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Existence and asymptotic analysis of Nash equilibria for the Navier-Stokes system

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Abstract

We show the existence of Nash equilibria for the Navier-Stokes system and for the Oseen system. We consider the cases of the Dirichlet boundary conditions and of the Navier slip boundary conditions. Then, we study the asymptotic behavior of the Nash equilibria as the friction coefficient goes to ∞ in the Navier slip boundary condition. More precisely, we show that the Nash equilibria for the Navier slip boundary condition converge towards a Nash equilibrium for the Dirichlet boundary condition. We also show the convergence of the corresponding direct and adjoint states.

Keywords: Navier-Stokes system, multi-objective optimization, Nash equilibrium.

2020 Mathematics Subject Classification. 76D55, 35Q30, 76D05, 93C20.

1 Introduction

In this article, we consider a multi-objective optimization problem for the motion of a viscous incompressible fluid. There are many studies in the literature of mono-objective optimization problems for the Navier-Stokes system or for different fluid models, see, for instance, [3], [17], [18], [28], [33], [31], etc. At the contrary, for multi-objective optimization problems, there are very few results for the Navier-Stokes systems. First, let us note that for multi-objective problems, there are several different strategies, coming from the game theory: see [24], [27], [32], etc. One can quote [15] where the authors consider a Stackelberg-Nash strategy for the Stokes system and [7, 8] where the authors show the existence of Nash equilibria for the stationary Navier-Stokes systems. Let us also the works [23], [29] for the case of the Stackelberg strategy for the Navier-Stokes system.

Up to our knowledge, there are no results for the existence of Nash equilibria for the Navier-Stokes system and our first objective is to show such a result. We consider the case of two different boundary conditions: the Dirichlet boundary condition and the Navier slip boundary condition. The first boundary condition is the most standard and corresponds to the case where the fluid adheres to the exterior boundary, that is a no-slip boundary condition. The Navier slip boundary condition is less standard but is important in some physical situations. This boundary condition was introduced by Navier in [25] and was used for instance in the following works: [14], [16], [19], [20], [21], etc. Note that in [6], one can find a rigorous derivation of this condition from the Boltzmann equation. These Navier slip boundary conditions model the possibility for the fluid to slip tangentially to the boundary with a friction. Formally, if the corresponding friction coefficient goes to infinity, one recover the standard no-slip boundary condition.

Our second objective is to show such a convergence result for the Nash equilibria. We will also consider the corresponding critical points called Nash quasi-equilibria and show a similar convergence result. We have done a similar work for a mono-objective optimization problem in [9]. As in the previous work, we need to choose carefully the functional framework. Working with weak solutions seems an interesting approach since in that case one can prove existence of global solutions without restrictions on the size of data and of the controls. Nevertheless, the uniqueness of weak solutions for the Navier-Stokes system is an open problem in dimension 3 and consequently the objective criterion are more complicated to handle in that case. We choose here to work with strong solutions and the drawbacks is that, in that case, we has to restrict either the the size of the data or the time of existence. Here, we consider solutions and a criterion written in $(0, T)$ where $T > 0$ is given, and we thus need to have small data and small controls. Moreover, in the idea to show the convergence result, we need to have a restriction that is uniform with respect to the friction coefficient in the Navier boundary condition.

We consider $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$, a bounded domain of class $C^{2,1}$ and we write the Navier-Stokes system with Dirichlet boundary conditions:

$$\left\{ \begin{array}{ll} \partial_t u + (u \cdot \nabla)u - \operatorname{div} \mathbb{T}(u, p) = f + g^{(1)}1_{\omega_1} + g^{(2)}1_{\omega_2} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0, T) \times \Omega, \\ u = b & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = a & \text{in } \Omega. \end{array} \right. \quad (1.1)$$

We have denoted by $\mathbb{T}(u, p)$ the Cauchy stress tensor:

$$\mathbb{T}(u, p) := 2\mathbb{D}(u) - pI_3, \quad \mathbb{D}(u) := \frac{1}{2} (\nabla u + (\nabla u)^\top).$$

Note that here, to simplify the notation, we have assumed that the viscosity of the fluid is equal to 1. In the above system, u and p are respectively the fluid velocity and the pressure of the fluid. The functions a and b are respectively the initial and the boundary conditions and f is a given source. The controls $g^{(1)}$ and $g^{(2)}$ are acting on the open nonempty sets $\omega_1, \omega_2 \subset \Omega$. We consider in this work the case where $g := (g^{(1)}, g^{(2)})$ aims at minimizing the following functionals

$$J_i(g^{(1)}, g^{(2)}) := \frac{1}{2} \int_0^T \int_{\mathcal{O}_i} |u_g - z^{(i)}|^2 dx dt + \frac{M}{2} \int_0^T \int_{\omega_i} |g^{(i)}|^2 dx dt \quad (1.2)$$

where \mathcal{O}_i ($i = 1, 2$) are nonempty open subsets of Ω , $z^{(i)} \in L^2(0, T; L^2(\mathcal{O}_i))^d$ ($i = 1, 2$) are given functions, $M > 0$ and u_g is the solution of (1.1) associated with $g = (g^{(1)}, g^{(2)})$.

We assume that the control $(g^{(1)}, g^{(2)}) \in \mathcal{U}_1 \times \mathcal{U}_2$, where

$$\mathcal{U}_i \text{ is a nonempty closed convex sets of } L^2(0, T; L^2(\omega_i))^d. \quad (1.3)$$

We are interested by a Nash equilibrium for the above functionals, as defined in the following definition:

Definition 1.1. *We say that*

$$(\widehat{g}^{(1)}, \widehat{g}^{(2)}) \in \mathcal{U}_1 \times \mathcal{U}_2$$

is a Nash equilibrium for (J_1, J_2) if

$$\left\{ \begin{array}{ll} J_1(\widehat{g}^{(1)}, \widehat{g}^{(2)}) \leq J_1(g^{(1)}, \widehat{g}^{(2)}) & (g^{(1)} \in \mathcal{U}_1), \\ J_2(\widehat{g}^{(1)}, \widehat{g}^{(2)}) \leq J_2(\widehat{g}^{(1)}, g^{(2)}) & (g^{(2)} \in \mathcal{U}_2). \end{array} \right. \quad (1.4)$$

In what follows, we will work in a framework for which J_1 and J_2 are Gateaux-differentiable and in that case if $(\widehat{g}^{(1)}, \widehat{g}^{(2)})$ is a Nash equilibrium for (J_1, J_2) , then we have the following standard relations:

$$\left\{ \begin{array}{ll} D_1 J_1(\widehat{g}^{(1)}, \widehat{g}^{(2)})(g^{(1)} - \widehat{g}^{(1)}) \geq 0 & (g^{(1)} \in \mathcal{U}_1), \\ D_2 J_2(\widehat{g}^{(1)}, \widehat{g}^{(2)})(g^{(2)} - \widehat{g}^{(2)}) \geq 0 & (g^{(2)} \in \mathcal{U}_2). \end{array} \right. \quad (1.5)$$

In the above relations and in what follows, we have denoted by $D_1 J_j(g)(h)$ and $D_2 J_j(g)(h)$ the Gateaux-differential of J_j at g respectively in the direction $(h, 0)$ and $(0, h)$.

Definition 1.2. We say that $(\widehat{g}^{(1)}, \widehat{g}^{(2)}) \in \mathcal{U}_1 \times \mathcal{U}_2$ is a Nash quasi-equilibrium for (J_1, J_2) if it satisfies (1.5).

As usual, in the case where $(\widehat{g}^{(1)}, \widehat{g}^{(2)}) \in \mathcal{U}_1 \times \mathcal{U}_2$ is an interior point of $\mathcal{U}_1 \times \mathcal{U}_2$, a Nash quasi-equilibrium for (J_1, J_2) is characterized by the following relations:

$$\begin{cases} D_1 J_1(\widehat{g}^{(1)}, \widehat{g}^{(2)})(h) = 0 & (h \in \mathcal{U}_1), \\ D_2 J_2(\widehat{g}^{(1)}, \widehat{g}^{(2)})(h) = 0 & (h \in \mathcal{U}_2). \end{cases}$$

1.1 Main results

Let us now consider the hypotheses on the data in (1.1). We assume

$$f \in L^2(0, T; L^2(\Omega))^d, \quad a \in H^1(\Omega)^d, \quad \operatorname{div} a = 0, \quad a = b(0, \cdot) \quad \text{on } \partial\Omega \quad (1.6)$$

and

$$b \in L^2(0, T; H^{3/2}(\partial\Omega))^d \cap H^{1/4}(0, T; L^2(\partial\Omega))^d, \quad \int_{\partial\Omega} b(t, \cdot) \cdot \nu \, d\gamma = 0 \quad (t \in [0, T]). \quad (1.7)$$

We denote by B_r the closed ball of radius $r > 0$ in $L^2(0, T; L^2(\omega_1) \times L^2(\omega_2))^d$:

$$B_r := \left\{ g = (g^{(1)}, g^{(2)}) \in L^2(0, T; L^2(\omega_1) \times L^2(\omega_2))^d : \|g^{(i)}\|_{L^2(0, T; L^2(\omega_i))^d} \leq r \quad (i = 1, 2) \right\}. \quad (1.8)$$

By standard results, if

$$\|f\|_{L^2(0, T; L^2(\Omega))^d} + \|a\|_{H^1(\Omega)^d} + \|b\|_{L^2(0, T; H^{3/2}(\partial\Omega))^d \cap H^{1/4}(0, T; L^2(\partial\Omega))^d} \leq r, \quad (1.9)$$

and

$$\mathcal{U}_1 \times \mathcal{U}_2 \subset B_r \quad (1.10)$$

with $r > 0$ small enough, then for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$, there exists a unique strong solution (u_g, p_g) of (1.1) with

$$u_g \in H^1(0, T; L^2(\Omega))^d \cap C^0([0, T]; H^1(\Omega))^d \cap L^2(0, T; H^2(\Omega))^d, \quad p_g \in L^2(0, T; H^1(\Omega)/\mathbb{R}).$$

In particular, with such hypotheses, then J_1 and J_2 are well-defined in $\mathcal{U}_1 \times \mathcal{U}_2$. Our first result is the existence of a Nash equilibrium for the system (1.1):

Theorem 1.3. Assume (1.9) and (1.10) for r small enough so that for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$, there exists a unique strong solution (u_g, p_g) of (1.1).

There exists $M_0 > 0$ such that if $M \geq M_0$ then there exists a unique Nash equilibrium for (J_1, J_2) in $\mathcal{U}_1 \times \mathcal{U}_2$. Moreover it is the unique Nash quasi-equilibrium for (J_1, J_2) in $\mathcal{U}_1 \times \mathcal{U}_2$.

We also show the existence of a Nash equilibrium for the following system similar to (1.1) but with Navier slip boundary condition instead of the Dirichlet boundary condition:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \operatorname{div} \mathbb{T}(u, p) = f + g^{(1)}1_{\omega_1} + g^{(2)}1_{\omega_2} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0, T) \times \Omega, \\ u \cdot \nu = b \cdot \nu & \text{on } (0, T) \times \partial\Omega, \\ [2(\mathbb{D}u)\nu + \alpha(u - b)]_\tau = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = a & \text{in } \Omega. \end{cases} \quad (1.11)$$

Here $\alpha \geq 0$ is a friction coefficient, assume to be constant and to satisfy

$$\alpha \geq \|b\|_{L^\infty(0, T; L^\infty(\partial\Omega))^d} + 1. \quad (1.12)$$

We add in particular in (1.7) that

$$b \in L^\infty(0, T; L^\infty(\partial\Omega))^d. \quad (1.13)$$

Similarly as for the Dirichlet case, we can define the functionals

$$J_{i,\alpha} \left(g^{(1)}, g^{(2)} \right) := \frac{1}{2} \int_0^T \int_{\mathcal{O}_i} \left| u_{\alpha,g} - z^{(i)} \right|^2 dx dt + \frac{M}{2} \int_0^T \int_{\omega_i} \left| g^{(i)} \right|^2 dx dt, \quad (1.14)$$

where $u_{\alpha,g}$ is the solution of (1.11) associated with $g = (g^{(1)}, g^{(2)})$. We can also define a Nash equilibrium or a Nash quasi-equilibrium for $(J_{1,\alpha}, J_{2,\alpha})$ in a similar way than for (J_1, J_2) (see Definition 1.1 and Definition 1.2): $(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)})$ is a Nash equilibrium for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$ if

$$\begin{cases} J_{1,\alpha} \left(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)} \right) \leq J_{1,\alpha} \left(g^{(1)}, \widehat{g}_\alpha^{(2)} \right) & (g^{(1)} \in \mathcal{U}_1), \\ J_{2,\alpha} \left(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)} \right) \leq J_{2,\alpha} \left(\widehat{g}_\alpha^{(1)}, g^{(2)} \right) & (g^{(2)} \in \mathcal{U}_2). \end{cases} \quad (1.15)$$

As for system (1.1), there exists $r > 0$ small enough such that if (1.9) and (1.10) hold, then for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$, there exists a unique strong solution $(u_{\alpha,g}, p_{\alpha,g})$ of (1.11) with

$$u_{\alpha,g} \in H^1(0, T; L^2(\Omega))^d \cap C^0([0, T]; H^1(\Omega))^d \cap L^2(0, T; H^2(\Omega))^d, \quad p_{\alpha,g} \in L^2(0, T; H^1(\Omega)/\mathbb{R}).$$

However the value of r may depend on α . In order to have a constant uniform in α , we need to strengthen our hypotheses on the data. We use a lift of the boundary condition and assume the existence of a solution $u^{(0)} \in H^1(0, T; H^1(\Omega))^d$ of the system

$$\begin{cases} \partial_t u^{(0)} - \operatorname{div} \mathbb{T}(u^{(0)}, p^{(0)}) = 0 & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u^{(0)} = 0 & \text{in } (0, T) \times \Omega, \\ u^{(0)} = b & \text{on } (0, T) \times \partial\Omega, \\ u^{(0)}(0, \cdot) = a & \text{in } \Omega. \end{cases} \quad (1.16)$$

Then we replace condition (1.9) by the condition

$$\|f\|_{L^2(0, T; L^2(\Omega))^d} + \left\| u^{(0)} \right\|_{H^1(0, T; H^1(\Omega))^d} \leq r. \quad (1.17)$$

We have shown in [9] that there exists $r > 0$ small enough such that if (1.10) and (1.17) hold then for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$ and for any α satisfying (1.12), there exists a unique strong solution $(u_{\alpha,g}, p_{\alpha,g})$ of (1.11) (see also Proposition 4.1). In particular, with such hypotheses, then for any α satisfying (1.12), $J_{1,\alpha}$ and $J_{2,\alpha}$ are well-defined in $\mathcal{U}_1 \times \mathcal{U}_2$. Then we have the following result

Theorem 1.4. *Assume (1.10) and (1.17) for r small enough so that for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$ and for any α satisfying (1.12), there exists a unique strong solution $(u_{\alpha,g}, p_{\alpha,g})$ of (1.11).*

There exists $M_0 > 0$ such that if $M \geq M_0$, then for any α satisfying (1.12), there exists a unique Nash equilibrium \widehat{g}_α for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$. Moreover it is the unique Nash quasi-equilibrium for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$.

