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Two-phase magma flow with phase exchange
– Part I –
Physical modeling of a volcanic conduit

Gladys Narbona-Reina*, Didier Bresch†, Alain Burgisser ‡, Marielle Collombet‡

August 24, 2024

Abstract

In a review paper in this same volume, we present the state of the art on modeling of compressible viscous flows ranging from single-phase to two-phase systems. It focuses on mathematical properties related to weak stability because they are important for numerical resolution and on the homogenization process that leads from a microscopic description of two separate phases to an averaged two-phase model. This review serves as the foundation for Parts I and II, which present averaged two-phase models with phase exchange applicable to magma flow during volcanic eruptions. Here, in Part I, after introducing the physical processes occurring in a volcanic conduit, we detail the steps needed at both microscopic and macroscopic scales to obtain a two-phase transient conduit flow model ensuring: 1) mass and volatile species conservation, 2) disequilibrium degassing considering both viscous relaxation and volatile diffusion, and 3) dissipation of total energy. The resulting compressible/incompressible system has 8 transport equations on 8 unknowns (gas volume fraction and density, dissolved water content, liquid pressure, and the velocity and temperature of both phases) as well as algebraic closures for gas pressure and bubble radius. We establish valid sets of boundary conditions such as imposing pressures and stress-free conditions at the conduit outlet and either velocity or pressure at the inlet. This model is then used to obtain a drift-flux system that isolates the effects of relative velocities, pressures, and temperatures. The dimensional analysis of this drift-flux system suggests that relative velocities can be captured with a Darcy equation and that gas–liquid pressure differences partly control magma acceleration. Unlike the vanishing small gas–liquid temperature differences, bulk magma temperature is expected to vary because of gas expansion. Mass exchange being a major control of flow dynamics, we propose a limit case of mass exchange by establishing a relaxed system at chemical equilibrium. This single-velocity, single-temperature system is a generalization of an existing volcanic conduit flow model. Finally, we compare our full compressible/incompressible system to another existing volcanic conduit flow model where both phases are compressible. This comparison illustrates that different two-phase systems may be obtained depending on the governing unknowns chosen. Part II presents a 1.5D version of the model established herein that is solved numerically. The numerical outputs are compared to those of another steady-state, equilibrium degassing, isothermal model under conditions typical of an effusive eruption at an andesitic volcano.

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Contents

1 Introduction

A magma is a three-phase mixture of gas bubbles, solid crystals and silicate liquid. During its deep storage, in magmatic reservoirs, the gas is not abundant and the magma is essentially two-phase with crystals in suspension in the silicate liquid. During a volcanic eruption, the magma rises to the surface, which creates gas bubbles by decompression when volatile components such as H_2O or CO_2 dissolved into the silicate liquid reach saturation. If the silicate liquid is viscous enough, the ascending magma can simply be viewed as gas bubbles suspended in a complex fluid where liquid and crystals move together.

During an eruption, the degree of coupling between the liquid and gas phases of the magma strongly determines the eruptive style. Explosive eruptions are a consequence of the strong coupling between gas and liquid, which yields magmas with large proportions of gas prone to fragmentation. Phase decoupling, on the other hand, enables the gas to escape gently out of the volcanic edifice and fosters effusive eruptions. There is no consensus

on what are the minimal elements to recover the transition between effusive and explosive eruptive regimes. A closely related issue is that many volcanoes with frequent effusive–explosive transitions also feature oscillations of mass flow rate at the vent. Both regime transitions and oscillating effusion rates are observed during the same eruptive period, which suggests that the peculiarities of the overall architecture of the plumbing system play a limited role in the matter. Instead, these observations point to a limited set of controlling processes, and two-phase flow modeling with phase exchange has been shown to potentially be a unifying framework to study them [Gonnermann and Manga, 2007, Burgisser and Degruyter, 2015].

Two-phase models have first been used to show that flow oscillations can result from rheology changes due to microlites (small crystals growing during ascent in response to decompression) [Melnik and Sparks, 1999, Costa et al., 2013]. Similar models have also shown that temperature changes play a role in modulating flow rates through rheology changes [Costa et al., 2007]. More recently, however, an isothermal two-phase model without mass exchange but with variable permeability has shown that gas–liquid coupling is sufficient to create flow rate oscillations [Michaut et al., 2009, Michaut et al., 2013]. Mass exchange, and temperature or rheology variations may thus not be necessary to capture flow instabilities.

Mass exchange is measured by the degassing rate, which is a combined effect of diffusion of dissolved water towards gas bubbles and pressure equalisation between the liquid and the gas. Simply dismissing it as a control of flow modulation is not possible; the transition between explosive and effusive regime has been shown to be fostered by slow diffusion of the dissolved water towards the gas bubbles [Mason et al., 2006]. Finding an appropriate exchange rate between liquid and gas is challenging. Although exchange rates are well-established at the micro-scale of a single gas bubble (e.g. [Proussevitch et al., 1993b, Lensky et al., 2004, Forestier-Coste et al., 2012]), the two-phase flow formulation is a macro-scale, homogenized model. The scale transition implies thus information loss. The assumption of equilibrium degassing (a diffusion fast enough to always keep the dissolved water content at the solubility value) has thus often been adopted in two-phase flow models (e.g., [Melnik, 2000, Degruyter et al., 2012]).

Often conduit flow models include many complexities [Gonnermann and Manga, 2007], and here we try to minimize them by setting a few initial simplifying assumptions such as modeling a pure silicate liquid, or assuming a constant bubble number density (thereby excluding nucleation processes). Our aim is to propose a two-phase system with appropriate exchange terms and to provide important properties such as dissipation and invariant region for the system. Using one of the homogenization techniques presented in the review paper [Bresch et al., 2024], we propose a new formulation of mass exchange rate and compare it with others rates from the literature. The review paper [Bresch et al., 2024] also established that if mass and momentum conservation are given, one can derive a energy balance from these two conservation relationships. We thus establish the energy balance of our system with the constraints that one phase is compressible and the other not and that the system should dissipate energy. The resulting equations describe changes in phase volume fraction, velocities, pressures, and temperatures in the system. As our final model still involves many processes, we draw from the review paper [Bresch et al., 2024] the techniques to establish a drift flux formulation of our system that is amenable to scaling. Using a wide range of natural values for the involved parameters, we establish a minimal system at the main order and we assess the consequences of the assumption of equilibrium degassing by proposing a relaxed version of our model. Finally, we compare our approach of deriving energy conservation from given mass and momentum equations of a compressible–incompressible system to an existing model [La Spina et al., 2014] derived by assuming energy conservation from thermodynamics in a system where both phases are compressible.

2 Physical model of volcanic conduit flow

The modeling of magma ascending within a vertical conduit from a magma reservoir to the Earth's surface implies peculiarities that the general models presented in the review paper [Bresch et al., 2024] do not contain. We set a few initial simplifying assumptions to keep the modeling tractable. The magma is composed of liquid and gas, which in turn are composed of several chemical components. It is usual to consider that the gas phase is just composed of water because it represents more than 95 vol% of this phase. We assume that the liquid phase (silicate liquid, or melt for short) is incompressible and composed of two components, H_2O and an average component representing all the other oxides. We denote by C_l the weight concentration of H_2O in the liquid, hence $(1-C_l)$ represents the concentration of the other oxides. Both phases may exchange water mass.

Crystals will be neglected, keeping the system as a two-phase flow sensu stricto. This is an acceptable assumption as long as the crystal load is small because melt viscosity is generally high enough for liquid and crystals to move together. Neglecting crystals and crystallization is a limitation as both also influence magma ascent dynamics (e.g., [La Spina et al., 2016, Arzilli et al., 2019, Bamber et al., 2022]) by affecting the rheology of the system. This is especially true at high crystal volume fraction, when the suspension becomes a granular system subject to jamming, which affects both solid and liquid velocities (see the review paper, Section 2.3 [Bresch et al., 2024]). The release of latent heat of crystallization also affects the temperature of the magma. As crystallization proceeds by cooling or by dehydration, the water concentration in the liquid melt increases, increasing volatile exsolution.

Fragmentation is not considered explicitly; the model can thus be used to approach the effusive–explosive transition from the effusive regime. Integrating fragmentation is generally done by a threshold-controlled switch of the constitutive relationship of the liquid viscosity so that the carrier phase becomes the gas instead of the liquid (e.g., [Melnik, 2000, Degruyter et al., 2012]). This approach does not change the structure of the transport equations, which means that adding fragmentation to the models presented herein would not affect the balances of mass, momentum, and energy. Finally, the bubble number density with respect to the liquid, N, is considered constant. Varying N to account for nucleation (e.g., [Chernov et al., 2014]) and coalescence (e.g., [Mancini et al., 2016]) was a tempting prospect but it significantly increases the complexity of the homogenization process. It was thus left aside for future works.

Even with the simplifications mentioned above, many processes and length scales remain involved in controlling flow dynamics (Figure 1). At the smallest scale of a single bubble, the main processes are the diffusion of water from the melt into the bubble and the viscous resistance of the melt to gas expansion. Bubbles are assumed to form a mono-disperse population homogeneously distributed in space at the mesoscale. Finally, at the scale of the volcanic conduit, the gas–liquid flow is controlled by viscous resistance against conduit wall over the whole length of the conduit. We describe the macroscale transport equations (Section 2.1) before focusing on closures equations, which are based on a microscale description (Section 2.2) that is averaged at the mesoscale (Section 2.3). These closure equations are integrated into the final macroscale system in Sections 2.4–2.6.

2.1 Mass and momentum transport equations

We consider a two-phase, three-component model of a liquid–gas mixture where φ represents the gas volume fraction in the system and then $(1-\varphi)$ is the liquid volume fraction. The gas has only one component, water vapor (which may be in supercritical conditions). The liquid phase is composed of H_2O with concentration C_l in the liquid phase, the other oxides (inert component) having the concentration $1-C_l$. The total mass of the liquid phase is thus decomposed as $(1-\varphi)\rho_l = (1-\varphi)\rho_l C_l + (1-\varphi)\rho_l(1-C_l)$. The velocities are denoted by u_k, and the densities by ρ_k, with $k = l, g$ for liquid and gas phases respectively.

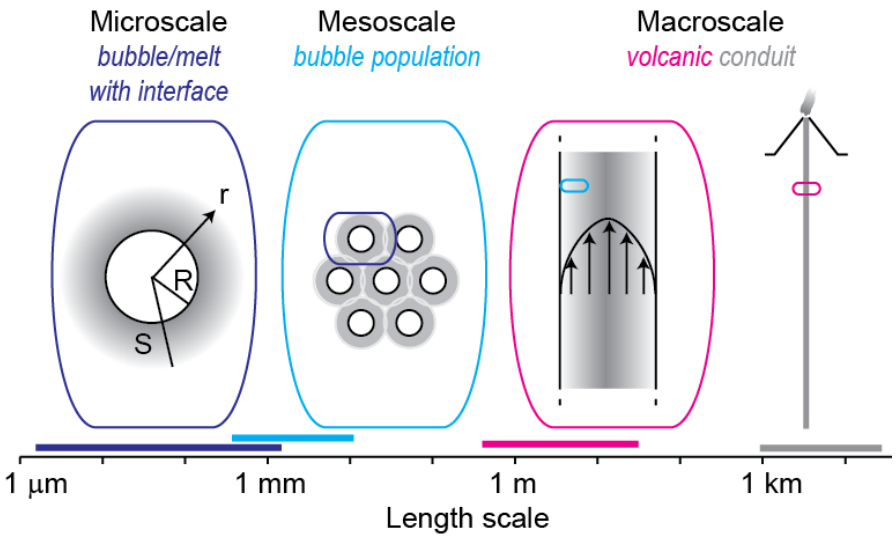

Figure 1: Scales over which the eruptive processes modeled herein occur: a single gas bubble (blue frame), a bubble population (cyan frame), and a volcanic conduit (magenta frame).

