{if } y+1 \in \{r, \dots, r+n\} \\ d(\eta, (r-1, n+1, G)) & \text{if } y+1 = r-1 \end{cases} \\
\eta_y^I \eta_{y+1}^S d(\eta, (r, n, i)) &= \begin{cases} 0 & \text{if } y \in \{r, \dots, r+n\} \\ d(\eta, (r, n, G)) & \text{if } y = r-1 \end{cases}
\end{aligned}$$

and so

$$\sum_{y \in \mathbb{Z}} A_y d(\eta, (r, n, i)) = (A_{r-2} + A_{r-1}) d(\eta, (r, n, i)) = \beta \left[d(\eta, (r-1, n+1, G)) - d(\eta, (r, n, G)) \right]$$

(ii) Action of B_y If $y-1 \notin \{r-1, \dots, r+n\}$ there is no action. Moreover,

$$\begin{aligned}
\eta_{y-1}^S \eta_y^I d(T_{y-1}^I \eta, (r, n, i)) &= \begin{cases} 0 & \text{if } y-1 \in \{r-1, \dots, r+n-1\} \\ d(\eta, (r, n+1, i)) & \text{if } y-1 = r+n \end{cases} \\
\eta_{y-1}^S \eta_y^I d(\eta, (r, n, i)) &= \begin{cases} 0 & \text{if } y \in \{r, \dots, r+n-1\} \\ d(\eta, (r, n, i)) & \text{if } y = r+n \end{cases}
\end{aligned}$$

and so

$$\sum_{y \in \mathbb{Z}} B_y d(\eta, (r, n, i)) = (B_{r+n+1} + B_{r+n}) d(\eta, (r, n, i)) = \beta \left[d(\eta, (r, n+1, i)) - d(\eta, (r, n, i)) \right]$$

(iii) Action of C_y If $y \notin \{r-1, \dots, r+n\}$ there is no action. Moreover,

$$\begin{aligned}
\eta_y^I d(T_y^R \eta, (r, n, i)) &= \begin{cases} 0 & \text{if } y \in \{r, \dots, r+n\} \\ d(\eta, (r, n, G)) \mathbb{1}_{i=J} & \text{if } y = r-1 \end{cases} \\
\eta_y^I d(\eta, (r, n, i)) &= \begin{cases} 0 & \text{if } y \in \{r, \dots, r+n-1\} \\ d(\eta, (r, n, G)) & \text{if } y = r-1 \\ d(\eta, (r, n, i)) & \text{if } y = r+n \end{cases}
\end{aligned}$$

and so

$$\begin{aligned} \sum_{y \in \mathbb{Z}} C_y d(\eta, (r, n, i)) &= (C_{r-1} + C_{r+n}) d(\eta, (r, n, i)) \\ &= \gamma \left[d(\eta, (r, n, G)) \mathbf{1}_{i=J} - d(\eta, (r, n, G)) - d(\eta, (r, n, i)) \right] \end{aligned}$$

□

4.3.2 Proofs for Subsection 4.2

Proof of Proposition 7. We have

$$\begin{aligned} \frac{d}{dt} G^\nu(n, t) &= \frac{d}{dt} \mathbb{E}_\nu \left[d(\cdot, (r, n, G))(\eta(t)) \right] = \mathbb{E}_\nu \left[\mathcal{L}^{\text{SIR}} d(\cdot, (r, n, G))(\eta(t)) \right] \\ &= \mathbb{E}_\nu \left[\mathcal{L}^{\text{Dual}} d(\eta(t), \cdot)(r, n, G) \right] \\ &= \beta \left[\mathbb{E}_\nu \left[d(\cdot, (r-1, n+1, G))(\eta(t)) \right] - \mathbb{E}_\nu \left[d(\cdot, (r, n, G))(\eta(t)) \right] \right] \\ &\quad + \beta \left[\mathbb{E}_\nu \left[d(\cdot, (r, n+1, G))(\eta(t)) \right] - \mathbb{E}_\nu \left[d(\cdot, (r, n, G))(\eta(t)) \right] \right] \\ &\quad - 2\gamma \mathbb{E}_\nu \left[d(\cdot, (r, n, G))(\eta(t)) \right] \\ &= 2\beta [G^\nu(n+1, t) - G^\nu(n, t)] - 2\gamma G^\nu(n, t), \end{aligned}$$

where we used the duality relation in the first line and the fact that G^ν does not depend on r , by translation invariance, in the last line. In the same way, we get:

$$\begin{aligned} \frac{d}{dt} J^\nu(n, t) &= \frac{d}{dt} \mathbb{E}_\nu \left[d(\cdot, (r, n, H))(\eta(t)) \right] = \mathbb{E}_\nu \left[\mathcal{L}^{\text{SIR}} d(\cdot, (r, n, H))(\eta(t)) \right] \\ &= \mathbb{E}_\nu \left[\mathcal{L}^{\text{Dual}} d(\eta(t), \cdot)(r, n, H) \right] \\ &= \beta \left[\mathbb{E}_\nu \left[d(\cdot, (r, n+1, H))(\eta(t)) \right] - \mathbb{E}_\nu \left[d(\cdot, (r, n, H))(\eta(t)) \right] \right] \\ &\quad + \gamma \left[\mathbb{E}_\nu \left[d(\cdot, (r, n, G))(\eta(t)) \right] - \mathbb{E}_\nu \left[d(\cdot, (r, n, H))(\eta(t)) \right] \right] \\ &= \beta [J^\nu(n+1, t) - J^\nu(n, t)] + \gamma [G^\nu(n, t) - J^\nu(n, t)]. \end{aligned}$$

