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K-THEORY OF RATIONAL QUADRATIC FORMS

IGOR NIKOLAEV¹

ABSTRACT. We compute the genus of a rational quadratic form in terms of the K-theory of a C^* -algebra attached to the adelic orthogonal group of the form. As a corollary, one gets a higher composition law for the rational quadratic forms. As an illustration, we consider the Gauss composition of the binary quadratic forms.

1. INTRODUCTION

The binary quadratic forms $q(u, v) = \{au^2 + buv + cv^2 \mid a, b, c \in \mathbf{Z}\}$ were studied by C.-F. Gauss. Two forms q and q' are said to be equivalent, if a substitution $\{u = \alpha u' + \beta v', v = \gamma u' + \delta v' \mid \alpha, \beta, \gamma, \delta \in \mathbf{Z}, \alpha\delta - \beta\gamma = 1\}$ transforms q into q' . It is easy to see that the discriminant $\Delta = b^2 - 4ac$ of the form $q(u, v)$ is an invariant of the substitution and, therefore, the equivalent forms have the same discriminant. But the converse is false in general. Gauss showed that there exists a finite number of the pairwise non-equivalent binary quadratic forms having the same discriminant. Moreover, the equivalence classes make an abelian group \mathcal{G} under a composition defined on these forms [Cassels 1978] [8, Chapter 14]. The cardinality $g = |\mathcal{G}|$ of such a group is called the genus of $q(u, v)$. An extension of the Gauss composition to the general quadratic forms $q(x) := q(x_1, \dots, x_n) = \{\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \mid a_{ij} \in \mathbf{Z}, n \geq 1\}$ is a difficult and important problem [Bhargava 2004] [1], [2], [3] and [Bhargava 2008] [4], see the survey [Bhargava 2006] [5].

Let $G(K)$ be a non-compact reductive algebraic group defined over a number field K . Denote by \mathbb{A} the ring of adèles of K and by \mathbb{A}_∞ a subring of the integer adèles. It is well known that $G(K)$ is a discrete subgroup of $G(\mathbb{A})$ and the double cosets $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$ is a finite set. The set $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$ is an important arithmetic invariant of the group $G(K)$. For instance, if $G \cong O(q)$ is the orthogonal group of a quadratic form, then $|G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)| = g$ [Platonov & Rapinchuk 1994] [14, Chapter 8].

Consider a Banach algebra $L^1(G(\mathbb{A})/G(K))$ of the integrable complex-valued functions on the homogeneous space $G(\mathbb{A})/G(K)$ endowed with the operator norm. Recall that the addition of functions $f_1, f_2 \in L^1(G(\mathbb{A})/G(K))$ is defined pointwise and multiplication is given by the convolution product:

$$(f_1 * f_2)(g) = \int_{G(\mathbb{A})/G(K)} f_1(gh^{-1})f_2(h)dh. \quad (1.1)$$

By \mathcal{A} we understand an enveloping C^* -algebra of the algebra $L^1(G(\mathbb{A})/G(K))$; we refer the reader to [Dixmier 1977] [9, Section 13.9] for details of this construction.

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In this note we construct a map from the set $G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K)$ to the K-theory of the C^* -algebra \mathcal{A} . Since the K-groups are abelian, one gets the structure of an abelian group on the set $G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K)$. As an application, we consider the case $G \cong O(q)$, where $O(q)$ is the orthogonal group of the rational quadratic form $q(x)$. In this case, one gets a higher composition law for the quadratic forms and a formula for their genera. In what follows, we use the following definitions.

It is known that \mathcal{A} is a stationary AF-algebra of rank $n = \text{rank } G(K)$ [13, Lemma 3.1]. Such an algebra is given by an $n \times n$ integer matrix A with the determinant $\det A = 1$ [Effros 1981] [10, Chapter 6]. Denote by $\mathcal{A} \rtimes_\sigma \mathbf{Z}$ the crossed product of the C^* -algebra \mathcal{A} by the shift automorphism σ of \mathcal{A} [Blackadar 1986] [6, Exercise 10.11.9]. It is known that $K_0(\mathcal{A} \rtimes_\sigma \mathbf{Z}) \cong \mathbf{Z}^n / (I - A)\mathbf{Z}^n$ and $|K_0(\mathcal{A} \rtimes_\sigma \mathbf{Z})| = |\det(I - A)|$, where I is the identity matrix and $K_0(\mathcal{A} \rtimes_\sigma \mathbf{Z})$ is the K_0 -group of the algebra $\mathcal{A} \rtimes_\sigma \mathbf{Z}$, see [Blackadar 1986] [6, Theorem 10.2.1]. Our main results can be formulated as follows.

