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# TIME SPLITTING AND ERROR ESTIMATES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH A POTENTIAL

RÉMI CARLES

ABSTRACT. We consider the nonlinear Schrödinger equation with a potential, also known as Gross-Pitaevskii equation. By introducing a suitable spectral localization, we prove low regularity error estimates for the time discretization corresponding to an adapted Lie-Trotter splitting scheme. The proof is based on tools from spectral theory and pseudodifferential calculus in order to obtain various estimates on the spectral localization, including discrete Strichartz estimates which support the nonlinear analysis.

## 1. INTRODUCTION

1.1. **Setting.** We consider time discretization for the nonlinear Schrödinger equation with a potential,

$$(1.1) \quad i\partial_t u + \Delta u = V(x)u + W(x)u + \varepsilon|u|^{2\sigma}u \quad ; \quad u|_{t=0} = u_0,$$

with  $x \in \mathbb{R}^d$ ,  $d \geq 1$ ,  $\varepsilon \in \{-1, +1\}$ , and  $\sigma > 0$ . More precisely, the nonlinearity is  $H^1$ -subcritical,  $0 < \sigma < \frac{2}{(d-2)_+}$ , that is,  $\sigma < \frac{2}{d-2}$  if  $d \geq 3$ . We denote

$$H = -\Delta + V(x).$$

A typical physically relevant case in dimension  $d \leq 3$ , related to the physics of superfluids or Bose-Einstein Condensation, is the cubic nonlinearity ( $\sigma = 1$ ), with a quadratic confining potential (possibly anisotropic),

$$(1.2) \quad V_{\text{quad}}(x) = \sum_{j=1}^d \omega_j^2 x_j^2, \quad \omega_j > 0.$$

Equation (1.1) in that case may be referred to as Gross-Pitaevskii equation; see e.g. [3, 26] and references therein. The potential  $W$  may be considered as a perturbation of  $V$ . A major difference between the two potentials, see Assumption 1.1, is that  $V$  is smooth, possibly unbounded, while  $W$  is in  $W^{2,\infty}(\mathbb{R}^d)$ . We emphasize that we make no assumption on the spectrum of the Hamiltonian  $H$ : for instance,  $V$  may correspond to a partial confinement, that is we may consider (1.2) in the case were some (or all)  $\omega_j$ 's are zero. The potential in (1.2) is not bounded, and cannot be addressed by perturbative arguments, typically when solving the Cauchy problem (1.1) and considering the associated dynamics. In this paper, and in agreement with the above example, we assume that  $V$  is bounded from below,

$$\exists C_0 > 0, \quad V(x) \geq -C_0, \quad \forall x \in \mathbb{R}^d.$$

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The gauge transform  $u \mapsto e^{-it(C_0+1)}u = \tilde{u}$  shows that  $\tilde{u}$  solves (1.1) with  $V$  replaced by  $V + C_0 + 1$ , so we may assume that  $V \geq 1$  without loss of generality. This will simplify some technical arguments, without discarding potentials such as in (1.2):

**Assumption 1.1.** *The potential  $V$  is real-valued, smooth,  $V \in C^\infty(\mathbb{R}^d; \mathbb{R})$ , with  $V \geq 1$ . Moreover, it is at most quadratic, in the sense that*

$$\forall \alpha \in \mathbb{N}^d, |\alpha| \geq 2, \quad \partial^\alpha V \in L^\infty(\mathbb{R}^d).$$

*The potential  $W$  is real-valued,  $W \in W^{2,\infty}(\mathbb{R}^d; \mathbb{R})$ , that is*

$$\forall \alpha \in \mathbb{N}^d, |\alpha| \leq 2, \quad \partial^\alpha W \in L^\infty(\mathbb{R}^d).$$

The operator  $H$  is self-adjoint, and we denote by

$$(1.3) \quad S(t) = e^{-itH}$$

the evolution group associated with the linear part in (1.1) ( $\varepsilon = 0$ ), in the case  $W = 0$ , which is well defined under Assumption 1.1 (see e.g. [32]).

**1.2. Time discretization.** This paper is focused on error estimates for the Lie-Trotter time splitting scheme associated to (1.1), see e.g. [2, 3] and references therein. As in (1.1),  $\partial_t u$  is the sum of four terms, several strategies appear naturally if one wants to write

$$\partial_t u = A_1 u + A_2 u,$$

where the equation  $\partial_t u = A_j u$  is convenient to solve numerically both for  $j = 1$  and  $j = 2$ . We do not consider more sophisticated splitting methods here, to avoid extra technicalities. One may set  $A_1 = i\Delta$ , and

$$A_2 u = -iVu - iWu - i\varepsilon|u|^{2\sigma}u.$$

This choice was made in, e.g., [4], in the case where  $V$  is quadratic like in (1.2):  $e^{tA_1}$  is computed by Fourier pseudospectral methods, and the nonlinear ordinary differential equation

$$\partial_t u = A_2 u$$

is solved explicitly, as it turns out to be a linear equation, since  $\partial_t |u|^2 = 0$  ( $\varepsilon$  is real). Error estimates are proved in [40], assuming large regularity for the initial data, but also considering a bounded spatial domain (hence the potential is bounded).

In the case  $W = 0$ , the other natural choice consists in setting  $A_1 = -iH$  and  $A_2 u = -i\varepsilon|u|^{2\sigma}u$ . This choice was made in, e.g., [5], again for  $V$  of the form (1.2), and using the Hermite eigenbasis to replace Fourier pseudospectral methods with Hermite pseudospectral methods: the operator  $e^{tA_1}$  is computed on each Hermite mode, see Example 1.4 below (with  $\chi \equiv 1$ ). In [41], the two decompositions were addressed numerically (for a higher order splitting method), and no crucial difference seemed to emerge regarding the size of the error. In the isotropic case of (1.2) ( $\omega_j = 1$  for all  $j$ ), an error analysis in the case of Strang splitting scheme was provided in [17] for the cubic nonlinearity  $\sigma = 1$ , in the spirit of the proof in [30] for the case without potential, requiring a high level of regularity for the initial data, and working in Sobolev spaces based on powers of  $H$  instead of powers of  $-\Delta$ , that is, accounting for the presence of the potential.

It is possible to mix the above two approaches, as in e.g. [41], in the following sense. Suppose for instance that  $V(x) = V_{\text{quad}}(x) + e^{-|x|^2} + 1$ . Then it is possible to consider  $A_1 = i\Delta - iV_{\text{quad}} - i$  and  $A_2 u = -ie^{-|x|^2}u - i\varepsilon|u|^{2\sigma}u$ . The flow generated

by  $A_1$  can be analyzed by spectral theory like in e.g. [17], while the potential in  $A_2$  is in the Schwartz class, and can be analyzed by perturbative arguments.

The framework that we consider here is in the spirit of the latter option, and we set  $A_1 = -iH$ ,  $A_2 = -iWu - i\varepsilon|u|^{2\sigma}u$ . This distinction can be of practical interest, even for  $W$  in the Schwartz class, as the spectral cutoff  $\Pi_\lambda$  defined in (1.4) is explicit for particular potentials only. We emphasize that even in the case where  $V$  is quadratic, the tools we present in Section 2 may be interesting in other contexts. Our main contribution, compared to the results evoked above, consists in decreasing the regularity required on the initial data, thanks to Strichartz estimates (but we stick to the Lie-Trotter discretization and do not address Strang splitting). The motivation is theoretical and practical: the nonlinearity need not be smooth ( $\sigma$  can be arbitrarily small), and we know, in some cases (typically when  $\sigma < 2/d - L^2$ -subcritical nonlinearity – or  $\varepsilon > 0$  – defocusing nonlinearity), that the solution to (1.1) is defined globally in time (at the level of regularity that we need), so the main error estimates are true on arbitrary time intervals (see Remark 1.7).

We define  $N(t)\phi$  as the solution of the flow

$$i\partial_t u = Wu + \varepsilon|u|^{2\sigma}u, \quad u|_{t=0} = \phi,$$

that is,  $N(t)\phi = \phi e^{-itW - it\varepsilon|\phi|^{2\sigma}}$ . As Equation (1.1) is time reversible, we shall only consider positive time, the case of negative time being similar. The standard Lie-Trotter approximation is defined, for  $\tau \in (0, 1)$ , as

$$Z(n\tau)\phi = (S(\tau)N(\tau))^n \phi, \quad n \in \mathbb{N},$$

where  $S$  is defined in (1.3). Note that in the case where  $V$  is harmonic, (1.2),  $S(\cdot)$  can be computed by using Hermite functions, see e.g. [3], and Example 1.4 below.

*Remark 1.2* (Time dependent potential). The potentials  $V$  and  $W$  may even depend on time: we just have to replace  $e^{-itH}$  with a more general semi-group, as the dispersive estimate from [16] that we use below remains valid, and in the formula for  $N(t)$ ,  $tW$  must be replaced by  $\int_0^t W$ . We emphasize that one has to be cautious with splitting methods in that case though, as the expression (1.3) must be replaced in order to account for the initial and final times,  $S(t_f, t_i)$ . The numerical cost is therefore different, as this semi-group is computed at each step, as well as  $\int_{t_i}^{t_f} W$ .

*Remark 1.3.* The potential  $W$  could include a singularity, in the sense that the assumption  $W \in L^\infty(\mathbb{R}^d)$  could be replaced by  $W \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ , for some  $p \geq 1$  such that  $p > d/2$ , typically (for suitable  $\gamma$  and  $p$ )

$$W(x) = \underbrace{\frac{1}{|x|^\gamma} \mathbf{1}_{|x| < 1}}_{\in L^p(\mathbb{R}^d)} + \underbrace{\frac{1}{|x|^\gamma} \mathbf{1}_{|x| \geq 1}}_{\in L^\infty(\mathbb{R}^d)},$$

like in [12], with similar assumptions on  $\partial^\alpha W$  for  $|\alpha| \leq 2$ . However, if  $W$  is singular, the nonlinear flow  $N(t)$  becomes more delicate to handle numerically (when space discretization is considered too), this is why we simply assume  $W \in W^{2,\infty}(\mathbb{R}^d)$ .

In the case  $V = W = 0$ , error estimates related to  $Z$  were first proven in [8], for globally Lipschitz nonlinearities. The proof was made more systematic, for other nonlinearities, in the more difficult case of Strang splitting, in [30]. In both papers, the regularity requested on the initial data, in order to obtain an error estimate in  $L^2$ , is rather high ( $H^2$  in [8] for Lie,  $H^4$  in [30] for Strang – in both papers, the space

dimension is sufficiently low so these Sobolev spaces are Banach algebras embedded into  $L^\infty$ ). This regularity constraint was relaxed in [23] and [14], by using discrete in time Strichartz estimates. We give more details regarding this tool in Section 3, which is crucial in our case too, where  $V$  need not be zero. As pointed out in [23, 24, 25, 35], in the case  $V \equiv 0$ , the group  $S(\cdot)$  does not satisfy discrete in time Strichartz estimates: a frequency cutoff is needed, see also [31]. When  $V$  is present, this frequency cutoff must be replaced by a more general operator.

To prove discrete in time Strichartz estimates, we need to remove the singularity at  $t = 0$  of the dispersive rate in  $e^{-itH}$ . In the case without potential, this is achieved by a frequency cutoff, see e.g. [23, 31]. In the present framework, the frequency cutoff is replaced by a spectral cutoff, in order typically to keep the commutation property between the cutoff  $\Pi$  and the exact flow  $S(t)$ : for potentials like in Assumption 1.1, the lack of commutation between the Fourier multiplier considered in [23] and  $H$  may generate error terms which cannot be controlled (typically in the case (1.2)).

Throughout this paper,  $\chi \in C_c^\infty(\mathbb{R}; [0, 1])$  denotes a smooth, compactly supported function which is one on  $[-1, 1]$ , and zero outside  $[-2, 2]$ . For  $\lambda > 0$ , we set

$$(1.4) \quad \Pi_\lambda = \chi^2 \left( \frac{H}{\lambda} \right),$$

the spectral localization defined by functional calculus, see Section 2. In particular, Proposition 2.3 implies that  $\Pi_\lambda$  and  $H$  commute, therefore so do  $\Pi_\lambda$  and  $S(t)$ .

We emphasize the fact that when  $V = W = 0$  (like in [23, 14, 31]), the Fourier multiplier  $\Pi_\lambda$  can be viewed as a convolution operator, which yields various estimates thanks to Young inequality. When  $V \neq 0$ ,  $\Pi_\lambda$  is no longer a Fourier multiplier: we shall use the fact that since  $\chi \in C_c^\infty$ ,  $\Pi_\lambda$  is a pseudodifferential operator, thanks to a result due to Helffer and Robert [20], and rely on properties of pseudodifferential operators (see Section 2). The generalization of the truncated free group from [23, 31] is then

$$(1.5) \quad S_\lambda(t) = \Pi_\lambda S(t) = S(t) \Pi_\lambda,$$

and the modified Lie-Trotter splitting scheme reads

$$(1.6) \quad u^n = \mathcal{Z}_\lambda(n\tau)u_0 := (S_\lambda(\tau)N(\tau))^n \Pi_\lambda u_0.$$

At this stage, we have not related  $\tau$  and  $\lambda$ . We will eventually consider  $\lambda = 1/\tau$ , which is the same choice as in [23] and [14]. Roughly speaking, this makes it possible to show  $L^2$  error estimates of order  $\tau^{1/2}$  on bounded time intervals, for  $H^1$  solutions, and of order  $\tau$  for  $H^2$  solutions. We point out however that the setting we introduce makes it possible to adapt the strategy developed in [31], where other relations between  $\lambda$  (denoted by  $K^2$  there) and  $\tau$  are proposed, for another integrator, in order to break the natural order barrier of  $\tau^{1/2}$  error estimates for  $H^1$  solutions.

*Example 1.4.* In the case of a harmonic potential as in (1.2), a well-known eigenbasis is given by Hermite functions, and the operator  $\Pi_\lambda$  can be written in terms of the eigendecomposition. If  $d = \omega_1 = 1$ , with  $V = V_{\text{quad}} + 1 = x^2 + 1$  (the constant 1 is added to be consistent with Assumption 1.1), Hermite functions  $\varphi_j$ ,  $j \in \mathbb{N}$ ,

solve  $H\varphi_j = (3 + 2j)\varphi_j$ , and, if they are normalized so that  $\|\varphi\|_{L^2} = 1$ , every  $L^2$  function  $\phi$  can be decomposed as

$$\phi(x) = \sum_{j \geq 0} \langle \phi, \varphi_j \rangle \varphi_j(x), \quad \langle \phi, \varphi_j \rangle = \int_{\mathbb{R}} \phi(y) \bar{\varphi}_j(y) dy.$$

Then we may also write

$$\Pi_\lambda \phi(x) = \sum_{j \geq 0} \chi \left( \frac{\lambda_j}{\lambda} \right)^2 \langle \phi, \varphi_j \rangle \varphi_j(x), \quad \lambda_j = 3 + 2j,$$

and

$$S_\lambda(t)\phi(x) = \sum_{j \geq 0} e^{-it\lambda_j} \chi \left( \frac{\lambda_j}{\lambda} \right)^2 \langle \phi, \varphi_j \rangle \varphi_j(x).$$

The expression of  $S(t)$  is obtained by setting  $\chi \equiv 1$ .

**1.3. Main results.** The first  $L^2$ -based function space we consider is

$$\mathcal{H}^1 = \{\phi \in H^1(\mathbb{R}^d); \phi\sqrt{V} \in L^2(\mathbb{R}^d)\},$$

equipped with the norm

$$\|\phi\|_{\mathcal{H}^1}^2 := \langle H\phi, \phi \rangle = \|\nabla\phi\|_{L^2(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d} V(x)|\phi(x)|^2 dx.$$

Note that in view of Assumption 1.1, since  $V \geq 1$ ,  $\|\phi\|_{\mathcal{H}^1}^2 \geq \|\phi\|_{L^2}^2$ . Also,  $\Sigma \subset \mathcal{H}^1$ , where

$$\Sigma := \left\{ \phi \in H^1(\mathbb{R}^d); \int_{\mathbb{R}^d} |x|^2 |\phi(x)|^2 dx < \infty \right\}.$$

In the case of a quadratic potential (1.2), the two spaces coincide, and correspond to the sharp analogue of the Sobolev space  $H^1(\mathbb{R}^d)$  compared to the case  $V = 0$ , see e.g. [10]. Under Assumption 1.1, (1.1) is locally well-posed in  $\Sigma$ , see e.g. [9]. In Section 4, we prove well-posedness results in the possibly larger space  $\mathcal{H}^1$ .

Like in [14], the statements of our results contain a restriction on the dimension, since when  $\sigma \geq 2/d$ , we assume in addition  $\sigma \geq 1/2$  for the nonlinearity in (1.1) to be of class  $C^2$  (or  $\dot{W}^{2,\infty}$  in the case  $\sigma = 1/2$ ), which implies  $d \leq 5$  since  $\sigma < 2/(d-2)_+$ . This extra condition is therefore only present when  $d = 5$ . Note that our results cover the physical case of a cubic nonlinearity in dimension  $d \leq 3$ .

**Theorem 1.5.** *Let Assumption 1.1 be verified. Assume that either  $0 < \sigma < 2/d$ , or  $d \leq 5$  and  $2/d \leq \sigma < 2/(d-2)_+$ , with in addition  $\sigma \geq 1/2$  when  $d = 5$ . Let  $u_0 \in \mathcal{H}^1$ , assume that (1.1) has a unique solution  $u \in C([0, T]; \mathcal{H}^1)$ , for some  $T > 0$ , and denote by  $u^n$  the sequence defined by the scheme (1.6), where we set  $\lambda = 1/\tau$ . There exist  $\tau_0 > 0$  and  $C = C(T) > 0$  such that for every  $\tau \in (0, \tau_0]$ , we have the error estimate*

$$\|u^n - u(n\tau)\|_{L^2(\mathbb{R}^d)} \leq C\tau^{1/2}, \quad 0 \leq n\tau \leq T.$$

To prove a better error estimate in terms of the time step, we assume higher regularity: let

$$\mathcal{H}^2 := \{\phi \in H^2(\mathbb{R}^d); V\phi \in L^2(\mathbb{R}^d)\},$$

equipped with the norm

$$\|\phi\|_{\mathcal{H}^2}^2 := \|\Delta\phi\|_{L^2(\mathbb{R}^d)}^2 + \|V\phi\|_{L^2(\mathbb{R}^d)}^2.$$

**Theorem 1.6.** *Let Assumption 1.1 be verified. Assume that  $1/2 \leq \sigma < 2/(d-2)_+$ . Let  $u_0 \in \mathcal{H}^2$ , assume that (1.1) has a unique solution  $u \in C([0, T]; \mathcal{H}^2)$ , for some  $T > 0$ , and denote by  $u^n$  the sequence defined by the scheme (1.6), where we set  $\lambda = 1/\tau$ . There exist  $\tau_0 > 0$  and  $C = C(T) > 0$  such that for every  $\tau \in (0, \tau_0]$ , we have the error estimate*

$$\begin{aligned} \|u^n - u(n\tau)\|_{L^2(\mathbb{R}^d)} &\leq C\tau, & 0 \leq n\tau \leq T, \\ \|u^n - u(n\tau)\|_{\mathcal{H}^1} &\leq C\tau^{1/2}, & 0 \leq n\tau \leq T. \end{aligned}$$

*Remark 1.7 (On the time  $T$ ).* As we will see in Section 4, the time  $T$  involved in Theorems 1.5 and 1.6 can be taken arbitrarily large (global solution) in two cases:  $0 < \sigma < 2/d$  ( $L^2$ -subcritical case) or  $\varepsilon = +1$  (defocusing case). When  $\sigma \geq 2/d$  and  $\varepsilon = -1$  (focusing case), finite time blow up may occur (see e.g. [9, 12]), so having a solution  $u$  well-defined up to time  $T$  becomes a nontrivial assumption, even though we know that a unique local in time solution always exists (for *some* time  $T > 0$ ). In the case  $\sigma \geq 2/d$  and  $\varepsilon = -1$ , we can prove that the solution is global provided that  $\|u_0\|_{\mathcal{H}^1}$  is sufficiently small. However, this theoretical statement does not yield an explicit smallness condition (unless  $\sigma = 2/d$ , see e.g. [12]), so it is not very convenient in practice.

*Remark 1.8 (Nonautonomous equation).* In the case of a nonautonomous nonlinearity of the form

$$i\partial_t u + \Delta u = V(x)u + W(x)u + h(t)|u|^{2\sigma}u,$$

where the function  $h$  is bounded on  $[0, T]$ , Theorems 1.5 and 1.6 remain valid, as it is easy to check that the introduction of  $h$  does not change the error analysis. Moreover, the local Cauchy problem can be handled like in Section 4; global existence is not straightforward though, as the conservation of energy is lost when  $h$  is not constant (see e.g. [12, Section 4.11]).

**1.4. Notations.** For  $Y$  a Banach space, we shall denote  $\|f\|_{L_T^q Y} = \|f\|_{L^q(0, T; Y)}$ . The Japanese bracket is classically defined as  $\langle z \rangle = (1 + |z|^2)^{1/2}$ .