Mass equations. In our system the components are conserved, that is, the oxides mass in the liquid phase and the total water mass,

$$\partial_t((1-\varphi)\rho_l(1-C_l)) + \operatorname{div}((1-\varphi)\rho_l(1-C_l)u_l) = 0, \tag{2.1}$$

$$\partial_t(\varphi\rho_g + (1-\varphi)\rho_l C_l) + \operatorname{div}(\varphi\rho_g u_g + (1-\varphi)\rho_l C_l u_l) = 0. \tag{2.2}$$

Summing up these two equations yields the conservation of the total mass in the system (see Remark 2.1 below). The exchange of mass between liquid and gas occurs between the gas phase $\varphi\rho_g$ and the dissolved gas in the liquid phase, $(1-\varphi)\rho_l C_l$. Denoting R^{H_2O} the gas-liquid mass transfer rate, the mass equations for these quantities read

$$\partial_t(\varphi\rho_g) + \operatorname{div}(\varphi\rho_g u_g) = R^{H_2O}, \tag{2.3}$$

$$\partial_t((1-\varphi)\rho_l C_l) + \operatorname{div}((1-\varphi)\rho_l C_l u_l) = -R^{H_2O}. \tag{2.4}$$

The sum of (2.1) (oxides in liquid phase) and (2.4) (water in liquid phase) gives the liquid phase mass equation

$$\partial_t((1-\varphi)\rho_l) + \operatorname{div}((1-\varphi)\rho_l u_l) = -R^{H_2O}. \tag{2.5}$$

Remark 2.1. *Conservation of the total mass is fulfilled. We introduce the mixture density and velocity as*

$$\rho = \varphi\rho_g + (1-\varphi)\rho_l, \qquad u = \frac{\varphi\rho_g u_g + (1-\varphi)\rho_l u_l}{\rho}. \tag{2.6}$$

Then from equations (2.3) *and* (2.5) *we get*

$$\partial_t\rho + \operatorname{div}(\rho u) = 0. \tag{2.7}$$

□

Momentum equations. Momentum conservation reads for both phases:

$$\begin{aligned}&\partial_t((1-\varphi)\rho_l u_l)+\mathrm{div}((1-\varphi)\rho_l u_l\otimes u_l)+\nabla((1-\varphi)p_l)-\mathrm{div}((1-\varphi)\mathcal{D}_l)-\langle p_{int,l}\mathrm{Id}-\mathcal{D}_{int,l}\rangle\nabla(1-\varphi)\\&\quad-f_{\mathrm{drag}}-(1-\varphi)\rho_l g+\langle u_I\rangle R^{H_2O}=0\end{aligned}\tag{2.8}$$

$$\begin{aligned}&\partial_t(\varphi\rho_g u_g)+\mathrm{div}(\varphi\rho_g u_g\otimes u_g)+\nabla(\varphi p_g)-\mathrm{div}(\varphi\mathcal{D}_g)-\langle p_{int,g}\mathrm{Id}-\mathcal{D}_{int,g}\rangle\nabla\varphi\\&\quad+f_{\mathrm{drag}}-\varphi\rho_g g-\langle u_I\rangle R^{H_2O}=0\end{aligned}\tag{2.9}$$

where $\mathcal{D}_l,\mathcal{D}_g$ are the diffusive quantities of the viscous stress tensors and f_{drag} is the liquid–gas drag. The transport equations above are the basis of well-known two-phase models, see for example [Gavrilyuk and Saurel, 2002, Guillemaud, 2007, Ishii, 1975, Ishii and Hibiki, 2011] and see also the review paper Section 3.1, [Bresch et al., 2024]. They are obtained by an averaging process over the mixture domain, and so jump conditions at the interface between phases must be prescribed. As a result, these relations contain the averaged values of the velocity and stress tensors at the gas/fluid interface, denoted here like in [Bresch et al., 2024] respectively as $\langle u_I\rangle, \langle p_{int,l}\mathrm{Id}-\mathcal{D}_{int,l}\rangle, \langle p_{int,g}\mathrm{Id}-\mathcal{D}_{int,g}\rangle$. In particular, the momentum exchange due to mass transfer between phases is a function of the exchange velocity $\pm\langle u_I\rangle R^{H_2O}$. Appropriate closures for these interface quantities are given later in the microscopic study (Section 2.3) and when establishing the energy balance of the system (Section 2.5).

The diffusive quantities. The gravity vector being $g=(0,0,-\bar{g})$ ($\bar{g}=9.81\mathrm{m/s^2}$), the viscous tensors $\mathcal{D}_l,\mathcal{D}_g$ are given by

$$\mathcal{D}_g=\lambda_g\mathrm{div}\,u_g\mathrm{Id}+2\eta_g D(u_g)\tag{2.10}$$

$$\mathcal{D}_l=\lambda_l\mathrm{div}\,u_l\mathrm{Id}+2\eta_l D(u_l)\tag{2.11}$$

where $D(u)=\frac{1}{2}(\nabla u+\nabla^t u)$ is the deformation rate tensor of a velocity field u. We neglect the viscosity at the gas/liquid interface. For each phase ($j=l,g$), $\lambda_j=\zeta_j-\frac{2}{3}\eta_j$ with ζ_j and η_j the volume and dynamic shear viscosities, respectively. For H_2O in either gas or supercritical state, $\zeta_g=11\eta_g/3$ and $\lambda_g=3\eta_g$. For the liquid, $\zeta_l=\eta_l$ and $\lambda_l=\frac{1}{3}\eta_l$ [Webb and Dingwell, 1990], η_l being a function of C_l and T_g (e.g., [Hess and Dingwell, 1996]).

The drag term. Following [Kozono and Koyaguchi, 2009, Kozono and Koyaguchi, 2010], the drag term f_{drag} depends on the region where the conduit the magma is. Here we will consider a bubbly and a permeable flow for the magma in the conduit, $\varphi_{bp}\sim 0.6$ being the threshold of the gas volume fraction between them. The critical volume fraction for fragmentation is denoted by $\varphi_c\sim 0.8$ [Melnik, 2000]. For the bubbly flow region ($\varphi<\varphi_{bp}$) the drag term is given by

$$f_b=\frac{3\eta_l}{R^2}\,\varphi(1-\varphi)(u_g-u_l)$$

and for the permeable flow region ($\varphi_{bp}<\varphi<\varphi_c$) in laminar flow regime it is instead

$$f_p=\frac{\eta_g}{k}\,\varphi(1-\varphi)(u_g-u_l)$$

for k the permeability coefficient of the magma. Then we introduce the following unified formulation for the drag term:

$$f_{\mathrm{drag}}=K_d\,\varphi(1-\varphi)(u_g-u_l)\qquad\text{with}\quad K_d=\begin{cases}\frac{3\eta_l}{R^2} & \text{bubbly flow}\\ \\ \frac{\eta_g}{k} & \text{permeable flow}\end{cases}.\tag{2.12}$$

Pressure law. The ideal gas law relates gas density to gas pressure:

$$\rho_g = \frac{p_g}{c_0 T_g} \tag{2.13}$$

where c_0 is the ideal gas constant for water (461.52 J/kg/K) and T_g is the gas temperature that becomes a new unknown of the system. Hence we must give a closure equation for T_g, which will be described when establishing the energy of the system in Section 2.5.

As ρ_l is assumed to be constant and thus the liquid is incompressible, we will provide in Section 2.3 a closure equation for the liquid pressure, p_l, that encodes the evolution of the bubbles, which will yield a formula of p_l in terms of the other unknowns.

The system has insofar 7 unknowns $(\varphi, \rho_g, C_l, p_l, u_l, u_g, T_g)$ for 5 equations: (2.5), (2.3), (2.1), (2.8), (2.9). Closing the system involves several steps. First, the physics of gas bubble growth at the microscopic scale detailed in Section 2.2 allows us to add an equation for the gas volume fraction, φ, and a closure for the mass exchange term R^{H_2O}. Next, scaling these processes up to the macroscopic scale in Section 2.3 yields in Section 2.4 a system without energy balance with 7 unknowns for 6 transport equations by adding the scaled-up equation for φ and an algebraic equation for the bubble radius, R. Finally, Section 2.5 shows how to use the energy balance to add the liquid temperature, T_l, as a final variable and establish the needed equations for both temperatures. Section 2.6 summarizes our final, closed system with 8 unknowns $(\varphi, \rho_g, C_l, p_l, u_l, u_g, T_g, T_l)$, as many transport equations, and algebraic closures for p_g and R.

2.2 Closure equations: bubble growth at the microscopic scale

To establish the closure equations for the gas volume fraction φ and the mass exchange term R^{H_2O}, we focus on the microscopic scale of one bubble submerged in the melt. According to previous work ([Plesset and Prosperetti, 1977, Arefmanesh and Advani, 1991, Proussevitch et al., 1993b, Leighton, 1994, Brennen, 1995, Toramaru, 1995, Lensky et al., 2001, Forestier-Coste et al., 2012, Mancini et al., 2016, Toramaru, 2022]), the description of bubble growth in response to pressure changes involve several processes at the microscopic level. On the other hand, scaling this description up to the macroscopic level involves additional assumptions. At the microscopic level, we adopt the common assumption that the submerged bubble is affected by two effects that controls its size. The first one is the deformation of the bubble itself by the gas expansion caused by the pressure jump at its interface with the melt. At the macroscopic level, this will yield the closure equations for φ. The pressure jump also causes an exchange of water mass between the bubble and the melt that is directly related to the evolution of the water concentration in the melt. At the macroscopic level, this will give us the definition of R^{H_2O}. An additional effect at that level is the bubble transport within the melt, which is assumed to be driven by the $-$macroscopic$-$ fluid velocity field u_l (cf. [Leighton, 2007]). Section 2.2.1 below focuses on the microscopic effects and the scaling up to the macroscopic level is presented in Section 2.3.

2.2.1 Gas volume fraction equation

Bubble volume changes due to gas expansion is described by the Rayleigh-Plesset equation that governs the dynamics of a gas bubble submerged in an infinite incompressible liquid with constant viscosity. The derivation of the Rayleigh-Plesset equation is made from the Navier-Stokes equations written in spherical coordinates, where r is the radial coordinate originating from the fixed bubble center. We thus consider the following Lagrangian

coordinates for the radial velocity $u_r = u_r(t, r(t))$,

$$\begin{cases} \dfrac{dr}{dt}(t, r_0) = u_r(t, r(t, r_0)), \\ r(t, r_0)_{|t=0} = r_0. \end{cases} \tag{2.14}$$

In this section all variables are written in these Lagrangian coordinates and for clarity we use the calligraphic notation for them to avoid confusion with the Eulerian variables. Variables represented with greek letters are adorned by a tilde and there are no changes in the notation of the constant variables. Thus, for any variable $\mathcal{F} = \mathcal{F}(t, r(t))$ we introduce the Lagrangian derivative

$$\frac{d\mathcal{F}}{dt} = \dot{\mathcal{F}} = \partial_t \mathcal{F}(t, r(t)) + u_r(t, r(t))\partial_r \mathcal{F}(t, r(t)). \tag{2.15}$$

In what follows the explicit dependence on $(t, r(t))$ is omitted for conciseness.

Rayleigh-Plesset with a bubble in an infinite medium. The Rayleigh-Plesset equation originally presented in [Scriven, 1959] is based on the analysis of the behavior of a bubble of radius $\mathcal{R}$, the center of which is assumed to be at rest. The derivation of the Rayleigh-Plesset equation is classic and can be found in the literature (e.g., [Toramaru, 2022]). For completeness we included it in Appendix A. It is obtained through the integration of the Navier-Stokes equation for the liquid, with radial velocity u_r and pressure $\mathcal{P}_l$. The most relevant point is that the dynamic mechanical equilibrium at the bubble/liquid interface given by the Young-Laplace equation is considered as the boundary condition at the bubble wall (see (A.4))

$$\mathcal{P}_g - 2\frac{\tilde{\sigma}}{\mathcal{R}} = \mathcal{P}_l - 2\eta_l \partial_r u_r, \qquad \text{at the bubble/liquid interface,} \tag{2.16}$$

where $\mathcal{P}_g$ denotes the gas pressure, $\tilde{\sigma} > 0$ is the surface tension coefficient and the gas viscosity is neglected at the interface. This condition will be also considered for our macroscopic model to determine the stress tensor at the interface appearing in the momentum equations (2.8)-(2.9), as we detail in Section 2.3.