We have thus derived (4.20). Note that indeed these are the analogues of equation (20) in [40]. The first equation is exactly the same one as for G in [40] and the solution is given by (4.21). The second equation can be solved similarly in a recursive way, by treating the terms $J^\nu(n+1, t)$ and $G^\nu(n, t)$ as inhomogeneities. The solution is given by (4.22). Then, using that $\eta_{r-1}^S + \eta_{r-1}^I + \eta_{r-1}^R = 1$, and noticing that

$$H^\nu(n, t) = H^\nu(n+1, t) + J^\nu(n, t) + G^\nu(n, t),$$

the equation for H^ν , which is the same as the one in [40], is

$$\frac{dH^\nu(n, t)}{dt} = -(\gamma + \beta)H^\nu(n, t) + \beta(H^\nu(n+1, t) - G^\nu(n, t)),$$

with solution

$$H^\nu(n, t) = e^{-(\gamma+\beta)t} \sum_{\ell \geq 0} \frac{(\beta t)^\ell}{\ell!} H(\ell + n, 0) - \beta \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{\ell \geq 0} \frac{(\beta(t-s))^\ell}{\ell!} G(\ell + n, s) ds.$$

Therefore, we can write the solution J^ν in terms of the H^ν cluster as (4.23). □

For the proofs of Theorems 4 and 5, we introduce two auxiliary dynamics. The first dynamics is defined on $\mathbb{Z} \times \mathbb{N} \times \{G\}$ by the following transition rates

$$(r, n, G) \rightarrow (r-1, n+1, G) : \text{ at rate } \beta, \quad \text{and} \quad (r, n, G) \rightarrow (r, n+1, G) : \text{ at rate } \beta. \quad (4.38)$$

Denote by $\mathbb{P}_{(r,n,G)}^G$, resp. $\mathbb{E}_{(r,n,G)}^{\text{Dual},G}$ the probability measure, resp. expectation, under this dynamics, when starting from (r, n, G) .

The second dynamics is defined on $\mathbb{Z} \times \mathbb{N} \times \{J\}$ by the following transition rates

$$(r, n, J) \rightarrow (r, n+1) : \text{ at rate } \beta. \quad (4.39)$$

Denote by $\mathbb{P}_{(r,n,J)}^J$, resp. $\mathbb{E}_{(r,n,J)}^J$ the probability measure, resp. expectation, under this dynamics when starting from (r, n, J) .

Proof of Theorem 4. Fix $(r, n) \in \mathbb{Z}^2$, thanks to duality we can write

$$G^\eta(r, n, t) = e^{-2\gamma t} \mathbb{E}_{(r,n,G)}^G [d(\eta, (r(t), n(t), G))] . \quad (4.40)$$

Indeed, for the left hand side we have

$$\begin{aligned} \mathbb{E}_\eta [d(\eta(t), (r, n, G))] &= \mathbb{E}_{(r,n,G)}^{\text{Dual}} [d(\eta, \xi(t))] \\ &= \mathbb{E}_{(r,n,G)}^{\text{Dual}} [d(\eta, \xi(t)) \cap (\xi(t) \neq \partial)] \\ &= \mathbb{E}_{(r,n,G)}^{\text{Dual}} [d(\eta, \xi(t)) \cap (\forall 0 \leq s \leq t, \xi_s \neq \partial)] \\ &= \mathbb{E}_{(r,n,G)}^{\text{Dual}} [\mathbb{1}_{\{\forall 0 \leq s \leq t, \xi_s \neq \partial\}} d(\eta, \xi(t))] \\ &= \mathbb{P}_{(r,n,G)} [\forall 0 \leq s \leq t, \xi_s \neq \partial] \mathbb{E}_{(r,n,G)}^{\text{Dual}} [d(\eta, \xi(t)) | \forall 0 \leq s \leq t, \xi_s \neq \partial] \\ &= \mathbb{P}_{(r,n,G)} [\forall 0 \leq s \leq t, \xi_s \neq \partial] \times \mathbb{E}_{(r,n,G)}^G [d(\eta, (r(t), n(t), G))] . \end{aligned}$$

Above we used the strong Markov property on the fifth line and the sixth one is due to the fact that when conditioning on not having reached the trap ∂ up to time t , the dual process starting from (r, n, G) has the same dynamics as the one defined in (4.38). Finally, to get the identity in (4.40) we used the fact that the time of reaching the trap ∂ from any (r, n, ∂) , is given by an exponential clock of parameter 2γ . Expanding the expectation in the right hand side of (4.40), we are left with

$$\begin{aligned} G^\eta(r, n, t) &= e^{-2\gamma t} \sum_{(\tilde{r}, \tilde{n}) \in \mathbb{Z} \times \mathbb{N}} d(\eta, (\tilde{r}, \tilde{n})) \mathbb{P}_{(r,n,G)}^G [(r(t), n(t), G) = (\tilde{r}, \tilde{n}, G)] \\ &= e^{-2\gamma t} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b}} d(\eta, (r-a, n+b)) \mathbb{P}_{(r,n,G)}^G [(r(t), n(t), G) = (r-a, n+b, G)] \\ &= e^{-2\gamma t} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b}} d(\eta, (r-a, n+b)) e^{-2\beta t} \frac{\binom{b}{a}}{2^b} \frac{(2\beta t)^b}{b!} \\ &= e^{-2(\gamma+\beta)t} \sum_{\substack{(a,b) \in \mathbb{N}^2 \\ a \leq b}} \frac{\binom{b}{a}}{2^b} \frac{(2\beta t)^b}{b!} G^\eta(r-a, n+b, 0). \end{aligned}$$