## 2. TECHNICAL TOOLS

In this section, we gather technical results which are useful in the rest of the paper, related to functional and pseudodifferential calculus. Most of the results that we shall use in the proof of Theorems 1.5 and 1.6 are presented in Subsection 2.6, and concern properties of the spectral localization  $\Pi_\lambda$ . We will also rely on the equivalence of norms stated in Lemma 2.12. The perturbative potential  $W$  does not appear in this section.

**2.1. Preliminaries.** We first state a consequence of Assumption 1.1, which is obvious in the case of a quadratic potential:

**Lemma 2.1.** *If  $V$  satisfies Assumption 1.1, then*

$$|\nabla V(x)|^2 \leq 2d\|\nabla^2 V\|_{L^\infty} V(x), \quad \forall x \in \mathbb{R}^d.$$

*Proof.* The property  $V \geq 1$  is not needed for this lemma, as  $V \geq 0$  suffices. Taylor's formula yields, for  $x, y \in \mathbb{R}^d$ ,

$$\begin{aligned} V(x+y) &= V(x) + y \cdot \nabla V(x) + \int_0^1 (1-\theta) \langle y, \nabla^2 V(x+\theta y) \rangle d\theta \\ &\leq V(x) + y \cdot \nabla V(x) + \frac{|y|^2}{2} \|\nabla^2 V\|_{L^\infty}. \end{aligned}$$

Since  $V(x+y) \geq 0$ , the polynomial in  $y$  on the right hand side is everywhere nonnegative. Taking for instance  $y = y_j e_j$ , where  $(e_k)_{1 \leq k \leq d}$  is the canonical basis of  $\mathbb{R}^d$ , the discriminant of  $V(x) + y_j \partial_j V(x) + \frac{y_j^2}{2} \|\nabla^2 V\|_{L^\infty}$  is nonpositive, hence

$$|\partial_j V(x)|^2 \leq 2 \|\nabla^2 V\|_{L^\infty} V(x), \quad \forall x \in \mathbb{R}^d.$$

The result follows by summing over  $j$ .  $\square$

We will also invoke the following result, known either as Schur test, or Young inequality for integral operators, as stated in [34, Theorem 0.3.1]:

**Proposition 2.2.** *For  $d \geq 1$ , consider an operator  $\mathcal{T}$  with Schwartz kernel  $K$ ,*

$$\mathcal{T}\phi(x) = \int_{\mathbb{R}^d} K(x, y) \phi(y) dy.$$

*Let  $1 \leq p \leq q \leq \infty$  and  $r \geq 1$  be related by*

$$\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}.$$

*If  $K$  satisfies*

$$\sup_{x \in \mathbb{R}^d} \|K(x, \cdot)\|_{L^r(\mathbb{R}^d)}, \quad \sup_{y \in \mathbb{R}^d} \|K(\cdot, y)\|_{L^r(\mathbb{R}^d)} \leq C,$$

*then  $\mathcal{T}$  is continuous from  $L^p(\mathbb{R}^d)$  to  $L^q(\mathbb{R}^d)$ ,*

$$\|\mathcal{T}\phi\|_{L^q(\mathbb{R}^d)} \leq C \|\phi\|_{L^p(\mathbb{R}^d)}, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

*Proof.* We give a direct proof, as it is simple. If  $r$  is finite, setting

$$\alpha = r \left(1 - \frac{1}{p}\right), \quad \beta = \frac{r}{q},$$

we have  $\alpha + \beta = 1$ ,  $\alpha, \beta \geq 0$ , and Hölder inequality yields, for  $\phi \in \mathcal{S}(\mathbb{R}^d)$ ,

$$\begin{aligned} \|\mathcal{T}\phi\|_{L^q}^q &\leq \int \left( \int |K(x, y)| |\phi(y)| dy \right)^q dx = \int \left( \int |K(x, y)|^{\alpha+\beta} |\phi(y)| dy \right)^q dx \\ &\leq \|\phi\|_{L^p}^q \int \sup_{y \in \mathbb{R}^d} |K(x, y)|^{\beta q} \left( \int |K(x, y)|^{\alpha p'} dy \right)^{q/p'} dx. \end{aligned}$$

As  $\beta q = \alpha p' = r$ , the assumption yields

$$\|\mathcal{T}\phi\|_{L^q}^q \lesssim C^q \|\phi\|_{L^p}^q.$$

and we note that

$$(r-1) \left(1 + \frac{q}{p'}\right) = (r-1) \left(1 + q \left(\frac{1}{r} - \frac{1}{q}\right)\right) = q \left(1 - \frac{1}{r}\right) = q \left(\frac{1}{p} - \frac{1}{q}\right).$$

The case  $r = \infty$  corresponds to the situation where  $p = 1$  and  $q = \infty$ , and the proposition is straightforward.  $\square$



**2.2. Functional calculus.** We recall properties presented in e.g. [33, Chapter VIII], or [13, Chapter 8]. The standard way to define functions of a self-adjoint operator (typically,  $H$ ) consists in using the spectral measure (see e.g. [33]), or, equivalently Helffer-Sjöstrand formula (see e.g. [15]). We note that  $H$  is self-adjoint on  $L^2(\mathbb{R}^d)$ , it is elliptic, and its spectrum  $\sigma(H)$  is included in  $[1, +\infty)$ , since  $V \geq 1$  by Assumption 1.1. The square root of  $H$  is defined in terms of quadratic forms, by

$$\langle H\phi, \phi \rangle = \|H^{1/2}\phi\|_{L^2}^2, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d),$$

and we have

$$[H^{1/2}, H] = 0,$$

where  $[A, B]$  denotes the Lie commutator  $[A, B] = AB - BA$ . We can also infer the following result from [33, Theorem VIII.5] and [13, Propositions 8.3 and 8.20]:

**Proposition 2.3.** *Let  $V$  satisfying Assumption 1.1, and  $H = -\Delta + V$ .*

- For any  $f \in \mathcal{S}(\mathbb{R})$ ,  $[f(H), H] = 0$ .
- For any  $f, g \in \mathcal{S}(\mathbb{R})$ ,  $f(H)g(H) = (fg)(H) = g(H)f(H)$ , on  $\{u \in \text{Dom}(g(H)) ; g(H)u \in \text{Dom}(f(H))\}$ .
- For any nonnegative bounded Borelian function  $f$ ,  $\|f(H)\|_{L^2 \rightarrow L^2} \leq \|f\|_{L^\infty}$ .
- If  $H\psi = \lambda\psi$ , then for any  $f \in \mathcal{S}(\mathbb{R})$ ,  $f(H)\psi = f(\lambda)\psi$ .

Recall that  $\Pi_\lambda$  is defined by (1.4), with  $\chi \in C_c^\infty(\mathbb{R}^d; [0, 1])$ . As a consequence of the first point of the above proposition,  $[\Pi_\lambda, H] = 0$ , and as  $[\Pi_\lambda, \partial_t] = 0$  since  $\Pi_\lambda$  does not depend on time,  $[\Pi_\lambda, e^{-itH}] = 0$ , hence (1.5). We insist on this aspect, since if we had defined initially  $\Pi_\lambda$  as a pseudodifferential operator of symbol  $\chi^2((|\xi|^2 + V(x))/\lambda)$  (the definition is recalled below, see (2.2)), the commutation relation  $[\Pi_\lambda, H] = 0$  would have become false, this bracket being only, in general, a (nontrivial) smoothing operator. It seems crucial, however, to know that  $\Pi_\lambda$  is a pseudodifferential operator, but whose symbol is not fully explicit, see Section 2.5.

**2.3. Weyl-Hörmander pseudodifferential calculus.** We present aspects of pseudodifferential calculus which can be found in [22, Chapter XVIII] and [19, Chapter 4]. This will be crucial to establish properties of the spectral localization  $\Pi_\lambda$ . Define the function  $m$  by

$$m(x, \xi) = \sqrt{|\xi|^2 + V(x)}.$$

We introduce two metrics,

$$\begin{aligned} g_0(x, \xi) &= dx^2 + d\xi^2, \quad \text{or, equivalently, } g_{0,(x,\xi)}(y, \eta) = |y|^2 + |\eta|^2, \\ g_1(x, \xi) &= dx^2 + \frac{d\xi^2}{|\xi|^2 + V(x)} = dx^2 + \frac{d\xi^2}{m^2(x, \xi)}, \quad \text{or, equivalently,} \\ g_{1,(x,\xi)}(y, \eta) &= |y|^2 + \frac{|\eta|^2}{m^2(x, \xi)}. \end{aligned}$$

We recall [22, Definitions 18.4.1 and 18.5.1] (the standard notation  $\sigma$  in the context of pseudodifferential calculus being used only in this subsection, no confusion with the exponent of the nonlinearity in (1.1) should be possible):

**Definition 2.4.** *The metric  $g$  is called  $\sigma$  temperate if it is slowly varying,*

$$\exists c, C > 0, \quad g_{(x,\xi)}(y, \eta) \leq c \implies g_{(x+y, \xi+\eta)}(z, \zeta) \leq C g_{(x,\xi)}(z, \zeta),$$

and there exist  $C, N > 0$  such that for all  $x_1, x, y, \xi_1, \xi, \eta \in \mathbb{R}^d$ ,

$$g_{(x_1, \xi_1)}(y, \eta) \leq C g_{(x, \xi)}(y, \eta) \left(1 + g_{(x_1, \xi_1)}^\sigma(x_1 - x, \xi_1 - \xi)^N\right),$$

where

$$g_{(x, \xi)}^\sigma(z, \zeta) = \sup_{(y, \eta) \neq 0} \frac{\sigma(y, \eta; z, \zeta)}{g_{(x, \xi)}(y, \eta)}, \quad \text{where } \sigma(y, \eta; z, \zeta) = y \cdot \zeta - z \cdot \eta.$$

A positive function  $w$  is  $\sigma, g$  temperate if it is  $g$  continuous,

$$\exists c, C > 0, \quad g_{(x, \xi)}(y, \eta) < c \implies \frac{w(x, \xi)}{C} \leq w(x + y, \xi + \eta) \leq C w(x, \xi),$$

and there exist  $C, N > 0$  such that

$$(2.1) \quad w(y, \eta) \leq C w(x, \xi) (1 + |x - y|^2 + |\xi - \eta|^2)^N, \quad \forall x, y, \xi, \eta \in \mathbb{R}^d.$$

We note that, classically (see e.g. [7, 19]),  $g_0^\sigma = g_0$ , and

$$g_1^\sigma(x, \xi) = m^2(x, \xi) dx^2 + d\xi^2.$$

**Lemma 2.5.** *Let  $m^2 = |\xi|^2 + V(x)$ , where  $V$  satisfies Assumption 1.1. Then  $m^2$  is  $\sigma, g$  temperate, for  $g = g_0$  and  $g = g_1$ .*

*Proof.* The term  $|\xi|^2$  is standard, and Peetre inequality (see e.g. [1]) yields

$$|\eta|^2 \leq (1 + |\xi|^2)(1 + |\xi - \eta|^2) \leq m^2(x, \xi)(1 + |\xi - \eta|^2),$$

where we have used  $V \geq 1$  for the last inequality. Regarding the term  $V$ , we argue like in the proof of Lemma 2.1, and write that from Taylor formula and Assumption 1.1,

$$\begin{aligned} V(y) &\leq V(x) + |y - x| |\nabla V(x)| + C |y - x|^2 \lesssim V(x) + |y - x| \sqrt{V(x)} + |x - y|^2 \\ &\lesssim V(x) + |x - y|^2, \end{aligned}$$

where we have used Lemma 2.1 for the second inequality, and Young inequality for the last one. Then (2.1) follows, with  $N = 1$ , since  $V \geq 1$ .  $\square$

The class of symbols  $S(w, g)$  is defined as follows:

**Definition 2.6** (Symbols associated with a weight and a metric). *Let  $w$  be  $\sigma$  temperate. The set  $S(w, g)$  consists of functions  $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  such that for every  $x, \xi \in \mathbb{R}^d$ , every  $\alpha, \beta \in \mathbb{N}^d$ ,*

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C(\alpha, \beta) \begin{cases} w(x, \xi) & \text{if } g = g_0, \\ w(x, \xi) m(x, \xi)^{-|\beta|} & \text{if } g = g_1. \end{cases}$$

The best constants  $C(\alpha, \beta)$  define the seminorms of  $a$ ,

$$|a|_{g_j, \alpha, \beta} = \sup_{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d} w(x, \xi)^{-1} m(x, \xi)^{j|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)|, \quad j = 0, 1.$$

We consider the Weyl quantization for pseudodifferential operators. For a symbol  $a \in S(w, g)$  and  $\phi \in \mathcal{S}(\mathbb{R}^d)$ , we set, with the usual convention  $D = -i\nabla$ ,

$$(2.2) \quad a^w(x, D)\phi = (\text{Op } a)^w \phi(x) = (2\pi)^{-d} \iint_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) \phi(y) dy d\xi.$$

We can also express  $(\text{Op } a)^w$  thanks to its Schwartz kernel,

$$(\text{Op } a)^w \phi(x) = \int_{\mathbb{R}^d} K(x, y) \phi(y) dy, \quad K(x, y) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) d\xi.$$

We denote

$$OPS(w, g) = \{(\text{Op } a)^w, \quad a \in S(w, g)\}.$$

The composition of pseudodifferential operators is described in [22, Theorem 18.5.4], and implies the following result, where we consider only the metric  $g_1$ :

**Proposition 2.7.** *If  $a_1 \in S(w_1, g_1)$  and  $a_2 \in S(w_2, g_1)$ , then  $(\text{Op } a_1)^w \circ (\text{Op } a_2)^w \in OPS(w_1 w_2, g_1)$ . The symbol of this pseudodifferential operator is denoted by  $a_1 \sharp a_2$ , and we have, for all  $N \in \mathbb{N}$ ,*

$$a_1 \sharp a_2(x, \xi) - \sum_{j=0}^{N-1} \frac{1}{j!} \left( \frac{i\sigma(D_x, D_\xi; D_y, D_\eta)}{2} \right)^j a_1(x, \xi) a_2(y, \eta) \Big|_{(y, \eta) = (x, \xi)} = r_N(x, \xi),$$

where  $r_N \in S(m^{-N} w_1 w_2, g_1)$ .

We note that  $H = (\text{Op } \mathbf{p})^w$  for

$$\mathbf{p}(x, \xi) = |\xi|^2 + V(x) \in S(m^2, g_1).$$

As  $V \geq 1$ ,  $H$  is elliptic (since  $\mathbf{p}(x, \xi) = m^2(x, \xi)$ ), and we emphasize two consequences:

**Proposition 2.8** (From Theorem 4.8 in [19]). *The operator  $\sqrt{H}$  is a pseudodifferential operator, belonging to  $OPS(m, g_1)$ , and*

$$\sqrt{H} - (\text{Op } \sqrt{\mathbf{p}})^w \in OPS(1, g_1).$$

Together with Proposition 2.7, and the proof of [22, Theorem 18.1.9], this implies:

**Proposition 2.9.** *There exists  $\mathbf{q} \in S(m^{-2}, g_1)$ , with  $\sqrt{\mathbf{q}} \in S(m^{-1}, g_1)$ , such that for all  $N \in \mathbb{N}$ ,*

$$\begin{aligned} H \circ (\text{Op } \mathbf{q})^w - \text{I}, \quad (\text{Op } \mathbf{q})^w \circ H - \text{I} &\in OPS(m^{-N}, g_1), \\ \sqrt{H} \circ (\text{Op } \sqrt{\mathbf{q}})^w - \text{I}, \quad (\text{Op } \sqrt{\mathbf{q}})^w \circ \sqrt{H} - \text{I} &\in OPS(m^{-N}, g_1). \end{aligned}$$

More precisely,

$$\mathbf{q}(x, \xi) - \frac{1}{\mathbf{p}(x, \xi)} \in S(m^{-3}, g_1), \quad \sqrt{\mathbf{q}(x, \xi)} - \frac{1}{\sqrt{\mathbf{p}(x, \xi)}} \in S(m^{-2}, g_1).$$

The metric  $g_1$  satisfies the assumption of [6], and we have, from [6, Theorem 3.1]:

**Proposition 2.10.** *Let  $1 < p < \infty$ . If  $a \in S(1, g_1)$ , then  $(\text{Op } a)^w$  is bounded in  $L^p$ , and its operator norm is controlled by finitely many seminorms of  $a$ .*

*Remark 2.11.* In the case  $V = 0$ , the set  $S(1, g_1)$  coincides with the class of symbols  $S^0$  from [22, Definition 18.1.1], and the above result meets e.g. [36, Chapter VI, Section 5.2], [39, Theorem 5.2], or [34, Theorem 3.1.6]. The proof is actually similar in our case, based on decay estimates for the Schwartz kernel  $K$  off the diagonal (obtained by integrations by parts, using the property  $m(x, \xi) \geq |\xi|$ ), which makes it possible to eventually invoke Proposition 2.2 (with  $r = 1$ ).

As a first application of these general results, we compare Sobolev norms:

**Lemma 2.12** (Equivalence of norms). *Let  $V$  satisfying Assumption 1.1. For any  $1 < p < \infty$ , there exist  $C_1, C_2 > 0$  such that for any  $\phi \in \mathcal{S}(\mathbb{R}^d)$ ,*

$$\begin{aligned} \|\nabla \phi\|_{L^p} + \|\phi \sqrt{V}\|_{L^p} &\leq C_1 \|H^{1/2} \phi\|_{L^p} \leq C_2 \left( \|\nabla \phi\|_{L^p} + \|\phi \sqrt{V}\|_{L^p} \right), \\ \|\Delta \phi\|_{L^p} + \|V \phi\|_{L^p} &\leq C_1 \|H \phi\|_{L^p} \leq C_2 (\|\Delta \phi\|_{L^p} + \|V \phi\|_{L^p}). \end{aligned}$$

*Proof.* We follow the same strategy as in the proof of [28, Lemma 2.7], and invoke analytic interpolation (from [38, Section V.4], see also [29, Theorem 2.7]). First, considering the function on  $\mathbb{R}_+ \setminus \{0\}$ ,  $x \mapsto x^{is}$  for  $s \in \mathbb{R}$ , the main result from [18] implies the bound

$$\|H^{is}\phi\|_{L^p} \lesssim \|\phi\|_{L^p}.$$

The estimate  $\|V^{is}\phi\|_{L^p} \lesssim \|\phi\|_{L^p}$  is straightforward, and  $\|(-\Delta)^{is}\phi\|_{L^p} \lesssim \|\phi\|_{L^p}$  follows from the Hörmander-Mikhlin multiplier theorem (see e.g. [21, Theorem 7.9.5], or [29, Theorem 2.8]), so

$$\|V^{is}\phi\|_{L^p} + \|(-\Delta)^{is}\phi\|_{L^p} + \|H^{is}\phi\|_{L^p} \lesssim \|\phi\|_{L^p}, \quad \forall s \in \mathbb{R}.$$

On the other hand, triangle inequality yields

$$\|H\phi\|_{L^p} \leq \|\Delta\phi\|_{L^p} + \|V\phi\|_{L^p}.$$

The maps

$$(x, \xi) \mapsto \frac{|\xi|^2}{|\xi|^2 + V(x)} \quad \text{and} \quad (x, \xi) \mapsto \frac{V(x)}{|\xi|^2 + V(x)}$$

define symbols, as can be checked directly, keeping Lemma 2.1 in mind for the second map. Proposition 2.10 implies

$$\|\Delta H^{-1}\phi\|_{L^p} + \|VH^{-1}\phi\|_{L^p} \lesssim \|\phi\|_{L^p},$$

hence the second inequality of the lemma by replacing  $\phi$  with  $H\phi$ . Analytic interpolation for the operator-valued functions  $z \mapsto (-\Delta)^z H^{-z}$  and  $V^z H^{-z}$  yields

$$\|(-\Delta)^s \phi\|_{L^p} + \|V^s \phi\|_{L^p} \lesssim \|H^s \phi\|_{L^p} \lesssim \|(-\Delta)^s \phi\|_{L^p} + \|V^s \phi\|_{L^p}, \quad \text{for all } 0 \leq s \leq 1,$$

as the case  $s = 0$  is trivial. The first inequality of the lemma corresponds to the case  $s = 1/2$ .  $\square$

*Remark 2.13.* We give an alternative proof of the main estimate in the proof of Lemma 2.12, in the spirit of [42, Lemma 2.4]:

$$(2.3) \quad \|\Delta\phi\|_{L^p} + \|V\phi\|_{L^p} \lesssim \|H\phi\|_{L^p}.$$

Setting  $Q = (\text{Op } \mathbf{q})^w$ , Proposition 2.9 yields

$$\Delta\phi = \Delta QH\phi + \Delta R\phi, \quad V\phi = VQH\phi + VR\phi, \quad R \in OPS(m^{-2}, g_1).$$

Since  $VQ$  and  $\Delta \circ Q$  belong to  $OPS(1, g_1)$ , Proposition 2.10 implies

$$\|\Delta QH\phi\|_{L^p} + \|VQH\phi\|_{L^p} \lesssim \|H\phi\|_{L^p}.$$

Also,  $\Delta \circ R, VR \in OPS(1, g_1)$ , hence

$$\|\Delta R\phi\|_{L^p} + \|VR\phi\|_{L^p} \lesssim \|\phi\|_{L^p} \lesssim \|H\phi\|_{L^p},$$

since  $H^{-1} \in OPS(m^{-2}, g_1)$ , and using Proposition 2.10 again, hence (2.3). Note that by considering Proposition 2.9 again (now for the parametrix of  $\sqrt{H}$ ), we also have, by following the same lines as above,

$$\|\nabla\phi\|_{L^p} + \|\phi\sqrt{V}\|_{L^p} \lesssim \|\sqrt{H}\phi\|_{L^p},$$

without invoking interpolation.