Theorem 1.1. *There exists a one-to-one map*

$$\phi : G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K) \rightarrow K_0(\mathcal{A} \rtimes_\sigma \mathbf{Z}). \quad (1.2)$$

In particular, the map (1.2) defines the structure of an abelian group on the set $G(\mathbb{A}_\infty)\backslash G(\mathbb{A})/G(K)$, so that ϕ becomes an isomorphism of the abelian groups.

Corollary 1.2. *If $G \cong O(q)$ is the orthogonal group of a rational quadratic form q , then (1.2) defines a composition law for the equivalence classes of the quadratic forms. In particular, the genus g of the quadratic form q is given by the formula*

$$g = |\det(I - A)|. \quad (1.3)$$

The article is organized as follows. In Section 2 we briefly review the AF-algebras, crossed products and algebraic groups over the ring of adeles. Theorem 1.1 and corollary 1.2 are proved in Section 3. An illustration of corollary 1.2 can be found in Section 4.

2. PRELIMINARIES

The C^* -algebras are covered in [Dixmier 1977] [9]. For the AF-algebras we refer the reader to [Effros 1981] [10]. The K-theory of C^* -algebras is the subject of [Blackadar 1986] [6]. An excellent introduction to the algebraic groups can be found in [Platonov & Rapinchuk 1994] [14].

2.1. AF-algebras.

2.1.1. *C^* -algebras.* The C^* -algebra is an algebra \mathcal{A} over \mathbf{C} with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $\|ab\| \leq \|a\| \|b\|$ and $\|a^*a\| = \|a\|^2$ for all $a, b \in \mathcal{A}$. Any commutative C^* -algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on a locally compact Hausdorff space X ; otherwise, the algebra \mathcal{A} represents a noncommutative topological space.

2.1.2. *Crossed products.* Let $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be an automorphism of the C^* -algebra \mathcal{A} . The idea of a crossed product C^* -algebra $\mathcal{A} \rtimes_\sigma \mathbf{Z}$ by σ is to embed \mathcal{A} into a larger C^* -algebra in which the automorphism σ becomes an inner automorphism. Namely, let G be a locally compact group and σ a continuous homomorphism from G into $\text{Aut}(\mathcal{A})$. A covariant representation of the triple (\mathcal{A}, G, σ) is a pair of

representations (π, ρ) of \mathcal{A} and G on a Hilbert space, such that $\rho(g)\pi(a)\rho(g)^* = \pi(\sigma_g(a))$ for all $a \in \mathcal{A}$ and $g \in G$. The covariant representation defines an algebra whose completion in the operator norm is called a crossed product C^* -algebra $\mathcal{A} \rtimes_{\sigma} G$.

2.1.3. AF-algebras. An AF-algebra (Approximately Finite C^* -algebra) is defined to be the norm closure of an ascending sequence of finite dimensional C^* -algebras M_n , where M_n is the C^* -algebra of the $n \times n$ matrices with entries in \mathbf{C} . The index $n = (n_1, \dots, n_k)$ represents a semi-simple matrix algebra $M_n = M_{n_1} \oplus \dots \oplus M_{n_k}$. The ascending sequence can be written as

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \dots, \quad (2.1)$$

where M_i are the finite dimensional C^* -algebras and φ_i the homomorphisms between such algebras. If $\varphi_i = \text{Const}$, then the AF-algebra \mathcal{A} is called stationary. The shift automorphism of a stationary AF-algebra corresponds to the map $i \mapsto i + 1$ in (2.1) [Effros 1981] [10, p.37].

The homomorphisms φ_i can be arranged into a graph as follows. Let $M_i = M_{i_1} \oplus \dots \oplus M_{i_k}$ and $M_{i'} = M_{i'_1} \oplus \dots \oplus M_{i'_k}$ be the semi-simple C^* -algebras and $\varphi_i : M_i \rightarrow M_{i'}$ the homomorphism. One has two sets of vertices V_{i_1}, \dots, V_{i_k} and $V_{i'_1}, \dots, V_{i'_k}$ joined by a_{rs} edges whenever the summand M_{i_r} contains a_{rs} copies of the summand $M_{i'_s}$ under the embedding φ_i . As i varies, one obtains an infinite graph called the Bratteli diagram of the AF-algebra. The $A = (a_{rs})$ is called a matrix of the partial multiplicities; an infinite sequence of A_i defines a unique AF-algebra. If $\text{rank}(A_i) = n = \text{Const}$, then the AF-algebra is said to be of *rank* n . If the AF-algebra \mathcal{A} is stationary, then $A_i = A$, where A is an $n \times n$ matrix with the non-negative entries.

