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# Isolated singularities of solutions of a 2-D diffusion equation with mixed reaction

Yimei Li\*  
Laurent Véron †

**Abstract** We study the local properties of positive solutions of the equation  $-\Delta u + ae^{bu} = m|\nabla u|^q$  in a punctured domain  $\Omega \setminus \{0\}$  of  $\mathbb{R}^2$  where  $m, a, b$  are positive parameters and  $q > 1$ . We study particularly the existence of solutions with an isolated singularity and the local behaviour of such singular solutions.

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*Key words.* elliptic equations; a priori estimates; isolated singularities.

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## 1 Introduction

The following type of nonlinear equation

$$-\Delta u + f(u) - g(|\nabla u|) = 0 \quad (1.1)$$

in a  $N$  dimensional domain when  $f$  and  $g$  are nondecreasing continuous functions is an interesting model for studying the interaction between the diffusion, the reaction and the absorption when the last two quantities are of a totally different nature. The most interesting case correspond to the case where the two reaction functions  $f$  and  $g$  increasing in their respective variable. In its full generality the problem is difficult to handle, but the model cases where  $f(u) = u^p$  if  $N \geq 3$

$$-\Delta u + u^p - m|\nabla u|^q = 0. \quad (1.2)$$

has already been studied in [5]. The results therein shows the rich interaction of the phenomena coming either from the interaction of the diffusion and the absorption, the diffusion and the reaction or the reaction and the absorption. A variant of this equation is the Chipot-Weissler equation

$$-\Delta u - u^p + m|\nabla u|^q = 0, \quad (1.3)$$

which presents also interesting interaction between the pairing of the three terms. This has been studied in [3] and [6]. For this last equation the main difficulty is to obtain a priori estimates of positive solutions near an isolated singularity. These estimates have been obtained in [3] and [6]. The aim of the present paper is to study a 2-dimensional problem of type (1.2) where naturally the power absorption is replaced by an exponential absorption. Our model equation is

$$\mathcal{E}(u) := -\Delta u + ae^{bu} - m|\nabla u|^q = 0 \quad (1.4)$$

in  $B_1 \setminus \{0\}$  where  $B_1$  is the unit disk of  $\mathbb{R}^2$ ,  $m, a, b$  are positive parameters and  $q > 1$ . For (1.2) and (1.1) the a priori estimate near an isolated singularity is easy to obtain, the difficulty lies in the full description of singularities and the classifications of singular solutions. Three particular equations are naturally associated to equation (1.4):

the Emden's equation

$$-\Delta u + ae^{bu} = 0, \quad (1.5)$$

the Riccati equation

$$-\Delta u - m|\nabla u|^q = 0, \quad (1.6)$$

and the eikonal equation

$$ae^{bu} - m|\nabla u|^q = 0. \quad (1.7)$$

The predominance of a specific equation of this type depends on the value of  $q$ , with critical value  $q = 2$ . When  $1 < q < 2$  the singularities are similar to the one of Emden's equation, an equation which has been completely described in [18]. We recall the main result therein:

**Vázquez theorem** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded smooth domain containing 0 and  $a, b > 0$ . Then for any  $\phi \in C^1(\partial\Omega)$  and  $\gamma \in [0, \frac{2}{b}]$  there exists a unique function  $v_\gamma \in C^1(\overline{\Omega} \setminus \{0\})$  such that  $e^{bv_\gamma} \in L^1(\Omega)$  satisfying*

$$-\Delta v_\gamma + ae^{bv_\gamma} = 2\pi\gamma\delta_0 \quad \text{in } \mathcal{D}'(\Omega). \quad (1.8)$$

coinciding with  $\phi$  on  $\partial\Omega$ . Furthermore

$$\lim_{x \rightarrow 0} \frac{v_\gamma(x)}{-\ln|x|} = \gamma. \quad (1.9)$$

The condition  $0 \leq \gamma \leq \frac{2}{b}$  is also a necessary condition for the solvability of (1.8). When  $q = 2$ , the change of unknown  $u = \frac{1}{m} \ln v$  transform (1.4) into

$$-\Delta v + mav^{\frac{b}{m}+1} = 0, \quad (1.10)$$

an equation which has been thoroughly studied (e.g. [11], [20]) and thus we will omit it. When  $q > 2$  the dominant equation is the eikonal one. There is a particular explicit solution which plays a key role in the studies of singularities

$$U(|x|) = \frac{q}{b} \left(\frac{m}{a}\right)^{\frac{1}{q}} \ln\left(\frac{q}{b|x|}\right). \quad (1.11)$$

We summarise below our three main results

**Theorem 1** *Let  $m, a, b > 0$  and  $1 < q < 2$ . If  $u \in C^1(\overline{B}_1 \setminus \{0\})$  is a positive function satisfying (1.4). Then there exists  $\gamma \in [0, \frac{2}{b}]$  such that*

$$\lim_{x \rightarrow 0} \frac{u(x)}{-\ln|x|} = \gamma. \quad (1.12)$$

Furthermore  $e^{bu}$  and  $|\nabla u|^q$  are integrable in  $B_1$  and there holds

$$-\Delta u + ae^{bu} - m|\nabla u|^q = 2\pi\delta_0 \quad \text{in } \mathcal{D}'(B_1). \quad (1.13)$$

If  $\gamma = 0$ ,  $u$  can be extended as a  $C^1(\overline{B}_1)$  function.

Conversely we prove

**Theorem 2** *Under the assumptions on  $m, a, b, q$  of Theorem 1, for any  $\gamma \in (0, \frac{2}{b}]$  and  $\phi \in C^1(\partial B_1)$  there exists a unique function  $u \in C^1(\overline{B}_1 \setminus \{0\})$  satisfying (1.13) and coinciding with  $\phi$  on  $\partial B_1$ . Furthermore  $e^{bu}$  and  $|\nabla u|^q$  are integrable in  $B_1$  and (1.13) holds.*

When  $q > 2$ , the classification of the solutions is totally different and present some similarity with the study of isolated singularities of Lane-Emden equation

$$-\Delta u - u^p = 0, \quad (1.14)$$

in the supercritical case  $N \geq 3$  and  $p > \frac{N}{N-2}$  (see [14]). We prove the following characterisation of singular solutions,

**Theorem 3** *Let  $m, a, b > 0$  and  $q > 2$ . If  $u \in C^1(\overline{B}_1 \setminus \{0\})$  is a positive solution of (1.4), then (i) either*

$$\lim_{x \rightarrow 0} \frac{u(x)}{-\ln|x|} = \frac{q}{b}, \quad (1.15)$$

(ii) or  $u$  can be extended to  $\overline{B_1}$  as a Hölder continuous function.

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## 2 A priori estimates

The next estimates are valid in any dimension

**Proposition 2.1** *Let  $m, a, b > 0$  and  $q > 1$ . Assume that for  $x_0 \in B_1 \subset \mathbb{R}^N$  and  $0 < R < |x_0|$   $u$  is a solution of (1.4) in  $B_R(x_0)$ . Then there exists  $\Lambda = \Lambda(m, q, N, a, b) > 0$  such that*

(i) If  $1 < q < 2$ ,

$$u(x_0) \leq \frac{2}{b} \ln \frac{1}{|x_0|} + \Lambda. \quad (2.1)$$

(ii) If  $q > 2$

$$u(x_0) \leq \frac{q}{b} \ln \frac{1}{|x_0|} + \tilde{\Lambda}. \quad (2.2)$$

*Proof.* Let  $R' < R$ . We set for  $\lambda, \mu > 0$ ,

$$\psi(x) = \lambda \ln(1/(R'^2 - |x - x_0|^2)) + \mu = \lambda \ln(1/(R'^2 - r^2)) + \mu,$$

with  $r = |x - x_0|$ . Then

$$\begin{aligned} ae^{b\psi} &= \frac{ae^{b\mu}}{(R'^2 - r^2)^{b\lambda}}, \\ |\nabla\psi(x)| &= |\psi_r(r)| = \frac{2\lambda r}{R'^2 - r^2}, \\ |\nabla\psi(x)|^q &= \frac{2^q \lambda^q r^q}{(R'^2 - r^2)^q}, \\ -\Delta\psi(x) &= -2\lambda \frac{NR'^2 - (N-2)r^2}{(R'^2 - r^2)^2}. \end{aligned}$$

Hence there holds in  $B_{R'}(x_0)$ ,

$$\mathcal{E}(\psi) = \frac{ae^{b\mu}}{(R'^2 - r^2)^{b\lambda}} - 2\lambda \frac{NR'^2 - (N-2)r^2}{(R'^2 - r^2)^2} - \frac{m2^q \lambda^q r^q}{(R'^2 - r^2)^q}.$$

We have  $\psi > u$  in a neighbourhood of  $\partial B_{R'}(x_0)$ . We encounter two cases

(i)  $1 < q \leq 2$ . Then we take  $b\lambda = 2$  and get

$$\begin{aligned} \mathcal{E}(\psi) &= (R'^2 - r^2)^{-2} \left[ ae^{b\mu} - \frac{4}{b} (NR'^2 - (N-2)r^2) - \frac{4^q m r^q}{b^q} (R'^2 - r^2)^{2-q} \right] \\ &\geq (R'^2 - r^2)^{-2} \left[ ae^{b\mu} - \Lambda^* \max\{R'^2, R'^{4-q}\} \right] \\ &\geq (R'^2 - r^2)^{-2} \left[ ae^{b\mu} - \Lambda^* R'^2 \right], \end{aligned} \quad (2.3)$$

where  $\Lambda^* = \Lambda^*(N, q, m, b) > 0$ . We choose

$$\mu = \frac{2}{b} \ln R' + \frac{1}{b} \ln \left( \frac{\Lambda^*}{a} \right) \quad (2.4)$$

and derive that  $\mathcal{E}(\psi) \geq 0$  in  $B_{R'}(x_0)$ . Hence  $u \leq \psi$  therein. This implies

$$u(x_0) \leq \psi(x_0) := \frac{2}{b} \ln \frac{1}{R'} + \frac{1}{b} \ln \left( \frac{\Lambda^*}{a} \right). \quad (2.5)$$

Letting  $R' \uparrow R = |x_0|$  yields (2.1).

(ii)  $q > 2$ . Then we take  $b\lambda = q$ . Hence

$$\begin{aligned} \mathcal{E}(\psi) &= (R^2 - r^2)^{-q} \left[ ae^{b\mu} - \frac{q}{b} (NR^2 - (N-2)r^2) (R^2 - r^2)^{q-2} - \frac{(2q)^q mr^q}{b^q} \right] \\ &\geq (R^2 - r^2)^{-q} \left[ ae^{b\mu} - \tilde{\Lambda}^* R^q \right] \end{aligned} \quad (2.6)$$

where  $\tilde{\Lambda}^* = \tilde{\Lambda}^*(N, q, m, a, b) > 0$ . We take

$$\mu = \frac{q}{b} \ln R + \frac{1}{b} \ln \left( \frac{\tilde{\Lambda}^*}{a} \right), \quad (2.7)$$

and derive as in case (i)

$$u(x_0) \leq \psi(x_0) := \frac{q}{b} \ln \left( \frac{1}{R'} \right) + \frac{1}{b} \ln \left( \frac{\tilde{\Lambda}^*}{a} \right). \quad (2.8)$$

Letting  $R' \uparrow R$  we obtain (2.2). □

In the next lemma we give some precise estimates on the function  $v_\gamma$  solution of (1.8)

**Lemma 2.2** *Let  $m, a, b > 0$  and  $0 < \gamma \leq \frac{2}{b}$ . If  $v_\gamma \in C^1(\overline{B}_1 \setminus \{0\})$  is the solution (1.8) which vanishes on  $\partial B_1$ , then*

1- *If  $0 \leq \gamma < \frac{2}{b}$ , there exists  $\phi^* \in C^1(\overline{B}_1)$  such that*

$$v_\gamma(x) = \gamma \ln \frac{1}{|x|} - \phi^*(x) \quad \text{for all } x \in \overline{B}_1 \setminus \{0\}. \quad (2.9)$$

2- *If  $\gamma = \frac{2}{b}$ , then for all  $x \in \overline{B}_1 \setminus \{0\}$  we have*

$$\frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) + \kappa_2 \leq v_{\frac{2}{b}}(x) \leq \frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) + \kappa_1 \quad (2.10)$$

where

$$\kappa_1 = \max \left\{ 0, \frac{1}{b} \ln \frac{2}{ab} \right\} \quad \text{and} \quad \kappa_2 = \min \left\{ 0, \frac{1}{b} \ln \frac{2}{ab} \right\}. \quad (2.11)$$

In particular

$$v_{\frac{2}{b}}(x) = \frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right), \quad (2.12)$$

if  $ab = 2$ .

