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# The KdV equation and solutions in terms of Wronskians

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## Abstract

A method to construct solutions to the Korteweg-de-Vries (KdV) equation in terms of wronskians is given.

For this, a particular type of polynomials is considered and we obtain for each positive integer  $n$ , rational solutions in terms of determinants of order  $n$ .

Explicit solutions can be easily constructed and rational solutions from order 1 until order 10 are given.

## 1 Introduction

In the following, the Korteweg-de-Vries (KdV) equation

$$u_t + \frac{3}{2}uu_x + u_{xxx} = 0, \quad (1)$$

is considered, with the usual notations, where the subscripts  $x$  and  $t$  denote partial derivatives.

Korteweg and de Vries [77] introduced this equation (1) for the first time in 1895. This KdV equation is the basis of the most common tool for the 2-dimensional modelling of shallow water waves.

Numerical modelling done by Zabusky and Kruskal revealed the presence of soliton solutions of this equation [?]. The inverse scattering technique developed in a work published in 1967 by Gardner et al. for the KdV equation enabled the derivation of analytic solutions for given initial conditions with zeros at infinity [74]. Being the historic first among the integrable nonlinear evolution equations, the KdV equation attracted significant attention from both physicists and mathematicians. Zakharov and Faddeev in 1971 [87] proved that this equation is a completely integrable.

Solutions were built by Hirota in 1971 by using the bilinear method [75].

Solutions in terms of Riemann theta functions [76] were constructed by Its and Matveev in 1975. In the same year the expressions of periodic and almost periodic solutions [79] were given by Lax. Among different works realized, we can mention for example Airault et al. in 1977 [3], Adler and Moser in 1978 [2],

Ablowitz and Cornille in 1979 [1], Freeman and Nimmo in 1984 [5], Matveev in 1992 [84], Ma in 2004 [81], Kovalyov in 2005 [78] and more recently Ma in 2015 [82].

Considering certain polynomials, we construct rational solutions in terms of wronkians of order  $n$ . So we get a very efficient method to construct rational solutions to the KdV equation. We give explicit solutions in the simplest cases for orders  $n = 1$  until 10.

## 2 Solutions to the KdV equation

Let  $p_n$  be the polynomials defined by

$$\begin{cases} p_{3k}(x, t) = \sum_{l=0}^k \frac{x^{3l} (-t)^{k-l}}{(3l)! (k-l)!}, & k \geq 0, \\ p_{3k+1}(x, t) = \sum_{l=0}^k \frac{x^{3l+1} (-t)^{k-l}}{(3l+1)! (k-l)!}, & k \geq 0, \\ p_{3k+2}(x, t) = \sum_{l=0}^k \frac{x^{3l+2} (-t)^{k-l}}{(3l+2)! (k-l)!}, & k \geq 0, \\ p_n(x, t) = 0, & n < 0. \end{cases} \quad (2)$$

Let  $W_n(x, t) = W(p_{2n}, \dots, p_{n+1})$  be the classical wronkian

$$W_n(x, t) = \det(\partial_x^{i-1}(p_{2n+1-j}))_{\{1 \leq i \leq n, 1 \leq j \leq n\}} \quad (3)$$

with  $\partial_x^0(p_{n+j})$  meaning  $p_{n+j}$ .

We get the following statement

**Theorem 2.1** *The function  $v_n(x, t)$  expressed as*

$$v_n(x, t) = 4\partial_x^2(\ln(W_n(x, t))) \quad (4)$$

*is a rational solution to the (KdV) equation (1)*

$$u_t + \frac{3}{2}uu_x + u_{xxx} = 0. \quad (5)$$

## 3 Rational solutions to the KdV equation

We construct explicitly rational solutions to the KdV equation.

We call rational solution to the KdV equation of order  $k$ , the following function

$$v_k(x, t) = 4\partial_x^2(\ln(W_k(x, t))).$$

In the following, we give some examples of these solutions.

### 3.1 Solutions of order 1

**Example 3.1** The function  $v_k(x, t)$  expressed as

$$v_k(x, t) = -\frac{8}{x^2} \quad (6)$$

is a rational solution to the KdV equation.

