



HAL
open science

On sums over prime ideals

Bruno Anglès, Huy Hung Le, Tuan Ngo Dac

► **To cite this version:**

Bruno Anglès, Huy Hung Le, Tuan Ngo Dac. On sums over prime ideals. Vietnam Journal of Mathematics, In press, 10.1007/s10013-024-00727-6 . hal-04677319

HAL Id: hal-04677319

<https://hal.science/hal-04677319v1>

Submitted on 26 Aug 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON SUMS OVER PRIME IDEALS

BRUNO ANGLÈS, HUY HUNG LE, AND TUAN NGO DAC

ABSTRACT. In 2017 Speyer proved a rational theorem for certain sums over irreducible polynomials of the polynomial ring $A = \mathbb{F}_q[\theta]$. In this paper we generalize Speyer's theorem for sums over prime ideals for an arbitrary coefficient ring A . Our approach is based on the theory of Goss's zeta values and that of Drinfeld-Hayes A -modules related to the class field theory.

CONTENTS

1. Introduction	1
2. Goss's zeta values and Drinfeld-Hayes modules	3
3. Symmetric polynomials	6
4. A generalization of Speyer's theorem	9
5. Vanishing and non-vanishing results for the first sum	13
References	16

1. INTRODUCTION

Throughout this paper, $\mathbb{Z}^{\geq 0}$ (resp. \mathbb{N}) denotes the set of non-negative integers (resp. the set of positive integers).

Let \mathbb{F}_q be a finite field of q elements, where q is a power of a prime number p . Let K/\mathbb{F}_q be a global function field (\mathbb{F}_q is algebraically closed in K) and ∞ be a place of K . Let A be the ring of elements of K which are regular outside ∞ . Let K_∞ be the ∞ -adic completion of K and \mathbb{F}_∞ be the residue field of K_∞ of degree $d_\infty := [K_\infty : \mathbb{F}_q]$. We denote by v_∞ the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(K_\infty^\times) = \mathbb{Z}$. Let \mathbb{C}_∞ be the completion of a fixed algebraic closure $\overline{K_\infty}$ of K_∞ . The unique valuation of \mathbb{C}_∞ which extends v_∞ will still be denoted by v_∞ .

1.1. Speyer's theorem for $A = \mathbb{F}_q[\theta]$.

In [14] Speyer considered the function field $K = \mathbb{F}_q(\theta)$ where θ is an indeterminate over \mathbb{F}_q and the place ∞ which is the pole of θ . Then $A = \mathbb{F}_q[\theta]$, $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$. The main result of [14] is the following remarkable theorem on sums over irreducible polynomials (see [14, Theorem 1.4]):

Date: August 26, 2024.

2010 Mathematics Subject Classification. Primary 11G09, 11S40.

Key words and phrases. Drinfeld modules, Goss's map, Goss's zeta values.

Theorem 1.1. *For $k \in \mathbb{N}$ and $k \equiv 0 \pmod{q-1}$, we have*

$$\sum_{P \in \mathcal{P}^+} \mathbf{G}_p \left(\frac{1}{P^k} \right) \in K$$

where \mathcal{P}^+ denotes the set of monic irreducible polynomials in $\mathbb{F}_q[\theta]$ and \mathbf{G}_p is the rational function given by

$$\mathbf{G}_p(X) = \frac{(1 - X^p) - (1 - X)^p}{p(1 - X)^p} \in \mathbb{F}_p(X).$$

We note that the numerator of $\mathbf{G}_p(X)$ is a polynomial in $\mathbb{Z}[X]$ with coefficients that are multiples of p . Hence $\mathbf{G}_p(X)$ is well-defined as a rational function in $\mathbb{F}_p(X)$.

Further, Speyer computed the previous sum for some small values of k . In particular, he proved a vanishing result for the first sum (see [14, Theorem 1.7]):

Theorem 1.2. *We keep the notation of Theorem 1.1. Then*

$$\sum_{P \in \mathcal{P}^+} \mathbf{G}_p \left(\frac{1}{P^{q-1}} \right) = 0.$$

1.2. Statement of the main results.

The aim of this paper is to generalize Theorem 1.1 of Speyer for an arbitrary coefficient ring A .

Theorem 1.3. *Let $k \in \mathbb{N}$ and $k \equiv 0 \pmod{q^{d_\infty} - 1}$. We consider the rational function \mathbf{G}_p as in Theorem 1.1. Then we have*

$$\sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^k} \right) \in F.$$

Here \mathcal{P} denotes the set of prime ideals of A , $[\cdot]$ is the Goss's map given as in Section 2.1, and F is a finite extension of K defined as in Eq. (2.5).

We also prove some surprising results for the first sum:

Theorem 1.4. *We keep the notation as above. We suppose that $d_\infty = 1$. Let h be the class number of A (i.e., $h = |\text{Pic}(A)|$ where $\text{Pic}(A)$ is given as in Section 2.1).*

(1) *If $h = 1$, then*

$$\sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^{q-1}} \right) = 0.$$

(2) *There exists a base ring A with $h = 2$ such that*

$$\sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^{q-1}} \right) \neq 0.$$

Our approach is based on two main ingredients. First, we use Goss's map $[\cdot]$ and Goss's zeta values $\zeta_A(\cdot)$ introduced by Goss in [8] (see also [9]), which play a primordial role in recent work on zeta values for arbitrary A (see for example [1–7, 10, 11]). Second, another important ingredient is the theory of class field theory for function fields via Drinfeld-Hayes modules as developed by Hayes (see [12, 13] and also [9]).

This paper is organized as follows. Section 2 is devoted to presenting some background on Goss's zeta values and Drinfeld-Hayes modules. In Section 3, we

briefly recall basic facts about symmetric polynomials and introduce sums over ideals. In Sections 4 and 5, we prove Theorems 1.3 and 1.4 (see Theorems 4.10, 5.3 and 5.4).

1.3. Acknowledgments.

T. Ngo Dac was partially supported by the Excellence Research Chair “*L*-functions in positive characteristic and applications” funded by the Normandy Region and the ANR Grant COLOSS ANR-19-CE40-0015-02. T. Ngo Dac would also like to thank to the Institut des Hautes Études Scientifiques (IHES) for the hospitality and excellent working conditions during the preparation of this work.

2. GOSS’S ZETA VALUES AND DRINFELD-HAYES MODULES

In this section we review Goss’s map and Goss’s zeta values associated to A . Then we briefly recall the theory of Drinfeld-Hayes modules or rank one Drinfeld A -modules related to the class field theory developed by Hayes. We refer the reader to [13] and also [2; 9, Chapters 4 and 7] for more details.

