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► To cite this version:

Nathan Chapelier-Laget, Christophe Hohlweg. Shi arrangements and low elements in affine Coxeter groups. Canadian Journal of Mathematics, 2024, pp.1-31. 10.4153/S0008414X24000130 . hal-04677006

HAL Id: hal-04677006

<https://hal.science/hal-04677006v1>

Submitted on 24 Aug 2024

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Shi arrangements and low elements in affine Coxeter groups

Nathan Chapelier-Laget¹ and Christophe Hohlweg²

Abstract. Given an affine Coxeter group W , the corresponding Shi arrangement is a refinement of the corresponding Coxeter hyperplane arrangements that was introduced by Shi to study Kazhdan–Lusztig cells for W . Shi showed that each region of the Shi arrangement contains exactly one element of minimal length in W . Low elements in W were introduced to study the word problem of the corresponding Artin–Tits (braid) group and turns out to produce automata to study the combinatorics of reduced words in W . In this article, we show, in the case of an affine Coxeter group, that the set of minimal length elements of the regions in the Shi arrangement is precisely the set of low elements, settling a conjecture of Dyer and the second author in this case. As a by-product of our proof, we show that the descent walls – the walls that separate a region from the fundamental alcove – of any region in the Shi arrangement are precisely the descent walls of the alcove of its corresponding low element.

1 Introduction

Let (W, S) be a Coxeter system with length function $\ell : W \rightarrow \mathbb{N}$. Let V be the $\mathbb{R}$ -vector space with basis the simple system $\Delta = \{\alpha_s \mid s \in S\}$. Let B be the symmetric bilinear form on V defined by

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos(\frac{\pi}{m_{st}}), & \text{if } m_{st} < \infty, \\ -1, & \text{if } m_{st} = \infty. \end{cases}$$

Denote by $O_B(V)$ the orthogonal group for the pair (V, B) . For each $s \in S$, we consider the reflection $\sigma_s : V \rightarrow V$ by $\sigma_s(x) = x - 2B(\alpha_s, x)\alpha_s$. The map $\sigma : W \rightarrow O_B(V)$ defined by $s \mapsto \sigma_s$ is a *geometrical representation* of (W, S) . The orbit $\Phi = W(\Delta)$ is a root system of (W, S) , with positive root system $\Phi^+ = \text{cone}(\Delta) \cap \Phi$, where $\text{cone}(X)$ is the set of nonnegative linear combinations of vectors in X . The *inversion set* of $w \in W$ is the set

$$N(w) = \Phi^+ \cap w(\Phi^-),$$

where $\Phi^- = -\Phi^+$. It is well known that $\ell(w) = |N(w)|$.

Received by the editors January 29, 2022; revised August 2, 2023; accepted August 17, 2023.

Published online on Cambridge Core February 12, 2024.

This work was supported by the NSERC grant *algebraic and geometric combinatorics of Coxeter groups* held by the second author.

AMS subject classification: 05E16, 20F55, 17B22, 52C35.

Keywords: Coxeter groups, low elements, Shi arrangements, affine Weyl groups, affine Coxeter groups.



1.1 Tits cone and Coxeter arrangement

The *contragredient representation* $\sigma^* : W \rightarrow \text{GL}(V^*)$ of σ is defined as follows: for $w \in W$ and $f \in V^*$, we have $wf := \sigma^*(w)(f) = f \circ \sigma(w^{-1})$. For $\alpha \in \Phi$, we consider the following hyperplane H_α and open half-space H_α^+ :

$$H_\alpha := \{f \in V^* \mid f(\alpha) = 0\} \text{ and } H_\alpha^+ := \{f \in V^* \mid f(\alpha) > 0\}.$$

The intersection C of all H_α^+ for $\alpha \in \Delta$ is called the *fundamental chamber*. Let $D = \overline{C}$ be the topological closure of C , the *Tits cone* $\mathcal{U}(W)$ of W is the following cone in V^* :

$$\mathcal{U}(W) := \bigcup_{w \in W} wD.$$

If $w \in W$, wC is called a *chamber* of $\mathcal{U}(W)$. The action of W on $\{wC, w \in W\}$ is simply transitive; the chambers of $\mathcal{U}(W)$ are, therefore, in bijection with the elements of W . The *fundamental chamber* C corresponds to the identity element e of W . The hyperplane arrangement $\mathcal{A}_{(W,S)} = \{H_\alpha \mid \alpha \in \Phi\}$ is called the *Coxeter arrangement* of (W, S) (see [19] for more details).

1.2 Small roots, Shi arrangement, and minimal elements

The *Shi arrangement* of an affine Weyl group (W, S) is an affine hyperplane arrangement introduced by Shi in [23, Chapter 7] to study Kazhdan–Lusztig cells in affine Weyl groups of type A and later extended to other affine types in [25] (see also Fishel's survey [15] for more information).

Shi arrangements are generalized to any Coxeter system as follows. Introduced by Brink and Howlett [3] to prove the automaticity of Coxeter groups, the *dominance order* is the partial order $\leq_{\text{dom}}$ on Φ^+ defined by

$$\alpha \leq_{\text{dom}} \beta \iff \forall w \in W, \beta \in N(w) \implies \alpha \in N(w).$$

We say that β is a *small root* if β dominates no other root than itself. The set of small roots is denoted by $\Sigma(W, S)$. One of the remarkable results of Brink and Howlett in [3] is that the set $\Sigma(W, S)$ is finite (if S is finite); we refer the reader to the book [2, Chapter 4] for more details.

The *Shi arrangement* of W is the subarrangement of the Coxeter arrangement $\mathcal{A}_{(W,S)}$ defined by

$$\text{Shi}(W, S) = \{H_\alpha \mid \alpha \in \Sigma(W, S)\}.$$

The *Shi regions* of W are the connected components of $\text{Shi}(W, S)$ in the Tits cone $\mathcal{U}(W)$, that is, the connected components of

$$\mathcal{U}(W) \setminus \bigcup_{\alpha \in \Sigma(W, S)} H_\alpha.$$

Each Shi region is uniquely determined by a union of chambers in the Tits cone, leading to an equivalence relation $\sim_{\text{Shi}}$ on W defined by: $u \sim_{\text{Shi}} w$ if and only if uC and wC are in the same Shi region. Let us define

$$L_{\text{Shi}}(W, S) := \{w \in W \mid u \sim_{\text{Shi}} w \implies \ell(w) \leq \ell(u)\}.$$

1.3 Low elements

A conjectural characterization of $L_{\text{Shi}}(W, S)$ is given in terms of *low elements*. Low elements were introduced by Dehornoy, Dyer, and the second author [12] in relation to Garside theory [11] in the case of Artin–Tits (braid) groups. Low elements are defined as follows: $w \in W$ is a *low element* if

$$N(w) = \text{cone}(\Sigma(W, S) \cap N(w)) \cap \Phi.$$

Denote by $L(W, S)$ the set of low elements of (W, S) . Dyer and the second author show in [13] that $L(W, S)$ is a finite *Garside shadow*, i.e., a subset of W containing S and closed under taking suffixes and under taking joins in the right weak order.

The following conjecture is a reformulation of [13, Conjecture 2] in the case of low elements (see also [18, Section 3.6]).

Conjecture 1 *We have $L_{\text{Shi}}(W, S) = L(W, S)$. Moreover, any Shi region $\mathcal{R}$ contains a unique low element, which is the unique element of minimal length in $\mathcal{R}$.*

Both statements in the above conjecture are equivalent (see Section 4.3). This conjecture implies also, if true, that the set $L_{\text{Shi}}(W, S)$ is a Garside shadow, since the set of low elements is a Garside shadow, as shown in [13]. In [18], the authors show that any Garside shadow is the set of states of an automaton that recognizes the language of reduced words for (W, S) . They also show that the Brink–Howlett automaton, whose states are indexed by $L_{\text{Shi}}(W, S)$, projects onto the Garside shadow automaton associated with $L(W, S)$.

Therefore, Conjecture 1 is known to be true in the cases where the Brink–Howlett automaton is minimal. For instance, this conjecture is true for Coxeter systems with complete Coxeter graphs and right-angled Coxeter groups (see [18, Theorem 1.3]); all cases of a minimal Brink–Howlett automaton are characterized by Parkinson and Yau in [21]. However, the sole affine Weyl groups for which the Brink–Howlett automaton is minimal are of type A . Recently, Charles [9] proved Conjecture 1 for any rank 3 Coxeter system.

1.4 Main results

In this article, we prove Conjecture 1 for any affine Coxeter systems.

Theorem 1.1 *If (W, S) is an affine Coxeter system, then $L_{\text{Shi}}(W, S) = L(W, S)$.*

Corollary 1.2 *If (W, S) is an affine Coxeter system, then $L_{\text{Shi}}(W, S)$ is a Garside shadow for (W, S) .*

The above corollary shows that the Brink–Howlett automaton is the Garside shadow automaton, in the sense of [18], associated with $L(W, S)$.

In the case of an (irreducible) affine Coxeter system (W, S) , Shi shows in [25, Proposition 7.2] that each equivalence class under $\sim_{\text{Shi}}$ contains exactly one element of minimal length, which means, by Theorem 1.1, that $|L(W, S)| = |L_{\text{Shi}}(W, S)|$ counts the number of regions of $\text{Shi}(W, S)$. Writing (W_0, S_0) for the underlying finite Coxeter system of (W, S) , Shi provides the following remarkable formula in

[25, Theorem 8.1]:

$$|L_{\text{Shi}}(W, S)| = (h + 1)^n,$$

where $n = |S_0|$ is the rank of W_0 and h is the Coxeter number of W_0 . This formula also appears in relation to diagonal invariants in the work of Haiman [17, Section 7]: Haiman conjectured that there exist a W_0 -stable quotient ring of the covariant ring $\mathbb{C}[V \oplus V^*]^{co W}$ of dimension $(h + 1)^n$; this conjecture was confirmed by Gordon in [16]. As a corollary of Theorem 1.1, we obtain the enumeration of the set of low elements in the case of affine Coxeter systems.

Corollary 1.3 *If (W, S) is an affine Coxeter system with underlying finite Weyl group W_0 of rank n and Coxeter number h , then the number of low elements in (W, S) is $|L(W, S)| = (h + 1)^n$.*

1.5 Outline of the article

The proof of Theorem 1.1 can be easily reduced to the case of irreducible affine Coxeter systems by standard technics (see, for instance, [14]). Therefore, all Coxeter systems considered in this article are irreducible.

In order to prove Theorem 1.1, we first recollect in Section 2 well-known facts about affine Coxeter systems and affine Weyl groups. In Section 3, we survey the notions of low elements. In Section 4, we survey Shi arrangements and Shi's admissible signs. In particular, in Section 4.2, we show that $L_{\text{Shi}}(W, S)$ is closed under taking suffixes (Proposition 4.5), and in Section 4.3, we reduce the proof of Theorem 1.1 to showing that $L_{\text{Shi}}(W, S) \subseteq L(W, S)$.

Then we tackle the core of the proof of Theorem 1.1 in Sections 5–7. First, we characterize the so-called *descent walls* of a Shi region $\mathcal{R}$, that is, the walls that separate $\mathcal{R}$ from the fundamental alcove (see Definition 5.1); this characterization is given in terms of Shi's admissible sign types and Shi coefficients (Proposition 5.3). This leads to the first key result, which is Lemma 5.9, which describes some descent walls of a Shi region.

Second, we observe that Cellini and Papi's set of minimal elements, which appears in their works [4, 5] on *ad*-nilpotent ideals of a Borel subalgebra, is precisely the set $L^0(W, S)$ of low elements in the dominant region of $\mathcal{A}(W, S)$, i.e., the fundamental region of the Coxeter arrangement of (W_0, S_0) (Corollary 6.4). This covers the initial case of our proof by induction. In order to conclude our inductive proof, we prove our second key result, the *descent-wall theorem*: the descent walls of $w \in L_{\text{Shi}}$ are precisely those of its corresponding Shi region (Theorem 7.1). We conclude the proof of Theorem 1.1 in Section 7.4.

The connection with Cellini and Papi's results implies, in particular, the following corollary where the second equality is already known via the last term, understood [5, Theorem 1] as the number of *ad*-nilpotent ideals of a Borel subalgebra $\mathfrak{b}$ of a simple Lie algebra $\mathfrak{g}$.

Corollary 1.4 *If (W, S) is an irreducible affine Coxeter system with underlying finite Weyl group W_0 , then the number of low elements in the fundamental chamber of*

$\mathcal{A}(W, S)$ is the W_0 -Catalan number:

$$|L^0(W, S)| = \text{Cat}(W_0) = \frac{1}{|W_0|} \prod_{i=1}^n (h + e_i + 1),$$

where $e_1, \dots, e_n$ are the exponents of W_0 .

For a survey of the history of Coxeter–Catalan numbers, we refer the reader to [1, p. 181]. For more details on Coxeter numbers and exponents, see [19, Chapter 3].

2 Affine Coxeter systems

For a Euclidean vector space V_0 with inner product $\langle \cdot, \cdot \rangle$, we denote by $O(V_0)$ the group of isometries on V_0 and $\text{Aff}(V_0)$ the group of affine isometries on V_0 . The aim of this section is to survey some well-known facts on affine Coxeter systems along the lines of [14, Section 3] (see also [20, Chapters 6 and 7]).

2.1 Weyl group

Let Φ_0 be an irreducible crystallographic root system in V_0 with simple system Δ_0 ; set $n = |\Delta_0|$. Let Φ_0^+ be the associated positive root system. The root poset on Φ_0^+ is the poset $(\Phi_0^+, \leq)$ defined by $\alpha \leq \beta$ if and only if $\beta - \alpha \in \text{cone}(\Delta_0)$.

For any $\alpha \in \Phi_0$, let $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha$ be the coroot of α . The reflection s_α associated with α is defined by

$$\begin{aligned} s_\alpha &: V_0 \longrightarrow V_0 \\ x &\longmapsto x - \langle \alpha, x \rangle \alpha^\vee. \end{aligned}$$

Notice that, for all $\alpha \in \Phi_0$, we have

$$(1) \quad \langle \alpha^\vee, \alpha \rangle = \left\langle \frac{2\alpha}{\langle \alpha, \alpha \rangle}, \alpha \right\rangle = 2.$$

The set of fixed points of s_α is the hyperplane $H_\alpha := \{x \in V_0 \mid \langle x, \alpha \rangle = 0\}$. Let W_0 be the Weyl group associated with Φ_0 . We identify the root lattice $Q := \mathbb{Z}\Phi_0$ and the coroot lattice $Q^\vee := \mathbb{Z}\Phi_0^\vee$ with their groups of associated translations and we denote by τ_x the translation corresponding to $x \in Q^\vee$. With $S_0 = \{s_\alpha \mid \alpha \in \Delta_0\}$, the pair (W_0, S_0) is a finite Coxeter system. The set of reflections T_0 of W_0 is

$$T_0 = \bigcup_{w \in W_0} w S_0 w^{-1}.$$

The *fundamental chamber* of W_0 , also called the *dominant region*, is the set:

$$C_0 := \{x \in V_0 \mid \langle x, \alpha \rangle > 0, \forall \alpha \in \Delta_0\}.$$

2.2 Affine Weyl group and affine Coxeter arrangement

The affine reflection $s_{\alpha, k} \in \text{Aff}(V_0)$ is defined by

$$(2) \quad s_{\alpha, k}(x) = x - (\langle \alpha, x \rangle - k) \alpha^\vee = \tau_{k\alpha^\vee} s_\alpha.$$

The *affine Weyl group* W associated with Φ_0 is the following discrete reflection group of affine isometries:

$$W = \langle s_{\alpha,k} \mid \alpha \in \Phi_0, k \in \mathbb{Z} \rangle = Q^\vee \rtimes W_0.$$

Let α_0 be the highest root of Φ_0 . The set $S := S_0 \cup \{s_{\alpha_0,1}\}$ is a set of simple reflections for W . The set of fixed points of the reflection $s_{\alpha,k}$ is the affine hyperplane

$$H_{\alpha,k} = \{x \in V \mid \langle x, \alpha \rangle = k\}.$$

The collection of hyperplanes $H_{\alpha,k}$, for $\alpha \in \Phi_0$ and $k \in \mathbb{Z}$, is denoted by $\mathcal{A}(W, S)$ and is called the *affine Coxeter arrangement* of (W, S) . An *alcove* of (W, S) in V_0 is a connected component of

$$V_0 \setminus \bigcup_{H \in \mathcal{A}(W, S)} H.$$

The group W acts transitively on the set of alcoves, i.e., the map $w \mapsto w \cdot A_\circ$ is a bijection between W and the set of alcoves of (W, S) in V_0 where $A_\circ$ is the alcove associated with the identity element.