### 3.2 Solutions of order 2

**Example 3.2** The function  $v_k(x, t)$  expressed as

$$v_k(x, t) = 24 \frac{(-x^3 + 24t)x}{x^6 + 24tx^3 + 144t^2} \quad (7)$$

is a rational solution to the KdV equation.

### 3.3 Solutions of order 3

**Example 3.3** The function  $v_k(x, t)$  expressed as

$$v_k(x, t) = -48 \frac{(x^9 + 5400t^2x^3 + 43200t^3)x}{x^{12} + 120tx^9 + 2160t^2x^6 - 86400t^3x^3 + 518400t^4} \quad (8)$$

is a rational solution to the KdV equation.

### 3.4 Solutions of order 4

**Example 3.4** The function  $v_k(x, t)$  expressed as

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (9)$$

with  $n(x, t) = -80(9144576000t^6 - 152409600t^4x^6 - 2116800t^3x^9 + 22680t^2x^{12} + 144tx^{15} + x^{18})$

and

$d(x, t) = (x^{18} + 360tx^{15} + 32400t^2x^{12} + 604800t^3x^9 + 108864000t^4x^6 + 91445760000t^6)x^2$   
is a rational solution to the KdV equation.

### 3.5 Solutions of order 5

**Example 3.5** The function  $v_k(x, t)$  expressed as

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (10)$$

with

$n(x, t) = 120(-x^{27} - 504tx^{24} - 120960t^2x^{21} - 1447891200t^4x^{15} - 512096256000t^5x^{12} - 17667320832000t^6x^9 - 184354652160000t^7x^6)$

$$\begin{aligned}
& - 19357238476800000 t^8 x^3 + 154857907814400000 t^9 x \\
& \text{and} \\
& d(x, t) = x^{30} + 840 t x^{27} + 226800 t^2 x^{24} + 25401600 t^3 x^{21} \\
& + 1905120000 t^4 x^{18} - 109734912000 t^5 x^{15} - 9601804800000 t^6 x^{12} \\
& - 1152216576000000 t^7 x^9 + 58071715430400000 t^8 x^6 \\
& + 774289539072000000 t^9 x^3 + 2322868617216000000 t^{10}
\end{aligned}$$

is a rational solution to the KdV equation.

### 3.6 Solutions of order 6

**Example 3.6** The function  $v_k(x, t)$  expressed as

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (11)$$

with

$$\begin{aligned}
n(x, t) = & -168 (x^{39} + 1200 t x^{36} + 604800 t^2 x^{33} + 120960000 t^3 x^{30} + 23278752000 t^4 x^{27} + \\
& 2112397056000 t^5 x^{24} - 1594448271360000 t^6 x^{21} - 294045670195200000 t^7 x^{18} \\
& - 17292927259238400000 t^8 x^{15} - 749512273821696000000 t^9 x^{12} \\
& + 66893970438586368000000 t^{10} x^9 - 404736627863715840000000 t^{11} x^6 \\
& + 70828909876150272000000000 t^{12} x^3 + 339978767405521305600000000 t^{13} x)
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & x^{42} + 1680 t x^{39} + 1038240 t^2 x^{36} + 313286400 t^3 x^{33} + 53317958400 t^4 x^{30} \\
& + 3990083328000 t^5 x^{27} + 441683020800000 t^6 x^{24} + 93283453992960000 t^7 x^{21} \\
& + 18418412410675200000 t^8 x^{18} - 534879213590937600000 t^9 x^{15} \\
& + 81739423771213824000000 t^{10} x^{12} + 10152143748914872320000000 t^{11} x^9 \\
& + 138824663357254533120000000 t^{12} x^6 - 2379851371838649139200000000 t^{13} x^3 \\
& + 7139554115515947417600000000 t^{14}
\end{aligned}$$

is a rational solution to the KdV equation.

