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Laurent Véron, Yimei Li

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Abstract We study the local properties of positive solutions of the equation $-\Delta u + ae^{bu} = m\left|\nabla u\right|^q$ in a punctured domain $\Omega \setminus \{0\}$ of $\mathbb{R}^2$ where m, a, b are positive parameters and $q > 1$. We study particularly the existence of solutions with an isolated singularity and the local behaviour of such singular solutions.

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Contents

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1 Introduction

The following type of nonlinear equation

$$-\Delta u + f(u) - g(|\nabla u|) = 0 \tag{1.1}$$

in a N dimensional domain when f and g are nondecreasing continuous functions is an interesting model for studying the interaction between the diffusion, the reaction and the absorption when the last two quantities are of a totally different nature. The most interesting case correspond to the case where the two reaction functions f and g increasing in their respective variable. In its full generality the problem is difficult to handle, but the model cases where $f(u) = u^p$ if $N \geq 3$

$$-\Delta u + u^p - m|\nabla u|^q = 0. \tag{1.2}$$

has already been studied in [2]. The results therein shows the rich interaction of the phenomena coming eiter from the interaction of the diffusion and the absorption, the diffusion and the reaction or the reaction and the absorption. A variant of this equation is the Chipot-Weissler equation

$$-\Delta u - u^p + m|\nabla u|^q = 0, \tag{1.3}$$

which presents also interesting interaction between the pairing of the three terms. This has been studied in [1] and [3]. For this last equation the main difficulty is to obtain a priori estimates of positive solutions near an isolated singularity. These estimates have been obtained in [1] and [3]. The aim of the present paper is to study a 2-dimensional problem of type (1.2) where naturally the power absorption is replaced by an exponential absorption. Our model equation is

$$\mathcal{E}(u) := -\Delta u + ae^{bu} - m|\nabla u|^q = 0 \tag{1.4}$$

in $B_1 \setminus \{0\}$ where B_1 is the unit disk of $\mathbb{R}^2$, m, a, b are positive parameters and $q > 1$. For (1.2) and (1.1) the a priori estimate near an isolated singularity is easy to obtain, the difficulty lies in the full description of singularities and the classifications of singular solutions . Three particular equations are naturally associated to equation (1.4):
the Emden's equation

$$-\Delta u + ae^{bu} = 0, \tag{1.5}$$

the Ricatti equation

$$-\Delta u - m|\nabla u|^q = 0, \tag{1.6}$$

and the eikonal equation

$$ae^{bu} - m|\nabla u|^q = 0. \tag{1.7}$$

The predominance of a specific equation of this type depends on the value of q, with critical value $q = 2$. When $1 < q < 2$ the singularities are similar to the one of Emden's equation, an equation which has been completely described in [12]. We recall the main result therein:

Vàzquez theorem *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain containing 0 and $a, b > 0$. Then for any $\phi \in C^1(\partial\Omega)$ and $\gamma \in [0, \frac{2}{b}]$ there exists a unique function $v_\gamma \in C^1(\overline{\Omega} \setminus \{0\})$ such that $e^{bv_\gamma} \in L^1(\Omega)$ satisfying*

$$-\Delta v_\gamma + ae^{bv_\gamma} = 2\pi\gamma\delta_0 \quad \text{in } \mathcal{D}'(\Omega). \tag{1.8}$$

coinciding with ϕ on $\partial\Omega$. Furthermore

$$\lim_{x\to 0}\frac{v_\gamma(x)}{-\ln|x|}=\gamma. \tag{1.9}$$

The condition $0\le\gamma\le\frac{2}{b}$ is also a necessary condition for the solvability of (1.8)
When $q=2$, the change of unknown $u=\frac{1}{m}\ln v$ transform (1.4) into

$$-\Delta v+mav^{\frac{b}{m}+1}=0, \tag{1.10}$$

an equation which has been thoroughly studied (e.g. [7], [14]) and thus we will omit it. When $q>2$ the dominent equation is the eikonal one. There is a particular explicit solution which plays a key role in the studies of singularities

$$U(|x|)=\frac{q}{b}\left(\frac{m}{a}\right)^{\frac{1}{q}}\ln\left(\frac{q}{b|x|}\right). \tag{1.11}$$

We summarise below our three main results

Theorem 1 *Let $m,a,b>0$ and $1<q<2$. If $u\in C^1(\overline{B}_1\setminus\{0\})$ is a positive function satisfying* (1.4). *Then there exists $\gamma\in[0,\frac{2}{b}]$ such that*

$$\lim_{x\to 0}\frac{u(x)}{-\ln|x|}=\gamma. \tag{1.12}$$

Furthermore e^{bu} and $|\nabla u|^q$ are integrable in B_1 and there holds

$$-\Delta u+ae^{bu}-m|\nabla u|^q=2\pi\delta_0\quad \text{in } \mathcal{D}'(B_1). \tag{1.13}$$

If $\gamma=0$, u can be extended as a $C^1(\overline{B}_1)$ function.

Conversely we prove

Theorem 2 *Under the assumptions on m,a,b,q of Theorem 1, for any $\gamma\in(0,\frac{2}{b}]$ and $\phi\in C^1(\partial B_1)$ there exists a unique function $u\in C^1(\overline{B}_1\setminus\{0\})$ satisfying* (1.13) *and coinciding with ϕ on ∂B_1. Furthermore e^{bu} and $|\nabla u|^q$ are integrable in B_1 and* (1.13) *holds.*

When $q>2$, the classification of the solutions is totally different and present some similarity with the study of isolated singularities of Lane-Emden equation

$$-\Delta u-u^p=0, \tag{1.14}$$

in the supercritical case $N\ge 3$ and $p>\frac{N}{N-2}$ (see [9]). We prove

Theorem 3 *Let $m,a,b>0$ and $q>2$. If $u\in C^1(\overline{B}_1\setminus\{0\})$ is a positive solution of* (1.4), *then (i) either*

$$\liminf_{x\to 0}\frac{u(x)}{-\ln|x|}\ge\frac{2}{b}, \tag{1.15}$$

(ii) or u can be extended to $\overline{B}_1$ as a Hölder continuous function.
Furthermore, if u is radial assertion (i) is replaced by

$$\lim_{x\to 0}\frac{u(x)}{-\ln|x|}=\frac{q}{b}. \tag{1.16}$$

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2 A priori estimates

The next estimates are valid in any dimension

Proposition 2.1 *Let $m,a,b>0$ and $q>1$. Assume that for $x_0\in B_1\subset\mathbb{R}^N$ and $0<R<|x_0|$ u is a solution of (1.4) in $B_R(x_0)$. Then there exists $\Lambda=\Lambda(m,q,N,a,b)>0$ such that*
(i) If $1<q<2$,

$$u(x_0)\le\frac{2}{b}\ln\frac{1}{|x_0|}+\Lambda. \tag{2.1}$$

(ii) If $q>2$

$$u(x_0)\le\frac{q}{b}\ln\frac{1}{|x_0|}+\tilde{\Lambda}. \tag{2.2}$$

Proof. Let $R'<R$. We set for $\lambda,\mu>0$,

$$\psi(x)=\lambda\ln(1/(R'^2-|x-x_0|^2))+\mu=\lambda\ln(1/(R'^2-r^2))+\mu,$$

with $r=|x-x_0|$. Then

$$ae^{b\psi}=\frac{ae^{b\mu}}{(R'^2-r^2)^{b\lambda}},$$

$$|\nabla\psi(x)|=|\psi_r(r)|=\frac{2\lambda r}{R'^2-r^2},$$

$$|\nabla\psi(x)|^q=\frac{2^q\lambda^q r^q}{(R'^2-r^2)^q},$$

$$-\Delta\psi(x)=-2\lambda\frac{NR'^2-(N-2)r^2}{(R'^2-r^2)^2}.$$

Hence there holds in $B_{R'}$,

$$\mathcal{E}(\psi)=\frac{ae^{b\mu}}{(R'^2-r^2)^{b\lambda}}-2\lambda\frac{NR'^2-(N-2)r^2}{(R'^2-r^2)^2}-\frac{m2^q\lambda^q r^q}{(R'^2-r^2)^q}.$$

We have $\psi>u$ in a neighbourhood of $\partial B_{R'}(x_0)$. We encounter two cases

(i) $1 < q \leq 2$. Then we take $b\lambda = 2$ and get

$$\begin{aligned}\mathcal{E}(\psi) &= (R'^2 - r^2)^{-2}\left[ae^{b\mu} - \frac{4}{b}\left(NR'^2 - (N-2)r^2\right) - \frac{4^q m r^q}{b^q}\left(R'^2 - r^2\right)^{2-q}\right] \\ &\geq (R'^2 - r^2)^{-2}\left[ae^{b\mu} - \Lambda^* \max\{R'^2, R'^{4-q}\}\right] \\ &\geq (R'^2 - r^2)^{-2}\left[ae^{b\mu} - \Lambda^* R'^2\right],\end{aligned} \tag{2.3}$$

where $\Lambda^* = \Lambda^*(N, q, m, b) > 0$. We choose

$$\mu = \frac{2}{b}\ln R' + \frac{1}{b}\ln\left(\frac{\Lambda^*}{a}\right) \tag{2.4}$$

and derive that $\mathcal{E}(\psi) \geq 0$ in $B_{R'}$. Hence $u \leq \psi$ therein. This implies

$$u(x_0) \leq \psi(x_0) := \frac{2}{b}\ln\frac{1}{R'} + \frac{1}{b}\ln\left(\frac{\Lambda^*}{a}\right). \tag{2.5}$$

Letting $R' \uparrow R = |x_0|$ yields (2.1).
(ii) $q > 2$. Then we take $b\lambda = q$. Hence

$$\begin{aligned}\mathcal{E}(\psi) &= (R^2 - r^2)^{-q}\left[ae^{b\mu} - \frac{q}{b}\left(NR^2 - (N-2)r^2\right)\left(R^2 - r^2\right)^{q-2} - \frac{(2q)^q m r^q}{b^q}\right] \\ &\geq (R^2 - r^2)^{-q}\left[ae^{b\mu} - \tilde{\Lambda}^* R^q\right]\end{aligned} \tag{2.6}$$

where $\tilde{\Lambda}^* = \tilde{\Lambda}^*(N, q, m, a, b) > 0$. We take

$$\mu = \frac{q}{b}\ln R + \frac{1}{b}\ln\left(\frac{\tilde{\Lambda}^*}{a}\right), \tag{2.7}$$

and derive as in case (i)

$$u(0) \leq \psi(0) := \frac{q}{b}\ln\left(\frac{1}{R'}\right) + \frac{1}{b}\ln\left(\frac{\tilde{\Lambda}^*}{a}\right). \tag{2.8}$$

Letting $R' \uparrow R$ we obtain (2.2). □

In the next lemma we give some precise estimates on the function v_γ solution of (1.8)

Lemma 2.2 *Let $m, a, b > 0$ and $0 < \gamma \leq \frac{2}{b}$. If $v_\gamma \in C^1(\overline{B}_1 \setminus \{0\})$ is the solution (1.8) which vanishes on ∂B_1, then*
1- If $0 \leq \gamma < \frac{2}{b}$, there exists $\phi^* \in C^1(\overline{B}_1)$ such that

$$v_\gamma(x) = \gamma \ln\frac{1}{|x|} - \phi^*(x) \quad \textit{for all } x \in \overline{B}_1 \setminus \{0\}. \tag{2.9}$$

2- If $\gamma = \frac{2}{b}$, then for all $x \in \overline{B}_1 \setminus \{0\}$ we have

$$\frac{2}{b}\left(\ln\frac{1}{|x|}-\ln(1-\ln|x|)\right)+\kappa_2\le v_{\frac{2}{b}}(x)\le\frac{2}{b}\left(\ln\frac{1}{|x|}-\ln(1-\ln|x|)\right)+\kappa_1 \tag{2.10}$$

where

$$\kappa_1=\max\left\{0,\frac{1}{b}\ln\frac{2}{ab}\right\}\quad \textit{and}\quad \kappa_2=\min\left\{0,\frac{1}{b}\ln\frac{2}{ab}\right\}. \tag{2.11}$$

In particular

$$v_{\frac{2}{b}}(x)=\frac{2}{b}\left(\ln\frac{1}{|x|}-\ln(1-\ln|x|)\right), \tag{2.12}$$

if $ab=2$.

