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RESEARCH ARTICLE

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ABHY Associahedra and Newton polytopes of F-polynomials for cluster algebras of simply laced finite type

Véronique Bazier-Matte | **Nathan Chapelier-Laget** | **Guillaume Douville** | **Kaveh Mousavand** | **Hugh Thomas** | **Emine Yildirim**

LaCIM, UQAM, Montreal, Quebec, Canada

Correspondence
Hugh Thomas, LaCIM, UQAM, Montreal, Quebec, Canada.
Email: hugh.ross.thomas@gmail.com

Present address
Véronique Bazier-Matte, Département de mathématiques et de statistique, Université Laval, Quebec City, Quebec, Canada
Nathan Chapelier-Laget, School of Mathematics and Statistics, University of Sydney, Sydney, Australia
Kaveh Mousavand, Representation Theory and Algebraic Combinatorics Unit, Okinawa Institute of Science and Technology, Kunigami-gun, Okinawa, Japan
Emine Yildirim, School of Mathematics, University of Leeds, Leeds, UK

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Abstract

A new construction of the associahedron was recently given by Arkani-Hamed, Bai, He, and Yan in connection with the physics of scattering amplitudes. We show that their construction (suitably understood) can be applied to construct generalized associahedra of any simply laced Dynkin type. Unexpectedly, we also show that this same construction produces Newton polytopes for all the F-polynomials of the corresponding cluster algebras. In addition, we show that the toric variety associated to the g-vector fan has the property that its nef cone is simplicial.

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1 | INTRODUCTION

Let Q be a Dynkin quiver. That is to say, Q is an orientation of a simply laced Dynkin diagram, with vertices numbered 1 to n. Let B_0 be the matrix with the property that the entry $(B_0)_{ij}$ equals the number of arrows from i to j minus the number of arrows from j to i. Starting from the matrix B_0, one can define a corresponding cluster algebra $\mathcal{A}(Q)$, which is a commutative ring with a distinguished set of generators, known as cluster variables, grouped together into overlapping sets of size n, known as clusters. Because of our particular choice of Q, there are only finitely many cluster variables. Each cluster variable has an associated vector in $\mathbb{Z}^n$, its g-vector. The g-vectors are the rays of the g-vector fan, the maximal-dimensional cones of which correspond to clusters. This fan is complete, in the sense that the union of its cones is all of $\mathbb{R}^n$.

Considerable attention has been given to the problem of constructing polytopes whose outer normal fan is the g-vector fan. Such polytopes are called *generalized associahedra*. The question of whether the g-vector fan can be realized in this way was first raised by Fomin and Zelevinsky in [16], and first solved by Chapoton, Fomin, and Zelevinsky [10]. In fact, g-vectors had not yet been defined at the time of these two papers, but the fan that they study would subsequently be recognized as the g-vector fan associated to a particular orientation of each Dynkin diagram. Subsequently, [18] gave a construction that solves the problem as described here. [19] solves a more general problem, where the initial quiver is assumed only to be mutation-equivalent to a Dynkin quiver, and not necessarily Dynkin itself. The papers we have cited actually work in greater generality, in that they also treat Dynkin types that are not simply laced. In this paper, we focus on the simply laced case because it is technically easier.

The prehistory of this problem goes back much further. The combinatorics of the face lattice of a generalized associahedron is not sensitive to the orientation of Q. When Q is of type A_n, the face lattice is that of the associahedron, as originally defined, as a cell complex, by Stasheff [33]. The first polytopal realization to appear in the literature is due to Lee [23], and many others followed. An excellent overview is provided by [8]. One particular realization of the associahedron has been studied by many authors [24, 28, 30, 32] (see also [27]), including Loday, whose name is most often associated to it. The associahedra constructed in this way turn out to be generalized associahedra corresponding to the linear orientation of the A_n diagram. Recently, yet another construction of this associahedron was given in the physics literature by Arkani-Hamed, Bai, He, and Yan [1], in connection with scattering amplitudes for bi-adjoint scalar φ^3 theory. We refer to this as the ABHY construction.

In this paper, we extend the ABHY construction to arbitrary (simply laced) Dynkin quivers. We further show that, quite surprisingly, this same construction realizes the Newton polytopes of the F-polynomials of the corresponding cluster algebras. (The F-polynomials are certain polynomials in n variables that are a reparameterization of the cluster variables; in particular, the cluster variables can be recovered from them.) In fact, the ABHY construction can be seen even more naturally as constructing Newton polytopes of certain universal F-polynomials, which we define. The universal F-polynomials bear the same relationship to the cluster algebra with universal coefficients as the usual F-polynomials do to the cluster algebra with principal coefficients. We also consider the toric variety associated to the g-vector fan. We show that our results on realizations of associahedra imply that the nef cone of this toric variety is simplicial.

After the appearance of the first version of this paper in 2018, Palu, Padrol, Pilaud, and Plamondon [26] showed that a very similar construction can be applied to construction of generalized associahedra for all seeds in all Dynkin-type cluster algebras. The first author of the present paper also extended the techniques in the present paper in her thesis [5], giving another proof that the

same construction can be applied to any seed of simply laced Dynkin type, and that this construction also gives Newton polytopes of F-polynomials in the same generality. Fei [13, 14] takes a less explicit approach but proves a very general result, to the effect that if A is any algebra with finitely many τ-rigid indecomposable modules, then a polytope dual to the τ tilting fan of A can be constructed as the Minkowski sum of the submodule polytopes of the τ-rigid indecomposables.

Our result that our polytopes also yield Newton polytopes of F-polynomials has subsequently been extended to nonsimply laced types by Arkani-Hamed, He, and Lam [2], by using a folding argument to reduce to the case we consider.

2 | CONSTRUCTION

In the interests of self-containedness, we will begin with a completely explicit, if somewhat unmotivated, description of our construction. We will then provide a more representation-theoretic description, which is needed for the proof of correctness. Statements that are not proved in this section will be proved in the following section (or will turn out to be equivalent to well-known facts from the theory of quiver representations).

We write Q_0 for the set of vertices of Q and Q_1 for the set of arrows. We assume that $Q_0 = \{1, \dots, n\}$. We write Q^{op} for the opposite quiver of Q, all of whose arrows are reversed compared to Q.

Draw $\mathbb{Z}_{\geqslant 0}$ many copies of Q. The vertices in this quiver are denoted (i, j) where $i \in \mathbb{Z}_{\geqslant 0}$ and $j \in Q_0$. We also add arrows between the copies of Q: if there is an arrow from j to k in Q, we put an arrow from (i, k) to $(i + 1, j)$. This infinite quiver we denote by $\mathbb{Z}_{\geqslant 0}Q$. See Example 1 for an example of the initial part of the quiver $\mathbb{Z}_{\geqslant 0}Q$ for Q the quiver $1 \longrightarrow 2 \longleftarrow 3$.

We associate to each vertex (i, j) a vector in $\mathbb{Z}^n$, which we call the dimension vector, and which we denote $\underline{\dim}(i, j)$. (What exactly it is the dimension of will be explained in the following section. For now, it is simply an integer vector.) To $(0, j)$, we associate the dimension vector $\underline{\dim}(0, j)$ obtained by putting a 1 at every vertex that can be reached from j by following arrows of Q^{op} (including the vertex j itself), and 0 at all other vertices. For (i, j) with $i > 0$, we associate the dimension vector that satisfies:

$$\underline{\dim}(i, j) + \underline{\dim}(i - 1, j) = \sum_{(i-1,j)\to(i',j')\to(i,j)} \underline{\dim}(i', j').$$

Here, the sum on the right-hand side runs over all vertices (i', j') on a path of length two from $(i - 1, j)$ to (i, j). Starting with the dimension vectors already defined for $(0, j)$, these equations allow us to deduce the value of $\underline{\dim}(i, j)$ for all (i, j) in $\mathbb{Z}_{\geqslant 0}Q$ inductively.

It turns out that the dimension vectors calculated in this way have the property that they are nonzero and sign-coherent, in the sense that each $\underline{\dim}(i, j)$ either has all entries nonnegative or all entries nonpositive. In these two cases, we simply say that $\underline{\dim}(i, j)$ is nonnegative or nonpositive, respectively.

A certain subset of the vertices of $\mathbb{Z}_{\geqslant 0}Q$ are in natural correspondence with the cluster variables of $\mathcal{A}(Q)$. For $1 \leqslant j \leqslant n$, define i_j to be the maximal index such that all of $\underline{\dim}(0, j), \underline{\dim}(1, j), \dots, \underline{\dim}(i_j, j)$ are nonnegative.

Define

$$\mathcal{I}^+ = \{(i, j) \mid 1 \leqslant j \leqslant n, 0 \leqslant i \leqslant i_j\}$$
$$\mathcal{I} = \{(i, j) \mid 1 \leqslant j \leqslant n, 0 \leqslant i \leqslant i_j + 1\}.$$

There is a natural bijection between the elements of $\mathcal{I}$ and the cluster variables of $A(Q)$. We recall the details of the correspondence in Section 4. We write x_{ij} for the cluster variable associated to $(i,j) \in \mathcal{I}$. We say that two cluster variables are *compatible* if there is some cluster that contains both of them. We say that two elements of $\mathcal{I}$ are compatible if the corresponding cluster variables are.

Consider a real vector space with basis indexed by the elements of $\mathcal{I}$, say $V = \mathbb{R}^{\mathcal{I}}$. For $(i,j) \in \mathcal{I}$, we write p_{ij} for the coordinate function on V indexed by (i,j).

Fix a collection of positive integers $\underline{c} = (c_{ij})_{ij \in \mathcal{I}^+}$. We construct an associahedron for each choice of $\underline{c}$.

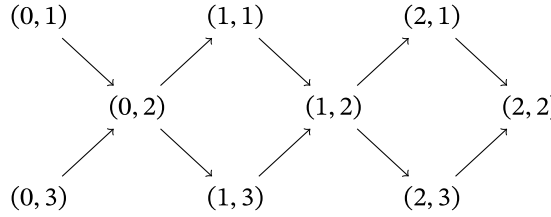
Consider the following collection of equations, one for each $(i,j) \in \mathcal{I}^+$.

$$p_{i,j} + p_{i+1,j} = c_{ij} + \sum_{(i,j)\to(i',j')\to(i+1,j)} p_{i',j'}$$

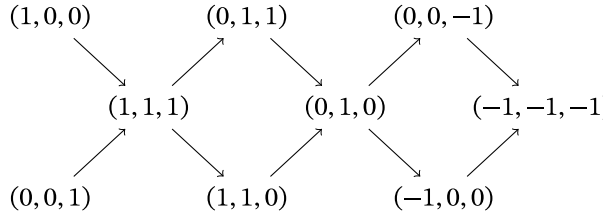
We call these equations the $\underline{c}$-*deformed mesh relations*. They define an n-dimensional affine space $\mathbb{E}_{\underline{c}}$ inside V.

Write $\mathbb{U}_{\underline{c}}$ for the region inside $\mathbb{E}_{\underline{c}}$ all of whose coordinates p_{ij} are nonnegative. That is to say, $\mathbb{U}_{\underline{c}}$ is the intersection of the positive orthant in V with $\mathbb{E}_{\underline{c}}$.

Example 1. Let us consider the quiver $Q : 1 \longrightarrow 2 \longleftarrow 3$. The following shows the part of $\mathbb{Z}_{\geqslant 0}Q$ whose vertices are in $\mathcal{I}$, labeled by the elements of $\mathcal{I}$.