*Proof.* Let  $\gamma \in (0, \frac{2}{b}]$  and  $\phi_1$  be the first eigenfunction of  $-\Delta$  in  $W_0^{1,2}(B_1)$ , with corresponding eigenvalue  $\lambda_1 > 0$  and supremum equal to 1 (at  $r = 0$ ). For  $\theta > 0$  and  $A > 0$ , we set

$$h_A(x) = \gamma \ln \frac{1}{|x|} - A|x|^\theta \phi_1(x) = \gamma \ln \frac{1}{r} - Ar^\theta \phi_1(r) \quad \text{with } r = |x|. \quad (2.13)$$

Then there holds in  $\mathcal{D}'(B_1)$ ,

$$-\Delta h_A(r) = 2\pi\gamma\delta_0 + A\theta^2 r^{\theta-2} \phi_1(r) - A\lambda_1 r^\theta \phi_1(r) + 2A\theta r^{\theta-1} \phi_1'(r),$$

thus

$$-\Delta h_A + ae^{bh_A} = 2\pi\gamma\delta_0 + r^{\theta-2} \left( A\theta^2 \phi_1(r) - A\lambda_1 r^2 \phi_1(r) + 2A\theta r \phi_1'(r) + ar^{2-\theta-b\gamma} e^{-bAr^\theta \phi_1(r)} \right).$$

*Step 1: the case  $b\gamma < 2$*

(i) *Estimate from above.* Take  $\theta = 2 - b\gamma$ . Since

$$ae^{-bAr^{2-b\gamma} \phi_1(r)} \geq a - abAr^{2-b\gamma} \phi_1(r),$$

we derive that for all  $0 < |x| < 1$ ,

$$\begin{aligned} -\Delta h_A + ae^{bh_A} &= r^{-b\gamma} \left( A(2 - b\gamma)^2 \phi_1(r) - A\lambda_1 r^2 \phi_1(r) + 2A(2 - b\gamma)r \phi_1'(r) + ae^{-bAr^{2-b\gamma} \phi_1(r)} \right) \\ &\geq r^{-b\gamma} \left( A(2 - b\gamma)^2 \phi_1(r) - A\lambda_1 r^2 \phi_1(r) + 2A(2 - b\gamma)r \phi_1'(r) + a - abAr^{2-b\gamma} \phi_1(r) \right). \end{aligned}$$

Then there exists  $A_0 > 0$  such that for  $A \leq A_0$  we have

$$-\Delta h_A + ae^{bh_A} \geq 2\gamma\pi\delta_0 \quad \text{in } \mathbf{D}'(B_1), \quad (2.14)$$

and  $A_0$  can be taken independent of  $\gamma$ . We derive by the comparison principle that

$$v_\gamma(x) \leq \gamma \ln \frac{1}{|x|} - A|x|^{2-b\gamma} \phi_1(x) \quad \text{in } B_1 \setminus \{0\}. \quad (2.15)$$

(ii) *Estimate from below.* We give an explicit expression for  $v_\gamma$ . We already know that  $v_\gamma(x) < \gamma \ln \frac{1}{|x|}$ . If we write

$$v_\gamma(x) = \gamma \ln \frac{1}{|x|} + \phi^*(x), \quad (2.16)$$

then  $\phi^*$  vanishes on  $\partial B_1$  and satisfies

$$-\Delta \phi^* + ar^{-b\gamma} e^{b\phi^*} = 0.$$

The convex functional

$$J(\phi) = \int_{B_1} \left( \frac{1}{2} |\nabla \phi|^2 + \frac{a}{b|x|^{b\gamma}} e^{b\phi} \right) dx$$

is well defined on the closed subset  $K_-$  of  $W_0^{1,2}(B_1)$  of nonpositive functions and it satisfies

$$J(\phi) \geq \frac{1}{2} \int_{B_1} |\nabla \phi|^2 dx.$$

Hence it achieves its minimum in  $K_-$  at  $\phi^*$ . There holds

$$-\Delta(\gamma \ln \frac{1}{|x|} - A|x|^{2-b\gamma}) = A\theta^2|x|^{-b\gamma}$$

and  $\phi^*$  vanishes on  $\partial B_1$ . Since  $|x|^{2-b\gamma} \in L^{1+\epsilon}(B_1)$ ,  $\phi^* \in W^{2,1+\epsilon}(B_1) \subset C^{0, \frac{2\epsilon}{1+\epsilon}}(\overline{B_1})$ . Consequently

$$v_\gamma(x) = \gamma \ln \frac{1}{|x|} + \phi^*(x) \quad \text{in } B_1 \setminus \{0\}, \quad (2.17)$$

and

$$v_\gamma(x) - \gamma \ln \frac{1}{|x|} \rightarrow \phi^*(0) < 0 \quad \text{as } x \rightarrow 0. \quad (2.18)$$

*Step 2: the case  $b\gamma = 2$ .* For  $\kappa \in \mathbb{R}$  we set

$$\psi_\kappa(r) = \frac{2}{b} \left( \ln \frac{1}{r} - \ln(1 - \ln r) \right) + \kappa.$$

Then  $\psi_\kappa(1) = \kappa$  and

$$-\Delta\psi_\kappa + ae^{b\psi_\kappa} = \frac{1}{r^2} \left( -\frac{2}{b(1 - \ln r)^2} + \frac{ae^{b\kappa}}{(1 - \ln r)^2} \right). \quad (2.19)$$

Since

$$\int_{B_1} \frac{dx}{|x|^2(1 - \ln|x|)^2} = 2\pi \int_0^1 \frac{dr}{r(1 - \ln r)^2} = 2\pi$$

the function  $e^{b\psi_\kappa}$  is integrable in  $B_1$  and  $\psi_\kappa$  satisfies in  $\mathcal{D}'(B_1)$

$$-\Delta\psi_\kappa + ae^{b\psi_\kappa} = \left( ae^{b\kappa} - \frac{2}{b} \right) \frac{1}{|x|^2(1 - \ln|x|)^2} + \frac{4\pi}{b} \delta_0. \quad (2.20)$$

Hence  $\psi_\kappa$  is a supersolution nonnegative on  $\partial B_1$  if we choose  $\kappa = \kappa_1$  where

$$\kappa_1 = \max \left\{ 0, \frac{1}{b} \ln \frac{2}{ab} \right\} \quad (2.21)$$

It is a subsolution if we take  $\kappa = \kappa_2$  where

$$\kappa_2 = \min \left\{ 0, \frac{1}{b} \ln \frac{2}{ab} \right\}. \quad (2.22)$$

Therefore

$$\frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) + \kappa_2 \leq v_{\frac{2}{b}}(x) \leq \frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) + \kappa_1, \quad (2.23)$$

with equality if  $b = \frac{2}{a}$  since in that case  $\kappa_1 = \kappa_2 = 0$ .

□



### 3 Singular solutions in the case $1 < q < 2$

**Definition 3.1** A function  $u \in C^1(\overline{B}_1 \setminus \{0\})$  is a solution of the singular Dirichlet problem

$$\begin{aligned} -\Delta u + ae^{bu} - m|\nabla u|^q &= 2\pi\gamma\delta_0 && \text{in } \mathcal{D}'(B_1) \\ u &= \phi && \text{on } \partial B_1, \end{aligned} \quad (3.1)$$

where  $\phi$  is a  $C^1$  function, if  $u$ ,  $e^{bu}$  and  $|\nabla u|^q$  are integrable in  $B_1$  and

$$\int_{B_1} \left( -u\Delta\zeta + \left( ae^{bu} - m|\nabla u|^q \right) \zeta \right) dx = 2\pi\gamma\zeta(0) \quad \text{for all } \zeta \in C_c^2(B_1). \quad (3.2)$$

In view of Vazquez theorem, any solution  $u_\gamma$  of (3.1) is bounded from below by the solution  $v_\gamma$

#### 3.1 Radial solutions

The expression of (1.2) for a radial function  $u := u(r)$  defined in  $B_1 \setminus \{0\}$  is

$$-u'' - \frac{1}{r}u' + ae^{bu} - m|u'|^q = 0 \quad \text{for } 0 < r < 1. \quad (3.3)$$

As a first gradient estimate for radial solutions we prove

**Lemma 3.2** Let  $1 < q < 2$  and  $u$  be a nonnegative solution of (3.3) in  $B_1 \setminus \{0\}$ . Then for any  $0 < r_0 < 1$  there exists  $c = c(m, q) > 0$  such that

$$|u_r(r)| \leq \frac{c}{r} \quad \text{for all } 0 < r \leq r_0. \quad (3.4)$$

*Proof.* Clearly  $u$  cannot have a local maximum and up to a translation (it changes only the value of  $a$ ) we can assume that  $u$  vanishes on  $\partial B_1$ , hence it is decreasing. There holds

$$-u'' - \frac{1}{r}u' - m|u'|^q \leq 0,$$

then

$$\left( \frac{1}{1-q}|u'|^{1-q} \right)' + \frac{1}{r}|u'|^{1-q} \leq m,$$

and thus

$$\left( r^{1-q}|u'|^{1-q} + m\frac{q-1}{2-q}r^{2-q} \right)' \geq 0.$$

Therefore the function  $r \mapsto r^{1-q}|u'|^{1-q} + m\frac{q-1}{2-q}r^{2-q}$  is increasing and nonnegative, and thus it admits a limit  $\ell \geq 0$  when  $r \rightarrow 0$ . This limit is the same as the one of  $r \mapsto r^{1-q}|u'|^{1-q}$ . If  $\ell > 0$

$$r^{1-q}|u'|^{1-q} = \ell(1 - o(1)) \quad \text{when } r \rightarrow 0.$$

Hence

$$\lim_{r \rightarrow 0} r^{1-q}|u'|^{1-q} = \ell$$

which implies (3.4). If  $\ell = 0$ , then

$$\lim_{r \rightarrow 0} r|u'(r)| = \infty.$$

This implies that for all  $A > 0$ , one has

$$\liminf_{r \rightarrow 0} \left( u(r) - A \ln \frac{1}{r} \right) = \infty.$$

This contradicts the a priori estimate obtained before.  $\square$

The next easy to prove result will be used several times in the sequel.

**Lemma 3.3** *Let  $\gamma \geq 0$ ,  $F \in L^1(B_1)$ ,  $\phi \in C(\partial B_1)$  and  $w \in L^1(B_1) \cap C(\bar{B}_1 \setminus \{0\})$  be the weak solution of*

$$\begin{aligned} -\Delta w &= F + 2\pi\gamma\delta_0 && \text{in } \mathcal{D}'(B_1) \\ w &= \phi && \text{in } \partial B_1. \end{aligned} \quad (3.5)$$

Then there holds

$$\bar{w}(r) = \gamma \ln \frac{1}{r} + o(\ln r) \quad \text{as } |x| = r \rightarrow 0 \quad (3.6)$$

where  $\bar{w}(r) = \frac{1}{2\pi} \int_0^{2\pi} w(r, \theta) d\theta$ . If we assume that  $F \in L^{1+\delta}(B_1)$  for some  $\delta > 0$ , then

$$w(x) = \gamma \ln \frac{1}{|x|} + O(1) \quad \text{as } |x| \rightarrow 0 \quad (3.7)$$

*Proof.* We write  $w(x) = \gamma E(x) + \psi(x)$  where  $E(x) = \ln \frac{1}{|x|}$  and  $\psi$  is the solution of

$$\begin{aligned} -\Delta \psi &= F && \text{in } \mathcal{D}'(B_1) \\ \psi &= \phi && \text{in } \partial B_1. \end{aligned} \quad (3.8)$$

Let  $\bar{\psi}(r)$ ,  $\bar{\phi}$  and  $\bar{F}(r)$  be the circular averages of  $\psi(r, \cdot)$ ,  $\phi(\cdot)$  and  $\bar{F}(r, \cdot)$ . Since

$$-(r\bar{\psi}'(r))' = r\bar{F}(r),$$

A standard approximation argument associated to uniqueness yields

$$\bar{\psi}(r) = \bar{\phi} + \int_r^1 s^{-1} \int_0^s t\bar{F}(t) dt ds.$$

By l'Hospital's rule we obtain (3.6). If  $F \in L^{1+\delta}(B_1)$ , then  $\psi \in W_{loc}^{2,1+\delta}(B_1)$ . By Morrey-Sobolev imbedding inequality,  $W_{loc}^{2,1+\delta}(B_1) \subset C^{0, \frac{2\delta}{1+\delta}}(B_1)$ . Hence  $\psi$  is Hölder continuous in  $B_1$ , which ends the proof.  $\square$

**Lemma 3.4** *Let  $1 < q < 2$ ,  $m, a, b > 0$  and  $\gamma \in (0, \frac{2}{b}]$ . If  $u := u_\gamma$  is a radial solution of*

$$-\Delta u + ae^{bu} - m|\nabla u|^q = 2\pi\gamma\delta_0 \quad \text{in } \mathcal{D}'(B_1)$$

which vanishes on  $\partial B_1$ , then

1- If  $0 < \gamma < \frac{2}{b}$ , there exist a positive constant  $A$  and a continuous positive function  $\phi^*$  in  $B_1$  which vanishes on  $\partial B_1$  such that in  $B_1 \setminus \{0\}$ ,

$$\gamma \ln \frac{1}{|x|} - \phi^*(x) \leq u_\gamma \leq \gamma \ln \frac{1}{|x|} - A|x|^{2-b\gamma} \phi_1(x) + mc^q \eta(x),$$

where  $\phi_1$  is the first eigenfunction of  $-\Delta$  in  $W_0^{1,2}(B_1)$  and  $\eta$  satisfies

$$\begin{aligned} -\Delta \eta &= |x|^{-q} & \text{in } B_1 \\ \eta &= 0 & \text{on } \partial B_1, \end{aligned} \tag{3.9}$$

with  $c$  is the constant in (3.4), and actually,  $\eta(x) = \frac{1}{(2-q)^2}(1 - |x|^{2-q})$ .

2- If  $\gamma = \frac{2}{b}$ , we have in  $B_1 \setminus \{0\}$ ,

$$\frac{2}{b} \left( \ln \frac{2}{|x|} - \ln(1 - \ln|x|) \right) + \kappa_2 \leq u_{\frac{2}{b}}(x) \leq \frac{2}{b} \left( \ln \frac{2}{|x|} - \ln(1 - \ln|x|) \right) + \kappa_1 + \eta(x),$$

where the  $\kappa_i$  have been defined in (2.11).

*Proof.* Because of (3.4)  $u_\gamma$  satisfies

$$-\Delta u_\gamma + ae^{bu_\gamma} \leq mc^q |x|^{-q} + 2\pi\gamma\delta_0 \quad \text{in } D'(B_1).$$

Since  $v_\gamma + mc^q \eta$  vanishes on  $\partial B_1$  and satisfies

$$-\Delta(v_\gamma + mc^q \eta) + ae^{b(v_\gamma + mc^q \eta)} \geq mc^q |x|^{-q} + 2\pi\gamma\delta_0 \quad \text{in } D'(B_1),$$

we deduce by the comparison principle as in [18], that

$$u_\gamma \leq v_\gamma + mc^q \eta \quad \text{in } B_1 \setminus \{0\}. \tag{3.10}$$

On the other hand,

$$-\Delta u_\gamma + ae^{bu_\gamma} \geq 2\pi\gamma\delta_0.$$

Since  $ae^{bu_\gamma} \in L^1(B_1)$ , it follows by the comparison principle again, that

$$v_\gamma \leq u_\gamma \quad \text{in } B_1 \setminus \{0\}. \tag{3.11}$$

Therefore, we conclude that

$$v_\gamma \leq u_\gamma \leq v_\gamma + mc^q \eta \quad \text{in } B_1 \setminus \{0\}. \tag{3.12}$$

Combining with Lemma 2.2, we finish the proof.  $\square$

## 3.2 General solutions

In this section we deal with the questions of existence, classification and uniqueness of solutions of (3.1).