The third line comes from the fact that in order to reach $(r-a, n+b)$ starting from (r, n) and with the dynamics (4.38), one has to perform $\binom{b}{a}$ steps, where r decreases by 1 and n increases by 1 and, the rest of the steps where it is only n that increases by 1. The time between two such jumps is an exponential clock with parameter 2β . \square

The other cluster can be found in a similar fashion.

Proof of Theorem 5. Similarly to the previous computation we have

$$\begin{aligned}
J^n(r, n, t) &= \mathbb{E}_\eta[d(\eta_t, (r, n, J))] = \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, \xi(t))] = \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, \xi(t)) \cap (\xi(t) \neq \partial)] \\
&= \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), i(t))) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)] \\
&= \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), G)) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)] \\
&\quad + \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), J)) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)],
\end{aligned} \tag{4.41}$$

where in the last line, we partitioned according to having or not a jump from J to G before time t . Defining:

$$\mathcal{A}_s^t := (\forall 0 \leq u \leq s^-, \xi_3(u) = J) \cap (\forall s^+ \leq u \leq t, \xi_3(u) = G) \cap (\forall 0 \leq u \leq t, \xi(u) \neq \partial),$$

the event that before time t , the walk does not reach the trap ∂ and at time $0 \leq s \leq t$, the walk jumps from lane J to G , we have:

$$\begin{aligned}
\mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), G)) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)] &= \int_0^t \mathbb{E}_{(r, n, J)}^{\text{Dual}}[\mathcal{A}_s^t] ds \\
&= \int_0^t d\mathbb{P}_{(r, n, J)}^{\text{Dual}}[\mathcal{A}_s^t] \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), i(t))) | \mathcal{A}_s^t].
\end{aligned} \tag{4.42}$$

Using that the law of the jump from state J to G is an exponential law of parameter γ , and that the rate of jump from state G to the trap ∂ is 2γ , we have

$$d\mathbb{P}_{(r, n, J)}^{\text{Dual}}[\mathcal{A}_s^t] = \gamma e^{-\gamma s} (1 - \int_0^{t-s} 2\gamma e^{-2\gamma u} du) ds = \gamma e^{-\gamma s} e^{-2\gamma(t-s)} ds = \gamma e^{-2\gamma t} e^{\gamma s} ds.$$

Now, we expand (4.42) by partitioning on the number of increases in n up to time s for the dual process conditioned on being of type J between times 0 and s , for a certain $s \in [0, t]$:

$$\begin{aligned}
&\mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), i(t))) | \mathcal{A}_s^t] \\
&= \sum_{k \geq 0} \mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), i(t))) \cap (r(s), n(s), i(s)) = (r, n+k, J) | \mathcal{A}_s^t] \\
&= \sum_{k \geq 0} \mathbb{P}_{(r, n, J)}^J[(r(s), n(s), J) = (r, n+k, J)] \mathbb{E}_{(r, n+k, G)}^G[d(\eta, (r(t-s), n(t-s), G))] \\
&= \sum_{k \geq 0} e^{-\beta s} \frac{(\beta s)^k}{k!} \sum_{\substack{a \leq b \\ (a, b) \in \mathbb{N}^2}} d(\eta, (r-a, n+k+b, G)) \\
&\quad \cdot \mathbb{P}_{(r, n+k, G)}^G[(r(t-s), n(t-s), G) = (r-a, r+k+b, G)] \\
&= \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a, b) \in \mathbb{N}^2}} e^{-\beta s} \frac{(\beta s)^k}{k!} e^{-2\beta(t-s)} \frac{\binom{b}{a}}{2^b} \frac{(2\beta(t-s))^b}{b!} G^\eta(r-a, n+k+b, 0),
\end{aligned}$$

where we used the strong Markov property in the second line, and expanded the expectation under the process (4.38) in the third line. Therefore,

$$\begin{aligned}
&\mathbb{E}_{(r, n, J)}^{\text{Dual}}[d(\eta, (r(t), n(t), G)) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)] = \\
&\gamma e^{-2\gamma t} \int_0^t e^{\gamma s} \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a, b) \in \mathbb{N}^2}} e^{-\beta s} \frac{(\beta s)^k}{k!} e^{-2\beta(t-s)} \frac{\binom{b}{a}}{2^b} \frac{(2\beta(t-s))^b}{b!} G^\eta(r-a, n+k+b, 0) ds.
\end{aligned} \tag{4.43}$$

Finally, the second expectation in the last line in (4.41) is given by:

$$\mathbb{E}_{(r,n,J)}^{\text{Dual}} [d(\eta, (r(t), n(t), J)) \cap (\forall 0 \leq s \leq t, \xi(s) \neq \partial)] = e^{-\gamma t} \sum_{k \geq 0} \frac{e^{-\beta t} (\beta t)^k}{k!} J^\eta(r, n+k, 0). \quad (4.44)$$