The corresponding dimension vectors are:



We fix a six-tuple of positive integers $\underline{c} = (c_{ij})_{(i,j)\in\mathcal{I}^+}$. The region $\mathbb{E}_{\underline{c}}$ is cut out by the following equations:

$$p_{01} + p_{11} = p_{02} + c_{01}$$

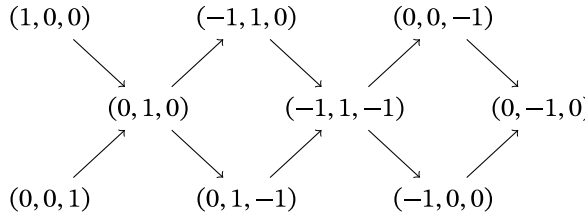
$$p_{03} + p_{13} = p_{02} + c_{03}$$

$$p_{02} + p_{12} = p_{11} + p_{13} + c_{02}$$

$$p_{11} + p_{21} = p_{12} + c_{11}$$
$$p_{13} + p_{23} = p_{12} + c_{13}$$
$$p_{12} + p_{22} = p_{21} + p_{23} + c_{12}.$$

Notice that, by construction, the dimension vectors satisfy the $\underline{0}$-deformed mesh relations (which are generally called the *mesh relations*), that is to say, the deformed mesh relations with the deformation parameters set to zero. There is another important collection of vectors that satisfy these equations: the *g-vectors*. By definition, $g(0, j)$ is the jth standard basis vector, and the other g-vectors are determined by the mesh relations. The *g-vector fan* is the fan whose rays are the g-vectors, and such that a collection of rays generates a cone of the fan if and only if the corresponding collection of cluster variables is compatible.

Example 2. We continue Example 1. In this case, the corresponding g-vectors are as follows:



Note that, in the example, the g-vectors corresponding to the elements of $\mathcal{I} \setminus \mathcal{I}^+$ are the negative standard basis vectors. This is a general phenomenon, and allows us to define an important projection $\pi : V \to \mathbb{R}^n$: the kth coordinate of the projection to $\mathbb{R}^n$ is given by p_{ij} for $(i, j) \in \mathcal{I} \setminus \mathcal{I}^+$ such that $g(i, j) = -e_k$. This projection defines a bijection between $\mathbb{E}_{\underline{c}}$ and $\mathbb{R}^n$. We define $\mathbb{A}_{\underline{c}} = \pi(\mathbb{U}_{\underline{c}})$.

Given a full-dimensional polytope P in $\mathbb{R}^n$, there is a fan associated to it called the *outer normal fan* of P, denoted Σ_P. For each facet F of P, let ρ_F be the ray pointing in the direction perpendicular to F and away from P. The collection of rays $\{\rho_F \mid F \text{ is a facet of } P\}$ are the rays of Σ_P; the set of rays $\{\rho_{F_1}, \dots, \rho_{F_j}\}$ generates a cone in Σ_P if and only if there is a face G of P such that the facets of P containing G are exactly $F_1, \dots, F_j$.

The following is our main theorem about realizing generalized associahedra.

Theorem 1.

(1) *Each facet of $\mathbb{U}_{\underline{c}}$ is defined by the vanishing of exactly one coordinate of V.*
(2) *The map sending the face G of $\mathbb{U}_{\underline{c}}$ to the set*

$$\{\alpha \in \mathcal{I} \mid G \text{ lies on the hyperplane } p_\alpha = 0\}$$

is an order-reversing bijection from the nonempty faces of $\mathbb{U}_{\underline{c}}$ to the faces of the cluster complex.
(3) *Under the map defined in the previous point, the vertices of $\mathbb{U}_{\underline{c}}$ correspond to clusters.*
(4) *π is an isomorphism of affine spaces between $\mathbb{E}_{\underline{c}}$ and $\mathbb{R}^n$. Consequently, the faces of $\mathbb{A}_{\underline{c}}$ also correspond bijectively to compatible sets in $\mathcal{I}$.*
(5) *If F_{ij} is the facet of $\mathbb{U}_{\underline{c}}$ given by $p_{ij} = 0$, the normal to $\pi(F_{ij})$ oriented away from $\mathbb{A}_{\underline{c}}$ is the ray generated by $g(i, j)$.*

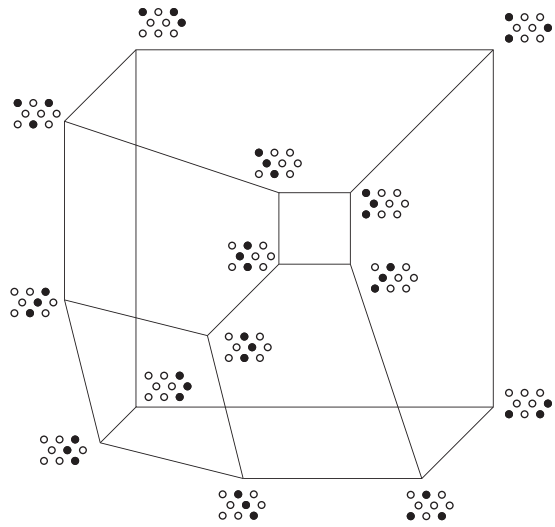


FIGURE 1 Associahedron corresponding to $1 \to 2 \leftarrow 3$.

Example 3. We continue Example 2.

In Figure 1 we show an illustration of $\mathbb{A}_{\underline{c}}$ in this case. At each vertex, we have drawn a small copy of the quiver with vertex set $\mathcal{I}$ (with the arrows omitted) on which we have marked the cluster corresponding to the vertex.

To represent the vertices of $\mathbb{U}_{\underline{c}}$ in V, it is convenient to write the values of p_{ij} at position (i, j), with the value c_{ij} positioned between (i, j) and $(i + 1, j)$. We write c_{ij} in red to make clear the distinction between the two kinds of entries. To see $\mathbb{A}_{\underline{c}}$, we forget everything in each row except the last entry.

In the above example, we set all the $c_{ij} = 1$. The vertices are as follows:

$$
\begin{array}{ccccccccccc}
0 & 1 & 1 & 1 & 3 & & 0 & 1 & 1 & 1 & 2 & & 1 & 1 & 0 & 1 & 3 & \\
 & 0 & 1 & 3 & 1 & 4 & & 0 & 1 & 2 & 1 & 4 & & 0 & 1 & 2 & 1 & 4 \\
0 & 1 & 1 & 1 & 3 & & 1 & 1 & 0 & 1 & 3 & & 0 & 1 & 1 & 1 & 2 & \\
\\
1 & 1 & 0 & 1 & 2 & & 2 & 1 & 0 & 1 & 1 & & 2 & 1 & 1 & 1 & 0 & \\
 & 0 & 1 & 1 & 1 & 4 & & 1 & 1 & 0 & 1 & 3 & & 2 & 1 & 0 & 1 & 2 \\
1 & 1 & 0 & 1 & 2 & & 2 & 1 & 0 & 1 & 1 & & 3 & 1 & 0 & 1 & 1 & \\
\\
3 & 1 & 0 & 1 & 1 & & 3 & 1 & 1 & 1 & 0 & & 3 & 1 & 2 & 1 & 0 & \\
 & 2 & 1 & 0 & 1 & 2 & & 3 & 1 & 0 & 1 & 1 & & 4 & 1 & 1 & 1 & 0 \\
2 & 1 & 1 & 1 & 0 & & 3 & 1 & 1 & 1 & 0 & & 3 & 1 & 2 & 1 & 0 & \\
\\
3 & 1 & 0 & 1 & 3 & & 0 & 1 & 3 & 1 & 0 & & 3 & 1 & 2 & 1 & 3 & \\
 & 2 & 1 & 2 & 1 & 2 & & 2 & 1 & 2 & 1 & 2 & & 4 & 1 & 4 & 1 & 0 \\
0 & 1 & 3 & 1 & 0 & & 3 & 1 & 0 & 1 & 3 & & 0 & 1 & 5 & 1 & 0 & \\
\\
 & & & & & & 0 & 1 & 5 & 1 & 0 & & 0 & 1 & 5 & 1 & 3 & \\
 & & & & & & & 4 & 1 & 4 & 1 & 0 & & 4 & 1 & 7 & 1 & 0 \\
 & & & & & & 3 & 1 & 2 & 1 & 3 & & 0 & 1 & 5 & 1 & 3 &
\end{array}
$$

3 | PROOF OF CORRECTNESS

Since the foundational work of [7, 25], it has been clear that representations of quivers are extremely useful in understanding the combinatorics of cluster algebras. We will use a setting that is inspired by [7], though described in somewhat different terms.

The quiver $\mathbb{Z}_{\geqslant 0}Q$ restricted to the vertices of $\mathcal{I}^+$, gives the Auslander–Reiten quiver for the category of representations of Q^{op}. If we restrict to $\mathcal{I}$ instead, we get n additional vertices added to the right-hand end. We understand this as the Auslander–Reiten quiver of a full subcategory of $D^b(\operatorname{rep} Q^{op})$, whose vertices correspond to the indecomposable quiver representations together with the $P_j[1]$ where P_j is the projective representation at vertex j and [1] is the shift functor. We write W_{ij} for the object in $D^b(\operatorname{rep} Q^{op})$ corresponding to the vertex $(i,j) \in \mathcal{I}$. The indecomposable projective P_j is W_{0j}. As is well-known, $\underline{\dim}(i,j)$, as defined in the previous section, is the dimension vector of W_{ij} if $(i,j) \in \mathcal{I}^+$; to include the cases of $(i,j) \in \mathcal{I} \setminus \mathcal{I}^+$ as well, we can say that $\underline{\dim}(i,j)$ is the class in the Grothendieck group of $D^b(\operatorname{rep} Q^{op})$ of W_{ij}.

As it will be useful later in the paper, we will begin by weakening the hypothesis on the tuple $\underline{c}$: we will begin by assuming that its entries are nonnegative, rather than all being strictly positive. Let

$$M_{\underline{c}} = \bigoplus_{(i,j)\in\mathcal{I}^+} W_{ij}^{\oplus c_{ij}}.$$

Representation theory gives us a natural point $\underline{v}_{\underline{c}}$ in V. The coordinates of this point are defined by:

$$v_{ij} = \dim \operatorname{Hom}(W_{ij}, M_{\underline{c}}).$$

Lemma 1. *The point $\underline{v}_{\underline{c}}$ is in $\mathbb{E}_{\underline{c}}$.*

Proof. Suppose that $(i,j) \in \mathcal{I}^+$. Let E_{ij} be the direct sum of all the $W_{i'j'}$ for (i',j') on a path of length two between (i,j) and $(i+1,j)$. We therefore have an Auslander–Reiten triangle in $D^b(\operatorname{rep} Q)$:

$$W_{ij} \to E_{ij} \to W_{i+1,j} \to W_{ij}[1]$$

We must verify, for each $(i,j) \in \mathcal{I}^+$ that

$$\dim(\operatorname{Hom}(W_{ij}, M_{\underline{c}})) - \dim(\operatorname{Hom}(E_{ij}, M_{\underline{c}})) + \dim(\operatorname{Hom}(W_{i+1,j}, M_{\underline{c}})) = c_{ij} \tag{1}$$

We know that the following sequence is exact except at the right-hand end:

$$0 = \operatorname{Hom}(W_{ij}[1], M_{\underline{c}}) \to \operatorname{Hom}(W_{i+1,j}, M_{\underline{c}}) \to \operatorname{Hom}(E_{ij}, M_{\underline{c}}) \to \operatorname{Hom}(W_{ij}, M_{\underline{c}})$$

Thus, the left-hand side of (1) is nothing but the dimension of the quotient of $\operatorname{Hom}(W_{ij}, M_{\underline{c}})$ by the image of $\operatorname{Hom}(E_{ij}, M_{\underline{c}})$. By the definition of Auslander–Reiten triangles, any map from W_{ij} to a summand of $M_{\underline{c}}$ which is not an isomorphism factors through E_{ij}. The left-hand side of (1) is therefore the multiplicity of W_{ij} in $M_{\underline{c}}$, which is exactly c_{ij}. □

We can now say

$$\mathbb{E}_{\underline{c}} = \underline{v}_{\underline{c}} + \mathbb{E}_{\underline{0}}.$$

Here, by $\mathbb{E}_{\underline{0}}$, we mean the points of V satisfying the ($\underline{0}$-deformed) mesh relations. We see that $\mathbb{E}_{\underline{0}}$ is an n-dimensional vector space. For $1 \leqslant k \leqslant n$, define a vector $\underline{d}^k = (d^k_{ij})_{ij\in\mathcal{I}}$ by setting $d^k_{ij} = \underline{\dim}(i,j)_k$. Then $\underline{d}^1, \dots, \underline{d}^n$ are a basis for $\mathbb{E}_{\underline{0}}$.