### 3.2.1 Existence of general solutions

**Theorem 3.5** *Let  $1 < q < 2$  and  $m, a, b > 0$ . Then for any  $\gamma \in (0, \frac{2}{b}]$  and  $\phi \in C^1(\partial B_1)$  there exists a function  $u := u_\gamma \in C^1(\overline{B_1} \setminus \{0\})$  such that  $e^{bu}$  and  $|\nabla u|^q$  are integrable in  $B_1$  satisfying (3.1) and*

$$\lim_{x \rightarrow 0} \frac{u(x)}{-\ln|x|} = \gamma \quad (3.13)$$

*Proof of Theorem 3.5. Step 1: existence.* For the sake of simplicity, we assume  $\phi \geq 0$ ; this can be achieved up to replacing  $a$  by some  $\tilde{a}$ . The function  $v_\gamma$  which satisfies (1.8) with boundary data  $\phi$  is a subsolution in  $B_1 \setminus \{0\}$ . We look for a supersolution under the form

$$\psi(x) = \gamma \ln \frac{1}{|x|} + c(1 - |x|^{2-q}) + M,$$

where  $c$  and  $M$  are to be found. Since  $-\Delta(1 - |x|^{2-q}) = (2 - q)^2|x|^{-q}$

$$\mathcal{E}(\psi) = a|x|^{-b\gamma} e^{bc(1-|x|^{2-q})+bM} + |x|^{-q} \left( c(2-q)^2 - m\gamma^q \left( 1 + \frac{c(2-q)}{\gamma} r^{2-q} \right)^q \right)$$

There exists  $r_0 \in (0, 1)$  and  $c_0 > 0$  such that

$$c_0(2-q)^2 - m\gamma^q \left( 1 + \frac{c_0(2-q)}{\gamma} r^{2-q} \right)^q \geq 0 \quad \text{for all } 0 < r \leq r_0.$$

Then we choose  $M > 0$  large enough such that

$$ae^{bM} \geq r_0^{-q} m\gamma^q \left( 1 + \frac{c_0(2-q)}{\gamma} \right)^q,$$

and

$$\psi(x) = M \geq \|\phi\|_{L^\infty(\partial B_1)}.$$

Hence  $\mathcal{E}(\psi) \geq 0$  in  $B_1 \setminus \{0\}$ , and  $\psi \geq \phi$  on  $\partial B_1$ . It follows that  $\psi$  is a supersolution, larger than  $v_\gamma$ . Therefore there exists  $u = u_\gamma$  coinciding with  $v_\gamma$ , such that

$$v_\gamma \leq u_\gamma \leq \psi \quad (3.14)$$

which satisfies (1.4) in  $B_1 \setminus \{0\}$ .

*Step 2: upper estimate.* The function  $u_\gamma$  may not be radial since it is not assumed that  $v_\gamma$  is so. But replacing  $v_\gamma$  by  $v_{rad,\gamma}$  which satisfies the same equation (1.8) with boundary data  $\|\phi\|_{L^\infty(\partial B_1)}$ , we have that  $v_{rad,\gamma} \geq v_\gamma$  with boundary data  $\|\phi\|_{L^\infty(\partial B_1)}$ . Hence there exists a radial solution  $u_{rad,\gamma}$  of (1.4) satisfying

$$v_\gamma \leq v_{rad,\gamma} \leq u_{rad,\gamma} \leq \psi \quad (3.15)$$

and  $u_{rad,\gamma} \geq u_\gamma$  by the standard construction iterative scheme. By Lemma 3.2 there holds

$$|u'_{rad,\gamma}(r)| \leq \frac{c}{r}.$$

Therefore

$$-\Delta u_{rad,\gamma} + ae^{bu_{rad,\gamma}} = m|\nabla u_{rad,\gamma}|^q \leq mc^q r^{-q},$$

and (3.15) gives that

$$\lim_{r \rightarrow 0} \frac{u_{rad,\gamma}(r)}{-\ln r} = \lim_{x \rightarrow 0} \frac{u_\gamma(x)}{-\ln |x|} = \gamma.$$

Let  $\{\theta_n\} \subset C^{1,1}(\mathbb{R}^2)$  defined by

$$\theta_n(x) = \begin{cases} 2n^2 \left( |x| - \frac{1}{n} \right)_+^2 & \text{if } |x| \leq \frac{3}{2n} \\ -2n^2|x|^2 + 8n|x| - 7 & \text{if } \frac{3}{2n} \leq |x| \leq \frac{2}{n} \\ 1 & \text{if } |x| \geq \frac{2}{n}, \end{cases}$$

then  $0 \leq \theta_n \leq 1$ ,  $\theta_n = 0$  on  $B_{\frac{1}{n}}$ ,  $\theta_n = 1$  on  $B_{\frac{2}{n}}^c$ ,  $|\nabla \theta_n| \leq cn \mathbf{1}_{B_{\frac{2}{n}} \setminus B_{\frac{1}{n}}}$  and

$$\Delta \theta_n(x) = 4n^2 \left( 2 - \frac{1}{n|x|} \right) \mathbf{1}_{B_{\frac{3}{2n}} \setminus B_{\frac{1}{n}}} - 4n^2 \left( 2 - \frac{2}{n|x|} \right) \mathbf{1}_{B_{\frac{2}{n}} \setminus B_{\frac{3}{2n}}}.$$

If  $\zeta \in C_c^\infty(B_1)$  is nonnegative with  $\zeta(0) = 1 = \max_{|\zeta| \leq 1} \zeta(z)$ . We have

$$-\int_{B_1} u_{rad,\gamma} \theta_n \Delta \zeta dx + a \int_{B_1} e^{bu_{rad,\gamma}} \zeta \theta_n dx - \int_{B_1} (\zeta \Delta \theta_n + 2\nabla \theta_n \cdot \nabla \zeta) u_{rad,\gamma} dx \leq mc^q \int_{B_1} \theta_n \zeta |x|^{-q} dx. \quad (3.16)$$

Clearly there holds when  $n \rightarrow \infty$ ,

$$\int_{B_1} u_{rad,\gamma} \theta_n \Delta \zeta dx \rightarrow \int_{B_1} u_{rad,\gamma} \Delta \zeta dx,$$

and

$$mc^q \int_{B_1} \theta_n \zeta |x|^{-q} dx \rightarrow mc^q \int_{B_1} \zeta |x|^{-q} dx.$$

Next

$$\int_{B_1} |\nabla \theta_n \cdot \nabla \zeta| u_{rad,\gamma} dx \leq cn \int_{B_{\frac{2}{n}} \setminus B_{\frac{1}{n}}} u_{rad,\gamma} |\nabla \zeta| dx \rightarrow 0,$$

since  $u_{rad,\gamma}(x) = \gamma \ln\left(\frac{1}{|x|}\right)(1 + o(1))$ . The last term is more delicate to estimate:

$$-\int_{B_1} \Delta \theta_n u_{rad,\gamma} \zeta dx = -\int_{B_{\frac{3}{2n}} \setminus B_{\frac{1}{n}}} \Delta \theta_n u_{rad,\gamma} \zeta dx - \int_{B_{\frac{2}{n}} \setminus B_{\frac{3}{2n}}} \Delta \theta_n u_{rad,\gamma} \zeta dx := I + II.$$

Since  $\zeta(x) = 1 + O(|x|)$  and  $u_{rad,\gamma}$  satisfies (3.15), estimates on  $v_\gamma$  and  $\psi$  yield

$$\gamma \ln \frac{1}{|x|} - M \leq u_{rad,\gamma}(x) \leq \gamma \ln \frac{1}{|x|} + M \quad (3.17)$$

for some  $M > 0$ , and therefore

$$I = -4n^2 \int_{B_{\frac{3}{2n}} \setminus B_{\frac{1}{n}}} \left(2 - \frac{1}{n|x|}\right) u_{rad,\gamma} dx + O\left(\frac{\ln n}{n}\right).$$

Hence

$$-6\pi \max \left\{ u_{rad,\gamma} : \frac{1}{n} \leq |x| \leq \frac{3}{2n} \right\} + O\left(\frac{\ln n}{n}\right) \leq I \leq -6\pi \min \left\{ u_{rad,\gamma} : \frac{1}{n} \leq |x| \leq \frac{3}{2n} \right\} + O\left(\frac{\ln n}{n}\right).$$

In the same way

$$II = 4n^2 \zeta(0) \int_{B_{\frac{2}{n}} \setminus B_{\frac{3}{2n}}} \left(2 - \frac{2}{n|x|}\right) u_{rad,\gamma} \zeta dx + O\left(\frac{\ln n}{n}\right).$$

Hence

$$6\pi \min \left\{ \tilde{u}_\gamma : \frac{3}{2n} \leq |x| \leq \frac{2}{n} \right\} + O\left(\frac{\ln n}{n}\right) \leq II \leq 6\pi \max \left\{ \tilde{u}_\gamma : \frac{3}{2n} \leq |x| \leq \frac{2}{n} \right\} + O\left(\frac{\ln n}{n}\right).$$

Using (3.17) we see that up to some constants  $M > 0$  Therefore on

$$-6\pi \ln n - M + O\left(\frac{\ln n}{n}\right) \leq I \leq -6\pi \ln n + M + O\left(\frac{\ln n}{n}\right),$$

and

$$6\pi \ln n - M + O\left(\frac{\ln n}{n}\right) \leq II \leq 6\pi \ln n + M + O\left(\frac{\ln n}{n}\right).$$

Thus  $I + II$  remains bounded independently of  $n$ . Combining these estimates with (3.16) we conclude that  $\int_{B_1} e^{bu_{rad,\gamma}} \zeta \theta_n dx$  remains bounded independently of  $n$  too. By Fatou's lemma  $e^{bu_{rad,\gamma}} \zeta \in L^1(B_1)$ . Since  $u_\gamma \leq u_{rad,\gamma}$  we infer that  $e^{bu_\gamma} \in L^1(B_1)$ . Because

$$-\Delta u_\gamma = m|\nabla u_\gamma|^q - ae^{bu_\gamma},$$

it follows by Brezis-Lions lemma that  $|\nabla u|^q \in L^1(B_1)$  and there exist  $\tilde{\gamma} \geq 0$  such that

$$-\Delta u_\gamma = m|\nabla u_\gamma|^q - ae^{bu_\gamma} + 2\pi\tilde{\gamma}\delta_0 \quad \text{in } \mathcal{D}'(B_1).$$

Set  $F = m|\nabla u_\gamma|^q - ae^{bu_\gamma}$ , Since  $F \in L^1(B_1)$  it follows by Lemma 3.3 that

$$\bar{u}_\gamma(r) = -\tilde{\gamma} \ln r + o(\ln r).$$

Combined with (3.14) we obtain that  $\tilde{\gamma} = \gamma$ . □

The next estimate compares  $v_\gamma$  and  $u_\gamma$

**Proposition 3.6** *Let  $\gamma \in (0, \frac{2}{b}]$  and  $\phi \in C^1(\partial B_1)$  is nonnegative. If  $u \in C^1(\bar{B}_1 \setminus \{0\})$  satisfies  $e^{bu} \in L^1(B_1)$  and  $|\nabla u|^q \in L^1(B_1)$  is a solution of (3.1), then there exists  $K > 0$  such that*

$$v_\gamma \leq u(x) \leq v_\gamma + K \quad \text{for } x \in B_1 \setminus \{0\}, \quad (3.18)$$

where  $v_\gamma$  is the solution of (1.8) which vanishes on  $\partial B_1$ .

*Proof.* Since  $u$  is a solution, it is a supersolution of (1.8), and by the order principle it is larger than solution  $v_\gamma$  with the same boundary value. Using the regularity estimates of the Laplacian in the space of bounded Radon measures  $\mathfrak{M}_b(B_1)$  we have that  $\nabla u \in M^2(B_1)$  where  $M^2(B_1)$  is the Marcinkiewicz space coinciding with the Lorentz space  $L^{2,\infty}(B_1)$ . What is important to know is that  $M^2(B_1) \subset L^{2-\epsilon}(B_1)$  for any  $\epsilon > 0$ . Therefore  $|\nabla u|^q \in L^{\frac{2-\epsilon}{q}}(B_1)$ . Since  $q < 2$  we take  $\epsilon > 0$  such that  $\frac{2-\epsilon}{q} = 1 + \delta > 1$ . Let  $u_1$  be the solution of

$$\begin{aligned} -\Delta u_1 &= m|\nabla u|^q && \text{in } B_1 \\ u_1 &= \phi && \text{on } B_1. \end{aligned}$$

The function  $u_1$  is nonnegative and belongs to  $W^{2,1+\delta}(B_1)$ . By Morrey-Sobolev imbedding theorem,  $W^{2,1+\delta}(B_1) \subset C^{0,\frac{2\delta}{1+\delta}}(\overline{B}_1)$ . This implies that  $u_1$  is bounded from above by some  $K > 0$ . The function  $v_\gamma + u_1$  satisfies

$$-\Delta(u_1 + v_\gamma) + ae^{b(u_1+v_\gamma)} > -\Delta(u_1 + v_\gamma) + ae^{bv_\gamma} = m|\nabla u|^q + 2\pi\gamma\delta_0.$$

Since  $ae^{b(u_1+v_\gamma)} \in L^1(B_1)$  we can apply Vazquez' monotonicity estimate [18] between  $u_1 + v_\gamma$  and  $u$  since the right-hand side of the respective equations coincide and belong to  $\mathfrak{M}_b(B_1)$ . Therefore

$$u(x) \leq v_\gamma(x) + K, \quad (3.19)$$

and (3.59) follows.  $\square$

As a consequence of Lemma 2.2 we obtain global estimates on the solutions

**Corollary 3.7** *Under the assumptions of Proposition 3.6 any function  $u \in C^1(\overline{B}_1 \setminus \{0\})$  such that  $e^{bu} \in L^1(B_1)$  and  $|\nabla u|^q \in L^1(B_1)$  of solution of (3.1) satisfies, for some  $K \geq 0$ , the following two-sided inequalities for all  $x \in \overline{B}_1 \setminus \{0\}$ ,*

1- *If  $0 \leq \gamma < \frac{2}{b}$ , then*

$$\gamma \ln \frac{1}{|x|} - K \leq u(x) \leq \gamma \ln \frac{1}{|x|} + K. \quad (3.20)$$

2- *If  $\gamma = \frac{2}{b}$ , then*

$$\frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) - K \leq u(x) \leq \frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) + K. \quad (3.21)$$

### 3.2.2 Classification of isolated singularities

In this section we prove the following

**Theorem 3.8** *Let  $1 < q < 2$ ,  $m, a, b > 0$ . If  $u \in C^1(\overline{B}_1 \setminus \{0\})$  is a solution of (1.4) in  $B_1 \setminus \{0\}$ , then there exists  $\gamma \in [0, \frac{2}{b}]$  such that*

$$\lim_{x \rightarrow 0} \frac{u(x)}{-\ln|x|} = \gamma. \quad (3.22)$$

*Furthermore  $e^{bu} \in L^1(B_1)$ ,  $|\nabla u|^q \in L^1(B_1)$  and  $u$  is a solution of (3.1).*

*Proof.* We can assume that  $u|_{\partial B_1} = \phi \in C^1(\partial B_1)$  is nonnegative.