Remark 2.1 The affine Coxeter arrangement is, in fact, the intersection of the usual Coxeter arrangement with an affine hyperplane in the Tits cone. The fundamental alcove is the intersection of the fundamental chamber with this affine hyperplane. For more details, see [19, Section 6.5], [14, Proof of Proposition 3], or [8, Section 1.3.5].

2.3 Shi coefficients and Shi parameterization

In [24, Proposition 5.1], Shi describes a correspondence $w \mapsto (k(w, \alpha))_{\alpha \in \Phi_0^+}$ between W (or equivalently the set of alcoves of (W, S)) and some Φ_0^+ -tuples over $\mathbb{Z}$. We refer to some of Shi's results from [24, 25]. However, there are three differences in the conventions between Shi's articles and our article:

- (1) In Shi's articles, W is denoted by W_a and the underlying Weyl group is denoted by W .
- (2) In Shi's articles, the root system of the affine Coxeter system (W, S) is associated with $\Phi_0^\vee$ instead of Φ_0 in this article. Since $(\Phi_0^\vee)^\vee = \Phi_0$, it is enough to apply the involution $\alpha \mapsto \alpha^\vee$ when applying Shi's results.
- (3) The action of W on the root system is from the right in Shi's articles, where we adopt the convention of acting by the left; for more details, see [7, Remark 2.1].

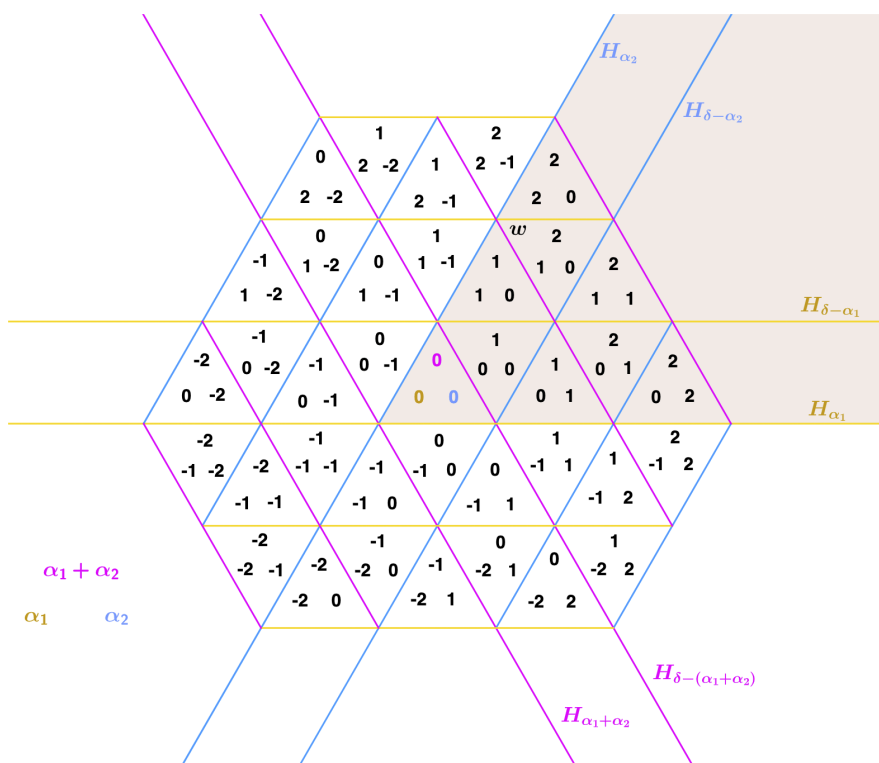
By convention, we have the following formula [24, Section 1] relating Shi coefficients on negative roots and Shi coefficients on positive roots:

$$(3) \quad k(w, -\alpha) = -k(w, \alpha) \quad \text{for any } \alpha \in \Phi_0.$$

Now the following statements give very important relations between Shi coefficients on positive roots.

Lemma 2.2 [24, Lemma 3.1] *Let w be an element of W_0 and $\alpha \in \Phi^+$. Then*

$$k(w, \alpha) = \begin{cases} 0, & \text{if } \alpha \notin N(w), \\ -1, & \text{if } \alpha \in N(w). \end{cases}$$



The following proposition is, in general, only stated for positive roots, but turns out to stand also for any root thanks to equation (3).

- (1) $k(sw, \alpha) = k(w, s(\alpha)) + k(s, \alpha)$ [24, Proposition 4.2].
- (2) $k(tw, \alpha) = k(w, t(\alpha)) + k(t, \alpha)$ (reformulation of [7, Proposition 3.2]).

Remark 2.5

- (1) Shi does not give a proof of Lemma 2.4 in [24]; instead, he refers the reader to [23, Chapter 7]. We point out that a direct proof of this result can be obtained using an analog of [13, Proposition 2.16] for “ p -inversions sets $N^p(w)$.”

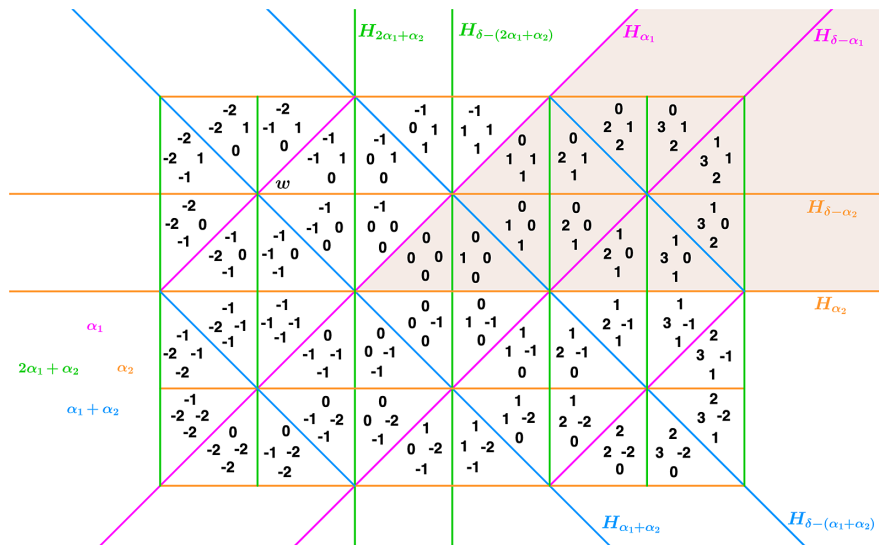


Figure 2: The Coxeter arrangement of type $\tilde{B}_2$. Each alcove is labeled with its Shi parameterization. The labels $k(w, \alpha)$ for $\alpha \in \Phi_0^+$ are indicated in each alcove with the parameterization given at the left-hand side of the figure. The shaded region is the dominant region of $\mathcal{A}(W, S)$, which is also the fundamental chamber for the finite Weyl group W_0 . The fundamental alcove A_o , which corresponds to e , is parameterized by 0's.

- (2) It is very useful to notice that, for an arbitrary $w \in W$ and $\alpha \in \Phi_0^+$, $|k(w, \alpha)|$ is given by the number of hyperplanes parallel to H_α that separates $w \cdot A_o$ from A_o ; the sign is then negative if H_α is one of these hyperplanes.

Example 2.6 (Type $\tilde{A}_2$) Consider (W, S) of type $\tilde{A}_2$. Alcoves and their labeling are shown in Figure 1. Write $S_0 = \{s_1, s_2\}$, so $\Delta_0 = \{\alpha_1, \alpha_2\}$, and $\Phi_0^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ (with $s_i = s_{\alpha_i}$).

Example 2.7 (Type $\tilde{B}_2$) Consider (W, S) of type $\tilde{B}_2$, with underlying finite Weyl group of type B_2 . We consider here $V = \mathbb{R}_2$ with orthonormal basis $\{e_1, e_2\}$. Set $\alpha_1 = e_1$ and $\alpha_2 = e_2 - e_1$. Write $S_0 = \{s_1, s_2\}$, so $\Delta_0 = \{\alpha_1, \alpha_2\}$, and $\Phi_0^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$ (with $s_i = s_{\alpha_i}$). The long roots are α_2 and $2\alpha_1 + \alpha_2$. Alcoves and their labeling are shown in Figure 2.

The example go type $\tilde{G}_2$ is given in Figure 7.

2.4 Affine crystallographic root systems

Set $V = V_0 \oplus \mathbb{R}\delta$ to be a real vector space with basis $\Delta_0 \sqcup \{\delta\}$, where δ is an indeterminate. We define a symmetric bilinear form $B : V \times V \rightarrow \mathbb{R}$ by extending $\langle \cdot, \cdot \rangle$ to V as follows: for $\alpha, \beta \in \Delta_0$, we set $B(\alpha, \beta) = \langle \alpha, \beta \rangle$, $B(\alpha, \delta) = 0$ and $B(\delta, \delta) = 0$. The pair (V, B) is a quadratic space for which the isotropic cone is $\mathbb{R}\delta = \{x \in V \mid B(x, x) = 0\}$.

We describe now a *geometric representation of (W, S) on (V, B)* . A simple system in (V, B) for (W, S) is $\Delta = \Delta_0 \sqcup \{\delta - \alpha_0\}$. The root system Φ and the positive root

system Φ^+ for (W, S) in (V, B) are defined by $\Phi = \Phi^+ \sqcup \Phi^-$, where $\Phi^- = -\Phi^+$ and $\Phi^+ = (\Phi_0^+ + \mathbb{N}\delta) \sqcup (\Phi_0^- + \mathbb{N}^*\delta)$. The pair (Φ, Δ) is called a (crystallographic) based root system of (W, S) in (V, B) . For $\alpha + k\delta \in \Phi^+$, the reflection $s_{\alpha+k\delta} : V \rightarrow V$ is defined as follows. Let $x = x_0 + a\delta \in V$, then

$$(4) \quad s_{\alpha+k\delta}(x) := x - 2 \frac{B(\alpha + k\delta, x)}{B(\alpha + k\delta, \alpha + k\delta)} (\alpha + k\delta) = s_\alpha(x_0) + (a - k\langle \alpha^\vee, x_0 \rangle) \delta.$$

Observe that $s_{\alpha+0\delta} = s_\alpha$. It follows that $S = S_0 \sqcup \{s_{\delta-\alpha_0}\}$. The correspondences between the positive root system Φ^+ , the reflections on V , and those on V_0 defined in Section 2.2 are as follows:

$$\begin{array}{ccccc} \Phi^+ & \longrightarrow & \{\text{reflections on } V\} & \longrightarrow & \{\text{reflections on } V_0\} \\ \alpha + k\delta & \longmapsto & s_{\alpha+k\delta} & \longmapsto & s_{-\alpha, k}. \end{array}$$

Therefore, the hyperplane associated with $\alpha + k\delta \in \Phi^+$ is $H_{\alpha+k\delta} := H_{-\alpha, k}$.

3 Small roots and low elements in affine Weyl groups

Let (W, S) be an affine Coxeter system with underlying Weyl group W_0 . We now survey the notions of small roots and low elements (see [13] for more details). We gave general definitions of small roots and low elements in the introduction. In the case of affine Weyl groups, these notions have easier interpretations, which we discuss below.

3.1 Small roots and low elements

A positive root $\alpha + k\delta \in \Phi^+$ (resp. a hyperplane $H_{\alpha+k\delta} \in \mathcal{A}(W, S)$) is *small* if there is no hyperplane parallel to $H_{\alpha+k\delta}$ that separates $H_{\alpha+k\delta}$ from A_0 . Denote by $\Sigma := \Sigma(W, S)$ the set of small roots. It turns out that

$$\Sigma = \Phi_0^+ \sqcup (\delta - \Phi_0^+) = \{\alpha, \delta - \alpha \mid \alpha \in \Phi_0^+\},$$

which is of cardinality $|\Phi_0|$. It is well known (see, for instance, [19, Proposition 1.4]) that $s \in S_0$ is a bijection on $\Phi_0^+ \setminus \{\alpha_s\}$, where α_s denotes the root in Δ_0 such that $s = s_{\alpha_s}$. The next proposition is an analog of this result for Σ .

Proposition 3.1 *Let $s \in S_0$, then $s(\Sigma \setminus \{\alpha_s, \delta - \alpha_s\}) = \Sigma \setminus \{\alpha_s, \delta - \alpha_s\}$.*

Proof Let $\alpha \in \Phi_0^+ \setminus \{\alpha_s\}$, then $s(\alpha) \in \Phi_0^+ \setminus \{\alpha_s\} \subseteq \Sigma \setminus \{\alpha_s, \delta - \alpha_s\}$. We also have by equation (4)

$$s(\delta - \alpha) = \delta - s(\alpha) \in \delta - (\Phi_0^+ \setminus \{\alpha_s\}) \subseteq \Sigma \setminus \{\alpha_s, \delta - \alpha_s\}.$$

Therefore, $s(\Sigma \setminus \{\alpha_s, \delta - \alpha_s\}) \subseteq \Sigma \setminus \{\alpha_s, \delta - \alpha_s\}$; the equality follows since s is injective. ■

An element $w \in W$ is a *low element* of (W, S) if there is $X \subseteq \Sigma$ such that

$$N(w) = \text{cone}_\Phi(X) := \text{cone}(X) \cap \Phi.$$

We denote by $L(W, S)$ the set of low elements of (W, S) . If there is no possible confusion, we write $L := L(W, S)$. Low elements were introduced in order to prove

that there is a finite Garside shadow, i.e., a finite subset of W containing S , closed by taking suffixes and closed under the join operator of the right weak order. It is proven in [13, Theorem 1.1].