Similarly, for $1 \leqslant k \leqslant n$, and $(i,j) \in \mathcal{I}$, let g^k_{ij} be the signed multiplicity of P_k in a projective resolution for W_{ij}. That is to say, we take a resolution

$$P^1 \to P^0 \to W_{ij} \to P^1[1] \tag{2}$$

and take g^k_{ij} to be the multiplicity of P_k in P^0 minus its multiplicity in P^1. Now define the vector $\underline{g}^k = (g^k_{ij})_{ij\in\mathcal{I}}$. Then the set of $\underline{g}^k$ are also a basis for $\mathbb{E}_0$. This is clear because (2) implies that $\underline{\dim} W_{ij} = \underline{\dim} P^0 - \underline{\dim} P^1$, from which we see that we can calculate the g-vector of W_{ij} by expressing $\underline{\dim} W_{ij}$ in the basis $\underline{\dim} P_1, \underline{\dim} P_2, \dots, \underline{\dim} P_n$ of $\mathbb{Z}^n$. It follows that the collection of vectors $\{\underline{g}^k\}$ are related to the vectors $\{\underline{d}^k\}$ by a change of basis.

As we have already mentioned, there is a relation of compatibility of cluster variables, which we can take as defining a notion of compatibility for elements of $\mathcal{I}$. Two elements α, β of $\mathcal{I}$ are compatible if and only if $\mathrm{Ext}^1(W_\alpha, W_\beta) = 0 = \mathrm{Ext}^1(W_\beta, W_\alpha)$. See [25, eq. (3.3)] and [7, Corollary 4.3].

The following is the key lemma.

Lemma 2. *Suppose the tuple $\underline{c}$ consists of strictly positive integers. Let $\alpha, \beta \in \mathcal{I}$. If they are incompatible, then there is no point of $\mathbb{U}_{\underline{c}}$ lying on the intersection of the hyperplanes $p_\alpha = 0$ and $p_\beta = 0$.*

Proof. If α and β are incompatible, then $\mathrm{Ext}^1(W_\alpha, W_\beta) \neq 0$ or $\mathrm{Ext}^1(W_\beta, W_\alpha) \neq 0$. Without loss of generality, suppose the former. This implies, in particular, that $\beta \in \mathcal{I}^+$. Choose a nonsplit triangle

$$W_\beta \to E \to W_\alpha \to W_\beta[1].$$

Suppose that $\underline{x} = (x_\gamma)_{\gamma\in\mathcal{I}} \in \mathbb{E}_{\underline{c}}$. As $\mathbb{E}_{\underline{c}} = \underline{v}_{\underline{c}} + \mathbb{E}_{\underline{0}}$, and $\underline{d}^1, \dots, \underline{d}^n$ span $\mathbb{E}_{\underline{0}}$, there is some n-tuple $(m_1, \dots, m_n)$ such that $\underline{x} = \underline{v}_{\underline{c}} + \sum_{k=1}^n m_k \underline{d}^k$ or in other words, for all $\gamma \in \mathcal{I}$, we have

$$x_\gamma = \dim \mathrm{Hom}(W_\gamma, M_{\underline{c}}) + \sum_{k=1}^n m_k \underline{\dim}(\gamma)_k.$$

Now suppose that $x_\alpha = x_\beta = 0$. Note that $\dim E_i = \dim(W_\beta)_i + \dim(W_\alpha)_i$. Also note that we have the Hom long exact sequence

$$\mathrm{Ext}^{-1}(W_\beta, M_{\underline{c}}) \to \mathrm{Hom}(W_\alpha, M_{\underline{c}}) \to \mathrm{Hom}(E, M_{\underline{c}}) \to \mathrm{Hom}(W_\beta, M_{\underline{c}}) \to \mathrm{Ext}^1(W_\alpha, M_{\underline{c}})$$

On the left-hand end $\mathrm{Ext}^{-1}(W_\beta, M_{\underline{c}}) = \mathrm{Hom}(W_\beta, M_{\underline{c}}[-1]) = 0$.

As $c_\beta > 0$, the map from $\mathrm{Hom}(W_\beta, M_{\underline{c}})$ to $\mathrm{Ext}^1(W_\alpha, M_{\underline{c}})$ is nonzero. Thus, $\dim \mathrm{Hom}(E, M_{\underline{c}}) < \dim \mathrm{Hom}(W_\alpha, M_{\underline{c}}) + \dim \mathrm{Hom}(W_\beta, M_{\underline{c}})$.

Therefore,

$$\begin{aligned}\dim \mathrm{Hom}(E, M_{\underline{c}}) + \sum_{k=1}^{n} m_k \dim E_k \\ < \dim \mathrm{Hom}(W_\alpha, M_{\underline{c}}) + \dim \mathrm{Hom}(W_\beta, M_{\underline{c}}) + \sum_{k=1}^{n} m_k(\dim(W_\alpha)_k + \dim(W_\beta)_k) \\ = x_\alpha + x_\beta = 0.\end{aligned}$$

But the quantity on the left-hand side is just a sum of the coordinates of $\underline{x}$ evaluated at the summands of E, weighted by their multiplicities. Thus, at least one of the coordinates of $\underline{x}$ is negative, so $\underline{x}$ is not in $\mathbb{U}_{\underline{c}}$. □

Lemma 3. *$\mathbb{U}_{\underline{c}}$ is bounded.*

Proof. If $\underline{x}$ is in $\mathbb{U}_{\underline{c}}$ then, as in the previous proof, there is an n-tuple $(m'_1, \dots, m'_n)$ such that

$$\underline{x} = \underline{v}_{\underline{c}} + \sum_{k=1}^{n} m'_k g^k.$$

Consider what this equation says at some coordinate $\alpha \in \mathcal{I} \setminus \mathcal{I}^+$. Note first that $(\underline{v}_{\underline{c}})_\alpha = 0$. Of course, $x_\alpha \geqslant 0$, as we assumed that $\underline{x} \in \mathbb{U}_{\underline{c}}$. As the g-vectors of the $P_j[1]$ are negative standard basis vectors, g^k_α is nonpositive, and it follows that the m'_k must be nonpositive.

Repeating the same argument with the projectives rather than the shifted projectives, and taking into account the fact that $(v_{\underline{c}})_{0j} > 0$, we conclude that each of the m'_k is bounded between 0 and some negative number B. Therefore, $\mathbb{U}_{\underline{c}}$ is bounded. □

Proof of Theorem 1. We begin by establishing that the vertices of $\mathbb{U}_{\underline{c}}$ correspond bijectively to maximal compatible sets in $\mathcal{I}$, with the bijection sending the vertex to the indices of the hyperplanes $p_\alpha = 0$ on which it lies.

As $\mathbb{E}_{\underline{c}}$ is n-dimensional, a vertex of $\mathbb{U}_{\underline{c}}$ must lie on at least n facets, and therefore on at least n hyperplanes of the form $p_\alpha = 0$. By Lemma 2, the collection of hyperplanes corresponding to a vertex must be compatible. The maximal compatible sets of $\mathcal{I}$ are exactly the compatible sets of size n. Thus, every vertex of $\mathbb{U}_{\underline{c}}$ corresponds to a cluster.

We now argue by induction that every cluster corresponds to a vertex of $\mathbb{U}_{\underline{c}}$. Let us suppose that we have a collection of n compatible indices from $\mathcal{I}$, such as $T = \{\alpha, \alpha_1, \dots, \alpha_{n-1}\}$, and another collection of n compatible indices $T' = \{\alpha', \alpha_1, \dots, \alpha_{n-1}\}$. Suppose that $\mathbb{U}_{\underline{c}}$ has a vertex q_T at the intersection of the facets corresponding to T. Consider moving along the ray from q_T where the facets corresponding to $\alpha_1, \dots, \alpha_{n-1}$ intersect. As $\mathbb{U}_{\underline{c}}$ is bounded by Lemma 3, we must eventually hit another hyperplane bounding $\mathbb{U}_{\underline{c}}$. By Lemma 2, it must be a hyperplane that is compatible with $\alpha_1, \dots, \alpha_{n-1}$; the only one is the hyperplane corresponding to α'. This intersection is a vertex of $\mathbb{U}_{\underline{c}}$ corresponding to T'.

We therefore know that T' also corresponds to a vertex of $\mathbb{U}_{\underline{c}}$. As the clusters are connected by mutations, every cluster corresponds to a vertex of $\mathbb{U}_{\underline{c}}$, and as by following edges, we only ever

get to vertices that correspond to clusters (never to an edge that goes to infinity, or to a vertex that doesn't correspond to a cluster), all the vertices of $\mathbb{U}_{\underline{c}}$ correspond to clusters. Further, the one-skeleton of $\mathbb{U}_{\underline{c}}$ is the cluster exchange graph.

We have therefore established point (3) of the theorem. By Lemma 2, each face of $\mathbb{U}_c$ corresponds to a compatible set (necessarily all different), and as all the maximal compatible sets correspond to vertices of $\mathbb{U}_{\underline{c}}$, every compatible set does correspond to a face. This establishes (1) and (2). (4) is straightforward. Given a point in $\underline{y} \in \mathbb{R}^n$, we can uniquely find an $\underline{x}$ such that $\pi(\underline{x}) = \underline{y}$ by inductively solving the $\underline{c}$-deformed mesh relations, working from right to left through the quiver for $\mathcal{I}$.

Finally, we establish (5). Let G be the $n \times |\mathcal{I}|$ matrix whose (i, j)th column is the g-vector $g(i, j)$. Let σ be the affine map from $\mathbb{R}^n$ to $\mathbb{E}_{\underline{c}} \subset V$ which is a section of π. This map is given by right multiplying by $-G$ and adding $\underline{v}_{\underline{c}}$. This induces a map from the tangent vectors of V to tangent vectors of $\mathbb{R}^n$ (which we identify with V and $\mathbb{R}^n$, respectively, via our fixed bases for each of them). Linear algebra tells us that the map on tangent vectors is given by *left* multiplying by $-G$. Write $\underline{e}_{ij}$ for the standard basis vector corresponding to $(i, j) \in \mathcal{I}$. As $-\underline{e}_{ij}$ is orthogonal to the hyperplane $p_{ij} = 0$, and points away from $\mathbb{U}_{\underline{c}}$, it image under left multiplication by $-G$ generates the outer normal ray corresponding to this facet. But clearly $(-G)(-\underline{e}_{ij}) = g(i, j)$, as desired. □

We close the section by commenting on the implications of Theorem 1 for different assumptions on $\underline{c}$.

Clearly, if we assume that the constants $\underline{c}$ are positive rational numbers, Theorem 1 still holds. By continuity, it also holds for positive real numbers. Now consider the case that the some of the constants $\underline{c}$ are zero.