1- If  $e^{bu} \in L^1(B_1)$ , it follows by Brezis-Lions Lemma [9] that  $|\nabla u|^q \in L^1(B_1)$  and there exists  $\gamma \geq 0$  such that (3.1) holds. By Lemma 3.3 the circular average  $\bar{u}$  of  $u$  satisfies

$$\bar{u}(r) = \gamma \ln \frac{1}{r} + o(\ln r) \quad \text{as } r \rightarrow 0.$$

By convexity

$$\overline{e^{bu}}(r) \geq e^{b\bar{u}(r)}.$$

Since for any  $\epsilon > 0$  there holds

$$\bar{u}(r) \geq (\gamma - \epsilon) \ln \frac{1}{r} - K_\epsilon,$$

we obtain

$$e^{b\bar{u}(r)} \geq \frac{e^{-bK_\epsilon}}{|x|^{b(\gamma-\epsilon)}}.$$

If  $\gamma > \frac{2}{b}$  we can choose  $\epsilon > 0$  such that  $b(\gamma - \epsilon) > 2$ . Hence  $e^{b\bar{u}(\cdot)}$  is not integrable in  $B_1$ , neither  $e^{bu}$ , contradiction. This implies that  $\gamma \in [0, \frac{2}{b}]$ . By Proposition 3.6 and Corollary 3.7 we obtain (3.22).

2- Next we assume that  $e^{bu} \notin L^1(B_1)$ , then using the inequality satisfied by  $\bar{u}$ ,

$$r\bar{u}'(r) + a \int_r^1 \overline{e^{bu}} t dt = \bar{u}'(1) + m \int_r^1 \overline{|\nabla u|^q} t dt \quad (3.23)$$

where  $\bar{u}$ ,  $\overline{E^{bu}}$  and  $\overline{|\nabla u|^q}$  denote the averages of  $u$ ,  $E^{bu}$  and  $|\nabla u|^q$  on  $S^1$ . Next we will distinguish according  $|\nabla u|^q$  is or is not integrable in  $B_1$ , although if  $u$  is radial Lemma 3.2 implies that  $|\nabla u|^q \in L^1(B_1)$ .

2-(i)- If  $|\nabla u|^q \in L^1(B_1)$  we obtain from (3.23) that

$$\lim_{r \rightarrow 0} r\bar{u}'(r) = -\infty.$$

By integration, it implies that

$$\lim_{r \rightarrow 0} \frac{\bar{u}(r)}{-\ln r} = \infty,$$

which again contradicts the a priori estimate (2.1).

2-(ii)- If  $|\nabla u|^q \notin L^1(B_1)$ , then for any  $\epsilon \in (0, 1)$ ,

$$\int_{B_\epsilon} |\nabla u|^q dx = \infty.$$

Then for any  $k > 0$  there exists  $\tau = \tau(\epsilon, k)$  such that

$$m \int_{B_\epsilon} |\nabla \min\{u, \tau\}|^q dx = 2\pi k.$$

Since

$$-\Delta \min\{u, \tau\} + ae^{b \min\{u, \tau\}} \geq m |\nabla \min\{u, \tau\}|^q \mathbf{1}_{B_\epsilon} \quad \text{in } B_1$$



$\min\{u, \tau\}$  is bounded from below by the solution  $v_{k,\epsilon}$  of

$$-\Delta v_{k,\epsilon} + ae^{bv_{k,\epsilon}} = m|\nabla \min\{u, \tau\}|^q \mathbf{1}_{B_\epsilon} \quad \text{in } B_1.$$

We have  $m|\nabla \min\{u, \tau\}|^q \mathbf{1}_{B_\epsilon} \rightarrow 2\pi k\delta_0$  in the weak sense of bounded Radon measures in  $B_1$ . By Vazquez' stability theorem, if  $k \leq \frac{2}{b}$ ,  $v_{k,\epsilon} \rightarrow v_k$  as  $\epsilon \rightarrow 0$ . Taking in particular  $k = \frac{2}{b}$  we obtain

$$u(x) \geq v_{\frac{2}{b}}(x) \quad \text{for all } x \in B_1 \setminus \{0\}. \quad (3.24)$$

Combining this estimate with (2.1) and the estimate on  $v_{\frac{2}{b}}$  yields, for some  $K \geq 0$ ,

$$\frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) - K \leq u(x) \leq \frac{2}{b} \ln \frac{1}{|x|} + K \quad \text{for all } x \in B_1 \setminus \{0\}. \quad (3.25)$$

We set  $w(t, \cdot) = u(r, \cdot) + \frac{2}{b} \ln r$  with  $t = \ln r$ . Then  $w$  satisfies in  $(-\infty, 0) \times S^1$

$$w_{tt} + w_{\theta\theta} - ae^{bw} + me^{(2-q)t} \left( \left( \frac{2}{b} - w_t \right)^2 + w_\theta^2 \right)^{\frac{q}{2}} = 0, \quad (3.26)$$

and the two-sided estimate

$$-\frac{2}{b} \ln(1-t) - K \leq w(t, \cdot) \leq K. \quad (3.27)$$

Thus the function  $z(t, \cdot) = \frac{w(t, \cdot)}{\ln(1-t)}$  satisfies in  $(-\infty, 0) \times S^1$ ,

$$\begin{aligned} z_{tt} + z_{\theta\theta} - \frac{2}{(1-t)\ln(1-t)} z_t - \frac{1}{(1-t)^2 \ln(1-t)} z - \frac{a}{\ln(1-t)} e^{b \ln(1-t)z} \\ + me^{(2-q)t} \ln^{q-1}(1-t) \left( \left( \frac{2}{b \ln(1-t)} - z_t + \frac{z}{(1-t)\ln(1-t)} \right)^2 + z_\theta^2 \right)^{\frac{q}{2}} = 0, \end{aligned} \quad (3.28)$$

and (3.27) yields for some  $T < 0$ ,

$$-\frac{2}{b} + o(1) \leq z(t, \cdot) \leq \frac{K}{\ln(1-t)} \leq o(1) \quad \text{in } (-\infty, T] \times S^1.$$

Since  $z$  is bounded and negative and all the coefficients are smooth and bounded, the standard regularity theory implies that  $z$  is bounded in  $C^{2,\alpha}([T' - 1, T' + 1] \times S^1)$ . Since  $w(t, \cdot) = z(t, \cdot) \ln(1-t)$  we have also

$$(w_t^2(t, \theta) + w_\theta^2(t, \theta))^{\frac{1}{2}} \leq c \ln(1-t). \quad (3.29)$$

Returning to  $u(r, \theta)$  implies

$$|\nabla u(x)| = |\nabla u(r, \theta)| = (u_r^2 + r^{-2} u_\theta^2)^{\frac{1}{2}} \leq c \frac{\ln(1 - \ln|x|)}{|x|} \quad \text{for all } x \in B_1 \setminus \{0\}. \quad (3.30)$$

Since  $q < 2$ , the right-hand side of (3.30) belongs to  $L^q(B_1)$  and we get a contradiction. This ends the proof.  $\square$

### 3.2.3 Uniqueness

Sharp estimates of the gradient of singular solutions near  $x = 0$  are the key-step for proving their uniqueness.

**Theorem 3.9** *Let  $q, m, a, b$  be as in Theorem 3.5. Then for any  $\gamma \in (0, \frac{2}{b}]$  and  $\phi \in C^1(\partial B_1)$  there exists a unique function  $u \in C^1(\overline{B_1} \setminus \{0\})$  satisfying equation (1.4) in  $B_1 \setminus \{0\}$  and*

$$u(x) = \gamma \ln \frac{1}{|x|} (1 + o(1)). \quad (3.31)$$

Furthermore  $e^{bu}$  and  $|\nabla u|^q$  are integrable in  $B_1$ , and  $u$  satisfies (3.1).

*Proof.* Let  $u$  be any solution nonnegative solution of (1.4) in  $B_1 \setminus \{0\}$  satisfying (3.31) (the sign requirement is not a restriction since we can change  $a$ ). By Theorem 3.8  $e^{bu}$  and  $|\nabla u|^q$  are integrable in  $B_1$  and  $u$  satisfies (3.1). Therefore the estimates of Corollary 3.7 hold.

1- The case  $0 < \gamma < \frac{2}{b}$ .

1-(i) *Convergence and gradient estimates.* Under the assumption (3.31) we first show that there holds

$$\lim_{|x| \rightarrow 0} \left( u(x) - \gamma \ln \frac{1}{|x|} \right) = \ell \quad \text{and} \quad \lim_{|x| \rightarrow 0} |x| \nabla u(x) = -\gamma \mathbf{i} \quad (3.32)$$

for some  $\ell \in [-K, K]$  and  $\mathbf{i} = \frac{x}{|x|}$ . For the proof, we set

$$w(t, \theta) = u(r, \theta) - \gamma \ln \frac{1}{r},$$

with  $t = \ln r \in (-\infty, 0]$  and  $\theta \in S^1$ . Then  $w$  is bounded in  $(-\infty, 0] \times S^1$  where there holds

$$w_{tt} + w_{\theta\theta} - ae^{(2-b\gamma)t} e^{bw} + me^{(2-q)t} \left( (\gamma - w_t)^2 + w_\theta^2 \right)^{\frac{q}{2}} = 0. \quad (3.33)$$

Since  $2 - b\gamma > 0$  and  $2 - q > 0$ , by the standard regularity theory [15, Theorem A1], we have the estimate

$$\begin{aligned} & \max\{|w_t(t, \theta)| + |w_\theta(t, \theta)| : (t, \theta) \in [T - 1, T + 1] \times S^1\} \\ & \leq c \max\{|w(t, \theta)| : (t, \theta) \in [T - 2, T + 2] \times S^1\} \leq cK, \end{aligned}$$

for all  $T \leq -2$ . Equivalently  $|u_\theta| + |ru_r + \gamma|$  is bounded. Now, by linear elliptic equation regularity theory  $w$  is bounded locally in  $C^{1,\alpha}([T - 1, T + 1] \times S^1)$  independently of  $T$  and thus in  $C^{2,\alpha}([T - 1, T + 1] \times S^1)$ , for some  $\alpha \in (0, 1)$ .

1-(ii) *Convergence and sharp gradient estimates.* Let  $\bar{w}(t) = \frac{1}{2\pi} \int_0^{2\pi} w(t, \theta) d\theta$  be the average of  $w(t, \cdot)$  on  $S^1$ . Then

$$-\bar{w}''(t) + ae^{(2-b\gamma)t} H_0(t) + e^{(2-q)t} G_0(t) = 0,$$

where

$$H_0(t) = \frac{1}{2\pi} \int_{S^1} e^{bw} d\theta \quad \text{and} \quad G_0(t) = -\frac{m}{2\pi} \int_{S^1} \left( (\gamma - w_t)^2 + w_\theta^2 \right)^{\frac{q}{2}} d\theta.$$

Since  $H_0$  and  $G_0$  are bounded,  $\bar{w}'(t)$  admits a limit when  $t \rightarrow -\infty$ , and this limit is necessarily equal to zero since  $\bar{w}$  is bounded. Thus

$$\bar{w}'(t) + a \int_{-\infty}^t e^{(2-b\gamma)\tau} H_0(\tau) d\tau + \int_{-\infty}^t e^{(2-q)\tau} G_0(\tau) d\tau = 0.$$

Therefore

$$|\bar{w}'(t)| \leq c \max\{e^{(2-b\gamma)t}, e^{(2-q)t}\}, \quad (3.34)$$

which implies that  $\bar{w}(t)$  admits a finite limit  $\ell$  when  $t \rightarrow -\infty$ . Set  $w^* = w - \bar{w}$ . Then

$$-w_{tt}^* - w_{\theta\theta}^* + ae^{(2-b\gamma)t} H^* + e^{(2-q)t} G^* = 0, \quad (3.35)$$

where

$$H^*(t, \cdot) = e^{bw} - \frac{1}{2\pi} \int_{S^1} e^{bw} d\theta \quad \text{and} \quad G^*(t) = \frac{m}{2\pi} \int_{S^1} |(\gamma - w_t)^2 + w_\theta^2|^{\frac{q}{2}} d\theta - m (\gamma - w_t)^2 + w_\theta^2)^{\frac{q}{2}}.$$

Since  $\int_{S^1} w^* d\theta = 0$ , we have by Wirtinger inequality,

$$- \int_{S^1} w_{\theta\theta} w^* d\theta = - \int_{S^1} w_{\theta\theta}^* w^* d\theta \geq \|w^*\|_{L^2(S^1)}^2,$$

and

$$\int_{S^1} H^*(t, \cdot) w^* d\theta = \int_{S^1} \left( e^{bw} - \frac{1}{2\pi} \int_{S^1} e^{bw} d\theta \right) (w - \bar{w}) d\sigma = \int_{S^1} \left( e^{bw} - ae^{b\bar{w}} \right) (w - \bar{w}) d\sigma \geq 0.$$

Using the inequality

$$- \|w^*\|_{L^2(S^1)} \frac{d^2}{dt^2} \|w^*\|_{L^2(S^1)} \leq - \int_{S^1} w^* w_{tt}^* d\theta$$

we obtain

$$- \frac{d^2}{dt^2} \|w^*\|_{L^2(S^1)} + \|w^*\|_{L^2(S^1)} \leq L_0(t) e^{\beta t},$$

where  $\beta = \min\{2 - q, 2 - b\gamma\} > 0$  and  $L_0(t)$  is some bounded function. Since  $\|w^*\|_{L^2(S^1)}$  is bounded and  $\beta < 1$ , it implies that

$$\|w^*(t, \cdot)\|_{L^2(S^1)} \leq C e^{\beta t}. \quad (3.36)$$

Combining this estimate with the fact that  $\bar{w}(t) \rightarrow \ell$  when  $t \rightarrow -\infty$  we deduce that

$$\lim_{t \rightarrow -\infty} \|w(t, \cdot) - \ell\|_{L^2(S^1)} = 0. \quad (3.37)$$