### 3.7 Solutions of order 7

**Example 3.7** The function  $v_k(x, t)$  defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (12)$$

with

$$\begin{aligned}
n(x, t) = & -224 x^{54} - 532224 t x^{51} - 533433600 t^2 x^{48} - 274743705600 t^3 x^{45} - 90973899878400 t^4 x^{42} - \\
& 17237159976960000 t^5 x^{39} - 341160574132224000 t^6 x^{36} - 1094458263478272000000 t^7 x^{33} \\
& - 741477502246327418880000 t^8 x^{30} - 154020874806534655180800000 t^9 x^{27} \\
& - 16034390275582650875904000000 t^{10} x^{24} - 177938087284701740924928000000 t^{11} x^{21} \\
& - 93996037692962389591326720000000 t^{12} x^{18} - 4714367526984320747757895680000000 t^{13} x^{15} \\
& - 862920380445981412545685094400000000 t^{14} x^{12} \\
& + 20603060246307153105741938688000000000 t^{15} x^9
\end{aligned}$$

$$\begin{aligned}
& + 1483420337734115023613419585536000000000 t^{16}x^6 \\
& - 29668406754682300472268391710720000000000 t^{18} \\
& \text{and} \\
& d(x, t) = (x^{54} + 3024 tx^{51} + 3646944 t^2x^{48} + 2336947200 t^3x^{45} + 895482604800 t^4x^{42} \\
& + 211662185011200 t^5x^{39} + 34736257188864000 t^6x^{36} + 3672731815796736000 t^7x^{33} \\
& - 240312372794081280000 t^8x^{30} - 285039517734390988800000 t^9x^{27} \\
& - 26606879994975849676800000 t^{10}x^{24} - 5005337403127787421696000000 t^{11}x^{21} - \\
& 341169543038359143972864000000 t^{12}x^{18} + 130784494154255779579822080000000 t^{13}x^{15} \\
& + 11439814652602935286431744000000000 t^{14}x^{12} \\
& + 73582358022525546806221209600000000 t^{15}x^9 \\
& + 13244824444054598425119817728000000000 t^{16}x^6 \\
& + 370855084433528755903354896384000000000 t^{18})x^2
\end{aligned}$$

is a rational solution to the KdV equation.

### 3.8 Solutions of order 8

**Example 3.8** The function  $v_k(x, t)$  expressed as

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (13)$$

with

$$\begin{aligned}
n(x, t) = & 288 (-x^{69} - 4200 tx^{66} - 7623000 t^2x^{63} - 7730553600 t^3x^{60} - 5000876496000 t^4x^{57} \\
& - 2135257886361600 t^5x^{54} - 609838468081920000 t^6x^{51} - 163518036732610560000 t^7x^{48} \\
& - 25169090600103690240000 t^8x^{45} + 24130294695297087897600000 t^9x^{42} \\
& + 1726822505974983222722560000 t^{10}x^{39} + 5035052987586864132194304000000 t^{11}x^{36} \\
& + 705713595558650622946050048000000 t^{12}x^{33} + 83184077095173887007853117440000000 t^{13}x^{30} \\
& + 3466115954227539946436186603520000000 t^{14}x^{27} \\
& - 515161125869404732022375621591040000000 t^{15}x^{24} \\
& - 289794992859322072928498262579609600000000 t^{16}x^{21} \\
& - 2969807516143698277274066010243072000000000 t^{17}x^{18} \\
& + 219447562532189704845715806256889856000000000 t^{18}x^{15} \\
& - 216142591024938360620006524225490780160000000000 t^{19}x^{12} \\
& - 5465663450786023746477963825911452139520000000000 t^{20}x^9 \\
& - 36692565823458620955376540369755203174400000000000 t^{21}x^6 \\
& - 240794963216447200019658546176518520832000000000000 t^{22}x^3 \\
& + 1155815823438946560094361021647288899993600000000000 t^{23})x
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = & x^{72} + 5040 tx^{69} + 10674720 t^2x^{66} + 12666931200 t^3x^{63} + 9461359353600 t^4x^{60} \\
& + 4710504608409600 t^5x^{57} + 1632339334112256000 t^6x^{54} + 397315117938106368000 t^7x^{51} \\
& + 65431718836877352960000 t^8x^{48} + 9437611212912434872320000 t^9x^{45} \\
& + 4685020977200251685437440000 t^{10}x^{42} + 1631982341617690935243571200000 t^{11}x^{39} \\
& + 419453485567536573208697241600000 t^{12}x^{36} + 56299529140380549326208761856000000 t^{13}x^{33} \\
& - 4653435919389780491390674796544000000 t^{14}x^{30} \\
& + 1234383299752144727965990775685120000000 t^{15}x^{27}
\end{aligned}$$