The following consequences of Lemma 2.12 will be convenient: if  $1 < p < \infty$ ,  $f, g \in \mathcal{S}(\mathbb{R}^d)$ , and  $\gamma > 0$ ,

$$\begin{aligned}
\|H^{1/2}(fg)\|_{L^p} &\lesssim \|f\|_{L^{a_1}} \|H^{1/2}g\|_{L^{a_2}} + \|\nabla f\|_{L^{a_3}} \|g\|_{L^{a_4}} \\
&\lesssim \|f\|_{L^{a_1}} \|H^{1/2}g\|_{L^{a_2}} + \|H^{1/2}f\|_{L^{a_3}} \|g\|_{L^{a_4}}, \\
(2.4) \quad \frac{1}{p} &= \frac{1}{a_1} + \frac{1}{a_2} = \frac{1}{a_3} + \frac{1}{a_4}, \\
\|H^{1/2}(|f|^\gamma f)\|_{L^p} &\lesssim \|f\|_{L^{a_1}}^\gamma \|H^{1/2}f\|_{L^{a_2}}, \quad \frac{1}{p} = \frac{\gamma}{a_1} + \frac{1}{a_2}.
\end{aligned}$$

**2.4. Semiclassical pseudodifferential calculus.** The spectral localization  $\Pi_\lambda$  is defined by functional calculus, in order to have the exact commutation relation  $[\Pi_\lambda, H] = 0$ . In the case  $V = 0$ ,  $\Pi_\lambda$  is a Fourier multiplier, which makes it possible to establish various estimates, needed for the analysis of error estimates for the discretization in time. When  $V \neq 0$ , the generalization of the notion of Fourier multiplier is the notion of pseudodifferential operator. It is therefore crucial to know that  $\Pi_\lambda$  is indeed a pseudodifferential operator: this follows from the initial result of Helffer and Robert [20], which was revisited by Dimassi and Sjöstrand [15]. For  $\hbar \in (0, 1]$ , the semiclassical version of (2.2) is

$$(2.5) \quad (\text{Op}_\hbar a)^w \phi(x) = (2\pi\hbar)^{-d} \iint_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi/\hbar} a\left(\frac{x+y}{2}, \xi\right) \phi(y) dy d\xi.$$

In view of Lemma 2.5, we have, from [20, Theorem 4.1] or [15, Theorem 8.7]:

**Proposition 2.14.** *Let  $P(\hbar) = (\text{Op}_\hbar \mathbf{p})^w$ , where  $\mathbf{p} \in S(m^2, g_0)$  is real-valued. If  $f \in C_c^\infty(\mathbb{R})$ , then  $f(P) \in \text{Op}_\hbar(S(m^{-k}, g_0))$  for every  $k \in \mathbb{N}$ . In addition,*

$$f(P) = \text{Op}_\hbar(a_0 + \hbar a_1 + \hbar^2 \rho^\hbar)^w,$$

where

$$a_0(x, \xi) = f(\mathbf{p}(x, \xi)), \quad a_1(x, \xi) = \mathbf{p}_1(x, \xi) f'(\mathbf{p}(x, \xi)),$$

for  $\mathbf{p}_1 \in S(m^2, g_0)$  and  $\rho^\hbar \in S(m^{-k}, g_0)$  uniformly in  $\hbar \in (0, 1]$  (the corresponding constants  $C(\alpha, \beta)$  from Definition 2.6 do not depend on  $\hbar \in (0, 1]$ ).

**2.5. The spectral cutoff as a pseudodifferential operator.** We can now prove that  $\Pi_\lambda$ , defined by functional calculus, is indeed a pseudodifferential operator, whose kernel is estimated like in the case  $V = 0$ :

**Proposition 2.15.** *Let  $\chi \in C_c^\infty(\mathbb{R}; [0, 1])$ , equal to one on  $[-1, 1]$ , and zero outside  $[-2, 2]$ , and  $\Pi_\lambda$  defined by (1.4), where  $V$  satisfies Assumption 1.1.*

- For every  $\lambda > 0$ ,  $\Pi_\lambda$  is a pseudodifferential operator,  $\Pi_\lambda \in OPS(1, g_1)$ .
- Its Schwartz kernel  $K_\lambda$  is such that

$$\Pi_\lambda \phi(x) = \int_{\mathbb{R}^d} K_\lambda(x, y) \phi(y) dy, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d),$$

and for all  $N > 0$ , there exists  $C_N > 0$  independent of  $\lambda \geq 1$  such that

$$(2.6) \quad |K_\lambda(x, y)| \leq C_N \frac{\lambda^{d/2}}{(1 + \sqrt{\lambda}|x - y|)^N}.$$

- We have the following decomposition:

$$\Pi_\lambda = (\text{Op } \pi_\lambda)^w, \quad \pi_\lambda = a_\lambda + \frac{1}{\sqrt{\lambda}}b_\lambda + \frac{1}{\lambda}r_\lambda,$$

where

$$a_\lambda(x, \xi) = \chi^2 \left( \frac{|\xi|^2 + V(x)}{\lambda} \right), \quad b_\lambda(x, \xi) = \mathbf{q}_1(x, \xi) \chi \chi' \left( \frac{|\xi|^2 + V(x)}{\lambda} \right),$$

for some  $\mathbf{q}_1 \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$  independent of  $\lambda$ , and  $r_\lambda \in \cap_k S(m^{-k}, g_0)$  whose seminorms are bounded uniformly in  $\lambda \geq 1$ . In particular,

$$\forall \alpha, \beta \in \mathbb{N}^d, \quad \exists C(\alpha, \beta), \quad |\partial_x^\alpha \partial_\xi^\beta r_\lambda(x, \xi)| \leq C(\alpha, \beta) m(x, \xi)^{-|\beta|}, \quad \forall \lambda \geq 1.$$

*Proof.* Set  $\hbar = 1/\sqrt{\lambda}$ , and  $p_\hbar = |\xi|^2 + \hbar^2 V$ . Then

$$\frac{H}{\lambda} = -\hbar^2 \Delta + \hbar^2 V(x) = (\text{Op}_{p_\hbar} p_\hbar)^w = \hbar^2 (\text{Op } \mathbf{p})^w,$$

and we have  $p_\hbar \in S(m^2, g_0)$ . We can therefore invoke Proposition 2.14, with  $f = \chi^2$ :

$$\Pi_\lambda = (\text{Op}_{p_\hbar} \alpha^\hbar)^w, \quad \alpha^\hbar = \alpha_0 + \frac{1}{\sqrt{\lambda}}\alpha_1 + \frac{1}{\lambda}\rho^\hbar,$$

with

$$\alpha_0(x, \xi) = \chi^2 (|\xi|^2 + \hbar^2 V(x)), \quad \alpha_1(x, \xi) = \mathbf{p}_1(x, \xi) \chi \chi' (|\xi|^2 + \hbar^2 V(x)),$$

and  $\mathbf{p}_1 \in S(m^2, g_0)$ ,  $\rho^\hbar \in S(m^{-k}, g_0)$  uniformly in  $\hbar \in (0, 1]$  for all  $k \in \mathbb{N}$ . The first claim of the proposition follows readily. In addition, we have

$$\begin{aligned} K_\lambda(x, y) &= (2\pi\hbar)^{-d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi/\hbar} \alpha^\hbar \left( \frac{x+y}{2}, \xi \right) d\xi \\ &= \left( \frac{\sqrt{\lambda}}{2\pi} \right)^d \int_{\mathbb{R}^d} e^{i\sqrt{\lambda}(x-y)\cdot\xi} \alpha^\hbar \left( \frac{x+y}{2}, \xi \right) d\xi. \end{aligned}$$

As  $\alpha^\hbar$  is integrable in  $\xi$ , uniformly in  $x \in \mathbb{R}^d$  and  $\hbar \in (0, 1]$ ,  $|K_\lambda(x, y)| \lesssim \hbar^{-d} = \lambda^{d/2}$ . Since for all  $\beta \in \mathbb{N}^d$ ,  $\partial_\xi^\beta \alpha^\hbar$  is also integrable in  $\xi$ , uniformly in  $x \in \mathbb{R}^d$  and  $\hbar \in (0, 1]$ ,  $N$  integrations by parts yield

$$|K_\lambda(x, y)| \lesssim \frac{\lambda^{d/2}}{(\sqrt{\lambda}|x-y|)^N},$$

hence (2.6) by combining these two estimates.

Going back to the definitions (2.2) and (2.5), and changing variables in  $\xi$ , we have

$$(\text{Op}_{p_\hbar} \alpha_0)^w = (\text{Op } a_\lambda)^w, \quad (\text{Op}_{p_\hbar} \alpha_1)^w = (\text{Op } b_\lambda)^w + \hbar (\text{Op } r_1^\hbar)^w, \quad (\text{Op}_{p_\hbar} \rho^\hbar)^w = (\text{Op } r_2^\hbar)^w,$$

where  $r_1^\hbar, r_2^\hbar$  satisfy

$$\forall \alpha, \beta \in \mathbb{N}^d, \quad \forall k \in \mathbb{N}, \quad |\partial_x^\alpha \partial_\xi^\beta r_j^\hbar(x, \xi)| \leq C m(x, \xi)^{-k}, \quad \forall x, \xi \in \mathbb{R}^d, \quad \forall \hbar \in (0, 1].$$

The proposition follows by setting  $r = r_1 + r_2$ .  $\square$

**2.6. Estimates on the spectral localization.** We can now establish some results which are the analogue of the tools used in [14, 23] for the case  $V = 0$ , where  $\Pi_\lambda$  is a Fourier multiplier.

**Lemma 2.16.** *For any  $1 < p < \infty$ , there exists  $C$  such that for any  $\phi \in \mathcal{S}(\mathbb{R}^d)$  and  $\lambda \geq 1$ ,*

$$(2.7) \quad \|\Pi_\lambda \phi\|_{L^p} \leq C \|\phi\|_{L^p},$$

$$(2.8) \quad \|\Pi_\lambda \phi - \phi\|_{L^p} \leq \frac{C}{\sqrt{\lambda}} \|H^{1/2} \phi\|_{L^p}.$$

$$(2.9) \quad \|\Pi_\lambda \phi - \phi\|_{L^p} \leq \frac{C}{\lambda} \|H \phi\|_{L^p}.$$

*Remark 2.17.* If we consider only the case  $p = 2$ , the above lemma is a direct consequence of Proposition 2.3, since  $\chi$  is bounded, as well as

$$g_j(z) = \frac{\chi^2(z/\lambda) - 1}{z^{j/2}} = \frac{1}{\lambda^{j/2}} \frac{\chi^2(\zeta) - 1}{\zeta^{j/2}} \Big|_{\zeta=z/\lambda}, \quad j = 1, 2,$$

is bounded, with  $\|g_j\|_{L^\infty} \lesssim \lambda^{-j/2}$ . However, we will need the above inequalities in the case  $p \neq 2$ , for which pseudodifferential calculus turns out to be very helpful.

*Proof.* The first inequality is a direct consequence of (2.6) and Proposition 2.2.

To prove (2.8) and (2.9), we invoke Proposition 2.9 (with  $N = 2$ ): if  $\beta = 1$  or  $2$ ,

$$\Pi_\lambda - 1 = \frac{1}{\lambda^{\beta/2}} \lambda^{\beta/2} \left( (\Pi_\lambda - 1) \left( \text{Op } \mathbf{q}^{\beta/2} \right)^w H^{\beta/2} + (\Pi_\lambda - 1) R_\beta \right),$$

with  $R_\beta \in OPS(m^{-2}, g_1)$ . In view of our assumption on  $\chi$ , the symbol  $\pi_\lambda$ , given in Proposition 2.15, satisfies:

$$\lambda^{\beta/2} (\pi_\lambda - 1) = \left( \frac{\sqrt{\lambda}}{m} \right)^\beta (\pi_\lambda - 1) m^\beta \in S(m^\beta, g_1), \quad \text{uniformly in } \lambda \geq 1.$$

Therefore, Proposition 2.7 implies that

$$\lambda^{\beta/2} (\Pi_\lambda - 1) \left( \text{Op } \mathbf{q}^{\beta/2} \right)^w \quad \text{and} \quad \lambda^{\beta/2} (\Pi_\lambda - 1) R_\beta$$

are pseudodifferential in  $OPS(1, g_1)$ , whose symbols have seminorms uniformly bounded for  $\lambda \geq 1$ . Proposition 2.10 then concludes the proof.  $\square$

**Lemma 2.18.** *For any  $1 < p < \infty$ , there exists  $C$  such that for any  $\phi \in \mathcal{S}(\mathbb{R}^d)$  and any  $\lambda \geq 1$ ,*

$$\left\| \Pi_\lambda H^{1/2} \phi \right\|_{L^p} \leq C \sqrt{\lambda} \|\phi\|_{L^p}.$$

*Proof.* In view of Proposition 2.8,  $H^{1/2} \in OPS(m, g_1)$ , and from Proposition 2.15,  $\Pi_\lambda \in OPS(1, g_1)$ , so Proposition 2.7 implies that the operator  $\Pi_\lambda H^{1/2} = H^{1/2} \Pi_\lambda$  is a pseudodifferential operator,  $\Pi_\lambda H^{1/2} \in OPS(m, g_1)$ . Propositions 2.7, 2.8 and 2.15 also entail that the symbol of  $\Pi_\lambda H^{1/2}$  is

$$\sqrt{\mathbf{p}} \chi^2 \left( \frac{\mathbf{p}}{\lambda} \right) + \mathbf{q}_1 \sqrt{\frac{\mathbf{p}}{\lambda}} \chi \chi' \left( \frac{\mathbf{p}}{\lambda} \right) + \frac{1}{\lambda} \rho_\lambda,$$

where  $\rho_\lambda \in S(1, g_1)$  has its seminorms bounded uniformly in  $\lambda \geq 1$ . Writing

$$\sqrt{\mathbf{p}}\chi^2\left(\frac{\mathbf{p}}{\lambda}\right) = \sqrt{\lambda}\sqrt{\frac{\mathbf{p}}{\lambda}}\chi^2\left(\frac{\mathbf{p}}{\lambda}\right),$$

Proposition 2.10 yields the result.  $\square$

We conclude this section with an easy generalization of Bernstein inequality:

**Lemma 2.19.** *For any  $1 \leq p \leq q \leq \infty$ , there exists  $C$  such that for any  $\phi \in \mathcal{S}(\mathbb{R}^d)$  and any  $\lambda \geq 1$ ,*

$$\|\Pi_\lambda \phi\|_{L^q(\mathbb{R}^d)} \leq C\lambda^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi\|_{L^p(\mathbb{R}^d)}.$$

*Proof.* This follows directly from Proposition 2.2 and (2.6): if  $r$  is finite,

$$\|K_\lambda(x, \cdot)\|_{L^r}^r + \|K_\lambda(\cdot, y)\|_{L^r}^r \lesssim \int_{\mathbb{R}^d} \frac{\lambda^{rd/2}}{\left(1 + (\sqrt{\lambda}|z|)^{d+1}\right)^r} dz \lesssim \lambda^{(r-1)d/2},$$

and, if  $r = \infty$ ,  $\|K_\lambda\|_{L_{x,y}^\infty} \lesssim \lambda^{d/2}$ .  $\square$

### 3. STRICHARTZ ESTIMATES

We recall Strichartz inequalities in the case of continuous time, and prove their analogue in the discrete case, thanks to the operator  $\Pi_\lambda$ .

#### 3.1. Continuous time.

**Definition 3.1.** *A pair  $(q, r)$  is admissible if  $2 \leq r < \frac{2d}{d-2}$  ( $2 \leq r \leq \infty$  if  $d = 1$ ,  $2 \leq r < \infty$  if  $d = 2$ ) and*

$$\frac{2}{q} = \delta(r) := d\left(\frac{1}{2} - \frac{1}{r}\right).$$

*Remark 3.2.* We note that the range for  $q$  is equivalent to:  $q \in (2, \infty]$  if  $d \geq 2$  (we do not consider the endpoint case), and  $q \in [4, \infty]$  if  $d = 1$ .

It is a direct consequence of the main result in [16] that the group  $S(t)$  satisfies local in time dispersive estimates, in the sense that there exists  $\delta > 0$  such for all  $\phi \in \mathcal{S}(\mathbb{R}^d)$ ,

$$(3.1) \quad \|S(t)\phi\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{|t|^{d/2}}\|\phi\|_{L^1(\mathbb{R}^d)}, \quad |t| \leq \delta.$$

The fact that such estimates are local in time may be sharp under Assumption 1.1, even though the restriction  $|t| \leq \delta$  can be removed when  $V \equiv 0$ . Typically in the harmonic case  $V(x) = |x|^2$ , Hermite functions  $\varphi_n$ ,  $n \geq 0$ , satisfy  $H\varphi_n = \lambda_n\varphi_n$  for  $\lambda_n \in d + 2\mathbb{N}$ , and  $S(t)\varphi_n(x) = e^{-it\lambda_n}\varphi_n(x)$  has a constant  $L^\infty$ -norm in space.

Together with the fact that  $S(t)$  is unitary on  $L^2(\mathbb{R}^d)$ , we infer from [27] the classical result (see e.g. [12, Section 9.2]):

**Proposition 3.3** (Strichartz estimates). *Let  $d \geq 1$  and  $S(t) = e^{-itH}$ .*

(1) Homogeneous estimates. *For any admissible pair  $(q, r)$ , there exists  $C_q$  such that for any  $T > 0$ ,*

$$\|S(t)\phi\|_{L^q([0, T]; L^r)} \leq C_q T^{1/q}\|\phi\|_{L^2}, \quad \forall \phi \in L^2.$$



(2) Inhomogeneous estimates. *Denote*

$$D(F)(t, x) = \int_0^t S(t-s)F(s, x)ds.$$

For all admissible pairs  $(a_1, b_1)$  and  $(a_2, b_2)$ , there exists  $C = C_{a_1, a_2}$  such that for any bounded interval  $I \ni 0$  such that  $|I| \leq 1$ ,

$$(3.2) \quad \|D(F)\|_{L^{a_1}(I; L^{b_1})} \leq C \|F\|_{L^{a_2'}(I; L^{b_2'})}, \quad \forall F \in L^{a_2'}(I; L^{b_2'}).$$

**3.2. Discrete time.** As pointed out in [23] in the case  $V = 0$ , discrete in time Strichartz estimates cannot be true without a frequency cutoff. One way to understand this consists in recalling that the proof of Strichartz estimates relies on Hardy-Littlewood-Sobolev inequality (fractional integration), and the discrete version of this result requires some care regarding the behavior of the kernel at the origin, as integration is replaced by summation. We give more details below to explain this phenomenon, by revisiting a result from [37]:

**Lemma 3.4.** *Let  $0 < \alpha < 1$ , and consider a kernel satisfying, for some  $c_0, c_1 > 0$ ,*

$$|k(n)| \leq \frac{1}{c_0 + c_1|n|^\alpha}, \quad n \in \mathbb{Z}.$$

Then the operator  $I$  defined by  $If = f * k$  is continuous from  $\ell^p(\mathbb{Z})$  to  $\ell^q(\mathbb{Z})$  provided that  $1 < p < q < \infty$  and

$$\frac{1}{q} = \frac{1}{p} - 1 + \alpha.$$

There exists  $C$  independent of  $c_0$  and  $c_1$  such that for all  $f \in \ell^p(\mathbb{Z})$ ,

$$\|If\|_{\ell^q(\mathbb{Z})} \leq C \max\left(\frac{1}{c_0}, \frac{1}{c_1}\right) \|f\|_{\ell^p(\mathbb{Z})}.$$

*Proof.* We single out the diagonal term as follows:

$$\begin{aligned} \|If\|_{\ell^q(\mathbb{Z})}^q &= \sum_{n \in \mathbb{Z}} |If(n)|^q = \sum_{n \in \mathbb{Z}} \left| \sum_{m \in \mathbb{Z}} f(n-m)k(m) \right|^q \\ &\lesssim \sum_{n \in \mathbb{Z}} \left( \left| \sum_{m \in \mathbb{Z} \setminus \{0\}} f(n-m)k(m) \right|^q + |f(n)k(0)|^q \right) \\ &\lesssim \sum_{n \in \mathbb{Z}} \left( \left| \sum_{m \in \mathbb{Z} \setminus \{0\}} f(n-m) \frac{1}{c_1|m|^\alpha} \right|^q + \frac{1}{c_0^q} |f(n)|^q \right) \end{aligned}$$

The first (double) sum is estimated thanks to [37, Proposition (a)], by

$$\sum_{n \in \mathbb{Z}} \left| \sum_{m \in \mathbb{Z} \setminus \{0\}} f(n-m) \frac{1}{c_1|m|^\alpha} \right|^q \lesssim \frac{1}{c_1^q} \|f\|_{\ell^p(\mathbb{Z})}^q.$$