Let

$$\mathcal{T}[w] = \bigcup_{t \leq 0} \{(w(t, \cdot), w_t)\}$$

be the negative trajectory of  $(w, w_t)$ . It is relatively compact in  $C^2(S^1) \times C^1(S^1)$  and because of the uniqueness of the limit, we have that  $w(t, \cdot) \rightarrow \ell$  in  $C^2(S^1)$  and thus  $w_\theta(t, \cdot) \rightarrow 0$  in  $C^1(S^1)$ . In order to make precise the behaviour of  $w_t$ , we differentiate (3.35) with respect to  $t$  and get

$$-w_{ttt}^* - w_{t\theta\theta}^* + e^{(2-b\gamma)t} \tilde{H}^* + e^{(2-q)t} \tilde{G}^* = 0 \quad (3.38)$$

where  $\tilde{H}^* = a(2 - b\gamma)H^* - aH_t^*$  and  $\tilde{G}^* = (2 - q)G^* - G_t^*$  are bounded functions (from the  $C^2$  a priori bounds on  $w$ ). Hence by Wirtinger inequality again,

$$-\frac{d^2}{dt^2} \|w_t^*\|_{L^2(S^1)} + \|w_t^*\|_{L^2(S^1)} \leq \tilde{L}_0(t)e^{\beta t},$$

where  $\tilde{L}_0$  is a bounded function. Since  $\|w_t^*\|_{L^2(S^1)}$  is bounded on  $(-\infty, 0]$  it implies again that

$$\|w_t^*(t, \cdot)\|_{L^2(S^1)} \leq C'e^{\beta t}. \quad (3.39)$$

By (3.34),  $\bar{w}_t(t) \rightarrow 0$  as  $t \rightarrow -\infty$ . This implies

$$\lim_{t \rightarrow -\infty} \|w_t(t, \cdot)\|_{L^2(S^1)} = 0. \quad (3.40)$$

As in the case of the function  $w(t, \cdot)$ , the relative compactness of the set  $\{w_t(t, \cdot)\}_{t \leq 0}$  in  $C^1(S^1)$  combined with (3.40) implies that  $w_t(t, \cdot) \rightarrow 0$  in  $C^1(S^1)$  as  $t \rightarrow -\infty$ . This implies (3.32).

1-(iii) *End of the proof.* Next we consider two solutions  $u_\gamma$  and  $\tilde{u}_\gamma$  satisfying (3.31) with  $\gamma \in (0, \frac{2}{b})$  and coinciding on  $\partial B_1$ . Denote  $u = u_\gamma$  and  $\tilde{u} = \tilde{u}_\gamma$ . For  $\epsilon \neq 0$  we recall that  $E(x) = \ln \frac{1}{|x|}$  and

$$U(r) = U_\epsilon(x) = u(x) + \epsilon E(x). \quad (3.41)$$

Then

$$\begin{aligned} -\Delta U + ae^{bU} - m|\nabla U|^q &= ae^{bu} \left( e^{b\epsilon E} - 1 \right) - m \left[ \left( |\nabla u|^2 + \frac{\epsilon^2}{|x|^2} - 2\frac{\epsilon}{|x|} \nabla u \cdot \mathbf{i} \right)^{\frac{q}{2}} - |\nabla u|^q \right] \\ &= ae^{bu} \left( e^{b\epsilon E} - 1 \right) - m|\nabla u|^q \left[ \left( 1 + \frac{\epsilon^2}{|x|^2|\nabla u|^2} - 2\frac{\epsilon}{|x||\nabla u|^2} \nabla u \cdot \mathbf{i} \right)^{\frac{q}{2}} - 1 \right]. \end{aligned}$$

Then, when  $r \rightarrow 0$ ,

$$1 + \frac{\epsilon^2}{|x|^2|\nabla u|^2} - 2\frac{\epsilon}{|x||\nabla u|^2} \nabla u \cdot \mathbf{i} = 1 + \frac{\epsilon^2}{\gamma^2}(1 + o(1)) + 2\frac{\epsilon}{\gamma}(1 + o(1)),$$

thus

$$|\nabla u|^q \left[ \left( 1 + \frac{\epsilon^2}{|x|^2|\nabla u|^2} - 2\frac{\epsilon}{|x||\nabla u|^2} \nabla u \cdot \mathbf{i} \right)^{\frac{q}{2}} - 1 \right] = \frac{\gamma^q}{|x|^q} \left( \frac{q\epsilon}{\gamma} + \frac{q(q-1)\epsilon^2}{2\gamma}(1 + o(1)) \right).$$

Since  $b\gamma \ln \frac{1}{|x|} + bK \geq bu(x) \geq b\gamma \ln \frac{1}{|x|} - bK$ , we have in the case where  $\epsilon > 0$ :

$$ae^{bK}|x|^{-b\gamma}(|x|^{-b\epsilon} - 1) \geq ae^{bu} \left( e^{b\epsilon E} - 1 \right) \geq ae^{-bK}|x|^{-b\gamma}(|x|^{-b\epsilon} - 1),$$

and in the case where  $\epsilon < 0$ :

$$ae^{bK}|x|^{-b\gamma}(|x|^{-b\epsilon} - 1) \leq ae^{bu} \left( e^{b\epsilon E} - 1 \right) \leq ae^{-bK}|x|^{-b\gamma}(|x|^{-b\epsilon} - 1).$$

1-(iii)-(i): Case  $\frac{q}{b} \leq \gamma < \frac{2}{b}$ . we take  $\epsilon > 0$  and get

$$\mathcal{E}(U) \geq ae^{-bK}|x|^{-b\gamma}(|x|^{-b\epsilon} - 1) - m\frac{\gamma^q}{|x|^q} \left( \frac{q\epsilon}{\gamma} + \frac{q(q-1)\epsilon^2}{2\gamma}(1+o(1)) \right).$$

Fix  $r_0 \in (0, 1)$  and  $\epsilon_0 > 0$  such that for any  $0 < r \leq r_0$  and  $0 < \epsilon \leq \epsilon_0$  there holds

$$m\gamma^q \left( \frac{q\epsilon}{\gamma} + \frac{q(q-1)\epsilon^2}{2\gamma}(1+o(1)) \right) \leq 2qm\gamma^{q-1}\epsilon.$$

Therefore

$$\mathcal{E}(U) \geq |x|^{-b\gamma} \left( ae^{-bK}(r_0^{-b\epsilon} - 1) - 2qm\gamma^{q-1}\epsilon|x|^{b\gamma-q} \right).$$

Up to reducing  $r_0$  we have  $\mathcal{E}(U) \geq 0$  in  $\bar{B}_{r_0} \setminus \{0\}$  for any  $\epsilon \in (0, \epsilon_0]$ . Next, in  $B_1 \setminus B_{r_0}$  we have that  $\nabla u_\gamma$  is bounded, therefore

$$\mathcal{E}(U) \geq -C\epsilon$$

for some constant  $C > 0$  whenever  $\epsilon$  is small enough, Consequently this last inequality holds in whole  $\bar{B}_1 \setminus \{0\}$ . Finally the function

$$x \mapsto U^*(x) = U(x) + \frac{1}{b} \ln(1 + C\epsilon)$$

is a supersolution of (3.1) larger than  $\tilde{u}$  in a neighbourhood of  $x = 0$  and near  $\partial B_1$ . Hence  $U^* \geq \tilde{u}_\gamma$ . Letting  $\epsilon \rightarrow 0$  yields  $u_\gamma \geq \tilde{u}$ . Similarly  $\tilde{u} \geq u$ , which implies that  $\tilde{u} = u$ , that is  $\tilde{u}_\gamma = u_\gamma$ .

1-(iii)-(ii): Case  $0 \leq \gamma < \frac{q}{b}$ . We denote by  $\eta$  the solution of

$$\begin{aligned} -\Delta\eta &= |x|^{-q} & \text{in } B_1 \\ \eta &= 0 & \text{on } \partial B_1, \end{aligned} \tag{3.42}$$

then  $\eta(x) = (2-q)^2(1-|x|^{2-q})$ . For  $\epsilon, \epsilon' > 0$ , set  $U(x) = u(x) - \epsilon E(x)$  and  $U_1(x) = U(x) - \epsilon' \eta(x)$ . Then

$$\nabla U_1 = \nabla U + (2-q)\epsilon'|x|^{1-q}\mathbf{i} \quad \text{and} \quad -\Delta U_1 = -\Delta U + \epsilon'\Delta\eta = -\Delta u - \epsilon'|x|^{-q},$$

$$\begin{aligned} |\nabla U_1|^q &= \left( |\nabla U|^2 + \frac{(2-q)^2\epsilon'^2}{|x|^{2(q-1)}} + \frac{2(2-q)\epsilon'}{|x|^{q-1}}\nabla U \cdot \mathbf{i} \right)^{\frac{q}{2}} \\ &= |\nabla U|^q \left( 1 + \frac{(2-q)^2\epsilon'^2}{|x|^{2(q-1)}|\nabla U|^2} + \frac{2(2-q)\epsilon'}{|x|^{q-1}|\nabla U|^2}\nabla U \cdot \mathbf{i} \right)^{\frac{q}{2}}, \\ e^{bU_1} &= e^{bU} e^{-b\epsilon'\eta}, \end{aligned}$$

and

$$-\Delta U = -\Delta u = m|\nabla u|^q - ae^{bu} = m|\nabla(U + \epsilon E)|^q - ae^{b(U + \epsilon E)}.$$

Since  $\eta \geq 0$  and  $0 < |x| \leq 1$ ,  $e^{bU} (e^{-be'\eta} - |x|^{-b\epsilon}) \leq 0$ . Therefore

$$\begin{aligned} -\Delta U_1 + ae^{bU_1} - m|\nabla U_1|^q &= -\epsilon'|x|^{-q} + ae^{bU} (e^{-be'\eta} - |x|^{-b\epsilon}) \\ -m|\nabla U|^q &\left[ \left( 1 + \frac{(2-q)^2\epsilon'^2}{|x|^{2(q-1)}|\nabla U|^2} + \frac{2(2-q)\epsilon'}{|x|^{q-1}|\nabla U|^2} \nabla U \cdot \mathbf{i} \right)^{\frac{q}{2}} - \left( 1 + \frac{\epsilon^2}{|x|^2|\nabla U|^2} - \frac{2\epsilon}{|x||\nabla U|^2} \nabla U \cdot \mathbf{i} \right)^{\frac{q}{2}} \right] \\ &\leq -\epsilon'|x|^{-q} + C|\nabla U|^q \left( \frac{\epsilon'}{|x|^{q-1}|\nabla U|} + \frac{\epsilon'^2}{|x|^{2(q-1)}|\nabla U|^2} + \frac{\epsilon}{|x||\nabla U|} + \frac{\epsilon^2}{|x|^2|\nabla U|^2} \right) (1 + o(1)), \end{aligned} \quad (3.43)$$

for some  $C > 0$  as  $r \rightarrow 0$ . In order  $U_1$  be a subsolution, the problem is reduced to proving that, under suitable conditions on  $\epsilon$  and  $\epsilon'$ , the expression

$$A(x) := -\epsilon'|x|^{-q} + 2C|\nabla U|^q \left( \frac{\epsilon'}{|x|^{q-1}|\nabla U|} + \frac{\epsilon'^2}{|x|^{2(q-1)}|\nabla U|^2} + \frac{\epsilon}{|x||\nabla U|} + \frac{\epsilon^2}{|x|^2|\nabla U|^2} \right)$$

is negative for  $r$  small enough. Because  $\frac{2\gamma}{|x|} \geq |\nabla U(x)| \geq \frac{\gamma}{2|x|}$  near 0, there holds

$$A(x) \leq -\epsilon'|x|^{-q} + 2^{q+1}C\gamma^q|x|^{-q} \left( \frac{2\epsilon'|x|^{2-q}}{\gamma} + \frac{4\epsilon'^2|x|^{4-2q}}{\gamma^2} + \frac{2\epsilon}{\gamma} + \frac{4\epsilon^2}{\gamma^2} \right).$$

We fix  $\epsilon' = 2^{q+1}C\gamma^q \left( \frac{2\epsilon'}{\gamma} + \frac{4\epsilon'^2}{\gamma^2} + \frac{2\epsilon}{\gamma} + \frac{4\epsilon^2}{\gamma^2} \right)$ , then

$$A(x) \leq 0 \quad \text{for } 0 < |x| \leq r_0.$$

Proceeding as in is the case (3-i) there exists  $\epsilon_0$  and  $C > 0$  such that, for any  $0 < \epsilon \leq \epsilon_0$  there holds

$$\mathcal{E}(U_1) \leq C\epsilon \quad \text{in } B_1 \setminus \{0\}. \quad (3.44)$$

The function  $x \mapsto \tilde{V}_\epsilon(x) := \tilde{u}(x) + \frac{1}{b} \ln(1 + C\epsilon)$  satisfies

$$\mathcal{E}(\tilde{V}_\epsilon) = C\epsilon e^{b\tilde{u}} \geq C.$$

Therefore  $\tilde{V}_\epsilon$  is a super solution of (3.44) which is larger than  $U_1$  near  $x = 0$  and  $|x| = 1$ . Then by the maximum principle,  $\tilde{V}_\epsilon \geq U_1$ . Letting  $\epsilon \rightarrow 0$  yields  $\tilde{u} \geq u$ . Similarly  $u \geq \tilde{u}$ , thus  $u_\gamma = \tilde{u}_\gamma$ .

2- The case  $\gamma = \frac{2}{b}$ .

2-(i) *Gradient estimates.* We first show that

$$\lim_{x \rightarrow 0} \left( u(x) - \frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) \right) = \ell \quad \text{and} \quad \lim_{x \rightarrow 0} |x| \nabla u(x) = -\frac{2}{b} \mathbf{i} \quad (3.45)$$

for some  $\ell \in [-K, K]$  with  $K$  from inequality (3.21) in Corollary 3.7 and  $\mathbf{i} = \frac{x}{|x|}$ .