$$\begin{aligned}
&+ 723770054157585558239884999655424000000000 t^{16} x^{24} \\
&+ 63065983896394392702398558917518950400000000 t^{17} x^{21} \\
&+ 463205120875098539875403609997626572800000000 t^{18} x^{18} \\
&- 66958128775409005273214599142943227904000000000 t^{19} x^{15} \\
&- 523286023959551923852244979136849772544000000000 t^{20} x^{12} \\
&- 61460047754293190100255705119339965317120000000000 t^{21} x^9 \\
&+ 1502560570470630528122669328141475569991680000000000 t^{22} x^6 \\
&+ 13869789881267358721132332259767466799923200000000000 t^{23} x^3 \\
&+ 27739579762534717442264664519534933599846400000000000 t^{24}
\end{aligned}$$

is a rational solution to the KdV equation.

### 3.9 Solutions of order 9

**Example 3.9** The function  $v_k(x, t)$  defined by

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (14)$$

with

$$\begin{aligned}
n(x, t) = &-360 (x^{87} + 6864 tx^{84} + 20845440 t^2 x^{81} + 36883123200 t^3 x^{78} + 42793643692800 t^4 x^{75} \\
&+ 34479450061056000 t^5 x^{72} + 20002051134793728000 t^6 x^{69} + 8731536922920075264000 t^7 x^{66} \\
&+ 2751355428529975787520000 t^8 x^{63} + 497883367475949628293120000 t^9 x^{60} \\
&+ 217830264215804373294120960000 t^{10} x^{57} + 325706259771597584319475875840000 t^{11} x^{54} \\
&+ 236136684508248983555530712678400000 t^{12} x^{51} + 90842180055466965502594778136576000000 t^{13} x^{48} \\
&+ 22464586263799027916853721618710528000000 t^{14} x^{45} + 3471147967080811533200910965060665344000000 t^{15} x^{42} \\
&+ 305398496769968754512043060254019158016000000 t^{16} x^{39} \\
&+ 17506664385513663140183161643807413370880000000 t^{17} x^{36} \\
&+ 23112992363226111560405467570860512929382400000000 t^{18} x^{33} \\
&+ 389754549604414714891502174564268060652339200000000 t^{19} x^{30} \\
&+ 355735131990323431740828949883178812563469107200000000 t^{20} x^{27} \\
&+ 93789856566957965633368930136922244647033294028800000000 t^{21} x^{24} \\
&- 6286020360053413047972505258268690471085133529088000000000 t^{22} x^{21} \\
&- 162329335579032153111484405548792940444728667668480000000000 t^{23} x^{18} \\
&- 75986637119012033976830106990619719139025700740136960000000000 t^{24} x^{15} \\
&- 2856556091043632926211314732491165804870086303730892800000000000 t^{25} x^{12} \\
&+ 130874642656649551250399943908988460516329488032097894400000000000 t^{26} x^9 \\
&- 63454372197163418788072700077085314189735509348895948800000000000 t^{27} x^6 \\
&+ 5829870445614389101154179319582213241181949921429815296000000000000 t^{28} x^3 \\
&+ 19988127242106476918242900524281873969766685444902223872000000000000 t^{29} x)
\end{aligned}$$

and

$$\begin{aligned}
d(x, t) = &x^{90} + 7920 tx^{87} + 27324000 t^2 x^{84} + 54752544000 t^3 x^{81} + 71622115488000 t^4 x^{78} + 65066476598323200 t^5 x^{75} \\
&+ 42677705283333120000 t^6 x^{72} + 20691357050667663360000 t^7 x^{69} + 7527123164747777310720000 t^8 x^{66} \\
&+ 2107511386632847215820800000 t^9 x^{63} + 469311169848026334959370240000 t^{10} x^{60} \\
&+ 40479537955403752535044915200000 t^{11} x^{57} - 38641983187081149702891896832000000 t^{12} x^{54} \\
&- 28229717733062389116522121396224000000 t^{13} x^{51} - 9755247601752680435560998198312960000000 t^{14} x^{48} \\
&- 1605415983686251630919982642723028992000000 t^{15} x^{45}
\end{aligned}$$