The Rayleigh-Plesset equation when the inertial terms are neglected reads

$$\dot{\mathcal{R}} = \frac{\mathcal{R}}{4\eta_l}\left(\mathcal{P}_g - \mathcal{P}_l - \frac{2\tilde{\sigma}}{\mathcal{R}}\right). \tag{2.17}$$

where $\dot{\mathcal{R}}$ represents the radial velocity at the bubble interface. This is the equation that we consider for the evolution of the bubble growth.

Remark 2.2. *We assume herein that the microscopic and macroscopic surface tension coefficients ($\tilde{\sigma}$ and σ, Section 2.3 below) are constant. Recent advances [Hajimirza et al., 2019], however, suggest that surface tension in magmatic liquids follow:*

$$\sigma_R = \frac{\sigma_i}{1 + 2\delta_T/R}$$

where σ_R is a size-dependent surface tension coefficient, σ_i is the constant surface tension coefficient of a planar surface, and δ_T is called the Tolman length (0.32 nm in rhyolitic liquids, [Hajimirza et al., 2019]). As a result, the Rayleigh-Plesset pressure term $\frac{2\tilde{\sigma}}{\mathcal{R}}$ becomes $\frac{2\sigma_i}{\mathcal{R}+2\delta_T}$ where both σ_i and δ_T are constants.

Differential equation on the volume fraction $\tilde{\varphi}$. To extend this single-bubble model to a magma containing many bubbles, we follow the works of [Proussevitch et al., 1993b, Toramaru, 1995, Proussevitch and Sahagian, 1998, Lensky et al., 2001, Lensky et al., 2004] that have since been widely adopted (e.g., [Forestier-Coste et al., 2012, Mancini et al., 2016, Su and Huber, 2017]). The magma is considered as a set of identical spherical cells

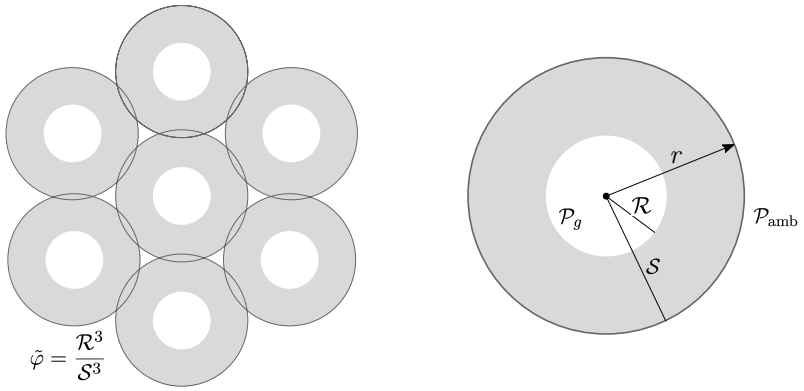

Figure 2: Configuration of a monosized bubble population in an infinite melt (adapted from [Lensky et al., 2004])

composed of a spherical gas bubble of radius $\mathcal{R}$ in a surrounding liquid melt of radius $\mathcal{S} > \mathcal{R}$, so the bubbles do not interact with each other (see Appendix A for more details and in particular, Figure 2 for a sketch and notation). The packing arrangement considers that these cells slightly overlap uniformly in the suspension in a way such that the gas volume fraction of magma is:

$$\tilde{\varphi} = \frac{\mathcal{R}^3}{\mathcal{S}^3}, \tag{2.18}$$

In order to find a continuity equation for the Lagrangian, microscopic gas volume fraction, $\tilde{\varphi}$, we use this definition to write

$$\dot{\tilde{\varphi}} = 3\frac{\mathcal{R}^2}{\mathcal{S}^2}\frac{\mathcal{S}\dot{\mathcal{R}} - \mathcal{R}\dot{\mathcal{S}}}{\mathcal{S}^2}.$$

As the liquid density ρ_l is constant, the volume of liquid at the microscopic level, $4\pi(S^3 - R^3)/3$, is conserved. We thus have $\dot{\mathcal{S}} = \frac{\mathcal{R}^2}{\mathcal{S}^2}\dot{\mathcal{R}}$, which gives:

$$\dot{\tilde{\varphi}} = 3\frac{\mathcal{R}^2}{\mathcal{S}^2}\frac{\dot{\mathcal{R}}}{\mathcal{S}}\left(1 - \frac{\mathcal{R}^3}{\mathcal{S}^3}\right) = 3\frac{\mathcal{R}^3}{\mathcal{S}^3}\left(1 - \frac{\mathcal{R}^3}{\mathcal{S}^3}\right)\frac{\dot{\mathcal{R}}}{\mathcal{R}} = 3\tilde{\varphi}(1-\tilde{\varphi})\frac{\dot{\mathcal{R}}}{\mathcal{R}}.$$

So embedding (2.17) into that last relationship yields the evolution equation of the gas volume fraction:

$$\dot{\tilde{\varphi}} = \frac{3}{4\eta_l}\tilde{\varphi}(1-\tilde{\varphi})\left(\mathcal{P}_g - \mathcal{P}_l - \frac{2\tilde{\sigma}}{\mathcal{R}}\right). \tag{2.19}$$

A discrete homogenization procedure to deduce the equation on the volume fraction $\tilde{\varphi}$ in one dimension. In this subsection, we explain how to use the method developed in the review paper, Section 3.2 [Bresch et al., 2024] (first introduced in [Burtea et al., 2023]) to deduce a volume fraction equation for a compressible mixture in one dimension (1D). We consider the 1D case with bubbles arranged in such a way that all their radii are aligned. We denote by $x(t)$ the position of the bubble interface at time t and consider the surrounding cell $[x_g(t), x_l(t)]$ with respective distances $\varepsilon_g(t) = x(t) - x_g(t)$, $\varepsilon_l(t) = x_l(t) - x(t)$ (see figure 3). We define $\alpha(t)$ as the volume fraction of the phase g in this cell that is

$$\alpha(t) = \frac{\varepsilon_g(t)}{\varepsilon_g(t) + \varepsilon_l(t)}.$$

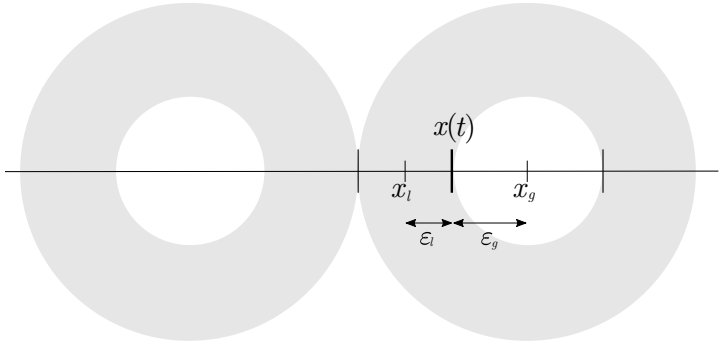

Figure 3: 1D arrangement of the bubbles for homogenization with $\varepsilon_g = R, \quad \varepsilon_l = S - R.$

Omitting the dependence on t for conciseness, we can write

$$D_t\alpha = \frac{(\varepsilon_g + \varepsilon_l)D_t\varepsilon_g - \varepsilon_g D_t(\varepsilon_g + \varepsilon_l)}{(\varepsilon_g + \varepsilon_l)^2}$$

If we adopt the notation used here, then the phase g corresponds to the gas and the phase l to the melt and

$$\varepsilon_g = \mathcal{R}, \quad \varepsilon_l = \mathcal{S} - \mathcal{R}, \quad \alpha = \frac{\mathcal{R}}{\mathcal{S}}$$

so

$$D_t\alpha = \frac{\mathcal{S}D_t(\mathcal{R}) - \mathcal{R}D_t(\mathcal{S})}{\mathcal{S}^2}$$

At the center of the cell x we consider, as for the Rayleigh-Plesset equation, the dynamic mechanical equilibrium:

$$\mathcal{P}_g - 2\eta_g(\partial_x u)_g - \frac{2\sigma}{\mathcal{R}} = \mathcal{P}_l - 2\eta_l(\partial_x u)_l. \tag{2.20}$$

Since we neglect the viscosity of the gas phase, we obtain

$$\mathcal{P}_g = \mathcal{P}_l - 2\eta_l(\partial_x u)_l + \frac{2\sigma}{\mathcal{R}} = \mathcal{P}_l - 2\eta_l \frac{D_t(\mathcal{S}) - D_t(\mathcal{R})}{\mathcal{S} - \mathcal{R}} + \frac{2\sigma}{\mathcal{R}}$$

where we approximated $(\partial_x u)_l \sim \frac{u_l - u}{\varepsilon_l}$ with $u_l = u(x_l)$. Assuming $D_t(\mathcal{R})$ is known, we obtain $D_t(S)$ from this expression to replace it in $D_t\alpha$. First,

$$D_t(\mathcal{S}) = D_t(\mathcal{R}) - \frac{\mathcal{S} - \mathcal{R}}{2\eta_l}\left(\mathcal{P}_g - \mathcal{P}_l - \frac{2\sigma}{\mathcal{R}}\right),$$

and embedding it in the previous expression of $D_t\alpha$ gives,

$$\begin{aligned} D_t\alpha &= \frac{D_t(\mathcal{R})}{\mathcal{S}} - \frac{\mathcal{R}}{\mathcal{S}^2}D_t(\mathcal{S}) = \frac{D_t(\mathcal{R})}{\mathcal{S}}\left(1 - \frac{\mathcal{R}}{\mathcal{S}}\right) + \frac{\mathcal{R}(\mathcal{S} - \mathcal{R})}{2\eta_l \mathcal{S}^2}\left(\mathcal{P}_g - \mathcal{P}_l - \frac{2\sigma}{\mathcal{R}}\right) \\ &= \alpha(1 - \alpha)\left(\frac{D_t(\mathcal{R})}{\mathcal{R}} + \frac{1}{2\eta_l}\left(\mathcal{P}_g - \mathcal{P}_l - \frac{2\sigma}{\mathcal{R}}\right)\right) \end{aligned}$$

If we identify $D_t(\mathcal{R}) = \dot{\mathcal{R}}$ and use the equation (2.17), we obtain

$$D_t\alpha = \alpha(1 - \alpha)\left(\frac{\dot{\mathcal{R}}}{\mathcal{R}} + \frac{1}{2\eta_l}\left(\mathcal{P}_g - \mathcal{P}_l - \frac{2\sigma}{\mathcal{R}}\right)\right) = \alpha(1 - \alpha)\frac{3}{4\eta_l}\left(\mathcal{P}_g - \mathcal{P}_l - \frac{2\sigma}{\mathcal{R}}\right) \tag{2.21}$$

Despite the different definition of the volume fraction α (1D) versus $\tilde{\varphi}$ (3D), we obtain the same kind of relationships (compare (2.19) and (2.21)). Remark that the two representations are equivalent when $\mathcal{R} \approx \mathcal{S}$ because the cubic configuration implied by the 1D approximation then becomes close to the spherical one and $\tilde{\varphi} \approx \alpha$.