Collecting (4.43) and (4.44) yields:

$$\begin{aligned} J^\eta(r, n, t) &= e^{-(\gamma+\beta)t} \sum_{k \geq 0} \frac{(\beta t)^k}{k!} J^\eta(r, n+k, 0) \\ &+ \gamma e^{-2\gamma t} \int_0^t e^{\gamma s} \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a,b) \in \mathbb{Z}^2}} e^{-\beta s} \frac{(\beta s)^k}{k!} e^{-2\beta(t-s)} \frac{\binom{b}{a}}{2^b} \frac{(2\beta(t-s))^b}{b!} G^\eta(r-a, n+k+b, 0) ds \\ &= e^{-(\gamma+\beta)t} \sum_{k \geq 0} \frac{(\beta t)^k}{k!} J^\eta(r, n+k, 0) \\ &+ \gamma e^{-(\gamma+\beta)t} \int_0^t e^{-(\gamma+\beta)(t-s)} \sum_{k \geq 0} \sum_{\substack{a \leq b \\ (a,b) \in \mathbb{Z}^2}} \frac{(\beta s)^k}{k!} \frac{\binom{b}{a}}{2^b} \frac{(2\beta(t-s))^b}{b!} G^\eta(r-a, n+k+b, 0) ds. \end{aligned}$$

□

A The diffusive contact process on small finite sets

A.1 Invariant measure of the DCP on small finite sets

Here, in the same spirit as [25], we compute the stationary distribution of the diffusive contact process for small N , where we recall that the dynamics is given by the generator \mathcal{L}^{DCP} , defined in (2.11), here we stress the dependence on the lattice size with the subscript N . This approach is based on the definition of the invariant measure as the probability distribution ν^{DCP} satisfying $\nu^{\text{DCP}} \mathcal{L}^{\text{DCP}} = 0$ cf. [30, Chapter 1, Proposition 1.8]). Here, configurations are represented in their vectorial form, for instance, for $N = 1$, (0) is the empty configuration and (1) the full one.

We notice that in our case the presence of boundary reservoirs allows to see the effect of the diffusive coefficient \mathcal{D} already when the bulk size is $N = 2$. This is in contrast to the scenario studied in Section 6.2 of [25] with boundary conditions $\eta_0 = \eta_{N+1} = 1$ when the results are equivalent to the case of $\mathcal{D} = 0$.

$N = 1$ site

Let us solve the stationary condition $\nu^{\text{DCP}} \mathcal{L}^{\text{DCP}} = 0$. For $N = 1$, the corresponding intensity matrix is

$$L^{\text{DCP}} = \begin{pmatrix} -(\alpha + \delta) & (\alpha + \delta) \\ (\gamma + \beta + 1) & -(\gamma + \beta + 1) \end{pmatrix}$$

and the stationary condition yields

$$\begin{cases} \nu^{\text{DCP}}((0)) &= \frac{\gamma + \beta + 1}{1 + \beta + \gamma + \delta + \alpha} \\ \nu^{\text{DCP}}((1)) &= \frac{\alpha + \delta}{1 + \beta + \gamma + \delta + \alpha}. \end{cases}$$

Notice that for just one bulk site there is no diffusive effect.

$N = 2$ sites

This is the first non trivial case where the diffusion parameter \mathcal{D} plays a role. In this case the intensity matrix is

$$L^{\text{DCP}} = \begin{pmatrix} -(\alpha + \delta) & \delta & \alpha & 0 \\ 1 + \beta & -(1 + \beta + \mathcal{D} + \lambda + \alpha) & \mathcal{D} & \lambda + \alpha \\ 1 + \gamma & \mathcal{D} & -(1 + \gamma + \mathcal{D} + \lambda + \delta) & \lambda + \delta \\ 0 & 1 + \gamma & 1 + \beta & -(2 + \beta + \gamma) \end{pmatrix}$$

and the stationary condition yields

$$\begin{cases} c(\mathcal{D})\nu^{\text{DCP}}((0, 0)) &= \mathcal{D}(\beta + \gamma + 2)^2 + (\alpha + \beta + \gamma + \delta + 2 + 2\lambda)(1 + \beta + \gamma + \beta\gamma) \\ c(\mathcal{D})\nu^{\text{DCP}}((1, 0)) &= \mathcal{D}(2 + \beta + \gamma)(\alpha + \delta) + (\gamma + 1)[\lambda(\alpha + \delta) + \delta(\alpha + \beta + \gamma + \delta + 2)] \\ c(\mathcal{D})\nu^{\text{DCP}}((0, 1)) &= \mathcal{D}(2 + \beta + \gamma)(\alpha + \delta) + (\beta + 1)[\lambda(\alpha + \delta) + \alpha(\alpha + \beta + \gamma + \delta + 2)] \\ c(\mathcal{D})\nu^{\text{DCP}}((1, 1)) &= \mathcal{D}(\alpha + \delta)(\alpha + \delta + 2\lambda) + \alpha\delta(2 + \alpha + \beta + \gamma + \delta) \\ &\quad + \lambda(\alpha + \delta)(\alpha + \delta + \lambda + 1) + \lambda(\alpha\beta + \gamma\delta), \end{cases} \quad (\text{A.1})$$

where

$$\begin{aligned} c(\mathcal{D}) &= \mathcal{D} [(\alpha + \beta + \gamma + \delta + 2)^2 + 2\lambda(\alpha + \delta)] \\ &\quad + [\alpha + \beta + \gamma + \delta + 1 + (\alpha + \gamma)(\beta + \delta)](\alpha + \beta + \gamma + \delta + 2) \\ &\quad + \lambda(\alpha + \delta)(\alpha + \delta + \lambda) + \lambda[(\alpha + \delta + 1)(\beta + \gamma + 2) + (\alpha + \gamma)(\beta + 1) + (\beta + \delta)(\gamma + 1)] \end{aligned} \quad (\text{A.2})$$