Proof. Let $\gamma\in(0,\frac{2}{b}]$ and ϕ_1 be the first eigenfunction of $-\Delta$ in $W_0^{12}(B_1)$, with corresponding eigenvalue $\lambda_1>0$ and supremum equal to 1 (at $r=0$). For $\theta>0$ and $A>0$, we set

$$h_A(x)=\gamma\ln\frac{1}{|x|}-A|x|^\theta\phi_1(x)=\gamma\ln\frac{1}{r}-Ar^\theta\phi_1(r)\quad\text{with } r=|x|. \tag{2.13}$$

Then there holds in $\mathcal{D}'(B_1)$,

$$-\Delta h_A(r)=2\pi\gamma\delta_0+A\theta^2r^{\theta-2}\phi_1(r)-A\lambda_1r^\theta\phi_1(r)+2A\theta r^{\theta-1}\phi_1'(r),$$

thus

$$-\Delta h_A+ae^{bh_A}=2\pi\gamma\delta_0+r^{\theta-2}\left(A\theta^2\phi_1(r)-A\lambda_1r^2\phi_1(r)+2A\theta r\phi_1'(r)+ar^{2-\theta-b\gamma}e^{-bAr^\theta\phi_1(r)}\right).$$

Step 1: the case $b\gamma<2$
(i) *Estimate from above.* Take $\theta=2-b\gamma$. Since

$$ae^{-bAr^{2-b\gamma}\phi_1(r)}\ge a-abAr^{2-b\gamma}\phi_1(r),$$

we derive that for all $0<|x|<1$,

$$\begin{aligned}-\Delta h_A+ae^{bh_A}&=r^{-b\gamma}\left(A(2-b\gamma)^2\phi_1(r)-A\lambda_1r^2\phi_1(r)+2A(2-b\gamma)r\phi_1'(r)+ae^{-bAr^{2-b\gamma}\phi_1(r)}\right)\\&\ge r^{-b\gamma}\left(A(2-b\gamma)^2\phi_1(r)-A\lambda_1r^2\phi_1(r)+2A(2-b\gamma)r\phi_1'(r)+a-abAr^{2-b\gamma}\phi_1(r)\right).\end{aligned}$$

Then there exists $A_0>0$ such that for $A\le A_0$ we have

$$-\Delta h_A+ae^{bh_A}\ge2\gamma\pi\delta_0\quad\text{in }\mathbf{D}'(B_1), \tag{2.14}$$

and A_0 can be taken independent of γ. We derive by the comparison principle that

$$v_\gamma(x)\le\gamma\ln\frac{1}{|x|}-A|x|^{2-b\gamma}\phi_1(x)\quad\text{in } B_1\setminus\{0\}. \tag{2.15}$$

(ii) *Estimate from below.* We give an explicit expression for v_γ. We already know that $v_\gamma(x)<\gamma\ln\frac{1}{|x|}$. If we write

$$v_\gamma(x)=\gamma\ln\frac{1}{|x|}+\phi^*(x), \tag{2.16}$$

then ϕ^* vanishes on ∂B_1 and satisfies

$$-\Delta\phi^* + ar^{-b\gamma}e^{b\phi^*} = 0.$$

The convex functional

$$J(\phi) = \int_{B_1} \left(\frac{1}{2}|\nabla\phi|^2 + \frac{a}{b|x|^{b\gamma}}e^{b\phi}\right) dx$$

is well defined on the closed subset K_- of $W_0^{1,2}(B_1)$ of nonpositive functions and it satisfies

$$J(\phi) \geq \frac{1}{2}\int_{B_1} |\nabla\phi|^2 dx.$$

Hence it achieves its minimum in K_- at ϕ^*. There holds

$$-\Delta(\gamma \ln \tfrac{1}{|x|} - A|x|^{2-b\gamma}) = A\theta^2|x|^{-b\gamma}$$

and ϕ^* vanishes on ∂B_1. Since $|x|^{2-b\gamma} \in L^{1+\epsilon}(B_1)$, $\phi^* \in W^{2,1+\epsilon}(B_1) \subset C^{0,\frac{2\epsilon}{1+\epsilon}}(\overline{B}_1)$. Consequently

$$v_\gamma(x) = \gamma \ln \frac{1}{|x|} + \phi^*(x) \quad \text{in } B_1 \setminus \{0\}, \tag{2.17}$$

and

$$v_\gamma(x) - \gamma \ln \frac{1}{|x|} \to \phi^*(0) < 0 \quad \text{as } x \to 0. \tag{2.18}$$

Step 2: the case $b\gamma = 2$. For $\kappa \in \mathbb{R}$ we set

$$\psi_\kappa(r) = \frac{2}{b}\left(\ln\frac{1}{r} - \ln(1 - \ln r)\right) + \kappa.$$

Then $\psi_\kappa(1) = \kappa$ and

$$-\Delta\psi_\kappa + ae^{b\psi_\kappa} = \frac{1}{r^2}\left(-\frac{2}{b(1-\ln r)^2} + \frac{ae^{b\kappa}}{(1-\ln r)^2}\right). \tag{2.19}$$

Since

$$\int_{B_1} \frac{dx}{|x|^2(1-\ln|x|)^2} = 2\pi\int_0^1 \frac{dr}{r(1-\ln r)^2} = 2\pi$$

the function $e^{b\psi_\kappa}$ is integrable in B_1 and ψ_κ satisfies in $\mathcal{D}'(B_1)$

$$-\Delta\psi_\kappa + ae^{b\psi_\kappa} = \left(ae^{b\kappa} - \frac{2}{b}\right)\frac{1}{|x|^2(1-\ln|x|)^2} + \frac{4\pi}{b}\delta_0. \tag{2.20}$$

Hence ψ_κ is a supersolution nonnegative on ∂B_1 if we choose $\kappa = \kappa_1$ where

$$\kappa_1 = \max\left\{0, \frac{1}{b}\ln\frac{2}{ab}\right\} \tag{2.21}$$

It is a subsolution if we take $\kappa = \kappa_2$ where

$$\kappa_2 = \min\left\{0, \frac{1}{b}\ln\frac{2}{ab}\right\}. \tag{2.22}$$

Therefore

$$\frac{2}{b}\left(\ln\frac{1}{|x|} - \ln(1-\ln|x|)\right) + \kappa_2 \le v_{\frac{2}{b}}(x) \le \frac{2}{b}\left(\ln\frac{1}{|x|} - \ln(1-\ln|x|)\right) + \kappa_1, \tag{2.23}$$

with equality if $b = \frac{2}{a}$ since in that case $\kappa_1 = \kappa_2 = 0$.

□

3 Singular solutions in the case $1 < q < 2$

Definition 3.1 *A function $u \in C^1(\overline{B}_1 \setminus \{0\})$ is a solution of the singular Dirichlet problem*

$$\begin{array}{ll} -\Delta u + ae^{bu} - m|\nabla u|^q = 2\pi\gamma\delta_0 & \text{in } \mathcal{D}'(B_1) \\ u = \phi & \text{on } \partial B_1, \end{array} \tag{3.1}$$

where ϕ is a C^1 function, if u, e^{bu} and $|\nabla u|^q$ are integrable in B_1 and

$$\int_{B_1}\left(-u\Delta\zeta + \left(ae^{bu} - m|\nabla u|^q\right)\zeta\right)dx = 2\pi\gamma\zeta(0) \quad \text{for all } \zeta \in C_c^2(B_1). \tag{3.2}$$

In view of Vazquez theorem, any solution u_γ of (3.1) is bounded from below by the solution v_γ

3.1 Radial solutions

The expression of (1.2) for a radial function $u := u(r)$ defined in $B_1 \setminus \{0\}$ is

$$-u'' - \frac{1}{r}u' + ae^{bu} - m|u'|^q = 0 \quad \text{for } 0 < r < 1. \tag{3.3}$$

As a first gradient estimate for radial solutions we prove

Lemma 3.2 *Let $1 < q < 2$ and u be a nonnegative solution of (3.3) in $B_1 \setminus \{0\}$. Then for any $0 < r_0 < 1$ there exists $c = c(m,q) > 0$ such that*

$$|u_r(r)| \le \frac{c}{r} \qquad \text{for all } 0 < r \le r_0. \tag{3.4}$$

Proof. Clearly u cannot have a local maximum and up to a translation (it changes only the value of a) we can assume that u vanishes on ∂B_1, hence it is decreasing. There holds

$$-u'' - \frac{1}{r}u' - m|u'|^q \le 0,$$

then

$$\left(\frac{1}{1-q}|u'|^{1-q}\right)' + \frac{1}{r}|u'|^{1-q} \le m,$$

and thus

$$\left(r^{1-q}|u'|^{1-q} + m\frac{q-1}{2-q}r^{2-q}\right)' \geq 0.$$

Therefore the function $r \mapsto r^{1-q}|u'|^{1-q} + m\frac{q-1}{2-q}r^{2-q}$ is increasing and nonnegative, and thus it admits a limit $\ell \geq 0$ when $r \to 0$. This limit is the same as the one of $r \mapsto r^{1-q}|u'|^{1-q}$. If $\ell > 0$

$$r^{1-q}|u'|^{1-q} = \ell(1 - o(1)) \quad \text{when } r \to 0.$$

Hence

$$\lim_{r\to 0} r^{1-q}|u'|^{1-q} = \ell$$

which implies (3.4). If $\ell = 0$, then

$$\lim_{r\to 0} r|u'(r)| = \infty.$$

This implies that for all $A > 0$, one has

$$\liminf_{r\to 0}\left(u(r) - A\ln\frac{1}{r}\right) = \infty.$$

This contradicts the a priori estimate obtained before. □

The next easy to prove result will be used several times in the sequel.