2.1.4. K_0 -groups. For a C^* -algebra \mathcal{A} , let $V(\mathcal{A})$ be the union of projections in the $k \times k$ matrix C^* -algebra with entries in \mathcal{A} taken over all $k = 1, 2, \dots, \infty$. The projections $p, q \in V(\mathcal{A})$ are equivalent, if there exists a partial isometry u such that $p = u^*u$ and $q = uu^*$. The equivalence class of projection p is denoted by $[p]$. The equivalence classes of orthogonal projections can be made to a semigroup by putting $[p] + [q] = [p + q]$. The Grothendieck completion of this semigroup to an abelian group is called the K_0 -group of the algebra \mathcal{A} . The functor $\mathcal{A} \rightarrow K_0(\mathcal{A})$ maps a category of the C^* -algebras into the category of abelian groups, so that projections in the algebra \mathcal{A} correspond to a positive cone $K_0^+ \subset K_0(\mathcal{A})$ and the unit element $1 \in \mathcal{A}$ corresponds to an order unit $u \in K_0(\mathcal{A})$.

Let \mathcal{A} be the stationary AF-algebra given by an $n \times n$ matrix A . Denote by $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ the shift automorphism of \mathcal{A} . Consider a crossed product C^* -algebra $\mathcal{A} \rtimes_{\sigma} \mathbf{Z}$ defined by the automorphism σ . The K_0 -group of the crossed product $\mathcal{A} \rtimes_{\sigma} \mathbf{Z}$ is given by the formula:

$$K_0(\mathcal{A} \rtimes_{\sigma} \mathbf{Z}) \cong \frac{\mathbf{Z}^n}{(I - A)\mathbf{Z}^n}, \quad \text{where } I = \text{diag}(1, 1, \dots, 1). \quad (2.2)$$

2.2. Algebraic groups over adeles.

2.2.1. Algebraic groups. An algebraic group is an algebraic variety G together with (i) an element $e \in G$, (ii) a morphism $\mu : G \times G \rightarrow G$ given by the formula $(x, y) \mapsto xy$ and (iii) a morphism $i : G \rightarrow G$ given by the formula $x \mapsto x^{-1}$ with respect to which the set G is a group. If G is a non-compact variety, the algebraic

group G is said to be non-compact. The G is called a K -group if G is a variety defined over the field K and if μ and i are defined over K . In what follows, we deal with the linear algebraic groups, i.e. the subgroups of the general linear group GL_n . An algebraic group G is called reductive if the unipotent radical of G is trivial. An informal equivalent definition says that G is reductive if and only if a representation of G is a direct sum of the irreducible representations.

2.2.2. Orthogonal group of a quadratic form. Let $q(x) = \{\sum_{i=1}^n \sum_{j=1}^n a_{ij}x_i x_j \mid a_{ij} \in \mathbf{Z}, n \geq 1\}$ be a quadratic form. Roughly speaking, the orthogonal group is a subgroup of the GL_n which preserves the form $q(x)$. Namely, let $A = (a_{ij})$ be a symmetric matrix attached to the quadratic form $q(x)$. The $O(q) = \{g \in GL_n \mid gAg = A\}$ is called an orthogonal group and $SO(q) = \{g \in O(q) \mid \det g = 1\}$ is called a special orthogonal group of the form $q(x)$.

2.2.3. Ring of adeles. The adeles is a powerful tool describing the Artin reciprocity for the abelian extensions of a number field K . In intrinsic terms, the ring of adeles \mathbb{A} of a field K is a subset of the direct product $\prod K_v$ taken over almost all places K_v of K endowed with the natural topology. The embeddings $K \hookrightarrow K_v$ induce a discrete diagonal embedding $K \hookrightarrow \mathbb{A}$; the image of such is a ring of the principal adeles of \mathbb{A} . The ring of integral adeles $\mathbb{A}_\infty := \prod \mathcal{O}_v$, where \mathcal{O}_v is a localization of the ring \mathcal{O}_K of the integers of the field K . The Artin reciprocity says that there exists a continuous homomorphism $\mathbb{A}^\times \rightarrow Gal(K^{ab}|K)$, where \mathbb{A}^\times is a group of the invertible adeles (the idele group) and $Gal(K^{ab}|K)$ is the absolute Galois group of the abelian extensions of K endowed with a profinite topology. On the other hand, there exists a canonical isomorphism $\mathbb{A}_\infty^\times \backslash \mathbb{A}^\times / K^\times \rightarrow Cl(K)$, where \mathbb{A}_∞^\times (\mathbb{A}^\times and K^\times , resp.) is a group of units of the ring \mathbb{A}_∞ (\mathbb{A} and \mathcal{O}_K , resp.) and $Cl(K)$ is the ideal class group of the field K .