In both cases, setting $\alpha = \delta = \lambda$ and $\gamma = \beta = 0$, our results correspond to the ones in [25], as expected. This strategy explicitly characterizes the stationary measure but only works when the size of the system is small.

From (A.1), recalling the notation $\rho_1^{\text{DCP}}(1) = \mathbb{E}_{\nu^{\text{DCP}}}[\eta_1] =: x$, $\rho_1^{\text{DCP}}(2) = \mathbb{E}_{\nu^{\text{DCP}}}[\eta_2] =: y$ and $\rho_2^{\text{DCP}}(1, 2) = \mathbb{E}_{\nu^{\text{DCP}}}[\eta_1\eta_2] =: z$, we deduce the following:

$$\begin{cases} c(\mathcal{D})x &= c(\mathcal{D})\nu^{\text{DCP}}((1, 0)) + c(\mathcal{D})\nu^{\text{DCP}}((1, 1)) \\ &= \mathcal{D}(\alpha + \delta)(2 + \beta + \gamma + \alpha + \delta + 2\lambda) + (\alpha + \delta)(\lambda + 2 + \alpha + \delta + \gamma)\lambda \\ &\quad + (\alpha\beta + \delta\gamma)\lambda + (\beta + \gamma + \alpha + \delta + 2)\delta(\gamma + \alpha + 1) \\ c(\mathcal{D})y &= c(\mathcal{D})\nu^{\text{DCP}}((0, 1)) + c(\mathcal{D})\nu^{\text{DCP}}((1, 1)) \\ &= \mathcal{D}(\alpha + \delta)(2 + \beta + \gamma + \alpha + \delta + 2\lambda) + (\alpha + \delta)(\lambda + 2 + \alpha + \delta + \beta)\lambda \\ &\quad + (\alpha\beta + \delta\gamma)\lambda + (\beta + 1 + \delta)\alpha(2 + \delta + \gamma + \beta + \alpha) \\ c(\mathcal{D})z &= c(\mathcal{D})\nu^{\text{DCP}}((1, 1)) \\ &= \mathcal{D}(\alpha + \delta)(\alpha + \delta + 2\lambda) + \alpha\delta(2 + \alpha + \beta + \gamma + \delta) \\ &\quad + \lambda(\alpha + \delta)(\alpha + \delta + \lambda + 1) + \lambda(\alpha\beta + \gamma\delta). \end{cases} \quad (\text{A.3})$$

Remark 8 (Fokker-Planck approach). *We could have derived directly x, y, z by solving the equation $\int \mathcal{L}^{\text{DCP}} f d\nu^{\text{DCP}} = 0$ for $f(\eta) = \eta_1$, $f(\eta) = \eta_2$, $f(\eta) = \eta_1\eta_2$.*

Then, taking the limit $\mathcal{D} \rightarrow \infty$, the solutions in (A.1) become

$$\begin{cases} c(\infty)\nu^{\text{DCP}}((0, 0))(\infty) &= (\beta + \gamma + 2)^2 \\ c(\infty)\nu^{\text{DCP}}((1, 0))(\infty) &= (2 + \beta + \gamma)(\alpha + \delta) \\ c(\infty)\nu^{\text{DCP}}((0, 1))(\infty) &= (2 + \beta + \gamma)(\alpha + \delta) \\ c(\infty)\nu^{\text{DCP}}((1, 1))(\infty) &= (\alpha + \delta)(\alpha + \delta + 2\lambda) \end{cases} \quad (\text{A.4})$$

where

$$c(\infty) = (\alpha + \beta + \gamma + \delta + 2)^2 + 2\lambda(\alpha + \delta) \quad (\text{A.5})$$

Due to the infinite diffusion, $\nu^{\text{DCP}}((1, 0))(\infty) = \nu^{\text{DCP}}((0, 1))(\infty)$, so this quantity only depends on the total number of particles, regardless of their position.

Finally, taking the limit $\mathcal{D} \rightarrow \infty$, the solutions in (A.3) simplifies to

$$\begin{cases} c(\infty)x(\infty) = (\alpha + \delta)(2 + \beta + \gamma + \alpha + \delta + 2\lambda) \\ c(\infty)y(\infty) = (\alpha + \delta)(2 + \beta + \gamma + \alpha + \delta + 2\lambda) \\ c(\infty)z(\infty) = (\alpha + \delta)(\alpha + \delta + 2\lambda). \end{cases} \quad (\text{A.6})$$

Notice that, as remarked at the end of Appendix A.2 for the absorption probabilities of one dual particle starting in site 1 or in site 2, also the corresponding correlations $\rho_1^{\text{DCP}}(1)$ and $\rho_1^{\text{DCP}}(2)$ are the same.