We set again  $w(t, \cdot) = u(r, \cdot) + \frac{2}{b} \ln r$  with  $t = \ln r$ . Thus  $w$  satisfies

$$-\frac{2}{b} \ln(1-t) - K \leq w(t, \cdot) \leq -\frac{2}{b} \ln(1-t) + K$$

and

$$w_{tt} + w_{\theta\theta} - ae^{bw} + me^{(2-q)t} \left( \left( \frac{2}{b} - w_t \right)^2 + w_\theta^2 \right)^{\frac{q}{2}} = 0. \quad (3.46)$$

Introducing again  $z(t, \cdot) = (\ln(1-t))^{-1}w(t, \cdot)$ , then

$$-\frac{2}{b} - \frac{K}{\ln(1-t)} \leq z(t, \cdot) \leq -\frac{2}{b} + \frac{K}{\ln(1-t)}$$

and  $z$  which is bounded satisfies equation (3.28); thus  $w_t$  satisfies (3.29). Since  $z$  is bounded and negative and all the coefficients are smooth and bounded, the standard regularity theory implies that  $z$  is bounded in  $C^{2,\alpha}([T' - 1, T' + 1] \times S^1)$  independently of  $T'$ , thus

$$\lim_{t \rightarrow -\infty} z(t, \cdot) = -\frac{2}{b} \quad \text{in } C^2(S^1) \quad (3.47)$$

Set

$$\lambda(t, \cdot) = w(t, \cdot) + \frac{2}{b} \ln(1-t)$$

Then  $\lambda$  is uniformly bounded and it satisfies in  $(-\infty, 0] \times S^1$ ,

$$-\lambda_{tt} - \lambda_{\theta\theta} - \frac{2}{b(1-t)^2} + \frac{a}{(1-t)^2} e^{b\lambda} - me^{(2-q)t} \left( \left( \frac{2}{b} - w_t \right)^2 + w_\theta^2 \right)^{\frac{q}{2}} = 0. \quad (3.48)$$

Let  $\bar{\lambda}(t)$  be the circular average of  $\lambda(t, \cdot)$  and  $\lambda^*(t, \cdot) = \lambda(t, \cdot) - \bar{\lambda}(t)$ . Then

$$-\lambda_{tt}^* - \lambda_{\theta\theta}^* + \frac{a}{(1-t)^2} \left( e^{b\lambda} - \overline{e^{b\lambda}} \right) + e^{(2-q-\epsilon)t} G_\epsilon^* = 0$$

where  $G_\epsilon^*$  is a bounded function. Again, by Wirtinger inequality we obtain that

$$\|\lambda^*(t, \cdot)\|_{L^2(S^1)} \leq C \max \left\{ e^t, e^{(2-q-\epsilon)t} \right\} = C e^{(2-q-\epsilon)t},$$

hence

$$\lim_{t \rightarrow -\infty} \|\lambda^*(t, \cdot)\|_{L^2(S^1)} = 0 \quad (3.49)$$

By Gagliardo-Nirenberg inequality we have for any  $p > 1$

$$\begin{aligned} \|\nabla \lambda^*(t, \cdot)\|_{W^{1,p}(S^1)} &\leq C \|\nabla \lambda^*(t, \cdot)\|_{W^{2,2}(S^1)}^{1-\frac{1}{p}} \|\nabla \lambda^*(t, \cdot)\|_{L^2(S^1)}^{\frac{1}{p}} \\ &\leq C \|\nabla \lambda^*(t, \cdot)\|_{C^2(S^1)}^{1-\frac{1}{p}} \|\nabla \lambda^*(t, \cdot)\|_{L^2(S^1)}^{\frac{1}{p}}. \end{aligned} \quad (3.50)$$

Since  $\lambda$  is bounded in  $C^2(S^1)$ , we combined (3.50) with Morrey-Sobolev inequality and obtain, by taking  $p > 2$ ,

$$\|\nabla \lambda^*(t, \cdot)\|_{C^{0,1-\frac{1}{p}}(S^1)} \leq \|\lambda^*(t, \cdot)\|_{L^2(S^1)}^{\frac{1}{p}} \leq C e^{\delta t}, \quad (3.51)$$

where  $\delta = \frac{2-q-\epsilon}{p} > 0$ . The function  $\bar{\lambda}$  satisfies

$$-\bar{\lambda}_{tt} - \frac{2}{b(1-t)^2} + \frac{a}{(1-t)^2} \overline{e^{b\lambda}} + e^{(2-q-\epsilon)t} G_\epsilon = 0 \quad (3.52)$$

If it is monotone at  $-\infty$ , then it converges to some  $\ell \in [-K, K]$ . If it oscillates infinitely many times when  $t \rightarrow -\infty$ , there exists a sequence  $\{t_n\}$  tending to  $-\infty$  such that  $\bar{\lambda}(t)$  achieves a local maximum when  $t = t_{2n}$  and a local minimum when  $t = t_{2n+1}$ . Hence

$$\bar{\lambda}_{tt}(t_{2n}) \leq 0 \quad \text{and} \quad \bar{\lambda}_{tt}(t_{2n+1}) \geq 0.$$

We have

$$e^{b\lambda(t, \cdot)} = e^{b\bar{\lambda}(t)} (e^{\lambda(t, \cdot) - \bar{\lambda}(t)}) = e^{b\bar{\lambda}(t)} \left(1 + D(t, \cdot) e^{\delta t}\right)$$

for some bounded function  $D$ , hence

$$\overline{e^{b\lambda}} = e^{b\bar{\lambda}(t)} \left(1 + \overline{D(t, \cdot)} e^{\delta t}\right)$$

Plugging this estimate into (3.52) we obtain that there exists a bounded function  $B(t)$  and  $\nu > 0$  such that

$$-\frac{2}{b} + ae^{b\bar{\lambda}(t_n)} + e^{\nu t_n} B(t_n) = (1 - t_n)^2 \bar{\lambda}_{tt}(t_n) \quad (3.53)$$

This yields

$$e^{b\bar{\lambda}(t_{2n})} \leq \frac{2}{ab} - e^{\nu t_{2n}} B(t_{2n}) \quad \text{and} \quad e^{b\bar{\lambda}(t_{2n+1})} \geq \frac{2}{ab} - e^{\nu t_{2n+1}} B(t_{2n+1}) \quad (3.54)$$

and finally

$$\limsup_{t \rightarrow -\infty} \bar{\lambda}(t) = \liminf_{t \rightarrow -\infty} \bar{\lambda}(t) = \lim_{t \rightarrow -\infty} \bar{\lambda}(t) = \frac{1}{b} \ln \frac{2}{ab} := \ell. \quad (3.55)$$

This implies

$$\lim_{x \rightarrow 0} \left( u(x) - \frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) \right) = \ell. \quad (3.56)$$

To end the estimate we have still to prove that  $w_t \rightarrow 0$ . Differentiating (3.46) with respect to  $t$  we obtain

$$-w_{ttt} - w_{t\theta\theta} + abe^{bw} w_t + e^{(2-q)t} \tilde{G} = 0 \quad (3.57)$$

where  $\tilde{G} = O(\ln(1-t))$ . Since  $e^{bw} \geq \frac{e^{-bK}}{(1-t)^2}$  we get

$$-\frac{d^2}{dt^2} \|w_t\|_{L^2(S^1)} + \frac{abe^{-bK}}{(1-t)^2} \|w_t\|_{L^2(S^1)} \leq Ae^{\mu t}$$

where  $\mu < 2 - q$  and  $A > 0$ . Put  $X(t) = \|w_t(t, \cdot)\|_{L^2(S^1)} + \frac{A'}{\mu^2} e^{\mu t}$  with  $A' > A$ . Then there exists  $T^* < 0$  such that for  $t \leq T^*$  we have

$$-X'' + \frac{e^{-bK} ab}{(1-t)^2} X \leq 0.$$

The corresponding equation

$$\mathcal{L}(x) := -x'' + \frac{e^{-bK} ab}{(1-t)^2} x = 0 \quad \text{with } t \leq 0$$



admits two linearly independent solutions

$$x_1(t) = (1 - t)^{\lambda_1} \quad \text{and} \quad x_2(t) = (1 - t)^{\lambda_2},$$

when  $t < 0$  arbitrary and  $\lambda_1 < 0 < \lambda_2$ . Since  $X(t) = o(x_2(t))$  when  $t \rightarrow -\infty$ , it follows that  $X(t) = O(x_1(t))$ . Hence

$$\|w_t\|_{L^2(S^1)} \leq B(1 - t)^{\lambda_1} \quad \text{for } t \leq 0 \quad (3.58)$$

for some  $B \geq 0$ . In order to have a uniform estimate, we use Poisson representation formula for the solution  $U$  of the Dirichlet problem for the Laplacian in the disk of radius  $R$

$$U(r, \theta) = \frac{R^2 - r^2}{2\pi R} \int_0^{2\pi} \frac{U(R, \phi) d\phi}{\sqrt{r^2 + R^2 - 2rR \cos(\theta - \phi)}} \quad (3.59)$$

Using the variable  $t = \ln r$  and  $\tilde{U}(t, \theta) = U(r, \theta)$ , it follows that the bounded solution of

$$\begin{aligned} -\tilde{U}_{tt} - \tilde{U}_{\theta\theta} &= 0 & \text{in } (-\infty, T) \times S^1 \\ \tilde{U}(T, \theta) &= \tilde{U}_T(\theta) & \text{in } S^1 \end{aligned} \quad (3.60)$$

is expressed by

$$\tilde{U}(t, \theta) = \frac{e^{2T} - e^{2t}}{2\pi e^T} \int_0^{2\pi} \frac{\tilde{U}(T, \phi) d\phi}{\sqrt{e^{2t} + e^{2T} - 2e^{t+T} \cos(\theta - \phi)}} \quad (3.61)$$

Hence, for  $T = \frac{t}{2}$

$$\tilde{U}(t, \theta) \leq \frac{1 + e^{\frac{t}{2}}}{2\pi} \int_0^{2\pi} \tilde{U}_+(\frac{t}{2}, \phi) d\phi \quad (3.62)$$

From (3.57) there exists some  $A^* \geq 0$  such that the function  $W(t, \theta) = w_t(t, \theta) + A^* e^{\mu t}$  satisfies

$$-W_{+tt} - W_{+\theta\theta} \leq 0 \quad \text{in } (-\infty, 0) \times S^1.$$

Since

$$\|W(T, \cdot)\|_{L^1(S^1)} = \|w_t(T, \cdot) + A^* e^{\mu T}\|_{L^1(S^1)} \leq 2\pi A^* e^{\mu T} + \sqrt{2\pi} \|w_t(T, \cdot)\|_{L^2(S^1)},$$

we derive from (3.58) and (3.62), that there holds for some  $C > 0$ ,

$$(w_t(t, \theta) + A^* e^{\mu t})_+ = W_+(t, \theta) \leq C((t + 1)^{\lambda_1}) \quad \text{for all } (t, \theta) \in (-\infty, 0] \times S^1. \quad (3.63)$$

This implies

$$\lim_{x \rightarrow 0} |x| \nabla u(x) = -\frac{2}{b} \mathbf{i}. \quad (3.64)$$

The end of the proof is similar to the case  $\frac{q}{b} < \gamma < \frac{2}{b}$ .  $\square$

## 4 Singular solutions in the case $q > 2$

In all this section we assume  $q > 2$  and  $m, b, a > 0$ .

#### 4.1 The eikonal equation

The eikonal equation associated to (1.4) is

$$ae^{bw} = m|\nabla w|^q \quad \text{in } \mathbb{R}^2 \setminus \{0\}. \quad (4.1)$$

This equation is completely integrable since it is equivalent to

$$\frac{b}{q} \left( \frac{a}{m} \right)^{\frac{1}{q}} = |\nabla(e^{-\frac{b}{q}w})|.$$

By [12]  $w$  is radial with respect to 0. Since any solution cannot have extremum, it is either decreasing or increasing with respect  $|x|$ . If it is decreasing on  $(0, 1]$  with  $u(0) = c$ , then

$$w(x) := w_c(x) = \frac{q}{b} \ln \left( \frac{qm^{\frac{1}{q}}}{ba^{\frac{1}{q}}|x| + qm^{\frac{1}{q}}e^{-\frac{bc}{q}}} \right) \quad (4.2)$$

Letting  $c \rightarrow \infty$  yields

$$\lim_{c \rightarrow \infty} w_c(x) := w_\infty(x) = \frac{q}{b} \ln \left( \frac{qm^{\frac{1}{q}}}{ba^{\frac{1}{q}}|x|} \right) = \frac{q}{b} \ln \left( \frac{1}{|x|} \right) + \frac{q}{b} \ln \left( \frac{qm^{\frac{1}{q}}}{ba^{\frac{1}{q}}} \right). \quad (4.3)$$

This is also a solution of

$$-\Delta u + ae^{bu} - m|\nabla u|^q = 0 \quad \text{in } \mathbb{R}^2 \setminus \{0\}. \quad (4.4)$$

It vanishes on  $\partial B_1$  if and only if  $qm^{\frac{1}{q}} = ba^{\frac{1}{q}}$ .

#### 4.2 Singular solutions

**Lemma 4.1** *If  $u$  is a radial nonnegative solution of (4.4) in  $B_1 \setminus \{0\}$  it is decreasing on some  $(0, r^*] \subset (0, 1]$  and there holds*

$$-u'(r) = |u'(r)| \geq cr^{-\frac{1}{q-1}} \quad \text{for all } r \in (0, r^*], \quad (4.5)$$

for some  $c > 0$ ,

*Proof.* If  $u$  is a positive radial function satisfying (4.4) in  $B_1 \setminus \{0\}$ , it cannot have any maximum there in, hence either it is decreasing on  $(0, 1)$  which implies that  $u(0) < u(1)$ , or it is decreasing in some interval  $(0, r^*)$  and increasing in  $(r^*, 1)$ . For simplicity and without loss of generality, we will assume that  $r^* = 1$  (this is always the case if  $u(1) = 0$ ). We have

$$-u'' - \frac{1}{r}u' - m|u'|^q \leq 0,$$

then with  $V = -u'$ , we obtain

$$V' + \frac{1}{r}V - mV^q \leq 0 \iff V'V^{-q} + \frac{1}{r}V^{1-q} - m \leq 0 \iff (r^{1-q}V^{1-q})' + m(q-1)r^{1-q} \geq 0$$

Therefore the function

$$r \mapsto r^{1-q}V^{1-q}(r) - m\frac{q-1}{q-2}r^{2-q}$$

is increasing, which implies

$$V^{1-q}(r) \leq m\frac{q-1}{q-2}r(1+o(1)) \quad \text{as } r \rightarrow 0. \quad (4.6)$$

Replacing  $V$  by  $-u'$  we obtain

$$u'(r) \leq -\left(\frac{q-2}{m(q-1)}\right)^{\frac{1}{q-1}} r^{-\frac{1}{q-1}}(1+o(1)) \quad \text{as } r \rightarrow 0. \quad (4.7)$$

□

**Proposition 4.2** *Let  $u$  be a positive solution of (4.4) in  $B_1 \setminus \{0\}$ . The following statements are equivalent:*

(i)  $e^{bu} \in L^1(B_1)$

(ii)  $|\nabla u|^q \in L^1(B_1)$ .