Note that in the above result the smallness conditions (1.10) and (1.17) are independent of α . We prove Theorem 1.4 in Section 6, we skip the proof of Theorem 1.3 since it is completely similar. For these proofs, we need in particular the adjoint system associated with (1.1) given by

$$\begin{cases} -\partial_t \varphi^{(i)} - (u_g \cdot \nabla) \varphi^{(i)} + (\nabla u_g)^\top \varphi^{(i)} - \operatorname{div} \mathbb{T}(\varphi^{(i)}, \pi_{\varphi^{(i)}}) = (u_g - z^{(i)}) 1_{\mathcal{O}_i} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} \varphi^{(i)} = 0 & \text{in } (0, T) \times \Omega, \\ \varphi^{(i)} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \varphi^{(i)}(T, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (1.18)$$

The adjoint system associated with (1.11) is given by

$$\left\{ \begin{array}{ll} -\partial_t \varphi^{(i)} - (u_{\alpha,g} \cdot \nabla) \varphi^{(i)} + (\nabla u_{\alpha,g})^\top \varphi^{(i)} - \operatorname{div} \mathbb{T}(\varphi^{(i)}, \pi_{\varphi^{(i)}}) = (u_{\alpha,g} - z^{(i)}) 1_{\mathcal{O}_i} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} \varphi^{(i)} = 0 & \text{in } (0, T) \times \Omega, \\ \varphi^{(i)} \cdot \nu = 0 & \text{on } (0, T) \times \partial\Omega, \\ [2(\mathbb{D}\varphi^{(i)})\nu + (\alpha + b \cdot \nu) \varphi^{(i)}]_\tau = 0 & \text{on } (0, T) \times \partial\Omega, \\ \varphi^{(i)}(T, \cdot) = 0 & \text{in } \Omega. \end{array} \right. \quad (1.19)$$

We denote in what follows by $\varphi_g^{(i)}$ and $\varphi_{\alpha,g}^{(i)}$ the solutions of the above systems.

In what follows, we write

$$V_\nu^0 := \{u \in L^2(\Omega)^d ; \operatorname{div} u = 0, \quad u \cdot \nu = 0 \text{ on } \partial\Omega\}, \quad (1.20)$$

$$V_0^1 := \{u \in H^1(\Omega)^d ; \operatorname{div} u = 0, \quad u = 0 \text{ on } \partial\Omega\}, \quad (1.21)$$

$$V_0^{-1} := (V_0^1)'$$

We denote by \mathbb{P} the Leray projector, that is the orthogonal projection $\mathbb{P} : L^2(\Omega)^d \rightarrow V_\nu^0$.

If M is not large enough, it is not clear if we have existence or uniqueness of Nash equilibria or of Nash quasi-equilibria. Moreover, a Nash quasi-equilibrium is not necessarily a Nash equilibria. In the case where we have, for any α large enough, a family of Nash quasi-equilibria or a family of Nash equilibria, one can nevertheless pass to the limit as $\alpha \rightarrow \infty$. We have the following convergence results:

Proposition 1.5. *Assume (1.10) and (1.17) for r small enough so that for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$ and for any α satisfying (1.12), there exists a unique strong solution $(u_{\alpha,g}, p_{\alpha,g})$ of (1.11).*

Let us consider a family of Nash equilibria $\widehat{g}_\alpha = (\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)})$ for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$. Then, there exists a Nash equilibrium $\widehat{g} = (\widehat{g}^{(1)}, \widehat{g}^{(2)})$ for (J_1, J_2) in $\mathcal{U}_1 \times \mathcal{U}_2$ and a subsequence $\alpha \rightarrow \infty$, such that

$$\left(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)}\right) \rightarrow \left(\widehat{g}^{(1)}, \widehat{g}^{(2)}\right) \quad \text{in } L^2(0, T; L^2(\omega_1) \times L^2(\omega_2))^d. \quad (1.22)$$

Moreover, the corresponding states (direct and adjoint) satisfy

$$u_{\alpha, \widehat{g}_\alpha} \rightharpoonup u_{\widehat{g}} \quad \text{weakly } * \text{ in } L^2(0, T; H^1(\Omega))^d \cap L^\infty(0, T; L^2(\Omega))^d, \quad (1.23)$$

$$u_{\alpha, \widehat{g}_\alpha} \rightarrow u_{\widehat{g}} \quad \text{strongly in } L^2(0, T; L^2(\Omega))^d, \quad (1.24)$$

$$u_{\alpha, \widehat{g}_\alpha} \rightarrow b \quad \text{strongly in } L^2(0, T; L^2(\partial\Omega))^d, \quad (1.25)$$

$$\partial_t u_{\alpha, \widehat{g}_\alpha} \rightharpoonup \partial_t u_{\widehat{g}} \quad \text{weakly in } L^{4/3}(0, T; V_0^{-1}), \quad (1.26)$$

$$\varphi_{\alpha, \widehat{g}_\alpha}^{(i)} \rightharpoonup \varphi_{\widehat{g}}^{(i)} \quad \text{weakly } * \text{ in } L^2(0, T; H^1(\Omega))^d \cap L^\infty(0, T; L^2(\Omega))^d, \quad (1.27)$$

$$\varphi_{\alpha, \widehat{g}_\alpha}^{(i)} \rightarrow \varphi_{\widehat{g}}^{(i)} \quad \text{strongly in } L^2(0, T; L^2(\Omega))^d, \quad (1.28)$$

$$\varphi_{\alpha, \widehat{g}_\alpha}^{(i)} \rightarrow 0 \quad \text{strongly in } L^2(0, T; L^2(\partial\Omega))^d. \quad (1.29)$$

$$\partial_t \varphi_{\alpha, \widehat{g}_\alpha}^{(i)} \rightharpoonup \partial_t \varphi_{\widehat{g}}^{(i)} \quad \text{weakly in } L^{4/3}(0, T; V_0^{-1}). \quad (1.30)$$

We have the following result of convergence for the Nash quasi-equilibria:

Proposition 1.6. *Assume the hypotheses of Proposition 1.5 and let us consider a family of Nash quasi-equilibria $\widehat{g}_\alpha = (\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)})$ for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$. Then, there exists a Nash quasi-equilibrium $\widehat{g} = (\widehat{g}^{(1)}, \widehat{g}^{(2)})$ for (J_1, J_2) in $\mathcal{U}_1 \times \mathcal{U}_2$ and a subsequence $\alpha \rightarrow \infty$, such that we have the convergence (1.22). Moreover, the corresponding states (direct and adjoint) satisfy (1.23)–(1.30).*

As a consequence of Proposition 1.5 or Proposition 1.6, we have the following result in the case where the Nash equilibria are unique:

Corollary 1.7. *Assume the hypotheses of Theorem 1.4. Then we have the convergence (1.22) for the Nash equilibria \widehat{g}_α of $(J_{1,\alpha}, J_{2,\alpha})$ towards the Nash equilibrium \widehat{g} of (J_1, J_2) . Moreover the corresponding states (direct and adjoint) satisfy (1.23)–(1.30).*

The above results of convergences in particular Corollary 1.7 can be seen as extensions of previous results obtained for other partial differential equations: [10], [11], [12], [13], in the case of elliptic problems and [4], [22], in the case of parabolic systems. We have also obtained in [9] a similar result for the Navier-Stokes system. Here the main difference is that we consider the case of a multi-objective optimization (Nash equilibrium).

1.2 The bidimensional case

The above results are valid for $d = 2$ or $d = 3$. In the case $d = 2$, we can improve these results. First, we can work here with weak solutions since one can show the uniqueness of such solutions. Moreover, we assume (instead of (1.6) and (1.7))

$$f \in L^2(0, T; L^2(\Omega))^d, \quad a \in L^2(\Omega)^d, \quad \operatorname{div} a = 0, \quad a \cdot \nu = b(0, \cdot) \cdot \nu \quad \text{on } \partial\Omega \quad (1.31)$$

with

$$b \in L^2(0, T; H^{3/2}(\partial\Omega))^d \cap H^{1/4}(0, T; L^2(\partial\Omega))^d, \quad \int_{\partial\Omega} b(t, \cdot) \cdot \nu \, d\gamma = 0 \quad (t \in [0, T]). \quad (1.32)$$

To simplify, we consider a lift of b on the boundary, that we still denote by b :

$$b \in H^1(0, T; L^2(\Omega))^d \cap C^0([0, T]; H^1(\Omega))^d \cap L^2(0, T; H^2(\Omega))^d, \quad \operatorname{div} b = 0 \quad \text{in } (0, T) \times \Omega. \quad (1.33)$$

We also assume relation (1.12) (and thus (1.13)). With such hypotheses, we can show the existence and uniqueness of weak solutions for systems (1.1) and (1.11). We can thus define (J_1, J_2) and $(J_{1,\alpha}, J_{2,\alpha})$ by (1.2) and (1.14) and the Nash equilibrium and Nash quasi-equilibrium by using Definition 1.1 and Definition 1.2. The main difference with the three-dimensional case is that we do not have any constraint on the size of the data or on the controls; nevertheless, we need to have bounded controls and we replace (1.3) by the following hypothesis:

$$\mathcal{U}_i \quad \text{is a nonempty closed bounded convex sets of } L^2(0, T; L^2(\omega_i))^d. \quad (1.34)$$

More precisely our results in that case are

Theorem 1.8. *Assume $d = 2$, (1.31), (1.33) and (1.34). There exists $M_0 > 0$ such that if $M \geq M_0$ then*

- *there exists a unique Nash equilibrium for (J_1, J_2) in $\mathcal{U}_1 \times \mathcal{U}_2$. Moreover it is the unique Nash quasi-equilibrium for (J_1, J_2) in $\mathcal{U}_1 \times \mathcal{U}_2$.*
- *Assume moreover (1.13). For any α satisfying (1.12), there exists a unique Nash equilibrium \widehat{g}_α for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$. Moreover it is the unique Nash quasi-equilibrium for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$.*

As for the tridimensional case, if M is not large enough, it is not clear that we have the existence or the uniqueness of Nash equilibria or if a Nash quasi-equilibrium is a Nash equilibrium. Below, we state general results of convergences that simplify for M large enough.

Proposition 1.9. *Assume (1.34). Let us consider a family of Nash equilibria $\widehat{g}_\alpha = (\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)})$ for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$. Then, there exists a Nash equilibrium $\widehat{g} = (\widehat{g}^{(1)}, \widehat{g}^{(2)})$ for (J_1, J_2) in $\mathcal{U}_1 \times \mathcal{U}_2$ and a subsequence $\alpha \rightarrow \infty$, such that we have the convergences (1.22), (1.23)–(1.25), (1.27)–(1.29), and*

$$\partial_t u_{\alpha, \widehat{g}_\alpha} \rightharpoonup \partial_t u_{\widehat{g}} \quad \text{weakly in } L^2(0, T; V_0^{-1}), \quad (1.35)$$

$$\partial_t \varphi_{\alpha, \widehat{g}_\alpha}^{(i)} \rightharpoonup \partial_t \varphi_{\widehat{g}}^{(i)} \quad \text{weakly in } L^2(0, T; V_0^{-1}). \quad (1.36)$$

Proposition 1.10. *Assume (1.34). Let us consider a family of Nash quasi-equilibria $\widehat{g}_\alpha = (\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)})$ for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$. Then, there exists a Nash quasi-equilibrium $\widehat{g} = (\widehat{g}^{(1)}, \widehat{g}^{(2)})$ for (J_1, J_2) in $\mathcal{U}_1 \times \mathcal{U}_2$ and a subsequence $\alpha \rightarrow \infty$, such that we have the convergences (1.22), (1.23)–(1.25), (1.27)–(1.29) and (1.35)–(1.36).*

The proofs of these results are completely similar and easier than in the tridimensional-case, and we thus skip them.