Lemma 3.3 *Let $\gamma \geq 0$, $F \in L^1(B_1)$, $\phi \in C(\partial B_1)$ and $w \in L^1(B_1) \cap C(\overline{B}_1 \setminus \{0\})$ be the weak solution of*

$$\begin{array}{rl} -\Delta w = F + 2\pi\gamma\delta_0 & \text{in } \mathcal{D}'(B_1) \\ w = \phi & \text{in } \partial B_1. \end{array} \tag{3.5}$$

Then there holds

$$\bar{w}(r) = \gamma\ln\frac{1}{r} + o(\ln r) \quad \text{as } |x| = r \to 0 \tag{3.6}$$

where $\bar{w}(r) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(r,\theta)d\theta$. If we assume that $F \in L^{1+\delta}(B_1)$ for some $\delta > 0$, then

$$w(x) = \gamma\ln\frac{1}{|x|} + O(1) \quad \text{as } |x| \to 0 \tag{3.7}$$

Proof. We write $w(x) = \gamma E(x) + \psi(x)$ where $E(x) = \ln\frac{1}{|x|}$ and ψ is the solution of

$$\begin{array}{rl} -\Delta\psi = F & \text{in } \mathcal{D}'(B_1) \\ \psi = \phi & \text{in } \partial B_1. \end{array} \tag{3.8}$$

Let $\bar{\psi}(r)$, $\bar{\phi}$ and $\bar{F}(r)$ be the circular averages of $\psi(r,.)$, $\phi(.)$ and $\bar{F}(r,.)$. Since

$$-(r\bar{\psi}'(r))' = r\bar{F}(r),$$

A standard approximation argument associated to uniqueness yields

$$\bar{\psi}(r) = \bar{\phi} + \int_r^1 s^{-1}\int_0^s t\bar{F}(t)dtds.$$

By l'Hospital's rule we obtain (3.6). If $F \in L^{1+\delta}(B_1)$, then $\psi \in W^{2,1+\delta}_{loc}(B_1)$. By Morrey-Sobolev imbedding inequality, $W^{2,1+\delta}_{loc}(B_1) \subset C^{0,\frac{2\delta}{1+\delta}}(B_1)$. Hence ψ is Hölder continuous in B_1, which ends the proof. □

Lemma 3.4 *Let $1 < q < 2$, $m, a, b > 0$ and $\gamma \in \left(0, \frac{2}{b}\right]$. If $u := u_\gamma$ is a radial solution of*

$$-\Delta u + ae^{bu} - m|\nabla u|^q = 2\pi\gamma\delta_0 \quad \text{in } D'(B_1)$$

which vanishes on ∂B_1, then
1- If $0 < \gamma < \frac{2}{b}$, there exist a positive constant A and a continuous positive function ϕ^* in B_1 which vanishes on ∂B_1 such that in $B_1 \setminus \{0\}$,

$$\gamma \ln \frac{1}{|x|} - \phi^*(x) \le u_\gamma \le \gamma \ln \frac{1}{|x|} - A|x|^{2-b\gamma}\phi_1(x) + mc^q\eta(x),$$

where ϕ_1 is the first eigenfunction of $-\Delta$ in $W^{1,2}_0(B_1)$ and η satisfies

$$\begin{array}{rl} -\Delta\eta = |x|^{-q} & \text{in } B_1 \\ \eta = 0 & \text{on } \partial B_1, \end{array} \tag{3.9}$$

with c is the constant in (3.4), and actually, $\eta(x) = \frac{1}{(2-q)^2}(1 - |x|^{2-q})$.
2- If $\gamma = \frac{2}{b}$, we have in $B_1 \setminus \{0\}$,

$$\frac{2}{b}\left(\ln\frac{2}{|x|} - \ln(1 - \ln|x|)\right) + \kappa_2 \le u_{\frac{2}{b}}(x) \le \frac{2}{b}\left(\ln\frac{2}{|x|} - \ln(1 - \ln|x|)\right) + \kappa_1 + \eta(x),$$

where the κ_i have been defined in (2.11).

Proof. Because of (3.4) u_γ satisfies

$$-\Delta u_\gamma + ae^{bu_\gamma} \le mc^q|x|^{-q} + 2\pi\gamma\delta_0 \quad \text{in } D'(B_1).$$

Since $v_\gamma + mc^q\eta$ vanishes on ∂B_1 and satisfies

$$-\Delta(v_\gamma + mc^q\eta) + ae^{b(v_\gamma + mc^q\eta)} \ge mc^q|x|^{-q} + 2\pi\gamma\delta_0 \quad \text{in } D'(B_1),$$

we deduce by the comparison principle as in [12], that

$$u_\gamma \le v_\gamma + mc^q\eta \quad \text{in} \quad B_1 \setminus \{0\}. \tag{3.10}$$

On the other hand,

$$-\Delta u_\gamma + ae^{bu_\gamma} \ge 2\pi\gamma\delta_0.$$

Since $ae^{bu_\gamma} \in L^1(B_1)$, it follows by the comparison principle again, that

$$v_\gamma \le u_\gamma \quad \text{in} \quad B_1 \setminus \{0\}. \tag{3.11}$$

Therefore, we conclude that

$$v_\gamma \le u_\gamma \le v_\gamma + \eta \quad \text{in} \quad B_1 \setminus \{0\}. \tag{3.12}$$

Combining with Lemma 2.2, we finish the proof. □

3.2 General solutions

In this section we deal with the questions of existence, classification and uniqueness of solutions of (3.1).

3.2.1 Existence of general solutions

Theorem 3.5 *Let $1 < q < 2$ and $m, a, b > 0$. Then for any $\gamma \in (0, \frac{2}{b}]$ and $\phi \in C^1(\partial B_1)$ there exists a function $u := u_\gamma \in C^1(\overline{B}_1 \setminus \{0\})$ such that e^{bu} and $|\nabla u|^q$ are integrable in B_1 satisfying (3.1) and*

$$\lim_{x\to 0} \frac{u(x)}{-\ln|x|} = \gamma \tag{3.13}$$

Proof of Theorem 3.5. Step 1: existence. For the sake of simplicity, we assume $\phi \geq 0$; this can be achieved up to replacing a by some $\tilde{a}$. The function v_γ which satisfies (1.8) with boundary data ϕ is a subsolution in $B_1 \setminus \{0\}$. We look for a supersolution under the form

$$\psi(x) = \gamma \ln \frac{1}{|x|} + c(1 - |x|^{2-q}) + M,$$

where c and M are to be found. Since $-\Delta(1 - |x|^{2-q}) = (2-q)^2|x|^{-q}$

$$\mathcal{E}(\psi) = a|x|^{-b\gamma} e^{bc(1-|x|^{2-q})+bM} + |x|^{-q}\left(c(2-q)^2 - m\gamma^q\left(1 + \frac{c(2-q)}{\gamma} r^{2-q}\right)^q\right)$$

There exists $r_0 \in (0,1)$ and $c_0 > 0$ such that

$$c_0(2-q)^2 - m\gamma^q\left(1 + \frac{c_0(2-q)}{\gamma} r^{2-q}\right)^q \geq 0 \quad \text{for all } 0 < r \leq r_0.$$

Then we choose $M > 0$ large enough such that

$$ae^{bM} \geq r_0^{-q} m\gamma^q \left(1 + \frac{c_0(2-q)}{\gamma}\right)^q,$$

and

$$\psi(x) = M \geq \|\phi\|_{L^\infty(\partial B_1)}.$$

Hence $\mathcal{E}(\psi) \geq 0$ in $B_1 \setminus \{0\}$, and $\psi \geq \phi$ on ∂B_1. It follows that ψ is a supersolution, larger than v_γ. Therefore there exists $u = u_\gamma$ coinciding with v_γ, such that

$$v_\gamma \leq u_\gamma \leq \psi \tag{3.14}$$

which satisfies (1.4) in $B_1 \setminus \{0\}$.

Step 2: upper estimate. The function u_γ may not be radial since it is not assumed that v_γ is so. But replacing v_γ by $v_{rad,\gamma}$ which satisfies the same equation (1.8) with boundary data $\|\phi\|_{L^\infty(\partial B_1)}$, we have that $v_{rad,\gamma} \geq v_\gamma$ with boundary data $\|\phi\|_{L^\infty(\partial B_1)}$. Hence there exists a radial solution $u_{rad,\gamma}$ of (1.4) satisfying

$$v_\gamma \leq v_{rad,\gamma} \leq u_{rad,\gamma} \leq \psi \tag{3.15}$$

and $u_{rad,\gamma} \geq u_\gamma$ by the standard construction iterative scheme. By Lemma 3.2 there holds

$$|u'_{rad,\gamma}|(r)| \leq \frac{c}{r}.$$

Therefore

$$-\Delta u_{rad,\gamma} + ae^{bu_{rad,\gamma}} = m|\nabla u_{rad,\gamma}|^q \leq mc^q r^{-q},$$

and (3.15) gives that

$$\lim_{r\to 0} \frac{u_{rad,\gamma}(r)}{-\ln r} = \lim_{x\to 0} \frac{u_\gamma(x)}{-\ln|x|} = \gamma.$$

Let $\{\theta_n\} \subset C^{1,1}(\mathbb{R}^2)$ defined by

$$\theta_n(x) = \begin{cases} 2n^2\left(|x| - \dfrac{1}{n}\right)_+^2 & \text{if } |x| \leq \frac{3}{2n} \\ -2n^2|x|^2 + 8n|x| - 7 & \text{if } \frac{3}{2n} \leq |x| \leq \frac{2}{n} \\ 1 & \text{if } |x| \geq \frac{2}{n}, \end{cases}$$

then $0 \leq \theta_n \leq 1$, $\theta_n = 0$ on $B_{\frac{1}{n}}$, $\theta_n = 1$ on $B^c_{\frac{2}{n}}$, $|\nabla\theta_n| \leq cn\mathbf{1}_{B_{\frac{2}{n}}\setminus B_{\frac{1}{n}}}$ and

$$\Delta\theta_n(x) = 4n^2\left(2 - \frac{1}{n|x|}\right)\mathbf{1}_{B_{\frac{3}{2n}}\setminus B_{\frac{1}{n}}} - 4n^2\left(2 - \frac{2}{n|x|}\right)\mathbf{1}_{B_{\frac{2}{n}}} \setminus B_{\frac{3}{2n}}.$$

If $\zeta \in C_c^\infty(B_1)$ is nonnegative with $\zeta(0) = 1 = \max_{|\zeta|\leq 1}\zeta(z)$. We have

$$-\int_{B_1} u_{rad,\gamma}\theta_n\Delta\zeta dx + a\int_{B_1} e^{bu_{rad,\gamma}}\zeta\theta_n dx - \int_{B_1}\left(\zeta\Delta\theta_n + 2\nabla\theta_n.\nabla\zeta\right)u_{rad,\gamma}dx \leq mc^q\int_{B_1}\theta_n\zeta|x|^{-q}dx. \tag{3.16}$$

Clearly there holds when $n \to \infty$,

$$\int_{B_1} u_{rad,\gamma}\theta_n\Delta\zeta dx \to \int_{B_1} u_{rad,\gamma}\Delta\zeta dx,$$

and

$$mc^q\int_{B_1}\theta_n\zeta|x|^{-q}dx \to mc^q\int_{B_1}\zeta|x|^{-q}dx.$$

Next

$$\int_{B_1}|\nabla\theta_n.\nabla\zeta|\,u_{rad,\gamma}dx \leq cn\int_{B_{\frac{2}{n}}\setminus B_{\frac{1}{n}}} u_{rad,\gamma}|\nabla\zeta|dx \to 0,$$

since $\tilde{u}_\gamma(x) = \gamma\ln(\frac{1}{|x|})(1 + o(1))$. The last term is more delicate to estimate:

$$-\int_{B_1}\Delta\theta_n u_{rad,\gamma}\zeta dx = -\int_{B_{\frac{3}{2n}}\setminus B_{\frac{1}{n}}}\Delta\theta_n u_{rad,\gamma}\zeta dx - \int_{B_{\frac{2}{n}}\setminus B_{\frac{3}{2n}}}\Delta\theta_n u_{rad,\gamma}\zeta dx := I + II.$$

Since $\zeta(x) = 1 + O(|x|)$., and $\tilde{u}_\gamma$ satisfies (3.15), estimates on v_γ and ψ yield

$$\gamma\ln\frac{1}{|x|} - M \leq u_{rad,\gamma}(x) \leq \gamma\ln\frac{1}{|x|} + M \tag{3.17}$$

for some $M > 0$, and therefore

$$I = -4n^2 \int_{B_{\frac{3}{2n}} \setminus B_{\frac{1}{n}}} \left(2 - \frac{1}{n|x|}\right) u_{rad,\gamma} dx + O\left(\frac{\ln n}{n}\right).$$

Hence

$$-6\pi \max\left\{u_{rad,\gamma} : \frac{1}{n} \le |x| \le \frac{3}{2n}\right\} + O\left(\frac{\ln n}{n}\right) \le I \le -6\pi \min\left\{u_{rad,\gamma} : \frac{1}{n} \le |x| \le \frac{3}{2n}\right\} + O\left(\frac{\ln n}{n}\right).$$