$$\begin{aligned}
& - 371454265116593208604981692729309265920000000 t^{16} x^{42} \\
& - 84128558826942877613705878234865231462400000000 t^{17} x^{39} \\
& + 28352032914692269976628454317964798407475200000000 t^{18} x^{36} \\
& + 13128049960975410746073137017106342394986496000000000 t^{19} x^{33} \\
& + 2430885024143419353984724033937664218012162457600000000 t^{20} x^{30} \\
& + 146410665507920500533143394324200504477481959424000000000 t^{21} x^{27} \\
& + 6602294506252984121623391996311398594267033108480000000000 t^{22} x^{24} \\
& + 634069343909296009543641088160215353258291033538560000000000 t^{23} x^{21} \\
& + 19508652154233834233358255478416252284354530010726400000000000 t^{24} x^{18} \\
& - 2947680626541557416229151402182284824997703481242419200000000000 t^{25} x^{15} \\
& + 20280161568695694925165280564409368881662627277701120000000000000 t^{26} x^{12} \\
& + 2577833870509763888265453440631590888958005067298897920000000000000 t^{27} x^9 \\
& + 274836749578964057625839882208875767084291924867405578240000000000000 t^{28} x^6 \\
& - 299821908631597153773643507864228109546500281673533358080000000000000 t^{29} x^3 \\
& + 599643817263194307547287015728456219093000563347066716160000000000000 t^{30}
\end{aligned}$$

is a rational solution to the KdV equation.

### 3.10 Solutions of order 10

**Example 3.10** The function  $v_k(x, t)$  expressed as

$$v_k(x, t) = \frac{n(x, t)}{d(x, t)} \quad (15)$$

with

$$\begin{aligned}
n(x, t) = & - 440 x^{108} - 4656960 t x^{105} - 2223936000 t^2 x^{102} - 63527473152000 t^3 x^{99} - 121936711792896000 t^4 x^{96} \\
& - 167286869336530944000 t^5 x^{93} - 170457089665294688256000 t^6 x^{90} - 132572493457040495738880000 t^7 x^{87} \\
& - 79773272960436964977868800000 t^8 x^{84} - 37509353281280939031016243200000 t^9 x^{81} \\
& - 14763698113159334472421736448000000 t^{10} x^{78} - 4978722500676502355093319843840000000 t^{11} x^{75} \\
& + 307544827106787687424323357769728000000 t^{12} x^{72} + 3062181416347097524628843272549171200000000 t^{13} x^{69} + \\
& 29999708228943136744320054622015822233600000000 t^{14} x^{66} \\
& + 1673514712689379266786652586236034049638400000000 t^{15} x^{63} \\
& + 614189063528038366650760463291115511794892800000000 t^{16} x^{60} \\
& + 155794693835149410545599807873777332894931353600000000 t^{17} x^{57} \\
& + 2902768310737004959527251853307297196031344640000000000 t^{18} x^{54} \\
& + 5218610071634201355190994147633369369868352094208000000000 t^{19} x^{51} \\
& + 78786645454296974763999242297079631825147527168000000000000 t^{20} x^{48} \\
& - 378545538586374130684752374182777086090607685208637440000000000 t^{21} x^{45} \\
& - 131329475080569841532200337554391930500452935693689159680000000000 t^{22} x^{42} \\
& - 3513566821938187068323458047568782732506359185178361856000000000000 t^{23} x^{39} \\
& - 1213277153327325812628585554718526296390804347854945202995200000000000 t^{24} x^{36} \\
& + 259542378662438531031176273828288551677962322567714166328524800000000000 t^{25} x^{33} \\
& - 161332608981727230241305201619521450571485883696858596226419916800000000000 t^{26} x^{30} \\
& - 38619657275477595326725531332288937493194588220016949924227710976000000000000 t^{27} x^{27} \\
& - 3949639858292160818760942549919713001064650682276256769927135887360000000000000 t^{28} x^{24} \\
& - 22398063092453206574593483244714613369097180484635029213910275194880000000000000 t^{29} x^{21}
\end{aligned}$$