The paper is structured as follows. In Section 2, we establish preliminaries results, which are necessary for the development of the following sections. In Section 3, we consider a simplified case where we replace the Navier-Stokes system by the linear Oseen systems. This part can be seen as a warm-up before the proof of the main result. We show the existence and uniqueness of Nash equilibrium and Nash quasi-equilibrium and prove the asymptotic behavior of the Nash controls and states (direct and adjoint), when the parameter $\alpha \rightarrow \infty$. In Section 4, we go back to the nonlinear problem and we show estimates on the direct and adjoint states for a Navier-Stokes systems with slip boundary conditions. Section 5 is devoted to the proof of the existence of Nash quasi-equilibria for the Navier-Stokes system. Finally, in Section 6, we prove the main results of this paper.

2 Preliminaries

2.1 Some useful inequalities

In what follows, we use many times the following Sobolev embeddings valid since $d = 2, 3$:

$$H^1(\Omega) \hookrightarrow L^6(\Omega), \quad W^{1,6}(\Omega) \subset L^\infty(\Omega). \quad (2.1)$$

We recall the Korn inequality (see, for instance, [26]): there exists $C > 0$ such that for any $w \in H^1(\Omega)^d$,

$$\|w\|_{H^1(\Omega)^d}^2 \leq C \left(\|w\|_{L^2(\Omega)^d}^2 + \|\mathbb{D}w\|_{L^2(\Omega)^{d \times d}}^2 \right). \quad (2.2)$$

In order to estimate the nonlinear term in the Navier-Stokes system, we have the following standard result:

Lemma 2.1. *There exists a positive constant $C > 0$ such that for any $t > 0$ and for any*

$$v \in L^2(0, t; H^1(\Omega)^d) \cap L^\infty(0, t; L^2(\Omega)^d), \quad u \in L^2(0, t; W^{1,6}(\Omega)^d),$$

we have the estimate

$$\left| \int_0^t \int_\Omega ((w \cdot \nabla)u) \cdot w \, dx \, ds \right| \leq C \int_0^t \left(\|u\|_{W^{1,6}(\Omega)^d} + \|u\|_{W^{1,6}(\Omega)^d}^2 \right) \|w\|_{L^2(\Omega)^d}^2 \, ds + \int_0^t \int_\Omega |\mathbb{D}w|^2 \, dx \, ds.$$

Proof. First, we use a Hölder inequality:

$$\left| \int_0^t \int_\Omega ((w \cdot \nabla)u) \cdot w \, dx \, ds \right| \leq C \int_0^t \|\nabla u\|_{L^6(\Omega)^{d \times d}} \|w\|_{L^6(\Omega)^d} \|w\|_{L^2(\Omega)^d} \, ds.$$

Then, using the Sobolev embedding in (2.1) and the Korn inequality (2.2), we deduce

$$\left| \int_0^t \int_\Omega ((w \cdot \nabla)u) \cdot w \, dx \, ds \right| \leq C \int_0^t \|\nabla u\|_{L^6(\Omega)^{d \times d}} \left(\|w\|_{L^2(\Omega)^d} + \|\mathbb{D}w\|_{L^2(\Omega)^{d \times d}} \right) \|w\|_{L^2(\Omega)^d} \, ds.$$

Combining the above estimate with a Young inequality, we conclude the proof of this lemma. \square

For the next result, we recall that V_ν^0 is defined by (1.20).

Lemma 2.2. *There exists a constant $C > 0$ independent of α such that for any $w \in H^2(\Omega)^d \cap V_\nu^0$, we have*

$$\|w\|_{W^{1,6}(\Omega)^d} \leq C \left(\|\mathbb{P}\Delta w\|_{L^2(\Omega)^d} + \|[2\mathbb{D}(w)\nu + \alpha w]_\tau\|_{H^{1/2}(\partial\Omega)^d} \right).$$

Proof. We write that w is solution of the following Stokes system:

$$\begin{cases} -\Delta w + \nabla Q = -\mathbb{P}\Delta w & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \\ w \cdot \nu = 0 & \text{on } \partial\Omega, \\ [2\mathbb{D}(w)\nu + \alpha w]_\tau = j_\tau & \text{on } \partial\Omega, \end{cases}$$

where $j := 2\mathbb{D}(w)\nu + \alpha w$. Then, we obtain the result by applying Theorem 2.2 in [2] (see also [1]). \square

Since \mathcal{U}_i is a nonempty closed convex subset of $L^2(0, T; L^2(\omega_i))^d$, we can consider the projection $\mathcal{P}_{\mathcal{U}_i}$ onto \mathcal{U}_i . Let us consider an element $e^{(i)} \in \mathcal{U}_i$. Then we have the relation for any $g^{(i)} \in L^2(0, T; L^2(\omega_i))^d$:

$$\left\| \mathcal{P}_{\mathcal{U}_i} g^{(i)} - g^{(i)} \right\|_{L^2(0, T; L^2(\omega_i))^d} \leq \left\| e^{(i)} - g^{(i)} \right\|_{L^2(0, T; L^2(\omega_i))^d}$$

and in particular, there exists a constant $C > 0$ such that for any $g^{(i)} \in L^2(0, T; L^2(\omega_i))^d$

$$\left\| \mathcal{P}_{\mathcal{U}_i} g^{(i)} \right\|_{L^2(0, T; L^2(\omega_i))^d} \leq C \left(1 + \left\| g^{(i)} \right\|_{L^2(0, T; L^2(\omega_i))^d} \right). \quad (2.3)$$

2.2 Derivatives of the states

As we recall it in the introduction, it is classical that for $r > 0$ small enough, (1.9)-(1.10) imply the existence and uniqueness of a strong solution (u_g, p_g) of (1.1) for any $g = (g^{(1)}, g^{(2)}) \in \mathcal{U}_1 \times \mathcal{U}_2$. We can thus consider the control-to-state mapping

$$\mathcal{L} : g = (g^{(1)}, g^{(2)}) \in \mathcal{U}_1 \times \mathcal{U}_2 \mapsto (u_g, p_g).$$

One can show (see, for instance [18]) that \mathcal{L} is Gateaux-differentiable in any direction $h \in L^2(0, T; L^2(\omega_i))^d$ and the corresponding derivative $(w_g^{(i)}, \pi_{w_g^{(i)}}) := D_i \mathcal{L}(g)(h)$ is the solution of

$$\begin{cases} \partial_t w_g^{(i)} + (u_g \cdot \nabla) w_g^{(i)} + (w_g^{(i)} \cdot \nabla) u_g - \operatorname{div} \mathbb{T}(w_g^{(i)}, \pi_{w_g^{(i)}}) = h \mathbf{1}_{\omega_i} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} w_g^{(i)} = 0 & \text{in } (0, T) \times \Omega, \\ w_g^{(i)} = 0 & \text{on } (0, T) \times \partial\Omega, \\ w_g^{(i)}(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

This shows that J_i defined by (1.2) is Gateaux-differentiable in any direction $h \in L^2(0, T; L^2(\omega_i))^d$ and the corresponding derivative writes

$$D_i J_i(g)(h) = \int_0^T \int_{\mathcal{O}_i} (u_g - z^{(i)}) \cdot w_g^{(i)} dx dt + M \int_0^T \int_{\omega_i} g^{(i)} \cdot h dx dt.$$

Note that we have the following useful relation with the adjoint system (1.18):

$$\int_0^T \int_{\mathcal{O}_i} (u_g - z^{(i)}) \cdot w_g^{(i)} dx dt = \int_0^T \int_{\omega_i} \varphi_g^{(i)} \cdot h dx dt.$$

Therefore,

$$D_i J_i(g)(h) = \int_0^T \int_{\omega_i} (\varphi_g^{(i)} + M g^{(i)}) \cdot h dx dt. \quad (2.4)$$

In particular, the relations in (1.5) write

$$\int_0^T \int_{\omega_i} (\varphi_{\hat{g}}^{(i)} + M \hat{g}^{(i)}) \cdot (g^{(i)} - \hat{g}^{(i)}) dx dt \geq 0 \quad (g^{(i)} \in \mathcal{U}_i).$$

We have shown the following result:

Lemma 2.3. *The couple $(\widehat{g}^{(1)}, \widehat{g}^{(2)})$ is a Nash quasi-equilibrium for (J_1, J_2) if and only if*

$$\left(\widehat{g}^{(1)}, \widehat{g}^{(2)}\right) = \left(\mathcal{P}_{\mathcal{U}_1} \left(-\frac{\left(\varphi_{\widehat{g}}^{(1)}\right)_{|\omega_1}}{M} \right), \mathcal{P}_{\mathcal{U}_2} \left(-\frac{\left(\varphi_{\widehat{g}}^{(2)}\right)_{|\omega_2}}{M} \right)\right) \quad (2.5)$$

where $\mathcal{P}_{\mathcal{U}_i}$ is the projection from $L^2(0, T; L^2(\omega_i))^d$ onto \mathcal{U}_i .

We can now consider the control-to-adjoint state mapping

$$\mathcal{M}^{(i)} : g = \left(g^{(1)}, g^{(2)}\right) \in \mathcal{U}_1 \times \mathcal{U}_2 \mapsto \left(\varphi_g^{(i)}, \pi_{\varphi_g^{(i)}}\right).$$

One can show that $\mathcal{M}^{(i)}$ is Gateaux-differentiable in any direction $h \in L^2(0, T; L^2(\omega_i))^d$ and the corresponding derivative $\left(\psi_g^{(i)}, \pi_{\psi_g^{(i)}}\right) := D_i \mathcal{M}^{(i)}(g)(h)$ is the solution of

$$\left\{ \begin{array}{l} -\partial_t \psi_g^{(i)} - (u_g \cdot \nabla) \psi_g^{(i)} + (\nabla u_g)^\top \psi_g^{(i)} - \operatorname{div} \mathbb{T} \left(\psi_g^{(i)}, \pi_{\psi_g^{(i)}} \right) \\ \quad = (w_g^{(i)} \cdot \nabla) \varphi_g^{(i)} - \left(\nabla w_g^{(i)} \right)^\top \varphi_g^{(i)} + w_g^{(i)} \mathbf{1}_{\mathcal{O}_i} \quad \text{in } (0, T) \times \Omega, \\ \quad \operatorname{div} \psi_g^{(i)} = 0 \quad \text{in } (0, T) \times \Omega, \\ \quad \psi_g^{(i)} = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ \quad \psi_g^{(i)}(T, \cdot) = 0 \quad \text{in } \Omega. \end{array} \right.$$

Using this property and (2.4), we can differentiate J_i a second time and we obtain

$$D_i^2 J_i(g)(h, h) = \int_0^T \int_{\omega_i} \left(\psi_g^{(i)} + Mh \right) \cdot h \, dx \, dt.$$

This formula is the cornerstone of the proof of the strict convexity that will be done in Section 6. We have similar formulas and results for the case of the Navier slip boundary conditions. We can define the control-to-state mapping and the control-to-adjoint state mappings:

$$\mathcal{L}_\alpha : g = \left(g^{(1)}, g^{(2)}\right) \in \mathcal{U}_1 \times \mathcal{U}_2 \mapsto (u_{\alpha, g}, p_{\alpha, g}), \quad \mathcal{M}_\alpha^{(i)} : g = \left(g^{(1)}, g^{(2)}\right) \in \mathcal{U}_1 \times \mathcal{U}_2 \mapsto \left(\varphi_{\alpha, g}^{(i)}, \pi_{\varphi_{\alpha, g}^{(i)}}\right).$$

They are Gateaux-differentiable in any direction $h \in L^2(0, T; L^2(\omega_i))^d$ and the corresponding derivatives

$$\left(w_{\alpha, g}^{(i)}, \pi_{w_{\alpha, g}^{(i)}}\right) := D_i \mathcal{L}_\alpha(g)(h) \quad \text{and} \quad \left(\psi_{\alpha, g}^{(i)}, \pi_{\psi_{\alpha, g}^{(i)}}\right) := D_i \mathcal{M}_\alpha^{(i)}(g)(h)$$

are respectively the solutions of

$$\left\{ \begin{array}{l} \partial_t w_{\alpha, g}^{(i)} + (u_{\alpha, g} \cdot \nabla) w_{\alpha, g}^{(i)} + (w_{\alpha, g}^{(i)} \cdot \nabla) u_{\alpha, g} - \operatorname{div} \mathbb{T}(w_{\alpha, g}^{(i)}, \pi_{w_{\alpha, g}^{(i)}}) = h \mathbf{1}_{\omega_i} \quad \text{in } (0, T) \times \Omega, \\ \quad \operatorname{div} w_{\alpha, g}^{(i)} = 0 \quad \text{in } (0, T) \times \Omega, \\ \quad w_{\alpha, g}^{(i)} \cdot \nu = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ \quad \left[2 \left(\mathbb{D} w_{\alpha, g}^{(i)} \right) \nu + \alpha w_{\alpha, g}^{(i)} \right]_\tau = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ \quad w_{\alpha, g}^{(i)}(0, \cdot) = 0 \quad \text{in } \Omega, \end{array} \right. \quad (2.6)$$

and

$$\left\{ \begin{array}{l} -\partial_t \psi_{\alpha, g}^{(i)} - (u_{\alpha, g} \cdot \nabla) \psi_{\alpha, g}^{(i)} + (\nabla u_{\alpha, g})^\top \psi_{\alpha, g}^{(i)} - \operatorname{div} \mathbb{T} \left(\psi_{\alpha, g}^{(i)}, \pi_{\psi_{\alpha, g}^{(i)}} \right) \\ \quad = (w_{\alpha, g}^{(i)} \cdot \nabla) \varphi_{\alpha, g}^{(i)} - \left(\nabla w_{\alpha, g}^{(i)} \right)^\top \varphi_{\alpha, g}^{(i)} + w_{\alpha, g}^{(i)} \mathbf{1}_{\mathcal{O}_i} \quad \text{in } (0, T) \times \Omega, \\ \quad \operatorname{div} \psi_{\alpha, g}^{(i)} = 0 \quad \text{in } (0, T) \times \Omega, \\ \quad \psi_{\alpha, g}^{(i)} \cdot \nu = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ \quad \left[2 \left(\mathbb{D} \psi_{\alpha, g}^{(i)} \right) \nu + (\alpha + b \cdot \nu) \psi_{\alpha, g}^{(i)} \right]_\tau = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ \quad \psi_{\alpha, g}^{(i)}(T, \cdot) = 0 \quad \text{in } \Omega. \end{array} \right. \quad (2.7)$$