Example 3.2 We continue Example 2.6. The small roots are

$$\Sigma = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \delta - \alpha_1, \delta - \alpha_2, \delta - (\alpha_1 + \alpha_2)\}.$$

Obviously, $k(e, \alpha_1) = k(e, \alpha_2) = k(e, \alpha_1 + \alpha_2) = 0$. Now, for an arbitrary $w \in W$, $|k(w, \alpha_1)|$ is given by the number of hyperplanes parallel to H_{α_1} that separates $w \cdot A_o$ from A_o ; the sign is then negative if H_{α_1} is one of these hyperplanes (see Remark 2.5). For instance, the w indicated in Figure 1 has $k(w, \alpha_1) = 1$ since only $H_{\delta - \alpha_1}$ separates the alcove $w \cdot A_o$ from A_o . We proceed similarly for α_2 and the highest root $\alpha_1 + \alpha_2$. So, for the particular w indicated in Figure 1, we have: $k(w, \alpha_2) = 0$, since no parallel to H_{α_2} separates $w \cdot A_o$ from A_o ; $k(w, \alpha_1 + \alpha_2) = 2$, since $H_{\delta - (\alpha_1 + \alpha_2)}$ and $H_{2\delta - (\alpha_1 + \alpha_2)}$ separates $w \cdot A_o$ from A_o .

Example 3.3 We continue Example 2.7. The small roots are

$$\Sigma = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \delta - \alpha_1, \delta - \alpha_2, \delta - (\alpha_1 + \alpha_2), \delta - (2\alpha_1 + \alpha_2)\}.$$

Now, for an arbitrary $w \in W$, $|k(w, \beta)|$ is, as in the preceding example, given by the number of hyperplanes parallel to H_β that separates $w \cdot A_o$ from A_o ; the sign is then negative if H_β is one of these hyperplanes. For instance, the w indicated in Figure 2 has $k(w, \alpha_1) = 1$ since only $H_{\delta - \alpha_1}$ separates $w \cdot A_o$ from A_o . We proceed similarly for α_2 , $\alpha_1 + \alpha_2$ and the highest root $2\alpha_1 + \alpha_2$. So we obtain $k(w, \alpha_2) = 1$, since only $H_{\delta - \alpha_2}$ separates $w \cdot A_o$ from A_o , $k(w, \alpha_1 + \alpha_2) = 0$, since no parallel to $H_{\alpha_1 + \alpha_2}$ separates the alcove $w \cdot A_o$ from A_o , and finally $k(w, 2\alpha_1 + \alpha_2) = -1$, since only $H_{2\alpha_1 + \alpha_2}$ separates $w \cdot A_o$ from A_o .

3.2 Small inversion sets

In order to check if an element $w \in W$ is a low element, it is often convenient to consider all small roots in the inversion set $N(w)$.

The *small inversion set* of $w \in W$ is $\Sigma(w) := N(w) \cap \Sigma$. Therefore, an element $w \in W$ is a low element if and only if $N(w) = \text{cone}_\Phi(\Sigma(w))$. The set of all small inversion sets is $\Lambda(W, S) = \{\Sigma(w) \mid w \in W\} \subseteq 2^\Sigma$. We write $\Lambda := \Lambda(W, S)$ if there is no possible confusion. Notice that, since Σ is finite, the set 2^Σ of subsets of Σ is finite, hence Λ is finite.

The following result, which defines the transitions in the so-called Brink–Howlett automaton, is [13, Lemma 3.21(1)]; we refer the reader to [18, 21, 22] for more information on the Brink–Howlett automaton.

Lemma 3.4 Let $w \in W$ and $s \in S$ such that $\ell(w) > \ell(sw)$. Then $\Sigma(w) = \{\alpha_s\} \sqcup s(\Sigma(sw)) \cap \Sigma$.

The set of low elements is finite since the map $\sigma : L \rightarrow \Lambda$, defined by $w \mapsto \Sigma(w)$, is injective [13, Proposition 3.26(ii)]. It is conjectured by Dyer and the second author that this map is surjective, which turns out to be equivalent to Conjecture 1 (see Section 4.3 for more details).

Conjecture 2 [13, Conjecture 2] *The map $\sigma : L \rightarrow \Lambda$ is a bijection.*

Remark 3.5 The map σ is denoted by Σ in [13], we change the notation to avoid confusion with the set of small roots Σ .

3.3 Basis of inversion sets

The most efficient way to check if an element $w \in W$ is low is to consider a basis for the inversion set $N(w)$ (see [13, Section 4.2] for details).

Let $w \in W$, and consider the set

$$\begin{aligned} N^1(w) &:= \{\alpha + k\delta \in \Phi^+ \mid \mathbb{R}(\alpha + k\delta) \text{ is an extreme ray of } \text{cone}(N(w))\} \\ &= \{\alpha + k\delta \in \Phi^+ \mid \ell(s_{\alpha+k\delta}w) = \ell(w) - 1\}. \end{aligned}$$

The set $N^1(w)$ is called the *basis of the inversion set* $N(w)$ and is the inclusion-minimal subset of $N(w)$ such that $N(w) = \text{cone}_{\Phi}(N^1(w))$.

Proposition 3.6 *Let $w \in W$, then $w \in L$ if and only if $N^1(w) \subseteq \Sigma$.*

Proof If $N^1(w) \subseteq \Sigma$, then $w \in L$ by definition of low elements. Assume now that $w \in L$, so $N(w) = \text{cone}_{\Phi}(\Sigma(w)) = \text{cone}_{\Phi}(N^1(w))$. Since $N^1(w)$ is the inclusion-minimal subset with that property, we have $N^1(w) \subseteq \Sigma(w) \subseteq \Sigma$. ■

3.4 Left and right descent sets

The basis of inversion sets gives a useful interpretation of the right and left descent sets. The *left descent set* of $w \in W$ is the set:

$$D_L(w) = \{s \in S \mid \ell(sw) = \ell(w) - 1\} = \{s \in S \mid \alpha_s \in \Delta \cap N(w)\},$$

where $\alpha_s \in \Delta$ is the simple root such that $s = s_{\alpha_s}$ in equation (4). The *set of left descent roots* of w is

$$ND_L(w) = \Delta \cap N(w) \subseteq N^1(w).$$

In other words, $\alpha + k\delta \in ND_L(w)$ if and only if $H_{\alpha+k\delta}$ is a wall of A_o that separates A_o from $w \cdot A_o$. Similarly, the *right descent set* of $w \in W$ is the set:

$$D_R(w) = \{s \in S \mid \ell(ws) = \ell(w) - 1\} = \{s \in S \mid w(\alpha_s) \in \Phi^-\}.$$

Let $s \in D_R(w)$, then there is a reduced word for w ending with s : $w = us$ with $u \in W$ and $\ell(w) = \ell(u) + 1$. Therefore,

$$N(u) = N(w) \setminus \{u(\alpha_s)\}.$$

The *set of right descent roots* of w , which is a subset of $N^1(w)$, is

$$\begin{aligned} ND_R(w) &= -w(\{\alpha_s \mid s \in D_R(w)\}) \\ &= \{\alpha + k\delta \in N(w) \mid \exists u \in W, N(u) = N(w) \setminus \{\alpha + k\delta\}\}. \end{aligned}$$

In other words, $\alpha + k\delta \in ND_R(w)$ if and only if $H_{\alpha+k\delta}$ is a wall of $w \cdot A_o$ that separates A_o from $w \cdot A_o$.

The following proposition gives a useful relationship between the sets of right descent roots of an element $w \in W$ and of one of its maximal suffixes.

Proposition 3.7 We have $ND_R(sw) = s(ND_R(w) \setminus \{\alpha_s\})$ for all $w \in W$ and $s \in D_L(w)$.

Proof Since $s \in D_L(w)$, we have $w = su$ for some $u \in W$ and $\ell(su) = \ell(u) + 1$. We first show that $D_R(u) = D_R(w) \setminus (\{w^{-1}sw\} \cap S)$. Let $r \in D_R(u)$, then $u = vr$ for some $v \in W$ and $\ell(u) = \ell(v) + 1$. So $w = svr$ with $\ell(w) = \ell(v) + 2$. Thus, $r \in D_R(w)$. If $r = w^{-1}sw$, then $wr = sw = s(su) = u$. Therefore, $w = ur$ and $r \notin D_R(u)$, a contradiction. So $D_R(u) \subseteq D_R(w) \setminus (\{w^{-1}sw\} \cap S)$. Now let $r \in D_R(w) \setminus (\{w^{-1}sw\} \cap S)$. By the exchange condition, we either have $wr = sur = u$ or $r \in D_R(u)$. The first case implies $w^{-1}sw = (u^{-1}s)sw = u^{-1}w = r \in S$, a contradiction. So $r \in D_R(u)$ and we have the desired equality. Now, by definition of $ND_R(\cdot)$, we have

$$\begin{aligned} ND_R(sw) &= ND_R(u) \\ &= -u(\{\alpha_r \mid r \in D_R(u)\}) = -u(\{\alpha_r \mid r \in D_R(w)\} \setminus (\{w^{-1}(\alpha_s)\} \cap \Delta)) \\ &= -sw(\{\alpha_r \mid r \in D_R(w)\} \setminus (\{w^{-1}(\alpha_s)\} \cap \Delta)) \\ &= s(ND_R(w) \setminus \{\alpha_s\}). \end{aligned}$$

4 Shi arrangements in affine Weyl groups

Let (W, S) be an affine Coxeter system with underlying Weyl group W_0 . The *Shi arrangement* of (W, S) is the affine hyperplane arrangement constituted of the small hyperplanes of (W, S) :

$$\text{Shi}(W, S) = \{H_\beta \mid \beta \in \Sigma\}.$$

A *Shi region* of (W, S) is a connected component of the complement of the Shi arrangement in V_0 , i.e., a connected component of

$$V_0 \setminus \bigcup_{H \in \text{Shi}(W, S)} H.$$

4.1 Separation sets of Shi regions

Shi regions correspond to equivalence classes of the relation $\sim_{\text{Shi}}$ over W given by $u \sim_{\text{Shi}} v$ if and only if both $u \cdot A_\circ$ and $v \cdot A_\circ$ are contained in the same Shi region. These equivalence classes have a useful interpretation in terms of separation sets.

Definition 4.1 The *separation set* of a Shi region $\mathcal{R}$ is the set of hyperplanes in $\text{Shi}(W, S)$ that separate $\mathcal{R}$ from $A_\circ$. We denote the *inversion set of the Shi region* $\mathcal{R}$ by

$$\Sigma(\mathcal{R}) = \{\alpha \in \Sigma \mid H_\alpha \text{ separates } \mathcal{R} \text{ from } A_\circ\}.$$

Proposition 4.2 Let $w \in W$ and $s \in S$. Let $\mathcal{R}$ be a Shi region such that $w \cdot A_\circ \subseteq \mathcal{R}$. Then

- (1) $\Sigma(\mathcal{R}) = \Sigma(w)$. In particular, $u \sim_{\text{Shi}} v$ if and only if $\Sigma(u) = \Sigma(v)$.
- (2) If $\alpha_s \in \Sigma(\mathcal{R})$, then there exists a Shi region $\mathcal{R}'$ such that $s \cdot \mathcal{R}' \subseteq \mathcal{R}$.

Proof (1) Since $w \cdot A_\circ \subseteq \mathcal{R}$, any hyperplane that separates $\mathcal{R}$ from $A_\circ$ also separates $w \cdot A_\circ$ from $A_\circ$. Therefore, $\Sigma(\mathcal{R}) \subseteq N(w) \cap \Sigma = \Sigma(w)$. Conversely, assume that H_α ,

$\alpha \in \Sigma$, separates $w \cdot A_o$ from A_o . Since $w \cdot A_o \subseteq \mathcal{R}$ and $\mathcal{R}$ is a connected component of $V_0 \setminus \bigcup_{H \in \text{Shi}(W, S)} H$, H_α separates $w \cdot A_o$ from $\mathcal{R}$.

(2) Let $w \in W$ such that $w \cdot A_o \subseteq \mathcal{R}$. Then, by (1), $\alpha_s \in \Sigma(\mathcal{R}) = \Sigma(w) \subseteq N(w)$. Therefore, $\ell(sw) < \ell(w)$. Let $\mathcal{R}'$ be the Shi region such that $sw \cdot A_o \subseteq \mathcal{R}'$. Let $u \in W$ such that $u \sim_{\text{Shi}} sw$, and we need to show that $su \sim_{\text{Shi}} w$. Since $\alpha_s \notin N(sw)$, we have $\alpha_s \notin \Sigma(sw) = \Sigma(u)$. In other words, $\ell(su) > \ell(u)$. Then, by Lemma 3.4, we have $\Sigma(w) = \{\alpha_s\} \sqcup s(\Sigma(sw)) \cap \Sigma$; moreover, since $u \sim_{\text{Shi}} sw$, it follows that $\{\alpha_s\} \sqcup s(\Sigma(sw)) \cap \Sigma = \{\alpha_s\} \sqcup s(\Sigma(u)) \cap \Sigma$. Then we obtain $\{\alpha_s\} \sqcup s(\Sigma(u)) \cap \Sigma = \{\alpha_s\} \sqcup s(\Sigma(su)) \cap \Sigma = \Sigma(su)$ since $\ell(su) > \ell(u)$. Thus, $su \sim_{\text{Shi}} w$. ■

4.2 Minimal elements in Shi regions

In [25, Proposition 7.1], Shi shows that any Shi region $\mathcal{R}$ on an affine Weyl group W contains a unique minimal element, that is, there is a unique element $w \in W$ with $w \cdot A_o \subseteq \mathcal{R}$ and such that $\ell(w) \leq \ell(u)$, for all $u \sim_{\text{Shi}} w$. We denote by L_{Shi} the set of minimal elements in Shi regions. Shi shows in [25, Proposition 7.2] that $w \in L_{\text{Shi}}$ if and only if for all $\alpha \in \Phi_0^+$ we have

$$(5) \quad |k(w, \alpha)| = \min\{|k(g, \alpha)| \mid g \sim_{\text{Shi}} w\}.$$

In [25, Proposition 7.3], Shi characterizes the minimal elements in Shi regions by

$$(6) \quad w \in L_{\text{Shi}} \iff w \not\sim_{\text{Shi}} ws, \quad \forall s \in D_R(w),$$

where $u \not\sim_{\text{Shi}} v$ means that u and v are not in the same Shi region. This characterization implies that any element in a given Shi region is greater or equal in the right weak order than its corresponding minimal element, i.e., the minimal element in a Shi region is the prefix of any other element in that Shi region.