$$\begin{aligned}
& - 9599260625185284618515095398119217595753908235129788366990376823685120000000000000000 t^{30}x^{18} \\
& - 244374061957435675906484370378509720841975249471428638815693042069012480000000000000000000000000 t^{31}x^{15} \\
& - 4608541109217035295960114787434198123641592451380593376086737533492264960000000000000000000000000 t^{32}x^{12} \\
& + 5199379712962809051852437196079608139493078663096054065328626960863068160000000000000000000000000 t^{33}x^9 \\
& + 37435533933322251733375478117731786043501663742915892703661141182140907520000000000000000000000000 t^{34}x^6 \\
& - 374355339333222517333754781177317860435016637429158927036611411821409075200000000000000000000000000 t^{36} \\
& \text{and} \\
& d(x, t) = (x^{108} + 11880 tx^{105} + 63082800 t^2 x^{102} + 199635321600 t^3 x^{99} + 423349568928000 t^4 x^{96} \\
& + 640877607943372800 t^5 x^{93} + 720356494667217408000 t^6 x^{90} + 617087988787839528960000 t^7 x^{87} \\
& + 410285748104906240286720000 t^8 x^{84} + 214683708353169921004339200000 t^9 x^{81} \\
& + 89188449953327630048644300800000 t^{10} x^{78} + 29191715419424489706841178112000000 t^{11} x^{75} \\
& + 7751684038952405989677244612608000000 t^{12} x^{72} + 2539364289397734716586557530177536000000 t^{13} x^{69} \\
& + 1689948975066768235981089928013414400000000 t^{14} x^{66} \\
& + 1130347213654414430863954455878455787520000000 t^{15} x^{63} \\
& + 506556111685474474049446101072644721868800000000 t^{16} x^{60} \\
& + 170205599051244573557277541381253103196569600000000 t^{17} x^{57} \\
& + 42383814923944123214485620080346613723378483200000000 t^{18} x^{54} + \\
& 177359494426191035376420187330956704562020352000000000 t^{19} x^{51} \\
& - 200783526308251655278234458571207724313002115072000000000 t^{20} x^{48} \\
& + 15707269194135167813143155649809134346239031115776000000000000 t^{21} x^{45} \\
& + 98060016736581296781691171288377731242277434636776243200000000000 t^{22} x^{42} \\
& + 2354802714065562148570473679063743107392403519798928998400000000000 t^{23} x^{39} \\
& + 33144668029747995763120450273639821218583553580325666816000000000000 t^{24} x^{36} \\
& + 32966193401700514519086283668796827258457603526842221645004800000000000 t^{25} x^{33} \\
& + 2835162318039931346255814548110509810564151862272965658804224000000000000 t^{26} x^{30} \\
& - 68928880057741411978666661846038373378046700602904509593681920000000000000 t^{27} x^{27} \\
& - 51377278748282079265812100726595897869050786810992623127195811840000000000000 t^{28} x^{24} \\
& - 631540059369508598822797386451140237938001415764898497226667458560000000000000 t^{29} x^{21} \\
& - 706399143837922099139889865939402543693176476409406132348522762076160000000000000 t^{30} x^{18} \\
& + 65932146650332942670872712529373985285363388094360016735129733326438400000000000000 t^{31} x^{15} \\
& + 572519824430265071938047106336545262612762738118317284826618915297689600000000000000 t^{32} x^{12} \\
& + 18569213260581460899472989985998600498189566653914478804745096288796672000000000000000 t^{33} x^9 \\
& + 334245838690466296190513819747974808967412199770460618485411733198340096000000000000000 t^{34} x^6 \\
& + 46794417416665281466671934764716473255437707967864486587957642647767613440000000000000000 t^{36} x^2
\end{aligned}$$

is a rational solution to the KdV equation.

## 4 Conclusion

New representation of the solutions to the KdV equation in terms of wronskians have been constructed. Precisely, these solutions can be written as the second derivative with respect to  $x$  of a logarithm of a wronskian involving particular polynomials. Surprisingly, although the representation of the solutions is different from this given in [62], we recover the same solutions.

In particular, rational solutions appear as the quotient of two polynomials in  $x$  and  $t$ ; the numerator is a polynomial of degree  $n(n+1) - 2$  in  $x$  and the denominator is a polynomial of degree  $n(n+1)$  in  $x$  have been constructed. We get a very efficient method to construct rational solutions to the KdV equation. Other recent works as [83], [85] or [80] can be quoted. The solutions presented in this paper are different from these presented in a previous work of the author [52].

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