Corollary 1. *If the constants $\underline{c}$ are nonnegative, then every vertex of $\mathbb{U}_{\underline{c}}$ lies on a collection of coordinate hyperplanes which is the union of one or more maximal compatible sets in $\mathcal{I}$. The facet normals to the facets of $\mathbb{A}_{\underline{c}}$ are a subset of the g-vectors.*

Proof. We can imagine what happens if we begin with $\underline{c}'$ where all values are positive, and then deform gradually to a nearby vector $\underline{c}$, where some values become zero. What can happen is that some vertices can merge, and some facets can collapse to something that is no longer codimension 1. The results are as described in the statement of the corollary. □

4 | BACKGROUND ON CLUSTER ALGEBRAS

Let Q be a quiver without loops or oriented two-cycles. As already mentioned, there is an associated cluster algebra $\mathcal{A}(Q)$, whose cluster variables are in natural bijection with the elements of $\mathcal{I}$, as we now explain.

A *slice* in $\mathcal{I}$ is a subset $\mathcal{T}$ of $\mathcal{I}$ such that:

- for each j with $1 \leqslant j \leqslant n$, there is exactly one element of $\mathcal{T}$ of the form (i, j) for some i, and
- if j and j' are adjacent vertices of Q, and (i, j) and (i', j') are the corresponding elements of $\mathcal{T}$, then (i, j) and (i', j') are adjacent in $\mathcal{I}$.

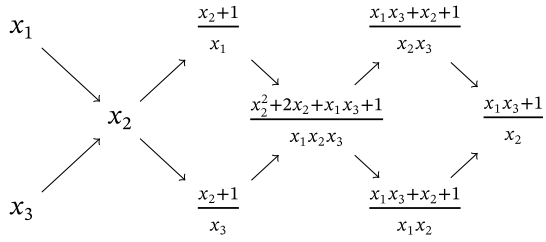
The collection of all $(0, j)$ form a slice, corresponding to the projective representations, as does the collection $\mathcal{I} \setminus \mathcal{I}^+$, corresponding to the shifted projectives. We refer to $\{(0, j)\}$ as the *initial slice* and $\mathcal{I} \setminus \mathcal{I}^+$ as the *final slice*.

Note that any slice in $\mathcal{I}$ has at least one element (i, j) such that all the arrows between it and the other elements of the slice are oriented from (i, j). We call such an (i, j) a *source* in the slice. Any slice other than the final slice has a source that is in $\mathcal{I}^+$.

We assign the initial cluster variable x_j to $(0, j) \in \mathcal{I}$. Suppose we have a slice $\mathcal{T}$ that is not the final slice, and the elements of $\mathcal{T}$ are associated to the cluster variables of a cluster. Let (i, j) be a source in the slice that is in $\mathcal{I}^+$. We can therefore replace (i, j) by $(i + 1, j)$ to obtain a new slice $\mathcal{T}'$. Mutate the cluster at the variable corresponding to (i, j), and associate the resulting cluster variable to the vertex $(i + 1, j)$. This provides a cluster associated to the slice $\mathcal{T}'$. As we proceed from the initial slice to the final slice, we may well have choices of source. However, these choices do not matter. We have the following theorem, essentially from [7]:

Theorem 2. *The above procedure results in a well-defined map from $\mathcal{I}$ to the cluster variables of $\mathcal{A}(Q)$, independent of choices.*

Example 4. In the setting of Example 1, the result is the following:

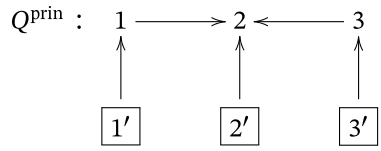


Let us emphasize that any slice in $\mathcal{I}$ corresponds to a cluster, though there are also further clusters that are not slices. Note that if we restrict the quiver on $\mathcal{I}$ to the vertices of the initial slice, we recover Q. In fact, it is easy to show by induction that for any slice, the quiver associated to that cluster is given by restricting the quiver on $\mathcal{I}$ to that slice.

One way to construct a more complicated cluster algebra that is still governed by the same Dynkin combinatorics is to define a new *ice quiver* Q^{ice} that is obtained from Q by adding some vertices that are designated as *frozen*. The new vertices may be connected to the vertices of Q in any way (provided that there are still no oriented two-cycles). It is standard to assume that there are also no arrows between frozen vertices, but this is not actually important because such arrows play no role. The cluster algebra associated to an ice quiver, $\mathcal{A}(Q^{\text{ice}})$ is the algebra generated by the initial cluster variables and all cluster variables obtained by all sequences of mutations at nonfrozen vertices only. Note that there is a cluster variable associated to each frozen vertex. To distinguish the frozen and unfrozen variables, we generally write $x_1, \dots, x_n$ for the initial unfrozen variables and $y_1, \dots, y_m$ for the frozen variables. A refined version of the Laurent Phenomenon says that every cluster variable is contained in $\mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}, y_1, \dots, y_m]$, see [15, Theorem 3.3.6]. The unfrozen cluster variables still correspond to the elements of $\mathcal{I}$ and can be calculated in the same way, by starting with the initial variables associated to the initial slice and then carrying out mutations as before.

There are two particular ice quivers obtained from Q that are of special importance, one of which we will explain now. For each vertex i of Q, add a frozen vertex i' and an arrow from i' to i. The resulting quiver is called the *framed quiver* of Q; we denote it Q^{prin}. To the new vertex i' we associate the frozen cluster variable y_i. The cluster algebra associated to Q^{prin} is called the *cluster algebra with principal coefficients*. We denote it $\mathcal{A}^{\text{prin}}(Q)$.

Example 5. The framed quiver of Q from Example 1 is as follows, where the frozen vertices appear in squares.



The cluster variables of the corresponding cluster algebra $\mathcal{A}^{\text{prin}}(Q)$ are listed in Section 7.

The significance of this choice of coefficients will be explained further in Section 7. Briefly, it turns out that from the cluster variables for the cluster algebra with principal coefficients, one can immediately calculate the cluster variables for any system of coefficients. The cluster algebra with principal coefficients is also essential for defining F-polynomials, as we shall explain shortly.

An ice quiver with n unfrozen vertices and m frozen vertices can also be represented as an $(n+m)\times n$ matrix of integers, $\widetilde{B}_0$. We associate the n columns and the corresponding first n rows to the n unfrozen vertices of Q, and we associate the remaining rows to the frozen vertices. The entry $(\widetilde{B}_0)_{ij}$ is the number of arrows from i to j minus the number of arrows from j to i. We see that the first n rows of $\widetilde{B}_0$ are simply the matrix B_0 that we have already seen. The matrix corresponding to Q^{prin} consists of B_0 with an $n\times n$ identity matrix below it.

Example 6. For our running example, the extended matrix $\widetilde{B}_0$ corresponding to principal coefficients is the following:

$$\left[\begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

5 | SUBMODULE POLYTOPES AND TORSION CLASSES

For the duration of this section, we work with an arbitrary finite-dimensional algebra A, thought of as a the path algebra of a quiver Q with relations, where the vertices of Q are numbered 1 to n. We can therefore still consider dimension vectors of such modules.

For X an A-module, we write $\mathcal{P}_X$ for the polytope in $\mathbb{R}^n$ which is the convex hull of the dimension vectors of subrepresentations of X. We call $\mathcal{P}_X$ the *submodule polytope* of X. Submodule polytopes for representations of preprojective algebras play an important rôle in the paper by Baumann, Kamnitzer, and Tingley [4] which we are essentially following in this section.

A *torsion class* in A-mod is a full subcategory closed under extensions and quotients. If $\mathcal{T}$ is a torsion class, any A-module M has a unique largest submodule that is contained in $\mathcal{T}$. We call this the *torsion part* of M with respect to $\mathcal{T}$. See [3, chapter VI] for more background on torsion classes.

The key fact about submodule polytopes is that, in order to find the vertices of a submodule polytope, we do not need to consider all submodules: it suffices to consider those submodules that are torsion parts with respect to some torsion class. The following proposition is established in [4, section 3]. Because we need rather less than is established in [4], we give the simple proof here.

Lemma 4 [4]. *Let M be an A-module. Then $\mathcal{P}_M$ equals the convex hull of the dimension vectors of the torsion parts of M. This amounts to saying that, for each vertex $\underline{x}$ of $\mathcal{P}_M$, there is some torsion class $\mathcal{T}$ with respect to which the dimension vector of the torsion part of M equals $\underline{x}$. Further, the submodule of M with this dimension vector is unique.*

Proof. Let $\underline{x}$ be a vertex of $\mathcal{P}_M$. It follows that there must be at least one submodule of M whose dimension vector is $\underline{x}$. Let N be such a submodule. We will show that there is a torsion class $\mathcal{T}$ such that the torsion part of M with respect to $\mathcal{T}$ is N.

Choose a linear form θ on the space of dimension vectors such that the unique point on $\mathcal{P}_M$ maximizing θ is x. Define a torsion class by

$$\mathcal{T}_\theta = \{X \mid \theta(\underline{\dim} Y) \geqslant 0 \text{ for all quotients } Y \text{ of } X\}$$

It is not hard to establish T_θ is a torsion class [4, Proposition 3.1]. Suppose L is a proper submodule of N. As L is also a submodule of M, we know that $\theta(\underline{\dim} L) < \theta(\underline{\dim} N)$, so $\theta(\underline{\dim} N/L) > 0$, and it follows that $N \in \mathcal{T}_\theta$. Let us write N' for the torsion part of M with respect to $\mathcal{T}_\theta$. As N is a submodule of M and is in $\mathcal{T}_\theta$, we know that N must be a submodule of N'. If it were a proper submodule, then $\theta(\underline{\dim} N') < \theta(\underline{\dim} N)$, so $\theta(\underline{\dim} N'/N) < 0$, contradicting the assumption that $N' \in \mathcal{T}$. Thus $N' = N$.

For the final statement, suppose that there were two distinct modules N_1, N_2 with dimension vector x. The above argument shows that both N_1 and N_2 are in $\mathcal{T}$, so the maximal torsion part of M with respect to $\mathcal{T}$ is larger than either of them, which we have established is impossible. □

6 | NEWTON POLYTOPES OF F-POLYNOMIALS

By definition, the F-polynomial F_α for $\alpha \in \mathcal{I}$ is obtained by taking x_α^{prin}, the cluster variable associated to α in $\mathcal{A}^{\text{prin}}(Q)$, and setting all the x_i to 1. It is therefore a polynomial in $y_1, \ldots, y_n$. It turns out that the cluster variable x_α^{prin} can be recovered from the F-polynomial (see [17, Corollary 6.3]), so no information has been lost, and at the same time, the F-polynomial turns out to be convenient for another reason: for $(i, j) \in \mathcal{I}$ with $i \geqslant 1$, the F-polynomial $F_{(i,j)}$ is the generating function for the submodules of the representation $W_{(i-1,j)}$ in the following sense:

$$F_{(i,j)}(y_1, \ldots, y_n) = \sum_{\underline{e} \leqslant \underline{\dim}(W_{(i-1,j)})} \chi(\mathrm{Gr}_{\underline{e}}(W_{(i-1,j)}) y^{\underline{e}}.$$

We refer to [12, eq. (1.6)] for the formula. Here $\underline{e} = (e_1, \ldots, e_n)$ is a dimension vector in $\mathbb{Z}^n_{\geqslant 0}$, $\mathrm{Gr}_{\underline{e}}(X)$ means the quiver Grassmannian of subrepresentations of X whose dimension vector is $\underline{e}$, and χ is the Euler characteristic. We write $y^{\underline{e}}$ for $y_1^{e_1} y_2^{e_2} \ldots y_n^{e_n}$.