2.1. Goss’s map and Goss’s zeta values.

We denote by $\mathcal{I}(A)$, I_A , and \mathcal{P} respectively the group of non-zero fractional ideals of A , the set of non-zero ideals of A , and the set of prime ideals of A .

We denote by $\deg : \mathcal{I}(A) \rightarrow \mathbb{Z}$ the natural homomorphism of groups such that for every prime ideal P of A , we have

$$\deg(P) = \dim_{\mathbb{F}_q}(A/P).$$

In particular, for $a \in A \setminus \{0\}$, $\deg a = \dim_{\mathbb{F}_q}(A/aA)$ and further for $x \in K^\times$, $\deg(xA) = -d_\infty v_\infty(x)$.

From now on we fix a uniformizer $\pi \in K_\infty$ and consider the sign function

$$(2.1) \quad \text{sgn} : K_\infty^\times \rightarrow \mathbb{F}_\infty^\times$$

given by $\text{sgn}\left(\sum_{i \geq i_0} a_i \pi^i\right) = a_{i_0}$, for $i_0 \in \mathbb{Z}$, $a_i \in \mathbb{F}_\infty$, $a_{i_0} \neq 0$. We set

$$\text{Pic}(A) := \frac{\mathcal{I}(A)}{\{xA, x \in K^\times\}}$$

and

$$\text{Pic}^+(A) := \frac{\mathcal{I}(A)}{\{xA, x \in K^\times, \text{sgn}(x) = 1\}}.$$

We define

$$h := |\text{Pic}^+(A)|.$$

We recall a lemma which will be necessary in the sequel.

Lemma 2.1. *For $x \in 1 + \pi\mathbb{F}_\infty[[\pi]]$, there exists a unique $y \in \overline{K_\infty}$ such that $v_\infty(y - 1) > 0$ and $y^h = x$.*

Proof. See [9, Lemma 8.2.2]. □

Let $\pi' \in \overline{K_\infty}$ such that $\pi'^{d_\infty} = \pi$. We define Goss's map

$$[\cdot] : \mathcal{I}(A) \rightarrow \overline{K_\infty}$$

as follows. For $I \in \mathcal{I}(A)$, there exists a unique $x \in K^\times$ such that $\text{sgn}(x) = 1$ and $I^h = xA$. Note that $\frac{x}{\pi^{v_\infty(x)}} \in 1 + \pi\mathbb{F}_\infty[[\pi]]$. Thus, $\left(\frac{x}{\pi^{v_\infty(x)}}\right)^{\frac{1}{h}}$ is well-defined by Lemma 2.1. We define

$$[I] := (\pi')^{-\deg I} \left(\frac{x}{\pi^{v_\infty(x)}}\right)^{\frac{1}{h}}.$$

We present some basic properties of Goss's map.

Proposition 2.2. *For all $I, J \in \mathcal{I}(A)$, we have*

- (1) $[I][J] = [IJ]$.
- (2) If $a \in K^\times$ and $I = aA$, then

$$[I] = \frac{a}{\text{sgn}(a)}.$$

- (3) $[\cdot]$ is injective.

Proof. See [9, Section 8.2]. □

We now define Goss's zeta values associated to A . Let $k \in \mathbb{N}$. As for all $n \in \mathbb{Z}^{\geq 0}$, $|\{I \in I_A : \deg I = n\}| < +\infty$ and for all $I \in I_A$, $v_\infty([I]) = -\frac{\deg I}{d_\infty}$, the sum $\sum_{I \in I_A} \frac{1}{[I]^k}$ converges in $\overline{K_\infty}$.

Definition 2.3. The Goss's zeta value at $k \in \mathbb{N}$ associated to A is given by

$$\zeta_A(k) := \sum_{I \in I_A} \frac{1}{[I]^k} \in \overline{K_\infty}.$$

We note that for $k \in \mathbb{N}$, the value $\zeta_A(k)$ is a 1-unit, i.e., $v_\infty(\zeta_A(k) - 1) > 0$. Hence $\zeta_A(k)$ is always non-zero.

2.2. Drinfeld-Hayes modules.

2.2.1. Drinfeld modules.

First, we recall the definition of Drinfeld A -modules and its related objects (see [9, Chapter 4]). Let $\tau : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ be the Frobenius map given by $\tau(x) = x^q$ for all $x \in \mathbb{C}_\infty$. A Drinfeld A -module over \mathbb{C}_∞ is an \mathbb{F}_q -algebra homomorphism $\phi : A \rightarrow \mathbb{C}_\infty\{\tau\}$ such that

$$\phi_a = a + a_1\tau + \dots + a_d\tau^d$$

for some $d \geq 1$, $a_1, \dots, a_d \in \mathbb{C}_\infty$, $a_d \neq 0$. By [9, Lemma 4.5.1], there exists $r \in \mathbb{N}$ such that $\deg_\tau \phi_a = r \deg a$ for all $a \in A$. This number r is called the rank of ϕ .

The exponential of ϕ is the unique element $\exp_\phi \in \mathbb{C}_\infty\{\{\tau\}\}$ such that $\exp_\phi \equiv 1 \pmod{\tau}$ and $\exp_\phi a = \phi_a \exp_\phi$ for all $a \in A$. If we write $\exp_\phi = \sum_{i \geq 0} e_i \tau^i$, $e_i \in \mathbb{C}_\infty$, then \exp_ϕ induces a homomorphism $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ such that for all $x \in \mathbb{C}_\infty$, we have

$$\exp_\phi(x) = \sum_{i \geq 0} e_i x^{q^i}.$$

We denote by $\Lambda_\phi \subset \mathbb{C}_\infty$ the kernel of $\exp_\phi : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$.

Proposition 2.4. *Recall that r is the rank of ϕ . Then Λ_ϕ is an A -lattice, i.e., Λ_ϕ is discrete in \mathbb{C}_∞ and Λ_ϕ is a finitely generated A -module of rank r .*

Further, for all $x \in \mathbb{C}_\infty$, we have the following factorization of the exponential \exp_ϕ :

$$\exp_\phi(x) = x \prod_{\lambda \in \Lambda_\phi \setminus \{0\}} \left(1 - \frac{x}{\lambda}\right).$$

Proof. See [9, Theorems 4.6.9 and 2.14]. □

2.2.2. Action of ideals on Drinfeld modules.

This section follows closely [9, Section 4.9]. We keep the notation of the previous section and work with a Drinfeld A -module $\phi : A \rightarrow \mathbb{C}_\infty\{\tau\}$ of rank $r \in \mathbb{N}$.