In the same way

$$II = 4n^2\zeta(0) \int_{B_{\frac{2}{n}} \setminus B_{\frac{3}{2n}}} \left(2 - \frac{2}{n|x|}\right) u_{rad,\gamma} \zeta dx + O\left(\frac{\ln n}{n}\right).$$

Hence

$$6\pi \min\left\{\tilde{u}_\gamma : \frac{3}{2n} \le |x| \le \frac{2}{n}\right\} + O\left(\frac{\ln n}{n}\right) \le II \le 6\pi \max\left\{\tilde{u}_\gamma : \frac{3}{2n} \le |x| \le \frac{2}{n}\right\} + O\left(\frac{\ln n}{n}\right).$$

Using (3.17) we see that up to some constants $M > 0$ Therefore on

$$-6\pi \ln n - M + O\left(\frac{\ln n}{n}\right) \le I \le -6\pi \ln n + M + O\left(\frac{\ln n}{n}\right),$$

and

$$6\pi \ln n - M + O\left(\frac{\ln n}{n}\right) \le II \le 6\pi \ln n + M + O\left(\frac{\ln n}{n}\right).$$

Thus $I + II$ remains bounded independently of n. Combining these estimates with (3.16) we conclude that $\int_{B_1} e^{bu_{rad,\gamma}} \zeta \theta_n dx$ remains bounded independently of n too. By Fatou's lemma $e^{bu_{rad,\gamma}} \zeta \in L^1(B_1)$. Since $u_\gamma \le u_{rad,\gamma}$ we infer that $e^{bu_\gamma} \in L^1(B_1)$. Because

$$-\Delta u_\gamma = m|\nabla u_\gamma|^q - a e^{bu_\gamma},$$

it follows by Brezis-Lions lemma that $|\nabla u|^q \in L^1(B_1)$ and there exist $\tilde{\gamma} \ge 0$ such that

$$-\Delta u_\gamma = m|\nabla u_\gamma|^q - a e^{bu_\gamma} + 2\pi\tilde{\gamma}\delta_0 \quad \text{in } \mathcal{D}'(B_1).$$

Set $F = m|\nabla u_\gamma|^q - a e^{bu_\gamma}$, Since $F \in L^1(B_1)$ it follows by Lemma 3.3 that

$$\bar{u}_\gamma(r) = -\tilde{\gamma} \ln r + o(\ln r).$$

Combined with (3.14) we obtain that $\tilde{\gamma} = \gamma$. □

The next estimate compares v_γ and u_γ

Proposition 3.6 *Let $\gamma \in (0, \frac{2}{b}]$ and $\phi \in C^1(\partial B_1)$ is nonnegative. If $u \in C^1(\overline{B}_1 \setminus \{0\})$ satisfies $e^{bu} \in L^1(B_1)$ and $|\nabla u|^q \in L^1(B_1)$ is a solution of (3.1), then there exists $K > 0$ such that*

$$v_\gamma \le u(x) \le v_\gamma + K \quad \text{for } x \in B_1 \setminus \{0\}, \tag{3.18}$$

where v_γ is the solution of (1.8) which vanishes on ∂B_1.

Proof. Since u is a solution, it is a supersolution of (1.8), and by the order principle it is larger than solution v_γ with the same boundary value. Using the regularity estimates of the Laplacian in the space of bounded Radon measures $\mathfrak{M}_b(B_1)$ we have that $\nabla u \in M^2(B_1)$ where $M^2(B_1)$ is the Marcinkiewicz space coinciding with the Lorentz space $L^{2,\infty}(B_1)$. What is important to know is that $M^2(B_1) \subset L^{2-\epsilon}(B_1)$ for any $\epsilon > 0$. Therefore $|\nabla u|^q \in L^{\frac{2-\epsilon}{q}}(B_1)$. Since $q < 2$ we take $\epsilon > 0$ such that $\frac{2-\epsilon}{q} = 1 + \delta > 1$. Let u_1 be the solution of

$$\begin{aligned} -\Delta u_1 &= m|\nabla u|^q \qquad \text{in } B_1 \\ u_1 &= \phi \qquad\qquad\quad \text{on } B_1. \end{aligned}$$

The function u_1 is nonnegative and belongs to $W^{2,1+\delta}(B_1)$. By Morrey-Sobolev imbedding theorem, $W^{2,1+\delta(B_1)} \subset C^{0,\frac{2\delta}{1+\delta}}(\overline{B}_1)$. This implies that u_1 is bounded from above by some $K > 0$. The function $v_\gamma + u_1$ satisfies

$$-\Delta(u_1 + v_\gamma) + ae^{b(u_1+v_\gamma)} > -\Delta(u_1 + v_\gamma) + ae^{bv_\gamma} = m|\nabla u|^q + 2\pi\gamma\delta_0.$$

Since $ae^{b(u_1+v_\gamma)} \in L^1(B_1)$ we can apply Vazquez' monotonicity estimate [12] between u_1+v_γ and u since the right-hand side of the respective equations coincide and belong to $\mathfrak{M}_b(B_1)$. Therefore

$$u(x) \le v_\gamma(x) + K, \tag{3.19}$$

and (3.59) follows. □

As a consequence of Lemma 2.2 we obtain global estimates on the solutions

Corollary 3.7 *Under the assumptions of Proposition 3.6 any function $u \in C^1(\overline{B}_1 \setminus \{0\})$ such that $e^{bu} \in L^1(B_1)$ and $|\nabla u|^q \in L^1(B_1)$ of solution of (3.1) satisfies, for some $K \ge 0$, the following two-sided inequalities for all $x \in \overline{B}_1 \setminus \{0\}$,*
1- If $0 \le \gamma < \frac{2}{b}$, then

$$\gamma \ln \frac{1}{|x|} - K \le u(x) \le \gamma \ln \frac{1}{|x|} + K. \tag{3.20}$$

2- If $\gamma = \frac{2}{b}$, then

$$\frac{2}{b}\left(\ln \frac{1}{|x|} - \ln(1 - \ln|x|)\right) - K \le u(x) \le \frac{2}{b}\left(\ln \frac{1}{|x|} - \ln(1 - \ln|x|)\right) + K. \tag{3.21}$$

Combining Proposition **??**, Proposition **??** and Corollary 3.7 we have a general existence and uniqueness result of solutions of (3.1).

3.2.2 Classification of isolated singularities

In this section we prove the following

Theorem 3.8 *Let $1 < q < 2$, $m, a, b > 0$. If $u \in C^1(\overline{B}_1 \setminus \{0\})$ is a solution of (1.4) in $B_1 \setminus \{0\}$, then there exists $\gamma \in [0, \frac{2}{b}]$ such that*

$$\lim_{x\to 0} \frac{u(x)}{-\ln|x|} = \gamma. \tag{3.22}$$

Furthermore $e^{bu} \in L^1(B_1)$, $|\nabla u|^q \in L^1(B_1)$ and u is a solution of (3.1).

Proof. We can assume that $u\lfloor_{\partial B_1} = \phi \in C^1(\partial B_1)$ is nonnegative.
1- If $e^{bu} \in L^1(B_1)$, it follows by Brezis-Lions Lemma [5] that $|\nabla u|^q \in L^1(B_1)$ and there exists $\gamma \geq 0$ such that (3.1) holds. By Lemma 3.3 the circular average $\bar{u}$ of u satisfies

$$\bar{u}(r) = \gamma \ln \frac{1}{r} + o(\ln r) \quad \text{as } r \to 0.$$

By convexity

$$\overline{e^{bu}}(r) \geq e^{b\bar{u}(r)}.$$

Since for any $\epsilon > 0$ there holds

$$\bar{u}(r) \geq (\gamma - \epsilon) \ln \frac{1}{r} - K_\epsilon,$$

we obtain

$$e^{b\bar{u}(r)} \geq \frac{e^{-bK_\epsilon}}{|x|^{b(\gamma-\epsilon)}}.$$

If $\gamma > \frac{2}{b}$ we can choose $\epsilon > 0$ such that $b(\gamma - \epsilon) > 2$. Hence $e^{b\bar{u}(.)}$ is not integrable in B_1, neither e^{bu}, contradiction. This implies that $\gamma \in [0, \frac{2}{b}]$. By Proposition 3.6 and Corollary 3.7 we obtain (3.22).
2- Next we assume that $e^{bu} \notin L^1(B_1)$, then using the inequality satisfied by $\bar{u}$,

$$r\bar{u}'(r) + a\int_r^1 \overline{e^{bu}} t dt = \bar{u}'(1) + m\int_r^1 \overline{|\nabla u|^q} t dt \tag{3.23}$$

where $\bar{u}$, $\overline{E^{bu}}$ and $\overline{|\nabla u|^q}$ denote the averages of u, E^{bu} and $|\nabla u|^q$ on S^1. Next we will distinguish according $|\nabla u|^q$ is or is not integrable in B_1, although if u is radial Lemma 3.2 implies that $|\nabla u|^q \in L^1(B_1)$.
2-(i)- If $|\nabla u|^q \in L^1(B_1)$ we obtain from (3.23) that

$$\lim_{r\to 0} r\bar{u}'(r) = -\infty.$$

By integration, it implies that

$$\lim_{r\to 0} \frac{\bar{u}(r)}{-\ln r} = \infty,$$

which again contradicts the a priori estimate (2.1).
2-(ii)- If $|\nabla u|^q \notin L^1(B_1)$, then for any $\epsilon \in (0,1)$,

$$\int_{B_\epsilon} |\nabla u|^q dx = \infty.$$

Then for any $k > 0$ there exists $\tau = \tau(\epsilon, k)$ such that

$$m\int_{B_\epsilon} |\nabla \min\{u, \tau\}|^q dx = 2\pi k.$$

Since

$$-\Delta \min\{u,\tau\} + ae^{b\min\{u,\tau\}} \geq m|\nabla \min\{u,\tau\}|^q \mathbf{1}_{B_\epsilon} \quad \text{in } B_1$$

$\min\{u,\tau\}$ is bounded from below by the solution $v_{k,\epsilon}$ of

$$-\Delta v_{k,\epsilon} + ae^{bv_{k,\epsilon}} = m|\nabla \min\{u,\tau\}|^q \mathbf{1}_{B_\epsilon} \quad \text{in } B_1.$$

We have $m|\nabla \min\{u,\tau\}|^q \mathbf{1}_{B_\epsilon} \to 2\pi k\delta_0$ in the weak sense of bounded Radon measures in B_1. By Vazquez' stability theorem, if $k \le \frac{2}{b}$, $v_{k,\epsilon} \to v_k$ as $\epsilon \to 0$. Taking in particular $k = \frac{2}{b}$ we obtain

$$u(x) \ge v_{\frac{2}{b}}(x) \quad \text{for all } x \in B_1 \setminus \{0\}. \tag{3.24}$$

Combining this estimate with (2.1) and the estimate on $v_{\frac{2}{b}}$ yields, for some $K \ge 0$,

$$\frac{2}{b}\left(\ln\frac{1}{|x|} - \ln(1-\ln|x|)\right) - K \le u(x) \le \frac{2}{b}\ln\frac{1}{|x|} + K \quad \text{for all } x \in B_1 \setminus \{0\}. \tag{3.25}$$