Let $f \in \mathbb{Z}[y_1, \dots, y_n]$ be a polynomial. Let P be the subset of $\mathbb{Z}^n$ such that f can be written as

$$f = \sum_{\underline{p} \in P} f_{\underline{p}} y^{\underline{p}}$$

with all $f_{\underline{p}}$ nonzero integers. That is to say P is the collection of the the n-tuples corresponding to exponents of terms that appear in f. The *Newton polytope* of f is then the convex hull of the points in P.

Newton polytopes of F-polynomials have been studied by Brodsky and Stump [6]. They give a description in type A_n and a uniform conjecture for all finite types. Subsequent to the first appearance of the present paper, this conjecture was proved in [20], relying in part on our results.

Newton polytopes of cluster variables have also been studied, by Sherman and Zelevinsky in rank 2 [31], by Cerulli Irelli for $\widetilde{A}_2$ [9], and by Kalman in type A_n [21, 22]. Note that, by [17, Corollary 6.3], the Newton polytope of a cluster variable is an affine transformation of that of the corresponding F-polynomial, so the two questions are quite close.

For $(i, j) \in \mathcal{I}^+$, let $\underline{e}_{ij}$ denote the standard basis vector in $\mathbb{R}^{\mathcal{I}^+}$ that has a 1 in position (i, j) and zeros elsewhere.

Theorem 3. *$\mathbb{A}_{\underline{e}_{ij}}$ is the Newton polytope of $F_{(i+1,j)}$.*

Subsequent to the appearance of the first version of the present paper, this result has been extended to the nonsimply laced case [2], by using a folding argument to reduce to the simply laced case, for which they rely on this result. Before we prove the theorem, we will state and prove a key lemma, and then a proposition.

Lemma 5. *Let T be a tilting object in the additive hull of $\{W_\alpha \mid \alpha \in \mathcal{I}\}$. Let $\mathcal{T}$ be the corresponding torsion class in* $\operatorname{rep} Q^{op}$*, consisting of all quotients of sums of summands of T that are contained in* $\operatorname{rep} Q^{op}$*. For $M \in \operatorname{rep} Q^{op}$, let tM denote the torsion part of M with respect to the torsion class $\mathcal{T}$. Let $\underline{q}_T$ be the vertex of $\mathbb{U}_{\underline{c}}$ at which the coordinates corresponding to summands of T are zero. Then $\pi(\underline{q}_T) = \underline{\dim}(tM)$.*

Example 7. To see examples of this lemma, we can revisit Example 3. The first vertex listed corresponds to $T_1 = kQ^{op}$. The torsion class $\mathcal{T}_1 = \operatorname{rep} Q^{op}$ is the full category of representations, so $t_1 M_{\underline{c}} = M_{\underline{c}}$, and $\pi(q_{T_1}) = (3, 4, 3)$, which is the dimension of $M_{\underline{c}}$.

Looking at the second vertex listed, we see that $\mathcal{T}_2$ consists of direct sums of all indecomposables except S_3. The corresponding torsion part $t_2 M_{\underline{c}}$ therefore has dimension $(3, 4, 2) = \pi(q_{T_2})$. (Recall that in the definition of π, the order in which the final slice of coordinates appear is determined by their corresponding g-vectors, which is why, in type A_3, π is in effect reading the final slice of coordinates from bottom to top.)

Proof. As before, let G be the $n \times |\mathcal{I}|$ matrix whose (i, j)th column consists of the g-vector $g(i, j)$.

Let the summands of T be $T_1, \dots, T_n$. Let us write $G|_T$ for the $n \times n$ matrix formed by taking the columns of G corresponding to $T_1, \dots, T_n$. We can interpret $G|_T$ as the change-of-basis matrix from the basis $\underline{\dim} T_1, \dots, \underline{\dim} T_n$ to the basis $\underline{\dim} P_1, \dots, \underline{\dim} P_n$. $G|_T$ is therefore invertible, with inverse given by the inverse change of basis. It follows that $(G|_T)^{-1} G$ is a matrix whose restriction to the columns corresponding to summands of T is an identity matrix.

The point in $\mathbb{E}_{\underline{c}}$ that has zeros in the columns corresponding to the T_i is therefore

$$\underline{v}_{\underline{c}} - [\dim \mathrm{Hom}(T_1, M_{\underline{c}}), \dots, \dim \mathrm{Hom}(T_n, M_{\underline{c}})](G|_T)^{-1}G.$$

Note that as T is a tilting object, each indecomposable projective module P_i admits a coresolution

$$P_i \to T_{i0} \to T_{i1} \to P_i[1],$$

where T_{i0} and T_{i1} are in add T. The entries in the ith column of $(G|_T)^{-1}$ encode the signed multiplicity of $T_1, \dots, T_n$ in this coresolution of P_i.

Note that the final n coordinates of $\underline{v}_{\underline{c}}$ are zero. The ith coordinate of $\mathbb{A}_{\underline{c}}$ is therefore $\dim \mathrm{Hom}(T_{i0}, M_{\underline{c}}) - \dim \mathrm{Hom}(T_{i1}, M_{\underline{c}})$.

From the coresolution of P_i, we obtain the following commutative diagram, with the rows exact:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(T_{i1}, M_{\underline{c}}) & \longrightarrow & \mathrm{Hom}(T_{i0}, M_{\underline{c}}) & \longrightarrow & \mathrm{Hom}(P_i, M_{\underline{c}}) & \longrightarrow & \mathrm{Ext}^1(T_{i1}, M_{\underline{c}}) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathrm{Hom}(T_{i1}, tM_{\underline{c}}) & \longrightarrow & \mathrm{Hom}(T_{i0}, tM_{\underline{c}}) & \longrightarrow & \mathrm{Hom}(P_i, tM_{\underline{c}}) & \longrightarrow & \mathrm{Ext}^1(T_{i1}, tM_{\underline{c}})
\end{array}$$

The zeros on the left-hand end follow from the fact that $\mathrm{Hom}(P_i, M_{\underline{c}}[-1]) = 0 = \mathrm{Hom}(P_i, tM_{\underline{c}}[-1])$. Further, $\mathrm{Ext}^1(T_{i1}, tM_{\underline{c}}) = 0$ because T_{i1} is Ext-projective in $\mathcal{T}$ while $tM_{\underline{c}}$ is in $\mathcal{T}$. Therefore, the map from $\mathrm{Hom}(T_{i0}, tM_{\underline{c}})$ to $\mathrm{Hom}(P_i, tM_{\underline{c}})$ is surjective.

The first three vertical maps are injective because they are induced from the inclusion of $tM_{\underline{c}}$ into $M_{\underline{c}}$.

Any map from a torsion module to $M_{\underline{c}}$ necessarily lands in the torsion part of $M_{\underline{c}}$, so factors through $tM_{\underline{c}}$. This means that the first two vertical arrows are also surjective.

Our goal is to understand the image of $\mathrm{Hom}(T_{i0}, M_{\underline{c}})$ inside $\mathrm{Hom}(P_i, M_{\underline{c}})$; by what we have already shown, it equals $\mathrm{Hom}(P_i, tM_{\underline{c}})$; in other words, the dimension of this image is the dimension of $tM_{\underline{c}}$ at vertex i, as desired. □

From the previous lemma, the following proposition is almost immediate.

Proposition 1. $\mathbb{A}_{\underline{c}} = \mathcal{P}_{M_{\underline{c}}}$.

Proof. Thanks to Corollary 1, we know that the vertices of $\mathbb{U}_{\underline{c}}$ are the set of points $\underline{q}_T$ for T a tilting object in the additive hull of the W_α, with $\alpha \in \mathcal{I}$. (Note that it is of course possible that $\underline{q}_T = \underline{q}_{T'}$ for two distinct tilting objects T and T'.) By the previous lemma, $\pi(\underline{q}_T)$ is the dimension vector of the torsion part of $M_{\underline{c}}$ with respect to the corresponding torsion class. All torsion classes are of this form, so $\mathbb{A}_{\underline{c}}$ is the convex hull of the dimension vectors of all possible torsion parts of $M_{\underline{c}}$. Lemma 4 now tells us that $\mathbb{A}_{\underline{c}}$ is therefore the submodule polytope of $M_{\underline{c}}$, as desired. □

We can now prove Theorem 3.

Proof of Theorem 3. As we are interested in $\mathbb{A}_{\underline{e}_{ij}}$, we set $\underline{c} = \underline{e}_{ij}$, $M_{\underline{c}} = W_{ij}$. The quiver Grassmannian $\mathrm{Gr}_{\underline{e}}(W_{ij})$ is empty if $\underline{e}$ is not the dimension vector of a submodule. Thus, the Newton polytope of $F_{(i+1,j)}$ is contained in the convex hull of the dimension vectors of submodules of W_{ij}, which we have established in Proposition 1 is $\mathbb{A}_{\underline{e}_{ij}}$. It remains to check that the vertices of $\mathbb{A}_{\underline{e}_{ij}}$ correspond to quiver Grassmannians with nonzero Euler characteristics. Lemma 4 tells us that for each vertex of $\mathbb{A}_{\underline{e}_{ij}}$, there is a unique submodule of the appropriate dimension vector. The quiver Grassmannian is therefore a single point, and the Euler characteristic of a single point is 1. □

7 | THE USE OF PRINCIPAL COEFFICIENTS

Let Q^{ice} be an ice quiver, whose unfrozen part is Q. We will explain, following [17], how the cluster variables of $\mathcal{A}(Q^{\mathrm{ice}})$ can be calculated directly from those of $\mathcal{A}^{\mathrm{prin}}(Q)$, rather than via mutation.

Let $f(x_1, \dots, x_n, y_1, \dots, y_n)$ be a cluster variable in $A^{\mathrm{prin}}(Q)$. By $F(y_1, \dots, y_n)$ we denote the associated F-polynomial, which is obtained by setting $x_1 = \cdots = x_n = 1$ in f.

Let $z_1, \dots, z_m$ be the coefficients corresponding to the frozen vertices of Q^{ice} (equivalently, these correspond to rows $n+1$ to $n+m$ of the matrix $\widetilde{B}_0$.) Define

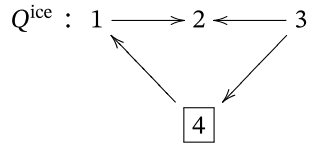
$$\tilde{y}_i = \prod_{j=1}^{m} z_j^{b_{n+j,i}}$$

We write $F^{\mathrm{trop}}(\tilde{y}_1, \dots, \tilde{y}_n)$ for the tropical evaluation of F at $\tilde{y}_1, \dots, \tilde{y}_n$. This is the monomial such that the power of z_i that appears in it is the minimum over all terms of F of the power of z_i in that term. (This is the gcd of the monomials that appear.)