We have the following result

Lemma 2.4. *The couple $(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)})$ is a Nash quasi-equilibrium for $(J_{1,\alpha}, J_{2,\alpha})$ if and only if*

$$(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)}) = \left(\mathcal{P}_{\mathcal{U}_1} \left(-\frac{(\varphi_{\alpha, \widehat{g}_\alpha^{(1)}})_{|\omega_1}}{M} \right), \mathcal{P}_{\mathcal{U}_2} \left(-\frac{(\varphi_{\alpha, \widehat{g}_\alpha^{(2)}})_{|\omega_2}}{M} \right) \right). \quad (2.8)$$

Moreover, $J_{i,\alpha}$ is two times Gateaux-differentiable and

$$D_i^2 J_{i,\alpha}(g)(h, h) = \int_0^T \int_{\omega_i} (\psi_{\alpha, g}^{(i)} + Mh) \cdot h \, dx \, dt. \quad (2.9)$$

3 The linear case

In this section, we consider a simplified case, where we replace the (nonlinear) Navier-Stokes system by the following linear Oseen systems. We can work here for $d = 2$ or $d = 3$.

$$\begin{cases} \partial_t u + (U \cdot \nabla)u + (u \cdot \nabla)U - \operatorname{div} \mathbb{T}(u, p) = f + g^{(1)}1_{\omega_1} + g^{(2)}1_{\omega_2} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0, T) \times \Omega, \\ u = b & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = a & \text{in } \Omega, \end{cases} \quad (3.1)$$

and

$$\begin{cases} \partial_t u + (U \cdot \nabla)u + (u \cdot \nabla)U - \operatorname{div} \mathbb{T}(u, p) = f + g^{(1)}1_{\omega_1} + g^{(2)}1_{\omega_2} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0, T) \times \Omega, \\ u \cdot \nu = b \cdot \nu & \text{on } (0, T) \times \partial\Omega, \\ [2(\mathbb{D}u)\nu + \alpha(u - b)]_\tau = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = a & \text{in } \Omega, \end{cases} \quad (3.2)$$

where

$$U \in L^2(0, T; H^2(\Omega))^d \cap H^1(0, T; L^2(\Omega)^d), \quad \operatorname{div} U = 0 \quad \text{in } (0, T) \times \Omega.$$

In that case, there are several simplifications with respect to the nonlinear case. First, we can work with weak solutions and we can weaken the hypotheses on the data. More precisely, we can consider the same hypotheses on the data than in the bidimensional nonlinear case: we assume relations (1.31), (1.32) and (1.33) instead of (1.6) and (1.7).

We also replace the relation (1.12) by

$$\alpha \geq \|U\|_{L^\infty(0, T; L^\infty(\partial\Omega))^d} + 1 \quad (3.3)$$

and the relation (1.13) by

$$U \in L^\infty(0, T; L^\infty(\partial\Omega))^d.$$

The weak solutions u_g of (3.1) are defined as functions satisfying

$$u_g \in L^2(0, T; H^1(\Omega)^d) \cap C^0([0, T]; L^2(\Omega)^d) \cap H^1(0, T; V_0^{-1}),$$

$$\begin{cases} \operatorname{div} u_g = 0 & \text{in } (0, T) \times \Omega, \\ u_g = b & \text{on } (0, T) \times \partial\Omega, \end{cases}$$

and

$$\begin{aligned} & - \int_0^T \int_\Omega \partial_t \varphi \cdot u_g \, dx \, dt - \int_0^T \int_\Omega [(U \cdot \nabla)\varphi] \cdot u_g \, dx \, dt + \int_0^T \int_\Omega [(u_g \cdot \nabla)U] \cdot \varphi \, dx \, dt \\ & + \int_0^T \int_\Omega 2\mathbb{D}(\varphi) : \mathbb{D}(u_g) \, dx \, dt = \int_0^T \int_\Omega f \cdot \varphi \, dx \, dt + \int_\Omega \varphi(0, \cdot) \cdot a \, dx \\ & \quad + \int_0^T \int_{\omega_1} g^{(1)} \cdot \varphi \, dx \, dt + \int_0^T \int_{\omega_2} g^{(2)} \cdot \varphi \, dx \, dt, \end{aligned}$$

for any $\varphi \in C_c^1([0, T]; V_0^1)$. We recall that V_0^1 is defined by (1.21).

We define similarly the weak solutions $u_{\alpha, g}$ of (3.2): they are the functions satisfying

$$u_{\alpha, g} \in L^2(0, T; H^1(\Omega)^d) \cap C^0([0, T]; L^2(\Omega)^d) \cap H^1(0, T; V_\nu^{-1}),$$

$$\begin{cases} \operatorname{div} u_{\alpha, g} = 0 & \text{in } (0, T) \times \Omega, \\ u_{\alpha, g} \cdot \nu = b \cdot \nu & \text{on } (0, T) \times \partial\Omega, \end{cases}$$

and

$$\begin{aligned} & - \int_0^T \int_\Omega \partial_t \varphi \cdot u_{\alpha, g} \, dx \, dt - \int_0^T \int_\Omega [(U \cdot \nabla) \varphi] \cdot u_{\alpha, g} \, dx \, dt + \int_0^T \int_\Omega [(u_{\alpha, g} \cdot \nabla) U] \cdot \varphi \, dx \, dt \\ & + \int_0^T \int_\Omega 2\mathbb{D}(\varphi) : \mathbb{D}(u_{\alpha, g}) \, dx \, dt + \int_0^T \int_{\partial\Omega} [(U \cdot \nu)(u_{\alpha, g})_\tau + \alpha(u_{\alpha, g} - b)_\tau] \cdot \varphi_\tau \, d\gamma \, dt \\ & = \int_0^T \int_\Omega f \cdot \varphi \, dx \, dt + \int_\Omega \varphi(0, \cdot) \cdot a \, dx \\ & + \int_0^T \int_{\omega_1} g^{(1)} \cdot \varphi \, dx \, dt + \int_0^T \int_{\omega_2} g^{(2)} \cdot \varphi \, dx \, dt, \end{aligned}$$

for any $\varphi \in C_c^1([0, T]; V_\nu^1)$ where

$$V_\nu^1 := \{u \in H^1(\Omega)^d; \operatorname{div} u = 0, \quad u \cdot \nu = 0 \text{ on } \partial\Omega\},$$

$$V_\nu^{-1} := (V_\nu^1)'$$

We also introduce the functionals

$$J_i(g^{(1)}, g^{(2)}) := \frac{1}{2} \int_0^T \int_{\mathcal{O}_i} |u_g - z^{(i)}|^2 \, dx \, dt + \frac{M}{2} \int_0^T \int_{\omega_i} |g^{(i)}|^2 \, dx \, dt \quad (3.4)$$

and

$$J_{i, \alpha}(g^{(1)}, g^{(2)}) := \frac{1}{2} \int_0^T \int_{\mathcal{O}_i} |u_{\alpha, g} - z^{(i)}|^2 \, dx \, dt + \frac{M}{2} \int_0^T \int_{\omega_i} |g^{(i)}|^2 \, dx \, dt. \quad (3.5)$$

As for the nonlinear case, we can define the Nash equilibrium and the Nash quasi-equilibrium for $(J_{1, \alpha}, J_{2, \alpha})$ and for (J_1, J_2) as in Definition 1.1 and Definition 1.2.

Theorem 3.1. *There exists $M_0 > 0$ such that for any $M \geq M_0$ and for any α satisfying (3.3), there exists a unique Nash equilibrium $(\hat{g}_\alpha^{(1)}, \hat{g}_\alpha^{(2)})$ for $(J_{1, \alpha}, J_{2, \alpha})$. There exists also a unique Nash equilibrium $(\hat{g}^{(1)}, \hat{g}^{(2)})$ for (J_1, J_2) .*

In order to show the above result, we introduce the adjoint systems associated with (3.1) and with (3.2):

$$\begin{cases} -\partial_t \varphi^{(i)} - (U \cdot \nabla) \varphi^{(i)} + (\nabla U)^\top \varphi^{(i)} - \operatorname{div} \mathbb{T}(\varphi^{(i)}, \pi_{\varphi^{(i)}}) = (u_g - z^{(i)}) 1_{\mathcal{O}_i} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} \varphi^{(i)} = 0 & \text{in } (0, T) \times \Omega, \\ \varphi^{(i)} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \varphi^{(i)}(T, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (3.6)$$

$$\begin{cases} -\partial_t \varphi^{(i)} - (U \cdot \nabla) \varphi^{(i)} + (\nabla U)^\top \varphi^{(i)} - \operatorname{div} \mathbb{T}(\varphi^{(i)}, \pi_{\varphi^{(i)}}) = (u_{\alpha, g} - z^{(i)}) 1_{\mathcal{O}_i} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} \varphi^{(i)} = 0 & \text{in } (0, T) \times \Omega, \\ \varphi^{(i)} \cdot \nu = 0 & \text{on } (0, T) \times \partial\Omega, \\ [2(\mathbb{D}\varphi^{(i)}) \nu + (\alpha + U \cdot \nu) \varphi^{(i)}]_\tau = 0 & \text{on } (0, T) \times \partial\Omega, \\ \varphi^{(i)}(T, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (3.7)$$

In what follows, we denote by $\varphi_g^{(i)}$ and $\varphi_{\alpha, g}^{(i)}$ the solutions of the above systems. As in the nonlinear case, we can use these adjoint systems to characterize the Nash equilibrium as the Nash quasi-equilibrium. We show below the existence and uniqueness of these quasi-equilibria.

Proposition 3.2. *There exists $M_0 > 0$ such that if $M \geq M_0$, for any α satisfying (3.3), then there exists a unique $\widehat{g}_\alpha \in \mathcal{U}_1 \times \mathcal{U}_2$ such that*

$$\left(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)}\right) = \left(\mathcal{P}_{\mathcal{U}_1} \left(-\frac{\left(\varphi_{\alpha, \widehat{g}_\alpha}^{(1)}\right)_{|\omega_1}}{M}\right), \mathcal{P}_{\mathcal{U}_2} \left(-\frac{\left(\varphi_{\alpha, \widehat{g}_\alpha}^{(2)}\right)_{|\omega_2}}{M}\right)\right). \quad (3.8)$$

There exists a unique $\widehat{g} \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$\left(\widehat{g}^{(1)}, \widehat{g}^{(2)}\right) = \left(\mathcal{P}_{\mathcal{U}_1} \left(-\frac{\left(\varphi_{\widehat{g}}^{(1)}\right)_{|\omega_1}}{M}\right), \mathcal{P}_{\mathcal{U}_2} \left(-\frac{\left(\varphi_{\widehat{g}}^{(2)}\right)_{|\omega_2}}{M}\right)\right). \quad (3.9)$$

Proof. We only perform the proof for the Navier boundary conditions since it is completely similar for the Dirichlet boundary conditions. We consider the following map

$$\Lambda_\alpha : \mathcal{U}_1 \times \mathcal{U}_2 \rightarrow \mathcal{U}_1 \times \mathcal{U}_2, \quad \left(\mathcal{P}_{\mathcal{U}_1} \left(-\frac{\left(\varphi_{\alpha, g}^{(1)}\right)_{|\omega_1}}{M}\right), \mathcal{P}_{\mathcal{U}_2} \left(-\frac{\left(\varphi_{\alpha, g}^{(2)}\right)_{|\omega_2}}{M}\right)\right). \quad (3.10)$$

We are going to show that for M large enough (independently of α) Λ_α is a strict contraction (and thus admits a unique fixed point). Let us consider

$$g_\#, g_b \in \mathcal{U}_1 \times \mathcal{U}_2$$

and let us denote by $(u_{\alpha, \#}, \varphi_{\alpha, \#}^{(1)}, \varphi_{\alpha, \#}^{(2)})$ and $(u_{\alpha, b}, \varphi_{\alpha, b}^{(1)}, \varphi_{\alpha, b}^{(2)})$ the corresponding solutions of (3.2), (3.7). We set

$$u := u_{\alpha, \#} - u_{\alpha, b}, \quad \varphi^{(i)} := \varphi_{\alpha, \#}^{(i)} - \varphi_{\alpha, b}^{(i)}, \quad g := g_\# - g_b$$

that satisfy the following systems

$$\left\{ \begin{array}{l} \partial_t u + (U \cdot \nabla)u + (u \cdot \nabla)U - \operatorname{div} \mathbb{T}(u, p) = g^{(1)}1_{\omega_1} + g^{(2)}1_{\omega_2} \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0 \quad \text{in } (0, T) \times \Omega, \\ u \cdot \nu = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ [2(\mathbb{D}u)\nu + \alpha u]_\tau = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = 0 \quad \text{in } \Omega, \end{array} \right. \quad (3.11)$$

$$\left\{ \begin{array}{l} -\partial_t \varphi^{(i)} - (U \cdot \nabla)\varphi^{(i)} + (\nabla U)^\top \varphi^{(i)} - \operatorname{div} \mathbb{T}(\varphi^{(i)}, \pi_{\varphi^{(i)}}) = u1_{\mathcal{O}_i} \quad \text{in } (0, T) \times \Omega, \\ \operatorname{div} \varphi^{(i)} = 0 \quad \text{in } (0, T) \times \Omega, \\ \varphi^{(i)} \cdot \nu = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ [2(\mathbb{D}\varphi^{(i)})\nu + (\alpha + U \cdot \nu)\varphi^{(i)}]_\tau = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ \varphi^{(i)}(T, \cdot) = 0 \quad \text{in } \Omega. \end{array} \right. \quad (3.12)$$