Proposition 4.3 *Let $w \in L_{\text{Shi}}$ and $g \in W$ such that $g \sim_{\text{Shi}} w$, then $N(w) \subseteq N(g)$.*

Proof We proceed by induction on $m = \ell(g) - \ell(w)$. Since $w \in L_{\text{Shi}}$ is of minimal length in its Shi region, $m \in \mathbb{N}$. If $m = 0$, then $g = w$ by unicity of $w \in L_{\text{Shi}}$, so $w = g$ and $N(g) = N(w)$. If $m > 0$, then $g \notin L_{\text{Shi}}$, so there is $s \in D_R(g)$ such that $gs \sim_{\text{Shi}} w$ by equation (6). Since $\ell(gs) - \ell(w) = \ell(g) - 1 - \ell(w) = m - 1$, we conclude by induction that $N(w) \subseteq N(gs)$. The result follows from the fact that $N(gs) = N(g) \setminus \{-g(\alpha_s)\}$ (see Section 3.4). ■

Proposition 4.4 *We have $L \subseteq L_{\text{Shi}}$.*

Proof Let $w \in L$, then $ND_R(w) \subseteq N^1(w) \subseteq \Sigma$, by Proposition 3.6. Let $s \in D_R(w)$, then $-w(\alpha_s) \in ND_R(w) \subseteq \Sigma$ and $N(ws) = N(w) \setminus \{-w(\alpha_s)\}$. Hence, $\Sigma(ws) = \Sigma(w) \setminus \{-w(\alpha_s)\}$. Therefore, by Proposition 4.2, $w \not\sim_{\text{Shi}} ws$. We conclude that $w \in L_{\text{Shi}}$ by equation (6). ■

The next proposition shows that L_{Shi} is closed under taking suffixes.

Proposition 4.5 *Let $w \in L_{\text{Shi}}$, then $sw \in L_{\text{Shi}}$ for all $s \in D_L(w)$. Moreover, if $g \sim_{\text{Shi}} sw$, then $sg \sim_{\text{Shi}} w$.*

Proof Let $s \in D_L(w)$. Let $w' \in L_{\text{Shi}}$ be such that $w' \sim_{\text{Shi}} sw$. Then $\Sigma(w') = \Sigma(sw)$. By Lemma 3.4, since $\ell(w) > \ell(sw)$, we have $\Sigma(w) = \{\alpha_s\} \cup (\Sigma \cap s(\Sigma(sw))) = \{\alpha_s\} \cup$

$(\Sigma \cap s(\Sigma(w'))) = \Sigma(sw')$. So $w \sim_{\text{Shi}} sw'$ by Proposition 4.2. But $\ell(sw') = \ell(w') + 1 \leq \ell(sw) + 1 = \ell(w)$ implying $sw' = w$ by minimality of w . Hence, $sw \in L_{\text{Shi}}$. Now let $g \sim_{\text{Shi}} sw$, then $\Sigma(g) = \Sigma(sw)$. Proceeding as above, we obtain $\Sigma(w) = \{\alpha_s\} \cup (\Sigma \cap s(\Sigma(sw))) = \{\alpha_s\} \cup (\Sigma \cap s(\Sigma(g))) = \Sigma(sg)$, which means that $sg \sim_{\text{Shi}} w$. ■

4.3 Theorem 1.1 implies Conjectures 1 and 2

To prove Theorem 1.1, we only have to prove the converse of Proposition 4.4, that is, to prove that $L_{\text{Shi}} \subseteq L$. Furthermore, this statement is also enough to show both conjectures.

First, as announced in the introduction, Theorem 1.1 implies Conjecture 1.

Proposition 4.6 Assume that $L_{\text{Shi}} \subseteq L$, then $L = L_{\text{Shi}}$ and any region $\mathcal{R}$ of $\text{Shi}(W, S)$ contains a unique low element, which is the unique element of minimal length in $\mathcal{R}$.

Proof The first statement follows from Proposition 4.4. Let $\mathcal{R}$ be a Shi region. Since $L_{\text{Shi}} \subseteq L$, $\mathcal{R}$ contains a low element w , which is of minimal length. The fact that w is the unique low element in $\mathcal{R}$ follows by definition and Proposition 4.2. Assume that $g \in L$ is another low element in $\mathcal{R}$, then $g \sim_{\text{Shi}} w$. Therefore, $\Sigma(w) = \Sigma(g)$ and by definition of low elements: $N(g) = \text{cone}_{\Phi}(\Sigma(g)) = \text{cone}_{\Phi}(\Sigma(w)) = N(w)$. Therefore, $g = w$, since any element of W is uniquely determined by its inversion set. ■

Second, to prove Conjecture 2, it is enough to show that any Shi region contains a low element, which is implied by the statement $L_{\text{Shi}} \subseteq L$.

Proposition 4.7 Assume that any Shi region contains a low element, then the map $\sigma : L \rightarrow \Lambda$ is a bijection.

Proof We know already that σ is injective by [13, Proposition 3.26(ii)]. Let $\Sigma(w) \in \Lambda$ with $w \in W$. Since any Shi region contains a low element, there is $u \in L$ such that $u \sim_{\text{Shi}} w$. Therefore, by Proposition 4.2, $\Sigma(u) = \Sigma(w)$. The map σ is, therefore, surjective. ■

4.4 Shi's admissible sign types and Shi regions

In order to show that $L_{\text{Shi}} \subseteq L$ and then to prove Theorem 1.1 (as explained in Section 4.3), we need now to survey Shi's admissible sign types.

In [25], Shi uses the parametrization of the alcoves of (W, S) in order to describe the Shi regions of $\text{Shi}(W, S)$. We follow here Shi's notations from [25]. Let $\mathcal{S}$ be the set of Φ_0^+ -tuples over the set $\{-, 0, +\}$; its elements are called *sign types*. For $w \in W$, the function $\zeta : W \rightarrow \mathcal{S}$ is defined as follows: $\zeta(w) = (X(w, \alpha))_{\alpha \in \Phi_0^+}$ where

$$(7) \quad X(w, \alpha) = \begin{cases} +, & \text{if } k(w, \alpha) > 0, \\ 0, & \text{if } k(w, \alpha) = 0, \\ -, & \text{if } k(w, \alpha) < 0. \end{cases}$$

Definition 4.8 The *sign type* $X(\mathcal{R}) = (X(\mathcal{R}, \alpha))_{\alpha \in \Phi_0^+}$ of a Shi region $\mathcal{R}$ is $X(\mathcal{R}) = \zeta(w)$ for some $w \in W$ such that $w \cdot A_0 \subseteq \mathcal{R}$. A sign type $X = (X_{\alpha})_{\alpha \in \Phi_0^+}$ is said to be *admissible* if X is in the image of ζ . We denote by $\mathcal{S}_{\Phi}$ the set of admissible sign types of Φ .

The following theorem sums up results from [25, Theorem 2.1 and Section 6].

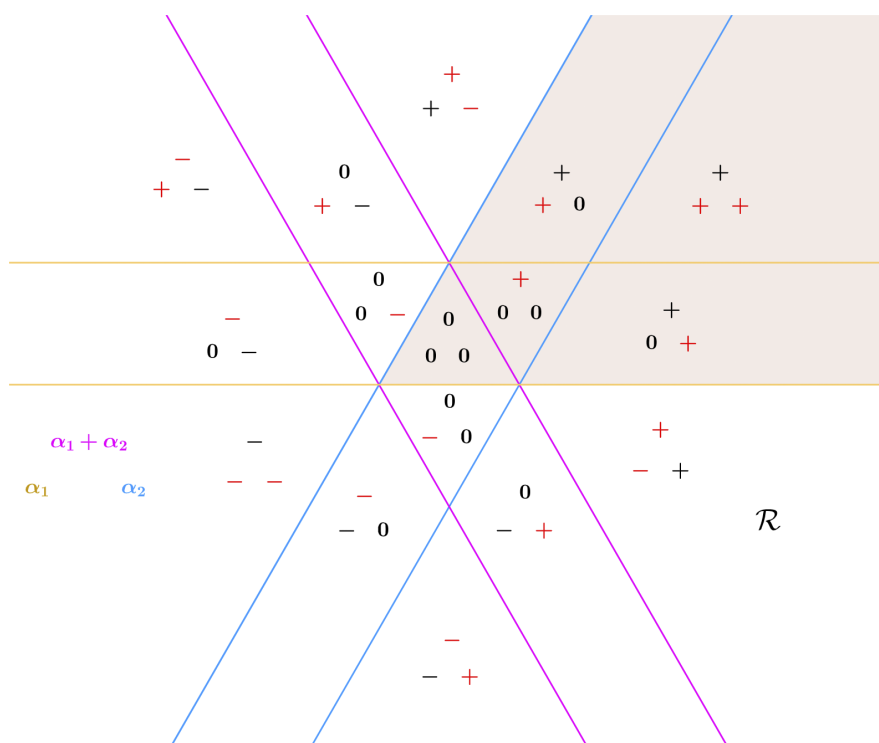


Figure 3: The Shi arrangement of type $\tilde{A}_2$. Each Shi region is labeled with its admissible sign type. The labels $X(\mathcal{R}, \alpha)$ for a Shi region $\mathcal{R}$ are indicated in each alcove with the parameterization given at the left-hand side of the figure. The signs colored in red indicate the descent roots (see Definition 5.1) of the corresponding Shi region.

Theorem 4.9 (Shi, 1987) *Let W be an irreducible affine Weyl group.*

- (1) *The rank 2 admissible sign types are precisely those in Figures 3, 4, and 8.*
- (2) *A sign type $X = (X_\alpha)_{\alpha \in \Phi_0^+}$ is admissible if and only if for any irreducible root subsystem Ψ of rank 2 in Φ_0 , the restriction $(X_\alpha)_{\alpha \in \Psi^+}$ of X to Ψ^+ is one of the rank 2 admissible sign types.*
- (3) *For $u, v \in W$, we have $u \sim_{\text{Shi}} v$ if and only if $\zeta(u) = \zeta(v)$.*

Example 4.10 For (W, S) of type $\tilde{A}$, the irreducible root subsystems of rank 2 are easy to describe. Set $V = \mathbb{R}^{n+1}$ with the usual orthonormal basis $\{e_1, \dots, e_{n+1}\}$. We abbreviate $e_i - e_j$ by $e_{ij} := e_i - e_j$. A way to describe the roots of A_n is by $\Phi = \{\pm(e_{ij}) \mid 1 \leq i < j \leq n+1\}$ with simple system $\Delta = \{e_{i, i+1} \mid 1 \leq i < n+1\}$, and positive roots $\Phi^+ = \{e_{ij} \mid 1 \leq i < j \leq n+1\}$. Thus, the irreducible root subsystems of rank 2 of Φ^+ are of the form $\{e_{ik}, e_{kj}, e_{ij} \mid 1 \leq i < k < j \leq n+1\}$. A convenient way to write the Shi coordinates $k(w, e_{ij})$ of an element $w \in W$ (or the admissible signs $X(\mathcal{R}, e_{ij})$ of a Shi region $\mathcal{R}$) is by placing them in a triangular shape as shown in Figure 5 for Type A_4 .

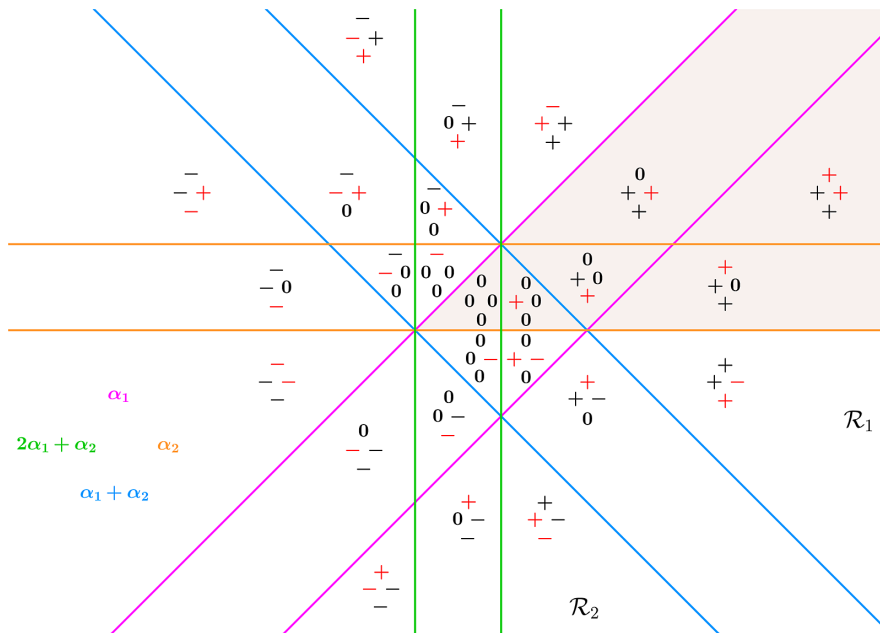


Figure 4: The Shi arrangement of type $\tilde{B}_2$. Each Shi region is labeled with its admissible sign type. The labels $X(\mathcal{R}, \alpha)$ for a Shi region $\mathcal{R}$ are indicated in each alcove with the parameterization given at the right-hand side of the figure. The signs colored in red indicate the descent roots (see Definition 5.1) of the corresponding Shi region.

$$\begin{array}{cccc}
 & e_{15} & & \\
 & e_{14} & e_{25} & \\
 & e_{13} & e_{24} & e_{35} \\
 e_{12} & e_{23} & e_{34} & e_{45}
 \end{array}$$

Figure 5: Triangular presentation of the positive root system in Type A_4 . The blue roots show a irreducible root subsystem of rank 2.

Then, using this presentation, it is easy to check if a sign is admissible using Theorem 4.9(1): we only have to look at all the subtriangles corresponding to irreducible root subsystems of rank 2 and check in Figure 3 if the sign type is admissible using Theorem 4.9. An example in rank 5 is given in Figure 6: on the left-hand side, the sign type is admissible, while on the right-hand side, the sign type is not admissible because the triplet in red does not belong to the rank 2 admissible sign types given in Figure 3.

Remark 4.11 The game of finding the admissible sign types in types B, C, D is more complicated because there is no triangular presentation of the positive roots. An alternative was recently provided by Charles [10].



Figure 6: On the left side, the sign type is admissible, while on the right side, it is not because the triplet in red does not belong to the rank 2 admissible sign types given in Figure 3.

Proposition 4.12 *If $\mathcal{R}$ is a Shi region, then the separation set of $\mathcal{R}$ is*

$$\Sigma(\mathcal{R}) = \{\alpha \in \Phi_0^+ \mid X(\mathcal{R}, \alpha) = -\} \sqcup \{\delta - \alpha \mid \alpha \in \Phi_0^+, X(\mathcal{R}, \alpha) = +\}.$$

Proof Let $w \in W$ such that $w \cdot A_o \subseteq \mathcal{R}$. By Proposition 4.2, we have $\Sigma(\mathcal{R}) = \Sigma(w) \subseteq N(w)$. Let $\alpha \in \Phi_0^+ \subseteq \Sigma$. By Lemma 2.2, we have $\alpha \in N(w) \cap \Sigma = \Sigma(w)$ if and only if $k(w, \alpha) < 0$. From equation (7), we get that $\alpha \in \Sigma(w)$ if and only if $X(\mathcal{R}, \alpha) = -$. The same line of reasoning shows that $\delta - \alpha \in \Sigma(w)$ if and only if $X(\mathcal{R}, \alpha) = +$, which concludes the proof. ■

Example 4.13 We continue Example 2.6. Consider the Shi region $\mathcal{R}$ in Figure 3. This region is the union of all the alcoves in Figure 1 bounded by the small hyperplanes H_{α_1} and $H_{\delta - (\alpha_1 + \alpha_2)}$. This region contains, for instance, the alcove $w \cdot A_o$ such that $k(w, \alpha_1) = -1$, $k(w, \alpha_2) = 2$ and $k(w, \alpha_1 + \alpha_2) = 2$. Therefore, $X(\mathcal{R}, \alpha_1)$ is the sign of $k(w, \alpha_1) = -1$, that is, $X(\mathcal{R}, \alpha_1) = -$. Similarly, we obtain $X(\mathcal{R}, \alpha_2) = +$ and $X(\mathcal{R}, \alpha_1 + \alpha_2) = +$. Finally, we have $\Sigma(\mathcal{R}) = \{\alpha_1, \delta - \alpha_2, \delta - (\alpha_1 + \alpha_2)\}$.