(iii)  $u$  can be extended to in  $\overline{B_1}$  as a Hölder continuous function and the extended function, still denoted by  $u$ , is a solution of

$$-\Delta u + ae^{bu} - m|\nabla u|^q = 0 \quad \text{in } \mathbf{D}'(B_1). \quad (4.8)$$

*Proof.* (i)  $\implies$  (ii). If  $e^{bu} \in L^1(B_1)$ , it follows by Brezis-Lions Lemma [9] that  $|\nabla u|^q \in L^1(B_1)$  and there exists  $\gamma \geq 0$  such that

$$-\Delta u + ae^{bu} - m|\nabla u|^q = 2\pi\gamma\delta_0 \quad \text{in } \mathbf{D}'(B_1). \quad (4.9)$$

The average function  $\bar{u}$  satisfies

$$-(r\bar{u}_r)_r = r \left( \overline{m(u_r^2 + r^{-2}u_\theta^2)^{\frac{q}{2}}} - a\overline{e^{bu}} \right). \quad (4.10)$$

Therefore  $(r\bar{u}_r)' \in L^1(0,1)$  which implies that  $r\bar{u}_r(r)$  admits a finite limit when  $r \rightarrow 0$ . If this limit is not zero, say  $\tilde{\gamma}$ , then

$$\bar{u}_r(r) = \frac{\tilde{\gamma}}{r}(1+o(1)) \implies |\bar{u}_r(r)|^q \sim \frac{|\tilde{\gamma}|^q}{r^q}.$$

Now

$$|u_r(r)| \leq \frac{1}{2\pi} \int_{S^1} |u_r(r, \cdot)| d\theta \leq \frac{1}{2\pi} \int_{S^1} |\nabla u(r, \cdot)| d\theta \leq \left( \frac{1}{2\pi} \int_{S^1} |\nabla u(r, \cdot)|^q d\theta \right)^{\frac{1}{q}}$$

Since  $q > 2$  we have  $|\nabla u|^q \notin L^1(B_1)$ , contradiction. Hence  $\tilde{\gamma} = 0$ . Therefore  $\bar{u}_r(r) = o(r^{-1})$ , and this implies (4.11).

(ii)  $\implies$  (i). We proceed by contradiction and assume that  $e^{bu} \notin L^1(B_1)$ . If  $|\nabla u|^q \in L^1(B_1)$ , then

$$r\bar{u}_r(r) - \bar{u}_r(1) = \int_r^1 \left( m|\nabla u|^q - ae^{bu} \right) t dt = -a \int_r^1 e^{bu} t dt + O(1) \rightarrow -\infty \quad \text{as } r \rightarrow 0.$$

Hence  $r\bar{u}_r(r) \rightarrow -\infty$  when  $r \rightarrow 0$ , which would imply

$$\lim_{r \rightarrow 0} \frac{u(r)}{-\ln r} = \infty,$$

which would contradict the a priori estimate. Therefore, if  $e^{bu}$  is not integrable in  $B_1$  implies that it is the same holds with  $|\nabla u|^q$ . This proves the claim.

(iii)  $\implies$  (i). Since  $u$  is continuous in  $\bar{B}_1$ ,  $e^{bu} \in L^1(B_1)$ .

(ii)  $\implies$  (iii). Since  $|\nabla u| \in L^q(B_1)$  and  $u$  is  $C^1$  in  $\bar{B}_1 \setminus \{0\}$ , it follows by Morrey-Sobolev theorem that it can be extended by continuity to  $x = 0$  and that the resulting function, still denoted by  $u$  belongs to  $C^{0,1-\frac{2}{q}}(\bar{B}_1)$ . In particular

$$|u(x) - u(0)| \leq c|x|^{1-\frac{2}{q}}. \quad (4.11)$$

Furthermore, by Brezis-Lions lemma, there exists  $\gamma \geq 0$  such that (4.9) holds. By the same averaging method that in the case (i)  $\implies$  (ii), we have also

$$\lim_{r \rightarrow 0} \frac{\bar{u}(r)}{-\ln r} = \gamma.$$

The previous estimate implies that  $\gamma = 0$ . This ends the proof.  $\square$

*Remark.* If  $u$  is a radial function estimate (4.11) can be improved. Actually, since by Lemma 4.1 one has (4.5), the integration of this inequality yields

$$|u(r) - u(0)| \leq cr^{\frac{q-2}{q-1}}, \quad (4.12)$$

and  $\frac{q-2}{q-1} > \frac{q-2}{q}$ .

### 4.3 Proof of Theorem 3

The proof of Theorem 3 is as follows: we first observe that the conclusion of Theorem 3 holds in the class of radial solutions. Then we prove that if  $u$  is a positive solution (4.4) in  $B_1 \setminus \{0\}$  which tends to  $\infty$  when  $x \rightarrow 0$ , then there exists a positive solution  $\tilde{u}$  which satisfies

$$(u - \max\{u(y) : |y| = 1\})_+ \leq \tilde{u} \leq u \quad \text{in } B_1 \setminus \{0\}$$

and vanishes on  $\partial B_1$ . This part of the proof is delicate and based upon a combination of Bernstein method for estimating the gradient of a solution and a subtle variation of Boccardo-Murat-Puel classical existence result of solution in presence of a couple of ordered supersolution and subsolution. Finally we prove by adaptating the moving plane method that the function  $\tilde{u}$  is radial and we apply the radial result and the above estimate.

The following result has already been obtained in the proof of assertion 2-(ii) in Theorem 3.8.

**Proposition 4.3** *Let  $q > 2$  and  $u$  be a positive solution of (4.4) in  $B_1 \setminus \{0\}$ . Then*

$$\int_{B_1} |\nabla u|^q dx = \infty \implies \liminf_{x \rightarrow 0} \frac{u(x)}{-\ln|x|} \geq \frac{2}{b}. \quad (4.13)$$

*Remark.* Using the estimates of Proposition 2 we have precisely

$$u(x) \geq \frac{2}{b} \left( \ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) - K. \quad (4.14)$$

Under the assumption of Proposition 4.3 a radial solution  $u$  vanishing on  $\partial B_1$  is decreasing. In the next result we prove that  $ru'(r)$  is monotone.

**Lemma 4.4** *Let  $u$  be a positive radial singular solution of (4.4) in  $B_1 \setminus \{0\}$ , then  $ru'(r)$  is monotone near  $r = 0$ .*

*Proof.* We set

$$\mathcal{W}(r) = ae^{bu(r)} - m|u'(r)|^q. \quad (4.15)$$

Using the equation (4.4) we have

$$\mathcal{W}' = u'ae^{bu} (b - qm|u'|^{q-2}) + qm|u'|^{q-2}u' \left( |u'|^q + \frac{u'}{r} \right).$$

Replacing  $ae^{bu}$  by  $\mathcal{W} + m|u'|^q$ , we obtain

$$\mathcal{W}' = u' (b - qm|u'|^{q-2}) \mathcal{W} + mb|u'|^q u' - (m-1)qm|u'|^{2q-2}u' + \frac{qm}{r}|u'|^q$$

Denote

$$\Theta(r) = u'(r) (b - qm|u'|^{q-2}(r)),$$

and

$$\Sigma(r) = mb|u'|^q u' - (m-1)qm|u'|^{2q-2}u' + \frac{qm}{r}|u'|^q$$

By (4.5),  $u'(r)$  is nonpositive near  $r = 0$  and tends to  $-\infty$  when  $r \rightarrow 0$ . Then

$$\Sigma(r) = (m-1)q|u'|^{2q-1}(1 + o(1)) + \frac{qm}{r}|u'|^q \quad \text{as } r \rightarrow 0,$$

since  $q > 2$ . By integration on  $[r, r_0] \subset (0, 1)$ ,

$$\frac{d}{dr} \left( e^{\int_r^{r_0} \Theta(s) ds} \mathcal{W}(r) \right) = e^{\int_r^{r_0} \Theta(s) ds} \Sigma(r),$$

hence

$$\mathcal{W}(r) = e^{-\int_r^{r_0} \Theta(s) ds} \mathcal{W}(r_0) - \int_r^{r_0} e^{-\int_r^s \Theta(t) dt} \Sigma(s) ds.$$

There exists  $r^* \in (0, 1)$  such that  $\Sigma(r) > 0$  for  $(0 < r \leq r^*)$ . Therefore if for some  $r_0 \in (0, r^*]$  there holds  $\mathcal{W}(r_0) \leq 0$ , it follows that  $\mathcal{W}(r) < 0$  for any  $r \in (0, r_0)$ . Since

$$(ru')' = r \left( ae^{bu} - m|u'|^q \right) = r\mathcal{W},$$

this implies that  $(ru')' \leq 0$  on  $(0, r_0]$ . On the contrary, for any if for any  $r \in (0, r^*]$  we have  $\mathcal{W}(r) > 0$ , we conclude that  $(ru')' \leq 0$  on  $(0, r_0]$ .  $\square$

**Proposition 4.5** *Let  $q > 2$  and  $u$  be a positive radial solution of (4.4) in  $B_1 \setminus \{0\}$  satisfying  $\int_{B_1} |\nabla u|^q dx = \infty$ . If  $ru'(r)$  is monotone then*

$$\lim_{x \rightarrow 0} \frac{u(x)}{-\ln|x|} = \frac{q}{b}. \quad (4.16)$$

*Proof.* If  $u$  satisfies the above assumption, there holds by Propositions 2 and 8,

$$\frac{2}{b} \ln \frac{1}{|x|} - K(b) \ln(1 - \ln|x|) \leq u(x) \leq \frac{q}{b} \ln \left( \frac{1}{|x|} \right) + \frac{q}{b} \left[ \left( \ln \left( \frac{b}{qm^{\frac{1}{q}}} \right) \right)_+ - \ln \left( \frac{b}{qm^{\frac{1}{q}}} \right) \right], \quad (4.17)$$

where  $K(b) = 1$  if  $b \geq 2$  and  $K(b) = \frac{2}{b}$  if  $0 < b \leq 2$ . We set  $t = \ln r$  and  $w(t) = u(r)$ . Then

$$-w_{tt} + ae^{bw+2t} - me^{(2-q)t}|w_t|^q = 0. \quad (4.18)$$

Moreover, if  $ru'(r)$  is monotone, then  $w_t$  is monotone too. From (4.17),

$$-\frac{2}{b}t - K(b) \ln(1 - t) \leq w(t) \leq -\frac{q}{b}t + \frac{q}{b} \left[ \left( \ln \left( \frac{b}{qm^{\frac{1}{q}}} \right) \right)_+ - \ln \left( \frac{b}{qm^{\frac{1}{q}}} \right) \right] \quad \text{on } (-\infty, 0).$$

If  $w_t$  is monotone, it admits a limit  $\ell$ , finite or not, when  $t \rightarrow -\infty$ . The only possibility is that

$$-\frac{q}{b} \leq \ell \leq -\frac{2}{b}.$$

and  $w(t) = \ell t(1 + o(1))$  when  $t \rightarrow -\infty$ . From (4.18),

$$w_{tt} = -me^{(2-q)t}|\ell|^q(1 + o(1)) + ae^{(b\ell+2)t+o(t)}.$$

There always holds  $2 - q \geq b\ell + 2$ . In order that  $w_{tt}$  does not tend to  $-\infty$  (which would be a contradiction), necessarily  $2 - q = b\ell + 2$  which implies  $\ell = -\frac{q}{b}$ . This yields (4.16).  $\square$

In next result we obtain an a priori estimate of the gradient of a solution by a combination of the Bernstein and the Keller-Osserman methods as in [1, Lemma 3.1] (see also [2, Lemma 2.2] for a simpler version).

**Proposition 4.6** *If  $q > 2$  and  $u$  is a positive solution of (4.4) in  $B_1 \setminus \{0\}$ , then for any  $\delta > 0$  there exists  $C^* = C^*(a, b, m, q, \delta) > 0$  such that*

$$|\nabla u(x)| \leq \frac{C^*}{|x|} \quad \text{for all } x \in B_{1-\delta} \setminus \{0\}. \quad (4.19)$$

*Proof.* If  $u$  is such a solution, we set

$$z = |\nabla u|^2.$$

Then by an explicit computation and using Bochner identity and Schwarz inequality

$$-\frac{1}{2}\Delta z = -|D^2u|^2 - \nabla(\Delta u) \cdot \nabla u \leq -\frac{1}{2}(\Delta u)^2 - \nabla(\Delta u) \cdot \nabla u. \quad (4.20)$$

Since  $\Delta u = ae^{bu} - mz^{\frac{q}{2}}$ , we have for any  $\epsilon > 0$  and explicit  $C(\epsilon) > 0$ ,

$$\begin{aligned} -\Delta z &\leq -(ae^{bu} - mz^{\frac{q}{2}})^2 - 2\nabla(ae^{bu} - mz^{\frac{q}{2}}) \cdot \nabla u \\ &\leq -a^2e^{2bu} - m^2z^q + 2ame^{bu}z^{\frac{q}{2}} - 2abe^{bu}z + qmz^{\frac{q}{2}}\frac{|\nabla z|}{\sqrt{z}} \\ &\leq -m^2z^q + 2am\left(\epsilon z^q + C(\epsilon)e^{2bu}\right) + qm\left(\epsilon z^q + C(\epsilon)\frac{|\nabla z|^2}{z}\right). \end{aligned}$$

Therefore

$$-\Delta z + (m^2 - \epsilon(2am + qm))z^q \leq 2amC(\epsilon)e^{2bu} + qmC(\epsilon)\frac{|\nabla z|^2}{z}.$$

We choose  $\epsilon$  such that  $m^2 - \epsilon(2am + qm) \geq \frac{m^2}{2}$ . By the a priori estimate (2.2),  $e^{2bu} \leq e^{2b\bar{\Lambda}}|x|^{-2q}$ . Therefore  $z$  satisfies

$$-\Delta z + \frac{m^2}{2}z^q \leq \frac{C_1}{|x|^{2q}} + C_2\frac{|\nabla z|^2}{z}, \quad (4.21)$$

for some  $C_j = C_j(a, b, m, q) > 0$ ,  $j = 1, 2$ . By [1, Lemma 3.1], it follows that

$$z(x) \leq C_1^*|x|^{-\frac{2}{q-1}} + C_2^*|x|^{-2} \leq (C_1 + C_2)|x|^{-2}. \quad (4.22)$$

This implies (4.19).  $\square$

*Remark.* When  $1 < q < 2$ , the above method leads to the estimate

$$|\nabla u(x)| \leq \frac{C^*}{|x|^{-\frac{1}{q-1}}} \quad \text{for all } x \in B_{1-\delta} \setminus \{0\}, \quad (4.23)$$

which is not optimal for radial functions as Lemma 3.2 shows it.