Then [17, Theorem 3.7] says that the cluster variable in $\mathcal{A}(Q)$ corresponding to f is equal to

$$f(x_1, \dots, x_n, \tilde{y}_1, \dots, \tilde{y}_n)/F^{\mathrm{trop}}(\tilde{y}_1, \dots, \tilde{y}_n).$$

Example 8. Let us consider the following example,



with vertex $\boxed{4}$ frozen, and associated to the variable z. The corresponding B-matrix is:

$$\left[\begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & -1 \end{array}\right]$$

Using the labeling of the vertices as in Example 1, the cluster variables as well as the cluster variables with coefficients associated to every vertex are as following:

	Cluster variables for Q^{ice}	Cluster variables with principal coefficients
$(0,1)$	x_1	x_1
$(0,2)$	x_2	x_2
$(0,3)$	x_3	x_3
$(1,1)$	$\frac{x_2+z}{x_1}$	$\frac{x_2+y_1}{x_1}$
$(1,2)$	$\frac{x_1x_3z+x_2^2z+x_2z^2+x_2+z}{x_1x_2x_3}$	$\frac{x_1x_3y_1y_2y_3+x_2^2+x_2y_1+x_2y_3+y_1y_3}{x_1x_2x_3}$
$(1,3)$	$\frac{x_2z+1}{x_3}$	$\frac{x_2+y_3}{x_3}$
$(2,1)$	$\frac{x_1x_3z+x_2+z}{x_1x_2}$	$\frac{x_1x_3y_1y_2+x_2+y_1}{x_1x_2}$
$(2,2)$	$\frac{x_1x_3+x_2z+1}{x_2x_3}$	$\frac{x_1x_3y_2y_3+x_2+y_3}{x_2x_3}$
$(2,3)$	$\frac{x_1x_3+1}{x_2}$	$\frac{x_1x_3y_2+1}{x_2}$

By the definition of $\tilde{y}_i$, in this example we have $\tilde{y}_1 = z$, while $\tilde{y}_2 = 1$ and $\tilde{y}_3 = z^{-1}$. Substituting them in the polynomials listed above, we obtain the polynomials $f(x_1, \dots, x_n, \tilde{y}_1, \dots, \tilde{y}_n)$. Moreover, the monomials $F^{\text{trop}}(\tilde{y}_1, \dots, \tilde{y}_n)$ associated to every polynomial f are as follows:

	$f(x_1, \dots, x_n, \tilde{y}_1, \dots, \tilde{y}_n)$	$F^{\text{trop}}(\tilde{y}_1, \dots, \tilde{y}_n)$
$(0,1)$	x_1	1
$(0,2)$	x_2	1
$(0,3)$	x_3	1
$(1,1)$	$\frac{x_2+z}{x_1}$	1
$(1,2)$	$\frac{x_1x_3+x_2^2+x_2z+x_2z^{-1}+1}{x_1x_2x_3}$	z^{-1}
$(1,3)$	$\frac{x_2+z^{-1}}{x_3}$	z^{-1}
$(2,1)$	$\frac{x_1x_3z+x_2+z}{x_1x_2}$	1
$(2,2)$	$\frac{x_1x_3z^{-1}+x_2+z^{-1}}{x_2x_3}$	z^{-1}
$(2,3)$	$\frac{x_1x_3+1}{x_2}$	1

Calculating $f(x_1, \dots, x_n, \tilde{y}_1, \dots, \tilde{y}_n)/F^{\text{trop}}(\tilde{y}_1, \dots, \tilde{y}_n)$ and comparing with the cluster variables above, we observe that the theorem holds in this case.

8 | UNIVERSAL COEFFICIENTS

As mentioned in Section 4, there are two choices of coefficients that are particularly interesting. One is the principal coefficients that we discussed in the previous section. The other is *universal coefficients*.

In fact, there are two closely related notions: universal coefficients, introduced by Fomin and Zelevinsky [17], and universal geometric coefficients, introduced by Reading [29]. In both cases, the goal is a cluster algebra with sufficiently general coefficients that it will admit a ring homomorphism (with certain good properties) to the cluster algebra defined for any other choice of coefficients. In the finite-type case, the two definitions yield the same system of coefficients, see [29]. We will not need any properties of universal coefficients, so we do not give the precise definitions.

Reading [29, Theorem 10.12] proves $\widetilde{B}_0$ provides universal coefficients if the coefficient rows of the extended exchange matrix are the g-vectors B_0^T, where T indicates transposition. As, for us, B_0 is skew-symmetric, $B_0^T = -B_0$. Thus, the desired coefficient rows are the g-vectors for Q^{op}. Reading the quiver for $\mathcal{I}$ from right to left instead of left to right, we see that the g-vectors for Q^{op} are simply the negatives of the g-vectors for Q. Thus, we shall be interested in the setting where we add a row to the exchange matrix B_0 for each element of $\mathcal{I}$, with the row corresponding to (i, j) being given by $-g(i, j)$. We will call the corresponding algebra $\mathcal{A}^{\text{univ}}(Q)$.

Example 9. Consider again Example 1. The extended exchange matrix $\widetilde{B}_0$ is

$$\left[\begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ \hline -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

The additional rows in $\widetilde{B}_0$ define the behavior of $z_{0,1}, \ldots, z_{2,3}$, where z_{ij} is the frozen variable corresponding to the row whose entries are $-g(i, j)$.

The cluster variables with universal coefficients are computed below.

	Cluster variables with universal coefficients
$(0, 1)$	x_1
$(0, 2)$	x_2
$(0, 3)$	x_3
$(1, 1)$	$\dfrac{x_2 z_{0,1} + z_{1,1} z_{1,2} z_{2,3}}{x_1}$
$(1, 2)$	$(x_2^2 z_{0,1} z_{0,2} z_{0,3} + x_2 z_{0,1} z_{0,2} z_{1,2} z_{1,3} z_{2,1} + x_2 z_{0,2} z_{0,3} z_{1,1} z_{1,2} z_{2,3}$ $+ z_{0,2} z_{1,1} z_{1,2}^2 z_{1,3} z_{2,1} z_{2,3} + x_1 x_3 z_{1,2} z_{2,1} z_{2,2} z_{2,3}) / x_1 x_2 x_3$

$$(1,3)\quad \frac{x_2z_{0,3}+z_{1,2}z_{1,3}z_{2,1}}{x_3}$$

$$(2,1)\quad \frac{x_2z_{0,2}z_{0,3}z_{1,1}+z_{0,2}z_{1,1}z_{1,2}z_{1,3}z_{2,1}+x_1x_3z_{2,1}z_{2,2}}{x_2x_3}$$

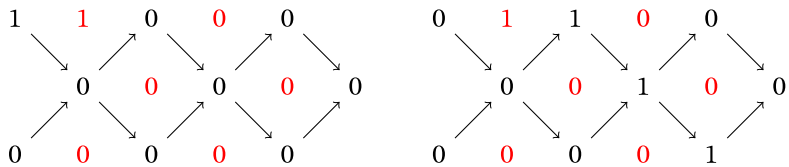
$$(2,2)\quad \frac{z_{0,2}z_{1,1}z_{1,2}z_{1,3}+x_1x_3z_{2,2}}{x_2}$$

$$(2,3)\quad \frac{x_2z_{0,1}z_{0,2}z_{1,3}+z_{0,2}z_{1,1}z_{1,2}z_{1,3}z_{2,3}+x_1x_3z_{2,2}z_{2,3}}{x_1x_2}$$

Define the universal F-polynomial $F^{\mathrm{univ}}_{(i,j)}$ to be the polynomial obtained starting from the cluster variable in position (i,j) in $\mathcal{A}^{\mathrm{univ}}(Q)$, and setting the initial cluster variables to one.

Theorem 4. *$\mathbb{U}_{\underline{e}_{(i,j)}}$ is the Newton polytope of $F^{\mathrm{univ}}_{(i+1,j)}$.*

Example 10. Consider, for example, the universal F-polynomial $F^{\mathrm{univ}}_{(1,1)}$. According to the above calculation, it is $z_{0,1}+z_{1,1}z_{1,2}z_{2,3}$. Thus, its Newton polytope is the line segment from (1,0,0,0,0,0,0,0,0) to (0,0,0,1,1,0,0,0,1). To find the vertices of $\mathbb{A}_{\underline{e}_{(0,1)}}$, we find solutions to the $\underline{e}_{(0,1)}$-deformed mesh relations that are all nonnegative and have at least three zeros. The results are the following, as expected.



Proof. We will use the strategy explained in Section 7 to calculate $F^{\mathrm{univ}}_{(i+1,j)}$ on the basis of $F_{(i+1,j)}$. Define $\tilde{y}_1,\dots,\tilde{y}_n$ as in Section 7 with respect to the matrix $\widetilde{B}_0$ as defined above. Then

$$F^{\mathrm{univ}}_{(i+1,j)}=\frac{F_{(i+1,j)}(\tilde{y}_1,\dots,\tilde{y}_n)}{F^{\mathrm{trop}}_{(i+1,j)}(\tilde{y}_1,\dots,\tilde{y}_n)}.$$

Let $\underline{x}$ be a vertex of $\mathbb{A}_{\underline{c}}$. It corresponds to a term $y^{\underline{x}}$ in $F_{(i+1,j)}$. There is a unique element $\underline{\tilde{x}}\in\mathbb{E}_{\underline{0}}$ such that $\pi(\underline{\tilde{x}})=\underline{x}$. This is precisely the exponent vector of the result of substituting $\tilde{y}_i$ for y_i in the monomial $y^{\underline{x}}$.

The element of $\mathbb{E}_{\underline{c}}$ that projects onto $\underline{x}$ is exactly $\underline{\tilde{x}}+\underline{v}_{\underline{c}}$. This, then, is the vertex of $\mathbb{U}_{\underline{c}}$ corresponding to $\underline{x}$. What remains to be verified is that $\underline{v}_{\underline{c}}$ equals negative the exponent of $F^{\mathrm{trop}}_{(i+1,j)}(\tilde{y}_1,\dots,\tilde{y}_n)$.

By definition, $F^{\mathrm{trop}}_{(i+1,j)}(\tilde{y}_1,\dots,\tilde{y}_n)$ is the least common multiple of all the terms in $F_{(i+1,j)}(\tilde{y}_1,\dots,\tilde{y}_n)$. Subtracting it from the Newton polytope of $F_{(i+1,j)}(\tilde{y}_1,\dots,\tilde{y}_n)$ translates the polytope so that for each z_{ij}, the minimal power that appears is zero. We know that $\mathbb{U}_{\underline{c}}$ also has the property that the minimum value that any coordinate takes on within $\mathbb{U}_{\underline{c}}$ is zero. Thus $\underline{v}_{\underline{c}}=-F^{\mathrm{trop}}_{(i+1,j)}(\tilde{y}_1,\dots,\tilde{y}_n)$, as desired. □

Example 11. We look at $F^{\mathrm{univ}}_{(1,1)}$ in our running example.

$$\tilde{y}_1 = z_{1,1}z_{1,2}z_{2,3}/z_{0,1}$$
$$\tilde{y}_2 = z_{2,2}/z_{0,2}z_{1,1}z_{1,2}z_{1,3}$$
$$\tilde{y}_3 = z_{1,2}z_{1,3}z_{2,1}/z_{0,3}$$

Now

$$F_{(1,1)}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = \frac{z_{0,1} + z_{1,1}z_{1,2}z_{2,3}}{z_{0,1}}, \quad F^{\mathrm{trop}}_{(1,1)}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = z_{0,1}^{-1}$$

and we indeed obtain

$$F^{\mathrm{univ}}_{(1,1)} = F_{(1,1)}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)/F^{\mathrm{trop}}_{(1,1)}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3).$$

9 | THE NEF CONE OF THE TORIC VARIETY ASSOCIATED TO THE g-VECTOR FAN IS SIMPLICIAL

9.1 | Brief reminder on toric varieties

Our main reference for toric varieties is [11]. We do not give specific references for the basic facts that are to be found in the first few chapters of that book.

Let N be a free abelian group of rank n. Write $M = \mathrm{Hom}(N, \mathbb{Z})$ for its dual. We write $\langle \cdot, \cdot \rangle$ for the duality pairing from $M \times N$ to $\mathbb{Z}$. We write $N_{\mathbb{R}}$ for $N \otimes_{\mathbb{Z}} \mathbb{R}$, and in general use a subscript $\mathbb{R}$ to denote tensoring by $\mathbb{R}$.