Example 4.14 (Type $\tilde{B}_2$) We continue Example 2.7. Consider the Shi regions $\mathcal{R}_1$ and $\mathcal{R}_2$ in Figure 4. We have $\Sigma(\mathcal{R}_1) = \{\alpha_1, \delta - \alpha_2, \delta - (\alpha_1 + \alpha_2), \delta - (2\alpha_1 + \alpha_2)\}$ together with $\Sigma(\mathcal{R}_2) = \{\alpha_1, \delta - \alpha_2, \alpha_1 + \alpha_2, \delta - (2\alpha_1 + \alpha_2)\}$.

5 Descent roots and descent walls of Shi regions

We are now interested in *descent wall* of a Shi region, that is, those walls of a Shi region $\mathcal{R}$ that separates $\mathcal{R}$ from the Shi region A_o . We show at the end of this section the first of the two core results (Lemma 5.9) for proving $L_{\text{Shi}} \subseteq L$ and therefore Theorem 1.1.

5.1 Walls and descents of Shi regions

Following classical terminology, a *wall of a Shi region* $\mathcal{R}$ is a hyperplane $H \in \text{Shi}(W, S)$ such that H supports a facet of $\mathcal{R}$.

Definition 5.1 (Descent wall and descent root) Let $\mathcal{R}$ be a Shi region. We say that a wall H of $\mathcal{R}$ is a *descent wall* of $\mathcal{R}$ if H separates $\mathcal{R}$ from A_o . In other words, they represent the last hyperplane crossed in a gallery from A_o to $\mathcal{R}$. The *set of descent roots of a Shi region* $\mathcal{R}$ is

$$\Sigma D(\mathcal{R}) = \{\alpha \in \Sigma(\mathcal{R}) \mid H_\alpha \text{ is a descent-wall of } \mathcal{R}\}.$$

Definition 5.2 Let $\mathcal{R}$ be a Shi region, and let $\alpha \in \Phi_0^+$. We define

$$X_{\mathcal{R}}^\alpha := (X_\beta^\alpha)_{\beta \in \Phi_0^+},$$

where $X_\alpha^\alpha = 0$ and $X_\beta^\alpha = X(\mathcal{R}, \beta)$ for all $\beta \in \Phi_0^+ \setminus \{\alpha\}$. If there is no possible confusion, we often omit the $\mathcal{R}$ as subscript of $X_{\mathcal{R}}^\alpha$ and we just write X^α .

The following statement, which is an immediate consequence of the definitions, gives a characterization of the descent walls of a Shi region in terms of walls, sign types, and Shi coefficients: a wall H_θ is a descent wall of a Shi region $\mathcal{R}$ if and only if there is a Shi region $\mathcal{R}'$ sharing a common wall with $\mathcal{R}$ such that one sign of $\mathcal{R}$ changes to a 0 in the sign type of $\mathcal{R}'$. We point out that the points (a) and (b) of Proposition 5.3 are of the utmost importance in the proof of our two key results: Lemma 5.9 and the descent-wall theorem in Section 7. Recall also that $\mathcal{S}_\Phi$ is the set of Shi admissible sign types.

Proposition 5.3 *Let $\mathcal{R}$ be a Shi region with minimal element w . Let $\alpha \in \Phi_0^+$, and let $\theta \in \{\alpha, \delta - \alpha\}$. The following statements are equivalent:*

- (1) H_θ is a descent wall of $\mathcal{R}$.
- (2) There is a Shi region $\mathcal{R}'$ such that $\Sigma(\mathcal{R}') = \Sigma(\mathcal{R}) \setminus \{\theta\}$.
- (3) The sign type $X_{\mathcal{R}}^\alpha$ is admissible, i.e., $X_{\mathcal{R}}^\alpha \in \mathcal{S}_\Phi$.

In particular, the set of descent roots of $\mathcal{R}$ is

$$\begin{aligned} \Sigma D(\mathcal{R}) = \{ \alpha \in \Phi_0^+ \mid X(\mathcal{R}, \alpha) = -, X_{\mathcal{R}}^\alpha \in \mathcal{S}_\Phi \} \\ \sqcup \{ \delta - \alpha \mid \alpha \in \Phi_0^+, X(\mathcal{R}, \alpha) = +, X_{\mathcal{R}}^\alpha \in \mathcal{S}_\Phi \}. \end{aligned}$$

Remark 5.4 In Proposition 5.3, since $X(\mathcal{R})$ is admissible, it follows from Theorem 4.9 that the sign type $X_{\mathcal{R}}^\alpha = (X_\beta^\alpha)_{\beta \in \Phi_0^+}$ is admissible if and only if $(X_\beta^\alpha)_{\beta \in \Psi^+}$ for any irreducible root subsystem $\Psi \subseteq \Phi_0$ of rank 2 such that $\alpha \in \Psi$.

Example 5.5 (Type $\tilde{A}_2$ – continuation of Example 4.4) For each admissible sign type of a Shi region in Figure 3, we colored in red the signs that correspond to descent roots in $\Sigma(\mathcal{R})$. For instance, for the Shi region $\mathcal{R}$ in Figure 3, we have $\Sigma D(\mathcal{R}) = \{\alpha_1, \delta - (\alpha_1 + \alpha_2)\}$, thanks to Proposition 5.3.

Example 5.6 (Type $\tilde{B}_2$ – continuation of Example 4.14) For each admissible sign type of a Shi region in Figure 4, we colored in red the signs that correspond to descent roots in $\Sigma(\mathcal{R})$. For instance, for the Shi regions $\mathcal{R}_1$ and $\mathcal{R}_2$ in Figure 4, we have, thanks to Proposition 5.3,

$$\Sigma D(\mathcal{R}_1) = \{\alpha_2, \delta - (\alpha_1 + \alpha_2)\} \text{ and } \Sigma D(\mathcal{R}_2) = \{\alpha_1 + \alpha_2, \delta - (2\alpha_1 + \alpha_2)\}.$$

Proposition 5.7 *Let $w \in W$, with Shi region $\mathcal{R}$, then $w \in L_{\text{Shi}}$ if and only if $ND_R(w) \subseteq \Sigma D(\mathcal{R})$. In particular, the set of descent walls of a Shi region contains those of its minimal element.*

Proof Assume first that $w \in L_{\text{Shi}}$. Let $\alpha \in ND_R(w)$. We show that $\alpha \in \Sigma D(\mathcal{R})$. Since $\alpha \in ND_R(w)$, there is by definition $s \in D_R(w)$ such that $\alpha = -w(\alpha_s)$. Moreover, we have $N(ws) = N(w) \setminus \{\alpha\}$ (see Section 3.4). Let $\mathcal{R}'$ be the Shi region containing ws . By Proposition 4.2(2), we know that $\Sigma(\mathcal{R}) = \Sigma(w)$ and $\Sigma(\mathcal{R}') = \Sigma(ws)$. Then it follows that

$$\begin{aligned} \Sigma(\mathcal{R}) \setminus \{\alpha\} &= \Sigma(w) \setminus \{\alpha\} = (N(w) \cap \Sigma) \setminus \{\alpha\} = (N(w) \setminus \{\alpha\}) \cap \Sigma \\ &= N(ws) \cap \Sigma = \Sigma(ws) = \Sigma(\mathcal{R}'). \end{aligned}$$

Since w is minimal in $\mathcal{R}$ and since $s \in D_R(w)$, it follows by equation (6) that $\mathcal{R}' \neq \mathcal{R}$. So $\alpha \in \Sigma$ and, therefore, α is a descent root of $\mathcal{R}$ by Proposition 5.3. Thus, we have $ND_R(w) \subseteq \Sigma D(\mathcal{R})$. The other direction of the statement follows immediately from equation (6) and the definitions of descent roots. ■

5.2 On the descent roots $\delta - \alpha_s$ for $s \in S_0$

We end this section with a result (Lemma 5.9), which we need in order to prove Theorem 1.1. We start by discussing the case of rank 2 affine Weyl groups.

Example 5.8 (On some descent roots in rank 2) Let Φ_0 be an irreducible finite crystallographic root system of rank 2, that is, Φ_0 is of type A_2 , B_2 , or G_2 . Set $\Delta_0 = \{\alpha_1, \alpha_2\}$. We are particularly interested in the descent roots corresponding to $\delta - \alpha_i$, for $i = 1, 2$. Let $\mathcal{R}$ be a Shi region of W . We know by Proposition 4.12 that $\delta - \alpha_i \in \Sigma(\mathcal{R})$ if and only if $X(\mathcal{R}, \alpha_i) = +$. By Proposition 5.3, we know that $\delta - \alpha_i \in \Sigma D(\mathcal{R})$ if and only if $X(\mathcal{R}, \alpha_i) = +$ and the sign type X^{α_i} , obtained from $X(\mathcal{R})$ by replacing $X(\mathcal{R}, \alpha_i) = +$ by 0, is admissible.

(Type A_2): for the Shi region $\mathcal{R}$ in Figure 3, $\delta - \alpha_2 \notin \Sigma D(\mathcal{R})$ since the sign type $X^{\alpha_2} = (X_{\alpha_1}^{\alpha_2}, X_{\alpha_1+\alpha_2}^{\alpha_2}, X_{\alpha_2}^{\alpha_2}) = (-, +, 0)$ obtained from $X(\mathcal{R})$ by replacing $X(\mathcal{R}, \alpha_2) = +$ by 0, is not admissible; recall that all admissible sign types for type A_2 are given in Figure 3.

More generally, let $\mathcal{R}$ be any Shi region in Figure 3 with $X(\mathcal{R}, \alpha_2) = +$, i.e., $\delta - \alpha_2 \in \Sigma(\mathcal{R})$ by Proposition 4.12. It is easy to check – case by case – that

$$\begin{aligned}\delta - \alpha_2 \in \Sigma D(\mathcal{R}) &\iff (X(\mathcal{R}, \alpha_1 + \alpha_2) = + \implies X(\mathcal{R}, \alpha_1) \in \{0, +\}), \\ \delta - \alpha_1 \in \Sigma D(\mathcal{R}) &\iff (X(\mathcal{R}, \alpha_1 + \alpha_2) = + \implies X(\mathcal{R}, \alpha_2) \in \{0, +\}).\end{aligned}$$

(Type B_2): for the Shi region $\mathcal{R}_1$ in Figure 4, $\delta - \alpha_2 \notin \Sigma D(\mathcal{R}_1)$ since the sign type $X^{\alpha_2} = (X_{\alpha_1}^{\alpha_2}, X_{2\alpha_1+\alpha_2}^{\alpha_2}, X_{\alpha_1+\alpha_2}^{\alpha_2}, X_{\alpha_2}^{\alpha_2}) = (0, +, +, -)$ obtained from $X(\mathcal{R}_1)$ by replacing $X(\mathcal{R}_1, \alpha_1) = +$ by 0, is not admissible; recall that all admissible sign types for type B_2 are given in Figure 4. Observe here, for example, that $X(\mathcal{R}_1, \alpha_1 + \alpha_2) = +$, but $X(\mathcal{R}_1, \alpha_2) \notin \{0, +\}$.

Now, for the Shi region $\mathcal{R}_2$ in Figure 4, $\delta - \alpha_2 \notin \Sigma D(\mathcal{R}_2)$ since the sign type

$$X^{\alpha_2} = (X_{\alpha_1}^{\alpha_2}, X_{2\alpha_1+\alpha_2}^{\alpha_2}, X_{\alpha_1+\alpha_2}^{\alpha_2}, X_{\alpha_2}^{\alpha_2}) = (0, +, -, -)$$

obtained from $X(\mathcal{R}_2)$ by replacing $X(\mathcal{R}_2, \alpha_1) = +$ by 0 is not admissible. Observe here that $X(\mathcal{R}_2, \alpha_2 + \alpha_1) = +$, but $X(\mathcal{R}_2, \alpha_2) \notin \{0, +\}$.

More generally, let $\mathcal{R}$ be any Shi region in Figure 4 with $X(\mathcal{R}, \alpha_2) = +$, i.e., $\delta - \alpha_2 \in \Sigma(\mathcal{R})$. It is easy to check case by case that $\delta - \alpha_1 \in \Sigma D(\mathcal{R})$ if and only if for any $\beta \in \Phi_0^+$ such that $\alpha_1 + \beta \in \Phi_0^+$ and $X(\mathcal{R}, \beta) = +$ we must have $X(\mathcal{R}, \beta) \in \{0, +\}$. The same remains valid for $\delta - \alpha_2$. We have, therefore, in this case the following condition: $\delta - \alpha_i \in \Sigma D(\mathcal{R})$ if and only if

$$(\beta \in \Phi_0^+ \text{ with } \alpha_i + \beta \in \Phi_0^+ \text{ and } X(\mathcal{R}, \alpha_i + \beta) = + \implies X(\mathcal{R}, \beta) \in \{0, +\}.$$

(Type G_2): The above condition remains true in this type for any Shi region $\mathcal{R}$ (Figure 7); we leave the reader check this assertion on Figure 8. For instance, for the region $\mathcal{R}$ indicated in Figure 8, $\delta - \alpha_1 \notin \Sigma D(\mathcal{R})$. Indeed, the sign type

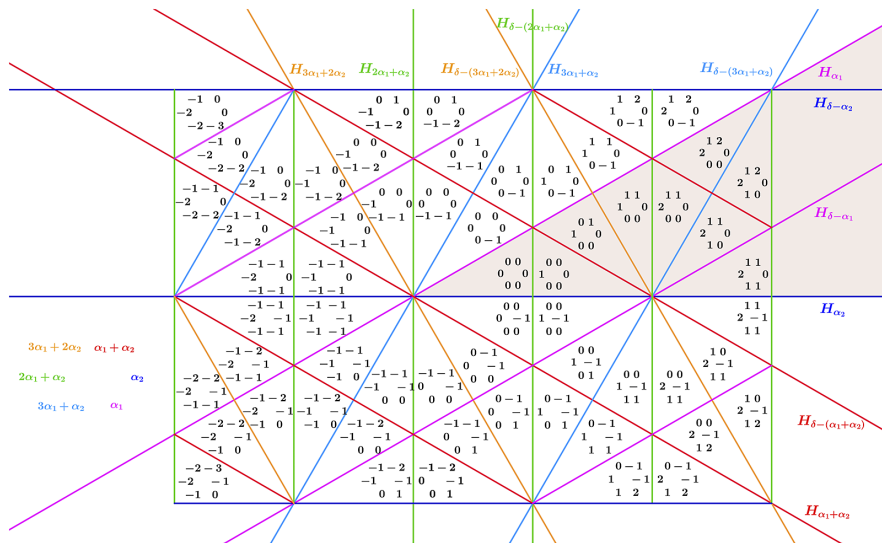


Figure 7: The Coxeter arrangement of type $\tilde{G}_2$. Each alcove is labeled with its Shi parameterization. The labels $k(w, \alpha)$ for $\alpha \in \Phi_0^+$ are indicated in each alcove with the parameterization of the finite root system given at the top of the figure. The long roots are α_2 , $3\alpha_1 + 2\alpha_2 = s_1(\alpha_2)$, and $3\alpha_1 + \alpha_2 = s_2s_1(\alpha_2)$. The shaded region is the dominant region of $\mathcal{A}(W, S)$, which is also the fundamental chamber for the finite Weyl group W_0 .