**Lemma 4.7** *Let  $q > 2$  and  $u \in C^1(B_1 \setminus \{0\}) \cap C(\overline{B_1} \setminus \{0\})$  be a nonnegative solutions of (4.4) in  $B_1 \setminus \{0\}$  which satisfies*

$$\liminf_{x \rightarrow 0} \frac{u(x)}{-\ln|x|} \geq \frac{2}{b}. \quad (4.24)$$

*If  $\mu = \max\{u(x) : |x| = 1\}$ , there exists a positive solution  $\tilde{u} \in C^1(B_1 \setminus \{0\}) \cap C(\overline{B_1} \setminus \{0\})$  of (4.4) in  $B_1 \setminus \{0\}$  which vanishes on  $\partial B_1$  and satisfies*

$$(u - \mu)_+ \leq \tilde{u} \leq u \quad \text{in } B_1 \setminus \{0\}. \quad (4.25)$$

*Proof. Step 1: Construction of an approximate solution.* The function  $u_\mu = (u - \mu)_+$  is a positive subsolution of (4.4) in  $B_1 \setminus \{0\}$  which vanishes on  $\partial B_1$  and it is dominated by the solution  $u$ . However we cannot apply directly the classical result of Boccardo, Murat and Puel [8] since  $q > 2$  (the limit is  $q = 2$ ) and a variation of the construction is needed. For  $\epsilon \in (0, 1)$  it follows by Proposition 4.6 that

$$|\nabla u(x)| \leq \frac{C^*}{\epsilon} \quad \text{for all } x \in B_1 \setminus B_\epsilon. \quad (4.26)$$

and  $C^*$  is independent of  $\epsilon$ . Up to changing  $C^*$  the above inequality holds if  $u$  is replaced by  $(u - \mu)_+$ . We define now the function  $\phi_\epsilon$  on  $\mathbb{R}_+$  by

$$\phi_\epsilon(\xi) = \begin{cases} \xi^{\frac{q}{2}} & \text{if } 0 \leq \xi \leq \left(\frac{C^*}{\epsilon}\right)^2 \\ \frac{q(q-1)}{2} \left(\frac{C^*}{\epsilon}\right)^{q-2} \xi + q(2-q) \left(\frac{C^*}{\epsilon}\right)^{q-1} \sqrt{\xi} + \frac{q^2 - 3q + 2}{2} \left(\frac{C^*}{\epsilon}\right)^q & \text{if } \xi \geq \left(\frac{C^*}{\epsilon}\right)^2. \end{cases} \quad (4.27)$$

The function  $\xi \mapsto \phi_\epsilon(|\xi|^2)$  belongs to  $C^2(\mathbb{R}^2)$  and is convex with a quadratic growth at infinity. The result from [8] applies. Since  $\phi_\epsilon(|\nabla u|^2) = |\nabla u|^q$  (resp.  $\phi_\epsilon(|\nabla(u - \mu)_+|^2) = |\nabla(u - \mu)_+|^q$ ), the problem

$$\begin{aligned} -\Delta w + ae^{bw} - m\phi_\epsilon(|\nabla w|^2) &= 0 & \text{in } B_1 \setminus \overline{B}_\epsilon \\ w &= 0 & \text{on } \partial B_1 \\ w &= (u - \mu)_+ & \text{on } \partial B_\epsilon, \end{aligned} \quad (4.28)$$

admits a solution  $w_\epsilon$  which satisfies

$$(u - \mu)_+ \leq w_\epsilon \leq u \quad \text{in } B_1 \setminus \overline{B}_\epsilon. \quad (4.29)$$

*Step 2: Local estimate of  $\nabla w_\epsilon$ .* In order to have a local estimate independent of  $\epsilon$ , we apply Bernstein method as in Proposition 4.6. Setting again  $z = |\nabla w_\epsilon|^2$  we have from the equation (4.29),

$$\begin{aligned} -\Delta z &\leq (ae^{bw_\epsilon} - m\phi_\epsilon(z))^2 - 2\nabla(ae^{bw_\epsilon} - m\phi_\epsilon(z)) \cdot \nabla w_\epsilon \\ &\leq -a^2 e^{2bw_\epsilon} - m^2 \phi_\epsilon^2(z) + 2ame^{bw_\epsilon} \phi_\epsilon(z) - 2abe^{bw_\epsilon} z + 2m\phi_\epsilon'(z) |\nabla z| \sqrt{z} \\ &\leq -(m^2 - 2am\delta^2) \phi_\epsilon^2(z) + \frac{2am}{\delta^2} e^{2bw_\epsilon} + 2m\delta^2 \phi_\epsilon'^2(z) z^2 + \frac{2m}{\delta^2} \frac{|\nabla z|^2}{z}. \end{aligned}$$

By the definition of  $\phi_\epsilon$  there exists  $\kappa > 0$  independent of  $\epsilon$  such that  $\phi_\epsilon'^2(z) z^2 \leq \kappa \phi_\epsilon^2(z)$ . Provided  $\delta > 0$  is small enough and using (2.8) and using estimate (4.29) we obtain,

$$-\Delta z + \frac{m^2}{2} \phi_\epsilon^2(z) \leq \frac{2am}{\delta^2} e^{2bw_\epsilon} + \frac{2m}{\delta^2} \frac{|\nabla z|^2}{z} \leq \frac{K}{|x|^{2q}} + \frac{2m}{\delta^2} \frac{|\nabla z|^2}{z}. \quad (4.30)$$

By the expression of  $\phi_\epsilon(\xi)$ , there exists constants  $\theta$  and  $C_\theta > 0$  such that

$$\phi_\epsilon^2(z) \geq 2\theta z^2 - C_\theta \quad (4.31)$$



Therefore (4.30) combined with (4.29) and (2.2) yields

$$-\Delta z + m^2 \theta z^2 \leq \frac{K}{|x|^{2q}} + \frac{C_\theta m^2}{2} + \frac{2m}{\delta^2} \frac{|\nabla z|^2}{z} \quad \text{in } B_1 \setminus B_\epsilon. \quad (4.32)$$

If  $\epsilon < |x| < 1 - \epsilon$  we apply [1, Lemma 3.1] in  $B_{|x|-\epsilon}(x)$  with  $q = 2$  and derive

$$z(x) := |\nabla w_\epsilon(x)|^2 \leq \frac{\tilde{C}_1}{|x|^q} + \frac{\tilde{C}_2}{(|x| - \epsilon)^2} \quad (4.33)$$

with  $\tilde{C}_j > 0$  independent of  $\epsilon$ . We now define  $\epsilon^* > \epsilon$  by the identity

$$\frac{\tilde{C}_1}{\epsilon^{*q}} + \frac{\tilde{C}_2}{(\epsilon^* - \epsilon)^2} = \left( \frac{C^*}{\epsilon} \right)^2 \quad (4.34)$$

where  $C^*$  is defined at (4.27) (such a  $\epsilon^*$  exists by the mean value theorem). The value of  $\epsilon^*$  cannot be explicitated but

$$\tilde{C}_1^{\frac{1}{q}} \left( \frac{\epsilon}{C^*} \right)^{\frac{2}{q}} \leq \epsilon^* \leq \tilde{C}_1^{\frac{1}{q}} \left( \frac{\epsilon}{C^*} \right)^{\frac{2}{q}} (1 + o(1)) \quad \text{as } \epsilon \rightarrow 0. \quad (4.35)$$

As a consequence the function  $w_\epsilon$  is also a solution of

$$\begin{aligned} -\Delta w + a e^{bw} - m |\nabla w|^q &= 0 \quad \text{in } B_1 \setminus \overline{B}_{\epsilon^*} \\ w &= 0 \quad \text{on } \partial B_1, \end{aligned} \quad (4.36)$$

and thus it satisfies the upper estimate obtained in Proposition 4.6,

$$|\nabla w_\epsilon(x)| \leq \frac{C^*}{|x| - \epsilon^*} \quad \text{for all } x \in B_1 \setminus \overline{B}_{\epsilon^*}. \quad (4.37)$$

*Step 3: End of the proof.* It follows from (4.37) that for any  $\epsilon \in (0, 1)$  small enough,  $w_\epsilon$  and  $\nabla w_\epsilon$  are uniformly bounded in  $B_1 \setminus \overline{B}_{2\epsilon^*}$ . By the standard elliptic equations regularity theory,  $w_\epsilon$  remains bounded in  $C^{2,\alpha}(K)$  for any  $K \subset B_1 \setminus \overline{B}_{3\epsilon^*}$ . Using a diagonal sequence, there exists  $\{\epsilon_n\}$  converging to 0 and  $\tilde{u} \in C^2(\overline{B}_1 \setminus \{0\})$  such that  $\{w_{\epsilon_n}\}$  converges to  $\tilde{u}$  in the  $C_{loc}^2$ -topology of  $\overline{B}_1 \setminus \{0\}$ . Therefore  $\tilde{u}$  is a solution of  $B_1 \setminus \{0\}$  which vanishes on  $\partial B_1$  and satisfies  $(u - \mu)_+ \leq \tilde{u} \leq u$  jointly with the gradient estimate (4.26).  $\square$

**Proposition 4.8** *Let  $q > 2$  and  $u \in C^1(B_1 \setminus \{0\}) \cap C(\overline{B}_1 \setminus \{0\})$  is a nonnegative solutions of (4.4) in  $B_1 \setminus \{0\}$  which vanishes on  $\partial B_1$  and satisfies*

$$\lim_{x \rightarrow 0} u(x) = \infty. \quad (4.38)$$

*Then  $u$  is radially symmetric.*

*Proof.* The proof is an easy variant of the moving plane method as it is introduced in [13] (see also [16, Theorem 8.2.2]). The only difference is the assumption that  $u$  satisfies (4.38) which actually allows the reflection through the singular point. The sketch of the proof is as follows. We set  $x = (x_1, x_2)$  and, if  $\lambda \in (0, 1)$ ,  $B^\lambda = \{x \in B_1 : x_1 > \lambda\}$ ,  $T^\lambda = \{x \in B_1 : x_1 = \lambda\}$  and  $C^\lambda = \partial B_1 \cap \overline{B^\lambda}$ . Put  $x_\lambda = (2\lambda - x_1, x_2)$  and  $u_\lambda(x) = u(x_\lambda)$ . If  $w = u_\lambda - u$ , there holds

$$-\Delta w(x) + A(x)w(x) - B(x) \cdot \nabla w(x) = 0 \quad \text{in } B^\lambda \quad (4.39)$$

where

$$A(x) = ab \int_0^1 e^{b(tu_\lambda(x) + (1-t)u(x))} dt$$

and

$$B(x) = mq \int_0^1 |\nabla(tu_\lambda(x) + (1-t)u(x))|^{q-2} \nabla(tu_\lambda(x) + (1-t)u(x)) dt$$

The two functions  $A$  and  $B$  are Hölder continuous in  $B^\lambda$  for  $\lambda > 0$ . Since  $u > 0$  in  $B_1 \setminus \{0\}$  and vanishes on  $\partial B_1$ , the outward normal derivative  $\frac{\partial u}{\partial \mathbf{n}}$  is negative on  $\partial B_1$ . For  $\lambda \in (\frac{1}{2}, 1)$  the function  $w$  satisfies (4.39) in  $B^\lambda$ . It vanishes on  $T^\lambda$  and is positive on  $C^\lambda$  except at the two end-points belonging to  $\partial B_1 \cap T^\lambda$ . In fact, if we assume now that there exists  $x^* \in \overline{B^\lambda}$  where  $w$  achieves a negative minimum, then  $x^* \in B^\lambda$  and because  $w$  is  $C^2$ ,  $\nabla w(x^*) = 0$  and  $-\Delta w(x^*) \leq 0$ . Since

$$-\Delta w(x^*) + A(x^*)w(x^*) - B(x^*) \cdot \nabla w(x^*) = 0$$

and  $A(x^*) > 0$  we have a contradiction. If  $\lambda \in (0, \frac{1}{2}]$  the function  $w$  is not continuous in whole  $\overline{B^\lambda}$ , but since  $u$  satisfies (4.38), the function  $w$  tends to infinity when  $x \rightarrow (2\lambda, 0)$  (this point is the reflected image of  $(0, 0)$  by the symmetry  $(x \mapsto x_\lambda)$ ). Hence the minimum of  $w$  in  $\overline{B^\lambda}$  coincides with the minimum in  $\overline{B^\lambda} \setminus B_\delta((2\lambda, 0))$ . Since  $w$  is  $C^2$  therein, this implies that it cannot achieve a negative minimum. As a consequence  $w \geq 0$  in  $B_+^0 := B_1 \cap \{x = (x_1, x_2) : x_1 > 0\}$ , and by continuity this inequality holds also in  $\overline{B_1} \cap \{x = (x_1, x_2) : x_1 \geq 0\} \setminus \{(0, 0)\}$ . This implies in particular  $u_{x_1}(0, x_2) \geq 0$ . Starting the reflection process from the point  $(-1, 0)$  we obtain in the same way  $u_{x_1}(0, x_2) \leq 0$ . Hence  $u_{x_1}(0, x_2) = 0$  for all  $x_2 \in (-1, 1)$ . Because the equation is invariant by rotation we can replace the axis  $(0, x_2)$  by any axis through  $(0, 0)$ . Consequently we have  $u_\theta(\theta, r) = 0$  in polar coordinates, which is the claim.  $\square$

*Proof of Theorem 3.* It follows from Proposition 4.2, Proposition 4.3, Proposition 4.5, Lemma 4.7 and Proposition 4.8.  $\square$

*Remark.* This type of dichotomy which occurs in this case between regular solutions and only one type of singular solutions is well known for semilinear Lane-Emden equations

$$-\Delta u = u^p \quad \text{in } B_1 \setminus \{0\} \quad (4.40)$$

when  $N \geq 3$  and  $p > \frac{N}{N-2}$ . To our knowledge it is a novelty for equation of (4.4) and in a 2-dimensional framework.

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