Let $I \in I_A$. We denote by I_ϕ the left ideal generated by $\{\phi_a\}_{a \in I}$. As $\mathbb{C}_\infty\{\tau\}$ is a principal ideal domain, the ideal I_ϕ is generated by one element. We denote by $\phi_I \in \mathbb{C}_\infty\{\tau\}$ the monic polynomial in τ generating I_ϕ , i.e.,

$$\sum_{a \in I} \mathbb{C}_\infty\{\tau\}\phi_a = \mathbb{C}_\infty\{\tau\}\phi_I.$$

We denote by $\psi(I)$ the constant coefficient of ϕ_I .

Since ϕ_I is monic, it is stable under the right multiplication with ϕ_a for all $a \in A$. Thus, for each $a \in A$, there exists a unique element in $\mathbb{C}_\infty\{\tau\}$, denoted by $I * \phi_a$, such that

$$\phi_I \phi_a = (I * \phi_a) \phi_I.$$

The map

$$\begin{aligned} I * \phi : A &\longrightarrow \mathbb{C}_\infty\{\tau\} \\ a &\longmapsto I * \phi_a \end{aligned}$$

is a Drinfeld module of rank r . The kernel of its exponential map (see [9, Corollary 4.9.5]) equals

$$\Lambda_{I*\phi} = \psi(I)I^{-1}\Lambda_\phi.$$

By [9, Theorem 2.14] we have

$$(2.2) \quad \exp_{I*\phi}(X) = X \prod_{\lambda \in I^{-1}\Lambda_\phi \setminus \{0\}} \left(1 - \frac{X}{\psi(I)\lambda}\right).$$

2.2.3. Standard Drinfeld-Hayes modules.

We review the notion of standard Drinfeld-Hayes modules, also called standard sgn-normalized Drinfeld modules of rank one. We follow closely [9, Chapter 7].

We recall that the sign function sgn is given as in Eq. (2.1).

Definition 2.5. 1) A Drinfeld-Hayes module also called sgn-normalized Drinfeld A -module of rank one is a homomorphism of \mathbb{F}_q -algebras $\rho : A \rightarrow \mathbb{C}_\infty\{\tau\}$ such that there exists $i \in \mathbb{N}$ satisfying

$$\forall a \in A \setminus \{0\}, \quad \rho_a = a + \dots + \text{sgn}(a)^{q^i} \tau^{\deg(a)}.$$

2) We say that a sgn-normalized rank one Drinfeld A -module ρ is standard if $\ker \exp_\rho$ is a free A -module of rank one.

There always exists a standard sgn-normalized rank one Drinfeld A -module (see [9, Chapter 7]). If $d_\infty = 1$, then the standard module is unique. In general, this is not true.

From now on, we fix a standard sgn-normalized Drinfeld A -module of rank one ϕ and denote by \exp_ϕ the exponential of ϕ . We can write

$$(2.3) \quad \exp_\phi = \sum_{n \geq 0} e_n \tau^n, \text{ with } e_n \in \mathbb{C}_\infty.$$

There exists $\tilde{\pi} \in \mathbb{C}_\infty$ such that

$$(2.4) \quad \ker \exp_\phi = \tilde{\pi}A.$$

We set

$$H_A^+ := K(e_n, n \geq 0),$$

and

$$(2.5) \quad F = H_A^+([I], I \in \mathcal{I}(A)).$$

We recall the following property of H_A^+ related to the class field theory developed by Hayes.

Proposition 2.6. *The extension H_A^+/K is a finite abelian extension which is ramified outside ∞ .*

Proof. See [9, Proposition 7.4.4]. □

We set $G := \text{Gal}(H_A^+/K)$ and let $\sigma : \mathcal{I}(A) \rightarrow G$ be the Artin map. Recall that if $P \in \mathcal{P}$ is a prime ideal of A , then the Frobenius map $\sigma_P \in G$ is the map such that

$$\forall b \in \mathcal{O}_{H_A^+}, \quad \sigma_P(b) \equiv b^{q^{\deg(P)}} \pmod{P\mathcal{O}_{H_A^+}}$$

where $\mathcal{O}_{H_A^+}$ is the integral closure of A in H_A^+ .

Proposition 2.7. *1) Let $I \in I_A$. Then $\sigma_I(\phi) = I * \phi$. In particular, we have*

$$\exp_{I*\phi} = \sigma_I(\exp_\phi)$$

where $\sigma_I(\exp_\phi) := \sum_{n \geq 0} \sigma_I(e_n) \tau^n$.

2) The Artin map induces an isomorphism of groups $\text{Pic}^+(A) \cong G$.

Proof. See [9, Theorem 7.4.8 and Corollary 7.4.7]. □

3. SYMMETRIC POLYNOMIALS

In this section we will discuss some combinatorial results about symmetric polynomials. We will represent some symmetric polynomials as expressions of elementary symmetric polynomials.

3.1. Symmetric polynomials.

Let S be a finite set of variables. We set $s := |S|$. For $n \in \mathbb{Z}^{\geq 0}$, we denote by $\mathbf{s}_n(X)_{X \in S}$ or \mathbf{s}_n the elementary symmetric polynomial of degree n in the variables $\{X\}_{X \in S}$. In particular, $\mathbf{s}_0 = 1$.

Let $\mathbf{f}(X)_{X \in S}$ or \mathbf{f} be a homogeneous symmetric polynomial of degree ℓ in the variables $\{X\}_{X \in S}$. Recall that the ring of symmetric polynomials is generated by elementary symmetric polynomials \mathbf{s}_n , $n \geq 0$. Hence, we can write \mathbf{f} as

$$\mathbf{f} = \sum_{|\underline{\lambda}|=\ell} c_{\underline{\lambda}} \mathbf{s}_{\underline{\lambda}},$$

where $|\underline{\lambda}| := \lambda_1 + \dots + \lambda_r$ with $\underline{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathbb{N}^r$, $\lambda_1 \geq \dots \geq \lambda_r \geq 1$ and $\mathbf{s}_{\underline{\lambda}} := \mathbf{s}_{\lambda_1} \cdots \mathbf{s}_{\lambda_r}$.

3.2. The polynomials \mathbf{p}_n .

We keep the notation of Section 3.1. Let $n \in \mathbb{N}$. We set

$$\mathbf{p}_n = \mathbf{p}_n(X)_{X \in S} := \sum_{X \in S} X^n.$$

We express

$$\mathbf{p}_n(X)_{X \in S} = \sum d_{\underline{\lambda}} \mathbf{s}_{\underline{\lambda}}(X)_{X \in S}$$

where the sum runs through $\underline{\lambda} = (\lambda_1, \dots, \lambda_r) \in (\mathbb{Z}^{\geq 0})^r$, $r \in \mathbb{Z}^{\geq 0}$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 1$ and $\lambda_1 + \dots + \lambda_r = n$. The coefficients $d_{\underline{\lambda}}$ are determined as follows (see for example [14, page 1240]).