$X^{\alpha_1} = (X_{\alpha_1}^{\alpha_1}, X_{3\alpha_1+\alpha_2}^{\alpha_1}, X_{2\alpha_1+\alpha_2}^{\alpha_1}, X_{\alpha_1+\alpha_2}^{\alpha_1}, X_{\alpha_2}^{\alpha_2}) = (0, +, +, +, -)$ obtained from $X(\mathcal{R})$ by replacing $X(\mathcal{R}, \alpha_1) = +$ by 0, is not admissible. Observe here that $X(\mathcal{R}, 2\alpha_1 + \alpha_2) = +$ and $2\alpha_1 + \alpha_2 = \alpha_1 + (\alpha_1 + \alpha_2) \in \Phi_0^+$, but $X(\mathcal{R}_1, \alpha_1 + \alpha_2) = - \notin \{0, +\}$.

Lemma 5.9 *Let $\mathcal{R}$ be a Shi region and $s \in S_0$ such that $\delta - \alpha_s \in \Sigma(\mathcal{R})$, i.e., $X(\mathcal{R}, \alpha_s) = +$. Then $H_{\delta-\alpha_s}$ is a descent wall of $\mathcal{R}$ if and only if for any irreducible root subsystem Ψ of rank 2 in Φ_0 such that $\alpha_s \in \Psi$ the following condition is verified by $\mathcal{R}$:*

(*) *If $\beta \in \Psi$ with $\alpha_s + \beta \in \Psi$ and $X(\mathcal{R}, \alpha_s + \beta) = +$, then $X(\mathcal{R}, \beta) \in \{0, +\}$.*

Proof First, the case of Φ_0 of rank 2 is done in Example 5.8. By Proposition 5.3, we know that $H_{\delta-\alpha_s}$ is a descent wall of $\mathcal{R}$ if and only if X^{α_s} is admissible. By Theorem 4.9, we know that X^{α_s} is admissible if and only if for any irreducible root subsystem Ψ of rank 2 in Φ_0 , the restriction $(X_y^{\alpha_s})_{y \in \Psi^+}$ of X^{α_s} to Ψ^+ is admissible. Since $X(\mathcal{R})$ is admissible and that the only difference between $X(\mathcal{R})$ and X^{α_s} is that $X(\mathcal{R}, \alpha_s) \neq 0$ and $X_{\alpha_s}^{\alpha_s} = 0$, one has to verify the admissibility of $(X_y^{\alpha_s})_{y \in \Psi^+}$ only for irreducible root subsystems Ψ of rank 2 in Φ_0 such that $\alpha_s \in \Psi$; it is well known that α_s is then also a simple root for Ψ .

But we know that the lemma is true for Ψ of rank 2, so X^{α_s} is admissible if and only if $(X(\mathcal{R}, \gamma))_{\gamma \in \Psi^+}$ satisfies Condition (*) for any irreducible root subsystem Ψ rank 2 in Φ_0 such that $\alpha_s \in \Psi$. ■

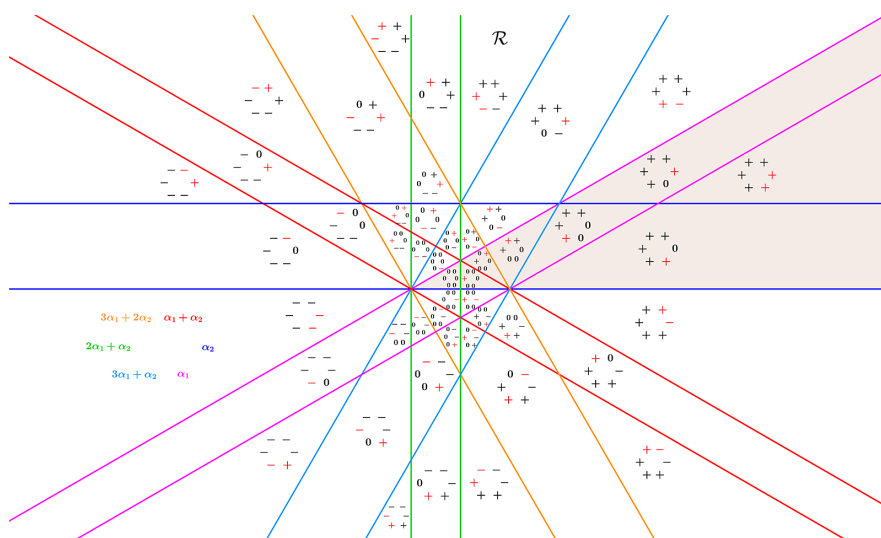


Figure 8: The Shi arrangement of type $\tilde{G}_2$. Each Shi region is labeled with its admissible sign type. The labels $X(\mathcal{R}, \alpha)$ for a Shi region $\mathcal{R}$ are indicated in each alcove with the parameterization given at the left-hand side of the figure. The shaded region is the dominant region C_0 of $\mathcal{A}(W, S)$. The signs colored in red indicate the descent roots of the corresponding Shi region.

6 Dominant Shi regions: the initial case for the proof of Theorem 1.1

Let (W, S) be an irreducible affine Coxeter system with underlying Weyl group W_0 . In Section 4.3, we explain that it is enough to show that $L_{\text{Shi}} \subseteq L$ in order to prove Theorem 1.1. The proof is by induction with initial case the case of Shi regions contained in the dominant region. In this section, we explain that any dominant Shi region contains a low element, which follows mainly by the works of Cellini and Papi [4–6] on ad-nilpotent ideals and Shi’s article on $\oplus$ sign-types [26].

6.1 Dominant Shi region

Recall from Section 2.1 that the dominant region of the affine Coxeter arrangement $\mathcal{A}(W, S)$ is denoted by C_0 . A Shi region $\mathcal{R}$ is called *dominant* if $\mathcal{R} \subseteq C_0$. An obvious remark is the fact that a Shi region $\mathcal{R}$ is dominant if and only if $X(\mathcal{R}, \alpha) \in \{+, 0\}$ for all $\alpha \in \Phi_0^+$.

The Weyl group W_0 is a standard parabolic subgroup of W , since W_0 is generated by $S_0 \subseteq S$. The set of minimal length coset representatives is

$${}^0W := \{v \in W \mid \ell(sv) > \ell(v), \forall s \in S_0\}.$$

The following well-known proposition states that an element is a minimal coset representative for $W_0 \backslash W$ if and only if its corresponding alcove is in the dominant region.

Proposition 6.1 *Let $w \in W$. We have $w \in {}^0W$ if and only if $w \cdot A_o \subseteq C_o$.*

The inductive step of the proof of Theorem 1.1, which is dealt with in Section 7, is based on the decomposition in minimal length coset representative. Let $w \in L_{\text{Shi}}$, then there is $u \in W_0$ and $v \in {}^0W$ such that $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$. Since L_{Shi} is closed under taking suffixes (Proposition 4.5), then $v \in L_{\text{Shi}}$. Therefore, if $\ell(u) = 0$, i.e., $u = e$, then $w = v \cdot A_o \subseteq C_o$, which is the initial case of our proof by induction.

6.2 Minimal elements in dominant Shi regions

Recall that $\Psi \subseteq \Phi_0^+$ is an ideal of the root poset $(\Phi_0^+, \leq)$ if for all $\alpha \in \Psi$ and $\gamma \in \Phi_0^+$ such that $\alpha \leq \gamma$ we have $\gamma \in \Psi$. By definition of the root poset, this is equivalent to the condition: $\Psi \subseteq \Phi_0^+$ is an ideal of $(\Phi_0^+, \leq)$ if for all $\alpha \in \Psi$ and $\beta \in \Phi_0^+$ such that $\alpha + \beta \in \Phi_0^+$, then $\alpha + \beta \in \Psi$.

Denote by $\mathcal{I}(\Phi_0)$ the set of ideals of $(\Phi_0^+, \leq)$ and by L_{Shi}^0 the set of minimal elements in dominant Shi regions. In [26, Theorem 1.4], Shi gave a bijection between dominant Shi regions and $\mathcal{I}(\Phi_0)$. Hence, the number of Shi regions is equal to the number of ideals of the roots poset. We recall here this bijection following the lines of [5, Section 4].

It is a general fact that ideals and antichains in $(\Phi_0^+, \leq)$ are in natural bijection, which is given by mapping an ideal Ψ to the set of its minimal elements $\Psi_{\min}$. Let $\Psi \in \mathcal{I}(\Phi_0)$, then the corresponding Shi region is

$$\mathcal{R}_\Psi = \{x \in C_o \mid \langle \beta, x \rangle > 1 \text{ if } \alpha \leq \beta \text{ for some } \alpha \in \Psi_{\min}; \langle \beta, x \rangle < 1 \text{ otherwise} \}.$$

In terms of admissible sign types, the bijection is

$$\Psi \in \mathcal{I}(\Phi_0) \longmapsto X(\mathcal{R}_\Psi, \beta) = \begin{cases} +, & \text{if } \beta \in \Psi, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, by Proposition 4.12, we have $\Sigma(\mathcal{R}_\Psi) = \{\delta - \beta \mid \beta \in \Psi\}$. Since $\Psi_{\min} \setminus \{\alpha\}$ is again an antichain for any $\alpha \in \Psi_{\min}$, it corresponds to an ideal Ψ' with $\Psi'_{\min} = \Psi_{\min} \setminus \{\alpha\}$. We deduce (Proposition 6.2) that the descent walls of a dominant Shi region correspond, therefore, to the roots in its corresponding antichain.

Proposition 6.2 *Let $\Psi \in \mathcal{I}(\Phi_0)$, then $\Sigma(\mathcal{R}_\Psi) = \{\delta - \beta \mid \beta \in \Psi\}$ and*

$$\Sigma D(\mathcal{R}_\Psi) = \{\delta - \alpha \mid \alpha \in \Psi_{\inf}\}.$$

We now describe the element $w_\Psi \in L_{\text{Shi}}^0$ with Shi region $\mathcal{R}_\Psi$.

Theorem 6.3 (Cellini–Papi) *Let $\Psi \subseteq \Phi_0^+$ be an ideal of the root poset $(\Phi_0^+, \leq)$. There is a unique element $w_\Psi \in W$ such that $N(w_\Psi) = \text{cone}_\Phi(\{\delta - \alpha \mid \alpha \in \Psi\})$. Moreover, $ND_R(w_\Psi) = \{\delta - \alpha \mid \alpha \in \Psi_{\inf}\}$.*

Proof As this result is not stated *Per Se* in [4, 6], we give, here, a brief proof that points out the relevant results in these articles. In [4, Section 2], the authors use the following notations: for $k \in \mathbb{N}^*$, write

$$\Psi^k := \{\alpha_1 + \cdots + \alpha_k \in \Phi_0^+ \mid \alpha_i \in \Psi\} \text{ and } \mathbb{L}_\Psi := \bigcup_{k \in \mathbb{N}^*} (k\delta - \Psi^k).$$

Observe that $\mathbb{L}_\Psi = \text{cone}_\Phi(\{\delta - \alpha \mid \alpha \in \Psi\})$. Now, by [4, Theorem 2.6], there is a unique $w_\Psi \in W$ such that $\mathbb{L}_\Psi = N(w_\Psi)$, that is, $N(w_\Psi) = \text{cone}_\Phi(\{\delta - \alpha \mid \alpha \in \Psi\})$. The statement $ND_R(w_\Psi) \supseteq \{\delta - \alpha \mid \alpha \in \Psi_{\text{inf}}\}$ is precisely [6, Proposition 3.4], while the reverse inclusion is precisely [4, Proposition 2.12(ii)]. ■

Cellini and Papi showed in [5] that $|L_{\text{Shi}}^0| = |\mathcal{J}(\Phi_0)| = \text{Cat}(\Phi_0)$. We obtain, therefore, the following corollary.

Corollary 6.4 *Let $\Psi \in \mathcal{J}(\Phi_0)$.*

- (1) *The element w_Ψ in Theorem 6.3 is a low element.*
- (2) *The Shi region containing $w_\Psi \cdot A_\circ$ is $\mathcal{R}_\Psi$ and is therefore dominant.*
- (3) *We have $ND_R(w_\Psi) = \Sigma D(\mathcal{R}_\Psi) = \{\delta - \alpha \mid \alpha \in \Psi_{\text{inf}}\}$.*
- (4) *We have $|L^0| = |L_{\text{Shi}}^0| = |\mathcal{J}(\Phi_0)| = \text{Cat}(\Phi_0)$.*
- (5) *In particular, $L^0 = L_{\text{Shi}}^0 = \{w_\Psi \mid \Psi \in \mathcal{J}(\Phi_0)\}$.*

Proof (i) We have $w_\Psi \in L$ by definition of low elements since $\delta - \alpha \in \Sigma$ for all $\alpha \in \Psi \subseteq \Phi_0^+$.

(ii) By Theorem 6.3, $\Sigma(w_\Psi) = N(w_\Psi) \cap \Sigma = \{\delta - \beta \mid \beta \in \Psi\}$. Therefore, $\Sigma(\mathcal{R}_\Psi) = \Sigma(w_\Psi)$ by Proposition 6.2. Hence, $w_\Psi \cdot A_\circ \subseteq \mathcal{R}_\Psi$ by Proposition 4.2(1). Now, by Proposition 6.1, $w_\Psi \cdot A_\circ$ is dominant, which implies that $\mathcal{R}_\Psi$ is also dominant.

(iii) follows immediately from Proposition 6.2 and Theorem 6.3. Let us prove (iv): From (i) and Theorem 6.3(1), there is an injective map from $\mathcal{J}(\Phi_0)$ into L^0 . By Proposition 4.4, we have $|\mathcal{J}(\Phi_0)| \leq |L^0| \leq |L_{\text{Shi}}^0|$. We conclude by Theorem 6.3(2) and [5]. Then, (v) is a consequence of (i) and (iv). ■

7 The descent-wall theorem and the inductive case of the proof of Theorem 1.1

In order to handle the inductive case of the proof of Theorem 1.1, we need to prove first that the descent walls of an element in L_{Shi} are precisely the descent walls of its corresponding Shi region, which takes the most part of this section. Then we conclude the proof of Theorem 1.1 in Section 7.4.

7.1 The descent-wall theorem

Theorem 7.1 *Let $w \in L_{\text{Shi}}$ with Shi region $\mathcal{R}$, then $ND_R(w) = \Sigma D(\mathcal{R})$.*

The proof of the so-called *descent-wall theorem* above is based on Proposition 3.7 and on its analog for descent roots of Shi regions.