Then, using the property of the projection, we deduce from (3.10) the existence of a constant $C > 0$ such that

$$\|\Lambda_\alpha(g_\#) - \Lambda_\alpha(g_b)\|_{L^2(0, T; L^2(\omega_1) \times L^2(\omega_2))^d} \leq \frac{C}{M} \left\| \left(\varphi^{(1)}, \varphi^{(2)}\right) \right\|_{L^2(0, T; L^2(\Omega))^{2d}} \quad (3.13)$$

Multiplying the first equation of (3.12) by $\varphi^{(i)}$, we deduce the existence of a constant $C > 0$ such that for α satisfying (3.3),

$$\left\| \left(\varphi^{(1)}, \varphi^{(2)}\right) \right\|_{L^2(0, T; H^1(\Omega))^{2d}} + \left\| \left(\varphi^{(1)}, \varphi^{(2)}\right) \right\|_{L^\infty(0, T; L^2(\Omega))^{2d}} \leq C \|u\|_{L^2(0, T; L^2(\Omega))^d}. \quad (3.14)$$

Multiplying the first equation of (3.11) by u , we deduce the existence of a constant $C > 0$ such that for α satisfying (3.3),

$$\|u\|_{L^2(0, T; H^1(\Omega))^d} + \|u\|_{L^\infty(0, T; L^2(\Omega))^d} \leq C \|g\|_{L^2(0, T; L^2(\Omega))^{2d}}. \quad (3.15)$$

Combining (3.13), (3.14), (3.15), we conclude the proof. \square

We are now in position to prove Theorem 3.1:

Proof of Theorem 3.1. We only show the proof in the case of the Navier slip boundary conditions, the proof in the case of the Dirichlet boundary condition is similar. We are going to show that $(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)})$ given by (3.8) in Proposition 3.2 is the unique Nash equilibrium for $(J_{1,\alpha}, J_{2,\alpha})$.

Assume $\theta \in [0, 1]$ and $g^{(1)} \in \mathcal{U}_1$. Let us also consider the system

$$\left\{ \begin{array}{ll} \partial_t w + (U \cdot \nabla)w + (w \cdot \nabla)U - \operatorname{div} \mathbb{T}(w, \pi_w) = (g^{(1)} - \widehat{g}_\alpha^{(1)}) 1_{\omega_1} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} w = 0 & \text{in } (0, T) \times \Omega, \\ w \cdot \nu = 0 & \text{on } (0, T) \times \partial\Omega, \\ [2(\mathbb{D}w)\nu + \alpha w]_\tau = 0 & \text{on } (0, T) \times \partial\Omega, \\ w(0, \cdot) = 0 & \text{in } \Omega. \end{array} \right. \quad (3.16)$$

Then, using the linearity of the Oseen system (3.2), we have the following relation

$$u_{\alpha, (\theta g^{(1)} + (1-\theta)\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)})} = u_{\alpha, \widehat{g}_\alpha} + \theta w.$$

In particular, from (3.5),

$$\begin{aligned} J_{1,\alpha}(\theta g^{(1)} + (1-\theta)\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)}) &= J_{1,\alpha}(\widehat{g}_\alpha) + \theta \int_0^T \int_{\mathcal{O}_1} (u_{\alpha, \widehat{g}} - z^{(1)}) \cdot w \, dx \, dt + \frac{1}{2} \int_0^T \int_{\mathcal{O}_1} \theta^2 |w|^2 \, dx \, dt \\ &\quad + M\theta \int_0^T \int_{\omega_1} \widehat{g}_\alpha^{(1)} \cdot (g^{(1)} - \widehat{g}_\alpha^{(1)}) \, dx \, dt + \frac{M}{2} \int_0^T \int_{\omega_1} \theta^2 |g^{(1)} - \widehat{g}_\alpha^{(1)}|^2 \, dx \, dt. \end{aligned} \quad (3.17)$$

Then multiplying the first equation of (3.16) by $\widehat{\varphi}_\alpha^{(1)}$ and the first equation of (3.7) by w , we obtain after some integrations by parts that

$$\int_0^T \int_{\mathcal{O}_1} (u_{\alpha, \widehat{g}} - z^{(1)}) \cdot w \, dx \, dt = \int_0^T \int_{\omega_1} \widehat{\varphi}_\alpha^{(1)} \cdot (g^{(1)} - \widehat{g}_\alpha^{(1)}) \, dx \, dt. \quad (3.18)$$

We deduce from the above relation and (3.17) that

$$J_{1,\alpha}(\theta g^{(1)} + (1-\theta)\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)}) \geq J_{1,\alpha}(\widehat{g}_\alpha) + M\theta \int_0^T \int_{\omega_1} \left(\widehat{g}_\alpha^{(1)} + \frac{\widehat{\varphi}_\alpha^{(1)}}{M} \right) \cdot (g^{(1)} - \widehat{g}_\alpha^{(1)}) \, dx \, dt.$$

Using (3.9), the above relation implies

$$J_{1,\alpha}(\theta g^{(1)} + (1-\theta)\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)}) \geq J_{1,\alpha}(\widehat{g}_\alpha)$$

and thus

$$J_{1,\alpha}(g^{(1)}, \widehat{g}_\alpha^{(2)}) \geq J_{1,\alpha}(\widehat{g}_\alpha).$$

With a similar proof for $J_{2,\alpha}$, we deduce that \widehat{g}_α is a Nash equilibrium.

Conversely, if \widehat{g}_α is a Nash equilibrium, then we deduce from (3.17) and (3.18) that

$$\int_0^T \int_{\omega_1} \left(\widehat{g}_\alpha^{(1)} + \frac{\widehat{\varphi}_\alpha^{(1)}}{M} \right) \cdot (g^{(1)} - \widehat{g}_\alpha^{(1)}) \, dx \, dt \geq 0$$

for any $g^{(1)} \in \mathcal{U}_1$ and this yields that

$$\widehat{g}_\alpha^{(1)} = \mathcal{P}_{\mathcal{U}_1} \left(- \frac{(\widehat{\varphi}_\alpha^{(1)})|_{\omega_1}}{M} \right).$$

By proceeding similarly for $i = 2$, we deduce that \widehat{g}_α is the solution of (3.8) which yields the uniqueness of the Nash equilibrium, by applying Proposition 3.2. \square

Theorem 3.3. *There exists $M_0 > 0$ such that for any $M \geq M_0$, we have convergence as $\alpha \rightarrow \infty$ of the Nash equilibria:*

$$\left(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)}\right) \rightarrow \left(\widehat{g}^{(1)}, \widehat{g}^{(2)}\right) \quad \text{in } L^2(0, T; L^2(\omega_1) \times L^2(\omega_2))^d.$$

Moreover, the corresponding states (direct and adjoint) satisfy (1.23)–(1.25), (1.27)–(1.29) and (1.35)–(1.36).

Proof. First, let us recall that $u_{\alpha, \widehat{g}_\alpha}$ and $\varphi_{\alpha, \widehat{g}_\alpha}$ are the solution of (3.2), (3.7) and (3.8). Setting

$$\widetilde{u} := u_{\alpha, \widehat{g}_\alpha} - b, \quad \widetilde{f} := f - \partial_t b + \Delta b - (U \cdot \nabla)b - (b \cdot \nabla)U,$$

we have

$$\begin{cases} \partial_t \widetilde{u} + (U \cdot \nabla)\widetilde{u} + (\widetilde{u} \cdot \nabla)U - \operatorname{div} \mathbb{T}(\widetilde{u}, p) = \widetilde{f} + \widehat{g}_\alpha^{(1)} 1_{\omega_1} + \widehat{g}_\alpha^{(2)} 1_{\omega_2} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} \widetilde{u} = 0 & \text{in } (0, T) \times \Omega, \\ \widetilde{u} \cdot \nu = 0 & \text{on } (0, T) \times \partial\Omega, \\ [2(\mathbb{D}\widetilde{u})\nu + \alpha\widetilde{u}]_\tau = -[2(\mathbb{D}b)\nu]_\tau & \text{on } (0, T) \times \partial\Omega, \\ \widetilde{u}(0, \cdot) = a - b(0, \cdot) & \text{in } \Omega. \end{cases}$$

By multiplying the first equation of the above system by \widetilde{u} and integrating in time and space, we obtain

$$\begin{aligned} & \frac{1}{2} \int_\Omega |\widetilde{u}(t, \cdot)|^2 dx + \int_0^t \int_\Omega 2|\mathbb{D}\widetilde{u}|^2 dx ds + \int_0^t \int_{\partial\Omega} \alpha |\widetilde{u}_\tau|^2 d\gamma ds \\ & \quad + \int_0^t \int_\Omega (\widetilde{u} \cdot \nabla)U \cdot \widetilde{u} dx ds + \int_0^t \int_{\partial\Omega} U \cdot \nu \frac{|\widetilde{u}|^2}{2} d\gamma \\ & = \int_0^t \int_\Omega \widetilde{f} \cdot \widetilde{u} dx ds + \int_0^t \int_{\omega_1} \widehat{g}_\alpha^{(1)} \cdot \widetilde{u} dx ds + \int_0^t \int_{\omega_2} \widehat{g}_\alpha^{(2)} \cdot \widetilde{u} dx ds \\ & \quad - \int_0^t \int_{\partial\Omega} [2(\mathbb{D}b)\nu]_\tau \cdot \widetilde{u}_\tau d\gamma ds + \frac{1}{2} \int_\Omega |a(x) - b(0, x)|^2 dx. \end{aligned} \quad (3.19)$$

Using Lemma 2.1, there exists a constant $C > 0$ such that

$$\left| \int_0^t \int_\Omega (\widetilde{u} \cdot \nabla)U \cdot \widetilde{u} dx ds \right| \leq C \int_0^t \left(\|U\|_{H^2(\Omega)^d} + \|U\|_{H^2(\Omega)^d}^2 \right) \|\widetilde{u}\|_{L^2(\Omega)^d}^2 ds + \int_0^t \int_\Omega |\mathbb{D}\widetilde{u}|^2 dx ds.$$

Combining the above estimate with (3.19) and the Grönwall inequality, we deduce that there exists a constant $C > 0$ such that

$$\begin{aligned} & \|u_{\alpha, \widehat{g}_\alpha}\|_{L^\infty(0, T; L^2(\Omega))^d}^2 + \|u_{\alpha, \widehat{g}_\alpha}\|_{L^2(0, T; H^1(\Omega))^d}^2 + \alpha \|u_{\alpha, \widehat{g}_\alpha} - b\|_{L^2(0, T; L^2(\partial\Omega))^d}^2 \\ & \leq C \left(1 + \|\widehat{g}_\alpha\|_{L^2(0, T; L^2(\omega_1) \times L^2(\omega_2))^d}^2 \right). \end{aligned} \quad (3.20)$$

Then we multiply the first equation of system (3.7) by $\varphi_{\alpha, \widehat{g}_\alpha}^{(i)}$ and integrating in time and space, we obtain

$$\begin{aligned} & \frac{1}{2} \int_\Omega \left| \varphi_{\alpha, \widehat{g}_\alpha}^{(i)}(t, x) \right|^2 dx + \int_t^T \int_\Omega 2 \left| \mathbb{D}\varphi_{\alpha, \widehat{g}_\alpha}^{(i)} \right|^2 dx ds + \frac{\alpha}{2} \int_t^T \int_{\partial\Omega} \left| \varphi_{\alpha, \widehat{g}_\alpha}^{(i)} \right| d\gamma ds \\ & \leq - \int_t^T \int_\Omega \left((\nabla U)^\top \varphi_{\alpha, \widehat{g}_\alpha}^{(i)} \right) \cdot \varphi_{\alpha, \widehat{g}_\alpha}^{(i)} dx ds + \int_t^T \int_{\mathcal{O}_i} \left(u_{\alpha, \widehat{g}_\alpha} - z^{(i)} \right) \cdot \varphi_{\alpha, \widehat{g}_\alpha}^{(i)} dx ds. \end{aligned}$$

Proceeding as above, we deduce

$$\begin{aligned} & \|\varphi_{\alpha, \widehat{g}_\alpha}\|_{L^\infty(0, T; L^2(\Omega))^d}^2 + \|\varphi_{\alpha, \widehat{g}_\alpha}\|_{L^2(0, T; H^1(\Omega))^d}^2 + \alpha \|\varphi_{\alpha, \widehat{g}_\alpha}\|_{L^2(0, T; L^2(\partial\Omega))^d}^2 \\ & \leq C \left(1 + \|u_{\alpha, \widehat{g}_\alpha}\|_{L^2(0, T; L^2(\Omega))^d}^2 \right). \end{aligned} \quad (3.21)$$

On the other hand, from (3.8) and (2.3), there exists a constant $C > 0$ such that

$$\|\widehat{g}_\alpha\|_{L^2(0,T;L^2(\omega_1)\times L^2(\omega_2))^d}^2 \leq C \left(1 + \frac{1}{M^2} \|\varphi_{\alpha,\widehat{g}_\alpha}\|_{L^2(0,T;L^2(\Omega))^d}^2\right).$$