2.2.4. Algebraic groups over adeles. The algebraic groups over the ring of adeles can be viewed as an analog of the Artin reciprocity for the non-abelian extensions of the field K and, therefore, we deal with a noncommutative arithmetic [Platonov & Rapinchuk 1994] [14, p. 243]. Such groups are a starting point of the Langlands program [Langlands 1978] [12].

Let G be an algebraic group. It is known that the double cosets $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$ is a finite set [Borel 1963] [7]. In particular, if $G \cong GL_n$ then $|G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)| = h_K$ and if $G \cong O(q)$ then $|G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)| = g$, where $h_K = |Cl(K)|$ is the class number of the field K and g is the genus of quadratic form $q(x)$, respectively. The cardinality of the set $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$ is a difficult open problem.

3. PROOFS

3.1. Proof of theorem 1.1. We shall split the proof in a series of lemmas.

Lemma 3.1. *Consider the group $Ext(\mathcal{A} \rtimes_\sigma \mathbf{Z})$ consisting of the equivalence classes of extensions*

$$1 \rightarrow \mathbf{C} \rightarrow E \rightarrow \mathcal{A} \rtimes_\sigma \mathbf{Z} \rightarrow 1 \quad (3.1)$$

of the $\mathcal{A} \rtimes_\sigma \mathbf{Z}$ by the complex numbers \mathbf{C} . Then $Ext(\mathcal{A} \rtimes_\sigma \mathbf{Z}) \cong K_0(\mathcal{A} \rtimes_\sigma \mathbf{Z})$.

Proof. The result follows from the equivalence of the **Ext** and the **K**₀-functors on the category of C^* -algebras. The special case of the C^* -algebra $\mathcal{A} \rtimes_\sigma \mathbf{Z} := O_A$ can be found in [Blackadar 1986] [6, Section 16.4.5]. \square

Lemma 3.2. *Denote by $E_0 \cong (\mathcal{A} \rtimes_\sigma \mathbf{Z}) \oplus \mathbf{C}$ the trivial extension of the crossed product $\mathcal{A} \rtimes_\sigma \mathbf{Z}$ corresponding to the null element of the group $\text{Ext}(\mathcal{A} \rtimes_\sigma \mathbf{Z})$. Then E_0 is a non-commutative coordinate ring of the group $G(\mathbb{A})$, i.e. there exists a covariant functor between the respective categories, such that the morphisms of $G(\mathbb{A})$ correspond to the morphisms of E_0 .*

Proof. (i) Recall that the AF-algebra \mathcal{A} is a non-commutative coordinate ring of the group $G(\mathbb{A})$ [13, Lemma 3.1]. Consider the crossed product $\mathcal{A} \rtimes_\sigma \mathbf{Z}$ by the shift automorphism σ of \mathcal{A} . It is well known, that there exists a canonical embedding of the C^* -algebras:

$$\mathcal{A} \hookrightarrow \mathcal{A} \rtimes_\sigma \mathbf{Z}. \quad (3.2)$$

Using (3.2) one can extend each morphism of \mathcal{A} to such of the crossed product $\mathcal{A} \rtimes_\sigma \mathbf{Z}$. Such an extension is well defined and unique. Thus one gets a functor between the category of groups $G(\mathbb{A})$ and the category of crossed products $\mathcal{A} \rtimes_\sigma \mathbf{Z}$. In other words, the $\mathcal{A} \rtimes_\sigma \mathbf{Z}$ is a non-commutative coordinate ring of the group $G(\mathbb{A})$.