The second sum is equal to

$$\frac{1}{c_0^q} \|f\|_{\ell^q(\mathbb{Z})}^q \leq \frac{1}{c_1^q} \|f\|_{\ell^p(\mathbb{Z})}^q,$$

since  $\ell^p(\mathbb{Z}) \subset \ell^q(\mathbb{Z})$  from the assumption  $p < q$ . The lemma easily follows.  $\square$

**Lemma 3.5.** *There exist  $C > 0$  and  $\delta > 0$  such that for all  $\lambda > 0$ ,  $S_\lambda$ , defined by (1.5), satisfies:*

$$\|S_\lambda(t)\phi\|_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{\lambda^{-d/2} + |t|^{d/2}} \|\phi\|_{L^1(\mathbb{R}^d)}, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d), \quad |t| \leq \delta.$$

*Proof.* First, we use the dispersive estimate (3.1) from [16], and write, for  $\phi \in \mathcal{S}(\mathbb{R}^d)$  and  $|t| \leq \delta$ ,

$$\|S_\lambda(t)\phi\|_{L^\infty(\mathbb{R}^d)} = \|S(t)\Pi_\lambda\phi\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \|\Pi_\lambda\phi\|_{L^1(\mathbb{R}^d)}.$$

Choosing for instance  $N = d + 1$  in (2.6) yields

$$\|K_\lambda\|_{L_y^\infty L_x^1} \lesssim 1.$$

Fubini Theorem implies

$$\|\Pi_\lambda\phi\|_{L^1(\mathbb{R}^d)} \lesssim \|\phi\|_{L^1(\mathbb{R}^d)}.$$

This yields the lemma for  $|t| > 1/\lambda$ . For small values of  $t$ , we write

$$S_\lambda(t) = \underbrace{\chi\left(\frac{H}{\lambda}\right)}_{L^2 \rightarrow L^\infty} \circ \underbrace{e^{-itH}}_{L^2 \rightarrow L^2} \circ \underbrace{\chi\left(\frac{H}{\lambda}\right)}_{L^1 \rightarrow L^2},$$

and invoke the classical  $TT^*$  argument, which entails, since  $\chi\left(\frac{H}{\lambda}\right)$  is self-adjoint,

$$\left\| \chi\left(\frac{H}{\lambda}\right) \right\|_{L^2 \rightarrow L^\infty} = \left\| \chi\left(\frac{H}{\lambda}\right) \right\|_{L^1 \rightarrow L^2} = \left\| \chi^2\left(\frac{H}{\lambda}\right) \right\|_{L^1 \rightarrow L^\infty}^{1/2} = \|\Pi_\lambda\|_{L^1 \rightarrow L^\infty}^{1/2},$$

so we can write, since  $e^{-itH}$  is unitary on  $L^2$ ,

$$\|S_\lambda(t)\phi\|_{L^\infty(\mathbb{R}^d)} \leq \|\Pi_\lambda\|_{L^1 \rightarrow L^\infty} \|\phi\|_{L^1(\mathbb{R}^d)}.$$

We readily have  $\|\Pi_\lambda\|_{L^1 \rightarrow L^\infty} \leq \|K_\lambda\|_{L_{x,y}^\infty} \lesssim \lambda^{d/2}$ , where the last inequality stems from (2.6). This completes the proof of the lemma.  $\square$

For any interval  $I \subset [0, \infty)$ , we define the space  $\ell^q(n\tau \in I; L^r(\mathbb{R}^d))$ , or simply  $\ell^q(I; L^r)$ , as consisting of functions defined on  $\tau\mathbb{Z} \cap I$  with values in  $L^r(\mathbb{R}^d)$ , the norm of which is given by

$$(3.3) \quad \|u\|_{\ell^q(I; L^r)} = \begin{cases} \left( \tau \sum_{n\tau \in I} \|u(n\tau)\|_{L^r(\mathbb{R}^d)}^q \right)^{1/q} & \text{if } 1 \leq q < \infty, \\ \sup_{n\tau \in I} \|u(n\tau)\|_{L^r(\mathbb{R}^d)} & \text{if } q = \infty. \end{cases}$$

As  $\lambda \geq 1$  (we will choose  $\lambda \rightarrow \infty$  as  $\tau \rightarrow 0$ ), Lemma 3.4 shows that in Lemma 3.5, the factor involving  $\lambda$  is dominant in the discrete Strichartz estimate: proceeding like in [31], we infer

**Proposition 3.6.** *Let  $(q, r)$ ,  $(a_1, b_1)$  and  $(a_2, b_2)$  be admissible pairs. Then, there exist  $C_{d,q}, C_{d,a_1,a_2} > 0$  such that for any bounded interval  $I$  such that  $|I| \leq 1$ , provided that  $\lambda\tau \geq 1$ ,*

$$(3.4) \quad \|S_\lambda(\cdot)\phi\|_{\ell^q(I; L^r)} \leq C_{d,q} (\lambda\tau)^{1/q} \|\phi\|_{L^2},$$

and

$$(3.5) \quad \left\| \tau \sum_{\substack{k=-\infty \\ k\tau \in I}}^{n-1} S_\lambda((n-k)\tau) f(k\tau) \right\|_{\ell^{a_1}(I; L^{b_1})} \leq C_{d, a_1, a_2} (\lambda\tau)^{\frac{1}{a_1} + \frac{1}{a_2}} \|f\|_{\ell^{a'_2}(I; L^{b'_2})}$$

hold for all  $\phi \in L^2$  and  $f \in \ell^{a'_2}(I; L^{b'_2})$ .

Similarly, the analogue of [23, Lemma 4.5] (see also [14, Corollary 2.4]) is:

**Corollary 3.7.** *Let  $(a_1, b_1)$  and  $(a_2, b_2)$  be admissible pairs, and  $\lambda = 1/\tau$ . There exists  $C = C(d, a_1, a_2)$  such that for any bounded interval  $I$  such that  $|I| \leq 1$ ,*

$$\left\| \int_{\substack{s < n\tau \\ s \in I}} S_\lambda(n\tau - s) f(s) ds \right\|_{\ell^{a_1}(I; L^{b_1})} \leq C \|f\|_{L^{a'_2}(I; L^{b'_2})}, \quad \forall f \in L^{a'_2}(I; L^{b'_2}).$$

#### 4. CAUCHY PROBLEM

**4.1. Well-posedness in  $\mathcal{H}^1$ .** We adapt well-posedness results known in  $\Sigma$  for (1.1), under Assumptions 1.1. For a time interval  $I$ , denote

$$(4.1) \quad \|u\|_{X^1(I)} = \max_{A \in \{\mathbf{1}, \nabla, \sqrt{V}\}} \sup_{(q,r) \text{ admissible}} \|Au\|_{L^q(I; L^r)},$$

and let  $X^1(I)$  be the corresponding (Banach) space. Note that  $X^1(I) \subset C(I; \mathcal{H}^1)$ . When  $I = [0, T]$ , we simply denote  $X^1([0, T])$  by  $X_T^1$ .

**Proposition 4.1.** *Let Assumption 1.1 be verified and  $0 < \sigma < \frac{2}{(d-2)_+}$ . For any  $u_0 \in \mathcal{H}^1$ , there exist  $T > 0$  a unique solution  $u \in X_T^1$  to (1.1). In addition, its mass and energy are independent of time,*

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |u(t, x)|^2 dx &= 0, \\ \frac{d}{dt} \int_{\mathbb{R}^d} \left( |\nabla u(t, x)|^2 + V(x)|u(t, x)|^2 + W(x)|u(t, x)|^2 + \frac{\varepsilon}{\sigma+1} |u(t, x)|^{2\sigma+2} \right) dx &= 0. \end{aligned}$$

Finally, the above statement is true for all  $T > 0$  (global solution) if  $0 < \sigma < 2/d$  ( $L^2$ -subcritical case), or  $\varepsilon = +1$  (defocusing case).

*Main steps of the proof.* Denote by

$$(q_0, r_0) = \left( \frac{4\sigma + 4}{d\sigma}, 2\sigma + 2 \right),$$

the pair present in the above statement, which is admissible (see Definition 3.1).

Local existence follows from a fixed point argument on Duhamel's formula associated to (1.1),

$$(4.2) \quad u(t) = S(t)u_0 - i \int_0^t S(t-s) (Wu)(s) ds - i\varepsilon \int_0^t S(t-s) (|u|^{2\sigma}u)(s) ds,$$

in a ball of the space

$$Z_T = \{u \in C([0, T]; \mathcal{H}^1); u, \nabla u, u\sqrt{V} \in L^{q_0}(0, T; L^{r_0}(\mathbb{R}^d))\},$$

which, equipped with the distance

$$d(u, v) = \|u - v\|_{L_T^{q_0} L^{r_0}} + \|u - v\|_{L_T^\infty L^2},$$

is a complete Banach space (see e.g. [12, Section 4.4]). Let

$$(4.3) \quad \theta = \frac{2\sigma(2\sigma + 2)}{2 - (d-2)\sigma}.$$

Then we have

$$\frac{1}{q'_0} = \frac{2\sigma}{\theta} + \frac{1}{q_0} \quad ; \quad \frac{1}{r'_0} = \frac{2\sigma}{r_0} + \frac{1}{r_0}.$$

Denoting by  $\Phi(u)(t)$  the right hand side of (4.2), Strichartz estimates (Proposition 3.3) yield, for  $(q, r)$  admissible,

$$(4.4) \quad \begin{aligned} \|\Phi(u)\|_{L_T^q L^r} &\lesssim \|u_0\|_{L^2} + \|Wu\|_{L_T^1 L^2} + \||u|^{2\sigma}u\|_{L_T^{q'_0} L^{r'_0}} \\ &\lesssim \|u_0\|_{L^2} + T\|W\|_{L^\infty}\|u\|_{L_T^\infty L^2} + \|u\|_{L_T^{q'_0} L^{r'_0}}^{2\sigma} \|u\|_{L_T^{q_0} L^{r_0}}, \end{aligned}$$

where we have used Hölder inequality. In view of Sobolev embedding,

$$(4.5) \quad \|\Phi(u)\|_{L_T^q L^r} \leq C\|u_0\|_{L^2} + T\|W\|_{L^\infty}\|u\|_{L_T^\infty L^2} + CT^{2\sigma/\theta}\|u\|_{L_T^\infty H^1}^{2\sigma} \|u\|_{L_T^{q_0} L^{r_0}}.$$

Unlike in the case without potential,  $V = 0$ , the group  $S(t)$  and the operator  $\nabla$  do not commute: we may either consider the commutators  $[\nabla, H] = \nabla V$  and  $[\sqrt{V}, H] = -\nabla\sqrt{V} \cdot \nabla - \frac{1}{2}\Delta\sqrt{V}$ , and get a closed system of estimates (see e.g. [10]), or use directly the operator  $\sqrt{H}$ , and rely on Lemma 2.12, which implies

$$Z_T = \{u \in C([0, T]; \mathcal{H}^1); \sqrt{H}u \in L^{q_0}(0, T; L^{r_0}(\mathbb{R}^d))\}.$$

We choose the latter option, as it paves the way to the analysis of error estimates in the discrete in time setting. We have

$$\begin{aligned} \sqrt{H}\Phi(u)(t) &= \sqrt{H}S(t)u_0 - i\sqrt{H} \int_0^t S(t-s)(Wu)(s)ds \\ &\quad - i\varepsilon\sqrt{H} \int_0^t S(t-s)(|u|^{2\sigma}u)(s)ds \\ &= S(t)\sqrt{H}u_0 - i \int_0^t S(t-s)\sqrt{H}(Wu)(s)ds \\ &\quad - i\varepsilon \int_0^t S(t-s)\sqrt{H}(|u|^{2\sigma}u)(s)ds. \end{aligned}$$

We resume the same Lebesgue exponents as above when applying Strichartz estimates:

$$\|\sqrt{H}\Phi(u)\|_{L_T^q L^r} \lesssim \|\sqrt{H}u_0\|_{L^2} + \|\sqrt{H}(Wu)\|_{L_T^1 L^2} + \left\| \sqrt{H}(|u|^{2\sigma}u) \right\|_{L_T^{q'_0} L^{r'_0}}.$$

In view of (2.4) and Hölder inequality like in (4.4),

$$\left\| \sqrt{H}(|u|^{2\sigma}u) \right\|_{L_T^{q'_0} L^{r'_0}} \lesssim \|u\|_{L_T^{q_0} L^{r_0}}^{2\sigma} \|\sqrt{H}u\|_{L_T^{q_0} L^{r_0}}.$$

Using Sobolev embedding, Lemma 2.12, and (2.4) again,

$$\begin{aligned} \|\sqrt{H}\Phi(u)\|_{L_T^q L^r} &\lesssim \|\sqrt{H}u_0\|_{L^2} + T\|\sqrt{H}u\|_{L_T^\infty L^2} + T^{1/\theta}\|u\|_{L_T^\infty H^1}^{2\sigma} \|\sqrt{H}u\|_{L_T^{q_0} L^{r_0}} \\ &\lesssim \|\sqrt{H}u_0\|_{L^2} + T\|\sqrt{H}u\|_{L_T^\infty L^2} + T^{1/\theta}\|\sqrt{H}u\|_{L_T^\infty L^2}^{2\sigma} \|\sqrt{H}u\|_{L_T^{q_0} L^{r_0}}. \end{aligned}$$

Choosing successively  $(q, r) = (\infty, 2)$  and  $(q_0, r_0)$ , we have a closed system of inequalities, and so picking  $T = T(\|u_0\|_{\mathcal{H}^1})$  sufficiently small, we can prove that  $\Phi$

maps a suitable ball in  $Z_T$  into itself. Contraction for the norm  $\|\cdot\|_{L_T^{q_0} L^{r_0}} + \|\cdot\|_{L_T^\infty L^2}$  is proved similarly, and local existence follows.

Uniqueness stems from the same estimates as above. We refer to [12] for the rigorous justification of the conservation of mass and energy. Globalization is a consequence of these conservations, and Gagliardo–Nirenberg inequality when  $\varepsilon = -1$  and  $\sigma < 2/d$ .  $\square$

**4.2. Well-posedness in  $\mathcal{H}^2$ .** For  $T > 0$ , denote

$$(4.6) \quad \|u\|_{X^2(I)} = \max_{A \in \{1, \Delta, V\}} \sup_{(q,r) \text{ admissible}} \|Au\|_{L^q(I; L^r)},$$

and let  $X^2(I)$  be the corresponding (Banach) space. Note that  $X^2(I) \subset C(I; \mathcal{H}^2)$ . When  $I = [0, T]$ , we simply denote  $X^2([0, T])$  by  $X_T^2$ .

**Proposition 4.2.** *Let Assumption 1.1 be verified and  $0 < \sigma < \frac{2}{(d-2)_+}$ . For any  $u_0 \in \mathcal{H}^2$ , there exist  $T > 0$  a unique solution  $u \in X_T^1 \cap C([0, T]; \mathcal{H}^2)$  to (1.1). If in addition  $\sigma \geq 1/2$ ,  $u \in X_T^2$ . The time  $T > 0$  can be chosen arbitrarily large (global solution) if  $0 < \sigma < 2/d$  ( $L^2$ -subcritical case), or  $\varepsilon = +1$  (defocusing case).*

*Proof.* Since  $\mathcal{H}^2 \subset \mathcal{H}^1$ , Proposition 4.1 provides a local solution in  $X_T^1$  for some  $T > 0$ , which is arbitrarily large in the cases addressed at the end of the statement of the proposition. Arguing as in [12, Section 5.3], we note that applying the operator  $\partial_t$  in (1.1), we find that  $\partial_t u$  solves

$$(i\partial_t - H)\partial_t u = W\partial_t u + \varepsilon\partial_t(|u|^{2\sigma}u), \quad \partial_t u|_{t=0} = -iHu_0 - iWu_0 - i\varepsilon|u_0|^{2\sigma}u_0.$$

We check that since  $u_0 \in \mathcal{H}^2 \subset H^2(\mathbb{R}^d)$ ,  $|u_0|^{2\sigma}u_0 \in L^2(\mathbb{R}^d)$  by Sobolev embedding  $H^2 \subset H^{d\sigma/(2\sigma+1)} \subset L^{4\sigma+2}$ . We also note that in view of Lemma 2.12,

$$\mathcal{H}^2 = \{\phi \in L^2(\mathbb{R}^d), \quad H\phi \in L^2(\mathbb{R}^d)\},$$

and, in the sense of the equivalence of norms,

$$\|u\|_{X^2(I)} \approx \sup_{(q,r) \text{ admissible}} \|Hu\|_{L^q(I; L^r)}.$$

Writing Duhamel's formula for  $\partial_t u$  and resuming the estimates from the proof of Proposition 4.1, we see that  $\partial_t u \in L_T^q L^r$  for all admissible pairs  $(q, r)$ . In particular,  $\partial_t u \in C([0, T]; L^2)$  (see e.g. [12] for continuity). At this stage, we know:

$$\underbrace{i\partial_t u}_{\in C([0, T]; L^2)} = Hu + \underbrace{Wu}_{\in C([0, T]; L^2)} + \varepsilon|u|^{2\sigma}u.$$

We show that the linear term  $Hu$  controls the nonlinear term, in  $L^2$ :

$$\|\phi\|_{L^{4\sigma+2}}^{2\sigma+1} \lesssim \|\phi\|_{H^{d\sigma/(2\sigma+1)}}^{2\sigma+1} \lesssim \|\phi\|_{H^1}^{2\sigma} \|\phi\|_{H^s},$$

with  $s = 1$  if  $d\sigma \leq 2\sigma + 1$ , and  $s = (d-2)\sigma < 2$  if  $d\sigma > 2\sigma + 1$ , so there is always  $\eta \in (0, 1)$  such that

$$\|\phi\|_{L^{4\sigma+2}}^{2\sigma+1} \lesssim \|\phi\|_{H^1}^{2\sigma+1-\eta} \|\phi\|_{H^2}^\eta \lesssim \|\sqrt{H}\phi\|_{L^2}^{2\sigma+1-\eta} \|H\phi\|_{L^2}^\eta.$$

In view of Young inequality, since  $u \in X_T^1$ , there exists  $C$  such that for all  $t \in [0, T]$ ,

$$\|u(t)\|_{L^{4\sigma+2}}^{2\sigma+1} \leq C + \frac{1}{4} \|Hu(t)\|_{L^2}.$$

We infer that  $Hu \in C([0, T]; L^2)$ , hence  $u \in C([0, T]; \mathcal{H}^2)$ . If in addition  $\sigma \geq 1/2$ , we can differentiate the nonlinearity twice, and so

$$(i\partial_t - H)Hu = H(Wu) + \varepsilon H(|u|^{2\sigma}u) = \Delta(Wu) + VWu + \varepsilon\Delta(|u|^{2\sigma}u) + \varepsilon V|u|^{2\sigma}u.$$

We can then prove that  $u \in X_T^2$ . We emphasize that unlike what is claimed in [14, 23], even in the case  $V = 0$ , the proof that  $\Delta u \in L_T^q L^r$  for admissible pairs with  $r \neq 2$  is not stated in [12, Section 5.3]. To simplify the presentation, we assume  $W = 0$ , as the corresponding term bears no extra difficulty. We distinguish two cases:

- If  $0 < \sigma < 2/d$ , we have  $1/q_0 < 1/\theta$ , and thus

$$\|u\|_{L_T^\theta L^{r_0}} \leq T^{1/\theta - 1/q_0} \|u\|_{L_T^{q_0} L^{r_0}}.$$

Strichartz estimates like in (4.4) Lemma 2.12 yield, on  $I = [t_j, t_{j+1}] \subset [0, T]$ ,

$$\begin{aligned} \|Hu\|_{L^q(I; L^r)} &\lesssim \|Hu(t_j)\|_{L^2} + \|u\|_{L^1(I; \mathcal{H}^2)} + \|u\|_{L_T^{2\sigma}(I; L^{r_0})} \|Vu\|_{L^{q_0}(I; L^{r_0})} \\ &+ \|u\|_{L_T^{2\sigma-1}(I; L^{r_0})} (\|\nabla u\|_{L^\theta(I; L^{r_0})} \|\nabla u\|_{L^{q_0}(I; L^{r_0})} + \|u\|_{L^\theta(I; L^{r_0})} \|\Delta u\|_{L^{q_0}(I; L^{r_0})}) \\ &\lesssim \|Hu(t_j)\|_{L^2} + |I| \|u\|_{L^\infty(I; \mathcal{H}^2)} + \|u\|_{L_T^{2\sigma}(I; L^{r_0})} \|Hu\|_{L^{q_0}(I; L^{r_0})} \\ &+ \|u\|_{L_T^{2\sigma-1}(I; L^{r_0})} \|\sqrt{H}u\|_{L^{q_0}(I; L^{r_0})}^2. \end{aligned}$$

If  $u \in X_T^1$ , then in particular  $u \in L^{q_0}([0, T]; L^{r_0})$ , so we can write  $[0, T]$  as the union of finitely many intervals on which  $\|u\|_{L^{q_0}(I; L^{r_0})}$  is sufficiently small so the nonlinear terms are absorbed by the left hand side when choosing  $(q, r) = (q_0, r_0)$ , up to doubling the constants on each intervals  $[t_j, t_{j+1}]$ . We infer  $Hu \in L_T^{q_0} L^{r_0}$ , and resuming the above estimate with  $(q, r)$  an arbitrary admissible pair, we conclude that  $u \in X_T^2$ .