We set $w(t,.) = u(r,.) + \frac{2}{b}\ln r$ with $t = \ln r$. Then w satisfies in $(-\infty,0)\times S^1$

$$w_{tt} + w_{\theta\theta} - ae^{bw} + me^{(2-q)t}\left(\left(\frac{2}{b} - w_t\right)^2 + w_\theta^2\right)^{\frac{q}{2}} = 0, \tag{3.26}$$

and the two-sided estimate

$$-\frac{2}{b}\ln(1-t) - K \le w(t,.) \le K. \tag{3.27}$$

Thus the function $z(t,.) = \frac{w(t,.)}{\ln(1-t)}$ satisfies in $(-\infty,0)\times S^1$,

$$\begin{aligned} z_{tt} + z_{\theta\theta} - \frac{2}{(1-t)\ln(1-t)}z_t - \frac{2}{(1-t)^2\ln(1-t)}z - \frac{a}{\ln(1-t)}e^{b\ln(1-t)z} \\ + me^{(2-q)t}\ln^{q-1}(1-t)\left(\left(\frac{2}{b\ln(1-t)} - z_t + \frac{z}{(1-t)\ln(1-t)}\right)^2 + z_\theta^2\right)^{\frac{q}{2}} = 0, \end{aligned} \tag{3.28}$$

and (3.27) yields for some $T < 0$,

$$-\frac{2}{b} + o(1) \le z(t,.) \le \frac{K}{\ln(1-t)} \le o(1) \quad \text{in } (-\infty,T]\times S^1.$$

Since z is bounded and negative and all the coefficients are smooth and bounded, the standard regularity theory implies that z is bounded in $C^{2,\alpha}([T'-1,T'+1]\times S^1)$. Since $w(t,.) = z(t,.)\ln(1-t)$ we have also

$$\left(w_t^2(t,\theta) + w_\theta^2(t,\theta)\right)^{\frac{1}{2}} \le c\ln(1-t). \tag{3.29}$$

Returning to $u(r,\theta)$ implies

$$|\nabla u(x)| = |\nabla u(r,\theta)| = (u_r^2 + r^{-2}u_\theta^2)^{\frac{1}{2}} \le c\frac{\ln(1-\ln|x|)}{|x|} \quad \text{for all } x \in B_1 \setminus \{0\}. \tag{3.30}$$

Since $q < 2$, the right-hand side of (3.30) belongs to $L^q(B_1)$ and we get a contradiction. This ends the proof. □

3.2.3 Uniqueness

Sharp estimates of the gradient of singular solutions near $x = 0$ are the key-step for proving their uniqueness.

Theorem 3.9 *Let q, m, a, b be as in Theorem 3.5. Then for any $\gamma \in (0, \frac{2}{b}]$ and $\phi \in C^1(\partial B_1)$ there exists a unique function $u \in C^1(\overline{B}_1 \setminus \{0\})$ satisfying equation (1.4) in $B_1 \setminus \{0\}$ and*

$$u(x) = \gamma \ln \frac{1}{|x|} \left(1 + o(1)\right). \tag{3.31}$$

Furthermore e^{bu} and $|\nabla u|^q$ are integrable in B_1, and u satisfies (3.1).

Proof. Let u be any solution nonnegative solution of (1.4) in $B_1 \setminus \{0\}$ satisfying (3.31) (the sign requirement is not a restriction since we can change a). By Theorem 3.8 e^{bu} and $|\nabla u|^q$ are integrable in B_1 and u satisfies (3.1). Therefore the estimates of Corollary 3.7 hold.

1- The case $0 < \gamma < \frac{2}{b}$.

1-(i) Convergence and gradient estimates. Under the assumption (3.31) we first show that there holds

$$\lim_{|x|\to 0} \left(u(x) - \gamma \ln \frac{1}{|x|} \right) = \ell \quad \text{and} \quad \lim_{|x|\to 0} |x| \nabla u(x) = -\gamma \mathbf{i} \tag{3.32}$$

for some $\ell \in [-K, K]$ and $\mathbf{i} = \frac{x}{|x|}$. For the proof, we set

$$w(t, \theta) = u(r, \theta) - \gamma \ln \frac{1}{r},$$

with $t = \ln r \in (-\infty, 0]$ and $\theta \in S^1$. Then w is bounded in $(-\infty, 0] \times S^1$ where there holds

$$w_{tt} + w_{\theta\theta} - a e^{(2-b\gamma)t} e^{bw} + m e^{(2-q)t} \left((\gamma - w_t)^2 + w_\theta^2 \right)^{\frac{q}{2}} = 0. \tag{3.33}$$

Since $2 - b\gamma > 0$ and $2 - q > 0$, by the standard regularity theory [10, Theorem A1], we have the estimate

$$\begin{aligned} \max\{|w_t(t,\theta)| + |w_\theta(t,\theta)| : (t,\theta) \in [T-1, T+1] \times S^1\} \\ \leq c \max\{|w(t,\theta)| : (t,\theta) \in [T-2, T+2] \times S^1\} \leq cK, \end{aligned}$$

for all $T \leq -2$. Equivalently $|u_\theta| + |r u_r + \gamma|$ is bounded. Now, by linear elliptic equation regularity theory w is bounded locally in $C^{1,\alpha}([T-1, T+1] \times S^1)$ independently of T and thus in $C^{2,\alpha}([T-1, T+1] \times S^1)$, for some $\alpha \in (0, 1)$.

1-(ii) Convergence and sharp gradient estimates. Let $\bar{w}(t) = \frac{1}{2\pi} \int_0^{2\pi} w(t, \theta) d\theta$ be the average of $w(t, .)$ on S^1. Then

$$-\bar{w}''(t) + a e^{(2-b\gamma)t} H_0(t) + e^{(2-q)t} G_0(t) = 0,$$

where

$$H_0(t) = \frac{1}{2\pi} \int_{S^1} e^{bw} d\theta \quad \text{and} \quad G_0(t) = -\frac{m}{2\pi} \int_{S^1} \left((\gamma - w_t)^2 + w_\theta^2 \right)^{\frac{q}{2}} d\theta.$$

Since H_0 and G_0 are bounded, $\bar{w}'(t)$ admits a limit when $t \to -\infty$, and this limit is necessarily equal to zero since $\bar{w}$ is bounded. Thus

$$\bar{w}'(t) + a\int_{-\infty}^{t} e^{(2-b\gamma)\tau} H_0(\tau)d\tau + \int_{-\infty}^{t} e^{(2-q)\tau} G_0(\tau)d\tau = 0.$$

Therefore

$$|\bar{w}'(t)| \leq c\max\{e^{(2-b\gamma)t}, e^{(2-q)t}\}, \tag{3.34}$$

which implies that $\bar{w}(t)$ admits a finite limit ℓ when $t \to -\infty$. Set $w^* = w - \bar{w}$. Then

$$-w^*_{tt} - w^*_{\theta\theta} + ae^{(2-b\gamma)t} H^* + e^{(2-q)t} G^* = 0, \tag{3.35}$$

where

$$H^*(t,.) = e^{bw} - \frac{1}{2\pi}\int_{S^1} e^{bw} d\theta \quad \text{and} \quad G^*(t) = \frac{m}{2\pi}\int_{S^1} \left|(\gamma - w_t)^2 + w_\theta^2\right|^{\frac{q}{2}} d\theta - m\left(\gamma - w_t)^2 + w_\theta^2\right)^{\frac{q}{2}}.$$

Since $\int_{S^1} w^* d\theta = 0$, we have by Wirtinger inequality,

$$-\int_{S^1} w_{\theta\theta} w^* d\theta = -\int_{S^1} w^*_{\theta\theta} w^* d\theta \geq \|w^*\|^2_{L^2(S^1)},$$

and

$$\int_{S^1} H^*(t,.)w^* d\theta = \int_{S^1} \left(e^{bw} - \frac{1}{2\pi}\int_{S^1} e^{bw} d\theta\right)(w - \bar{w})d\sigma = \int_{S^1} \left(e^{bw} - ae^{b\bar{w}}\right)(w - \bar{w})d\sigma \geq 0.$$

Using the inequality

$$-\|w^*\|_{L^2(S_1)} \frac{d^2}{dt^2}\|w^*\|_{L^2(S^1)} \leq -\int_{S^1} w^* w^*_{\theta\theta} d\theta$$

we obtain

$$-\frac{d^2}{dt^2}\|w^*\|_{L^2(S^1)} + \|w^*\|_{L^2(S^1)} \leq L_0(t)e^{\beta t},$$

where $\beta = \min\{2-q, 2-b\gamma\} > 0$ and $L_0(t)$ is some bounded function. Since $\|w^*\|_{L^2(S^1)}$ is bounded and $\beta < 1$, it implies that

$$\|w^*(t,.)\|_{L^2(S^1)} \leq Ce^{\beta t}. \tag{3.36}$$

Combining this estimate with the fact that $\bar{w}(t) \to \ell$ when $t \to -\infty$ we deduce that

$$\lim_{t\to-\infty} \|w(t,.) - \ell\|_{L^2(S^1)} = 0. \tag{3.37}$$

Let

$$\mathcal{T}[w] = \bigcup_{t\leq 0}\{(w(t,.), w_t)\}$$

be the negative trajectory of (w, w_t). It is relatively compact in $C^2(S^1)\times C^1(S^1)$ and because of the uniqueness of the limit, we have that $w(t,.) \to \ell$ in $C^2(S^1)$ and thus $w_\theta(t,.) \to 0$ in $C^1(S^1)$. In order to make precise the behaviour of w_t, we differentiate (3.35) with respect to t and get

$$-w^*_{ttt} - w^*_{t\theta\theta} + e^{(2-b\gamma)t}\tilde{H}^* + e^{(2-q)t}\tilde{G}^* = 0 \tag{3.38}$$

where $\tilde{H}^* = a(2-b\gamma)H^* - aH_t^*$ and $\tilde{G}^* = (2-q)G^* - G_t^*$ are bounded functions (from the C^2 a priori bounds on w). Hence by Wirtinger inequality again,

$$-\frac{d^2}{dt^2}\|w_t^*\|_{L^2(S^1)} + \|w_t^*\|_{L^2(S^1)} \leq \tilde{L}_0(t)e^{\beta t},$$

where $\tilde{L}_0$ is a bounded function. Since $\|w_t^*\|_{L^2(S^1)}$ is bounded on $(-\infty, 0]$ it implies again that

$$\|w_t^*(t,.)\|_{L^2(S^1)} \leq C'e^{\beta t}. \tag{3.39}$$

By (3.34), $\bar{w}_t(t) \to 0$ as $t \to -\infty$. This implies

$$\lim_{t\to-\infty}\|w_t(t,.)\|_{L^2(S^1)} = 0. \tag{3.40}$$

As in the case of the function $w(t,.)$, the relative compactness of the set $\{w_t(t,.)\}_{t\leq 0}$ in $C^1(S^1)$ combined with (3.40) implies that $w_t(t,.) \to 0$ in $C^1(S^1)$ as $t \to -\infty$. This implies (3.32).