Proposition 3.1. *Let $n \in \mathbb{N}$. Recall that S is a finite set of s variables. Then we have*

$$(3.1) \quad \mathbf{p}_n(X)_{X \in S} = n \sum_{\substack{i_1, \dots, i_s \geq 0 \\ i_1 + 2i_2 + \dots + si_s = n}} \frac{(-1)^{n+i_1+\dots+i_s}}{i_1 + \dots + i_s} \binom{i_1 + \dots + i_s}{i_1, i_2, \dots, i_s} \mathbf{s}_1^{i_1} \cdots \mathbf{s}_s^{i_s}.$$

Proof. Let Z be an indeterminate. We write

$$\sum_{X \in S} \sum_{j \geq 1} \frac{(-1)^{j-1} X^j Z^j}{j} = \sum_{X \in S} \log(1 + XZ).$$

Hence

$$\sum_{j \geq 1} \frac{(-1)^{j-1} \mathbf{p}_j(X)_{X \in S} Z^j}{j} = \log \left(\prod_{X \in S} (1 + XZ) \right).$$

We note that $\prod_{X \in S} (1 + XZ) = 1 + \sum_{i=1}^s \mathbf{s}_i Z^i$. Expanding the log on the right-hand side as a Taylor series, we get

$$\begin{aligned} & \sum_{j \geq 1} \frac{(-1)^{j-1} \mathbf{p}_j(X)_{X \in S} Z^j}{j} \\ &= \sum_{j \geq 1} \frac{(-1)^{j-1}}{j} \left(\sum_{i=1}^s \mathbf{s}_i Z^i \right)^j \\ &= \sum_{j \geq 1} \frac{(-1)^{j-1}}{j} \sum_{\substack{i_1, \dots, i_s \geq 0 \\ i_1 + \dots + i_s = j}} \binom{j}{i_1, \dots, i_s} \mathbf{s}_1^{i_1} \mathbf{s}_2^{i_2} \cdots \mathbf{s}_s^{i_s} Z^{i_1 + 2i_2 + \dots + si_s}. \end{aligned}$$

We consider the coefficients of Z^n in both sides. It follows that

$$\frac{(-1)^{n-1} \mathbf{p}_n(X)_{X \in S}}{n} = \sum_{\substack{i_1, \dots, i_s \geq 0 \\ i_1 + 2i_2 + \dots + si_s = n}} \frac{(-1)^{i_1 + \dots + i_s - 1}}{i_1 + \dots + i_s} \binom{i_1 + \dots + i_s}{i_1, \dots, i_s} \mathbf{s}_1^{i_1} \dots \mathbf{s}_s^{i_s}.$$

Hence, we have

$$\mathbf{p}_n(X)_{X \in S} = n \sum_{\substack{i_1, \dots, i_s \geq 0 \\ i_1 + 2i_2 + \dots + si_s = n}} \frac{(-1)^{n+i_1+\dots+i_s}}{i_1 + \dots + i_s} \binom{i_1 + \dots + i_s}{i_1, i_2, \dots, i_s} \mathbf{s}_1^{i_1} \dots \mathbf{s}_s^{i_s}. \quad \square$$

3.3. The polynomial \mathbf{g}_p .

We keep the notation of Section 3.1. We consider the following symmetric polynomials

$$(3.2) \quad \mathbf{g}_p = \mathbf{g}_p(X)_{X \in S} := \frac{1}{p} \left(\left(\sum_{X \in S} X \right)^p - \sum_{X \in S} X^p \right) \in \mathbb{Z}[X].$$

We want to express this polynomial in terms of elementary symmetric polynomials \mathbf{s}_n , $n \geq 0$.

As a consequence of Proposition 3.1, we have the following corollary.

Corollary 3.2. *We have*

$$\mathbf{g}_p(X)_{X \in S} = \sum_{\substack{i_1, \dots, i_s \geq 0 \\ i_1 + 2i_2 + \dots + si_s = p \\ i_1 < p}} \frac{(-1)^{p+i_1+\dots+i_s-1}}{i_1 + \dots + i_s} \binom{i_1 + \dots + i_s}{i_1, i_2, \dots, i_s} \mathbf{s}_1^{i_1} \dots \mathbf{s}_s^{i_s}.$$

In particular, we have $\mathbf{g}_2 = \mathbf{s}_2$.

Proof. We see that $\mathbf{g}_p = \frac{1}{p}(\mathbf{s}_1^p - \mathbf{p}_p)$. The coefficient of \mathbf{s}_1^p in \mathbf{p}_p in Eq. (3.1) corresponds to $i_1 = p, i_2 = \dots = i_s = 0$. The corollary follows immediately. \square

3.4. Sums over ideals.

We now describe the values of a homogeneous symmetric polynomial \mathbf{f} at the value of Goss map. We denote by $I_{A, \leq n}$ the set of non-zero ideals of A of degree less than or equal to n . As $|I_{A, \leq n}| < +\infty$ and for all $I \in I_A$, $v_\infty([I]) = -\frac{\deg I}{d_\infty}$, the sequence $\{\mathbf{f}\left(\frac{1}{[I]^{q^{d_\infty}-1}}\right)\}_{I \in I_{A, \leq n}}$ converges in \mathbb{C}_∞ . Thus we can define

$$\mathbf{f} \left(\frac{1}{[I]^{q^{d_\infty}-1}} \right)_{I \in I_A} := \lim_{n \rightarrow +\infty} \mathbf{f} \left(\frac{1}{[I]^{q^{d_\infty}-1}} \right)_{I \in I_{A, \leq n}}.$$

Similarly, we can define

$$\mathbf{s}_\ell \left(\frac{1}{[I]^{q^{d_\infty}-1}} \right)_{I \in I_A} := \lim_{n \rightarrow \infty} \mathbf{s}_\ell \left(\frac{1}{[I]^{q^{d_\infty}-1}} \right)_{I \in I_{A, \leq n}} \in \mathbb{C}_\infty$$

and

$$(3.3) \quad \mathbf{g}_p \left(\frac{1}{[I]^k} \right)_{I \in I_A} := \lim_{n \rightarrow \infty} \mathbf{g}_p \left(\frac{1}{[I]^k} \right)_{I \in I_{A, \leq n}} \in \mathbb{C}_\infty.$$

Thus, as a consequence of Corollary 3.2, when passing to the limit, we obtain the following proposition.