2.2.2 Mass exchange term R^{H_2O}

Equation on the bubble number density per unit volume of melt. It is usually introduced the bubble number density per unit volume of melt, denoted by $\mathcal{N}$, that is given in terms of the total bubble number density $\tilde{n}$, as

$$\mathcal{N}(1-\tilde{\varphi}) = \tilde{n}. \tag{2.22}$$

Using previous definitions we write $\tilde{n} = \left(\frac{4\pi}{3}\mathcal{S}^3\right)^{-1} = \left(\frac{4\pi}{3}\frac{\mathcal{R}^3}{\tilde{\varphi}}\right)^{-1}$, which yields

$$\mathcal{N} = \frac{3}{4\pi\mathcal{R}^3}\frac{\tilde{\varphi}}{1-\tilde{\varphi}}. \tag{2.23}$$

The derivative of this expression is:

$$\dot{\tilde{\varphi}} = \tilde{\varphi}(1-\tilde{\varphi})\left(3\frac{\dot{\mathcal{R}}}{\mathcal{R}} + \frac{\dot{\mathcal{N}}}{\mathcal{N}}\right),$$

so thanks to (2.17) and (2.19), we obtain that the bubble number density with respect to the melt is conserved:

$$\dot{\mathcal{N}} = 0. \tag{2.24}$$

The microscopic definition of gas volume fraction implies that the magma is arranged as adjacent spherical cells where there always is melt surrounding a bubble, which stops gas bubbles to interact directly with each other. As a consequence, no coalescence/nucleation effect is considered in the system. The limit case $\tilde{\varphi} \to 1$ thus implies an infinite $\mathcal{N}$. In detail, however, the arrangement of melt cells is such that a certain overlap can occur (see Figure 2). There is thus an opportunity for two bubbles to come into contact with each other depending on the overlap distance. In the Lagrangian microscopic framework, however, the assumption of a uniform distribution (i.e. all bubbles have the same radius $\mathcal{R}$) implies that all bubbles are in contact at the same time. So, introducing coalescence/nucleation effects can only be done in the Lagrangian framework if bubbles have different sizes. For coalescence, this is discussed in [Mancini et al., 2016] for a magma containing a polydisperse distribution of bubbles.

Water mass balance at the interface of a bubble. A description of the mass exchange between the bubbles and the melt at the microscopic level is needed to establish the macroscopic closure for R^{H_2O}. In the Lagrangian framework, the driving force is the changes experimented by the pressures. In magmas, mass diffusion at the bubble interface plays a large role in controlling bubble size in the melt cells (e.g. [Plesset and Prosperetti, 1977, Arefmanesh and Advani, 1991, Proussevitch et al., 1993b, Lyakhovsky et al., 1996, Lensky et al., 2004]). Interesting macroscopic closures for the mass diffusion are found in [Lyakhovsky et al., 1996] and [Mancini et al., 2016]. The comparison of these relationships with the closure we propose and the resulting macroscopic mass exchange variable R^{H_2O} is analyzed in Appendix A.2.

Henry's law, which relates pressure and the concentration of the dissolved water in the melt ($\mathcal{C}$), is valid for silicate melts (e.g., [Burnham, 1975]). In the fluid adjacent to the bubble, Henry's law reads:

$$\mathcal{C}^{\mathrm{eq}} = k_h \mathcal{P}^{1/2}, \tag{2.25}$$

where k_h is Henry's constant. $\mathcal{P}$ at the interface is equal to $\mathcal{P}_g$ so that when there is a pressure disequilibrium in the gas, the concentration of dissolved water at the bubble wall changes [Lensky et al., 2004]. When $\mathcal{C}_{|r=\mathcal{R}} > \mathcal{C}^{\text{eq}}$, the bubble grows and when $\mathcal{C}_{|r=\mathcal{R}} < \mathcal{C}^{\text{eq}}$, the bubble shrinks [Plesset and Prosperetti, 1977, Proussevitch et al., 1993b]. The moving boundary diffusion equation of $\mathcal{C}$ is:

$$\frac{d\mathcal{C}}{dt} = \frac{1}{r^2}\partial_r(r^2 D\partial_r\mathcal{C}); \qquad \mathcal{R} \leq r \leq \mathcal{S}; \tag{2.26}$$

where, as in (2.15), $\frac{d\mathcal{C}}{dt} = \partial_t\mathcal{C} + u_r\partial_r\mathcal{C}$, and D is the volatile diffusivity, which we here consider constant in a given melt cell (see [Forestier-Coste et al., 2012] for an example of variable diffusion). This equation is completed by the following boundary conditions:

$$(\mathcal{C})_{|r=\mathcal{R}} = k_h\sqrt{\mathcal{P}_g} \quad \text{and} \quad (\partial_r\mathcal{C})_{|r=\mathcal{S}} = 0. \tag{2.27}$$

The first condition is the equilibrium at the bubble interface established by (2.25), and the second condition stipulates that the water concentration flux is zero at the outer limit of the melt cell. Notice that considering the quasi-static approximation, which is valid at small Peclet number (Peclet number is the ratio between the diffusion and viscous time scales $Pe = \frac{\theta_d}{\theta_v}$, where $\theta_d = R^2/D$, $\theta_v = \eta/\Delta P.$), equation (2.26) reduces to $\frac{1}{r^2}\partial_r(r^2 D\partial_r\mathcal{C}) = 0$, which in turn leads to the equilibrium solution (2.25).

To study the water mass exchange at the interface bubble level, we define the bubble gas mass as:

$$\mathcal{M}_g = \frac{4\pi}{3}\tilde{\rho}_g\mathcal{R}^3. \tag{2.28}$$

The conservation of the total water in the cell states that the sum of the water in the bubble and the water dissolved in the melt must remain constant in time:

$$\mathcal{M}_g + \int_{\mathcal{R}}^{\mathcal{S}} 4\pi r^2\rho_l\mathcal{C}dr = \frac{4\pi}{3}\mathcal{S}_0^3\mathcal{C}_0\rho_l. \tag{2.29}$$

Here $\mathcal{S}_0$ represents the radius of the influence region for a zero radius bubble, $\mathcal{S}_0^3 = \mathcal{S}^3 - \mathcal{R}^3$, that is constant in time, and $\mathcal{C}_0$ is the constant water concentration in the melt for a bubble of radius zero. We rewrite this equation in radial coordinates:

$$\frac{d\mathcal{M}_g}{dt} + \int_{\mathcal{R}}^{\mathcal{S}} \frac{d}{dt}(4\pi r^2\rho_l\mathcal{C})dr = 0.$$

Using equation (2.26) to write $\frac{d\mathcal{C}}{dt} = \frac{1}{r^2}\partial_r(r^2 D\partial_r\mathcal{C})$ yields that

$$\frac{d\mathcal{M}_g}{dt} + 4\pi\mathcal{S}^2\rho_l D(\partial_r\mathcal{C})_{|r=\mathcal{S}} - 4\pi\mathcal{R}^2\rho_l D(\partial_r\mathcal{C})_{|r=\mathcal{R}} = 0.$$

Using the previous boundary condition at the limit of the cell $r = \mathcal{S}$ in (2.27), we obtain the water mass balance at the bubble interface:

$$\frac{d\mathcal{M}_g}{dt} = 4\pi\mathcal{R}^2\rho_l D(\partial_r\mathcal{C})_{|r=\mathcal{R}}, \tag{2.30}$$

which represents the accumulation of water in the bubble by diffusion. On the right hand side, the term $D(\partial_r\mathcal{C})_{|r=\mathcal{R}}$ represents the diffusion flux of dissolved water from the melt to the gas bubble at the interface (Fick's law) [Arefmanesh and Advani, 1991]. So $\rho_l D(\partial_r\mathcal{C})_{|r=\mathcal{R}}$ is the mass flux at the interface per unit area and it depends on the concentration gradient at the bubble interface.

The concentration gradient at the interface is given by equation (2.26)-(2.27). This is an issue because the resolution of this equation is not direct. As a result, several analysis have been performed in order to find a profile

for the water concentration $\mathcal{C}$ in different dynamic regimes [Proussevitch et al., 1993b, Arefmanesh and Advani, 1991, Lensky et al., 2004, Lyakhovsky et al., 1996, Forestier-Coste et al., 2012, Mancini et al., 2016].

Here we propose a simple approximation of the concentration gradient using Taylor's formula:

$$\mathcal{C}(r) = (\mathcal{C})_{|r=\mathcal{R}} + (\partial_r \mathcal{C})_{|r=\mathcal{R}}(r-\mathcal{R}) + \mathcal{O}((r-\mathcal{R})^2) \Rightarrow (\partial_r \mathcal{C})_{|r=\mathcal{R}} \sim \frac{\mathcal{C}(r) - (\mathcal{C})_{|r=\mathcal{R}}}{r-\mathcal{R}}.$$

By evaluating this expression at the outer melt boundary ($r = \mathcal{S}$), we find

$$(\partial_r \mathcal{C})_{|r=\mathcal{R}} \sim \frac{(\mathcal{C})_{|r=\mathcal{S}} - k_h\sqrt{\mathcal{P}_g}}{\mathcal{S}-\mathcal{R}}. \tag{2.31}$$

The value $(\mathcal{C})_{|r=S}$ can be considered as a water concentration independent of r that we must define. The closure of the concentration gradient given by (2.31) determines the water mass balance at the bubble interface thanks to equation (2.30). See Appendix A.2 for comparison with the closures proposed in [Lyakhovsky et al., 1996] and [Mancini et al., 2016].

Remark 2.3. *We would like to analyze the behavior of the rate of mass exchange $\frac{d\mathcal{M}_g}{dt}$ at limit cases that are relevant at the macroscopic scale. It is desirable that $\frac{d\mathcal{M}_g}{dt}$ tends to 0 when $\mathcal{R}$ tends to 0 since in this case there would be no gas in the cell and thus no exchange happens. Conversely, when $\mathcal{R}$ tends to $\mathcal{S}$, there is only gas in the cell, so it is convenient that $\frac{d\mathcal{M}_g}{dt}$ to tends to ∞. In other words, we want the following limits to be satisfied:*

$$\lim_{\mathcal{R}\to 0} \frac{d\mathcal{M}_g}{dt} = 0 \qquad \textit{and} \qquad \lim_{\mathcal{R}\to \mathcal{S}} \frac{d\mathcal{M}_g}{dt} = \infty. \tag{2.32}$$

For the proposed approach (2.31), (2.30) *reads*

$$\frac{d\mathcal{M}_g}{dt} = 4\pi\rho_l D \frac{\mathcal{R}^2}{\mathcal{S}-\mathcal{R}}\Big((\mathcal{C})_{|r=\mathcal{S}} - k_h\sqrt{\mathcal{P}_g}\Big)$$

for any fixed value of $(\mathcal{C})_{|r=S}$. We easily check that both limits are achieved. □

Macroscopic mass exchange rate. The gas mass exchange at the bubble level needs to be scaled up to the macroscopic framework in order to find an expression for R^{H_2O}, which appears for instance in the mass balance (2.4) of the concentration of water in the melt C_l. As all the calculations done in this section involve variables written in Lagrangian coordinates (2.14), we also introduce the notation $\mathcal{R}^{H_2O}$ for the exchange rate in Lagrangian coordinates. Since $\frac{d\mathcal{M}_g}{dt}$ is the water mass exchange rate over the surface of one bubble, we may identify

$$\mathcal{R}^{H_2O} = \tilde{n}\frac{d\mathcal{M}_g}{dt} \tag{2.33}$$

where $\tilde{n}$ is the total bubble number density, which for uniform cells − the layout considered above − is defined as $\tilde{n} = \left(\frac{4\pi}{3}\mathcal{S}^3\right)^{-1}$. Equation (2.30) describes the rate of $\mathcal{M}_g$ that we remind here, $\frac{d\mathcal{M}_g}{dt} = 4\pi\mathcal{R}^2\rho_l D(\partial_r\mathcal{C})_{r=\mathcal{R}}$, where D is the volatile diffusivity in the melt cell. Thus we have a first expression for $\mathcal{R}^{H_2O}$:

$$\mathcal{R}^{H_2O} = 4\pi\rho_l D\mathcal{R}^2\tilde{n}\frac{(\mathcal{C})_{|r=\mathcal{S}} - k_h\sqrt{\mathcal{P}_g}}{\mathcal{S}-\mathcal{R}}.$$

Using (2.18) and $\tilde{n} = \left(\frac{4\pi}{3}\mathcal{S}^3\right)^{-1}$, we get

$$\mathcal{R}^{H_2O} = 4\pi\rho_l D\tilde{n}\left(\frac{3}{4\pi\tilde{n}}\right)^{1/3}\tilde{\varphi}^{2/3}\frac{(\mathcal{C})_{|r=\mathcal{S}} - k_h\sqrt{\mathcal{P}_g}}{1-\tilde{\varphi}^{1/3}}.$$