Combining the above relation with (3.20) and (3.21), we deduce the existence of two constants M_0 and $C > 0$ such that for any $M \geq M_0$ and for any α satisfying (3.3),

$$\begin{aligned} & \|u_{\alpha,\widehat{g}_\alpha}\|_{L^\infty(0,T;L^2(\Omega))^d}^2 + \|u_{\alpha,\widehat{g}_\alpha}\|_{L^2(0,T;H^1(\Omega))^d}^2 + \alpha \|u_{\alpha,\widehat{g}_\alpha} - b\|_{L^2(0,T;L^2(\partial\Omega))^d}^2 \\ & + \|\varphi_{\alpha,\widehat{g}_\alpha}\|_{L^\infty(0,T;L^2(\Omega))^d}^2 + \|\varphi_{\alpha,\widehat{g}_\alpha}\|_{L^2(0,T;H^1(\Omega))^d}^2 + \alpha \|\varphi_{\alpha,\widehat{g}_\alpha}\|_{L^2(0,T;L^2(\partial\Omega))^d}^2 \\ & + \|\widehat{g}_\alpha\|_{L^2(0,T;L^2(\omega_1)\times L^2(\omega_2))^d}^2 \leq C. \end{aligned}$$

We can now show the convergence of the controls and of the state. First we deduce from the above bounds and the linearity of the system the existence of $\widetilde{g} \in \mathcal{U}_1 \times \mathcal{U}_2$ that

$$\widehat{g}_\alpha \rightharpoonup \widetilde{g} \quad \text{in } L^2(0,T;L^2(\omega_1)\times L^2(\omega_2))^d, \quad (3.22)$$

$$u_{\alpha,\widehat{g}_\alpha} \rightharpoonup u_{\widetilde{g}} \quad \text{weakly } * \text{ in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))^d, \quad (3.23)$$

$$\partial_t u_{\alpha,\widehat{g}_\alpha} \rightharpoonup \partial_t u_{\widetilde{g}} \quad \text{weakly in } L^2(0,T;V_0^{-1}), \quad (3.24)$$

$$u_{\alpha,\widehat{g}_\alpha} \rightarrow u_{\widetilde{g}} \quad \text{strongly in } L^2(0,T;L^2(\Omega))^d, \quad (3.25)$$

$$u_{\alpha,\widehat{g}_\alpha} \rightarrow b \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega))^d, \quad (3.26)$$

$$\varphi_{\alpha,\widehat{g}_\alpha}^{(i)} \rightharpoonup \varphi_{\widetilde{g}}^{(i)} \quad \text{weakly } * \text{ in } L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))^d, \quad (3.27)$$

$$\partial_t \varphi_{\alpha,\widehat{g}_\alpha}^{(i)} \rightharpoonup \partial_t \varphi_{\widetilde{g}}^{(i)} \quad \text{weakly in } L^2(0,T;V_0^{-1}), \quad (3.28)$$

$$\varphi_{\alpha,\widehat{g}_\alpha}^{(i)} \rightarrow \varphi_{\widetilde{g}}^{(i)} \quad \text{strongly in } L^2(0,T;L^2(\Omega))^d, \quad (3.29)$$

$$\varphi_{\alpha,\widehat{g}_\alpha}^{(i)} \rightarrow 0 \quad \text{strongly in } L^2(0,T;L^2(\partial\Omega))^d. \quad (3.30)$$

Using the continuity of the projection $\mathcal{P}_{\mathcal{U}_i}$, we deduce from (3.29), (3.8) and (3.9) that

$$\widehat{g}_\alpha \rightarrow \widetilde{g} \quad \text{in } L^2(0,T;L^2(\omega_1)\times L^2(\omega_2))^d,$$

and thus we obtain that $\widetilde{g} = \widehat{g}$, which concludes the proof of the theorem. \square

4 Estimates on the direct and the adjoint states

In this section, and all that follows, we go back to the study of the nonlinear problem, that is the study of the Nash equilibria for (1.1) and (1.11). This section is devoted to several estimates, independent of α of the systems (1.11), (1.19), and their derivatives, (2.6) and (2.7).

We define the following norm:

$$\|u\|_{\mathcal{W}} := \left(\|u\|_{L^\infty(0,T;H^1(\Omega))^d}^2 + \|\mathbb{P}\Delta u\|_{L^2(0,T;L^2(\Omega))^d}^2 + \|u\|_{L^2(0,T;W^{1,6}(\Omega))^d}^2 \right)^{1/2},$$

where we recall that \mathbb{P} is the Leray projection. First we recall a result obtained in [9]:

Proposition 4.1. *Assume $T > 0$ and that (1.16) admits a solution $u^{(0)} \in H^1(0,T;H^2(\Omega))^d$. There exist positive constants r and K such that if (1.10) and (1.17) hold, then for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$ and for any $\alpha \geq \|b\|_{L^\infty(0,T;L^\infty(\partial\Omega))^d} + 1$, system (1.11) admits a unique strong solution*

$$(u_{\alpha,g}, p_{\alpha,g}) \in (H^1(0,T;L^2(\Omega))^d \cap L^2(0,T;H^2(\Omega))^d) \times L^2(0,T;H^1(\Omega)/\mathbb{R}).$$

Moreover,

$$\|u_{\alpha,g}\|_{\mathcal{W}}^2 + \alpha \|u_{\alpha,g} - b\|_{L^\infty(0,T;L^2(\partial\Omega))^d}^2 \leq K. \quad (4.1)$$

Remark 4.2. Note that in [9], we consider a different lifting than $u^{(0)}$ for the nonhomogeneous boundary condition in system (1.11). Nevertheless, the proof is completely similar and the additional regularity hypothesis $u^{(0)} \in H^1(0, T; H^2(\Omega))^d$ only involves a and b here (see (1.16)), which is why we choose to consider such a lifting instead of the lifting in [9].

Let us state now the result for the adjoint system (1.19):

Proposition 4.3. *With the hypotheses of Proposition 4.1, there exists a constant $C > 0$ such that if (1.10) and (1.17) hold, then for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$ and for any $\alpha \geq \|b\|_{L^\infty(0, T; L^\infty(\partial\Omega))^d} + 1$, system (1.19) admits a unique strong solution*

$$\left(\varphi_{\alpha, g}^{(i)}, \pi_{\varphi_{\alpha, g}^{(i)}} \right) \in (H^1(0, T; L^2(\Omega))^d \cap L^2(0, T; H^2(\Omega))^d) \times L^2(0, T; H^1(\Omega)/\mathbb{R}).$$

Moreover,

$$\left\| \varphi_{\alpha, g}^{(i)} \right\|_{\mathcal{W}}^2 + \alpha \left\| \varphi_{\alpha, g}^{(i)} \right\|_{L^\infty(0, T; L^2(\partial\Omega))^d}^2 \leq C e^{CK}, \quad (4.2)$$

where K is the constant appearing in Proposition 4.1.

The proof is similar to the proof of Proposition 4.1 and also to the proof of Proposition 4.5 below, and we thus skip it.

For the system (2.6), we have the following result:

Proposition 4.4. *With the hypotheses of Proposition 4.1, there exists a constant $C > 0$ such that if (1.10) and (1.17) hold, then for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$, for any $\alpha \geq \|b\|_{L^\infty(0, T; L^\infty(\partial\Omega))^d} + 1$, and for any $h \in L^2(0, T; L^2(\omega_i))^d$, system (2.6) admits a unique strong solution*

$$\left(w_{\alpha, g}^{(i)}, \pi_{w_{\alpha, g}^{(i)}} \right) \in (H^1(0, T; L^2(\Omega))^d \cap L^2(0, T; H^2(\Omega))^d) \times L^2(0, T; H^1(\Omega)/\mathbb{R}).$$

Moreover,

$$\left\| w_{\alpha, g}^{(i)} \right\|_{\mathcal{W}}^2 + \alpha \left\| w_{\alpha, g}^{(i)} \right\|_{L^\infty(0, T; L^2(\partial\Omega))^d}^2 \leq C e^{CK} \|h\|_{L^2(0, T; L^2(\omega_i))^d}^2, \quad (4.3)$$

where K is the constant appearing in Proposition 4.1.

Proof. The proof can be done by a standard Galerkin method. The key point in a such a proof is the a priori estimates for the approximated solutions obtained by the Galerkin method. Since these proofs are classical (see, for instance, [30, pp. 255–260, pp. 283–289]) we show here only the a priori estimates on the solutions (without approximation).

We multiply the first equation of (2.6) by $w_{\alpha, g}^{(i)}$ and we integrate in time and space:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \left| w_{\alpha, g}^{(i)} \right|^2 dx + \int_0^t \int_{\Omega} 2 \left| \mathbb{D} w_{\alpha, g}^{(i)} \right|^2 dx ds + \int_0^t \int_{\partial\Omega} \frac{\alpha}{2} \left| w_{\alpha, g}^{(i)} \right|^2 d\gamma ds \\ \leq \int_0^t \int_{\Omega} \left(h 1_{\omega_i} - (w_{\alpha, g}^{(i)} \cdot \nabla) u_{\alpha, g} \right) \cdot w_{\alpha, g}^{(i)} dx ds. \end{aligned} \quad (4.4)$$

Using Lemma 2.1, we have

$$\begin{aligned} \int_0^t \int_{\Omega} \left((w_{\alpha, g}^{(i)} \cdot \nabla) u_{\alpha, g} \right) \cdot w_{\alpha, g}^{(i)} dx ds \\ \leq C \int_0^t \left(\|u_{\alpha, g}\|_{W^{1,6}(\Omega)^d} + \|u_{\alpha, g}\|_{W^{1,6}(\Omega)^d}^2 \right) \left\| w_{\alpha, g}^{(i)} \right\|_{L^2(\Omega)}^2 ds + \int_0^t \int_{\Omega} \left| \mathbb{D} w_{\alpha, g}^{(i)} \right|^2 dx ds. \end{aligned}$$

Combining the above estimate with (4.4) and (4.1) and using the Grönwall lemma, we deduce that

$$\left\| w_{\alpha,g}^{(i)} \right\|_{L^\infty(0,T;L^2(\Omega))^d}^2 + \left\| w_{\alpha,g}^{(i)} \right\|_{L^2(0,T;H^1(\Omega))^d}^2 + \alpha \left\| w_{\alpha,g}^{(i)} \right\|_{L^2(0,T;L^2(\partial\Omega))^d}^2 \leq C \|h\|_{L^2(0,T;L^2(\omega_i))^d}^2 e^{CK}.$$

Then, we multiply the first equation of (2.6) by $-\mathbb{P}\Delta w_{\alpha,g}^{(i)}$ and we integrate in time and space:

$$\begin{aligned} & \int_{\Omega} \left| \mathbb{D}w_{\alpha,g}^{(i)}(t,x) \right|^2 dx + \int_0^t \int_{\Omega} \left| \mathbb{P}\Delta w_{\alpha,g}^{(i)} \right|^2 dx ds + \frac{\alpha}{2} \int_{\partial\Omega} \left| w_{\alpha,g}^{(i)} \right|^2 d\gamma \\ & \leq C \int_0^t \int_{\Omega} \left| (u_{\alpha,g} \cdot \nabla) w_{\alpha,g}^{(i)} + (w_{\alpha,g}^{(i)} \cdot \nabla) u_{\alpha,g} \right|^2 dx ds + C \|h\|_{L^2(0,T;L^2(\omega_i))^d}^2. \end{aligned} \quad (4.5)$$

Then, from Hölder's inequality,

$$\begin{aligned} & \int_0^t \int_{\Omega} \left| (u_{\alpha,g} \cdot \nabla) w_{\alpha,g}^{(i)} + (w_{\alpha,g}^{(i)} \cdot \nabla) u_{\alpha,g} \right|^2 dx ds \\ & \leq C \int_0^t \left(\|u_{\alpha,g}\|_{L^\infty(\Omega)^d}^2 \left\| \nabla w_{\alpha,g}^{(i)} \right\|_{L^2(\Omega)^{d \times d}}^2 + \left\| w_{\alpha,g}^{(i)} \right\|_{L^6(\Omega)^d}^2 \left\| \nabla u_{\alpha,g} \right\|_{L^6(\Omega)^{d \times d}}^2 \right) ds. \end{aligned} \quad (4.6)$$

Applying Lemma 2.2, the Korn inequality and Sobolev embeddings, we deduce from (4.5) and (4.6) that

$$\begin{aligned} & \left\| w_{\alpha,g}^{(i)}(t, \cdot) \right\|_{H^1(\Omega)^d}^2 + \int_0^t \int_{\Omega} \left| \mathbb{P}\Delta w_{\alpha,g}^{(i)} \right|^2 dx ds + \int_0^t \left\| w_{\alpha,g}^{(i)} \right\|_{W^{1,6}(\Omega)^d}^2 ds + \frac{\alpha}{2} \int_{\partial\Omega} \left| w_{\alpha,g}^{(i)} \right|^2 d\gamma \\ & \leq C \int_0^t \|u_{\alpha,g}\|_{W^{1,6}(\Omega)^d}^2 \left\| w_{\alpha,g}^{(i)}(t, \cdot) \right\|_{H^1(\Omega)^d}^2 ds + C \|h\|_{L^2(0,T;L^2(\omega_i))^d}^2. \end{aligned}$$

Thus, combining (4.5), (4.6), and the Grönwall lemma we deduce the result. \square

Proposition 4.5. *With the hypotheses of Proposition 4.1, there exists a constant $C > 0$ such that if (1.10) and (1.17) hold, then for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$, for any $\alpha \geq \|b\|_{L^\infty(0,T;L^\infty(\partial\Omega))^d} + 1$, and for any $h \in L^2(0,T;L^2(\omega_i))^d$, system (2.7) admits a unique strong solution*

$$\left(\psi_{\alpha,g}^{(i)}, \pi_{\psi_{\alpha,g}^{(i)}} \right) \in (H^1(0,T;L^2(\Omega))^d \cap L^2(0,T;H^2(\Omega))^d) \times L^2(0,T;H^1(\Omega)/\mathbb{R}).$$

Moreover,

$$\left\| \psi_{\alpha,g}^{(i)} \right\|_{\mathcal{W}}^2 + \alpha \left\| \psi_{\alpha,g}^{(i)} \right\|_{L^\infty(0,T;L^2(\partial\Omega))^d}^2 \leq C e^{CK} \|h\|_{L^2(0,T;L^2(\omega_i))^d}^2. \quad (4.7)$$

where K is the constant appearing in Proposition 4.1.