A cone in a real vector space is a semigroup closed under multiplication by nonnegative reals. A strongly convex, rational, polyhedral cone in $N_{\mathbb{R}}$ is a cone which is generated by a finite collection of vectors from N, all lying in a proper half-space. A fan in $N_{\mathbb{R}}$ is a collection of strongly convex, rational, polyhedral cones such that the intersection of any two cones is necessarily a face of each. (The examples to which we will apply this theory are the outer normal fan to the generalized associahedra $\mathbb{A}_{\underline{c}}$ that we have constructed, which are particular instances of g-vector fans of finite-type cluster algebras. They are indeed fans in the above sense.)

Let Σ be a fan in $N_{\mathbb{R}}$. For simplicity of exposition, we will assume that the n-dimensional cones of Σ cover $\mathbb{R}^n$, as this is true in particular for the outer normal fans that we are considering. We write Σ^i for the i-dimensional cones of Σ.

Associated to a fan Σ, there is a normal toric variety X_Σ. The field of rational functions on X_Σ is the fraction field of the group ring $\mathbb{C}[M]$. For $\underline{m} \in M$, we write $\chi^{\underline{m}}$ for the corresponding function field element.

There is a torus $T \simeq (\mathbb{C}^*)^n$ acting on X_Σ. There is a bijection between cones of Σ and T-orbits in X_Σ. We write $\mathcal{O}_\sigma$ for the orbit corresponding to the cone $\sigma \in \Sigma$. The dimension of $\mathcal{O}_\sigma$ is n minus the dimension of the span of σ.

A divisor on a normal variety is a formal $\mathbb{Z}$-linear combination of irreducible codimension 1 subvarieties. On a toric variety, we are particularly interested in those divisors that are torus invariant.

For $\rho \in \Sigma^1$, we define D_ρ to be the closure of $\mathcal{O}_\rho$. This is an irreducible codimension 1 subvariety that is torus-invariant. The torus-invariant divisors of X_Σ are $\mathrm{Div}(X_\Sigma) = \bigoplus_{\rho\in\Sigma^1} \mathbb{Z}[D_\rho]$.

Given a normal variety X and an element of the function field of X, say f, there is an associated divisor, $\mathrm{div}(f)$. Informally, it consists of the zero locus of f minus the locus where f blows up. We shall shortly define this notion precisely in the setting of toric varieties. A divisor D on a normal variety X is called a Cartier divisor if there exists an open cover $\{U_i\}$ of X such that $D|_{U_i}$ is principal for each $i \in I$, that is to say, there exists an element f_i of the function field of U_i such that $\mathrm{div}(f_i)$ equals the restriction of D to U_i.

In the case of toric varieties, we are interested in torus-invariant Cartier divisors. It turns out that there is a canonical choice of open cover that works for any torus-invariant Cartier divisor. For each maximal cone σ, let X_σ denote the toric variety associated to the fan consisting of σ and its faces. X_σ is open in X_Σ, and X_Σ is covered by the varieties X_σ. Any torus-invariant Cartier divisor on X_Σ is given by a collection of functions, one on each X_σ. In fact, we can take the function on X_σ to be of the form $\chi^{\underline{m}_\sigma}$ for some $\underline{m}_\sigma \in M$. We call the collection $\{\underline{m}_\sigma\}_{\sigma\in\Sigma^n}$ the local data of the Cartier divisor. For each ray $\rho \in \Sigma^1$, let $\underline{u}_\rho$ be the first lattice point along the ray ρ. The multiplicity of D_ρ in the divisor corresponding to the local data $\{\underline{m}_\sigma\}_{\sigma\in\Sigma^n}$ is given by $-\langle \underline{m}_\sigma, \underline{u}_\rho\rangle$ for σ any cone of Σ containing ρ.

A collection $\{\underline{m}_\sigma\}_{\sigma\in\Sigma^n}$ is not necessarily the local data of any Cartier divisor. The following condition provides a necessary and sufficient condition to verify that it is.

Lemma 6 [11, Theorem 4.2.8, Exercise 4.2.3]. *The collection $\{\underline{m}_\sigma\}_{\sigma\in\Sigma^n}$ forms the local data for a Cartier divisor if and only if $\langle \underline{m}_\sigma, \underline{u}_\rho\rangle$ is equal for all maximal cones σ containing the ray ρ.*

As we have already remarked, $-\langle \underline{m}_\sigma, \underline{u}_\rho\rangle$ is the multiplicity of D_ρ in the divisor; the condition for $\{\underline{m}_\sigma\}$ to be local data amounts to saying that the formula for the multiplicity of D_ρ does not depend on which σ is used, among those containing ρ.

We write $\mathrm{CDiv}(X_\Sigma)$ for the torus invariant Cartier divisors on X_Σ. We write $\mathrm{Div}_0(X_\Sigma)$ for the principal divisors on X_Σ. These are the Cartier divisors for which all m_σ are equal. The Picard group of X_Σ, denoted $\mathrm{Pic}(X_\Sigma)$, is defined to be $\mathrm{CDiv}(X_\Sigma)/\mathrm{Div}_0(X_\Sigma)$. Under our assumptions on the fan Σ, $\mathrm{CDiv}(X_\Sigma)$ is a free abelian group whose rank is $|\Sigma^1| - n$ [11, Theorem 4.2.1].

Let D be a Cartier divisor on a normal variety X and let C a complete curve in X. We write $D \cdot C$ for the intersection product of D and C. We will not define it in full generality, but the following can be taken as a definition in the toric setting.

Definition 1. On the toric variety X_Σ, let D be a torus-invariant Cartier divisor, which is thus given by a collection of local data $\{\underline{m}_\sigma\}_{\sigma\in\Sigma^n}$. Let C be a complete, irreducible torus-invariant curve in X_Σ; it is therefore the closure of $\mathcal{O}_\tau$ for some codimension 1 τ in Σ. The cone τ separates two maximal cones σ and σ'. Let u be an element of σ' that maps to a generator of N/N_τ. (Here N_τ is the lattice generated by τ, so N/N_τ is isomorphic to $\mathbb{Z}$.) Then

$$D \cdot C = \langle \underline{m}_\sigma - \underline{m}_{\sigma'}, \underline{u}\rangle$$

We take this as the definition of $D \cdot C$; see also [11, Proposition 6.3.8], which shows that this definition agrees with the definition for general varieties.

A Cartier divisor on a normal variety X is called nef ("numerically effective") if $D \cdot C \geqslant 0$ for every irreducible complete curve C in X. In the toric case, D is nef if and only if $D \cdot C \geqslant 0$ for every irreducible torus-invariant complete curve, that is, for those curves that are the closure of

$\mathcal{O}_\tau$ for some $\tau \in \Sigma^{n-1}$ [11, Theorem 6.3.12]. Thus, the definition of $D \cdot C$ that we have given above is sufficient to determine, for any Cartier divisor on X_Σ, whether or not it is nef.

A Cartier divisor D on a normal variety is said to be numerically equivalent to zero if and only if $D \cdot C = 0$ for all irreducible complete curves C; two Cartier divisors are numerically equivalent if their difference is numerically equivalent to zero. The nef cone of a variety is defined in the vector space of its Cartier divisors modulo numerical equivalence, and tensored by $\mathbb{R}$. For a toric variety, a torus-invariant Cartier divisor is numerically equivalent to zero if and only if it is principal [11, Proposition 6.3.15]. Thus, we can view $\mathrm{Nef}(X_\Sigma)$ as contained in $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$. Then, $\mathrm{Nef}(X_\Sigma)$ is the cone generated by the classes of the nef Cartier divisors in $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$.

Associated to a torus-invariant Cartier divisor $D = \sum_{\rho \in \Sigma^1} a_\rho [D_\rho]$ on X_Σ, there is a polytope defined by

$$P_D = \{\underline{m} \in M_\mathbb{R} \mid \langle \underline{m}, \underline{u}_\rho \rangle \geqslant -a_\rho \text{ for all } \rho \in \Sigma^1\}.$$

See [11, (6.1.1)].

Combining [11, Theorems 6.1.7 and 6.3.12], we obtain the following useful criterion for a Cartier divisor's being nef:

Proposition 2. *A Cartier divisor D on X_Σ with local data $\{\underline{m}_\sigma\}_{\sigma \in \Sigma^n}$ is nef if and only if $\underline{m}_\sigma \in P_D$ for all $\sigma \in \Sigma^n$.*

A cone in a real vector space is said to be simplicial if its number of generating rays is equal to the dimension of its span.

We can now state the theorem that we seek to prove about the toric variety associated to the g-vector fan.

Theorem 5. *Let Σ be the outer normal fan of $\mathbb{A}_{\underline{c}}$ (or equivalently the g-vector fan corresponding to a Dynkin quiver). The nef cone of the toric variety X_Σ is simplicial.*

9.2 | Combinatorics of $\mathrm{Pic}(X_\Sigma)$

Let $N = \mathbb{Z}^n$ be a rank n free abelian group with a fixed basis $\underline{b}_1, \ldots, \underline{b}_n$, and let $M = \mathrm{Hom}(N, \mathbb{Z})$, equipped with the dual basis $\underline{b}_1^*, \ldots, \underline{b}_n^*$. Let Σ be the g-vector fan realized in $N_\mathbb{R}$ with respect to the basis $\{\underline{b}_i\}$, and let X_Σ be the corresponding toric variety. The irreducible divisors correspond to rays of Σ, which themselves correspond to elements $(i, j) \in \mathcal{I}$. We will write D_{ij} for the divisor corresponding to the ray $g(i, j)$.

We identify $M_\mathbb{R}$ with the vector space $\mathbb{R}^n$ which is the image of the projection π, identifying $\underline{b}_i^*$ with the standard basis of $\mathbb{R}^n$.

We will now construct a bijection between $\mathbb{Z}^{|\mathcal{I}^+|}$ and $\mathrm{Pic}(X_\Sigma)$, the Cartier divisors of X_Σ up to linear equivalence.

Fix $\underline{c} = (c_{ij})_{(i,j) \in \mathcal{I}^+}$, with all $c_{ij} \in \mathbb{Z}$. Unlike earlier in the paper, we do not assume that the entries of $\underline{c}$ are nonnegative. Nonetheless, as earlier, we get a well-defined affine subspace $\mathbb{E}_{\underline{c}}$ of V. There is a section of π, which we denote $i_{\underline{c}}$, sending $\mathbb{R}^n$ to $\mathbb{E}_{\underline{c}}$.

The definition we gave of $\underline{v}_{\underline{c}}$ in Section 3 does not make sense any longer, as some of the c_{ij} are negative. However, it can be extended to the more general setting in the following way.

Define a matrix A, whose rows are indexed by $\mathcal{I}$ and whose columns are indexed by $\mathcal{I}^+$, and such that the $(i, j), (k, l)$ entry is $\dim \mathrm{Hom}(M_{ij}, M_{kl})$. If we order the rows and columns of A in the

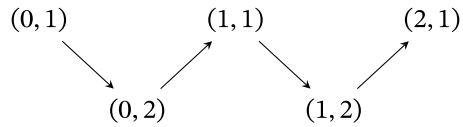
same way, respecting the left-to-right order of the Auslander–Reiten quiver of Q and keeping the elements of $\mathcal{I} \setminus \mathcal{I}^+$ for last, we obtain a matrix with 1's on the diagonal and zeros below it, with the last n rows consisting entirely of zeros.

We then define $\underline{v} = A\underline{c}$. Clearly, this recovers the definition of Section 3 when the c_{ij} are nonnegative. The proof that $\underline{v} \in \mathbb{E}_{\underline{c}}$ goes through without essential alterations.