This rate may also be written in terms of bubble number density per unit melt $\mathcal{N}$ (see equation (2.22)):

$$\mathcal{R}^{H_2O} = 4\pi\mathcal{R}^2\mathcal{N}(1-\varphi)\rho_l D\frac{(\mathcal{C})_{|r=\mathcal{S}} - k_h\sqrt{\mathcal{P}_g}}{\mathcal{S}-\mathcal{R}}, \tag{2.34}$$

which can be understood as the diffusion rate of water over the interface surface $4\pi R^2\mathcal{N}(1-\varphi)$ divided by a measure of the interbubble spacing $S-R$. In order to reduce the number of variables in the macroscopic model, we use the following form that no longer involves R and S:

$$\mathcal{R}^{H_2O} = 3^{1/3}\rho_l D\left(4\pi\mathcal{N}\tilde{\varphi}(1-\tilde{\varphi})\right)^{2/3}\frac{(\mathcal{C})_{|r=\mathcal{S}} - k_h\sqrt{\mathcal{P}_g}}{1-\tilde{\varphi}^{1/3}}. \tag{2.35}$$

Similarly to $\frac{d\mathcal{M}_g}{dt}$, we would like the same behavior at the limits for $\mathcal{R}^{H_2O}$ (see equation (2.32)). So we search for

$$\lim_{\tilde{\varphi}\to 0}\mathcal{R}^{H_2O} = 0 \qquad \text{and} \qquad \lim_{\tilde{\varphi}\to 1}\mathcal{R}^{H_2O} = \infty. \tag{2.36}$$

Thanks to (2.33) and taking into account that $\tilde{n} = (1-\tilde{\varphi})\mathcal{N}$, the limit when $\tilde{\varphi}$ tends to zero (or equivalently $\mathcal{R}\to 0$) still holds. Conversely, when $\tilde{\varphi}$ tends to one (or equivalently $\mathcal{R}\to\mathcal{S}$), we have:

$$\begin{aligned}
\lim_{\tilde{\varphi}\to 1}\mathcal{R}^{H_2O} &= 3^{1/3}\rho_l D(4\pi\mathcal{N})^{2/3}((\mathcal{C})_{|r=\mathcal{S}} - k_h\sqrt{\mathcal{P}_g})\lim_{\tilde{\varphi}\to 1}\frac{(1-\tilde{\varphi})^{2/3}}{1-\tilde{\varphi}^{1/3}} \\
&= 3^{1/3}\rho_l D(4\pi\mathcal{N})^{2/3}((\mathcal{C})_{|r=\mathcal{S}} - k_h\sqrt{\mathcal{P}_g})\lim_{\tilde{\varphi}\to 1}\frac{(1+\tilde{\varphi}^{1/3}+\tilde{\varphi}^{2/3})^{2/3}}{(1-\tilde{\varphi}^{1/3})^{1/3}} \\
&= \infty
\end{aligned}$$

since $(1-\tilde{\varphi}) = (1-\tilde{\varphi}^{1/3})(1+\tilde{\varphi}^{1/3}+\tilde{\varphi}^{2/3})$.

In Appendix A.2 we present the equivalent exchange terms R^{H_2O} when the approaches in [Lyakhovsky et al., 1996] and [Mancini et al., 2016] are considered. In Part II, Section 6.1 [Burgisser et al., 2024], we compare the influence on flow dynamics of the proposed mass exchange R^{H_2O} to those proposed by [Lyakhovsky et al., 1996] and [Mancini et al., 2016], which are given respectively in equations (A.23) and (A.24). We find that, in the flow conditions studied, the [Lyakhovsky et al., 1996] approach and ours yield similar results. In comparison, the [Mancini et al., 2016] approach overestimates the mass exchange rate at small gas volume fraction and overestimates it at high gas volume fraction. This result is consistent with the limits established in Remarks 2.3 and Appendix A.2.

2.3 From microscopic to macroscopic closures

In the macroscopic model, all quantities are averaged locally, so for example, R is the averaged local radius of the bubbles and M_g is the averaged local bubble mass. We use the microscopic equations developed above to complete the system (2.5)-(2.13) by determining closure equations for φ and R^{H_2O}. As established microscopically, these quantities depend on $\mathcal{R}$ and $\mathcal{N}$, which need to be expressed as macroscopic variables of the system. The microscopic bubble growth equations were established in terms of the radial coordinate, so all the variables involved were functions of $(t, r(t))$. To write such variables in terms of the Eulerian coordinates, $(t, x(t))$, we assume that the bubbles of the microscopic configuration are transported through the velocity field of the liquid phase, u_l (see [Leighton, 2007]). Thus, for example, for the bubble radius $\mathcal{R}(t)$, we define

$$R(t, x(t)) = \mathcal{R}(t) \tag{2.37}$$

and then $\dot{\mathcal{R}} = \frac{d\mathcal{R}}{dt} = \partial_t R + u_l \cdot \nabla R$. The same reasoning for the pressures, bubble number densities and volume fraction yields:

$$p_g(t, x(t)) = \mathcal{P}_g(t), \quad p_l(t, x(t)) = \mathcal{P}_l(t), \quad N(t, x(t)) = \mathcal{N}(t), \quad n(t, x(t)) = \tilde{n}(t),$$
$$\rho_g(t, x(t)) = \tilde{\rho}_g(t), \quad \varphi(t, x(t)) = \tilde{\varphi}(t), \quad \sigma(t, x(t)) = \tilde{\sigma}(t), \quad R^{H_2O}(t, x(t)) = \mathcal{R}^{H_2O}(t).$$

The closure equations are then given by (2.17), (2.19), (2.22), and (2.35), which we write here for clarity in Cartesian coordinates:

$$\partial_t R + u_l \cdot \nabla R = \frac{R}{4\eta_l} \left(p_g - p_l - \frac{2\sigma}{R} \right), \tag{2.38a}$$

$$\partial_t \varphi + u_l \cdot \nabla \varphi = \frac{3}{4\eta_l} \varphi(1-\varphi) \left(p_g - p_l - \frac{2\sigma}{R} \right), \tag{2.38b}$$

$$n = N(1-\varphi), \tag{2.38c}$$

and the expression of R^{H_2O} is

$$R^{H_2O} = 3^{1/3} \rho_l D \left(4\pi N \varphi(1-\varphi)\right)^{2/3} \frac{C_l - k_h\sqrt{p_g}}{1-\varphi^{1/3}}. \tag{2.38d}$$

To write R^{H_2O}, we identified that $(\mathcal{C})_{|r=\mathcal{S}}$ is C_l, the dissolved water concentration in the liquid. The equivalent equilibrium water concentration at saturation associated with Henry's law (2.25) is:

$$\mathcal{C}_l^{\text{eq}} = k_h \sqrt{p_g}. \tag{2.39}$$

The microscopic bubble number density, $\mathcal{N}$, becomes at the macroscopic scale a local number density, N, that thanks to (2.24) is a transported variable. We assume that N is constant, so if initially

$$(R^3)_{|t=0} = \Big(\frac{3}{4\pi N} \frac{\varphi}{1-\varphi} \Big)_{|t=0},$$

then equation (2.23) yields an algebraic relationship for the bubble radius:

$$R^3 = \frac{3}{4\pi N} \frac{\varphi}{1-\varphi}, \tag{2.40}$$

This removes the need to include R as a variable controlled by (2.38a). If N is not constant, R and (2.38a) need to be considered instead.

Remark 2.4. *When $\varphi \to 1$, equation* (2.40) *implies that $R \to \infty$. At high φ values, however, the monosized bubble population configuration of Figure 2 is no longer valid. Bubbles are so close to each other that they deform, adopting a geometry made of polyhedral shapes made of inter-bubble films (nearly planar surfaces) connected by thicker wedges of liquid called Plateau borders. These geometries have been described in natural magmas; such magma fragments (pyroclasts) are called reticulite and can reach φ values of 0.95–0.99 [Mangan and Cashman, 1996]. There is thus no longer a single length scale involved, and the dynamics of such high porosity foams is generally described by considering planar films and Plateau borders separately [Proussevitch et al., 1993a]. Equation* (2.40) *has thus a limited applicability above the critical volume fraction for fragmentation $\varphi_c \sim 0.8$ (see bubbly and permeable flow drag relationships* (2.12)*). Our focus on permeable flow implies that our model best represents situations propitious to bubbles interconnection and percolation (e.g., [Castro et al., 2012]), which prevents reticulite formation.*

Now the system has 7 unknowns $(\varphi, \rho_g, C_l, p_l, u_l, u_g, T_g)$ for 6 equations ((2.5), (2.3), (2.1), (2.8), (2.9), (2.38b)) with N constant and the variables n, R^{H_2O}, and R given by equations (2.38c), (2.38d), and (2.40), respectively. The new equation (2.38b) for φ is important because it will later (cf. Section 2.6) allow us to characterize p_l in terms of the other unknowns using that ρ_l is constant.

We must still define $\langle p_{int,l}\mathrm{Id} - \mathcal{D}_{int,l}\rangle$, $\langle p_{int,g}\mathrm{Id} - \mathcal{D}_{int,g}\rangle$, $\langle u_I\rangle$, and an equation for the gas temperature T_g. To establish a closure relationship of the tensor at the gas–liquid interface, we follow the same condition as that used in the microscopy study, which is the dynamic mechanical equilibrium stipulating the balance of the interface forces as in the Rayleight-Plesset equation (A.4), and the homogenization procedure (2.20). For the macroscopic situation, this condition is written as:

$$\langle p_{int,g}\mathrm{Id} - \mathcal{D}_{int,g}\rangle - 2\sigma\langle H\rangle = \langle p_{int,l}\mathrm{Id} - \mathcal{D}_{int,l}\rangle \tag{2.41}$$

where $\langle H\rangle$ is the mean interface curvature. As bubbles are spherical, the radius of curvature is taken as R and the mean curvature becomes $\langle H\rangle = 1/R$. Ignoring as before the gas viscosity at the interface, we get

$$\langle p_{int,g}\mathrm{Id} - \mathcal{D}_{int,g}\rangle = \langle p_{int,g}\rangle, \qquad \langle p_{int,l}\mathrm{Id} - \mathcal{D}_{int,l}\rangle = \langle p_{int,g}\rangle - \frac{2\sigma}{R}. \tag{2.42}$$

where $\langle p_{int,g}\rangle$ must still be specified. We assume that the interface pressure $\langle p_{int,g}\rangle$ and exchange velocity $\langle u_I\rangle$ are convex combinations of gas and liquid pressures and velocities, respectively, through two constants, a and b:

$$\langle p_{int,g}\rangle = ap_g + (1-a)p_l. \tag{2.43}$$

By analogy, the exchange velocity must be such that $\langle u_I\rangle = u_g = u_l$ when $u_g = u_l$:

$$\langle u_I\rangle = bu_g + (1-b)u_l. \tag{2.44}$$

The values of the coefficients a, b and the closure equation for T_g will be defined in Section 2.5 to ensure a dissipative energy balance of the system.