Proof. As the proof of Proposition 4.4, the proof of this proposition can be done by a standard Galerkin method where the main point is to show the a priori estimates for the approximated solutions obtained by the Galerkin method. We refer for instance to [30, pp. 255–260, pp. 283–289] where more details are given. Here, we show only the a priori estimates on the solutions (without approximation).

We multiply the first equation of (2.7) by $\psi_{\alpha,g}^{(i)}$

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \left| \psi_{\alpha,g}^{(i)}(t,x) \right|^2 dx + \int_t^T \int_{\Omega} 2 \left| \mathbb{D}\psi_{\alpha,g}^{(i)} \right|^2 dx ds + \int_t^T \int_{\partial\Omega} \frac{\alpha}{2} \left| \psi_{\alpha,g}^{(i)} \right|^2 d\gamma ds \\ & \leq \int_t^T \int_{\Omega} \left((\nabla u_{\alpha,g})^\top \psi_{\alpha,g}^{(i)} \right) \cdot \psi_{\alpha,g}^{(i)} dx ds \\ & + C \int_t^T \int_{\Omega} \left| (w_{\alpha,g}^{(i)} \cdot \nabla) \varphi_{\alpha,g}^{(i)} - (\nabla w_{\alpha,g}^{(i)})^\top \varphi_{\alpha,g}^{(i)} + w_{\alpha,g}^{(i)} 1_{\mathcal{O}_i} \right|^2 dx ds + C \int_t^T \int_{\Omega} \left| \psi_{\alpha,g}^{(i)} \right|^2 dx ds. \end{aligned}$$

From Lemma 2.1, we have

$$\begin{aligned} \int_t^T \int_{\Omega} \left((\nabla u_{\alpha,g})^\top \psi_{\alpha,g}^{(i)} \right) \cdot \psi_{\alpha,g}^{(i)} dx ds \leq C \int_t^T \left(\|u_{\alpha,g}\|_{W^{1,6}(\Omega)^d}^2 + \|u_{\alpha,g}\|_{W^{1,6}(\Omega)^d} \right) \left\| \psi_{\alpha,g}^{(i)} \right\|_{L^2(\Omega)^d}^2 ds \\ + \int_t^T \int_{\Omega} \left| \mathbb{D} \psi_{\alpha,g}^{(i)} \right|^2 dx ds. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_t^T \int_{\Omega} \left| (w_{\alpha,g}^{(i)} \cdot \nabla) \varphi_{\alpha,g}^{(i)} - \left(\nabla w_{\alpha,g}^{(i)} \right)^\top \varphi_{\alpha,g}^{(i)} \right|^2 dx ds \\ \leq C \left(\left\| w_{\alpha,g}^{(i)} \right\|_{L^\infty(0,T;H^1(\Omega)^d)}^2 \left\| \varphi_{\alpha,g}^{(i)} \right\|_{L^2(0,T;W^{1,6}(\Omega)^d)}^2 + \left\| w_{\alpha,g}^{(i)} \right\|_{L^2(0,T;W^{1,6}(\Omega)^d)}^2 \left\| \varphi_{\alpha,g}^{(i)} \right\|_{L^\infty(0,T;H^1(\Omega)^d)}^2 \right). \end{aligned}$$

Applying the Grönwall lemma, we deduce

$$\begin{aligned} \left\| \psi_{\alpha,g}^{(i)} \right\|_{L^\infty(0,T;L^2(\Omega)^d)}^2 + \left\| \psi_{\alpha,g}^{(i)} \right\|_{L^2(0,T;H^1(\Omega)^d)}^2 + \alpha \left\| \psi_{\alpha,g}^{(i)} \right\|_{L^2(0,T;L^2(\partial\Omega)^d)}^2 \\ \leq C \left(\left\| w_{\alpha,g}^{(i)} \right\|_{L^\infty(0,T;H^1(\Omega)^d)}^2 \left\| \varphi_{\alpha,g}^{(i)} \right\|_{L^2(0,T;W^{1,6}(\Omega)^d)}^2 + \left\| w_{\alpha,g}^{(i)} \right\|_{L^2(0,T;W^{1,6}(\Omega)^d)}^2 \left\| \varphi_{\alpha,g}^{(i)} \right\|_{L^\infty(0,T;H^1(\Omega)^d)}^2 \right. \\ \left. + \left\| w_{\alpha,g}^{(i)} \right\|_{L^2(0,T;L^2(\Omega)^d)}^2 \right) \exp \left(C \|u_{\alpha,g}\|_{L^2(0,T;W^{1,6}(\Omega)^d)}^2 \right). \end{aligned}$$

Combining the above relation with (4.1), (4.2) and (4.3), we deduce

$$\left\| \psi_{\alpha,g}^{(i)} \right\|_{L^\infty(0,T;L^2(\Omega)^d)}^2 + \left\| \psi_{\alpha,g}^{(i)} \right\|_{L^2(0,T;H^1(\Omega)^d)}^2 + \alpha \left\| \psi_{\alpha,g}^{(i)} \right\|_{L^2(0,T;L^2(\partial\Omega)^d)}^2 \leq C e^{CK} \|h\|_{L^2(0,T;L^2(\omega_i)^d)}^2.$$

Now, we multiply the first equation of (2.7) by $-\mathbb{P}\Delta\psi_{\alpha,g}^{(i)}$ and we integrate in time and space:

$$\begin{aligned} \int_{\Omega} \left| \mathbb{D} \psi_{\alpha,g}^{(i)}(t, x) \right|^2 dx + \alpha \int_{\partial\Omega} |\psi_{\tau}(t, \cdot)|^2 d\gamma + \int_t^T \int_{\Omega} \left| \mathbb{P}\Delta \psi_{\alpha,g}^{(i)} \right|^2 dx ds \\ \leq C \left(\left\| w_{\alpha,g}^{(i)} \right\|_{\mathcal{W}}^2 \left\| \varphi_{\alpha,g}^{(i)} \right\|_{\mathcal{W}}^2 + \left\| w_{\alpha,g}^{(i)} \right\|_{\mathcal{W}}^2 \right) + C \int_t^T \int_{\Omega} \|u_{\alpha,g}\|_{W^{1,6}(\Omega)^d}^2 \left\| \psi_{\alpha,g}^{(i)} \right\|_{H^1(\Omega)^d}^2 ds. \end{aligned}$$

Combining the above relation with (4.1), (4.2) and (4.3), we deduce (4.7). \square

5 Existence of the Nash quasi-equilibria

This section is devoted to the proof of the existence of Nash quasi-equilibria for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$. More precisely, we show the following result:

Theorem 5.1. *Assume (1.10) and (1.17) with $r > 0$ given in Proposition 4.1. There exists a constant $C > 0$ such that if*

$$M \geq C e^{CK} \tag{5.1}$$

with K in Proposition 4.1, then, for any $\alpha \geq \|b\|_{L^\infty(0,T;L^\infty(\partial\Omega)^d)} + 1$, there exists a unique Nash quasi-equilibrium \widehat{g}_α for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$. Moreover, the corresponding (direct and adjoint) states satisfy

$$\left(u_{\alpha,\widehat{g}_\alpha}, \varphi_{\alpha,\widehat{g}_\alpha}^{(1)}, \varphi_{\alpha,\widehat{g}_\alpha}^{(2)} \right) \in \left(H^1(0,T;L^2(\Omega)^d) \cap L^2(0,T;H^2(\Omega)^d) \right)^3$$

and there exists a constant $C > 0$ independent of α such that

$$\|u_{\alpha,\widehat{g}_\alpha}\|_{\mathcal{W}}^2 + \sum_{i=1}^2 \left\| \varphi_{\alpha,\widehat{g}_\alpha}^{(i)} \right\|_{\mathcal{W}}^2 + \alpha \|u_{\alpha,\widehat{g}_\alpha} - b\|_{L^\infty(0,T;L^2(\partial\Omega)^d)}^2 + \alpha \left\| \varphi_{\alpha,\widehat{g}_\alpha}^{(i)} \right\|_{L^\infty(0,T;L^2(\partial\Omega)^d)}^2 \leq C e^{CK}.$$

Proof. Assume $\alpha \geq \|b\|_{L^\infty(0,T;L^\infty(\partial\Omega))}^d + 1$. We define the following mapping:

$$\Lambda_\alpha : \mathcal{U}_1 \times \mathcal{U}_2 \rightarrow \mathcal{U}_1 \times \mathcal{U}_2, \quad \left(\mathcal{P}_{\mathcal{U}_1} \left(-\frac{\left(\varphi_{\alpha,g}^{(1)} \right)_{|\omega_1}}{M} \right), \mathcal{P}_{\mathcal{U}_2} \left(-\frac{\left(\varphi_{\alpha,g}^{(2)} \right)_{|\omega_2}}{M} \right) \right),$$

where $(\varphi_{\alpha,g}^{(1)}, \varphi_{\alpha,g}^{(2)})$ are the solutions of (1.19) obtained in Proposition 4.3. From Lemma 2.4, the fixed points g_α of Λ_α are the Nash quasi-equilibria for $(J_{1,\alpha}, J_{2,\alpha})$. We thus have to show the existence and the uniqueness of a fixed point of Λ_α in $\mathcal{U}_1 \times \mathcal{U}_2$.

Assume

$$g_\#, g_b \in \mathcal{U}_1 \times \mathcal{U}_2$$

and let us denote by $(u_{\alpha,\#}, \varphi_{\alpha,\#}^{(1)}, \varphi_{\alpha,\#}^{(2)})$ and $(u_{\alpha,b}, \varphi_{\alpha,b}^{(1)}, \varphi_{\alpha,b}^{(2)})$ the corresponding solutions of (1.11), (1.19). We set

$$u := u_{\alpha,\#} - u_{\alpha,b}, \quad \varphi^{(i)} := \varphi_{\alpha,\#}^{(i)} - \varphi_{\alpha,b}^{(i)}, \quad g := g_\# - g_b$$

We can check that

$$\begin{cases} \partial_t u + (u \cdot \nabla) u_{\alpha,\#} + (u_{\alpha,b} \cdot \nabla) u - \operatorname{div} \mathbb{T}(u, p) = g^{(1)} 1_{\omega_1} + g^{(2)} 1_{\omega_2} & \text{in } (0, T) \times \Omega, \\ \operatorname{div} u = 0 & \text{in } (0, T) \times \Omega, \\ u \cdot \nu = 0 & \text{on } (0, T) \times \partial\Omega, \\ [2(\mathbb{D}u)\nu + \alpha u]_\tau = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (5.2)$$

Multiplying the first equation by u and integrating in time and space, we deduce that

$$\begin{aligned} \frac{1}{2} \int_\Omega |u(t, \cdot)|^2 dx + \int_0^t \int_\Omega 2|\mathbb{D}u|^2 dx ds + \int_0^t \int_{\partial\Omega} \alpha |u|^2 d\gamma ds \\ + \int_0^t \int_\Omega (u \cdot \nabla) u_{\alpha,\#} \cdot u dx ds + \int_0^t \int_{\partial\Omega} b \cdot \nu \frac{|u|^2}{2} d\gamma \\ = \int_0^t \int_{\omega_1} g^{(1)} \cdot u dx ds + \int_0^t \int_{\omega_2} g^{(2)} \cdot u dx ds. \end{aligned}$$

Using the Grönwall lemma, we obtain

$$\|u\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|u\|_{L^2(0,T;H^1(\Omega))}^2 \leq C \|g\|_{L^2(0,T;L^2(\omega_1) \times L^2(\omega_2))^d}^2 e^{C \|u_{\alpha,\#}\|_{L^2(0,T;W^{1,6}(\Omega))^d}^2}.$$

Multiplying the first equation of (5.2) by $-\mathbb{P}\Delta u$ and integrating in time and space, we deduce that

$$\begin{aligned} \int_\Omega |\mathbb{D}u(t, x)|^2 dx + \int_0^t \int_\Omega |\mathbb{P}\Delta u|^2 dx ds + \int_{\partial\Omega} \alpha |u_\tau(t, \cdot)|^2 d\gamma ds \\ \leq C \int_0^t \int_\Omega |(u \cdot \nabla) u_{\alpha,\#} + (u_{\alpha,b} \cdot \nabla) u|^2 dx ds + C \|g\|_{L^2(0,T;L^2(\omega_1) \times L^2(\omega_2))^d}^2. \end{aligned} \quad (5.3)$$

On the other hand, there exists a positive constant $C > 0$ such that

$$\begin{aligned} \int_0^t \int_\Omega |(u \cdot \nabla) u_{\alpha,\#} + (u_{\alpha,b} \cdot \nabla) u|^2 dx ds \\ \leq C \int_0^t \left(\|u\|_{L^6(\Omega)^d}^2 \|u_{\alpha,\#}\|_{W^{1,6}(\Omega)^d}^2 + \|u_{\alpha,b}\|_{L^\infty(\Omega)^d}^2 \|\nabla u\|_{L^2(\Omega)^{d \times d}}^2 \right) ds. \end{aligned} \quad (5.4)$$