- If  $2/d \leq \sigma < 2/(d-2)_+$ , we resume the argument from [11, Section 2.2], and change the above estimates to

$$\begin{aligned} \|Hu\|_{L^q(I; L^r)} &\lesssim \|Hu(t_j)\|_{L^2} + \|Hu\|_{L_T^\infty L^2} + \|u\|_{L_T^{2\sigma}(I; L^{r_0})} \|Vu\|_{L^{q_0}(I; L^{r_0})} \\ &+ \|u\|_{L_T^{2\sigma-1}(I; L^{r_0})} (\|\nabla u\|_{L^\theta(I; L^{r_0})} \|\nabla u\|_{L^{q_0}(I; L^{r_0})} + \|u\|_{L^\theta(I; L^{r_0})} \|\Delta u\|_{L^{q_0}(I; L^{r_0})}) \\ &\lesssim \|Hu(t_j)\|_{L^2} + \|Hu\|_{L_T^\infty L^2} + \|u\|_{L_T^{2\sigma}(I; L^{r_0})} \|Vu\|_{L^{q_0}(I; L^{r_0})} \\ &+ \|u\|_{L_T^{2\sigma-1}(I; L^{r_0})} (\|u\|_{L^\theta(I; W^{2, b_1})} \|\nabla u\|_{L^{q_0}(I; L^{r_0})} + \|u\|_{L^\theta(I; L^{r_0})} \|\Delta u\|_{L^{q_0}(I; L^{r_0})}), \end{aligned}$$

where  $b_1 \geq 2$  is such that  $(\theta, b_1)$  is admissible: indeed,  $\theta > 2$  when  $\sigma \geq 2/d$ , and moreover  $W^{s, b_1}(\mathbb{R}^d) \hookrightarrow L^{r_0}(\mathbb{R}^d)$  with

$$s = d \left( \frac{1}{b_1} - \frac{1}{r_0} \right) = \frac{2}{q_0} - \frac{2}{\theta} = \frac{d\sigma - 2}{2\sigma} \in [0, 1).$$

If  $u \in X_T^1$ , then in particular  $u, \nabla u \in L^{q_0}([0, T]; L^{r_0})$ , and we can follow essentially the same lines as in the case  $\sigma < 2/d$ .  $\square$

The following corollary explains why, in the statements of Theorems 1.5 and 1.6, we do not assume that  $u \in X_T^1$  or  $u \in X_T^2$ , even though the proof of these results will rely on such properties:

**Corollary 4.3.** *Let  $u_0 \in \mathcal{H}^1$ . Either the solution provided by Proposition 4.1 is global, in the sense that  $T > 0$  is arbitrary, or there exists  $T^* > 0$  such that*

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \xrightarrow[t \rightarrow T^*]{} +\infty.$$

In particular, if  $u \in C([0, T]; \mathcal{H}^1)$ , then  $u \in X_T^1$ .

If in addition  $u_0 \in \mathcal{H}^2$  and  $\sigma \geq 1/2$ , a similar statement holds: if  $u \in C([0, T]; \mathcal{H}^1)$ , then  $u \in X_T^2$ .

*Proof.* The construction of a solution in the proof of Proposition 4.1 relies on a fixed point argument, which provides a local existence time  $T = T(\|u_0\|_{\mathcal{H}^1})$ . Standard ODE arguments imply that either the solution is global,  $u \in X_T^1$  for all  $T > 0$ , or its  $\mathcal{H}^1$ -norm becomes infinite in finite time,

$$\exists T^* > 0, \quad \|u(t)\|_{\mathcal{H}^1} \xrightarrow{t \rightarrow T^*} +\infty.$$

Recall that the  $L^2$ -norm of  $u$  is independent of time. We now examine the conservation of the energy. The conservation of mass implies  $Wu \in L^\infty([0, T^*]; L^2)$ . If  $\nabla u \in L^\infty([0, T^*]; L^2)$ , then the Sobolev embedding  $H^1(\mathbb{R}^d) \hookrightarrow L^{2\sigma+2}(\mathbb{R}^d)$  implies that  $u \in L^\infty([0, T^*]; L^{2\sigma+2})$ . In the conserved energy, three terms out of four are bounded, so we infer  $u\sqrt{V} \in L^\infty([0, T^*]; L^2)$ , and thus  $u \in L^\infty([0, T^*]; \mathcal{H}^1)$ : therefore, either the solution is global, or

$$\exists T^* > 0, \quad \|\nabla u(t)\|_{L^2} \xrightarrow{t \rightarrow T^*} +\infty.$$

In view of [12, Proposition 4.2.1], uniqueness holds for solutions of (1.1) which belong to  $C([0, T]; \mathcal{H}^1)$ , hence the first part of the corollary. In the proof of Proposition 4.2, we have seen that if  $u \in X_T^1$ , then  $u \in X_T^2$ , hence the corollary.  $\square$

## 5. STABILITY IMPLIES CONVERGENCE

The goal of this section is to establish the convergence results of Theorems 1.5 and 1.6, when stability is assumed:

**Theorem 5.1.** *Let Assumption 1.1 be verified. Suppose that for some  $T > 0$ , (1.1) has a unique solution  $u \in X_T^1$ . Denote by  $u^n$  the sequence defined by the scheme (1.6), where we set  $\lambda = 1/\tau$ . Suppose that there exist  $\gamma, M > 0$  such that for all interval  $I \subset [0, T]$ , the numerical solution satisfies*

$$(5.1) \quad \|u^n\|_{\ell^\theta(I; L^{r_0})} \leq |I|^\gamma M, \quad \forall \tau \in (0, 1), \quad \theta = \frac{2\sigma(2\sigma+2)}{2-(d-2)\sigma}, \quad r_0 = 2\sigma+2,$$

where the above  $\ell^\theta$  norm is defined in (3.3). Let  $(q, r)$  be an admissible pair.

• Assume that either  $0 < \sigma < 2/d$ , or  $d \leq 5$  and  $2/d \leq \sigma < 2/(d-2)_+$ , with in addition  $\sigma \geq 1/2$  when  $d = 5$ . There exists  $C$  such that

$$\|u^n - u(n\tau)\|_{\ell^q([0, T]; L^r)} \leq C\tau^{1/2}.$$

• If  $1/2 \leq \sigma < 2/(d-2)_+$ , and  $u \in X_T^2$ , then there exists  $C$  such that

$$\|u^n - u(n\tau)\|_{\ell^q([0, T]; L^r)} \leq C\tau.$$

• If  $1/2 \leq \sigma < 2/(d-2)_+$ ,  $u \in X_T^2$ , and, up to increasing  $M$ ,

$$(5.2) \quad \|u^n\|_{\ell^\infty(I; L^{r_0})} \leq M, \quad \forall \tau \in (0, 1),$$

then there exists  $C$  such that

$$\left\| H^{1/2}(u^n - u(n\tau)) \right\|_{\ell^q([0, T]; L^r)} \leq C\tau^{1/2}.$$

Setting  $(q, r) = (\infty, 2)$ , we get the conclusions of Theorems 1.5 and 1.6.

We note that the condition (5.2) is stronger than (5.1), since (5.2) and Hölder inequality imply

$$\|u^n\|_{\ell^\theta(I; L^{r_0})} \leq |I|^{1/\theta} \|u^n\|_{\ell^\infty(I; L^{r_0})} \leq |I|^{1/\theta} M.$$

The proof of this result is very similar to the proof of [23, Theorem 1.2] and [14, Theorem 1.4]. We note however that the assumption (5.1) is weaker than the one made in [14, Theorem 1.4]: it turns out that in the proof of [14, Theorem 1.4], it is precisely (5.1) which is used. We recall the main steps of the arguments, with enough details so it should be clear that only the condition (5.1) is required on the numerical solution, and emphasize the modifications due to the presence of the potential  $V$  here.

**5.1. Preliminary lemmas.** We first note that (2.7), (2.8), Lemmas 2.18 and 2.19 provide the analogue of [14, Lemma 2.6]. Also, contrary to what happens in [14, 23], the operators  $\Pi_\lambda$  and  $\nabla$  do not commute: this is why we consider the operator  $H^{1/2}$  instead of  $\nabla$ , using (2.4). The next result corresponds essentially to [14, Lemma 2.5], except for the last statement, which appears in the proof of [23, Lemma 4.3], and is easily deduced from explicit computations and (5.4):

**Lemma 5.2.** *Denote by  $N_0$  the map  $N$  when  $W = 0$ ,  $N_0(t)\phi = \phi^{-i\epsilon t|\phi|^{2\sigma}}$ . There exists  $c > 0$  such that*

$$(5.3) \quad \left| \frac{N_0(\tau) - \mathbf{1}}{\tau} v - \frac{N_0(\tau) - \mathbf{1}}{\tau} w \right| \leq c (|v|^{2\sigma} + |w|^{2\sigma}) |v - w|$$

and

$$(5.4) \quad \left| \frac{N_0(\tau) - \mathbf{1}}{\tau} v \right| = \left| \frac{\exp(-i\tau\epsilon|v|^{2\sigma}) - 1}{\tau} v \right| \leq |v|^{2\sigma+1}$$

hold for all  $v, w \in \mathbb{C}$ . Furthermore, for  $f$  smooth enough, we have the pointwise estimates

$$(5.5) \quad \left| \nabla \left( \frac{N_0(\tau) - \mathbf{1}}{\tau} f \right) \right| \lesssim |f|^{2\sigma} |\nabla f|,$$

and, if  $\sigma \geq 1/2$  and  $0 < \tau \leq 1$ ,

$$(5.6) \quad \left| \Delta \left( \frac{N_0(\tau) - \mathbf{1}}{\tau} f \right) \right| \lesssim |f|^{2\sigma} |\Delta f| + |f|^{2\sigma-1} |\nabla f|^2 + \tau |f|^{4\sigma-1} |\nabla f|^2.$$

Together with Lemma 2.12 and Hölder inequality, the above lemma entails:

**Lemma 5.3.** *Let  $1 < p < \infty$ . There exist constants such that for all  $f \in \mathcal{S}(\mathbb{R}^d)$ ,*

$$(5.7) \quad \left\| \sqrt{H} \left( \frac{N_0(\tau) - \mathbf{1}}{\tau} f \right) \right\|_{L^p} \lesssim \|f\|_{L^{a_1}}^{2\sigma} \|\sqrt{H}f\|_{L^{a_2}}, \quad \frac{1}{p} = \frac{2\sigma}{a_1} + \frac{1}{a_2},$$

and, if  $\sigma \geq 1/2$  and  $0 < \tau \leq 1$ ,

$$(5.8) \quad \left\| H \left( \frac{N_0(\tau) - \mathbf{1}}{\tau} f \right) \right\|_{L^p} \lesssim \|f\|_{L^{a_1}}^{2\sigma} \|Hf\|_{L^{a_2}} + \|f\|_{L^{a_3}}^{2\sigma-1} \|\sqrt{H}f\|_{L^{a_3}}^2 \\ + \tau \|f\|_{L^{a_5}}^{4\sigma-1} \|\sqrt{H}f\|_{L^{a_6}}^2, \\ \frac{1}{p} = \frac{2\sigma}{a_1} + \frac{1}{a_2} = \frac{2\sigma-1}{a_3} + \frac{2}{a_4} = \frac{4\sigma-1}{a_5} + \frac{2}{a_6},$$

with, for all  $j$ 's,  $a_j \in [1, \infty]$ .



The analogue of [23, Lemma 4.6] and [14, Lemma 2.7] is rather straightforward:

**Lemma 5.4.** *For any admissible pairs  $(a_1, b_1)$  and  $(a_2, b_2)$ , any interval  $I = [0, T]$  of length  $T \leq 1$ , and  $\beta \in \{1, 2\}$ , there is a constant  $C = C_{d, a_1, a_2} > 0$  such that if  $\lambda = 1/\tau$ , for any test function  $f \in \mathcal{S}(\mathbb{R}^{d+1})$ ,*

$$(5.9) \quad \left\| \int_0^{n\tau} S_\lambda(n\tau - s)f(s)ds - \tau \sum_{k=0}^{n-1} S_\lambda(n\tau - k\tau)f(k\tau) \right\|_{\ell^{a_1}(I; L^{b_1})} \\ \leq C\tau^{\beta/2} \|H^{\beta/2}f\|_{L^{a'_2}(I; L^{b'_2})} + C\tau \|\partial_t f\|_{L^{a'_2}(I; L^{b'_2})}.$$

*Main steps of the proof.* In the case  $V = 0$ , this result appears in [23, Lemma 4.6] when  $\beta = 2$ , and in [14, Lemma 2.7] when  $\beta = 1$ . Following the proof of [23, Lemma 4.6], the quantity to estimate is rewritten as

$$\begin{aligned} & \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} (S_\lambda(n\tau - s)f(s) - S_\lambda(n\tau - k\tau)f(k\tau)) ds \\ &= \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} \int_{k\tau}^s \frac{d}{dt} (S_\lambda(n\tau - t)f(t)) dt ds \\ &= \sum_{k=0}^{n-1} \iint_{k\tau < t < s < (k+1)\tau} (iS_\lambda(n\tau - t)Hf(t) + S_\lambda(n\tau - t)\partial_t f(t)) dt ds \\ &= \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} ((k+1)\tau - t) (iS_\lambda(n\tau - t)Hf(t) + S_\lambda(n\tau - t)\partial_t f(t)) dt \\ &= \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} S_\lambda(n\tau - t) ((k+1)\tau - t) (iHf(t) + \partial_t f(t)) dt. \end{aligned}$$

Then (5.9) follows from the inhomogeneous discrete Strichartz estimate from Corollary 3.7, applied to

$$g(t) = \sum_0^{n-1} ((k+1)\tau - t) (iHf(t) + \partial_t f(t)) \mathbf{1}_{(k\tau, (k+1)\tau)}(t),$$

and from triangle inequality. Passing from  $\beta = 2$  to  $\beta = 1$  is a direct application of Lemma 2.18.  $\square$

**5.2. Stability implies convergence in  $L^2$ .** In this section, we prove the first two points of Theorem 5.1. The argument is similar to the one introduced in [23] and extended in [14], so we describe the main steps, and emphasize the main differences due to the present framework.

Denote  $I = [0, T]$ . We estimate  $\mathcal{Z}_\lambda(n\tau)u_0 - \Pi_\lambda u(n\tau)$  instead of  $\mathcal{Z}_\lambda(n\tau)u_0 - u(n\tau)$ . Indeed, Lemma 2.16 yields, for any admissible pair  $(q, r)$ , since  $\lambda = 1/\tau$ ,

$$\|u(n\tau) - \Pi_\lambda u(n\tau)\|_{\ell^q(I; L^r)} \lesssim \frac{1}{\lambda^{\beta/2}} \|H^{\beta/2}u(n\tau)\|_{\ell^q(I; L^r)} \lesssim \tau^{\beta/2},$$

where we assume  $u \in X_T^\beta$ , with  $\beta \in \{1, 2\}$ . Indeed, Corollary 3.7 implies

$$\|H^{\beta/2}u(n\tau)\|_{\ell^q(I; L^r)} \lesssim \|H^{\beta/2}u_0\| + \|H^{\beta/2}Wu\|_{L_T^1 L^2} + \left\| H^{\beta/2}(|u|^{2\sigma}u) \right\|_{L_T^{q'_0} L^{r'_0}},$$

and the right hand side is bounded if  $u \in X_T^\beta$ , as we have seen in the proofs of Proposition 4.1 and Proposition 4.2. To lighten notations, we now write  $Z_\tau(n\tau)$  for  $\mathcal{Z}_\lambda(n\tau)u_0 = u^n$  with  $\lambda = 1/\tau$ . In view of (3.3) and Proposition 4.1,

$$\|u(k\tau)\|_{\ell^\theta(I;L^{r_0})} \lesssim |I|^{1/\theta} \|u\|_{L_T^\infty H^1} \lesssim |I|^{1/\theta} \|u\|_{X_T^1}.$$

Keeping (5.1) into account, we infer that for any  $\eta > 0$ , we can find a finite number  $K = K(\eta)$ , and  $\rho = \rho(\eta) > 0$  with  $\rho/\tau \in \mathbb{N}$  such that, if we set

$$I_j = [j\rho, (j+1)\rho] =: [m_j\tau, m_{j+1}\tau], \quad 0 \leq j \leq K-1,$$

we have

$$[0, T] = \bigcup_{j=0}^{K-1} I_j \cup [K\rho, T] = \bigcup_{j=0}^K I_j,$$

and for  $0 \leq j \leq K$ ,

$$\|u(k\tau)\|_{\ell^\theta(I_j;L^{r_0})} + \|Z_\tau(k\tau)\|_{\ell^\theta(I_j;L^{r_0})} \leq \eta.$$

On each interval  $I_j$ , the discrete Duhamel's formula can be written as

$$(5.10) \quad Z_\tau(m_j\tau + n\tau) = S_\lambda(n\tau)Z_\tau(m_j\tau) + \tau \sum_{k=0}^{n-1} S_\lambda(n\tau - k\tau) \frac{N(\tau) - \mathbf{1}}{\tau} Z_\tau(m_j\tau + k\tau),$$

for  $0 \leq n \leq \rho/\tau$ . Combining this with (4.2), we obtain the following decomposition:

$$(5.11) \quad Z_\tau(m_j\tau + n\tau) - \Pi_\lambda u(m_j\tau + n\tau) = \mathcal{A}_1(j) + \mathcal{A}_2(j) + \mathcal{A}_3(j) + \mathcal{A}_4(j),$$

where

$$\mathcal{A}_1(j) := S_\lambda(n\tau) (Z_\tau(m_j\tau) - \Pi_\lambda u(m_j\tau)),$$

$$\mathcal{A}_2(j) := S_\lambda(n\tau) (\Pi_\lambda u(m_j\tau) - u(m_j\tau)),$$

$$\mathcal{A}_3(j) := \tau \sum_{k=0}^{n-1} S_\lambda(n\tau - k\tau) \left( \frac{N(\tau) - \mathbf{1}}{\tau} Z_\tau(m_j\tau + k\tau) - \frac{N(\tau) - \mathbf{1}}{\tau} \Pi_\lambda u(m_j\tau + k\tau) \right),$$

$$\begin{aligned} \mathcal{A}_4(j) &:= \tau \sum_{k=0}^{n-1} S_\lambda(n\tau - k\tau) \frac{N(\tau) - \mathbf{1}}{\tau} \Pi_\lambda u(m_j\tau + k\tau) \\ &\quad + i \int_0^{n\tau} S_\lambda(n\tau - s) (Wu)(m_j\tau + s) ds \\ &\quad + i\varepsilon \int_0^{n\tau} S_\lambda(n\tau - s) (|u|^{2\sigma} u)(m_j\tau + s) ds, \end{aligned}$$

and we omit the dependence of the  $\mathcal{A}_k$ 's upon  $n$  to ease notations. The goal is to show that in the estimates, the term  $\mathcal{A}_3$  can be absorbed by the left hand side of (5.11),  $\mathcal{A}_2$  and  $\mathcal{A}_4$  are  $\mathcal{O}(\tau^{1/2})$  or  $\mathcal{O}(\tau)$ , according to the case considered in the theorem, and  $\mathcal{A}_1$  is then estimated by induction on  $j$ .