1-(iii) End of the proof. Next we consider two solutions u_γ and $\tilde{u}_\gamma$ satisfying (3.31) with $\gamma \in (0, \frac{2}{b})$ and coinciding on ∂B_1. Denote $u = u_\gamma$ and $\tilde{u} = \tilde{u}_\gamma$. For $\epsilon \neq 0$ we recall that $E(x) = \ln\frac{1}{|x|}$ and

$$U(r) = U_\epsilon(x) = u(x) + \epsilon E(x). \tag{3.41}$$

Then

$$\begin{aligned}-\Delta U + ae^{bU} - m|\nabla U|^q &= ae^{bu}\left(e^{b\epsilon E} - 1\right) - m\left[\left(|\nabla u|^2 + \frac{\epsilon^2}{|x|^2} - 2\frac{\epsilon}{|x|}\nabla u.\mathbf{i}\right)^{\frac{q}{2}} - |\nabla u|^q\right]\\ &= ae^{bu}\left(e^{b\epsilon E} - 1\right) - m|\nabla u|^q\left[\left(1 + \frac{\epsilon^2}{|x|^2|\nabla u|^2} - 2\frac{\epsilon}{|x||\nabla u|^2}\nabla u.\mathbf{i}\right)^{\frac{q}{2}} - 1\right].\end{aligned}$$

Then, when $r \to 0$,

$$1 + \frac{\epsilon^2}{|x|^2|\nabla u|^2} - 2\frac{\epsilon}{|x||\nabla u|^2}\nabla u.\mathbf{i} = 1 + \frac{\epsilon^2}{\gamma^2}(1+o(1)) + 2\frac{\epsilon}{\gamma}(1+o(1)),$$

thus

$$|\nabla u|^q\left[\left(1 + \frac{\epsilon^2}{|x|^2|\nabla u|^2} - 2\frac{\epsilon}{|x||\nabla u|^2}\nabla u.\mathbf{i}\right)^{\frac{q}{2}} - 1\right] = \frac{\gamma^q}{|x|^q}\left(\frac{q\epsilon}{\gamma} + \frac{q(q-1)\epsilon^2}{2\gamma}(1+o(1))\right).$$

Since $b\gamma\ln\frac{1}{|x|} + bK \geq bu(x) \geq b\gamma\ln\frac{1}{|x|} - bK$, we have in the case where $\epsilon > 0$:

$$ae^{bK}|x|^{-b\gamma}(|x|^{-b\epsilon} - 1) \geq ae^{bu}\left(e^{b\epsilon E} - 1\right) \geq ae^{-bK}|x|^{-b\gamma}(|x|^{-b\epsilon} - 1),$$

and in the case where $\epsilon < 0$:

$$ae^{bK}|x|^{-b\gamma}(|x|^{-b\epsilon} - 1) \leq ae^{bu}\left(e^{b\epsilon E} - 1\right) \leq ae^{-bK}|x|^{-b\gamma}(|x|^{-b\epsilon} - 1).$$

1-(iii)-(i): Case $\frac{q}{b} \leq \gamma < \frac{2}{b}$. we take $\epsilon > 0$ and get

$$\mathcal{E}(U) \geq ae^{-bK}|x|^{-b\gamma}(|x|^{-b\epsilon} - 1) - m\frac{\gamma^q}{|x|^q}\left(\frac{q\epsilon}{\gamma} + \frac{q(q-1)\epsilon^2}{2\gamma}(1 + o(1))\right).$$

Fix $r_0 \in (0,1)$ and $\epsilon_0 > 0$ such that for any $0 < r \leq r_0$ and $0 < \epsilon \leq \epsilon_0$ there holds

$$m\gamma^q\left(\frac{q\epsilon}{\gamma} + \frac{q(q-1)\epsilon^2}{2\gamma}(1 + o(1))\right) \leq 2qm\gamma^{q-1}\epsilon.$$

Therefore

$$\mathcal{E}(U) \geq |x|^{-b\gamma}\left(ae^{-bK}(r_0^{-b\epsilon} - 1) - 2qm\gamma^{q-1}\epsilon|x|^{b\gamma-q}\right).$$

Up to reducing r_0 we have $\mathcal{E}(U) \geq 0$ in $\overline{B}_{r_0} \setminus \{0\}$ for any $\epsilon \in (0, \epsilon_0]$. Next, in $B_1 \setminus B_{r_0}$ we have that ∇u_γ is bounded, therefore

$$\mathcal{E}(U) \geq -C\epsilon$$

for some constant $C > 0$ whenever ϵ is small enough, Consequently this last inequality holds in whole $\overline{B}_1 \setminus \{0\}$. Finally the function

$$x \mapsto U^*(x) = U(x) + \frac{1}{b}\ln(1 + C\epsilon)$$

is a supersolution of (3.1) larger than $\tilde{u}$ in a neighbourhood of $x = 0$ and near ∂B_1. Hence $U^* \geq \tilde{u}_\gamma$. Letting $\epsilon \to 0$ yields $u_\gamma \geq \tilde{u}$. Similarly $\tilde{u} \geq u$, which implies that $\tilde{u} = u$, that is $\tilde{u}_\gamma = u_\gamma$.

1-(iii)-(ii): Case $0 \leq \gamma < \frac{q}{b}$. We denote by η the solution of

$$\begin{array}{rl} -\Delta\eta = |x|^{-q} & \text{in } B_1 \\ \eta = 0 & \text{on } \partial B_1, \end{array} \tag{3.42}$$

then $\eta(x) = (2-q)^2(1-|x|^{2-q})$. For $\epsilon, \epsilon' > 0$, set $U(x) = u(x) - \epsilon E(x)$ and $U_1(x) = U(x) - \epsilon'\eta(x)$. Then

$$\nabla U_1 = \nabla U + (2-q)\epsilon'|x|^{1-q}\mathbf{i} \quad \text{and} \quad -\Delta U_1 = -\Delta U + \epsilon'\Delta\eta = -\Delta u - \epsilon'|x|^{-q},$$

$$\begin{aligned} |\nabla U_1|^q &= \left(|\nabla U|^2 + \frac{(2-q)^2\epsilon'^2}{|x|^{2(q-1)}} + \frac{2(2-q)\epsilon'}{|x|^{q-1}}\nabla U.\mathbf{i}\right)^{\frac{q}{2}} \\ &= |\nabla U|^q\left(1 + \frac{(2-q)^2\epsilon'^2}{|x|^{2(q-1)}|\nabla U|^2} + \frac{2(2-q)\epsilon'}{|x|^{q-1}|\nabla U|^2}\nabla U.\mathbf{i}\right)^{\frac{q}{2}}, \\ e^{bU_1} &= e^{bU}e^{-b\epsilon'\eta}, \end{aligned}$$

and

$$-\Delta U = -\Delta u = m|\nabla u|^q - ae^{bu} = m|\nabla(U + \epsilon E)|^q - ae^{b(U+\epsilon E)}.$$

Since $\eta \geq 0$ and $0 < |x| \leq 1$, $e^{bU}\left(e^{-b\epsilon'\eta} - |x|^{-b\epsilon}\right) \leq 0$. Therefore

$$\begin{aligned}
&-\Delta U_1 + ae^{bU_1} - m|\nabla U_1|^q = -\epsilon'|x|^{-q} + e^{bU}\left(e^{-b\epsilon'\eta} - |x|^{-b\epsilon}\right) \\
&- m|\nabla U|^q\left[\left(1 + \frac{(2-q)^2\epsilon'^2}{|x|^{2(q-1)}|\nabla U|^2} + \frac{2(2-q)\epsilon'}{|x|^{q-1}|\nabla U|^2}\nabla U.\mathbf{i}\right)^{\frac{q}{2}} - \left(1 + \frac{\epsilon^2}{|x|^2|\nabla U|^2} - \frac{2\epsilon}{|x||\nabla U|^2}\nabla U.\mathbf{i}\right)^{\frac{q}{2}}\right] \\
&\leq -\epsilon'|x|^{-q} + C|\nabla U|^q\left(\frac{\epsilon'}{|x|^{q-1}|\nabla U|} + \frac{\epsilon'^2}{|x|^{2(q-1)}|\nabla U|^2} + \frac{\epsilon}{|x||\nabla U|} + \frac{\epsilon^2}{|x|^2|\nabla U|^2}\right)(1 + o(1)),
\end{aligned} \tag{3.43}$$

for some $C > 0$ as $r \to 0$. In order U_1 be a subsolution, the problem is reduced to proving that, under suitable conditions on ϵ and ϵ', the expression

$$A(x) := -\epsilon'|x|^{-q} + 2C|\nabla U|^q\left(\frac{\epsilon'}{|x|^{q-1}|\nabla U|} + \frac{\epsilon'^2}{|x|^{2(q-1)}|\nabla U|^2} + \frac{\epsilon}{|x||\nabla U|} + \frac{\epsilon^2}{|x|^2|\nabla U|^2}\right)$$

is negative for r small enough. Because $\frac{2\gamma}{|x|} \geq |\nabla U(x)| \geq \frac{\gamma}{2|x|}$ near 0, there holds

$$A(x) \leq -\epsilon'|x|^{-q} + 2^{q+1}C\gamma^q|x|^{-q}\left(\frac{2\epsilon'|x|^{2-q}}{\gamma} + \frac{4\epsilon'^2|x|^{4-2q}}{\gamma^2} + \frac{2\epsilon}{\gamma} + \frac{4\epsilon^2}{\gamma^2}\right).$$

We fix $\epsilon' = 2^{q+1}C\gamma^q\left(\frac{2\epsilon'}{\gamma} + \frac{4\epsilon'^2}{\gamma^2} + \frac{2\epsilon}{\gamma} + \frac{4\epsilon^2}{\gamma^2}\right)$, then

$$A(x) \leq 0 \quad \text{for } 0 < |x| \leq r_0.$$

Proceeding as in is the case (3-i) there exists ϵ_0 and $C > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$ there holds

$$\mathcal{E}(U_1) \leq C\epsilon \quad \text{in } B_1 \setminus \{0\}. \tag{3.44}$$

The function $x \mapsto \tilde{V}_\epsilon(x) := \tilde{u}(x) + \frac{1}{b}\ln(1 + C\epsilon)$ satisfies

$$\mathcal{E}(\tilde{V}_\epsilon) = C\epsilon e^{b\tilde{u}} \geq C.$$

Therefore $\tilde{V}_\epsilon$ is a super solution of (3.44) which is larger than U_1 near $x = 0$ and $|x| = 1$. Then by the maximum principle, $\tilde{V}_\epsilon \geq U_1$. Letting $\epsilon \to 0$ yields $\tilde{u} \geq u$. Similarly $u \geq \tilde{u}$, thus $u_\gamma = \tilde{u}_\gamma$.

2- The case $\gamma = \frac{2}{b}$.

2-(i) Gradient estimates. We first show that

$$\lim_{x\to 0}\left(u(x) - \frac{2}{b}\left(\ln\frac{1}{|x|} - \ln(1 - \ln|x|)\right)\right) = \ell \quad \text{and} \quad \lim_{x\to 0}|x|\nabla u(x) = -\frac{2}{b}\mathbf{i} \tag{3.45}$$

for some $\ell \in [-K, K]$ with K from inequality (3.21) in Corollary 3.7 and $\mathbf{i} = \frac{x}{|x|}$.