A.2 Absorption probabilities

In this subsection, related to Section 2.3, we explicitly compute the absorption probabilities of the dual process given in Proposition 2 for $N = 1$, which is simpler because there is neither diffusion nor birth effects, then for $N = 2$. Recall that for the sake of simplicity, we took $\beta = \delta = 0$ in Section 2.3, and we will also do this in what follows.

$N = 1$ site

We perform a conditioning on the first step of the dual dynamics. The behavior of the bulk is conservative as the contact dynamics has no effect here (no possible birth). This provides a finite sum in the expression of the correlation function (2.40) obtained through duality, allowing us to compute it conveniently. By (2.32), we have

$$\mathbb{E}_{\nu^{\text{DCP}}}[1 - \eta_1] = 1 - \rho_1^{\text{DCP}}(1) = \mathbb{P}_{\delta_1}(\xi_0(\infty) = 0)c_-^0 + \mathbb{P}_{\delta_1}(\xi_0(\infty) = 1)c_-.$$

To compute the absorption probabilities $\mathbb{P}_{\delta_1}(\xi_0(\infty) = 0)$ and $\mathbb{P}_{\delta_1}(\xi_0(\infty) = 1)$, we perform a conditioning on the first possible outcome of the process, starting from a particle at site 1:

$$\begin{aligned} \mathbb{P}_{\delta_1}(\xi_0(\infty) = 0) &= \mathbb{P}_{\delta_1}(\xi_0(\infty) = 0 \mid \text{left jump})\mathbb{P}_{\delta_1}(\text{left jump}) \\ &\quad + \mathbb{P}_{\delta_1}(\xi_0(\infty) = 0 \mid \text{it dies})\mathbb{P}_{\delta_1}(\text{it dies}) \\ &= 0 \cdot \frac{\alpha + \gamma}{\alpha + \gamma + 1} + 1 \cdot \frac{1}{\alpha + \gamma + 1} \\ &= \frac{1}{\alpha + \gamma + 1} \end{aligned}$$

while if the particle is absorbed,

$$\mathbb{P}_{\delta_1}(\xi_0(\infty) = 1) = 1 - \mathbb{P}_{\delta_1}(\xi_0(\infty) = 0) = \frac{\alpha + \gamma}{\alpha + \gamma + 1}.$$

Therefore,

$$\rho_1^{\text{DCP}}(1) = \frac{\alpha}{\alpha + \gamma + 1},$$

$N = 2$ sites

We now explicitly compute the absorption probabilities of the dual process when $N = 2$, as well as their fast stirring limits. Notice that, in computing correlations, this is the first non trivial case. In addition to the dynamics for one bulk site, the effect of the exclusion process (with rate \mathcal{D}) as well as the effect of the contact process (with rate λ) are present, therefore we have an infinite sum in equation (2.40), as expressed below in (A.7). Recall the expressions of the correlation functions (2.28) provided by the duality relations (2.32), and (2.40):

$$\begin{cases} \rho_1^{\text{DCP}}(1) &= 1 - \sum_{k \geq 0} \mathbb{P}_{\delta_1}(\xi_0(\infty) = k) c_-^k \\ \rho_2^{\text{DCP}}(1, 2) &= 1 - \sum_{k \geq 0} \mathbb{P}_{\delta_{1,2}}(\xi_0(\infty) = k) c_-^k \\ \rho_1^{\text{DCP}}(2) &= 1 - \sum_{k \geq 0} \mathbb{P}_{\delta_2}(\xi_0(\infty) = k) c_-^k . \end{cases} \quad (\text{A.7})$$

To compute the absorption probabilities appearing in the infinite sums, as in the case $N = 1$, we condition on the first possible outcome of the process, starting from a particle at site 1, 2 or at both sites. For $k \geq 0$, we define

$$x_1^k := \mathbb{P}_{\delta_1}[\xi_0(\infty) = k], \quad x_2^k := \mathbb{P}_{\delta_{1,2}}[\xi_0(\infty) = k], \quad \text{and} \quad x_3^k := \mathbb{P}_{\delta_2}[\xi_0(\infty) = k],$$

the probabilities that $k \geq 0$ dual particles are absorbed in the left reservoir starting from a particle at site 1 (x_1^k), at site 2 (x_3^k) or at both sites (x_2^k). To alleviate notation we do not write the dependence on the model parameters α, γ, λ and \mathcal{D} on these absorption probabilities. For $k \geq 0$, define the column vector

$$C_k = \begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \end{pmatrix}. \quad (\text{A.8})$$

- For $k = 0$, the conditioning on the first step leads to the following system:

$$\begin{cases} x_1^0 - \frac{\lambda}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_2^0 - \frac{\mathcal{D}}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_3^0 = \frac{1}{\alpha + \gamma + \mathcal{D} + \lambda + 1} \\ \frac{-1}{\alpha + \gamma + 2} x_1^0 + x_2^0 - \frac{1}{\alpha + \gamma + 2} x_3^0 = 0 \\ \frac{-\mathcal{D}}{\mathcal{D} + \lambda + 1} x_1^0 - \frac{\lambda}{\mathcal{D} + \lambda + 1} x_2^0 + x_3^0 = \frac{1}{\mathcal{D} + \lambda + 1}, \end{cases} \quad (\text{A.9})$$

that is,

$$M_2 C_0 = \begin{pmatrix} \frac{1}{\alpha + \gamma + \mathcal{D} + \lambda + 1} \\ 0 \\ \frac{1}{\mathcal{D} + \lambda + 1} \end{pmatrix}, \quad (\text{A.10})$$

where

$$M_2 := \begin{pmatrix} 1 & \frac{-\lambda}{\alpha + \gamma + \mathcal{D} + \lambda + 1} & \frac{-\mathcal{D}}{\alpha + \gamma + \mathcal{D} + \lambda + 1} \\ \frac{-1}{\alpha + \gamma + 2} & 1 & \frac{-1}{\alpha + \gamma + 2} \\ \frac{-\mathcal{D}}{\mathcal{D} + \lambda + 1} & \frac{-\lambda}{\mathcal{D} + \lambda + 1} & 1 \end{pmatrix}. \quad (\text{A.11})$$