For future use, let us write A' for the submatrix of A consisting of the rows indexed by elements of $\mathcal{I}^+$. As it is upper triangular with 1's on the diagonal, it is invertible in $GL_{|\mathcal{I}^+|}(\mathbb{Z})$.

Example 12. Consider the case that $Q = 1 \longrightarrow 2$.

The elements of $\mathcal{I}$ are as follows:



In this case, there is only one ordering of $\mathcal{I}$ consistent with the left-to-right ordering of the Auslander–Reiten quiver, namely (0,1),(0,2),(1,1),(1,2),(2,1). With respect to this ordering the matrix A is given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix A' consists of the top three rows of A.

Let G be the $n \times |\mathcal{I}|$ matrix whose (i, j)th column is the g-vector corresponding to $(i, j) \in \mathcal{I}$. As already noted in the proof of point (5) of Theorem 1,

$$i_c(\underline{y}) = -\underline{y} \cdot G + \underline{v}_{\underline{c}}.$$

Pullback along $i_{\underline{c}}$ defines a map from functions on V to functions on $\mathbb{R}^n = M_{\mathbb{R}}$. In particular, we can consider the pullbacks of the coordinate functions $i_c^*(p_{ij})$. From the formula for $i_{\underline{c}}(\underline{y})$, we obtain:

$$i_{\underline{c}}^*(p_{ij})(\underline{y}) = -\underline{y} \cdot g(i, j) + v_{ij}. \tag{3}$$

The zero locus of $i_{\underline{c}}^*(p_{ij})$ is an affine hyperplane in $\mathbb{R}^n = M_{\mathbb{R}}$.

The maximal cones in X_Σ correspond to maximal compatible sets in $\mathcal{I}$. For a maximal cone σ, define $-\underline{m}_\sigma(\underline{c})$ to be the intersection of the zero loci of the pullbacks by $i_{\underline{c}}$ of the coordinate functions corresponding to the rays of σ. As the corresponding g-vectors are linearly independent, this intersection is a well-defined point.

Lemma 7. *The collection $\{\underline{m}_\sigma(\underline{c})\}$ provides local data for a Cartier divisor on X_Σ. This Cartier divisor is $D(\underline{c}) = \sum_{(i,j)\in\mathcal{I}^+} v_{ij}[D_{ij}]$.*

Proof. As defined, the points $\underline{m}_\sigma(\underline{c})$ lie in $M_\mathbb{R}$. We must first check that they are in fact elements of M. This follows from the fact that the g-vectors corresponding to the rays of Σ form a basis for $\mathbb{Z}^n$.

Now we check that the $\underline{m}_\sigma(\underline{c})$ satisfy the necessary condition to be local data of a Cartier divisor, as recalled in Lemma 6. Let ρ be a ray of Σ corresponding to $(i,j) \in \mathcal{I}$. We must check that the value of $\langle \underline{m}_\sigma, \underline{u}_\rho \rangle$ is independent of the choice of σ containing ρ. This is so because by construction all the points $-\underline{m}_\sigma$ lie on the hyperplane in $M_\mathbb{R}$ where $i_c^*(p_{ij}) = 0$. Now $\underline{u}_\rho = g(i,j)$, and $i_c^*(p_{ij})(-\underline{m}_\sigma) = \langle \underline{m}_\sigma, g(i,j) \rangle + v_{ij}$, so $\langle \underline{m}_\sigma, \underline{u}_\rho \rangle = -v_{ij}$, independent of σ, as desired. □

We now establish the following converse to the previous lemma:

Lemma 8. *Up to linear equivalence, any Cartier divisor on X_Σ is given by local data $\{m_\sigma(\underline{c})\}$ for some $\underline{c}$.*

Proof. Suppose we have a Cartier divisor $D = \sum_{(i,j)\in\mathcal{I}} v_{ij}[D_{ij}]$. There is a principal divisor whose coefficients with respect to the rays corresponding to $(i,j) \in \mathcal{I} \setminus \mathcal{I}^+$ take any integer values, so, by subtracting it from D, we may assume that $v_{ij} = 0$ for $(i,j) \notin \mathcal{I}^+$. Consider the vector $\underline{v}$ which is the $|\mathcal{I}^+|$-tuple consisting of the entries v_{ij} for $(i,j) \in \mathcal{I}^+$. To construct D as a Cartier divisor, we must show that there exists a $|\mathcal{I}^+|$-tuple of integers $\underline{c}$ such that $\underline{v} = A'\underline{c}$. As, as we have already commented, A' is invertible, we find that $\underline{c} = (A')^{-1}(\underline{v})$ gives us the necessary $\underline{c}$. □

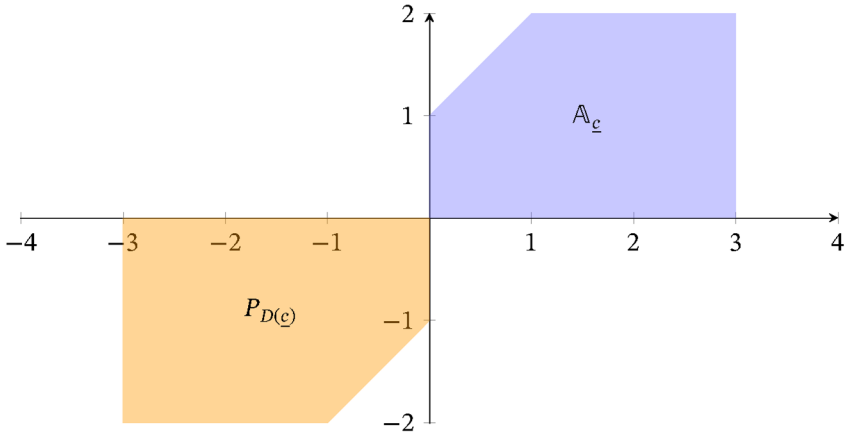
From the two previous lemmas, we deduce:

Proposition 3. *There is an isomorphism of abelian groups between $\mathbb{Z}^{\mathcal{I}^+}$ and $\mathrm{Pic}(X_\Sigma)$, sending $\underline{c}$ to $D(\underline{c})$.*

Proof. Lemma 7 establishes the existence of the desired map, which is clearly a morphism of groups, and Lemma 8 shows that it is surjective. By [11, Theorem 4.2.1], which we have already cited, $\mathrm{Pic}(X_\Sigma)$ is a free abelian group of rank $|\mathcal{I}| - n = |\mathcal{I}^+|$. As $\mathbb{Z}^{\mathcal{I}^+}$ and $\mathrm{Pic}(X_\sigma)$ are free abelian groups of the same rank, a surjective map from one to the other must be an isomorphism. □

Example 13. We continue the setting of Example 12.

Let us set $c_{01} = 2$, $c_{02} = 1$, $c_{11} = 1$. The figure below shows $\mathbb{A}_{\underline{c}}$ and $P_{D(\underline{c})}$.



9.3 | Proof of Theorem 5

We are now almost ready to prove Theorem 5.

Lemma 9. *If all the entries of $\underline{c}$ are nonnegative, then $D(\underline{c})$ is nef.*

Proof. Let $D(\underline{c}) = \sum_{(i,j)\in I} v_{ij}[D_{ij}]$. By Proposition 2, it suffices to show that, provided the entries of $\underline{c}$ are nonnegative integers, then $\underline{m}_\sigma(\underline{c}) \in P_{D(\underline{c})}$ for all $\sigma \in \Sigma^n$.

$\mathbb{A}_{\underline{c}}$ is essentially by definition the region cut out by the inequalities $i_{\underline{c}}^*(p_{ij})(\underline{y}) \geqslant 0$. By (3), this is equivalent to $-\underline{y} \cdot g(i,j) + \underline{v}_{ij} \geqslant 0$. Thus, $\mathbb{A}_{\underline{c}}$ is cut out by the inequalities $-\underline{y} \cdot g(i,j) \geqslant -v_{ij}$. This says that $\mathbb{A}_{\underline{c}} = -P_{D(\underline{c})}$.

Now, by Theorem 1, we know that, assuming all the entries of $\underline{c}$ are nonnegative, the points $-\underline{m}_\sigma(\underline{c})$ are the vertices of $\mathbb{A}_{\underline{c}}$, and thus the points $\underline{m}_\sigma(\underline{c})$ lie in $P_{D(\underline{c})}$. □

Lemma 10. *If $\underline{c}$ has a negative entry, then $D(\underline{c})$ is not nef.*

Proof. Recall that, as defined in Section 4, a slice of the AR quiver of Q is a choice, for each $1 \leqslant i \leqslant n$ of a vertex $(i, s(i))$ such that if vertices i and i' of Q are adjacent, then $(i, s(i))$ and $(i+1, s(i+1))$ are adjacent in the AR quiver. Any slice is a compatible set.

Suppose that $c_{kl} < 0$. Choose two slices, s and s', so that s contains (k,l), and s' contains $(k, l+1)$, while the other vertices in the two slices are the same. Write σ and σ' for the corresponding cones of Σ. As the two cones differ only in one ray, they share a common codimension 1 face, τ. Write C for the corresponding curve. We will show that $D(\underline{c}) \cdot C < 0$, showing that $D(\underline{c})$ is not nef.

We know that $i_{\underline{c}}(\underline{m}_\sigma(\underline{c}))_{i,s(i)} = 0$ for $1 \leqslant i \leqslant n$. This implies that $i_{\underline{c}}(\underline{m}_\sigma(\underline{c}))_{k,l+1} = -c_{kl}$, by the $\underline{c}$-deformed mesh relation, while $i_{\underline{c}}(\underline{m}_{\sigma'}(\underline{c}))_{i,s'(i)} = 0$, so $i_{\underline{c}}(\underline{m}_{\sigma'}(\underline{c}))_{k,l+1} = 0$.

To apply Definition 1, we may take $\underline{u} = g(k, l+1)$. We find that $D(\underline{c}) \cdot C = \langle \underline{m}_\sigma(\underline{c}) - \underline{m}_{\sigma'}(\underline{c}), g(k,l+1)\rangle = i_c^*(p_{k,l+1})(-\underline{m}_\sigma) - i_c^*(p_{k+1,l})(-\underline{m}_{\sigma'}) = c_{kl} < 0$ □

Proof of Theorem 5. Proposition 3 establishes an isomorphism of abelian groups from $\mathbb{Z}^{|I|}$ and $\mathrm{Pic}(X_\Sigma)$, and thus a linear transformation from $\mathbb{R}^{|I|}$ to $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$. By the previous two lemmas, the cone $\mathrm{Nef}(X_\Sigma)$ is the image in $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$ of the positive orthant in $\mathbb{R}^{|I|}$, and is thus simplicial. □

The proof of Theorem 5 tells us that the extreme rays of the nef cone of X_Σ are generated by divisors $D(\underline{c})$ where exactly one of the entries of $\underline{c}$ equals 1 and the others are 0. The corresponding polytope $P_{D(\underline{c})}$ is the negative of the submodule polytopes of the indecomposable representations of Q. (That is to say, one passes from one polytope to the other by negating all the coordinates of all its vertices, as in Lemma 9 and Example 13.)

Translated into more polytopal language, this says that the negative of any polytope corresponding to a nef divisor on X_Σ can be written in a unique way as the Minkowski sum of dilations of the submodule polytopes of the indecomposable representations of Q. It is natural to ask how to recover classic realizations of the associahedron in this way, such as the Loday associahedron mentioned earlier. By results from [28, section 8], up to a change of basis, the Loday associahedron can be realized as the sum of each of the submodule polytopes of the indecomposable representations of the linearly oriented A_n quiver, each appearing with no dilation.

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ORCID

Hugh Thomas https://orcid.org/0000-0003-1177-9972

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