(ii) On the other hand, each morphism h of $\mathcal{A} \rtimes_\sigma \mathbf{Z}$ can be extended to $E_0 \cong (\mathcal{A} \rtimes_\sigma \mathbf{Z}) \oplus \mathbf{C}$ by the formula

$$h(E_0) = h(\mathcal{A} \rtimes_\sigma \mathbf{Z}) \oplus \mathbf{C}. \quad (3.3)$$

Therefore we get a functor from the category of groups $G(\mathbb{A})$ and such of the C^* -algebras $(\mathcal{A} \rtimes_\sigma \mathbf{Z}) \oplus \mathbf{C}$. In other words, the E_0 is a non-commutative coordinate ring of the group $G(\mathbb{A})$. \square

Lemma 3.3. *There exists a one-to-one map*

$$\phi : G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K) \rightarrow K_0^{\text{set}}(\mathcal{A} \rtimes_\sigma \mathbf{Z}), \quad (3.4)$$

where $K_0^{\text{set}}(\mathcal{A} \rtimes_\sigma \mathbf{Z})$ is the set of elements of the group $K_0(\mathcal{A} \rtimes_\sigma \mathbf{Z})$.

Proof. Consider the exact sequence (3.1) of the C^* -algebras:

$$1 \rightarrow \mathbf{C} \rightarrow E_0 \rightarrow \mathcal{A} \rtimes_\sigma \mathbf{Z} \rightarrow 1, \quad (3.5)$$

where $E_0 \cong (\mathcal{A} \rtimes_\sigma \mathbf{Z}) \oplus \mathbf{C}$. The exact sequence (3.5) gives rise to the exact sequence of the K_0 -groups:

$$1 \rightarrow \mathbf{Z} \rightarrow K_0(E_0) \rightarrow K_0(\mathcal{A} \rtimes_\sigma \mathbf{Z}) \rightarrow 1, \quad (3.6)$$

where $K_0(\mathbf{C}) \cong \mathbf{Z}$.

In view of lemma 3.2, we have a functor F acting by the formula $G(\mathbb{A}) \mapsto E_0$. On the other hand, the K_0 is a covariant functor from the category of C^* -algebras to the category of abelian groups. Thus, the composition $K_0 \circ F$ defines a functor acting by the formula $G(\mathbb{A}) \mapsto K_0(E_0)$.

Using the exact sequence (3.6), one can define a map $K_0(E_0) \rightarrow K_0^{\text{set}}(\mathcal{A} \rtimes_\sigma \mathbf{Z})$ from the abelian group to a finite set. On the other hand, we have a double coset map $G(\mathbb{A}) \rightarrow G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$ from an algebraic group to a finite set, see Section 2.2. Bringing these facts together, one gets a commutative diagram in Figure 1.

It remains to notice, that Φ is a functor from the category of finite sets to itself. This observation implies that Φ is a trivial functor, i.e. $|G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)| = |K_0^{\text{set}}(\mathcal{A} \rtimes_\sigma \mathbf{Z})|$. In particular, there exists a one-to-one map

$$\begin{array}{ccc}
G(\mathbb{A}) & \longrightarrow & G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K) \\
\downarrow K_0 \circ F & & \downarrow \Phi \\
K_0(E_0) & \longrightarrow & K_0^{\text{set}}(\mathcal{A} \rtimes_\sigma \mathbf{Z})
\end{array}$$

FIGURE 1. Double cosets $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$

$\phi : G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K) \rightarrow K_0^{\text{set}}(\mathcal{A} \rtimes_\sigma \mathbf{Z})$ between the corresponding finite sets. Lemma 3.3 follows. \square

Corollary 3.4. *The map ϕ^{-1} defines the structure of an abelian group on the set $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$ thus extending ϕ to an isomorphism of the abelian groups.*

Proof. Recall that

$$K_0(\mathcal{A} \rtimes_\sigma \mathbf{Z}) \cong \frac{\mathbf{Z}^n}{(I - A)\mathbf{Z}^n} \cong \frac{\mathbf{Z}}{p_1^{n_1}\mathbf{Z}} \oplus \cdots \oplus \frac{\mathbf{Z}}{p_k^{n_k}\mathbf{Z}}, \quad (3.7)$$

where A is an $n \times n$ integer matrix defining the AF-algebra \mathcal{A} , the p_i are prime and n_i are positive integer numbers. One can take a generator x_i in each cyclic group $\mathbf{Z}/p_k^{n_k}\mathbf{Z}$ and let $\phi^{-1}(x_i)$ be a generator of the abelian group structure on the set $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$. It is clear that the map ϕ defines an isomorphism between the two abelian groups. Corollary 3.4 is proved. \square

Theorem 1.1 follows from lemma 3.3 and corollary 3.4.