Proposition 3.3. *We have*

$$\begin{aligned} & \mathbf{g}_p \left(\frac{1}{[I]^k} \right)_{I \in I_A} \\ &= \sum_{\substack{i_1, \dots, i_s > 0 \\ i_1 + 2i_2 + \dots + si_s = p \\ i_1 < p}} \frac{(-1)^{p+i_1+\dots+i_s-1}}{i_1 + \dots + i_s} \binom{i_1 + \dots + i_s}{i_1, i_2, \dots, i_s} \mathbf{s}_1^{i_1} \left(\frac{1}{[I]^k} \right)_{I \in I_A} \cdots \mathbf{s}_s^{i_s} \left(\frac{1}{[I]^k} \right)_{I \in I_A}. \end{aligned}$$

4. A GENERALIZATION OF SPEYER'S THEOREM

In this section we prove Theorem 1.3 (see Section 4.3).

4.1. Speyer's identity.

For $j \in \mathbb{Z}^{\geq 0}$, we set

$$\varphi(j) := \left| \left\{ (m_1, \dots, m_p) \in \mathbb{Z}^p : \sum_{i=1}^p m_i = j, \min\{m_i, i = 1, \dots, p\} = 0 \right\} \right|.$$

Then we have the following equality in $\mathbb{Z}[[X]]$ (see [14, page 1237]):

$$\sum_{j=0}^{\infty} \varphi(j) X^j = \frac{1 - X^p}{(1 - X)^p}.$$

Thus, $\varphi(0) = 1$ and for $j \geq 1$, we have

$$(4.1) \quad \varphi(j) \equiv 0 \pmod{p}.$$

As a corollary, we get

Proposition 4.1. *We have*

$$\mathbf{G}_p(X) = \sum_{j \geq 1} \frac{\varphi(j)}{p} X^j \text{ in } \mathbb{Z}[[X]].$$

For $I \in I_A \setminus \{A\}$, we set

$$(4.2) \quad N(I) := |\{(I_1, \dots, I_p) \in I_A^p, \gcd(I_1, \dots, I_p) = A, I_1 \cdots I_p = I\}|.$$

We see that if $I = P_1^{k_1} \cdots P_r^{k_r}$, $P_i \in \mathcal{P}$, $k_i \in \mathbb{Z}$ is the prime decomposition of I , then

$$(4.3) \quad N(I) = \varphi(k_1) \cdots \varphi(k_r).$$

As a consequence of Proposition 4.1, we obtain the following corollary (see [14, page 1237]).

Corollary 4.2. *We have the following equality in \mathbb{C}_∞ :*

$$\sum_{I \in I_A \setminus \{A\}} \frac{N(I)/p}{[I]^k} = \sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^k} \right).$$

Proof. We consider Eq. (4.3). If $r \geq 2$, by Eq. (4.1), p^2 divides $\varphi(k_1)\varphi(k_2)$. Hence $N(I)/p = 0$ in \mathbb{F}_p . It implies

$$\sum_{I \in I_A \setminus \{A\}} \frac{N(I)/p}{[I]^k} = \sum_{P \in \mathcal{P}} \sum_{j \geq 0} \frac{N(P^j)/p}{[P^j]^k}$$

$$\begin{aligned}
&= \sum_{P \in \mathcal{P}} \sum_{j \geq 0} \frac{\varphi(j)/p}{[P^k]^j} \\
&= \sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^k} \right).
\end{aligned}$$

The last equality follows from Proposition 4.1. \square

We deduce the following equality in \mathbb{C}_∞ .

Proposition 4.3. *With the notation in Eq. (3.3), for every $k \in \mathbb{N}$, we have*

$$(4.4) \quad \sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^k} \right) = \frac{\mathbf{g}_p \left(\frac{1}{[I]^k} \right)_{I \in I_A}}{\zeta_A(pk)}.$$

Proof. For all $n \in \mathbb{N}$, we set $\Delta_n := \{(I, \dots, I) \in I_{A, \leq n}^p\}$. Let C be the cyclic group of degree p that acts on $I_{A, \leq n}^p$ by rotating coordinates. We have

$$\begin{aligned}
\mathbf{g}_p \left(\frac{1}{[I]^k} \right)_{I \in I_{A, \leq n}} &= \sum_{(I_1, \dots, I_p) \in (I_{A, \leq n} \setminus \Delta_n)/C} \frac{1}{[I_1]^k \cdots [I_p]^k} \\
&= \sum_{D \in I_{A, \leq n}} \frac{1}{[D]^{pk}} \times \sum_{\substack{(I_1, \dots, I_p) \in (I_{A, \leq n} \setminus \{(A, \dots, A)\})/C \\ \gcd(I_1, \dots, I_p) = A}} \frac{1}{[I_1]^k \cdots [I_p]^k}.
\end{aligned}$$

Letting $n \rightarrow +\infty$, we get

$$\begin{aligned}
\mathbf{g}_p \left(\frac{1}{[I]^k} \right)_{I \in I_A} &= \zeta_A(pk) \times \sum_{\substack{(I_1, \dots, I_p) \in (I_A \setminus \{(A, \dots, A)\})/C \\ \gcd(I_1, \dots, I_p) = A}} \frac{1}{[I_1]^k \cdots [I_p]^k} \\
&= \zeta_A(pk) \times \sum_{I \in I_A \setminus \{A\}} \frac{N(I)/p}{[I]^k} \\
&= \zeta_A(pk) \times \sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^k} \right).
\end{aligned}$$

Here $N(I)$ is given as in Eq. (4.2) and the last equality follows from Corollary 4.2. We note that $\zeta_A(pk) \neq 0$ by the discussion just after Definition 2.3. Thus, the proof follows. \square

4.2. A rationality result.

In this section we first prove a rationality result (see Proposition 4.5). Then we use this rationality result to show that both the numerator and denominator of (4.4) are rational up to a suitable power of $\tilde{\pi}$ (see Corollaries 4.8 and 4.9).

The following proposition will be necessary in the sequel.

Proposition 4.4. *Let $I \in I_A$. Recall that $\psi(I)$, F are given as in Section 2. We denote by \mathcal{O}_F the integral closure of A in F . Then the element $\frac{\psi(I)}{[I]}$ belongs to \mathcal{O}_F^\times and depends only on the class of I in $\text{Pic}^+(A)$.*

Proof. This proposition is a combination of results in [9, Lemma 4.9.2, Theorems 7.4.8 and 7.6.2, Proposition 8.2.10]. \square

Let $g \in G$ and $I \in I_A$ such that $g = \sigma_I$. By Proposition 4.4 the following element is well-defined:

$$\alpha(g) := \frac{\psi(I)}{[I]} \in \mathcal{O}_F^\times.$$

Proposition 4.5. *For $I \in I_A$, we denote the class of I in $\text{Pic}^+(A)$ by \bar{I} and put $g = \sigma_I \in G$. Let X be an indeterminate, then*

$$\prod_{\xi \in \mathbb{F}_\infty^\times / \mathbb{F}_q^\times} \frac{g(\exp_\phi)(\xi \alpha(g)X)}{\xi \alpha(g)X} = \prod_{J \in I_A \cap [I]} \left(1 - \left(\frac{X}{\tilde{\pi}[J]} \right)^{q^{d_\infty} - 1} \right).$$

Proof. We set

$$F_g(X) := \frac{g(\exp_\phi(\alpha(g)X))}{\alpha(g)X}.$$

Then $F_g(X)$ belongs to $F[[X]]$.