Let  $(q, r) \in \{(q_0, r_0), (\infty, 2)\}$ . The homogeneous Strichartz estimate (3.4) yields

$$\|\mathcal{A}_1(j)\|_{\ell^q(I_j;L^r)} \leq C_{d,q} \|Z_\tau(m_j\tau) - \Pi_\lambda u(m_j\tau)\|_{L^2}.$$

The term  $\mathcal{A}_2$  is controlled again via the homogeneous discrete Strichartz estimate (3.4), and (2.8) or (2.9): it is  $\mathcal{O}(\tau^{1/2})$  in the first case of Theorem 5.1,  $\mathcal{O}(\tau)$  in the second case. To estimate  $\mathcal{A}_3$ , we use (5.3), the inhomogeneous Strichartz estimate (3.5), and Hölder inequality like in (4.4), to obtain

$$\|\mathcal{A}_3(j)\|_{\ell^q(I_j;L^r)} \leq C|I_j| \|Z_\tau - \Pi_\lambda u\|_{\ell^\infty(I_j;L^2)} + C\eta^{2\sigma} \|Z_\tau - \Pi_\lambda u\|_{\ell^{q_0}(I_j;L^{r_0})},$$

where we have used the definition of the intervals  $I_j$  in terms of  $\eta$ . We now choose  $\eta > 0$  and  $|I_j|$  sufficiently small so that, for all  $0 \leq j \leq K$ ,

$$\|\mathcal{A}_3(j)\|_{\ell^{q_0}(I_j; L^{r_0})} + \|\mathcal{A}_3(j)\|_{\ell^\infty(I_j; L^2)} \leq \frac{1}{2} (\|Z_\tau - \Pi_\lambda u\|_{\ell^{q_0}(I_j; L^{r_0})} + \|Z_\tau - \Pi_\lambda u\|_{\ell^\infty(I_j; L^2)}).$$

The estimate of  $\mathcal{A}_4$  is postponed to Lemma 5.6 below,

$$(5.12) \quad \max_{0 \leq j \leq K} (\|\mathcal{A}_4(j)\|_{\ell^\infty(I_j; L^2)} + \|\mathcal{A}_4(j)\|_{\ell^{q_0}(I_j; L^{r_0})}) \lesssim \tau^\alpha,$$

where  $\alpha \in \{1/2, 1\}$ . Thus, we get

$$\begin{aligned} \|Z_\tau - \Pi_\lambda u\|_{\ell^{q_0}(I_j; L^{r_0})} + \|Z_\tau - \Pi_\lambda u\|_{\ell^\infty(I_j; L^2)} &\leq C \|Z_\tau(m_j\tau) - \Pi_\lambda u(m_j\tau)\|_{L^2} \\ &\quad + C\tau^\alpha + \frac{1}{2} (\|Z_\tau - \Pi_\lambda u\|_{\ell^{q_0}(I_j; L^{r_0})} + \|Z_\tau - \Pi_\lambda u\|_{\ell^\infty(I_j; L^2)}), \end{aligned}$$

hence, for all  $0 \leq j \leq K$ ,

$$\|Z_\tau - \Pi_\lambda u\|_{\ell^{q_0}(I_j; L^{r_0})} + \|Z_\tau - \Pi_\lambda u\|_{\ell^\infty(I_j; L^2)} \leq 2C \|Z_\tau(m_j\tau) - \Pi_\lambda u(m_j\tau)\|_{L^2} + 2C\tau^\alpha.$$

Now by construction  $m_0 = 0$ ,  $Z_\tau(m_0\tau) - \Pi_\lambda u(m_0\tau) = 0$ , and for  $1 \leq j \leq K$ ,

$$\|Z_\tau(m_j\tau) - \Pi_\lambda u(m_j\tau)\|_{L^2} \leq \|Z_\tau - \Pi_\lambda u\|_{\ell^\infty(I_{j-1}; L^2)},$$

hence, by induction,

$$\|Z_\tau - \Pi_\lambda u\|_{\ell^{q_0}([0, T]; L^{r_0})} + \|Z_\tau - \Pi_\lambda u\|_{\ell^\infty([0, T]; L^2)} \lesssim \tau^\alpha.$$

Using Strichartz estimates again, for any admissible pair  $(q, r)$ ,

$$\|Z_\tau - \Pi_\lambda u\|_{\ell^q([0, T]; L^r)} \lesssim \tau^\alpha,$$

hence the first two estimates in Theorem 5.1.

**5.3. Proof of (5.12).** We start with the following lemma, which is an adaptation of [14, Lemma 2.8].

**Lemma 5.5.** *Set  $\lambda = 1/\tau$ . There exists  $C$  independent of  $\tau \in (0, 1)$  and the time interval  $I$  of length at most one, such that*

$$\| |\Pi_\lambda u|^{4\sigma+1} \|_{L^{q'_0}(I; L^{r'_0})} + \| |\Pi_\lambda u|^{2\sigma} \Pi_\lambda(|u|^{2\sigma} u) \|_{L^{q'_0}(I; L^{r'_0})} \leq C\tau^{-1/2} \|u\|_{X^1(I)}^{4\sigma+1},$$

and,

$$\| |\Pi_\lambda u|^{4\sigma+1} \|_{L^{q'_0}(I; L^{r'_0})} + \| |\Pi_\lambda u|^{2\sigma} \Pi_\lambda(|u|^{2\sigma} u) \|_{L^{q'_0}(I; L^{r'_0})} \leq C \|u\|_{X^2(I)}^{4\sigma+1}.$$

*Proof.* The first case is proven in [14, Lemma 2.8], by combining Bernstein inequality, Hölder inequality, and Sobolev embedding. We can thus mimic the proof, up to the modifications provided by (2.7), (2.8), Lemmas 2.12, 2.18 and 2.19, which provide the analogue of [14, Lemma 2.6].

For the second case, like in the proof of [14, Lemma 2.8], we write

$$\| |\Pi_\tau u|^{4\sigma+1} \|_{L^{q'_0} L^{r'_0}} = \| |\Pi_\tau u|^{4\sigma+1} \|_{L^{(4\sigma+1)q'_0} L^{(4\sigma+1)r'_0}},$$

and we distinguish two cases:

- If  $0 < \sigma \leq 1/d$ , we use the same embedding as in [14],

$$H^s(\mathbb{R}^d) \hookrightarrow L^{(4\sigma+1)r'_0}(\mathbb{R}^d), \quad s = \frac{d}{2} - \frac{d(2\sigma+1)}{(2\sigma+2)(4\sigma+1)} \in [0, 1],$$

hence  $X^2(I) \subset L^\infty(I; H^1) \subset L^{(4\sigma+1)q'_0}(I; L^{(4\sigma+1)r'_0})$ .

• Now if  $1/d \leq \sigma < 2/(d-2)_+$ , we define the pair  $(a_2, b_2)$  by

$$a_2 = (4\sigma + 1)q'_0, \quad \frac{1}{b_2} = \frac{1}{2} - \frac{2}{da_2},$$

that is, we consider the Lebesgue exponent in time in the last estimate, and pick  $b_2$  so that  $(a_2, b_2)$  is admissible. We check that for such value of  $\sigma$ ,

$$W^{2, b_2}(\mathbb{R}^d) \hookrightarrow L^{(4\sigma+1)r'_0}(\mathbb{R}^d),$$

as

$$s := d \left( \frac{1}{b_2} - \frac{1}{b_1} \right) = \frac{d}{2} - \frac{d}{4\sigma+1} \left( \frac{2}{d} + \frac{1}{2} \right) \in [0, 2].$$

The second inequality follows, using (2.7).  $\square$

The main result of this subsection is the following, in which the last case will be used for convergence in  $\mathcal{H}^1$  instead of merely  $L^2$ :

**Lemma 5.6.** *Let  $\lambda = 1/\tau$ , and  $\beta = 1$  or  $2$ . For  $u \in X_T^\beta$ ,  $\tau \in (0, 1)$ , denote*

$$\begin{aligned} A(u)(n\tau) &= \tau \sum_{k=0}^{n-1} S_\lambda(n\tau - k\tau) \frac{N(\tau) - \mathbf{1}}{\tau} \Pi_\lambda u(k\tau) + i \int_0^{n\tau} S_\lambda(n\tau - s) W u(s) ds \\ &\quad + i\varepsilon \int_0^{n\tau} S_\lambda(n\tau - s) |u|^{2\sigma} u(s) ds. \end{aligned}$$

*In the case  $\beta = 2$ , we assume in addition  $\sigma \geq 1/2$  (hence  $d \leq 5$ ). Then for all admissible pairs  $(q, r)$  and all  $T \in [0, 1]$ ,*

$$\|A(u)\|_{\ell^q([0, T]; L^r)} \lesssim \tau^{\beta/2} \left( \|u\|_{X_T^\beta} + \|u\|_{X_T^{2\sigma+1}} + \|u\|_{X_T^{4\sigma+1}} \right).$$

*We also have the higher order estimate: if  $\sigma \geq 1/2$ ,*

$$\|H^{1/2} A(u)\|_{\ell^q([0, T]; L^r)} \lesssim \tau^{1/2} \left( \|u\|_{X_T^2} + \|u\|_{X_T^{2\sigma+1}} + \|u\|_{X_T^{4\sigma+1}} \right).$$

*Proof.* In the case  $V = 0$  and  $\beta = 1$ , this result is exactly [14, Lemma 3.1]. We emphasize how this result is adapted to the case where  $V$  satisfies Assumption 1.1, and to the case  $\beta = 2$ . The last case of the lemma corresponds to (5.19) in [14], when  $V = 0$ .

The assumptions of Lemma 5.6 imply, by Hölder inequality and Sobolev embedding,

$$u \in L^\infty([0, T]; H^1) \subset L^\theta([0, T]; L^{r_0}),$$

where  $\theta$  is given by (4.3). Decompose  $A(u)(n\tau)$  as

$$A(u)(n\tau) = A_1(u)(n\tau) + A_2(u)(n\tau),$$

where

$$\begin{aligned} A_1(u)(n\tau) &= \tau \sum_{k=0}^{n-1} S_\lambda(n\tau - k\tau) \mathcal{B}_1(u)(k\tau) - \int_0^{n\tau} S_\lambda(n\tau - s) \mathcal{B}_1(u)(s) ds, \\ A_2(u)(n\tau) &= \int_0^{n\tau} S_\lambda(n\tau - s) \mathcal{B}_1(u)(s) ds + i \int_0^{n\tau} S_\lambda(n\tau - s) W u(s) ds \\ &\quad + i\varepsilon \int_0^{n\tau} S_\lambda(n\tau - s) |u|^{2\sigma} u(s) ds, \end{aligned}$$

with

$$\mathcal{B}_1(u)(s) := \frac{N_0(\tau) - \mathbf{1}}{\tau} \Pi_\lambda u(s).$$

Recall that  $N_0$  denotes the nonlinear flow in the case without  $W$ ,  $N_0(t)\phi = \phi e^{-i\epsilon t|\phi|^{2\sigma}}$ , and decompose  $\mathcal{B}_1(u) = \mathcal{B}_2(u) + \mathcal{B}_3(u)$ , with

$$\mathcal{B}_2(u)(s) = e^{-i\tau W} \frac{N_0(\tau) - \mathbf{1}}{\tau} \Pi_\lambda u(s), \quad \mathcal{B}_3(u)(s) = \frac{e^{-i\tau W} - \mathbf{1}}{\tau} \Pi_\lambda u(s).$$

By Lemma 5.4, we can estimate

$$\begin{aligned} \|A_1(u)\|_{\ell^q([0,T];L^r)} &\lesssim \tau^{\beta/2} \left\| H^{\beta/2} \mathcal{B}_2(u) \right\|_{L_T^{q_0'} L^{r_0'}} + \tau \|\partial_t \mathcal{B}_2(u)\|_{L_T^{q_0'} L^{r_0'}} \\ &\quad + \tau^{\beta/2} \left\| H^{\beta/2} \mathcal{B}_3(u) \right\|_{L_T^1 L^2} + \tau \|\partial_t \mathcal{B}_3(u)\|_{L_T^1 L^2}, \end{aligned}$$

for all admissible pair  $(q, r)$ . In view of (5.7), Hölder inequality (4.4), and (2.7),

$$\begin{aligned} \left\| \sqrt{H} \mathcal{B}_2(u) \right\|_{L_T^{q_0'} L^{r_0'}} &\lesssim \|\Pi_\lambda u\|_{L_T^\theta L^{r_0}}^{2\sigma} \|\sqrt{H} \Pi_\lambda u\|_{L_T^{q_0} L^{r_0}} \\ &\lesssim \|u\|_{L_T^\theta L^{r_0}}^{2\sigma} \|\Pi_\lambda \sqrt{H} u\|_{L_T^{q_0} L^{r_0}} \lesssim \|u\|_{X_T^1}^{2\sigma+1}. \end{aligned}$$

In the case  $\beta = 2$ , we invoke (5.8): like above,

$$\begin{aligned} \|\Pi_\lambda u\|_{L_T^\theta L^{r_0}}^{2\sigma} \|H \Pi_\lambda u\|_{L_T^{q_0} L^{r_0}} + \|\Pi_\lambda u\|_{L_T^\theta L^{r_0}}^{2\sigma-1} \|\sqrt{H} \Pi_\lambda u\|_{L_T^\theta L^{r_0}} \|\sqrt{H} \Pi_\lambda u\|_{L_T^{q_0} L^{r_0}} \\ \lesssim \|u\|_{X_T^2}^{2\sigma+1}. \end{aligned}$$

For the last term estimating  $H\mathcal{B}_2(u)$  in (5.8), Lemma 2.19 yields

$$\|\Pi_\lambda u\|_{L^{(4\sigma+1)r_0'}^\theta}^{4\sigma-1} \left\| \sqrt{H} \Pi_\lambda u \right\|_{L^{(4\sigma+1)r_0'}}^2 \lesssim \tau^{-\frac{d}{2} \left( \frac{1}{r_1} - \frac{1}{r_0'} \right)} \|\Pi_\lambda u\|_{L^{(4\sigma+1)r_1}}^{4\sigma-1} \left\| \sqrt{H} \Pi_\lambda u \right\|_{L^{(4\sigma+1)r_1}}^2,$$

where  $r_1$  is chosen like in the proof of the first inequality in Lemma 5.5 (see [14, Lemma 2.8]), so we get eventually

$$\|H\mathcal{B}_2(u)\|_{L_T^{q_0'} L^{r_0'}} \lesssim \|u\|_{X_T^2}^{2\sigma+1} + \sqrt{\tau} \|u\|_{X_T^2}^{4\sigma+1}.$$

For the other term, Lemma 5.2 yields

$$\|\partial_t \mathcal{B}_2(u)\|_{L_T^{q_0'} L^{r_0'}} = \left\| \partial_t \left( \frac{N_0(\tau) - \mathbf{1}}{\tau} \Pi_\lambda u \right) \right\|_{L_T^{q_0'} L^{r_0'}} \lesssim \left\| |\Pi_\lambda u|^{2\sigma} \partial_t \Pi_\lambda u \right\|_{L_T^{q_0'} L^{r_0'}}.$$

Observing that  $\partial_t$  and  $\Pi_\tau$  commute, (1.1) implies

$$\|\partial_t \mathcal{B}_2(u)\|_{L_T^{q_0'} L^{r_0'}} \lesssim \left\| |\Pi_\lambda u|^{2\sigma} \Pi_\lambda (Hu) \right\|_{L_T^{q_0'} L^{r_0'}} + \left\| |\Pi_\lambda u|^{2\sigma} \Pi_\lambda (|u|^{2\sigma} u) \right\|_{L_T^{q_0'} L^{r_0'}}.$$

For the first term of the right side, Hölder inequality (4.4) implies

$$\left\| |\Pi_\lambda u|^{2\sigma} \Pi_\lambda Hu \right\|_{L_T^{q_0'} L^{r_0'}} \leq \|\Pi_\lambda u\|_{L_T^\theta L^{r_0}}^{2\sigma} \|\Pi_\lambda Hu\|_{L_T^{q_0} L^{r_0}} \lesssim \|u\|_{X_T^1}^{2\sigma} \|\Pi_\lambda Hu\|_{L_T^{q_0} L^{r_0}}.$$

In the case  $\beta = 1$ , we also invoke Lemma 2.18, which introduces an extra factor  $\tau^{-1/2}$ . The second term of the right hand side is controlled thanks to Lemma 5.5.

For  $\mathcal{B}_3(u)$ , we easily have

$$\left\| H^{\beta/2} \mathcal{B}_3(u) \right\|_{L_T^1 L^2} \lesssim \|u\|_{X_T^\beta}, \quad \|\partial_t \mathcal{B}_3(u)\|_{L_T^1 L^2} \lesssim \|\Pi_\lambda Hu\|_{L_T^\infty L^2} + \|\Pi_\lambda (|u|^{2\sigma} u)\|_{L_T^1 L^2}.$$

Only the last term requires some extra care: when  $\beta = 2$ , we invoke (2.7) and the embeddings  $H^2 \subset H^{d\sigma/(2\sigma+1)} \subset L^{4\sigma+2}$  used in Section 4.2,

$$\|\Pi_\lambda (|u|^{2\sigma} u)\|_{L_T^1 L^2} \lesssim \|u\|_{L_T^\infty L^{4\sigma+2}}^{2\sigma+1} \lesssim \|u\|_{X_T^2}^{2\sigma+1}.$$

In the case  $\beta = 1$ , we invoke Lemma 2.19: for  $1 \leq p \leq 2$ ,

$$\|\Pi_\lambda (|u|^{2\sigma} u)\|_{L_T^1 L^2} \lesssim \tau^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{2})} \|u\|_{L_T^\infty L^{(2\sigma+1)p}}^{2\sigma+1}.$$

If  $\sigma \leq 1/d$ , we pick  $(2\sigma+1)p = 2$ , and

$$\|\Pi_\lambda (|u|^{2\sigma} u)\|_{L_T^1 L^2} \lesssim \tau^{-\frac{d\sigma}{2}} \|u\|_{L_T^\infty L^2}^{2\sigma+1} \lesssim \tau^{-\frac{1}{2}} \|u_0\|_{L^2}^{2\sigma+1}.$$

If  $\sigma > 1/d$ , we use the Sobolev embedding  $H^s(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d+2s}}(2\sigma+1)(\mathbb{R}^d)$  where

$$s = \frac{d\sigma - 1}{2\sigma + 1} < 1,$$

and this Lebesgue index is chosen so that the power of  $\tau$  is exactly  $-1/2$ ,

$$\|\Pi_\lambda (|u|^{2\sigma} u)\|_{L_T^1 L^2} \lesssim \tau^{-1/2} \|u\|_{L_T^\infty H^s}^{2\sigma+1} \lesssim \tau^{-1/2} \|u\|_{X_T^1}^{2\sigma+1}.$$

We come up with

$$\|A_1(u)\|_{\ell^q([0,T];L^r)} \lesssim \tau^{\beta/2} \left( \|u\|_{X_T^\beta} + \|u\|_{X_T^\beta}^{2\sigma+1} + \|u\|_{X_T^\beta}^{4\sigma+1} \right).$$

To complete the proof, we perform another decomposition, for the term  $A_2$ ,

$$\mathcal{B}_1(u) + iWu + i\varepsilon|u|^{2\sigma}u = \mathcal{C}_1(u) + \mathcal{C}_2(u) + \mathcal{C}_3(u) + i\mathcal{C}_4(u) + i\varepsilon\mathcal{C}_5(u),$$

where

$$\begin{aligned} \mathcal{C}_1(u) &= (e^{-i\tau W} - 1) \frac{N_0(\tau) - \mathbf{1}}{\tau} \Pi_\lambda u, \\ \mathcal{C}_2(u) &= \frac{N_0(\tau) - \mathbf{1}}{\tau} \Pi_\lambda u + i\varepsilon |\Pi_\lambda u|^{2\sigma} \Pi_\lambda u, \\ \mathcal{C}_3(u) &= \frac{e^{-i\tau W} - \mathbf{1}}{\tau} \Pi_\lambda u + iW \Pi_\lambda u, \\ \mathcal{C}_4(u) &= W(u - \Pi_\lambda u), \\ \mathcal{C}_5(u) &= |u|^{2\sigma} u - |\Pi_\lambda u|^{2\sigma} \Pi_\lambda u. \end{aligned}$$

The discrete inhomogeneous Strichartz estimates (3.5) yields, for  $(q, r)$  admissible,

$$\begin{aligned} \|\mathcal{A}_2(u)\|_{\ell^q([0,T];L^r)} &\lesssim \|\mathcal{C}_1(u)\|_{L_T^{q'_0} L^{r'_0}} + \|\mathcal{C}_2(u)\|_{L_T^{q'_0} L^{r'_0}} + \|\mathcal{C}_3(u)\|_{L_T^1 L^2} + \|\mathcal{C}_4(u)\|_{L_T^1 L^2} \\ &\quad + \|\mathcal{C}_5(u)\|_{L_T^{q'_0} L^{r'_0}}. \end{aligned}$$

In view of (5.4), we have the pointwise estimate

$$|\mathcal{C}_1(u)| \lesssim \tau \|W\|_{L^\infty} |\Pi_\lambda u|^{2\sigma+1},$$

hence

$$\|\mathcal{C}_1(u)\|_{L_T^{q'_0} L^{r'_0}} \lesssim \tau \|u\|_{X_T^1}^{2\sigma+1}.$$

Recall that  $N_0(\tau)z = ze^{-i\varepsilon\tau|z|^{2\sigma}}$ , Taylor formula yields the pointwise estimate

$$|\mathcal{C}_2(u)| = \left| \frac{N_0(\tau) - \mathbf{1}}{\tau} \Pi_\lambda u + i\varepsilon |\Pi_\lambda u|^{2\sigma} \Pi_\lambda u \right| \lesssim \tau |\Pi_\lambda u|^{4\sigma+1}.$$

Thus, we have

$$\|\mathcal{C}_2(u)\|_{L_T^{q'_0} L^{r'_0}} \lesssim \tau \|\Pi_\lambda u\|_{L_T^{q'_0} L^{r'_0}}^{4\sigma+1},$$

and this last term is estimated by Lemma 5.5. We readily have

$$\|\mathcal{C}_3(u)\|_{L_T^1 L^2} \lesssim \tau \|\Pi_\lambda u\|_{L_T^\infty L^2} \lesssim \tau \|u_0\|_{L^2}, \quad \|\mathcal{C}_4(u)\|_{L_T^1 L^2} \lesssim \tau^{\beta/2} \|u\|_{X_T^\beta}.$$