Proposition 2.1. *Let consider a constant liquid density ρ_l. Then we have the following relation*

$$\operatorname{div} u_l = \varphi\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right) - \frac{R^{H_2O}}{\rho_l(1-\varphi)} \tag{2.45}$$

where from (2.38d)

$$R^{H_2O} = \alpha\rho_l\frac{((1-\varphi)\varphi)^{2/3}}{1-\varphi^{1/3}}(C_l - k_h\sqrt{p_g}) \quad \textit{with} \quad \alpha = 3^{1/3}(4\pi N)^{2/3}D. \tag{2.46}$$

Proof. From equations (2.5) and (2.38b) the result comes directly using that $\rho_l = \mathrm{cte}$. □

Remark 2.5. *If we neglect surface tension and set $R^{H_2O} = 0$ in* (2.45)*, we get a dilatancy-like constraint*

$$\operatorname{div} u_l = \frac{3\varphi}{4\eta_l}(p_g - p_l) \tag{2.47}$$

where $p_g = c_0\rho_g T_g$ and p_l is an unknown encoding that one phase is incompressible at the macroscopic scale. More precisely, if $p_l > p_g$ then $\operatorname{div} u_l < 0$ and if $p_l \le p_g$ then $\operatorname{div} u_l \ge 0$. □

Bounds on the volume fraction φ. Even if equation (2.40) is not an accurate descriptor of the bubble network geometry above a critical gas volume fraction (see Remark 2.4), it is important that our system maintains a consistent behavior even when $\varphi \to 1$. We start studying the bounds of φ by assuming that at time $t = 0$ there is $\varepsilon > 0$ small enough such that $\varphi_0 \in [\varepsilon, 1 - \varepsilon]$. Assuming then that the flow related to u_l is regular and that $p_g - p_l$ bounded, two cases arise:

- If $\sigma \geq 0$, then for all T fixed there exists $\delta > 0$ such that $\varphi \in [0, 1 - \delta]$ on $(0, T) \times \Omega$.
- If $\sigma > 0$, then for all T fixed, there exists $\delta > 0$ such that $\varphi \in [\delta, 1 - \delta]$ almost everywhere on $(0, T) \times \Omega$.

Formal proof. The quantity $\varphi/(1 - \varphi)$ plays a crucial role in the proof. Remark first that

$$\frac{1}{\varphi(1-\varphi)} = \frac{1}{\varphi} + \frac{1}{1-\varphi}.$$

Dividing (2.38b) by $\varphi(1 - \varphi)$ yields

$$\partial_t \log\left(\frac{\varphi}{1-\varphi}\right) + u_l \cdot \nabla \log\left(\frac{\varphi}{1-\varphi}\right) = \frac{3}{4\eta_l}\left(p_g - p_l - 2\frac{\sigma}{R}\right).$$

A change of variable can be done by defining that

$$\Phi = \log\Big(\varphi/(1-\varphi)\Big)$$

When depending on Φ, the equation to solve becomes:

$$\partial_t \Phi + u_l \cdot \nabla \Phi = \frac{3}{4\eta_l}\Big(p_g - p_l - 2\frac{\sigma}{R}\Big) \qquad \text{with} \qquad R^3 = \frac{3}{4\pi N}\exp\Phi. \tag{2.48}$$

Recall that the characteristics $\xi_l(t, x)$ associated to u_l are defined as being a solution of

$$\partial_t \xi_l(t, x) = u_l(t, \xi_l(t, x)), \qquad \xi_l(0, x) = x.$$

This definition allows us switch to Lagrangian coordinates related to the velocity u_l. In other words, the characteristics $\xi_l(t, x)$ related to u_l allows us to rewrite equation (2.48) (and thus (2.38b)) with the new unknown Φ as:

$$\partial_t(\Phi(t, \xi_l(t, x))) = \frac{3}{4\eta_l}\left(p_g - p_l - 2\frac{\sigma}{R}\right)(t, \xi_l(t, x)) \tag{2.49}$$

with

$$R^3(t, \xi_l(t, x)) = \frac{3}{4\pi N}(\exp\Phi)(t, \xi_l(t, x)).$$

This is a nonlinear ODE in time governing Φ. As $\sigma/R \geq 0$, integrating (2.49) in time from 0 to any t yields

$$\Phi(t, \xi_l(t, x)) \leq \Phi(0, x) + \int_0^t \frac{3}{4\eta_l}\left(p_g - p_l\right)(\tau, \xi_l(\tau, x))\, d\tau.$$

for $\Phi_0 = \Phi_{|t=0} = \log(\varphi_0/(1 - \varphi_0)) \leq C < +\infty$. This bound on Φ_0 is obtained because we assume there exists $\varepsilon > 0$ such that $\varphi_0 \in (\varepsilon, 1 - \varepsilon)$. Therefore

$$\sup_{t\in(0,T)} \sup_{x\in\Omega} \Phi(t, \xi_l(t, x)) \leq C < +\infty \tag{2.50}$$

if initially we assume $\int_0^T (\sup_x |p_g - p_l|)dt < +\infty$. Assume that we have solved the equation related to Φ in the Lagrangian coordinates. Defining φ such that

$$\varphi/(1 - \varphi) = \exp\Phi \tag{2.51}$$

then $\varphi \in [0,1]$ because $\exp\Phi \geq 0$. Using that Φ is bounded by (2.50), then $\exp\Phi$ is bounded, which provides from 2.51 that φ is far from 1 (i.e. there exits $\delta > 0$ small enough such that $\varphi < 1-\delta$).

We now assume that $\sigma > 0$. Going back to (2.48), integrating it in time from 0 to any t yields

$$\Phi(t,\xi_l(t,x)) + 2\sigma \int_0^t \frac{1}{R(\tau,\xi_l(\tau,x))}\, d\tau = \Phi(0,x) + \int_0^t \frac{3}{4\eta_l}\left(p_g - p_l\right)(\tau,\xi_l(\tau,x))\, d\tau,$$

which gives more constraints than (2.50). Using the expression of R with respect to Φ provides the bound

$$\int_0^t \sup_x \Big(\exp(-\Phi)(t,\xi_l(t,x))\Big)^{1/3} d\tau \leq C < +\infty,$$

which implies that there exists $M{>}0$ such that $(-\Phi) < M < +\infty$ almost everywhere. Therefore, coming back to the definition $\varphi/(1-\varphi) = \exp\Phi$, we deduce that there exists $\delta > 0$ such that $\varphi > \delta$ almost everywhere.

Importantly, (2.38b) is not used to compute φ but to express p_l as a function of R^{H_2O}, $\operatorname{div} u_l$ and p_g. Equation (2.5) is used to calculate φ with the expression (2.38d). To ensure that $\varphi \in [0,1]$ starting with $\varphi_{|t=0} \in (0,1)$, we have to consider the relationship (2.45) and therefore when φ is close to 1 we have

$$R^{H_2O} = \mathcal{O}((1-\varphi)).$$

Conversely, when φ is close to 0, (2.38d) implies that $R^{H_2O} = \mathcal{O}(\varphi)$. This property is ensured at the continuous level.

Remark 2.6. *As noted in Remark 2.2, recent results suggest that the surface tension coefficient depends on bubble size, which would change the Rayleigh-Plesset pressure term from $\frac{2\sigma}{R}$ to $\frac{2\sigma_i}{R+2\delta_T}$ where both σ_i and δ_T are constants. This does change the reasoning above and we cannot conclude φ to be far from 0.*

2.4 Resulting macroscopic system without energy balance

The system at this point has 7 unknowns $(\varphi, \rho_g, C_l, p_l, u_l, u_g, T_g)$ for 6 equations ((2.5), (2.3), (2.1), (2.8), (2.9), (2.40), (2.38b)) with R^{H_2O} given by (2.38d), $\langle p_{int,l}\mathrm{Id} - \mathcal{D}_{int,l}\rangle$ given by (2.42), and $\langle p_{int,g}\rangle, \langle u_I\rangle$ given by equations (2.43)-(2.44). The coefficients a, b of these last two equations need to be determined and we must also give a closure for the gas temperature T_g in (2.13). Variable p_g is given by (2.13), and R is given by (2.40). In summary, the system is thus:

$$\partial_t((1-\varphi)\rho_l) + \operatorname{div}((1-\varphi)\rho_l u_l) = -R^{H_2O} \tag{2.52a}$$

$$\partial_t(\varphi\rho_g) + \operatorname{div}(\varphi\rho_g u_g) = R^{H_2O} \tag{2.52b}$$

$$\partial_t((1-\varphi)\rho_l(1-C_l)) + \operatorname{div}((1-\varphi)\rho_l(1-C_l)u_l) = 0 \tag{2.52c}$$

$$\begin{aligned}&\partial_t((1-\varphi)\rho_l u_l) + \operatorname{div}((1-\varphi)\rho_l u_l \otimes u_l) + \nabla((1-\varphi)p_l) - \operatorname{div}((1-\varphi)\mathcal{D}_l)\\ &- \left(\langle p_{int,g}\rangle - \frac{2\sigma}{R}\right)\nabla(1-\varphi) - K_d\,\varphi(1-\varphi)(u_g - u_l) - (1-\varphi)\rho_l g + \langle u_I\rangle R^{H_2O} = 0\end{aligned} \tag{2.52d}$$

$$\begin{aligned}&\partial_t(\varphi\rho_g u_g) + \operatorname{div}(\varphi\rho_g u_g \otimes u_g) + \nabla(\varphi p_g) - \operatorname{div}(\varphi\mathcal{D}_g)\\ &- \langle p_{int,g}\rangle\nabla\varphi + K_d\,\varphi(1-\varphi)(u_g - u_l) - \varphi\rho_g g - \langle u_I\rangle R^{H_2O} = 0\end{aligned} \tag{2.52e}$$

$$\partial_t \varphi + u_l \cdot \nabla \varphi = \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right) \tag{2.52f}$$

Locally, the gas volume fraction φ can also be related to the amount of water dissolved in the melt, C_l, and to the amount of gas exsolved. Defining the total amount of water, C_T, as:

$$C_T = \frac{\varphi\rho_g + (1-\varphi)\rho_l C_l}{\rho}, \tag{2.53}$$

allows us to express the gas volume fraction with a classical mass balance that will be useful when establishing a relaxed system in Section 5:

$$\varphi = \frac{\rho_l(C_l - C_T)}{\rho_l(C_l - C_T) + \rho_g(C_T - 1)}.$$

From the definition of C_T we can find a nice relation with C_l that serves to write the continuity equation for the total water content. We have that

$$(1-\varphi)\rho_l(1-C_l) = (1-\varphi)\rho_l - \rho C_T + \varphi\rho_g = \rho(1-C_T).$$

Thus from equation (2.52c) we write

$$\partial_t(\rho(1-C_T)) + \mathrm{div}(\rho(1-C_T)u_l) = 0. \tag{2.54}$$

The convective part can also be developed using the total mass conservation as follows

$$\begin{aligned}\partial_t(\rho(1-C_T)) + \mathrm{div}(\rho(1-C_T)u_l) &= (1-C_T)\mathrm{div}(\rho(u_l-u)) - \rho(\partial_t C_T + u_l\cdot\nabla C_T)\\ &= (1-C_T)\mathrm{div}(\varphi\rho_g(u_l-u_g)) - \rho(\partial_t C_T + u_l\cdot\nabla C_T),\end{aligned}$$

so we can write

$$\partial_t C_T + u_l\cdot\nabla C_T = \frac{1-C_T}{\rho}\mathrm{div}(\varphi\rho_g(u_l-u_g)). \tag{2.55}$$

Equation (2.54) or (2.55) may replace (2.52c) when the unknown C_T is considered instead of C_l.

2.5 Energy of the system

We use the energy balance to constrain $\langle p_{int,g}\rangle$, $\langle u_I\rangle$ and thus find a closure for the gas temperature T_g. Note that our approach differs from the classical one that uses fundamental principles of thermodynamics to establish the heat equation. We refer the reader to Appendix B for a justification of the temperature equations proposed herein and how they relate to the thermodynamic theory. This is done by proving the following proposition.