Proposition 7.2 *Let $w \in L_{\text{Shi}}$ with Shi region $\mathcal{R}$. Let $s \in D_L(w) \cap S_0$, and let $\mathcal{R}_1$ be the Shi region associated with $sw \in L_{\text{Shi}}$. Then $\Sigma D(\mathcal{R}_1) = s(\Sigma D(\mathcal{R}) \setminus \{\alpha_s\})$.*

Before proving Proposition 7.2, we prove the descent-wall theorem. The proof of Proposition 7.2 is given in Section 7.3.

Proof of Theorem 7.1 We decompose $w = uv$ with $u \in W_0$ and $v \in {}^0W$. We show by induction on $\ell(u)$ that $ND_R(w) = \Sigma D(\mathcal{R})$. Notice that $ND_R(w) \subseteq \Sigma D(\mathcal{R})$ by Proposition 5.7.

If $\ell(u) = 0$, then $\mathcal{R}$ is a dominant Shi region. Therefore, $ND_R(w) = \Sigma D(\mathcal{R})$ by Proposition 6.4(iii). Assume now that $\ell(u) > 0$. Then there is $s \in D_L(u) \subseteq S_0$. Therefore, $\alpha_s \in \Sigma(\mathcal{R}) = \Sigma(w)$, but $\delta - \alpha_s \notin \Sigma(\mathcal{R}) = \Sigma(w)$. Set $w' = (su)v$ and $\mathcal{R}'$ its Shi region. By Proposition 4.5, we know that $w' \in L_{\text{Shi}}$ and $\ell(w') = \ell(w) - 1$. By induction, we have $ND_R(w') = \Sigma D(\mathcal{R}')$. By Theorem 7.2 and Proposition 3.7, we obtain that $\Sigma D(\mathcal{R}) = \{\alpha_s\} \sqcup s(\Sigma D(\mathcal{R}')) = \{\alpha_s\} \sqcup s(ND_R(w')) = ND_R(w)$. ■

7.2 Suffix decomposition of some Shi regions

By Proposition 4.2, we know that, for a Shi region $\mathcal{R}$ such that $\alpha_s \in \Sigma(\mathcal{R})$, there is a Shi region $\mathcal{R}_1$ such that $s \cdot \mathcal{R}_1 \subseteq \mathcal{R}$; or equivalently $\mathcal{R}_1 \subseteq s \cdot \mathcal{R}$. In order to prove Proposition 7.2, we need first to describe precisely, in terms of sign types, the image $s \cdot \mathcal{R}$ of Shi region $\mathcal{R}$ with $s \in S_0$.

Proposition 7.3 *Let $w \in L_{\text{Shi}}$ with Shi region $\mathcal{R}$, that is, $w \in \widetilde{\mathcal{R}}$. Let $s \in S_0$ such that $X(\mathcal{R}, \alpha_s) = -$; so $\alpha_s \in \Sigma(\mathcal{R})$ and $s \in D_L(w)$. Denote $\mathcal{R}_1$ the Shi region of $sw \in L_{\text{Shi}}$, that is, $sw \in \widetilde{\mathcal{R}}_1$. Consider the two following subsets of W :*

$$\widetilde{\mathcal{R}}_0^s = \{x \in s\widetilde{\mathcal{R}} \mid k(x, \alpha_s) = 0\} \text{ and } \widetilde{\mathcal{R}}_+^s = \{x \in s\widetilde{\mathcal{R}} \mid k(x, \alpha_s) > 0\}.$$

- (1) We have $s\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}_0^s \sqcup \widetilde{\mathcal{R}}_+^s$.
- (2) If $k(w, \alpha_s) = -1$, then $\widetilde{\mathcal{R}}_0^s \neq \emptyset$ is an equivalence class for $\sim_{\text{Shi}}$ that corresponds to the Shi region $\mathcal{R}_0^s = \mathcal{R}_1$. Moreover, $X(\mathcal{R}_1, \alpha_s) = 0$ and $\widetilde{\mathcal{R}}_+^s$ corresponds to a Shi region $\mathcal{R}_2$ if it is nonempty.
- (3) If $k(w, \alpha_s) < -1$, then $\widetilde{\mathcal{R}}_0^s = \emptyset$ and $\widetilde{\mathcal{R}}_+^s \neq \emptyset$ is an equivalence class for $\sim_{\text{Shi}}$ that corresponds to the Shi region $\mathcal{R}_+^s = \mathcal{R}_1$. Moreover, $\mathcal{R}_1 = s \cdot \mathcal{R}$ and $X(\mathcal{R}_1, \alpha_s) = +$.
- (4) The signs of the elements of $s\widetilde{\mathcal{R}}$ can only differ over the position α_s . In particular, $X(\mathcal{R}_1, \gamma) = X(\mathcal{R}, s(\gamma))$ for all $\gamma \in \Phi_0^+ \setminus \{\alpha_s\}$.

See Table 1 or Table 2 for examples in rank 2.

Proof First, observe that $\alpha_s \in \Sigma(w) = \Sigma(\mathcal{R})$ by Propositions 4.12 and 4.2. In particular, $s \in D_L(w)$ and $sw \in L_{\text{Shi}}$ by Proposition 4.5.

(1). Let $u \in \widetilde{\mathcal{R}}$, then by Lemma 2.2 and Proposition 2.3, we obtain

$$(7.1) \quad k(su, \alpha_s) = k(u, s(\alpha_s)) + k(s, \alpha_s) = k(u, -\alpha_s) - 1 = -(k(u, \alpha_s) + 1).$$

Moreover, since $X(\mathcal{R}, \alpha_s) = -$, we have $k(u, \alpha_s) \leq -1$. If $k(u, \alpha_s) = -1$, then $k(su, \alpha_s) = 0$. Now, if $k(u, \alpha_s) < -1$, then $k(su, \alpha_s) > 0$. Therefore, $su \in \widetilde{\mathcal{R}}_0^s \sqcup \widetilde{\mathcal{R}}_+^s$, which implies that $s\widetilde{\mathcal{R}} \subseteq \widetilde{\mathcal{R}}_0^s \sqcup \widetilde{\mathcal{R}}_+^s$. The converse inclusion is obvious.

(4). Let $\gamma \in \Phi_0^+ \setminus \{\alpha_s\}$ and $u \in \widetilde{\mathcal{R}}$. By Proposition 2.3, we have $k(su, \gamma) = k(u, s(\gamma)) + k(s, \gamma)$. Since $\gamma \in \Phi_0^+ \setminus \{\alpha_s\}$, we have $s(\gamma) \in \Phi_0^+ \setminus \{\alpha_s\}$. Therefore, by Lemma 2.2, we have $k(s, \gamma) = 0$. It follows that $k(su, \gamma) = k(u, s(\gamma))$. Therefore, by equation (7), we obtain $X(su, \gamma) = X(u, s(\gamma)) = X(\mathcal{R}, s(\gamma))$, that is, the signs of the elements in $s\widetilde{\mathcal{R}}$ can only differ over α_s . In particular, for $w \in L_{\text{Shi}}$ and its suffix $sw \in L_{\text{Shi}}$, we obtain

$$X(\mathcal{R}_1, \gamma) = X(sw, \gamma) = X(w, s(\gamma)) = X(\mathcal{R}, s(\gamma)),$$

for all $\gamma \in \Phi_0^+ \setminus \{\alpha_s\}$.

s_i	$X(\mathcal{R})$	$X(\mathcal{R}_1)$	$X(\mathcal{R}_2)$	$\beta \in \Phi_0^+ \setminus \{\alpha_i\}$ with $s_i(\beta) \in \Phi_0^+ \setminus \{\alpha_i\}$ with $X(\mathcal{R}_1, \beta)$ red	$X(\mathcal{R}, s_i(\beta))$ red
s_1	+	+	+		
	- +	0 +	+ +	α_2	$\alpha_1 + \alpha_2$
	0	+	+	$\alpha_1 + \alpha_2$	α_2
	- +	0 0	+ 0		
	-	0	0	α_2	$\alpha_1 + \alpha_2$
	- 0	0 -	+ -		
	-	-	-	$\alpha_1 + \alpha_2$	α_2
	- -	0 -	+ -		
	+	+	+	α_1	$\alpha_1 + \alpha_2$
	+ -	+ 0	+ +	$\alpha_1 + \alpha_2$	α_1
s_2	0	+	+	α_1	$\alpha_1 + \alpha_2$
	+ -	0 0	0 +	$\alpha_1 + \alpha_2$	α_1
	-	0	0	α_1	$\alpha_1 + \alpha_2$
	0 -	- 0	- +	$\alpha_1 + \alpha_2$	α_1
	-	-	-		
	- -	- 0	- +		

Table 1: Type A_2 .

(2) and (3). By the same arguments as in the proof of (1) applied to $w \in \widetilde{\mathcal{R}}$, we obtain two cases: $k(w, \alpha_s) = -1$ and $k(w, \alpha_s) < -1$. The details are similar to the cases (1) and (4) above and are left to the reader. ■

7.3 Proof of Proposition 7.2

Let $w \in L_{\text{Shi}}$ with Shi region $\mathcal{R}$. Let $s \in D_L(w)$, and let $\mathcal{R}_1$ be the Shi region associated with $sw \in L_{\text{Shi}}$. Notice that $X(\mathcal{R}, \alpha_s) = -$.

Thanks to Proposition 7.3, we have two cases to consider: either $s \cdot \mathcal{R}_1 = \mathcal{R}$ or there is another Shi region $\mathcal{R}_2$ distinct from $\mathcal{R}_1$ such that $s \cdot \mathcal{R}_2 \subseteq \mathcal{R}$.

The case where $s \cdot \mathcal{R}_1 = \mathcal{R}$ is settled by the following lemma.

Lemma 7.4 *Let $\mathcal{R}$ be a Shi region and $s \in S_0$ such that $X(\mathcal{R}, \alpha_s) = -$. If a Shi region $\mathcal{R}_1$ is such that $s \cdot \mathcal{R}_1 = \mathcal{R}$, then $\Sigma D(\mathcal{R}_1) = s(\Sigma D(\mathcal{R}) \setminus \{\alpha_s\})$.*

s_i	$X(\mathcal{R})$		$X(\mathcal{R}_1)$		$X(\mathcal{R}_2)$		$\beta \in \Phi_0^+ \setminus \{\alpha_i\}$ with $X(\mathcal{R}_1, \beta)$ red	$s_i(\beta) \in \Phi_0^+ \setminus \{\alpha_i\}$ with $X(\mathcal{R}, s_i(\beta))$ red
s_1		−		0		+		
	0	+	+	0	+	0	$\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2$
		+		+		+		
		−		0		+		
	+	+	+	+	+	+	α_2	$2\alpha_1 + \alpha_2$
		+		+		+		
		−		0		+		
	−	+	+	−	+	−	$\alpha_2, 2\alpha_1 + \alpha_2$	$2\alpha_1 + \alpha_2, \alpha_2$
		0		0		0		
		−		0		+		
	−	0	0	−	0	−	$\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2$
		−		−		−		
		−		0		+		
	−	−	−	−	−	−	$2\alpha_1 + \alpha_2$	α_2
		−		−		−		
	+		+	+		+		
s_2	+	−	+	0	+	+	α_1	$\alpha_1 + \alpha_2$
		+		+		+		
		+		0		0		
	+	−	+	0	+	+	$\alpha_1 + \alpha_2$	α_1
		0		+		+		
		0		−		−		
	0	−	0	0	0	+	α_1	$\alpha_1 + \alpha_2$
		−		0		0		
		0		−		−		
	−	−	−	0	−	+	$2\alpha_1 + \alpha_2$	$2\alpha_1 + \alpha_2$
		−		0		0		
		−		−		−		
	−	−	−	0	−	+	$\alpha_1 + \alpha_2$	α_1
		−		−		−		

Table 2: Type B_2 .

Proof Since $s \cdot \mathcal{R}_1 = \mathcal{R}$, the walls of $\mathcal{R}_1$ are the image of the walls of $\mathcal{R}$ under s . So the corollary follows from the following observations: (1) a wall H separates $\mathcal{R}_1$ from A_o if and only if $s \cdot H$ separates $s \cdot \mathcal{R}_1 = \mathcal{R}$ from $s \cdot A_o$; (2) the only wall separating $s \cdot A_o$ from A_o is H_{α_s} and so A_o and $s \cdot A_o$ are on the same side of any other hyperplane in the Coxeter arrangement. ■

Consider now the case for which there is another Shi region $\mathcal{R}_2$ distinct from $\mathcal{R}_1$ such that $s \cdot \mathcal{R}_2 \subseteq \mathcal{R}$. By Proposition 7.3(1) and (4), we have

$$X(\mathcal{R}_1, \alpha_s) = 0 \text{ and } X(\mathcal{R}_1, \gamma) = X(\mathcal{R}, s(\gamma)) \text{ for all } \gamma \in \Phi_0^+ \setminus \{\alpha_s\}.$$

In particular, $\alpha_s \in \Sigma D(\mathcal{R})$ but $\alpha_s \notin \Sigma D(\mathcal{R}_1)$.

For $\alpha \in \Phi_0^+ \setminus \{\alpha_s\}$, we consider the following sign types:

- $X^{s(\alpha)} = (X_\gamma^{s(\alpha)})_{\gamma \in \Phi_0^+}$ is defined by $X_{s(\alpha)}^{s(\alpha)} = 0$ and $X_\gamma^{s(\alpha)} = X(\mathcal{R}, \gamma)$ for $\gamma \in \Phi_0^+ \setminus \{s(\alpha)\}$.
- $Y^\alpha = (Y_\gamma^\alpha)_{\gamma \in \Phi_0^+}$ is defined by $Y_\alpha^\alpha = 0$ and $Y_\gamma^\alpha = X(\mathcal{R}_1, \gamma)$ for $\gamma \in \Phi_0^+ \setminus \{\alpha\}$.

Thanks to Proposition 5.3, we know that X^α is admissible if and only if either α or $\delta - \alpha$ is a descent root of $\mathcal{R}$. Similarly, Y^α is admissible if and only if either α or $\delta - \alpha$ is a descent root of $\mathcal{R}_1$.

Lemma 7.5 Assume that, for any $\alpha \in \Phi_0^+ \setminus \{\alpha_s\}$, we have Y^α is admissible if and only if $X^{s(\alpha)}$ is admissible. Then $\Sigma D(\mathcal{R}_1) = s(\Sigma D(\mathcal{R}) \setminus \{\alpha_s\})$.

Proof Let $\alpha \in \Phi_0^+ \setminus \{\alpha_s\}$. Notice that $s(\alpha) \in \Phi_0^+ \setminus \{\alpha_s\}$.

We first show that $\Sigma D(\mathcal{R}_1) \subseteq s(\Sigma D(\mathcal{R}) \setminus \{\alpha_s\})$. Assume first that $\alpha \in \Sigma D(\mathcal{R}_1)$, i.e., $X(\mathcal{R}_1, \alpha) = -$, by Proposition 5.3. Then Y^α is admissible, so is $X^{s(\alpha)}$ by assumption. Therefore, either $s(\alpha)$ or $\delta - s(\alpha)$ is in $\Sigma D(\mathcal{R}) \setminus \{\alpha_s\}$. Since $X(\mathcal{R}, s(\alpha)) = X(\mathcal{R}_1, \alpha) = -$, we have $s(\alpha) \in \Sigma D(\mathcal{R}) \setminus \{\alpha_s\}$, by Proposition 5.3 again. Therefore, $\alpha \in s(\Sigma D(\mathcal{R}) \setminus \{\alpha_s\})$. If $\delta - \alpha \in \Sigma D(\mathcal{R}_1)$, then by similar arguments we show that $\delta - \alpha \in s(\Sigma D(\mathcal{R}) \setminus \{\alpha_s\})$. Hence, $\Sigma D(\mathcal{R}_1) \subseteq s(\Sigma D(\mathcal{R}) \setminus \{\alpha_s\})$.