Finally, Hölder inequality (4.4) yields

$$\begin{aligned} \|\mathcal{C}_5(u)\|_{L_T^{q'_0} L^{r'_0}} &= \|\Pi_\lambda u\|_{L_T^\theta L^{r_0}}^{2\sigma} \|\Pi_\lambda u - |u|^{2\sigma} u\|_{L_T^{q'_0} L^{r'_0}} \\ &\lesssim \left( \|\Pi_\lambda u\|_{L_T^\theta L^{r_0}}^{2\sigma} + \|u\|_{L_T^\theta L^{r_0}}^{2\sigma} \right) \|\Pi_\lambda u - u\|_{L_T^{q_0} L^{r_0}} \lesssim \tau^{\beta/2} \|u\|_{X_T^\beta}^{2\sigma+1}, \end{aligned}$$

where we have used Lemma 2.16. This yields the announced estimate for  $A(u)$  in  $\ell^q([0, T]; L^r)$ . We now emphasize the modifications needed to estimate  $\sqrt{H}A(u)$  in the same space, and thus conclude the proof of Lemma 5.6. Invoking discrete Strichartz estimates like above, we have

$$\begin{aligned} \left\| \sqrt{H}A_1(u) \right\|_{\ell^q([0, T]; L^r)} &\lesssim \tau^{1/2} \|H\mathcal{B}_2(u)\|_{L_T^{q'_0} L^{r'_0}} + \tau \left\| \sqrt{H}\partial_t \mathcal{B}_2(u) \right\|_{L_T^{q'_0} L^{r'_0}} \\ &\quad + \tau^{1/2} \|H\mathcal{B}_3(u)\|_{L_T^1 L^2} + \tau \left\| \sqrt{H}\partial_t \mathcal{B}_3(u) \right\|_{L_T^1 L^2}. \end{aligned}$$

The previous computations, in the case  $\beta = 2$ , can then be resumed with essentially no modification, except maybe for the term involving  $\partial_t \mathcal{B}_2(u)$ . The factor  $e^{-i\tau W}$  is obviously discarded, and  $\partial_t \mathcal{B}_2(u)$  appears as a linear combination of the terms

$$\frac{N_0(\tau) - 1}{\tau} |\Pi_\lambda u|^{2\sigma} \partial_t \Pi_\lambda u \quad \text{and} \quad \frac{N_0(\tau) - 1}{\tau} |\Pi_\lambda u|^{2\sigma-2} (\Pi_\lambda u)^2 \partial_t \overline{\Pi_\lambda u}.$$

Like before, we use the commutation between  $\partial_t$  and  $\Pi_\lambda$ , and (1.1). Invoking Lemma 5.3 and (2.4), the most delicate terms to control are

$$\begin{aligned} I &= \|\Pi_\lambda u\|_{L_T^\theta L^{r_0}}^{2\sigma} \left\| \sqrt{H}\Pi_\lambda H u \right\|_{L_T^{q_0} L^{r_0}}, \\ II &= \|\Pi_\lambda u\|_{L_T^\theta L^{r_0}}^{2\sigma-1} \left\| \sqrt{H}\Pi_\lambda u \right\|_{L_T^\theta L^{r_0}} \|\Pi_\lambda H u\|_{L_T^{q_0} L^{r_0}}, \\ &\quad \text{when differentiating the power of } \Pi_\lambda u, \\ III &= \|\Pi_\lambda u\|_{L_T^{2\theta} L^{2r_0}}^{4\sigma-1} \left\| \sqrt{H}\Pi_\lambda u \right\|_{L_T^{2\theta} L^{2r_0}} \|\Pi_\lambda H u\|_{L_T^{q_0} L^{r_0}}, \\ &\quad \text{when differentiating } \frac{N_0(\tau) - 1}{\tau} \text{ again.} \end{aligned}$$

Proceeding like before, Lemma 2.18 and (2.7) yield

$$I \lesssim \tau^{-1/2} \|u\|_{X_T^2}^{2\sigma+1}.$$

For the term  $II$ , when  $1/2 \leq \sigma < 2/d$ , we write

$$\left\| \sqrt{H}\Pi_\lambda u \right\|_{L_T^\theta L^{r_0}} \leq T^{1/\theta-1/q_0} \left\| \sqrt{H}\Pi_\lambda u \right\|_{L_T^{q_0} L^{r_0}} \lesssim \|u\|_{X_T^1}.$$

When  $\sigma \geq 2/d$ , we resume the estimate from the proof of Proposition 4.2, with  $(\theta, b_1)$  admissible,

$$\left\| \sqrt{H}\Pi_\lambda u \right\|_{L_T^\theta L^{r_0}} \lesssim \|H\Pi_\lambda u\|_{L_T^\theta L^{b_1}} \lesssim \|u\|_{X_T^2}.$$

We obtain, in both cases,

$$II \lesssim \|u\|_{X_T^2}^{2\sigma+1}.$$

The last term is estimated differently from the proof of Lemma 5.5:

$$III \lesssim \|\Pi_\lambda u\|_{L_T^{2\theta} L^{2r_0}}^{4\sigma-1} \|\sqrt{H}\Pi_\lambda u\|_{L_T^{2\theta} L^{2r_0}} \|u\|_{X_T^2}.$$

We check that since  $\sigma < 2/(d-2)_+$  and  $d \leq 5$ ,  $H^2(\mathbb{R}^d) \hookrightarrow L^{2r_0}(\mathbb{R}^d) = L^{4\sigma+4}(\mathbb{R}^d)$ ,

$$\|\Pi_\lambda u\|_{L_T^{2\theta} L^{2r_0}} \lesssim \|u\|_{L_T^{2\theta} L^{2r_0}} \lesssim \|u\|_{L_T^\infty H^2} \lesssim \|u\|_{X_T^2},$$

and, using Lemma 2.18,

$$\|\Pi_\lambda \sqrt{H}u\|_{L_T^\infty H^2} \lesssim \tau^{-1/2} \|u\|_{X_T^2}.$$

We leave out the details of the other estimates.  $\square$

**5.4. Stability implies convergence in  $\mathcal{H}^1$ .** We now get to the last estimate in Theorem 5.1. We therefore assume  $1/2 \leq \sigma < 2/(d-2)_+$ . Again, in view of Lemma 2.16, it suffices to consider the difference  $Z_\tau(n\tau) - \Pi_\lambda u(n\tau)$ . The scheme is essentially the same as in Subsection 5.2: we decompose  $[0, T]$  into a finite union of intervals  $I_j$ , on which some convenient norms will be sufficiently small so the corresponding term on the right hand side is absorbed by the left hand side, when using Strichartz estimates. The core of the proof of convergence in  $\mathcal{H}^1$  then relies on higher order estimates of the terms involved in Subsection 5.2.

We resume Duhamel formula (5.10) and the decomposition (5.11). The terms  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are estimated thanks to the homogeneous discrete Strichartz estimate,

$$\begin{aligned} \|\sqrt{H}\mathcal{A}_1(j)\|_{\ell^q(I_j; L^r)} &= \left\| S_\lambda \sqrt{H} (Z_\tau(m_j\tau) - \Pi_\lambda u(m_j\tau)) \right\|_{\ell^q(I_j; L^r)} \\ &\lesssim \left\| \sqrt{H} (Z_\tau(m_j\tau) - \Pi_\lambda u(m_j\tau)) \right\|_{L^2} \\ &\lesssim \|Z_\tau(m_j\tau) - \Pi_\lambda u(m_j\tau)\|_{\mathcal{H}^1}, \end{aligned}$$

and the term  $\mathcal{A}_2$  is estimated similarly,

$$\begin{aligned} \|\sqrt{H}\mathcal{A}_2(j)\|_{\ell^q(I_j; L^r)} &\lesssim \left\| \sqrt{H} (\Pi_\lambda u(m_j\tau) - u(m_j\tau)) \right\|_{L^2} \\ &\lesssim \left\| \Pi_\lambda \sqrt{H}u(m_j\tau) - \sqrt{H}u(m_j\tau) \right\|_{L^2} \\ &\lesssim \tau^{1/2} \|u(m_j\tau)\|_{\mathcal{H}^2} \lesssim \tau^{1/2} \|u\|_{X_T^2}, \end{aligned}$$

where we have used (2.8). The term  $\mathcal{A}_4$  is estimated thanks to the last case of Lemma 5.6,

$$\|\sqrt{H}\mathcal{A}_4(j)\|_{\ell^q(I_j; L^r)} \lesssim \tau^{1/2} \left( \|u\|_{X_T^2} + \|u\|_{X_T^2}^{2\sigma+1} + \|u\|_{X_T^2}^{4\sigma+1} \right).$$

Therefore, the new estimate needed at this stage is the estimate of  $\mathcal{A}_3$ . To simplify the presentation, we treat the case  $W = 0$  only, as we have seen before how to handle the presence of this term. We prove the analogue of [14, Lemma 5.2]:



**Lemma 5.7.** *Let  $1/2 \leq \sigma < 2/(d-2)_+$ . There exists  $C$  such that for all time interval  $I$  and all  $w_1, w_2$  sufficiently regular,*

$$\begin{aligned} & \left\| \sqrt{H} \left( \frac{N_0(\tau) - \mathbf{1}}{\tau} w_1 - \frac{N_0(\tau) - \mathbf{1}}{\tau} w_2 \right) \right\|_{\ell^{q'_0}(I; L^{r'_0})} \\ & \leq C \|w_1\|_{\ell^\theta(I; L^{r_0})}^{2\sigma} \|\sqrt{H}(w_1 - w_2)\|_{\ell^{q_0}(I; L^{r_0})} \\ & \quad + C \left( \tau \|w_1\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} + \tau \|w_2\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} \right. \\ & \quad \left. + \|w_1\|_{\ell^\theta(I; L^{r_0})}^{2\sigma-1} + \|w_2\|_{\ell^\theta(I; L^{r_0})}^{2\sigma-1} \right) \|\sqrt{H}w_2\|_{\ell^\theta(I; L^{r_0})} \|w_1 - w_2\|_{\ell^{q_0}(I; L^{r_0})} \\ & \quad + C \tau \|w_1\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} \|w_1\|_{\ell^\theta(I; L^{r_0})} \|\sqrt{H}(w_1 - w_2)\|_{\ell^{q_0}(I; L^{r_0})}. \end{aligned}$$

*Proof.* When  $\sqrt{H}$  is replaced by  $J \in \{\sqrt{V}, \mathbf{1}\}$  on the left hand side, the result is a direct consequence of (5.3) and Hölder inequality (4.4), recalling Lemma 2.12. We thus consider the case  $J = \nabla$ , and compute

$$\begin{aligned} & \nabla \left( \frac{N_0(\tau) - \mathbf{1}}{\tau} w_1 - \frac{N_0(\tau) - \mathbf{1}}{\tau} w_2 \right) \\ & = \frac{e^{-i\varepsilon\tau|w_1|^{2\sigma}} - 1}{\tau} \nabla w_1 - \frac{e^{-i\varepsilon\tau|w_2|^{2\sigma}} - 1}{\tau} \nabla w_2 \pm \frac{e^{-i\varepsilon\tau|w_1|^{2\sigma}} - 1}{\tau} \nabla w_2 \\ & \quad - i\tau\varepsilon w_1 \nabla |w_1|^{2\sigma} \frac{e^{-i\varepsilon\tau|w_1|^{2\sigma}} - 1}{\tau} + i\tau\varepsilon w_2 \nabla |w_2|^{2\sigma} \frac{e^{-i\varepsilon\tau|w_2|^{2\sigma}} - 1}{\tau} \\ & \quad \pm i\tau\varepsilon w_2 \nabla |w_2|^{2\sigma} \frac{e^{-i\varepsilon\tau|w_1|^{2\sigma}} - 1}{\tau}, \end{aligned}$$

where the symbol “ $\pm$ ” means that we add and subtract the same term in order to group suitable terms together. We have, in view of (5.4),

$$\left| \frac{e^{-i\varepsilon\tau|w_1|^{2\sigma}} - 1}{\tau} \nabla(w_1 - w_2) \right| \leq |w_1|^{2\sigma} |\nabla(w_1 - w_2)|,$$

and (4.4) yields, together with Lemma 2.12,

$$\begin{aligned} \left\| \frac{e^{-i\varepsilon\tau|w_1|^{2\sigma}} - 1}{\tau} \nabla(w_1 - w_2) \right\|_{\ell^{q'_0}(I; L^{r'_0})} & \leq \|w_1\|_{\ell^\theta(I; L^{r_0})}^{2\sigma} \|\nabla(w_1 - w_2)\|_{\ell^{q_0}(I; L^{r_0})} \\ & \lesssim \|w_1\|_{\ell^\theta(I; L^{r_0})}^{2\sigma} \|\sqrt{H}(w_1 - w_2)\|_{\ell^{q_0}(I; L^{r_0})}. \end{aligned}$$

We next write, like for (5.3),

$$\begin{aligned} \left| \left( \frac{e^{-i\varepsilon\tau|w_1|^{2\sigma}} - 1}{\tau} - \frac{e^{-i\varepsilon\tau|w_2|^{2\sigma}} - 1}{\tau} \right) \nabla w_2 \right| & = \left| \left( \frac{e^{-i\varepsilon\tau|w_1|^{2\sigma}} - e^{-i\varepsilon\tau|w_2|^{2\sigma}}}{\tau} \right) \nabla w_2 \right| \\ & \lesssim \left| |w_1|^{2\sigma} - |w_2|^{2\sigma} \right| |\nabla w_2| \\ & \lesssim (|w_1|^{2\sigma-1} + |w_2|^{2\sigma-1}) |w_1 - w_2| \times |\nabla w_2|, \end{aligned}$$

and (4.4) yields

$$\begin{aligned} & \left\| \left( \frac{e^{-i\varepsilon\tau}|w_1|^{2\sigma} - 1}{\tau} - \frac{e^{-i\varepsilon\tau}|w_2|^{2\sigma} - 1}{\tau} \right) \nabla w_2 \right\|_{\ell^{q'_0}(I; L^{r'_0})} \\ & \lesssim \left( \|w_1\|_{\ell^\theta(I; L^{r_0})}^{2\sigma-1} + \|w_2\|_{\ell^\theta(I; L^{r_0})}^{2\sigma-1} \right) \|\nabla w_2\|_{\ell^\theta(I; L^{r_0})} \|w_1 - w_2\|_{\ell^{q_0}(I; L^{r_0})}. \end{aligned}$$

Writing in a rather loose fashion, that is, proceeding as if  $w_1, w_2 \geq 0$ ,

$$w_1 \nabla |w_1|^{2\sigma} - w_2 \nabla |w_2|^{2\sigma} \approx 2\sigma \left( w_1^{2\sigma} \nabla w_1 - w_2^{2\sigma} \nabla w_2 \pm w_1^{2\sigma} \nabla w_2 \right),$$

we also have, in view of (5.4),

$$\begin{aligned} \left| (w_1 \nabla |w_1|^{2\sigma} - w_2 \nabla |w_2|^{2\sigma}) \frac{e^{-i\varepsilon\tau}|w_1|^{2\sigma} - 1}{\tau} \right| & \lesssim |w_1|^{4\sigma} |\nabla(w_1 - w_2)| \\ & \quad + |w_1|^{2\sigma} (|w_1|^{2\sigma-1} + |w_2|^{2\sigma-1}) |w_1 - w_2| |\nabla w_2|. \end{aligned}$$

Finally, like for (5.3),

$$\begin{aligned} & \left| \left( \frac{e^{-i\varepsilon\tau}|w_1|^{2\sigma} - 1}{\tau} - \frac{e^{-i\varepsilon\tau}|w_2|^{2\sigma} - 1}{\tau} \right) w_2 \nabla |w_2|^{2\sigma} \right| \\ & \lesssim \left| |w_1|^{2\sigma} - |w_2|^{2\sigma} \right| |w_2|^{2\sigma} |\nabla w_2|, \end{aligned}$$

so the sum of the last two terms considered is controlled by

$$|w_1|^{4\sigma} |\nabla(w_1 - w_2)| + (|w_1|^{4\sigma-1} + |w_2|^{4\sigma-1}) |w_1 - w_2| |\nabla w_2|.$$

The  $\ell^{q'_0}(I; L^{r'_0})$ -norm of this quantity is estimated, in view of (4.4), by

$$\begin{aligned} & \|w_1\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} \|w_1\|_{\ell^\theta(I; L^{r_0})} \|\nabla(w_1 - w_2)\|_{\ell^{q_0}(I; L^{r_0})} \\ & + \left( \sum_{j=1}^2 \|w_j\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} \right) \|\nabla w_2\|_{\ell^\theta(I; L^{r_0})} \|w_1 - w_2\|_{\ell^{q_0}(I; L^{r_0})}, \end{aligned}$$

hence the lemma.  $\square$

We apply Lemma 5.7 to  $\mathcal{A}_3$ , with  $w_1 = Z_\tau$  and  $w_2 = \Pi_\lambda u$ . We have, thanks to Strichartz estimates (3.5), (2.4) and Hölder inequality,

$$\begin{aligned} & \|\sqrt{H}\mathcal{A}_3(j)\|_{\ell^q(I_j; L^r)} \lesssim \|Z_\tau\|_{\ell^\theta(I_j; L^{r_0})}^{2\sigma} \|\sqrt{H}(Z_\tau - \Pi_\lambda u)\|_{\ell^{q_0}(I_j; L^{r_0})} \\ & + \left( \tau \|Z_\tau\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I_j; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} + \tau \|\Pi_\lambda u\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I_j; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} \right. \\ & \quad \left. + \|Z_\tau\|_{\ell^\theta(I_j; L^{r_0})}^{2\sigma-1} + \|\Pi_\lambda u\|_{\ell^\theta(I_j; L^{r_0})}^{2\sigma-1} \right) \|\sqrt{H}\Pi_\lambda u\|_{\ell^\theta(I_j; L^{r_0})} \|Z_\tau - \Pi_\lambda u\|_{\ell^{q_0}(I_j; L^{r_0})} \\ & + \tau \|Z_\tau\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I_j; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} \|Z_\tau\|_{\ell^\theta(I_j; L^{r_0})} \|\sqrt{H}(Z_\tau - \Pi_\lambda u)\|_{\ell^{q_0}(I_j; L^{r_0})} \\ & \lesssim |I_j|^{2\sigma\gamma} \|\sqrt{H}(Z_\tau - \Pi_\lambda u)\|_{\ell^{q_0}(I_j; L^{r_0})} \\ & + \left( \tau \|Z_\tau\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I_j; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} + \tau \|\Pi_\lambda u\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I_j; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} + 1 \right) \\ & \quad \times |I_j|^{1/\theta} \|\sqrt{H}u\|_{\ell^\infty(I_j; H^1)} \|Z_\tau - \Pi_\lambda u\|_{\ell^{q_0}(I_j; L^{r_0})} \\ & + \tau \|Z_\tau\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I_j; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} |I_j|^\gamma M \|\sqrt{H}(Z_\tau - \Pi_\lambda u)\|_{\ell^{q_0}(I_j; L^{r_0})}, \end{aligned}$$

where we have used (5.1). Noticing that the assumptions on  $\chi$  imply

$$\chi(z) = \chi\left(\frac{z}{4}\right) \chi(z), \quad \text{hence} \quad \Pi_\lambda = \Pi_{4\lambda} \Pi_\lambda,$$

we have

$$Z_\tau = \Pi_{4\lambda} Z_\tau.$$

To control the  $L^{\frac{4\sigma-1}{2\sigma-1}r_0}$ -norms in space, we now invoke Lemma 2.19, with

$$p = r_0 < q = \frac{4\sigma-1}{2\sigma-1}r_0,$$

applied to  $\phi = Z_\tau$  or  $\Pi_\lambda u$ , so we have  $\phi = \Pi_{4\lambda} \phi$  (leaving out the time dependence at this stage), and thus

$$\|f\|_{L^{\frac{4\sigma-1}{2\sigma-1}r_0}}^{4\sigma-1} \lesssim \tau^{-\frac{d}{2}\left(\frac{1}{r_0} - \frac{2\sigma-1}{(4\sigma-1)r_0}\right)(4\sigma-1)} \|f\|_{L^{r_0}}^{4\sigma-1}.$$

We compute

$$\frac{d}{2} \left( \frac{1}{r_0} - \frac{2\sigma-1}{(4\sigma-1)r_0} \right) (4\sigma-1) = \frac{d\sigma}{2\sigma+2} \leq 1, \quad \text{since} \quad \sigma < \frac{2}{(d-2)_+},$$

so the loss induced by this Bernstein-type inequality is compensated by the factor  $\tau$  in front of the corresponding norm,

$$\tau \|Z_\tau\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I_j; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} + \tau \|\Pi_\lambda u\|_{\ell^{\frac{4\sigma-1}{2\sigma-1}\theta}(I_j; L^{\frac{4\sigma-1}{2\sigma-1}r_0})}^{4\sigma-1} \lesssim M^{4\sigma-1} + \|u\|_{X_T^1}^{4\sigma-1},$$

where we have used (5.2) and the Sobolev embedding  $H^1(\mathbb{R}^d) \hookrightarrow L^{r_0}(\mathbb{R}^d)$ . We can then simplify the above estimate to

$$\begin{aligned} \|\sqrt{H} \mathcal{A}_3(j)\|_{\ell^q(I_j; L^r)} &\lesssim |I_j|^{2\sigma\gamma} \|\sqrt{H}(Z_\tau - \Pi_\lambda u)\|_{\ell^{q_0}(I_j; L^{r_0})} \\ &+ |I_j|^{1/\theta} \left( M^{4\sigma-1} + \|u\|_{X_T^1}^{4\sigma-1} + 1 \right) \|u\|_{X_T^2} \|Z_\tau - \Pi_\lambda u\|_{\ell^{q_0}(I_j; L^{r_0})} \\ &+ |I_j|^\gamma M^{4\sigma} \|\sqrt{H}(Z_\tau - \Pi_\lambda u)\|_{\ell^{q_0}(I_j; L^{r_0})}. \end{aligned}$$

We can then proceed like in Subsection 5.2, and leave out the details.