3.2. Proof of corollary 1.2. Recall that if $G \cong O(q)$ is the orthogonal group of a quadratic form $q(x)$, then the equivalence classes of $q(x)$ correspond one-to-one to the double cosets $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$, see Section 2.2.4. But in view of the corollary 3.4, the set $G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$ has the natural structure of an abelian group defined by the formula (3.7). In particular, the group operation defines a composition law for the equivalence classes of quadratic form $q(x)$. The first part of corollary 1.2 is proved.

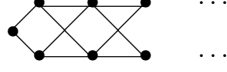
To express the genus g of the form $q(x)$ in terms of an invariant of the algebra \mathcal{A} , recall that $g = |G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)|$, see Section 2.2.4. But $|G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)| = |K_0(\mathcal{A} \rtimes_\sigma \mathbf{Z})|$ and one gets from the formulas (3.7):

$$\begin{aligned}
g &= |K_0(\mathcal{A} \rtimes_\sigma \mathbf{Z})| = \left| \frac{\mathbf{Z}^n}{(I - A)\mathbf{Z}^n} \right| = \left| \frac{\mathbf{Z}}{p_1^{n_1}\mathbf{Z}} \oplus \cdots \oplus \frac{\mathbf{Z}}{p_k^{n_k}\mathbf{Z}} \right| = \\
&= p_1^{n_1} \cdots p_k^{n_k} = |\det(I - A)|.
\end{aligned} \quad (3.8)$$

The genus formula of corollary 1.2 follows from the equations (3.8).

4. BINARY QUADRATIC FORMS

Let $n = 2$ and $\Delta > 0$ be a square-free integer, i.e. $q_\Delta(x)$ is an indefinite binary quadratic form. In view of corollary 1.2, we are looking for a matrix A with

FIGURE 2. Bratteli diagram of the AF-algebra \mathcal{A} .

$\det A = 1$, such that

$$g(q_{f^2\Delta}(x)) = |\det(I - A^k)|, \quad (4.1)$$

where $f \geq 1$ is a conductor of the quadratic form and $k \geq 1$ is an integer.

Remark 4.1. The left and right sides of equation (4.1) depend only on the integers f and k , respectively. Indeed, the left side depends only on the conductor f , if the discriminant Δ is a fixed square-free integer. For the right side, we have easily verified equalities $|\det(I - A^k)| = \text{tr}(A^k) - 2 = 2T_k(\frac{1}{2}\text{tr}(A)) - 2 = 2T_k(\frac{1}{2}\sqrt{\Delta+4}) - 2$, where $T_k(x)$ is the Chebyshev polynomial of degree k .

Let f and k be the least integers satisfying equation (4.1). Recall that the equivalence classes of the quadratic form $q_{f^2\Delta}(x)$ correspond one-to-one to the (narrow) ideal classes of the order $R_f := \mathbf{Z} + fO_K$ in the real quadratic field $K := \mathbf{Q}(\sqrt{\Delta})$. The number h_{R_f} of such classes is calculated by the formula:

$$h_{R_f} = h_K \frac{f}{e_f} \prod_{p|f} \left(1 - \left(\frac{\Delta}{p}\right) \frac{1}{p}\right), \quad (4.2)$$

where h_K is the class number of the field K , e_f is the index of the group of units of the order R_f in the group of units of the ring O_K , p is a prime number and $\left(\frac{\Delta}{p}\right)$ is the Legendre symbol [Hasse 1950] [11, pp. 297 and 351]. Thus the genus of the binary quadratic form of discriminant $\Delta > 0$ is given by the formula:

$$g(q_\Delta(x)) = \frac{|\det(I - A^k)|}{\frac{f}{e_f} \prod_{p|f} \left(1 - \left(\frac{\Delta}{p}\right) \frac{1}{p}\right)}. \quad (4.3)$$

Example 4.2. Let $G \cong O(q)$ be the orthogonal group of the quadratic form

$$q(u, v) = u^2 + 3uv + v^2. \quad (4.4)$$

The discriminant is $\Delta = 5$ and therefore the $q(u, v)$ is an indefinite quadratic form. It is not hard to see, that the AF-algebra \mathcal{A} corresponding to the group $G(\mathbb{A})$ is represented by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad (4.5)$$

since the determinant of the characteristic polynomial of A must coincide with $\Delta = 5$. The Bratteli diagram of algebra \mathcal{A} is shown in Figure 2. To calculate the genus of $q(u, v)$, we shall use formula (4.3) with $f = k = 1$. Namely, one gets

$$g = |\det(I - A)| = \left| \det \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} \right| = 1. \quad (4.6)$$

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