Assume now that $\alpha \in s(\Sigma D(\mathcal{R}) \setminus \{\alpha_s\})$. Therefore, $s(\alpha) \in \Sigma D(\mathcal{R}) \setminus \{\alpha_s\}$ and $- = X(\mathcal{R}, s(\alpha)) = X(\mathcal{R}_1, \alpha)$. Hence, $X^{s(\alpha)}$ is admissible, so is Y^α by assumption. Then $\alpha \in \Sigma D(\mathcal{R}_1)$. The case of $\delta - \alpha \in s(\Sigma D(\mathcal{R}) \setminus \{\alpha_s\})$ is similar. Finally, we get $\Sigma D(\mathcal{R}_1) = s(\Sigma D(\mathcal{R}) \setminus \{\alpha_s\})$. ■

Thanks to Lemma 7.5, we only need to show now that, for any $\alpha \in \Phi_0^+ \setminus \{\alpha_s\}$, we have Y^α is admissible if and only if $X^{s(\alpha)}$ is admissible.

Lemma 7.6 Let $\alpha \in \Phi_0^+ \setminus \{\alpha_s\}$. The following statements are equivalent.

- (1) Y^α is admissible if and only if $X^{s(\alpha)}$ is admissible.
- (2) For any rank 2 irreducible root subsystem $\Psi \subseteq \Phi_0$ such that $\alpha, \alpha_s \in \Psi$, we have $(Y_\gamma^\alpha)_{\gamma \in \Psi^+}$ is admissible if and only if $(X_\gamma^{s(\alpha)})_{\gamma \in \Psi^+}$ is admissible.

Proof (i) implies (ii) follows directly from Theorem 4.9. Assume (ii) to be true for all $\alpha \in \Phi_0^+$. We describe first four cases that are valid for any rank 2 irreducible rank 2 subsystem Ψ and any $\alpha \in \Phi_0^+ \setminus \{\alpha_s\}$. Let Ψ be a rank 2 irreducible root subsystem $\Psi \subseteq \Phi_0$ and $\alpha \in \Phi_0^+ \setminus \{\alpha_s\}$.

Case 1. If $\alpha, \alpha_s \in \Psi^+$, then $(Y_\gamma^\alpha)_{\gamma \in \Psi^+}$ is admissible if and only if $(X_\gamma^{s(\alpha)})_{\gamma \in \Psi^+}$ is admissible by assumption.

Case 2. If $\alpha_s \in \Psi$ and $\alpha \notin \Psi$. Then $s(\Psi) = \Psi$ since Ψ a root system containing α_s . Therefore, $s(\alpha) \notin \Psi$ since $s(\alpha) = \alpha - 2\langle \alpha, \alpha_s^\vee \rangle \alpha_s$. By definition, we have for all $\gamma \in \Psi \subseteq \Phi_0^+ \setminus \{\alpha, s(\alpha)\}$:

$$Y_\gamma^\alpha = X(\mathcal{R}_1, \gamma) \text{ and } X_\gamma^{s(\alpha)} = X(\mathcal{R}, \gamma).$$

Therefore, both $(Y_\gamma^\alpha)_{\gamma \in \Psi^+}$ and $(X_\gamma^{s(\alpha)})_{\gamma \in \Psi^+}$ are admissible in this case.

Case 3. If $\alpha \in \Psi$ and $\alpha_s \notin \Psi$, then $s(\Psi^+) \subseteq \Phi_0^+ \setminus \{\alpha_s\}$ is the positive root system associated with root system $s(\Psi)$ and $s(\alpha) \in s(\Psi^+)$. By definition of Y^α , we have $Y_\alpha^\alpha = 0$, and for all $\gamma \in \Psi^+ \setminus \{\alpha\}$,

$$Y_\gamma^\alpha = X(\mathcal{R}_1, \gamma) = X(\mathcal{R}, s(\gamma)).$$

Notice that $\gamma \in \Psi^+ \setminus \{\alpha\}$ if and only if $s(\gamma) \in s(\Psi^+) \setminus \{s(\alpha)\}$. Now, by definition of $X^{s(\alpha)}$, we have $X_{s(\alpha)}^{s(\alpha)} = 0$ and for all $\nu = s(\gamma) \in s(\Psi^+) \setminus \{s(\alpha)\}$, or in other words for all $\gamma \in \Psi^+ \setminus \{\alpha\}$, we have

$$X_\nu^{s(\alpha)} = X(\mathcal{R}, \nu) = X(\mathcal{R}, s(\gamma)) = Y_\gamma^\alpha.$$

Therefore, the restriction of Y^α to Ψ is admissible if and only if the restriction of $X^{s(\alpha)}$ to $s(\Psi)$ is admissible.

Case 4. If $\alpha \notin \Psi$ and $\alpha_s \notin \Psi$, then $s(\alpha) \notin s(\Psi)$. By the same line of reasoning than in Case 3, we have, for all $\gamma \in \Psi^+ \setminus \{\alpha\}$,

$$Y_\gamma^\alpha = X(\mathcal{R}_1, \gamma) = X(\mathcal{R}, s(\gamma)).$$

Moreover, for all $\nu = s(\gamma) \in s(\Psi^+)$, or in other words for all $\gamma \in \Psi^+$, we have

$$X_\nu^{s(\alpha)} = X(\mathcal{R}, \nu) = X(\mathcal{R}, s(\gamma)) = Y_\gamma^\alpha.$$

Therefore, the restriction of Y^α to Ψ is admissible if and only if the restriction of $X^{s(\alpha)}$ to $s(\Psi)$ is admissible.

Conclusion. By Theorem 4.9, we need to prove that for any rank 2 irreducible root subsystem $\Psi \subseteq \Phi_0$, we have $(Y_\gamma^\alpha)_{\gamma \in \Psi^+}$ is admissible if and only if $(X_\gamma^{s(\alpha)})_{\gamma \in \Psi^+}$ is admissible. Let $\alpha \in \Phi_0^+ \setminus \{\alpha_s\}$.

Assume first that Y^α is admissible. By Theorem 4.9, we have to show that the restriction of $X^{s(\alpha)}$ to any rank 2 irreducible root subsystem Ψ is admissible. Let Ψ be a rank 2 irreducible root subsystem. If $\alpha_s \in \Psi$, then by Case 1 the restriction of $X^{s(\alpha)}$ to Ψ is admissible since Y^α is, and by Case 2, the restriction of $X^{s(\alpha)}$ to Ψ is always admissible. Assume now that $\alpha_s \notin \Psi$.

If $s(\alpha) \in \Psi$, then we apply Case 3 with $s(\Psi)$ and α . Therefore, the restriction of Y^α to $s(\Psi)$ is admissible if and only if the restriction of $X^{s(\alpha)}$ to $s(s(\Psi)) = \Psi$ is admissible, which settles that case since Y^α is admissible. If $s(\alpha) \notin \Psi$, then we conclude by Case 4 with $s(\Psi)$ and α . So $X^{s(\alpha)}$ is admissible.

The same line of reasoning using Cases 1–4 shows that if $X^{s(\alpha)}$ then Y^α is admissible. ■

Thanks to the above lemmas, Proposition 7.2 is a consequence of the following lemma.

Lemma 7.7 Let Φ_0 be an irreducible finite crystallographic root system of rank 2, that is, Φ_0 is of type A_2 , B_2 , or G_2 . Set $\Delta_0 = \{\alpha_1, \alpha_2\}$ and $S_0 = \{s_1, s_2\}$. Let $i \in \{1, 2\}$ and $\mathcal{R}$ be a Shi region such that $X(\mathcal{R}, \alpha_i) = -$. Let $\mathcal{R}_1, \mathcal{R}_2$ be two distinct Shi regions such that $s(\mathcal{R}_1), s(\mathcal{R}_2) \subseteq \mathcal{R}$. Assume that $X(\mathcal{R}_1, \alpha_i) = 0$. Let $\alpha \in \Phi_0^+ \setminus \{\alpha_i\}$, then Y^α is admissible if and only if $X^{s_i(\alpha)}$ is admissible.

Proof We have $X(\mathcal{R}_1, \alpha_i) = 0$, $X(\mathcal{R}_2, \alpha_i) = +$, and $X(\mathcal{R}_j, \gamma) = X(\mathcal{R}, s_i(\gamma))$ for all $j = 1, 2$ and $\gamma \in \Phi_0^+ \setminus \{\alpha_i\}$ by Proposition 7.3(1). We prove this lemma in type A_2 and B_2 and leave the case G_2 to the reader. The proof works as follows. Let $\alpha \in \Phi_0^+ \setminus \{\alpha_i\}$. We know, thanks to Proposition 5.3, how to identify the descent roots of a Shi region within its sign type (see Examples 5.5 and 5.6). More precisely, in Figures 3 and 4, we have:

- a sign $X(\mathcal{R}_1, \alpha)$ is colored in red if and only if Y^α is admissible;
- equivalently, a sign $X(\mathcal{R}, s_i(\alpha))$ is colored in red if and only if $X^{s_i(\alpha)}$ is admissible.

So in order to prove our claim and since $X(\mathcal{R}_j, \gamma) = X(\mathcal{R}, s_i(\gamma))$ for all $j = 1, 2$ and $\gamma \in \Phi_0^+ \setminus \{\alpha_i\}$, it is enough in each case to:

- (1) list the Shi regions $\mathcal{R}$ such that $s_i \cdot \mathcal{R}_1, s_i \cdot \mathcal{R}_2 \subseteq \mathcal{R}$ for two distinct Shi regions $\mathcal{R}_1, \mathcal{R}_2$;
- (2) check that, for all $\beta \in \Phi_0^+ \setminus \{\alpha_i\}$, the sign $X(\mathcal{R}_1, \beta)$ is colored in red if and only if $X(\mathcal{R}, s_i(\beta))$ is also colored in red.

The result of this case-by-case analysis, which confirms our statement, is provided in Tables 1 and 2. Recall that $s_i(\Phi_0^+ \setminus \{\alpha_i\}) = \Phi_0^+ \setminus \{\alpha_i\}$. These tables are obtained with the help of Figures 3 and 4, with the same notations. ■

7.4 Conclusion of the proof of Theorem 1.1

In Section 4.3, we explain that it is enough to show that $L_{\text{Shi}} \subseteq L$ in order to prove Theorem 1.1. We already know that this statement is true in the dominant region: $L_{\text{Shi}}^0 = L^0$ by Corollary 6.4.

We first state the following corollary of the descent-wall theorem: if a hyperplane in $\text{Shi}(W, S)$ is parallel to a hyperplane H_α with $\alpha \in \Delta_0$ and is also in the basis of the inversion set $N^1(w)$ for $w \in L_{\text{Shi}}$, then this hyperplane must be a descent wall of the alcove of w .

Corollary 7.8 (Of the descent-wall theorem) Let $w \in L_{\text{Shi}}$ and $s \in S_0$ such that $\delta - \alpha_s \in N^1(w)$. We have $\delta - \alpha_s \in ND_R(w)$.

Proof Let $\mathcal{R}$ be the Shi region associated with w . By Theorem 7.1, we know that $ND_R(w) = \Sigma D(\mathcal{R})$. Assume that $\delta - \alpha_s \notin \Sigma D(\mathcal{R})$. Then, by Lemma 5.9, there is an irreducible root subsystem Ψ of rank 2 in Φ_0 such that $\alpha_s \in \Psi$ and there is $\beta \in \Psi$ with $\alpha_s + \beta \in \Psi$, $X(\mathcal{R}, \alpha_s + \beta) = +$ and $X(\mathcal{R}, \beta) = -$. So $\delta - (\alpha_s + \beta), \beta \in N(w)$ by equation (7). Therefore, $\delta - \alpha_s = \delta - (\alpha_s + \beta) + \beta$, which contradicts the fact that $\delta - \alpha_s \in N^1(w)$ spans a ray of $\text{cone}(N(w))$. So $\delta - \alpha_s \notin \Sigma D(\mathcal{R}) = ND_R(w)$. ■

Proof of Theorem 1.1 Let $w \in L_{\text{Shi}}$. We decompose $w = uv$ with $u \in W_0$ and $v \in {}^0W$. We show by induction on $\ell(u)$ that $w \in L$.

Assume first that $\ell(u) = 0$, then $u = e$ and $w = v \in L_{\text{Shi}}^0 = L^0$ by Proposition 6.4. Now assume that $\ell(u) \geq 1$. Then there is $s \in D_L(u) \subseteq D_L(w) \cap S_0$. So $w' = u'v = sw \in L_{\text{Shi}}$ by Proposition 4.5. By induction, we have $sw \in L$, since $\ell(u') < \ell(u)$.

We know that $\alpha_s \notin N(sw)$, so $\alpha_s \notin N^1(sw)$. Assume $\delta - \alpha_s \in N^1(sw)$, then, by Corollary 7.8, $\delta - \alpha_s \in ND_R(sw)$. Since $ND_R(sw) \subseteq s(ND_R(w))$ by Proposition 3.7, we have $\delta + \alpha_s = s(\delta - \alpha_s) \in ND_R(w)$. Since $w \in L_{\text{Shi}}$, we obtain by Proposition 5.7 that $ND_R(w) \subseteq \Sigma$, which contradicts $\Sigma = \{\alpha, \delta - \alpha \mid \alpha \in \Phi_0^+\}$. Therefore, $\delta - \alpha_s \notin N^1(sw)$ and $N^1(sw) \subseteq \Sigma \setminus \{\alpha_s, \delta - \alpha_s\}$. By Proposition 3.1, we obtain, therefore, that $s(N^1(sw)) \subseteq \Sigma \setminus \{\alpha_s, \delta - \alpha_s\}$. We conclude by [13, Theorem 4.10] that

$$N^1(w) \subseteq \{\alpha_s\} \cup s(N^1(sw)) \subseteq \{\alpha_s\} \cup (\Sigma \setminus \{\alpha_s, \delta - \alpha_s\}) \subseteq \Sigma.$$

In other words, $w \in L$. ■

Acknowledgments The authors warmly thank Antoine Abram, Balthazar Charles, and Nathan Williams for numerous interesting discussions on the subject. The authors are also grateful to Nathan Williams for having pointed us the results in [4, 5]. The second author thanks François Bergeron for the references on diagonal invariants. Finally, we are indebted to the anonymous referees: their thorough reading of the first version of this article and the comments and suggestions made helped us to considerably improve the content of this article.

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Faculté des sciences et techniques, Institut Denis Poisson, CNRS, Université de Tours, Parc de Grandmont,
37200 Tours, France

e-mail: nathan.chapelier@gmail.com

LaCIM et Département de Mathématiques, Université du Québec à Montréal, CP 8888 Succ. Centre-Ville,
Montréal, QC H3C 3P8, Canada

e-mail: hohlweg.christophe@uqam.ca