## 6. STABILITY

To complete the proof of Theorems 1.5 and 1.6, we now have to prove that the stability conditions (5.1), and (5.2) in the corresponding case, are verified. Throughout this section,  $T > 0$  is fixed, such that  $u \in C([0, T]; \mathcal{H}^1)$ . We distinguish two cases, for which the roadmap is the same, but the technical details are different. Indeed, we emphasize the property

$$\theta < q_0 \iff \sigma < \frac{2}{d}.$$

In the  $L^2$ -subcritical case  $0 < \sigma < 2/d$ , we use the Hölder inequality

$$\|u^n\|_{\ell^\theta(I; L^{r_0})} \leq |I|^{1/\theta-1/q_0} \|u^n\|_{\ell^{q_0}(I; L^{r_0})},$$

and the goal is to prove that  $\|u^n\|_{\ell^{q_0}([0, T]; L^{r_0})} \leq M$  for some  $M$  independent of  $\tau \in (0, 1)$ .

If  $\sigma \geq 2/d$ , we use Sobolev embedding and Hölder inequality,

$$\begin{aligned} \|u^n\|_{\ell^\theta(I;L^{r_0})} &\lesssim \|u^n\|_{\ell^\theta(I;H^1)} \lesssim |I|^{1/\theta} \|u^n\|_{\ell^\infty(I;H^1)}, \\ \|u^n\|_{\ell^\infty(I;L^{r_0})} &\lesssim \|u^n\|_{\ell^\infty(I;H^1)}, \end{aligned}$$

and the goal is to prove that  $\|u^n\|_{\ell^\infty([0,T];H^1)} \leq M$  for some  $M$  independent of  $\tau \in (0,1)$ .

**6.1.  $L^2$ -subcritical case:**  $0 < \sigma < 2/d$ . In the absence of potential,  $V = 0$ , (5.1) was proven in [23, Theorem 1.1] and resumed in [14, Proposition 4.1]. We slightly modify the argument here, in a lemma where we do not try to make the constants sharp:

**Lemma 6.1.** *Let  $0 < \sigma < 2/d$ ,  $u_0 \in \mathcal{H}^1$  and  $T > 0$  such that  $u \in X_T^1$ . There exists  $\tau_0 > 0$  such that for all  $\tau \in (0, \tau_0]$ ,  $Z_\tau(n\tau)u_0 \in \ell^{q_0}([0, T]; L^{r_0})$ , and there exists  $C > 0$  such that*

$$\|Z_\tau(n\tau)u_0\|_{\ell^{q_0}([0,T];L^{r_0})} \leq C \|u\|_{L^\infty \mathcal{H}^1}.$$

*Proof.* The idea of the proof is a bootstrap argument: as long as  $Z_\tau(k\tau)u_0$  is bounded in  $\ell^{q_0}L^{r_0}$ , Theorem 5.1 implies that its size is the same as that of  $u$  modulo  $\mathcal{O}(\tau^{1/2})$ , hence a uniform bound for  $\tau > 0$  sufficiently small. Let  $\Lambda = \Lambda(\tau)$  defined by

$$\Lambda = \left\{ N \in \mathbb{N}, \|Z_\tau(k\tau)u_0\|_{\ell^{q_0}(0 \leq n\tau \leq N\tau; L^{r_0})} \leq (\underline{C} + C_1) \|u_0\|_{L^2} + C_1 \|u(n\tau)\|_{\ell^{q_0}([0,T];L^{r_0})} \right\},$$

where  $\underline{C} = \underline{C}(d, \sigma, T)$  is the best constant provided by the homogeneous discrete Strichartz estimate (3.4) in the case  $\lambda = 1/\tau$ , on the time interval  $[0, T]$ ,

$$\underline{C} = \sup_{\tau \in (0,1)} \sup_{\phi \in L^2(\mathbb{R}^d)} \frac{\|S_{1/\tau}\phi\|_{\ell^{q_0}([0,T];L^{r_0})}}{\|\phi\|_{L^2}},$$

and  $C_1 > 0$  is determined below. Note that Sobolev embedding and the finiteness of  $T$  yield (even though this estimate is far from being sharp)

$$\|u(n\tau)\|_{\ell^{q_0}([0,T];L^{r_0})} \lesssim \|u\|_{L^\infty \mathcal{H}^1}.$$

First, regardless of the value of  $C_1$ ,  $\Lambda$  is not empty, as  $0 \in \Lambda$ :

$$\tau^{1/q_0} \|Z_\tau(0)u_0\|_{L^{r_0}} = \tau^{1/q_0} \|S_{1/\tau}(0)u_0\|_{L^{r_0}} \leq \underline{C} \|u_0\|_{L^2}.$$

If  $\Lambda$  is infinite, then (5.1) holds, in view of the above presentation. If  $\Lambda$  is bounded, let  $N_* = N_*(\tau)$  be its largest element: we suppose that  $N_* + 1 < T/\tau$ . Like in the proof of Lemma 5.6, we decompose

$$\frac{N(\tau) - \mathbf{1}}{\tau} Z_\tau(n\tau)u_0 = e^{-i\tau W} \frac{N_0(\tau) - \mathbf{1}}{\tau} Z_\tau(n\tau)u_0 + \frac{e^{-i\tau W} - \mathbf{1}}{\tau} Z_\tau(n\tau)u_0.$$

In view of the discrete Duhamel's formula (see (5.10)), discrete Strichartz estimates yield

$$\begin{aligned} \left( \tau \sum_{k=0}^{N_*+1} \|Z_\tau(k\tau)u_0\|_{L^{r_0}}^{q_0} \right)^{1/q_0} &\leq \underline{C} \|u_0\|_{L^2} + C \left\| \frac{N_0(\tau) - \mathbf{1}}{\tau} Z_\tau(n\tau)u_0 \right\|_{\ell^{q_0'}(0 \leq n\tau \leq N_*\tau; L^{r_0'})} \\ &\quad + C \left\| \frac{e^{-i\tau W} - \mathbf{1}}{\tau} Z_\tau(n\tau)u_0 \right\|_{\ell^1(0 \leq n\tau \leq N_*\tau; L^2)}, \end{aligned}$$

for some  $C = C(d, \sigma, T)$ . Using (5.4), we infer

$$\begin{aligned} \left( \tau \sum_{k=0}^{N_*+1} \|Z_\tau(k\tau)u_0\|_{L^{q_0}}^{q_0} \right)^{1/q_0} &\leq \underline{C} \|u_0\|_{L^2} + C \|Z_\tau(n\tau)u_0\|_{\ell^{(2\sigma+1)q'_0}(0 \leq n\tau \leq N_*\tau; L^{(2\sigma+1)r'_0})}^{2\sigma+1} \\ &\quad + C \|Z_\tau(n\tau)u_0\|_{\ell^1(0 \leq n\tau \leq N_*\tau; L^2)}. \end{aligned}$$

We note that  $(2\sigma+1)r'_0 = r_0$ , and, since

$$(2\sigma+1) \left( \frac{1}{(2\sigma+1)q'_0} - \frac{1}{q_0} \right) = 1 - \frac{2\sigma+2}{q_0} = 1 - \frac{d\sigma}{2},$$

Hölder inequality yields

$$\|Z_\tau(n\tau)u_0\|_{\ell^{(2\sigma+1)q'_0}(I; L^{(2\sigma+1)r'_0})}^{2\sigma+1} \leq |I|^{1-\frac{d\sigma}{2}} \|Z_\tau(n\tau)u_0\|_{\ell^{q_0}(I; L^{r_0})}^{2\sigma+1}.$$

We infer

$$\begin{aligned} \|Z_\tau(n\tau)u_0\|_{\ell^{q_0}(0 \leq n\tau \leq (N_*+1)\tau; L^{r_0})} &\leq \underline{C} \|u_0\|_{L^2} + C(T) \|Z_\tau(n\tau)u_0\|_{\ell^{q_0}(0 \leq n\tau \leq N_*\tau; L^{r_0})}^{2\sigma+1} \\ &\quad + C(T) \|u_0\|_{L^2}. \end{aligned}$$

The definition of  $N_*$  and Theorem 5.1 imply, for  $\tau \leq 1$ ,

$$\begin{aligned} \|Z_\tau(n\tau)u_0\|_{\ell^{q_0}(0 \leq n\tau \leq (N_*+1)\tau; L^{r_0})} &\leq \underline{C} \|u_0\|_{L^2} + C(T) \|u(n\tau)\|_{\ell^{q_0}(0 \leq n\tau \leq N_*\tau; L^{r_0})}^{2\sigma+1} \\ &\quad + C(T) \|u_0\|_{L^2} + C_2\tau^{1/2}. \end{aligned}$$

We can then set

$$C_1 = 2C(T) + 2C(T) \|u(n\tau)\|_{\ell^{q_0}(0 \leq n\tau \leq T; L^{r_0})}^{2\sigma},$$

so

$$\begin{aligned} \|Z_\tau(n\tau)u_0\|_{\ell^{q_0}(0 \leq n\tau \leq (N_*+1)\tau; L^{r_0})} &\leq \underline{C} \|u_0\|_{L^2} + \frac{C_1}{2} \|u_0\|_{L^2} \\ &\quad + \frac{C_1}{2} \|u(n\tau)\|_{\ell^{q_0}(0 \leq n\tau \leq N_*\tau; L^{r_0})} + C_2\tau^{1/2}. \end{aligned}$$

Taking  $\tau > 0$  sufficiently small then contradicts the maximality of  $N_*$ , hence the lemma.  $\square$

**6.2. Case  $\sigma \geq 2/d$ .** When  $V = 0$ , the case  $u \in X_T^1$  is considered in [14]: a central role is played by Proposition 5.1 there, where the authors prove a local stability result at the  $H^1$  level, based on continuity arguments and the density of  $H^2(\mathbb{R}^d)$  in  $H^1(\mathbb{R}^d)$ . Typically, (5.5) in [14] is exactly the convergence stated in the last point of Theorem 5.1 here. In the present context, we shall therefore prove that there exists  $M$  such that

$$(6.1) \quad \|H^{1/2}u^n\|_{\ell^\infty([0, T]; L^2)} = \|H^{1/2}Z_\tau(n\tau)\|_{\ell^\infty([0, T]; L^2)} \leq M.$$

We first consider the case  $u \in X_T^2$ :

**Lemma 6.2** (Local stability,  $L^2$ -(super)critical case). *Let  $2/d \leq \sigma < 2/(d-2)_+$ , with  $\sigma \geq 1/2$ ,  $u_0 \in \mathcal{H}^2$ , and  $T > 0$  such that  $u \in X_T^2$ . There exists  $\tau_0 > 0$  such that for all  $\tau \in (0, \tau_0]$ ,  $Z_\tau(n\tau)u_0 \in \ell^\infty([0, T]; \mathcal{H}^1)$ , and there exists  $C > 0$  such that*

$$\|H^{1/2}Z_\tau(n\tau)u_0\|_{\ell^\infty([0, T]; L^2)} \leq C \|u\|_{X_T^1},$$

*Proof.* The spirit of the proof is the same as for Lemma 6.1. Set

$$\begin{aligned} \Lambda = \left\{ N \in \mathbb{N}, \quad & \|H^{1/2}Z_\tau(n\tau)u_0\|_{\ell^{q_0}(0, N\tau; L^{r_0})} + \|H^{1/2}Z_\tau(n\tau)u_0\|_{\ell^\infty(0, N\tau; L^2)} \right. \\ & \left. \leq \underline{K}\|H^{1/2}u_0\|_{L^2} + C_1\|u\|_{X_T^1} \right\}, \end{aligned}$$

where  $\underline{K} = \underline{K}(d, \sigma, T)$  is defined by

$$\underline{K} = \sup_{\tau \in (0, 1)} \sup_{\phi \in \mathcal{H}^1} \frac{\|H^{1/2}S_{1/\tau}\phi\|_{\ell^{q_0}([0, T]; L^{r_0})} + \|H^{1/2}S_{1/\tau}\phi\|_{\ell^\infty([0, T]; L^2)}}{\|H^{1/2}\phi\|_{L^2}},$$

and  $C_1$  will be given in the course of the proof. This set is not empty, as

$$H^{1/2}Z_\tau(0)u_0 = H^{1/2}S_{1/\tau}u_0,$$

and so  $0 \in \Lambda$ . If  $\Lambda$  is infinite, then (5.1) holds, in view of the above presentation. If  $\Lambda$  is bounded, let  $N_* = N_*(\tau)$  be its largest element and we suppose that  $N_* + 1 < T/\tau$ .

Strichartz estimates yield, for  $(q, r) \in \{(q_0, r_0), (\infty, 2)\}$ ,

$$\begin{aligned} \|H^{1/2}Z_\tau(n\tau)u_0\|_{\ell^q([0, (N_*+1)\tau]; L^r)} &\leq \underline{K}\|H^{1/2}u_0\|_{L^2} \\ &+ C \left\| H^{1/2}e^{-i\tau W} \frac{N_0(\tau) - \mathbf{1}}{\tau} Z_\tau(n\tau)u_0 \right\|_{\ell^{q'_0}([0, N_*\tau]; L^{r'_0})} \\ &+ C \left\| H^{1/2} \frac{e^{-i\tau W} - \mathbf{1}}{\tau} Z_\tau(n\tau)u_0 \right\|_{\ell^1([0, N_*\tau]; L^2)}. \end{aligned}$$

Invoking Lemmas 2.12 and 5.2,

$$\begin{aligned} &\left\| H^{1/2} \frac{N_0(\tau) - \mathbf{1}}{\tau} Z_\tau(n\tau)u_0 \right\|_{\ell^{q'_0}([0, N_*\tau]; L^{r'_0})} \\ &\lesssim \sum_{J \in \{\nabla, \sqrt{\nabla}, \mathbf{1}\}} \| |Z_\tau(n\tau)u_0|^{2\sigma} J Z_\tau(n\tau)u_0 \|_{\ell^{q'_0}([0, N_*\tau]; L^{r'_0})} \\ &\lesssim \|Z_\tau(n\tau)u_0\|_{\ell^\theta([0, N_*\tau]; L^{r_0})}^{2\sigma} \sum_{J \in \{\nabla, \sqrt{\nabla}, \mathbf{1}\}} \|J Z_\tau(n\tau)u_0\|_{\ell^{q_0}([0, N_*\tau]; L^{r_0})} \\ &\lesssim \|Z_\tau(n\tau)u_0\|_{\ell^\theta([0, N_*\tau]; L^{r_0})}^{2\sigma} \|H^{1/2}Z_\tau(n\tau)u_0\|_{\ell^{q_0}([0, N_*\tau]; L^{r_0})} \end{aligned}$$

On the other hand,

$$\begin{aligned} \|Z_\tau(n\tau)u_0\|_{\ell^\theta([0, N_*\tau]; L^{r_0})} &\lesssim \|Z_\tau(n\tau)u_0\|_{\ell^\theta([0, N_*\tau]; H^1)} \\ &\leq C(T) \|Z_\tau(n\tau)u_0\|_{\ell^\infty([0, N_*\tau]; H^1)} \\ &\leq C(T) \|H^{1/2}Z_\tau(n\tau)u_0\|_{\ell^\infty([0, N_*\tau]; L^2)}. \end{aligned}$$

The factor  $e^{-i\tau W}$  does not change the nature of this estimate, and we find, setting  $T_* = (N_* + 1)\tau$ , for some constants depending on  $T$ ,

$$\begin{aligned} \|H^{1/2}Z_\tau(n\tau)u_0\|_{\ell^q([0, T_*]; L^r)} &\leq \underline{K}\|H^{1/2}u_0\|_{L^2} \\ &+ C \|H^{1/2}Z_\tau(n\tau)u_0\|_{\ell^\infty([0, N_*\tau]; L^2)}^{2\sigma} \|H^{1/2}Z_\tau(n\tau)u_0\|_{\ell^{q_0}([0, N_*\tau]; L^{r_0})} \\ &+ C \|H^{1/2}Z_\tau(n\tau)u_0\|_{\ell^\infty([0, N_*\tau]; L^2)}. \end{aligned}$$

Summing the cases  $(q, r) = (q_0, r_0)$  and  $(q, r) = (\infty, 2)$ , we infer

$$\begin{aligned} \sum_{(q,r) \in \{(q_0, r_0), (\infty, 2)\}} \|H^{1/2} Z_\tau(n\tau) u_0\|_{\ell^q([0, T_*]; L^r)} &\leq 2\underline{K} \|H^{1/2} u_0\|_{L^2} \\ &+ C \|H^{1/2} Z_\tau(n\tau) u_0\|_{\ell^\infty([0, N_*\tau]; L^2)}^{2\sigma} \|H^{1/2} Z_\tau(n\tau) u_0\|_{\ell^{q_0}([0, N_*\tau]; L^{r_0})} \\ &+ C \|H^{1/2} Z_\tau(n\tau) u_0\|_{\ell^\infty([0, N_*\tau]; L^2)}. \end{aligned}$$

We then set

$$C_1 = 2C + 2C \|H^{1/2} u(n\tau), \|_{\ell^\infty([0, T]; L^2)}^{2\sigma},$$

so the definition of  $N_*$  and Theorem 5.1 yield, for all  $\tau \leq 1$ ,

$$\begin{aligned} \sum_{(q,r) \in \{(q_0, r_0), (\infty, 2)\}} \|H^{1/2} Z_\tau(n\tau) u_0\|_{\ell^q([0, T_*]; L^r)} &\leq 2\underline{K} \|H^{1/2} u_0\|_{L^2} \\ &+ \frac{C_1}{2} \|H^{1/2} u(n\tau)\|_{\ell^{q_0}([0, N_*\tau]; L^{r_0})} \\ &+ \frac{C_1}{2} \|H^{1/2} u(n\tau)\|_{\ell^\infty([0, N_*\tau]; L^2)} + C\tau^{1/2}. \end{aligned}$$

For  $\tau > 0$  sufficiently small, the maximality of  $N_*$  leads to a contradiction, hence the lemma.  $\square$

We have assumed so far  $u \in X_T^2$ . In Theorem 1.5, we assume only  $u \in X_T^1$ , and the nonlinearity is supposed to be  $C^2$  (or almost) when  $\sigma \geq 2/d$ : like in [14], passing from the  $X_T^2$  case to the  $X_T^1$  case relies on a density argument:

**Proposition 6.3.** *Let Assumption 1.1 be verified and  $\frac{1}{2} \leq \sigma < \frac{2}{(d-2)_+}$ . For any  $M \geq 1$ , there exists  $T > 0$  and  $C = C(M, d, \sigma)$  such that if  $u_0, v_0 \in \mathcal{H}^1$  with  $\|u_0\|_{\mathcal{H}^1}, \|v_0\|_{\mathcal{H}^1} \leq M$ , then the solutions  $u$  and  $v$  to (1.1) with respective initial data  $u_0$  and  $v_0$  are such that  $u, v \in X_T^1$ , and*

$$\|u - v\|_{X_T^1} \leq C \|u_0 - v_0\|_{\mathcal{H}^1}.$$

We refer to [11, Theorem 2.3] for a proof which is readily adapted to the present framework. We also invoke the following result, whose proof is essentially the same as for [14, Proposition 5.1, (5.4)] or [11, Proposition 6.2, (6.4)]:

**Proposition 6.4.** *Let Assumption 1.1 be verified and  $\frac{1}{2} \leq \sigma < \frac{2}{(d-2)_+}$ . For any  $M \geq 1$ , there exists  $T > 0$  and  $C = C(M, d, \sigma)$  such that if  $u_0, v_0 \in \mathcal{H}^1$  with  $\|u_0\|_{\mathcal{H}^1}, \|v_0\|_{\mathcal{H}^1} \leq M$ , then for all  $\tau \in (0, 1]$ , and all admissible pair  $(q, r)$ ,*

$$\left\| H^{1/2} (Z_\tau(n\tau) u_0 - Z_\tau(n\tau) v_0) \right\|_{\ell^q([0, T]; L^r)} \leq C \|u_0 - v_0\|_{\mathcal{H}^1}.$$

In view of Theorem 5.1, Propositions 6.3 and 6.4, we have (like in [14, Proposition 5.1, (5.6)]), by approaching  $u_0 \in \mathcal{H}^1$  by  $v_0 \in \mathcal{H}^2$  with  $\|u_0 - v_0\|_{\mathcal{H}^1} \leq \varepsilon$  and eventually letting  $\varepsilon$  go to zero:

$$\lim_{\tau \rightarrow 0} \|Z_\tau(n\tau) u_0 - u(n\tau)\|_{\ell^\infty([0, T]; \mathcal{H}^1)} = 0.$$

In particular, for  $\tau_0 > 0$  sufficiently small and  $\tau \in (0, \tau_0]$ , we know that

$$\|Z_\tau(n\tau) u_0\|_{\ell^\infty([0, T]; \mathcal{H}^1)} \leq M$$

for some  $M$  independent of  $\tau \in (0, \tau_0]$ , which is (slightly more than) what we wanted to prove.

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