As $g = \sigma_I$, by Proposition 2.7 and Eq. (2.2), we have

$$F_g(X) = \prod_{a \in I^{-1} \setminus \{0\}} \left(1 - \frac{\alpha(g)X}{\tilde{\pi}\psi(I)a} \right).$$

Replacing $\alpha(g) = \frac{\psi(I)}{[I]}$ yields

$$F_g(X) = \prod_{a \in I^{-1} \setminus \{0\}} \left(1 - \frac{X}{\tilde{\pi}[I]a} \right).$$

By Proposition 2.7, $\sigma: \mathcal{I}(A) \rightarrow G$ induces an isomorphism $\text{Pic}^+(A) \xrightarrow{\sim} G$. We see that if $\sigma_I = \sigma_J$, then $J = Ia$ for some $a \in K^\times$, $\text{sgn}(a) = 1$. On the other hand, for $a, b \in I^{-1} \setminus \{0\}$, then $Ia = Ib$ if and only if $\frac{a}{b} \in \mathbb{F}_q^\times$. Hence

$$F_g(X) = \prod_{J \subset A, J=Ia} \prod_{\delta \in \mathbb{F}_q^\times} \left(1 - \frac{X}{\tilde{\pi}[I]\delta a} \right).$$

It follows that

$$\begin{aligned} \prod_{\xi \in \mathbb{F}_\infty^\times / \mathbb{F}_q^\times} F_g(\xi X) &= \prod_{J \subset A, J=Ia} \prod_{\delta \in \mathbb{F}_\infty^\times} \left(1 - \frac{X}{\tilde{\pi}[I]\delta a} \right) \\ &= \prod_{J \in I_A \cap \bar{I}} \prod_{\delta \in \mathbb{F}_\infty^\times} \left(1 - \frac{X}{\tilde{\pi}[J]\delta} \right) \\ &= \prod_{J \in I_A \cap \bar{I}} \left(1 - \left(\frac{X}{\tilde{\pi}[J]} \right)^{q^{d_\infty} - 1} \right). \end{aligned}$$

□

Proposition 4.6. *With above notation we have*

$$\prod_{I \in I_A} \left(1 - \left(\frac{X}{\tilde{\pi}[I]} \right)^{q^{d_\infty} - 1} \right) \in F[[X]].$$

Proof. We have

$$\prod_{I \in I_A} \left(1 - \left(\frac{X}{\tilde{\pi}[I]} \right)^{q^{d_\infty} - 1} \right) = \prod_{\bar{I} \in \text{Pic}(A)} \prod_{J \in I_A \cap \bar{I}} \left(1 - \left(\frac{X}{\tilde{\pi}[J]} \right)^{q^{d_\infty} - 1} \right).$$

The proposition follows from Proposition 4.5. \square

Corollary 4.7. *For all $\ell \in \mathbb{Z}^{\geq 0}$, we have*

$$\mathbf{s}_\ell \left(\frac{1}{[I]^{q^{d_\infty} - 1}} \right)_{I \in I_A} \in \tilde{\pi}^{\ell(q^{d_\infty} - 1)} F.$$

Proof. We have

$$\prod_{I \in I_A} \left(1 - \left(\frac{X}{\tilde{\pi}[I]} \right)^{q^{d_\infty} - 1} \right) = \sum_{\ell \geq 0} \frac{(-1)^\ell}{\tilde{\pi}^{\ell(q^{d_\infty} - 1)}} \mathbf{s}_\ell \left(\frac{1}{[I]^{q^{d_\infty} - 1}} \right)_{I \in I_A} X^{\ell(q^{d_\infty} - 1)}.$$

By Proposition 4.6, the left-hand side of Eq. (4.2) is in $F[[X]]$. Hence, we have

$$\frac{(-1)^\ell}{\tilde{\pi}^{\ell(q^{d_\infty} - 1)}} \mathbf{s}_\ell \left(\frac{1}{[I]^{q^{d_\infty} - 1}} \right)_{I \in I_A} \in F.$$

The proof follows. \square

Corollary 4.8. *Recall the notation in Eqs. (3.2) and (3.3). For $k \geq 0$, $k \equiv 0 \pmod{q^{d_\infty} - 1}$, we have*

$$\mathbf{g}_p \left(\frac{1}{[I]^k} \right)_{I \in I_A} \in \tilde{\pi}^{pk} F.$$

Proof. The corollary follows from Corollary 4.7 and the fact that the polynomial $\mathbf{g}_p \left(\frac{1}{[I]^k} \right)_{I \in I_A}$ is symmetric in the variables $\left\{ \frac{1}{[I]^{q^{d_\infty} - 1}} \right\}_{I \in I_A}$. \square

To end this section we recall a rationality result concerning Goss's zeta values.

Proposition 4.9. *For $k \in \mathbb{N}$ such that $k \equiv 0 \pmod{q^{d_\infty} - 1}$, we have*

$$\zeta_A(k) \in \tilde{\pi}^k F.$$

Proof. See [9, Theorem 8.19.4]. \square

4.3. The rationality result.

We prove the main result of this section which generalizes the work of Speyer [14].

Theorem 4.10. *Let $k \in \mathbb{N}$ such that $k \equiv 0 \pmod{q^{d_\infty} - 1}$. Then we have*

$$\sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^k} \right) \in F.$$

Proof. The proof follows directly from Proposition 4.3, Corollaries 4.8 and 4.9. \square

5. VANISHING AND NON-VANISHING RESULTS FOR THE FIRST SUM

In this section, we will find an explicit formula of Theorem 4.10 in the case $d_\infty = 1$ and $k = q - 1$ (see Proposition 5.2). Then we show Theorem 1.4 (see Theorems 5.3 and 5.4) which proves some vanishing and non-vanishing results in several examples presented in [11, 12, 14, 15].