We set again $w(t, .) = u(r, .) + \frac{2}{b}\ln r$ with $t = \ln r$. Thus w satisfies

$$-\frac{2}{b}\ln(1 - t) - K \leq w(t, .) \leq -\frac{2}{b}\ln(1 - t) + K$$

and

$$w_{tt} + w_{\theta\theta} - ae^{bw} + me^{(2-q)t}\left((\tfrac{2}{b} - w_t)^2 + w_\theta^2\right)^{\frac{q}{2}} = 0. \tag{3.46}$$

Introducing again $z(t,.) = (\ln(1-t))^{-1}w(t,.)$, then

$$-\frac{2}{b} - \frac{K}{\ln(1-t)} \leq z(t,.) \leq -\frac{2}{b} + \frac{K}{\ln(1-t)}$$

and z which is bounded satisfies equation (3.28); thus w_t satisfies (3.29). Since z is bounded and negative and all the coefficients are smooth and bounded, the standard regularity theory implies that z is bounded in $C^{2,\alpha}([T'-1, T'+1] \times S^1)$ independently of T', thus

$$\lim_{t\to-\infty} z(t,.) = -\frac{2}{b} \quad \text{in } C^2(S^1) \tag{3.47}$$

Set

$$\lambda(t,.) = w(t,.) + \frac{2}{b}\ln(1-t)$$

Then λ is uniformly bounded and it satisfies in $(-\infty, 0] \times S^1$,

$$-\lambda_{tt} - \lambda_{\theta\theta} - \frac{2}{b(1-t)^2} + \frac{a}{(1-t)^2}e^{b\lambda} - me^{(2-q)t}\left((\tfrac{2}{b} - w_t)^2 + w_\theta^2\right)^{\frac{q}{2}} = 0. \tag{3.48}$$

Let $\bar{\lambda}(t)$ be the circular average of $\lambda(t,.)$ and $\lambda^*(t,.) = \lambda(t,.) - \bar{\lambda}(t)$. Then

$$-\lambda^*_{tt} - \lambda^*_{\theta\theta} + \frac{a}{(1-t)^2}\left(e^{b\lambda} - \overline{e^{b\lambda}}\right) + e^{(2-q-\epsilon)t}G^*_\epsilon = 0$$

where G^*_ϵ is a bounded function. Again, by Wirtinger inequality we obtain that

$$\|\lambda^*(t,.)\|_{L^2(S^1)} \leq C\max\left\{e^t, e^{(2-q-\epsilon)t}\right\} = Ce^{(2-q-\epsilon)t},$$

hence

$$\lim_{t\to-\infty} \|\lambda^*(t,.)\|_{L^2(S^1)} = 0 \tag{3.49}$$

By Gagliardo-Nirenberg inequality we have for any $p > 1$

$$\begin{aligned} \|\nabla\lambda^*(t,.)\|_{W^{1,p}(S^1)} &\leq C\,\|\nabla\lambda^*(t,.)\|^{1-\frac{1}{p}}_{W^{2,2}(S^1)}\,\|\nabla\lambda^*(t,.)\|^{\frac{1}{p}}_{L^2(S^1)} \\ &\leq C\,\|\nabla\lambda^*(t,.)\|^{1-\frac{1}{p}}_{C^2(S^1)}\,\|\nabla\lambda^*(t,.)\|^{\frac{1}{p}}_{L^2(S^1)}. \end{aligned} \tag{3.50}$$

Since λ is bounded in $C^2(S^1)$, we combined (3.50) with Morrey-Sobolev inequality and obtain, by taking $p > 2$,

$$\|\nabla\lambda^*(t,.)\|_{C^{0,1-\frac{1}{p}}(S^1)} \leq \|\lambda^*(t,.)\|^{\frac{1}{p}}_{L^2(S^1)} \leq Ce^{\delta t}, \tag{3.51}$$

where $\delta = \frac{2-q-\epsilon}{p} > 0$. The function $\bar{\lambda}$ satisfies

$$-\bar{\lambda}_{tt} - \frac{2}{b(1-t)^2} + \frac{a}{(1-t)^2}\overline{e^{b\lambda}} + e^{(2-q-\epsilon)t}G_\epsilon = 0 \tag{3.52}$$

If it is monotone at $-\infty$, then it converges to some $\ell \in [-K, K]$. If it oscillates infinitely many times when $t \to -\infty$, there exists a sequence $\{t_n\}$ tending to $-\infty$ such that $\bar{\lambda}(t)$ achieves a local maximum when $t = t_{2n}$ and a local minimum when $t = t_{2n+1}$. Hence

$$\bar{\lambda}_{tt}(t_{2n}) \leq 0 \text{ and } \bar{\lambda}_{tt}(t_{2n+1}) \geq 0.$$

We have

$$e^{b\lambda(t,.)} = e^{b\bar{\lambda}(t)}(e^{\lambda(t,.)-\bar{\lambda}(t)}) = e^{b\bar{\lambda}(t)}\left(1 + D(t,.)e^{\delta t}\right)$$

for some bounded function D, hence

$$\overline{e^{b\lambda}} = e^{b\bar{\lambda}(t)}\left(1 + \overline{D(t,.)}e^{\delta t}\right)$$

Plugging this estimate into (3.52) we obtain that there exists a bounded function $B(t)$ and $\nu > 0$ such that

$$-\frac{2}{b} + ae^{b\bar{\lambda}(t_n)} + e^{\nu t_n}B(t_n) = (1-t_n)^2\bar{\lambda}_{tt}(t_n) \tag{3.53}$$

This yields

$$e^{b\bar{\lambda}(t_{2n})} \leq \frac{2}{ab} - e^{\nu t_{2n}}B(t_{2n}) \quad \text{and} \quad e^{b\bar{\lambda}(t_{2n+1})} \geq \frac{2}{ab} - e^{\nu t_{2n+1}}B(t_{2n+1}) \tag{3.54}$$

and finally

$$\limsup_{t\to-\infty} \bar{\lambda}(t) = \liminf_{t\to-\infty} \bar{\lambda}(t) = \lim_{t\to-\infty} \bar{\lambda}(t) = \frac{1}{b}\ln\frac{2}{ab} := \ell. \tag{3.55}$$

This implies

$$\lim_{x\to 0}\left(u(x) - \frac{2}{b}\left(\ln\frac{1}{|x|} - \ln(1 - \ln|x|)\right)\right) = \ell. \tag{3.56}$$

To end the estimate we have still to prove that $w_t \to 0$. Differentiating (3.46) with respect to t we obtain

$$-w_{ttt} - w_{t\theta\theta} + abe^{bw}w_t + e^{(2-q)t}\tilde{G} = 0 \tag{3.57}$$

where $\tilde{G} = O(\ln(1-t))$. Since $e^{bw} \geq \frac{e^{-bK}}{(1-t)^2}$ we get

$$-\frac{d^2}{dt^2}\|w_t\|_{L^2(S^1)} + \frac{abe^{-bK}}{(1-t)^2}\|w_t\|_{L^2(S^1)} \leq Ae^{\mu t}$$

where $\mu < 2 - q$ and $A > 0$. Put $X(t) = \|w_t(t,.)\|_{L^2(S^1)} + \frac{A'}{\mu^2}e^{\mu t}$ with $A' > A$. Then there exists $T^* < 0$ such that for $t \leq T^*$ we have

$$-X'' + \frac{e^{-bK}ab}{(1-t)^2}X \leq 0.$$

The corresponding equation

$$\mathcal{L}(x) := -x'' + \frac{e^{-bK}ab}{(1-t)^2}x = 0 \quad \text{with } t \leq 0$$

admits two linearly independent solutions

$$x_1(t) = (1-t)^{\lambda_1} \quad \text{and} \quad x_2(t) = (1-t)^{\lambda_2},$$

when $t < 0$ arbitrary and $\lambda_1 < 0 < \lambda_2$. Since $X(t) = o(x_2(t))$ when $t \to -\infty$, it follows that $X(t) = O(x_1(t))$. Hence

$$\|w_t\|_{L^2(S^1)} \leq B(1-t)^{\lambda_1} \quad \text{for } t \leq 0 \tag{3.58}$$

for some $B \geq 0$. In order to have a uniform estimate, we use Poisson representation formula for the solution U of the Dirichlet problem for the Laplacian in the disk of radius R

$$U(r,\theta) = \frac{R^2 - r^2}{2\pi R} \int_0^{2\pi} \frac{U(R,\phi))d\phi}{\sqrt{r^2 + R^2 - 2rR\cos(\theta - \phi)}} \tag{3.59}$$

Using the variable $t = \ln r$ and $\tilde{U}(t,\theta) = U(r,\theta)$, it follows that the bounded solution of

$$\begin{array}{rl} -\tilde{U}_{tt} - \tilde{U}_{\theta\theta} = 0 & \text{in } (-\infty, T) \times S^1 \\ \tilde{U}(T,\theta) = \tilde{U}_T(\theta) & \text{in } S^1 \end{array} \tag{3.60}$$

is expressed by

$$\tilde{U}(t,\theta) = \frac{e^{2T} - e^{2t}}{2\pi e^T} \int_0^{2\pi} \frac{\tilde{U}(T,\phi))d\phi}{\sqrt{e^{2t} + e^{2T} - 2e^{t+T}\cos(\theta - \phi)}} \tag{3.61}$$

Hence, for $T = \frac{t}{2}$

$$\tilde{U}(t,\theta) \leq \frac{1 + e^{\frac{t}{2}}}{2\pi} \int_0^{2\pi} \tilde{U}_+(\tfrac{t}{2},\phi)d\phi \tag{3.62}$$

From (3.57) there exists some $A^* \geq 0$ such that the function $W(t,\theta) = w_t(t,\theta) + A^* e^{\mu t}$ satisfies

$$-W_{+\,tt} - W_{+\,\theta\theta} \leq 0 \qquad \text{in } (-\infty, 0) \times S^1.$$

Since

$$\|W(T,.)\|_{L^1(S^1)} = \left\|w_t(T,.) + A^* e^{\mu T}\right\|_{L^1(S^1)} \leq 2\pi A^* e^{\mu T} + \sqrt{2\pi}\,\|w_t(T,.)\|_{L^2(S^1)},$$

we derive from (3.58) and (3.62), that there holds for some $C > 0$,

$$(w_t(t,\theta) + A^* e^{\mu t})_+ = W_+(t,\theta) \leq C((t+1)^{\lambda_1} \qquad \text{for all } (t,\theta) \in (-\infty, 0] \times S^1. \tag{3.63}$$

This implies

$$\lim_{x \to 0} |x| \nabla u(x) = -\frac{2}{b}\mathbf{i}. \tag{3.64}$$

The end of the proof is similar to the case $\frac{q}{b} < \gamma < \frac{2}{b}$. □

4 Singular solutions in the case $q > 2$

In all this section we assume $q > 2$ and $m, b, a > 0$.

4.1 The eikonal equation

The eikonal equation associated to (1.4) is

$$ae^{bw} = m|\nabla w|^q \quad \text{in } \mathbb{R}^2 \setminus \{0\}. \tag{4.1}$$

This equation is completely integrable since it is equivalent to

$$\frac{b}{q}\left(\frac{a}{m}\right)^{\frac{1}{q}} = |\nabla(e^{-\frac{b}{q}w})|.$$

By [8] w is radial with respect to 0. Since any solution cannot have extremum, it is either decreasing or increasing with respect $|x|$. If it is decreasing on $(0,1]$ with $u(0) = c$, then

$$w(x) := w_c(x) = \frac{q}{b}\ln\left(\frac{qm^{\frac{1}{q}}}{ba^{\frac{1}{q}}|x| + qm^{\frac{1}{q}}e^{-\frac{bc}{q}}}\right) \tag{4.2}$$

Letting $c \to \infty$ yields

$$\lim_{c\to\infty} w_c(x) := w_\infty(x) = \frac{q}{b}\ln\left(\frac{qm^{\frac{1}{q}}}{ba^{\frac{1}{q}}|x|}\right) = \frac{q}{b}\ln\left(\frac{1}{|x|}\right) + \frac{q}{b}\ln\left(\frac{qm^{\frac{1}{q}}}{ba^{\frac{1}{q}}}\right). \tag{4.3}$$

This is also a solution of

$$-\Delta u + ae^{bu} - m|\nabla u|^q = 0 \qquad \text{in } \mathbb{R}^2 \setminus \{0\}. \tag{4.4}$$

It vanishes on ∂B_1 if and only if $qm^{\frac{1}{q}} = ba^{\frac{1}{q}}$.