We have

$$d := \det M_2 = \frac{\tilde{d}}{(\mathcal{D} + \lambda + 1)(\alpha + \gamma + 2)(\mathcal{D} + \alpha + \gamma + \lambda + 1)}, \quad (\text{A.12})$$

where

$$\tilde{d} := \mathcal{D}A + (\alpha + \gamma)[(\alpha + \gamma + \lambda + 2)(\lambda + 1) + 1] + 2(\lambda + 1), \quad (\text{A.13})$$

with

$$A := (\alpha + \gamma + \lambda + 2)^2 - \lambda(\lambda + 4) = (\alpha + \gamma + 2)^2 + 2\lambda(\alpha + \gamma) \quad (\text{A.14})$$

So $\det M_2 \neq 0$, and solving (A.10) by inverting M_2 yields

$$\begin{cases} x_1^0 = \tilde{d}^{-1}(2\mathcal{D} + \lambda + 1)(\alpha + \gamma + 2) \\ x_2^0 = \tilde{d}^{-1}[4\mathcal{D} + \alpha + \gamma + 2\lambda + 2] \\ x_3^0 = \tilde{d}^{-1}(2\mathcal{D} + \alpha + \gamma + \lambda + 1)(\alpha + \gamma + 2). \end{cases} \quad (\text{A.15})$$

- For $k = 1$, the same conditioning leads to

$$\begin{cases} x_1^1 - \frac{\lambda}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_2^1 - \frac{\mathcal{D}}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_3^1 = \frac{\alpha + \gamma}{\alpha + \gamma + \mathcal{D} + \lambda + 1} \\ \frac{-1}{\alpha + \gamma + 2} x_1^1 + x_2^1 - \frac{1}{\alpha + \gamma + 2} x_3^1 = \frac{\alpha + \gamma}{\alpha + \gamma + 2} x_3^0 \\ \frac{-\mathcal{D}}{\mathcal{D} + \lambda + 1} x_1^1 - \frac{\lambda}{\mathcal{D} + \lambda + 1} x_2^1 + x_3^1 = 0, \end{cases} \quad (\text{A.16})$$

that is,

$$M_2 C_1 = \begin{pmatrix} \frac{\alpha + \gamma}{\alpha + \gamma + \mathcal{D} + \lambda + 1} \\ \frac{\alpha + \gamma}{\alpha + \gamma + 2} x_3^0 \\ 0 \end{pmatrix} \quad (\text{A.17})$$

Again, as $\det M_2 \neq 0$, we can invert this and, to lighten the formulas of the solution, we introduce the notation

$$\varphi := \mathcal{D}B + \lambda^2 + \lambda(\alpha + \gamma + 2) + \alpha + \gamma + 1, \quad (\text{A.18})$$

where

$$B := (\alpha + \gamma + 2\lambda + 2). \quad (\text{A.19})$$

We get:

$$\begin{cases} x_1^1 = \tilde{d}^{-1}(\alpha + \gamma)(\alpha + \gamma + 2) \left[\mathcal{D} + 1 + \frac{\lambda(\alpha + \gamma + 1)}{\alpha + \gamma + 2} \right. \\ \quad \left. + \lambda \tilde{d}^{-1}(2\mathcal{D} + \lambda + 1)(2\mathcal{D} + \alpha + \gamma + \lambda + 1) \right] \\ x_2^1 = \tilde{d}^{-1}(\alpha + \gamma)(\alpha + \gamma + 2) \left[\frac{2\mathcal{D} + \lambda + 1}{\alpha + \gamma + 2} + \tilde{d}^{-1}\varphi(2\mathcal{D} + \alpha + \gamma + 1 + \lambda) \right] \\ x_3^1 = \tilde{d}^{-1}(\alpha + \gamma)(\alpha + \gamma + 2) \left[\mathcal{D} + \frac{\lambda}{\alpha + \gamma + 2} + \tilde{d}^{-1}\lambda(2\mathcal{D} + \lambda + \alpha + \gamma + 1)^2 \right]. \end{cases} \quad (\text{A.20})$$

- For $k = 2$, the same conditioning leads to

$$\begin{cases} x_1^k - \frac{\lambda}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_2^k - \frac{\mathcal{D}}{\alpha + \gamma + \mathcal{D} + \lambda + 1} x_3^k = 0 \\ \frac{-1}{\alpha + \gamma + 2} x_1^k + x_2^k - \frac{1}{\alpha + \gamma + 2} x_3^k = \frac{\alpha + \gamma}{\alpha + \gamma + 2} x_3^{k-1} \\ \frac{-\mathcal{D}}{\mathcal{D} + \lambda + 1} x_1^k - \frac{\lambda}{\mathcal{D} + \lambda + 1} x_2^k + x_3^k = 0 \end{cases} \quad (\text{A.21})$$

that is,

$$M_2 C_k = \begin{pmatrix} 0 \\ \frac{\alpha+\gamma}{\alpha+\gamma+2} x_3^{k-1} \\ 0 \end{pmatrix} \quad (\text{A.22})$$