5.1. An explicit formula for $d_\infty = 1$ and $k = q - 1$.

We suppose that $d_\infty = 1$, $k = q - 1$. It follows that $\text{Pic}(A) = \text{Pic}^+(A)$. Recall that $h = |\text{Pic}^+(A)| = |\text{Pic}(A)|$.

For $g \in G$ and $I \in I_A$ such that $\sigma_I = g$, by Proposition 4.5, we have

$$\frac{g(\exp_\phi)(\alpha(g)X)}{\alpha(g)X} = \prod_{J \in I_A \cap \bar{I}} \left(1 - \left(\frac{X}{\tilde{\pi}[J]} \right)^{q-1} \right).$$

Here \bar{I} is the class of I in the group $\text{Pic}(A)$. Hence

$$\prod_{g \in G} \frac{g(\exp_\phi)(\alpha(g)X)}{\alpha(g)X} = \prod_{I \in I_A} \left(1 - \left(\frac{X}{\tilde{\pi}[I]} \right)^{q-1} \right).$$

Expanding the right-hand side, we get

$$\prod_{g \in G} \frac{g(\exp_\phi)(\alpha(g)X)}{\alpha(g)X} = \sum_{\ell \geq 0} (-1)^\ell \frac{1}{\tilde{\pi}^{\ell(q-1)}} \mathbf{s}_\ell \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} X^{\ell(q-1)}.$$

Recall that

$$\exp_\phi = \sum_{i \geq 0} e_i \tau^i$$

as in Eq. (2.3). We have

$$\prod_{g \in G} \sum_{i \geq 0} g(e_i) \alpha(g)^{q^i-1} X^{q^i-1} = \sum_{\ell \geq 0} (-1)^\ell \frac{1}{\tilde{\pi}^{\ell(q-1)}} \mathbf{s}_\ell \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} X^{\ell(q-1)}.$$

Comparing the coefficients of $X^{\ell(q-1)}$ in both sides, we have the following lemma.

Lemma 5.1. *For all $\ell \in \mathbb{N}$, we have*

$$\mathbf{s}_\ell \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} = (-1)^\ell \tilde{\pi}^{\ell(q-1)} \sum_{\substack{i_1, \dots, i_h \geq 0 \\ (q^{i_1}-1) + \dots + (q^{i_h}-1) = \ell(q-1)}} \prod_{j=1}^h g_j(e_{i_j}) \alpha(g_j)^{q^{i_j}-1}.$$

In particular, we have

$$\mathbf{s}_1 \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} = -\tilde{\pi}^{q-1} \sum_{g \in G} g(e_1) \alpha(g)^{q-1}.$$

Proposition 5.2. *Suppose that $d_\infty = 1$, $k = q - 1$. We have*

$$\sum_{P \in \mathcal{P}} \mathbf{G}_P \left(\frac{1}{[P]^{q-1}} \right) = \sum_{\substack{i_1, \dots, i_s > 0 \\ i_1 + \dots + i_s = p \\ i_1 < p}} \frac{(-1)^{p+i_1+\dots+i_s-1}}{i_1 + \dots + i_s} \binom{i_1 + \dots + i_s}{i_1, i_2, \dots, i_s} \frac{\mathbf{s}_2^{i_2} \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} \cdots \mathbf{s}_s^{i_s} \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A}}{\mathbf{s}_1^{p-i_1} \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A}} \in F.$$

Proof. On one hand,

$$\begin{aligned} & \mathbf{g}_p \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} \\ &= \sum_{\substack{i_1, \dots, i_s > 0 \\ i_1 + \dots + s i_s = p \\ i_1 < p}} \frac{(-1)^{p+i_1+\dots+i_s-1}}{i_1 + \dots + i_s} \binom{i_1 + \dots + i_s}{i_1, i_2, \dots, i_s} \mathbf{s}_1^{i_1} \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} \cdots \mathbf{s}_s^{i_s} \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A}. \end{aligned}$$

On the other hand,

$$\zeta_A(p(q-1)) = \zeta_A(q-1)^p = \mathbf{s}_1^p \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A}.$$

Thus,

$$\begin{aligned} \sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^{q-1}} \right) &= \frac{\mathbf{g}_p \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A}}{\zeta_A(p(q-1))} \\ &= \sum_{\substack{i_1, \dots, i_s > 0 \\ i_1 + \dots + s i_s = p \\ i_1 < p}} \frac{(-1)^{p+i_1+\dots+i_s-1}}{i_1 + \dots + i_s} \binom{i_1 + \dots + i_s}{i_1, i_2, \dots, i_s} \frac{\mathbf{s}_2^{i_2} \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} \cdots \mathbf{s}_s^{i_s} \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A}}{\mathbf{s}_1^{p-i_1} \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A}}. \end{aligned}$$

Since $i_1 + \dots + s i_s = p$, the factor $\tilde{\pi}$ in the above formula is canceled when we substitute Lemma 5.1 into Proposition 5.2. The remaining terms are in F , so $\sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^{q-1}} \right) \in F$. \square

5.2. The genus 0 example.

The first example is studied by Speyer in [14]. We consider the rational function field $K = \mathbb{F}_q(\theta)$ with an indeterminate θ . Let ∞ be the infinity place of degree $d_\infty = 1$. Then $A = \mathbb{F}_q[\theta]$, which is a principal ideal domain, thus $h = 1$. The set \mathcal{P} of prime ideals of A is in bijection with the set of monic irreducible polynomials in A . We choose $\pi = \frac{1}{\theta}$ and $K_\infty = \mathbb{F}_q\left(\left(\frac{1}{\theta}\right)\right)$ equipped with the discrete valuation v_∞ .

Speyer showed that the sum $\sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^{q-1}} \right)$ vanishes (see [14, Theorem 1.7]). In the next section, we will generalize this vanishing result for all base rings A whose class number is 1.

5.3. The class number 1 examples.

We consider an important class of rings A consisting of principal ideal domains, i.e., $h = 1$. In particular, $d_\infty = 1$. Besides the polynomial ring $A = \mathbb{F}_q[\theta]$ mentioned above, there are only four other rings (see for example [5, 7, 11, 15]).

- (1) $A = \mathbb{F}_3[x, y]/(y^2 - x^3 + x + 1)$.
- (2) $A = \mathbb{F}_4[x, y]/(y^2 + y - x^3 - \alpha)$. Here α is an element of \mathbb{F}_4 satisfying $\alpha^2 + \alpha + 1 = 0$.
- (3) $A = \mathbb{F}_2[x, y]/(y^2 + y - x^3 - x - 1)$.
- (4) $A = \mathbb{F}_2[x, y]/(y^2 + y - x^5 - x^3 - 1)$.