4.2 Singular solutions

Lemma 4.1 *If u is a radial nonnegative solution of (4.4) in $B_1 \setminus \{0\}$ it is decreasing on some $(0, r^*] \subset (0, 1]$ and there holds*

$$-u'(r) = |u'(r)| \geq cr^{-\frac{1}{q-1}} \qquad \textit{for all } r \in (0, r^*], \tag{4.5}$$

for some $c > 0$,

Proof. If u is a positive radial function satisfying (4.4) in $B_1 \setminus \{0\}$, it cannot have any maximum there in, hence either it is decreasing on $(0, 1)$ which implies that $u(0) < u(1)$, or it is decreasing in some interval $(0, r^*)$ and increasing in $(r^*, 1)$. For simplicity and without loss of generality, we will assume that $r^* = 1$ (this is always the case if $u(1) = 0$). We have

$$-u'' - \frac{1}{r}u' - m|u'|^q \leq 0,$$

then with $V = -u'$, we obtain

$$V' + \frac{1}{r}V - mV^q \leq 0 \Longleftrightarrow V'V^{-q} + \frac{1}{r}V^{1-q} - m \leq 0 \Longleftrightarrow (r^{1-q}V^{1-q})' + m(q-1)r^{1-q} \geq 0$$

Therefore the function

$$r \mapsto r^{1-q}V^{1-q}(r) - m\frac{q-1}{q-2}r^{2-q}$$

is increasing, which implies

$$V^{1-q}(r) \leq m\frac{q-1}{q-2}r(1+o(1)) \quad \text{as } r \to 0. \tag{4.6}$$

Replacing V by $-u'$ we obtain

$$u'(r) \leq -\left(\frac{q-2}{m(q-1)}\right)^{\frac{1}{q-1}} r^{-\frac{1}{q-1}}(1+o(1)) \quad \text{as } r \to 0. \tag{4.7}$$

□

Proposition 4.2 *Let u be a positive solution of (4.4) in $B_1 \setminus \{0\}$. The following statements are equivalent:*
(i) $e^{bu} \in L^1(B_1)$
(ii) $|\nabla u|^q \in L^1(B_1)$.
(iii) u can be extended to in $\overline{B}_1$ as a Hölder continuous function and the extended function, still denoted by u, is a solution of

$$-\Delta u + ae^{bu} - m|\nabla u|^q = 0 \qquad \text{in } \mathbf{D}'(B_1). \tag{4.8}$$

Proof. $(i) \Longrightarrow (ii)$. If $e^{bu} \in L^1(B_1)$, it follows by Brezis-Lions Lemma [5] that $|\nabla u|^q \in L^1(B_1)$ and there exists $\gamma \geq 0$ such that

$$-\Delta u + ae^{bu} - m|\nabla u|^q = 2\pi\gamma\delta_0 \qquad \text{in } \mathbf{D}'(B_1). \tag{4.9}$$

The average function $\bar{u}$ satisfies

$$-(r\bar{u}_r)_r = r\left(m\overline{(u_r^2 + r^{-2}u_\theta^2)^{\frac{q}{2}}} - a\overline{e^{bu}}\right). \tag{4.10}$$

Therefore $(r\bar{u}_r)' \in L^1(0,1)$ which implies that $r\bar{u}_r(r)$ admits a finite limit when $r \to 0$. If this limit is not zero, say $\tilde{\gamma}$, then

$$\bar{u}_r(r) = \frac{\tilde{\gamma}}{r}(1+o(1)) \Longrightarrow |\bar{u}_r(r)|^q \sim \frac{|\tilde{\gamma}|^q}{r^q}.$$

Now

$$|u_r(r)| \leq \frac{1}{2\pi}\int_{S^1} |u_r(r,.)|\, d\theta \leq \frac{1}{2\pi}\int_{S^1} |\nabla u(r,.)| d\theta \leq \left(\frac{1}{2\pi}\int_{S^1} |\nabla u(r,.)|^q\, d\theta\right)^{\frac{1}{q}}$$

Since $q > 2$ we have $|\nabla u|^q \notin L^1(B_1)$, contradiction. Hence $\tilde{\gamma} = 0$. Therefore $\bar{u}_r(r) = o(r^{-1})$, and this implies (4.11).

$(ii) \Longrightarrow (i)$. By contradiction assume that $e^{bu} \notin L^1(B_1)$. If $|\nabla u|^q \in L^1(B_1)$, then

$$r\bar{u}_r(r) - \bar{u}_r(1) = \int_r^1 \left(m\overline{|\nabla u|^q} - a\overline{e^{bu}} \right) tdt = -a \int_r^1 \overline{e^{bu}} tdt + O(1) \to -\infty \quad \text{as } r \to 0.$$

Hence $r\bar{u}_r(r) \to -\infty$ when $r \to 0$, which implies

$$\lim_{r\to 0} \frac{u(r)}{-\ln r} = \infty,$$

which contradicts the a priori estimate. Hence e^{bu} is not integrable in B_1 which implies that the same holds with $|\nabla u|^q$. This proves the claim.
$(iii) \Longrightarrow (i)$. Since u is continuous in $\overline{B}_1$, $e^{bu} \in L^1(B_1)$.
$(ii) \Longrightarrow (iii)$. Since $|\nabla u| \in L^q(B_1)$ and u is C^1 in $\overline{B}_1 \setminus \{0\}$, it follows by Morrey-Sobolev theorem that it can be extended by continuity to $x = 0$ and that the resulting function, still denoted by u belongs to $C^{0,1-\frac{2}{q}}(\overline{B}_1)$. In particular

$$|u(x) - u(0)| \leq c|x|^{1-\frac{2}{q}}. \tag{4.11}$$

Furthermore, by Brezis-Lions lemma, there exists $\gamma \geq 0$ such that (4.9) holds. By the same averaging method that in the case $(i) \Longrightarrow (ii)$, we have also

$$\lim_{r\to 0} \frac{\bar{u}(r)}{-\ln r} = \gamma.$$

The previous estimate implies that $\gamma = 0$. This ends the proof. □

Remark. If u is a radial function estimate (4.11) can be improved. Actually, since by Lemma 4.1 one has (4.5), the integration of this inequality yields

$$|u(r) - u(0)| \leq cr^{\frac{q-2}{q-1}}, \tag{4.12}$$

and $\frac{q-2}{q-1} > \frac{q-2}{q}$.

4.3 Classification of solutions with an isolated singularity

The following result has already been proved in the proof of assertion 2-(ii) in Theorem 3.8.
Proposition 8 *Let $q > 2$ and u be a positive solution of* (4.4) *in $B_1 \setminus \{0\}$. Then*

$$\int_{B_1} |\nabla u|^q dx = \infty \Longrightarrow \liminf_{x\to 0} \frac{u(x)}{-\ln|x|} \geq \frac{2}{b}. \tag{4.13}$$

Remark Using the estimates of Proposition 2 we have precisely

$$u(x) \geq \frac{2}{b} \left(\ln \frac{1}{|x|} - \ln(1 - \ln|x|) \right) - K. \tag{4.14}$$

Under the assumption of Proposition 8 a solution u vanishing on ∂B_1 is decreasing. In the next result we prove that u' is monotone.

Lemma 2 *Let u be a positive radial singular solution of (4.4) in $B_1\setminus\{0\}$, then $ru'(r)$ is monotone near $r=0$*

Proof. We set

$$\mathcal{W}(r) = ae^{bu(r)} - m|u'(r)|^q. \tag{4.15}$$

Using the equation (4.4) we have

$$\mathcal{W}' = u'ae^{bu}\left(b - qm|u'|^{q-2}\right) + qm|u'|^{q-2}u'\left(|u'|^q + \frac{u'}{r}\right).$$

Replacing ae^{bu} by $\mathcal{W} + m|u'|^q$, we obtain

$$\mathcal{W}' = u'\left(ab - qm|u'|^{q-2}\right)\mathcal{W} + mb|u'|^q u' - (m-1)qm|u'|^{2q-2}u' + \frac{qm}{r}|u'|^q$$

Denote

$$\Theta(r) = u'(r)\left(b - qm|u'|^{q-2}(r)\right),$$

and

$$\Sigma(r) = mb|u'|^q u' - (m-1)qm|u'|^{2q-2}u' + \frac{qm}{r}|u'|^q$$

By (4.5), $u'r)$ is nonpositive near $r=0$ and tends to $-\infty$ when $r\to 0$. Then

$$\Sigma(r) = (m-1)q|u'|^{2q-1}(1+o(1)) + \frac{qm}{r}|u'|^q \quad \text{as } r\to 0,$$

since $q>2$. By integration on $[r,r_0]\subset(0,1)$,

$$\frac{d}{dr}\left(e^{\int_r^{r_0}\Theta(s)ds}\mathcal{W}(r)\right) = e^{\int_r^{r_0}\Theta(s)ds}\Sigma(r)$$

hence

$$\mathcal{W}(r) = e^{-\int_r^{r_0}\Theta(s)ds}\mathcal{W}(r_0) - \int_r^{r_0} e^{-\int_r^s\Theta(t)dt}\Sigma(s)ds$$

There exists $r^*\in(0,1)$ such that $\Sigma(r)>0$ for $(0<r\le r^*$. Therefore if for some $r_0\in(0,r^*]$ there holds $\mathcal{W}(r_0)\le 0$, it follows that $\mathcal{W}(r)<0$ for any $r\in(0,r_0)$. Since

$$(ru')' = r\left(ae^{bu} - m|u'|^q\right) = r\mathcal{W},$$

this implies that $(ru')'\le 0$ on $(0,r_0]$. On the contrary, for any if for any $r\in(0,r^*]$ we have $\mathcal{W}(r)>0$, we conclude that $(ru')'\le 0$ on $(0,r_0]$. □

Proposition 9 *Let $q>2$ and u be a positive radial solution of (4.4) in $B_1\setminus\{0\}$ satisfying $\int_{B_1}|\nabla u|^q dx=\infty$. If $ru'(r)$ is monotone then*

$$\lim_{x\to 0}\frac{u(x)}{-\ln|x|} = \frac{q}{b}. \tag{4.16}$$

Proof. If u satisfies the above assumption, there holds by Propositions 2 and 8,

$$\frac{2}{b}\ln\frac{1}{|x|}-K(b)\ln(1-\ln|x|)\le u(x)\le\frac{q}{b}\ln\left(\frac{1}{|x|}\right)+\frac{q}{b}\left[\left(\ln\left(\frac{b}{qm^{\frac{1}{q}}}\right)\right)_+-\ln\left(\frac{b}{qm^{\frac{1}{q}}}\right)\right], \quad (4.17)$$

where $K(b)=1$ if $b\ge 2$ and $K(b)=\frac{2}{b}$ if $0<b\le 2$. We set $t=\ln r$ and $w(t)=u(r)$. Then

$$-w_{tt}+ae^{bw+2t}-me^{(2-q)t}|w_t|^q=0. \quad (4.18)$$

Moreover, if $ru'(r)$ is monotone, then w_t is monotone too. From (4.17),

$$-\frac{2}{b}t-K(b)\ln(1-t)\le w(t)\le-\frac{q}{b}t+\frac{q}{b}\left[\left(\ln\left(\frac{b}{qm^{\frac{1}{q}}}\right)\right)_+-\ln\left(\frac{b}{qm^{\frac{1}{q}}}\right)\right]\quad\text{on }(-\infty,0).$$

If w_t is monotone, it admits a limit ℓ, finite or not, when $t\to-\infty$. The only possibility is that

$$-\frac{q}{b}\le\ell\le-\frac{2}{b}.$$

and $w(t)=\ell t(1+o(1))$ when $t\to-\infty$. From (4.18),

$$w_{tt}=-me^{(2-q)t}|\ell|^q(1+o(1))+ae^{(b\ell+2)t+o(t)}.$$

There always holds $2-q\ge b\ell+2$. In order that w_{tt} does not tend to $-\infty$ (which would be a contradiction), necessarily $2-q=b\ell+2$ which implies $\ell=-\frac{q}{b}$. This yields (4.16). □

The proof of **Theorem 3** follows by combining Proposition 7, Proposition 8, Lemma 2 and Proposition 9.

Remark We conjecture that in case (ii) the assumption of radiality is unnecessary.
This type of dichotomy which occurs in this case between regular solutions and only one type of singular solutions is wellknown for semilinear Lane-Emden equations

$$-\Delta u=u^p\quad\text{in }B_1\setminus\{0\} \quad (4.19)$$

when $N\ge 3$ and $p>\frac{N}{N-2}$. To our knowledge it is a novelty for equation of (4.4) and in a 2-dimensional framework.

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