Proposition 2.2. *Let $(\varphi, \rho_g, C_l, N, p_l, u_l, u_g, T_g, T_l)$ be a smooth enough solution of the system* (2.52a)–(2.52f) *with the following equations*

$$\begin{aligned}\partial_t(\varphi c_{pg}\rho_g T_g) + \mathrm{div}(\varphi c_{pg}\rho_g T_g u_g) - \varphi\left(\partial_t\left(p_g - \frac{\boldsymbol{2\sigma}}{\boldsymbol{R}}\right) + u_g\cdot\nabla p_g\right) - \gamma(T_g - T_l) - \mathrm{div}(\varphi\kappa_g\nabla T_g)\\ -\varphi\left(2\eta_g D(u_g):\nabla u_g + \lambda_g(\mathrm{div}u_g)^2\right) = 0\end{aligned} \tag{2.56}$$

and

$$\begin{aligned}\partial_t((1-\varphi)c_{p_l}\rho_l T_l) + \mathrm{div}((1-\varphi)c_{p_l}\rho_l T_l u_l) - \frac{p_l}{\rho_l}R^{H_2O} + \gamma(T_g - T_l) - \mathrm{div}((1-\varphi)\kappa_l\nabla T_l)\\ -(1-\varphi)\left(2\eta_l D(u_l):\nabla u_l + \lambda_l(\mathrm{div}u_l)^2\right) = 0,\end{aligned} \tag{2.57}$$

where c_{pg}, c_{pl} are the constant heat capacities (at constant pressure), γ is the heat exchange coefficient (either constant or function of the existing variables), and κ_g, κ_l are the thermal conductivity coefficients. Then we have the following conservation of energy

$$
\begin{aligned}
&\partial_t \Bigg((1-\varphi)\rho_l \frac{|u_l|^2}{2} + \varphi\rho_g \frac{|u_g|^2}{2} \\
&- (g\cdot X)((1-\varphi)\rho_l + \varphi\rho_g) + \varphi(c_{pg}-c_0)\rho_g T_g + (1-\varphi)\rho_l c_{pl} T_l + \varphi\frac{2\sigma}{R} \Bigg) \\
&+ \operatorname{div}\Big(\varphi c_{pg}\rho_g T_g u_g + (1-\varphi)c_{p_l}\rho_l T_l u_l + (1-\varphi)p_l u_l\Big) \\
&+ \operatorname{div}\left((1-\varphi)\rho_l u_l \frac{|u_l|^2}{2} + \varphi\rho_g u_g \frac{|u_g|^2}{2} - (g\cdot X)((1-\varphi)\rho_l u_l + \varphi\rho_g u_g) \right) \\
&- \operatorname{div}\Big(\varphi\kappa_g \nabla T_g + (1-\varphi)\kappa_l \nabla T_l\Big) \\
&- \operatorname{div}\Big((1-\varphi)u_l(\lambda_l \operatorname{div} u_l \mathrm{Id} + 2\eta_l D(u_l)) + \varphi u_g(\lambda_g \operatorname{div} u_g \mathrm{Id} + 2\eta_g D(u_g))\Big) \\
&+ K_d\varphi(1-\varphi)|u_g-u_l|^2 + \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right)^2 + \frac{1}{2}|u_g-u_l|^2 |R^{H_2O}|\delta = 0.
\end{aligned}
\tag{2.58}
$$

where $\delta \in \{0,1\}$ is a unifying coefficient defined below in (2.67)*. Recall that σ is a positive constant and that R is related to φ through the formula* (2.40)*,*

$$R^3 = \frac{3}{4\pi N}\frac{\varphi}{1-\varphi}$$

and therefore $2\sigma\varphi/R$ is a positive quantity.

Proof. To study the associated energy balance we multiply the momentum equations for respective velocities and sum up the results. We detail for instance the calculations for the fluid phase, by multiplying equation (2.52d) by u_l. For the convective terms we develop (2.52d) $\cdot\, u_l$ as follows:

$$u_l\cdot\Big(\partial_t((1-\varphi)\rho_l u_l) + \operatorname{div}((1-\varphi)\rho_l u_l \otimes u_l)\Big) = (1-\varphi)\rho_l \frac{1}{2}(\partial_t|u_l|^2 + u_l\cdot\nabla|u_l|^2) + |u_l|^2(\partial_t((1-\varphi)\rho_l + \operatorname{div}((1-\varphi)\rho_l u_l))$$

We use (2.52a) in the last term above and in the time derivative to obtain:

$$
\begin{aligned}
&u_l\cdot\Big(\partial_t((1-\varphi)\rho_l u_l) + \operatorname{div}((1-\varphi)\rho_l u_l\otimes u_l)\Big) \\
&\quad = \partial_t((1-\varphi)\rho_l \tfrac{1}{2}|u_l|^2) - \tfrac{1}{2}|u_l|^2(-R^{H_2O} - \operatorname{div}((1-\varphi)\rho_l u_l)) + (1-\varphi)\rho_l \tfrac{1}{2}u_l\cdot\nabla|u_l|^2 - |u_l|^2 R^{H_2O},
\end{aligned}
$$

and we finally have

$$u_l\cdot\Big(\partial_t((1-\varphi)\rho_l u_l) + \operatorname{div}((1-\varphi)\rho_l u_l\otimes u_l)\Big) = \partial_t\left((1-\varphi)\rho_l\frac{|u_l|^2}{2}\right) + \operatorname{div}\left((1-\varphi)\rho_l u_l\frac{|u_l|^2}{2}\right) - \frac{|u_l|^2}{2}R^{H_2O} \tag{2.59}$$

The viscosity term can be written as:

$$
\begin{aligned}
u_l\operatorname{div}((1-\varphi)\mathcal{D}_l) &= u_l\nabla((1-\varphi)\lambda_l\operatorname{div} u_l) + 2u_l\operatorname{div}((1-\varphi)\eta_l D(u_l)) \\
&= \operatorname{div}((1-\varphi)\lambda_l u_l \operatorname{div} u_l) \\
&\quad - (1-\varphi)\lambda_l|\operatorname{div} u_l|^2 + 2\operatorname{div}((1-\varphi)\eta_l u_l D(u_l)) - 2(1-\varphi)\eta_l D(u_l):\nabla u_l \\
&= \operatorname{div}((1-\varphi)u_l(\lambda_l\operatorname{div} u_l\mathrm{Id} + 2\eta_l D(u_l))) - (1-\varphi)(\lambda_l|\operatorname{div} u_l|^2 + 2\eta_l D(u_l):\nabla u_l)
\end{aligned}
\tag{2.60}
$$

To develop the gravity term, we use the space variable denoted by X and (2.52a):

$$\begin{aligned}(1-\varphi)\rho_l u_l g &= (1-\varphi)\rho_l u_l \mathrm{div}(g\cdot X)\\ &= \mathrm{div}((g\cdot X)(1-\varphi)\rho_l u_l) - (g\cdot X)(-R^{H_2O} - \partial_t((1-\varphi)\rho_l))\\ &= \partial_t((g\cdot X)(1-\varphi)\rho_l) + \mathrm{div}((g\cdot X)(1-\varphi)\rho_l u_l) + (g\cdot X)R^{H_2O}\end{aligned} \tag{2.61}$$

We use (2.59), (2.60) and (2.61) and gather together the terms in the time derivative and the divergence, respectively. We add the remaining terms of the relation (2.52d) $\cdot\, u_l$ (i.e. the pressure, the interface tensor, and the drag and exchange terms), so that the energy balance from the liquid momentum equation becomes

$$\begin{aligned}&\partial_t\left((1-\varphi)\rho_l\frac{|u_l|^2}{2} - (g\cdot X)(1-\varphi)\rho_l\right)\\ &+\mathrm{div}\left((1-\varphi)\rho_l u_l\frac{|u_l|^2}{2} - (g\cdot X)(1-\varphi)\rho_l u_l - (1-\varphi)u_l(\lambda_l \mathrm{div}\, u_l \mathrm{Id} + 2\eta_l D(u_l))\right)\\ &+u_l\cdot\nabla((1-\varphi)p_l) - \left(\langle p_{int,g}\rangle - \frac{2\sigma}{R}\right)u_l\cdot\nabla(1-\varphi) + (1-\varphi)(\lambda_l|\mathrm{div}\, u_l|^2 + 2\eta_l D(u_l):\nabla u_l)\\ &-K_d\,\varphi(1-\varphi)(u_g-u_l)u_l - (g\cdot X)R^{H_2O} + \Big(u_l\langle u_I\rangle - \frac{|u_l|^2}{2}\Big)R^{H_2O} = 0\end{aligned} \tag{2.62}$$

Multiplying (2.52e) by u_g and working similarly, we get for the gas momentum equation:

$$\begin{aligned}&\partial_t\left(\varphi\rho_g\frac{|u_g|^2}{2} - (g\cdot X)\varphi\rho_g\right) + \mathrm{div}\left(\varphi\rho_g u_g\frac{|u_g|^2}{2} - (g\cdot X)\varphi\rho_g u_g - \varphi u_g(\lambda_g \mathrm{div}\, u_g \mathrm{Id} + 2\eta_g D(u_g))\right)\\ &+u_g\cdot\nabla(\varphi p_g) - \langle p_{int,g}\rangle u_g\cdot\nabla\varphi + \varphi(\lambda_g|\mathrm{div}\, u_g|^2 + 2\eta_g D(u_g):\nabla u_g) + K_d\,\varphi(1-\varphi)(u_g-u_l)u_g\\ &+(g\cdot X)R^{H_2O} + \Big(\frac{|u_g|^2}{2} - \langle u_I\rangle u_g\Big)R^{H_2O} = 0\end{aligned} \tag{2.63}$$

We can then write a preliminary expression of the energy balance for the total system as follows,

$$\begin{aligned}&\partial_t\left((1-\varphi)\rho_l\frac{|u_l|^2}{2} + \varphi\rho_g\frac{|u_g|^2}{2} - (g\cdot X)((1-\varphi)\rho_l + \varphi\rho_g)\right)\\ &+\mathrm{div}\left((1-\varphi)\rho_l u_l\frac{|u_l|^2}{2} + \varphi\rho_g u_g\frac{|u_g|^2}{2} - (g\cdot X)((1-\varphi)\rho_l u_l + \varphi\rho_g u_g)\right)\\ &-\mathrm{div}\Big((1-\varphi)u_l(\lambda_l \mathrm{div}\, u_l \mathrm{Id} + 2\eta_l D(u_l)) + \varphi u_g(\lambda_g \mathrm{div}\, u_g \mathrm{Id} + 2\eta_g D(u_g))\Big)\\ &+u_l\cdot\nabla((1-\varphi)p_l) - \left(\langle p_{int,g}\rangle - \frac{2\sigma}{R}\right)u_l\cdot\nabla(1-\varphi) + u_g\cdot\nabla(\varphi p_g) - \langle p_{int,g}\rangle u_g\cdot\nabla\varphi\\ &+(1-\varphi)\left(\lambda_l|\mathrm{div}\, u_l|^2 + 2\eta_l D(u_l):\nabla u_l\right) + \varphi\left(\lambda_g|\mathrm{div}\, u_g|^2 + 2\eta_g D(u_g):\nabla u_g\right)\\ &+K_d\varphi(1-\varphi)|u_g-u_l|^2\\ &+\Big(\langle u_I\rangle(u_l-u_g) - \frac{|u_l|^2}{2} + \frac{|u_g|^2}{2}\Big)R^{H_2O} = 0.\end{aligned} \tag{2.64}$$

Viscosity terms of the fifth line can be simplified by using $\lambda_g = 3\eta_g$ and $\lambda_l = \frac{1}{3}\eta_l$ (see (2.10)–(2.11)):

$$\begin{aligned}&(1-\varphi)(\lambda_l|\mathrm{div}\, u_l|^2 + 2\eta_l D(u_l):\nabla u_l) + \varphi(\lambda_g|\mathrm{div}\, u_g|^2 + 2\eta_g D(u_g):\nabla u_g)\\ &\quad= (1-\varphi)\eta_l\left(\frac{1}{3}|\mathrm{div}\, u_l|^2 + 2D(u_l):\nabla u_l\right) + \varphi\eta_g\Big(3|\mathrm{div}\, u_g|^2 + 2D(u_g):\nabla u_g\Big)\end{aligned} \tag{2.65}$$

All these terms are giving dissipative contributions because $\eta_l > 0$, $\eta_g > 0$, and $D(u_k) : \nabla u_k \geq 0$.