We prove that

Theorem 5.3. *The sum $\sum_{P \in \mathcal{P}} \mathbf{G}_p \left(\frac{1}{[P]^{q-1}} \right)$ vanishes.*

Proof. Since $h = 1$, the Galois group G is trivial and $F = K$. By Lemma 5.1, for all $\ell \in \mathbb{N}$, we have

$$\mathbf{s}_\ell \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} = (-1)^\ell \tilde{\pi}^{\ell(q-1)} \sum_{\substack{i \geq 0 \\ q^i - 1 = \ell(q-1)}} e_i.$$

In particular, $\mathbf{s}_\ell \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} = 0$ for all $2 \leq \ell \leq q$. We recall that $\mathbf{s}_1 \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} = \zeta_A(q-1) \neq 0$. Hence, Proposition 5.2 implies $\sum_{P \in \mathcal{P}} \mathbf{G}_P \left(\frac{1}{[P]^{q-1}} \right) = 0$. \square

5.4. The class number 2 examples.

In this section, we consider the class of rings A such that $d_\infty = 1$ and $h = 2$. We suppose that $q > 2$. We recall that $k = q - 1$. By Lemma 5.1, for all $\ell \in \mathbb{N}$, we have

$$\mathbf{s}_\ell \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} = (-1)^\ell \tilde{\pi}^{\ell(q-1)} \sum_{\substack{i_1, i_2 \geq 0 \\ (q^{i_1} - 1) + (q^{i_2} - 1) = \ell(q-1)}} g_1(e_{i_1}) \alpha(g_1)^{q^{i_1} - 1} g_2(e_{i_2}) \alpha(g_2)^{q^{i_2} - 1}.$$

By [16], we know that $e_1 \neq 0$. Thus

$$\begin{aligned} \mathbf{s}_1 \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} &= \zeta_A(q-1) \neq 0, \\ \mathbf{s}_2 \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} &= \tilde{\pi}^{2(q-1)} g_1(e_1) \alpha(g_1)^{q-1} g_2(e_1) \alpha(g_2)^{q-1} \neq 0, \\ \mathbf{s}_\ell \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A} &= 0, \quad \text{for all } 3 \leq \ell \leq q. \end{aligned}$$

Proposition 5.2 implies

$$\sum_{P \in \mathcal{P}} \mathbf{G}_P \left(\frac{1}{[P]^{q-1}} \right) = \sum_{\substack{i_1, i_2 > 0 \\ i_1 + 2i_2 = p \\ i_1 < p}} \frac{(-1)^{p+i_1+i_2-1}}{i_1 + i_2} \binom{i_1 + i_2}{i_1, i_2} \frac{\mathbf{s}_2^{i_2} \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A}}{\mathbf{s}_1^{p-i_1} \left(\frac{1}{[I]^{q-1}} \right)_{I \in I_A}}.$$

5.5. A non-vanishing result.

There exists a global function field K such that $d_\infty = 1$, $q = 3$, and $h = 2$. An example is given in [12, Example 11.7] (see also [11, Example 9.2]): we consider the elliptic curve $y^2 = \theta^3 - \theta^2 - 1$ over \mathbb{F}_3 equipped with the rational point ∞ .

Since $q = 3$, we get

$$\mathbf{G}_3(X) = \frac{1 - X^3 - (1 - X)^3}{3(1 - X)^3} = \frac{X}{(1 - X)^2}.$$

Theorem 5.4. *We keep the notation as above. Then*

$$\sum_{P \in \mathcal{P}} \mathbf{G}_3 \left(\frac{1}{[P]^2} \right) = \frac{\mathbf{s}_2 \left(\frac{1}{[I]^2} \right)_{I \in I_A}}{\mathbf{s}_1^2 \left(\frac{1}{[I]^2} \right)_{I \in I_A}} \neq 0.$$

Proof. The proposition follows immediately from the calculations given in Section 5.4. \square

REFERENCES

- [1] G. Anderson. Rank one elliptic A -modules and A -harmonic series. *Duke Math. J.*, 73(3):491–542, 1994.
- [2] G. Anderson. Log-algebraicity of twisted A -harmonic series and special values of L -series in characteristic p . *J. Number Theory*, 60(1):165–209, 1996.
- [3] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro. Twisted characteristic p zeta function. *J. Number Theory*, 168:180–214, 2016.
- [4] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro. Stark units in positive characteristic. *Proc. Lond. Math. Soc. (3)*, 115(4):763–812, 2017.
- [5] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro. Recent developments in the theory of Anderson modules. *Acta Math. Vietnamica*, 45:199–216, 2020.
- [6] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro. A class formula for admissible Anderson modules. *Invent. Math.*, 229(2):563–606, 2022.
- [7] K. Chung, T. Ngo Dac, and F. Pellarin. Universal families of Eulerian multiple zeta values in positive characteristic. *Adv. Math.*, 422:109003, 32 pp., 2023.
- [8] D. Goss. v -adic zeta functions, L -series and measures for function fields. *Invent. Math.*, 55(2):107–119, 1979.
- [9] D. Goss. *Basic Structures of function field arithmetic*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1996.
- [10] N. Green and T. Ngo Dac. Algebraic relations among Goss’s zeta values on elliptic curves. *Forum Math. Sigma*, 11:Paper No. e90, 40, 2023.
- [11] N. Green and M. Papanikolas. Special L -values and shtuka functions for Drinfeld modules on elliptic curves. *Res. Math. Sci.*, 5(1):Paper No. 4, 47pp, 2018.
- [12] D. Hayes. Explicit class field theory in global function fields. In *Studies in algebra and number theory*, volume 6 of *Adv. in Math. Suppl. Stud.*, pages 173–217. Academic Press, New York-London, 1979.
- [13] D. Hayes. A brief introduction to Drinfeld modules. In *The arithmetic of function fields (Columbus, OH, 1991)*, volume 2 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 1–32. de Gruyter, Berlin, 1992.
- [14] D. Speyer. Some sums over irreducible polynomials. *Algebra Number Theory*, 11(5):1231–1241, 2017.
- [15] D. Thakur. Drinfeld modules and arithmetic in function fields. *Int. Math. Res. Not.*, 1992(9):185–197, 1992.
- [16] D. Thakur. Shtukas and Jacobi sums. *Invent. Math.*, 111:557–570, 1993.

NORMANDIE UNIVERSITÉ, UNIVERSITÉ DE CAEN NORMANDIE - CNRS, LABORATOIRE DE MATHÉMATIQUES
NICOLAS ORESME (LMNO), UMR 6139, 14000 CAEN, FRANCE.

Email address: `bruno.angles@unicaen.fr`

Email address: `lehuyhung94@gmail.com`

Email address: `tuan.ngodac@unicaen.fr`