Thus, combining (5.3), (5.4), and the Grönwall lemma we deduce

$$\begin{aligned} & \|u\|_{\mathcal{W}}^2 + \alpha \|u\|_{L^\infty(0,T;L^2(\partial\Omega))}^d \\ & \leq C \|g\|_{L^2(0,T;L^2(\omega_1)\times L^2(\omega_2))}^d \exp\left(C \int_0^t \left(\|u_{\alpha,\#}\|_{W^{1,6}(\Omega)^d}^2 + \|u_{\alpha,b}\|_{W^{1,6}(\Omega)^d}^2\right) ds\right). \end{aligned}$$

Using (4.1), we obtain

$$\|u\|_{\mathcal{W}}^2 + \alpha \|u\|_{L^\infty(0,T;L^2(\partial\Omega))}^d \leq C e^{CK} \|g\|_{L^2(0,T;L^2(\omega_1)\times L^2(\omega_2))}^d. \quad (5.5)$$

Using (1.19), we obtain the system satisfied by φ :

$$\left\{ \begin{array}{l} -\partial_t \varphi^{(i)} - (u_{\alpha,\#} \cdot \nabla) \varphi^{(i)} + (\nabla u_{\alpha,\#})^\top \varphi^{(i)} - (u \cdot \nabla) \varphi_b^{(i)} + (\nabla u)^\top \varphi_b^{(i)} \\ \quad - \operatorname{div} \mathbb{T}(\varphi^{(i)}, \pi_{\varphi^{(i)}}) = u \mathbf{1}_{\mathcal{O}_i} \quad \text{in } (0, T) \times \Omega, \\ \quad \operatorname{div} \varphi^{(i)} = 0 \quad \text{in } (0, T) \times \Omega, \\ \quad \varphi^{(i)} \cdot \nu = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ \quad [2(\mathbb{D}\varphi^{(i)}) \nu + (\alpha + b \cdot \nu) \varphi^{(i)}]_\tau = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ \quad \varphi^{(i)}(T, \cdot) = 0 \quad \text{in } \Omega. \end{array} \right.$$

Multiplying the first equation by $\varphi^{(i)}$ and integrating in time and space, we deduce that

$$\begin{aligned} & \frac{1}{2} \int_\Omega |\varphi^{(i)}(t, x)|^2 dx + \int_t^T \int_\Omega 2 |\mathbb{D}\varphi^{(i)}|^2 dx ds + \frac{\alpha}{2} \int_t^T \int_{\partial\Omega} |\varphi^{(i)}| d\gamma ds \\ & \leq - \int_t^T \int_\Omega \left((\nabla u_{\alpha,\#})^\top \varphi^{(i)} - (u \cdot \nabla) \varphi_b^{(i)} + (\nabla u)^\top \varphi_b^{(i)} \right) \cdot \varphi^{(i)} dx ds + \int_t^T \int_{\mathcal{O}_i} u \cdot \varphi^{(i)} dx ds. \end{aligned} \quad (5.6)$$

In order to estimate the right-hand side of (5.6), we first use an integration by parts to obtain

$$\int_\Omega \left((\nabla u)^\top \varphi_b^{(i)} \right) \cdot \varphi^{(i)} dx = - \int_\Omega \left((\varphi^{(i)} \cdot \nabla) \varphi_b^{(i)} \right) \cdot u dx.$$

Then, using the above relation and combining it with Lemma 2.1 and with Hölder inequalities, we find

$$\begin{aligned} & \left| \int_t^T \int_\Omega \left((\nabla u_{\alpha,\#})^\top \varphi^{(i)} - (u \cdot \nabla) \varphi_b^{(i)} + (\nabla u)^\top \varphi_b^{(i)} \right) \cdot \varphi^{(i)} dx ds \right| \\ & \leq C \int_t^T \left(\|\nabla u_{\alpha,\#}\|_{L^6(\Omega)^d}^2 + \|\nabla u_{\alpha,\#}\|_{L^6(\Omega)^d} \right) \|\varphi^{(i)}\|_{L^2(\Omega)^d}^2 ds + \int_t^T \|\mathbb{D}\varphi^{(i)}\|_{L^2(\Omega)^{d \times d}}^2 ds \\ & \quad + C \int_t^T \|u\|_{H^1(\Omega)^d}^2 \|\nabla \varphi_b^{(i)}\|_{L^6(\Omega)^{d \times d}}^2 ds + C \int_t^T \|\varphi^{(i)}\|_{L^2(\Omega)^d}^2 ds. \end{aligned} \quad (5.7)$$

Combining (5.6) and (5.7), we deduce

$$\begin{aligned} & \frac{1}{2} \int_\Omega |\varphi^{(i)}(t, x)|^2 dx + \int_t^T \int_\Omega |\mathbb{D}\varphi^{(i)}|^2 dx ds + \frac{\alpha}{2} \int_t^T \int_{\partial\Omega} |\varphi^{(i)}| d\gamma ds \\ & \leq C \int_t^T \left(\|\nabla u_{\alpha,\#}\|_{L^6(\Omega)^{d \times d}}^2 + 1 \right) \|\varphi^{(i)}\|_{L^2(\Omega)^d}^2 ds + C \int_t^T \|u\|_{H^1(\Omega)^d}^2 \left(\|\nabla \varphi_b^{(i)}\|_{L^6(\Omega)^{d \times d}}^2 + 1 \right) ds. \end{aligned}$$

Applying the Grönwall lemma, we obtain

$$\begin{aligned} & \|\varphi^{(i)}\|_{L^\infty(0,T;L^2(\Omega)^d)}^2 + \|\varphi^{(i)}\|_{L^2(0,T;H^1(\Omega)^d)}^2 + \alpha \|\varphi^{(i)}\|_{L^2(0,T;L^2(\partial\Omega)^d)}^2 \\ & \leq C \|u\|_{L^\infty(0,T;H^1(\Omega)^d)}^2 \left(\int_t^T \left(\|\nabla \varphi_b^{(i)}\|_{L^6(\Omega)^{d \times d}}^2 + 1 \right) ds \right) \exp\left(C \int_t^T \left(\|\nabla u_{\alpha,\#}\|_{L^6(\Omega)^{d \times d}}^2 + 1 \right) ds\right). \end{aligned}$$

Using (4.1), (4.2) and (5.5), the above estimate implies

$$\left\| \varphi^{(i)} \right\|_{L^\infty(0,T;L^2(\Omega))^d}^2 + \left\| \varphi^{(i)} \right\|_{L^2(0,T;H^1(\Omega))^d}^2 + \alpha \left\| \varphi^{(i)} \right\|_{L^2(0,T;L^2(\partial\Omega))^d}^2 \leq C e^{CK} \|g\|_{L^2(0,T;L^2(\omega_1) \times L^2(\omega_2))^d}^2.$$

The above estimate and the property of the projection yield

$$\begin{aligned} \|\Lambda_\alpha(g_\#) - \Lambda_\alpha(g_b)\|_{L^2(0,T;L^2(\omega_1) \times L^2(\omega_2))^d} &\leq \frac{C}{M} \left\| \left(\varphi^{(1)}, \varphi^{(2)} \right) \right\|_{L^2(0,T;L^2(\Omega))^{2d}} \\ &\leq \frac{C e^{CK}}{M} \|g\|_{L^2(0,T;L^2(\omega_1) \times L^2(\omega_2))^d}^2. \end{aligned}$$

Thus, we deduce that if M satisfies (5.1) for a constant C sufficiently large, then Λ_α is a strict contraction on $\mathcal{U}_1 \times \mathcal{U}_2$ and thus admits a unique fixed point. \square

6 Proof of the main results

6.1 Proof of Theorem 1.4

Proof of Theorem 1.4. Assume $\alpha \geq \|b\|_{L^\infty(0,T;L^\infty(\partial\Omega))^d} + 1$. First, from (2.9) and Proposition 4.5, we deduce that if M satisfies (5.1) for C large enough, then for any $g \in \mathcal{U}_1 \times \mathcal{U}_2$, and for any $h \in L^2(0,T;L^2(\omega_i))^d$,

$$D_i^2 J_{i,\alpha}(g)(h, h) \geq C \|h\|_{L^2(0,T;L^2(\omega_i))^d}^2.$$

By assuming (5.1) for C possibly larger, we can also apply Theorem 5.1 and deduce the existence and uniqueness of a Nash quasi-equilibrium \hat{g}_α for $(J_{1,\alpha}, J_{2,\alpha})$ in $\mathcal{U}_1 \times \mathcal{U}_2$. It satisfies in particular

$$\begin{cases} D_1 J_{1,\alpha}(\hat{g}_\alpha^{(1)}, \hat{g}_\alpha^{(2)})(g^{(1)} - \hat{g}_\alpha^{(1)}) \geq 0 & (g^{(1)} \in \mathcal{U}_1), \\ D_2 J_{2,\alpha}(\hat{g}_\alpha^{(1)}, \hat{g}_\alpha^{(2)})(g^{(2)} - \hat{g}_\alpha^{(2)}) \geq 0 & (g^{(2)} \in \mathcal{U}_2). \end{cases}$$

We thus deduce (1.15) and this ends the proof of the theorem. \square

6.2 Proof of Proposition 1.5

Proof. Since $(\hat{g}_\alpha^{(1)}, \hat{g}_\alpha^{(2)}) \in B_r$ (where we recall that B_r is defined by (1.8)), we can use the Banach-Alaoglu theorem: there exists $\hat{g} := (\hat{g}^{(1)}, \hat{g}^{(2)}) \in B_r$ such that, up to a subsequence, as $\alpha \rightarrow \infty$,

$$(\hat{g}_\alpha^{(1)}, \hat{g}_\alpha^{(2)}) \rightharpoonup \hat{g} \quad \text{in } L^2(0,T;L^2(\omega_1) \times L^2(\omega_2))^d. \quad (6.1)$$

Since $\mathcal{U}_1 \times \mathcal{U}_2$ is a closed convex sets of $L^2(0,T;L^2(\omega_1) \times L^2(\omega_2))^d$, then using the Mazur theorem (see, for instance, [5, Theorem 3.7, p.60]) we deduce that $\hat{g} \in \mathcal{U}_1 \times \mathcal{U}_2$. Then, using [9, Proposition 3.1], we deduce (1.23)-(1.30). Assume now that $g^{(1)} \in L^2(0,T;L^2(\omega_1))$. Then as $\alpha \rightarrow \infty$,

$$(g^{(1)}, \hat{g}_\alpha^{(2)}) \rightharpoonup (g^{(1)}, \hat{g}^{(2)}) \quad \text{in } L^2(0,T;L^2(\omega_1) \times L^2(\omega_2))^d.$$

We deduce again from [9, Proposition 3.1] that if $\alpha \rightarrow \infty$,

$$u_{\alpha, (g^{(1)}, \hat{g}_\alpha^{(2)})} \rightarrow u_{(g^{(1)}, \hat{g}^{(2)})} \quad \text{strongly in } L^2(0,T;L^2(\Omega))^d.$$

In particular, taking the lim sup of the relation

$$\begin{aligned} J_{1,\alpha}(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)}) &= \frac{1}{2} \int_0^T \int_{\mathcal{O}_i} \left| u_{\alpha, (\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)})} - z^{(1)} \right|^2 dx dt + \frac{M}{2} \int_0^T \int_{\omega_1} \left| \widehat{g}_\alpha^{(1)} \right|^2 dx dt \\ &\leq J_{1,\alpha}(\widehat{g}^{(1)}, \widehat{g}^{(2)}) = \frac{1}{2} \int_0^T \int_{\mathcal{O}_i} \left| u_{\alpha, (\widehat{g}^{(1)}, \widehat{g}^{(2)})} - z^{(1)} \right|^2 dx dt + \frac{M}{2} \int_0^T \int_{\omega_1} \left| \widehat{g}^{(1)} \right|^2 dx dt \end{aligned}$$

we deduce

$$\limsup_{\alpha \rightarrow \infty} \int_0^T \int_{\omega_1} \left| \widehat{g}_\alpha^{(1)} \right|^2 dx dt \leq \int_0^T \int_{\omega_1} \left| \widehat{g}^{(1)} \right|^2 dx dt.$$

From (6.1), we also have

$$\int_0^T \int_{\omega_1} \left| \widehat{g}^{(1)} \right|^2 dx dt \leq \liminf_{\alpha \rightarrow \infty} \int_0^T \int_{\omega_1} \left| \widehat{g}_\alpha^{(1)} \right|^2 dx dt$$

and we deduce that

$$\widehat{g}_\alpha^{(1)} \rightarrow \widehat{g}^{(1)} \quad \text{in } L^2(0, T; L^2(\omega_1))^d.$$

We have similarly

$$\widehat{g}_\alpha^{(2)} \rightarrow \widehat{g}^{(2)} \quad \text{in } L^2(0, T; L^2(\omega_2))^d.$$

Then we can pass to the limit in (1.15) and we obtain (1.4). This implies that \widehat{g} is a Nash equilibrium for (J_1, J_2) in $\mathcal{U}_1 \times \mathcal{U}_2$. \square

6.3 Proof of Proposition 1.6

Proof. We proceed as in the proof of Proposition 1.5: there exists $\widehat{g} := (\widehat{g}^{(1)}, \widehat{g}^{(2)}) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that, up to a subsequence, as $\alpha \rightarrow \infty$,

$$\left(\widehat{g}_\alpha^{(1)}, \widehat{g}_\alpha^{(2)} \right) \rightharpoonup \widehat{g} \quad \text{in } L^2(0, T; L^2(\omega_1) \times L^2(\omega_2))^d. \quad (6.2)$$

Then, using [9, Proposition 3.1], we deduce (1.23)-(1.30). From (2.8), (1.28) and (6.2), we deduce that \widehat{g} satisfies (2.5). Using Lemma 2.3, we conclude that \widehat{g} is a Nash quasi-equilibrium and from (1.28), we also obtain (1.22). \square

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