Again, inverting M_2 , using the definition of φ given in (A.18), and introducing

$$\psi := \mathcal{D}(\alpha + \gamma + 2) + \lambda + \tilde{d}^{-1} \lambda (\alpha + \gamma + 2) (2\mathcal{D} + \alpha + \gamma + \lambda + 1)^2 \quad (\text{A.23})$$

we get:

$$\begin{cases} x_1^2 = \tilde{d}^{-2} \lambda (\alpha + \gamma)^2 (2\mathcal{D} + \lambda + 1) \psi \\ x_2^2 = \tilde{d}^{-2} (\alpha + \gamma)^2 \varphi \psi \\ x_3^2 = \tilde{d}^{-2} \lambda (\alpha + \gamma)^2 (2\mathcal{D} + \alpha + \gamma + \lambda + 1) \psi. \end{cases} \quad (\text{A.24})$$

- For $k = 3$, the same reasoning as for the cases above yields

$$\begin{cases} x_1^3 = \tilde{d}^{-3} \lambda^2 (\alpha + \gamma)^3 (2\mathcal{D} + \lambda + 1) (2\mathcal{D} + \alpha + \gamma + \lambda + 1) \psi \\ x_2^3 = \tilde{d}^{-3} \lambda (\alpha + \gamma)^3 (2\mathcal{D} + \alpha + \gamma + \lambda + 1) \varphi \psi \\ x_3^3 = \tilde{d}^{-3} \lambda^2 (\alpha + \gamma)^3 (2\mathcal{D} + \alpha + \gamma + \lambda + 1)^2 \psi. \end{cases} \quad (\text{A.25})$$

- For $k > 3$, using the same conditioning and reasoning recursively, we find that

$$\begin{cases} x_1^k = \tilde{d}^{-k} \lambda^{(k-2)} (\alpha + \gamma)^k (2\mathcal{D} + \lambda + 1) (2\mathcal{D} + \alpha + \gamma + \lambda + 1)^{(k-3)} \varphi \psi \\ x_2^k = \tilde{d}^{-k} \lambda^{(k-3)} (\alpha + \gamma)^k (2\mathcal{D} + \alpha + \gamma + \lambda + 1)^{(k-3)} \varphi^2 \psi \\ x_3^k = \tilde{d}^{-k} \lambda^{(k-2)} (\alpha + \gamma)^k (2\mathcal{D} + \alpha + \gamma + \lambda + 1)^{(k-2)} \varphi \psi. \end{cases} \quad (\text{A.26})$$

We now consider the fast stirring limit of the absorption probabilities just found. For all $k \geq 0$ and $i = 1, 2, 3$ we denote

$$x_i^k(\infty) := \lim_{\mathcal{D} \rightarrow \infty} x_i^k.$$

The solutions obtained in (A.15), (A.20), (A.24), (A.25), (A.26) simplify (recall the definitions (A.14) of A and (A.19) of B) to

- For $k = 0$,

$$\begin{cases} x_1^0(\infty) = x_3^0(\infty) = 2A^{-1}(\alpha + \gamma + 2) \\ x_2^0(\infty) = 4A^{-1} \end{cases} \quad (\text{A.27})$$

- For $k = 1$, we get:

$$\begin{cases} x_1^1(\infty) = x_3^1(\infty) = A^{-1}(\alpha + \gamma)(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_2^1(\infty) = A^{-1}2(\alpha + \gamma) [1 + A^{-1}B(\alpha + \gamma + 2)] \end{cases} \quad (\text{A.28})$$

- For $k = 2$, we get:

$$\begin{cases} x_1^2(\infty) = x_3^2(\infty) = 2A^{-2}\lambda(\alpha + \gamma)^2(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_2^2(\infty) = B(2\lambda)^{-1}x_1^2(\infty) \end{cases} \quad (\text{A.29})$$

- For $k = 3$, the same reasoning as for the cases above yields

$$\begin{cases} x_1^3(\infty) = x_3^3(\infty) = 4A^{-3}\lambda^2(\alpha + \gamma)^3(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_2^3(\infty) = B(2\lambda)^{-1}x_1^3(\infty) \end{cases} \quad (\text{A.30})$$

- For $k > 3$, we find that

$$\begin{cases} x_1^k(\infty) = x_3^k(\infty) = A^{-k}(2\lambda)^{k-2}B(\alpha + \gamma)^k(\alpha + \gamma + 2) [1 + 4A^{-1}\lambda] \\ x_2^k(\infty) = B(2\lambda)^{-1}x_1^k(\infty) \end{cases} \quad (\text{A.31})$$

These formulas induce simple expressions for the values (A.7) when $\mathcal{D} \rightarrow \infty$, where in particular for all $k \geq 0$, $x_1^k(\infty) = x_3^k(\infty)$, that is, the absorption probability to the left reservoirs of one particle at site 1 is the same as the one of a particle at site 2.

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