The drag term on the sixth line of (2.64), $K_d\varphi(1-\varphi)|u_g - u_l|^2$, has a dissipative contribution. The term on the last line does not have a fixed sign as R^{H_2O} can be either positive or negative. The simplest energy equation is obtained by setting $\langle u_I\rangle(u_l - u_g) - \frac{1}{2}|u_l|^2 + \frac{1}{2}|u_g|^2 = 0$, which implies that $b = \frac{1}{2}$ in (2.44) and:

$$\langle u_I\rangle = \frac{u_g + u_l}{2}. \tag{2.66}$$

This is the natural choice of b value in two-phase or two-layer models with mass exchange because when the velocity of the exchange is given as the averaged of the two phase velocities, it provides an exact balance of this effect in terms of energy. Note that the exchange of mass must be exactly compensated since the amount of mass released by one phase is fully absorbed by the other one. This is not strictly true for the velocity of the transfer $\langle u_I\rangle$ involved in the momentum exchange. This leads us to define the exchange velocity $\langle u_I\rangle$ taking into account the sign of the transfer R^{H_2O}. When $R^{H_2O} \geq 0$ the transfer is made from the liquid phase to the gas phase, so the appropriate exchange velocity would be u_l. On the contrary, when $R^{H_2O} \leq 0$ the transfer is made from the gas phase to the liquid phase, so the velocity of the exchange would be u_g. This is summarized with the following definition of the coefficient b in (2.44),

$$b = \frac{1}{2} - \frac{1}{2}\mathrm{sgn}(R^{H_2O}), \qquad \langle u_I\rangle = bu_g + (1-b)u_l = \left\{ \begin{array}{ll} u_l & R^{H_2O} \geq 0 \\ u_g & R^{H_2O} \leq 0 \end{array} \right. .$$

Notice that $R^{H_2O} \geq 0$ leads to $b = 0$ and $R^{H_2O} \leq 0$ leads to $b = 1$. A unified formulation considering the previous case $b = \frac{1}{2}$ is achieved under the following definition

$$b = \frac{1}{2} - \frac{1}{2}\mathrm{sgn}(R^{H_2O})\delta, \qquad \delta = \left\{ \begin{array}{ll} 0 & \text{if averaged} \\ 1 & \text{if binary} \end{array} \right. . \tag{2.67}$$

For the binary case, the momentum exchange term does not disappear from the energy balance and it has a dissipative contribution, $\frac{1}{2}|u_g - u_l|^2|R^{H_2O}|$.

We must still analyze the pressure terms on the fourth line of (2.64) that we write as:

$$\begin{aligned} Q &= u_l \cdot \nabla((1-\varphi)p_l) - \left(\langle p_{int,g}\rangle - \frac{2\sigma}{R}\right)u_l \cdot \nabla(1-\varphi) + u_g \cdot \nabla(\varphi p_g) - \langle p_{int,g}\rangle u_g \cdot \nabla\varphi \qquad (2.68)\\ &= \left(u_l\left(\langle p_{int,g}\rangle - p_l - \frac{2\sigma}{R}\right) + u_g(p_g - \langle p_{int,g}\rangle)\right)\cdot\nabla\varphi + (1-\varphi)u_l\cdot\nabla p_l + \varphi u_g \cdot \nabla p_g \end{aligned}$$

In order to relate the first term with (2.52f), we cancel the term in u_g by setting $\langle p_{int,g}\rangle = p_g$ (equivalently $a = 1$ in (2.43)). This value coincides with that considered by [Guillemaud, 2007] when mass transfer occurs.

Then using (2.52f) to replace $u_l \cdot \nabla\varphi$ in the first term we find

$$\begin{aligned} Q &= \left(p_g - p_l - \frac{2\sigma}{R}\right)\left(-\partial_t\varphi + \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right)\right) + (1-\varphi)u_l\cdot\nabla p_l + \varphi u_g\cdot\nabla p_g \\ &= -\partial_t\left(\varphi\left(p_g - \frac{\mathbf{2\sigma}}{\boldsymbol{R}}\right)\right) + \varphi\left(\partial_t\left(p_g - \frac{\mathbf{2\sigma}}{\boldsymbol{R}}\right) + u_g\cdot\nabla p_g\right) + \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right)^2 \qquad (2.69)\\ &\quad + \mathrm{div}((1-\varphi)u_l p_l) - p_l(\partial_t(1-\varphi) + \mathrm{div}((1-\varphi)u_l)). \end{aligned}$$

We use again (2.52f) and (2.52a) to simplify the last term, which yields:

$$\begin{aligned} Q &= -\partial_t\left(\varphi\left(p_g - \frac{\boldsymbol{2\sigma}}{\boldsymbol{R}}\right)\right) + \mathrm{div}((1-\varphi)u_l p_l) + \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right)^2 \\ &\quad +\varphi\left(\partial_t\left(p_g - \frac{\boldsymbol{2\sigma}}{\boldsymbol{R}}\right) + u_g\cdot\nabla p_g\right) + p_l\frac{R^{H_2O}}{\rho_l} \end{aligned}$$

The first two terms contribute directly to the time and spatial derivatives of the total energy balance, respectively. The third term has a dissipative contribution. To write the last two terms in Q, we use the two equations related respectively to the gas temperature T_g and to the liquid temperature T_l, (2.56) and (2.57). From (2.56), we get:

$$\begin{aligned} \varphi\left(\partial_t\left(p_g - \frac{\boldsymbol{2\sigma}}{\boldsymbol{R}}\right) + u_g\cdot\nabla p_g\right) = \partial_t(\varphi c_{pg}\rho_g T_g) + \mathrm{div}(\varphi c_{pg}\rho_g T_g u_g) - \gamma(T_g - T_l) - \mathrm{div}(\varphi\kappa_g\nabla T_g) \\ -\varphi\left(2\eta_g D(u_g) : \nabla u_g + \lambda_g(\mathrm{div} u_g)^2\right), \end{aligned}$$

and from (2.57) we get

$$\begin{aligned} p_l\frac{R^{H_2O}}{\rho_l} = \partial_t((1-\varphi)c_{p_l}\rho_l T_l) + \mathrm{div}((1-\varphi)c_{p_l}\rho_l T_l u_l) + \gamma(T_g - T_l) - \mathrm{div}((1-\varphi)\kappa_l\nabla T_l) \\ -(1-\varphi)\left(2\eta_l D(u_l) : \nabla u_l + \lambda_l(\mathrm{div} u_l)^2\right). \end{aligned} \tag{2.70}$$

Embedding these quantities into Q yields:

$$\begin{aligned} Q &= \partial_t\left(\varphi c_{pg}\rho_g T_g - \varphi\left(p_g - \frac{\boldsymbol{2\sigma}}{\boldsymbol{R}}\right) + (1-\varphi)c_{p_l}\rho_l T_l\right) \\ &\quad +\mathrm{div}(\varphi c_{pg}\rho_g T_g u_g + (1-\varphi)c_{p_l}\rho_l T_l u_l + (1-\varphi)u_l p_l) - \mathrm{div}(\varphi\kappa_g\nabla T_g + (1-\varphi)\kappa_l\nabla T_l) \\ &\quad -(1-\varphi)\left(2\eta_l D(u_l) : \nabla u_l + \lambda_l(\mathrm{div} u_l)^2\right) - \varphi\left(2\eta_g D(u_g) : \nabla u_g + \lambda_g(\mathrm{div} u_g)^2\right) \\ &\quad +\varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right)^2. \end{aligned} \tag{2.71}$$

Finally, we can bring the expression of Q given by (2.71) back into (2.64):

$$\begin{aligned} &\partial_t\left((1-\varphi)\rho_l\frac{|u_l|^2}{2} + \varphi\rho_g\frac{|u_g|^2}{2}\right. \\ &\left. - (g\cdot X)((1-\varphi)\rho_l + \varphi\rho_g) + \varphi(c_{pg} - c_0)\rho_g T_g + (1-\varphi)c_{p_l}\rho_l T_l + \varphi\frac{2\sigma}{R}\right) \\ &+ \mathrm{div}\left((1-\varphi)\rho_l u_l\frac{|u_l|^2}{2} + \varphi\rho_g u_g\frac{|u_g|^2}{2} - (g\cdot X)((1-\varphi)\rho_l u_l + \varphi\rho_g u_g)\right) \\ &+ \mathrm{div}\Big(\varphi c_{pg}\rho_g T_g u_g + (1-\varphi)c_{p_l}\rho_l T_l u_l + (1-\varphi)u_l p_l\Big) \\ &- \mathrm{div}(\varphi\kappa_g\nabla T_g + (1-\varphi)\kappa_l\nabla T_l) \\ &- \mathrm{div}\Big((1-\varphi)u_l(\lambda_l\mathrm{div}\, u_l\mathrm{Id} + 2\eta_l D(u_l)) + \varphi u_g(\lambda_g\mathrm{div}\, u_g\mathrm{Id} + 2\eta_g D(u_g))\Big) \\ &+ K_d\varphi(1-\varphi)|u_g - u_l|^2 + \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right)^2 + \frac{1}{2}|u_g - u_l|^2|R^{H_2O}|\delta = 0. \end{aligned} \tag{2.72}$$

Remark 2.7. *The term $e_g = (c_{pg} - c_0)\rho_g T_g$ in the time derivative of* (2.58) *involves both the gas heat capacity, c_{pg}, and the gas constant for water, c_0. The heat-conducting Navier-Stokes equations are usually written in terms of (p, ρ, T) expressing ρ and the internal energy e in functions of (P, T). Denoting S the entropy, the specific heats at constant volume and pressure are given by [Gaskell and Laughlin, 2017]*

$$c_p := T\frac{\partial S}{\partial T}, \qquad c_v = T\frac{(\partial S/\partial T)(\partial \rho/\partial P) - (\partial S/\partial P)(\partial S/\partial T)}{\partial \rho/\partial P}$$

with the identity

$$\partial S/\partial P = \rho^{-2}(\partial \rho/\partial T)$$

which implies that

$$\frac{c_p}{c_v} = \frac{(\partial S/\partial T)(\partial \rho/\partial P)}{(\partial S/\partial T)(\partial \rho/\partial P) - \rho^{-2}(\partial \rho/\partial T)^2}.$$

This proves that $c_p > c_v$. In our system, we consider the gas to be ideal so we can define $c_{vg} = c_{pg} - c_0$ [Gaskell and Laughlin, 2017] and thus $c_{pg} - c_0 > 0$.

Remark 2.8. *The temperature equations,* (2.56) *and* (2.57)*, do not have the same structure. The inertial term appearing in the gas temperature equation, $\varphi(\partial_t p_g + u_g \cdot \nabla p_g)$, is not present in that of the liquid, where we find instead the term $\frac{p_l}{\rho_l} R^{H_2O}$. A different option is to propose similar equations for both phases , which of course modifies the conserved energy. This option comes from a different way to write the term Q in* (2.69)*. Instead of using the equation of φ* (2.52f) *to simplify the last term, we write it in conservative form, which yields $\mathrm{div}((1-\varphi)u_l p_l) - p_l(\partial_t(1-\varphi) + \mathrm{div}((1-\varphi)u_l)) = -\partial_t((1-\varphi)p_l) + (1-\varphi)(\partial_t p_l + u_l \cdot \nabla p_l)$. The resulting liquid temperature equation is:*

$$(1-\varphi)c_{p_l}\rho_l(\partial_t T_l + u_l \cdot \nabla T_l) - (1-\varphi)(\partial_t p_l + u_l \cdot \nabla p_l) - c_{pl}T_l R^{H_2O} + \gamma(T_g - T_l) - \mathrm{div}((1-\varphi)\kappa_l \nabla T_l) = 0.$$

In this case

$$\begin{aligned} Q \;=\; & \partial_t\Big(\varphi\rho_g(c_{pg}T_g - p_g) + (1-\varphi)\rho_l(c_{p_l}T_l - p_l) + \varphi\frac{2\sigma}{R}\Big) + \mathrm{div}(\varphi c_{pg}\rho_g T_g u_g + (1-\varphi)c_{p_l}\rho_l T_l u_l) \\ & -\mathrm{div}((1-\varphi)\kappa_l\nabla T_l + \varphi\kappa_g \nabla T_g) + \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right)^2 \\ & -(1-\varphi)\left(2\eta_l D(u_l) : \nabla u_l + \lambda_l(\mathrm{div} u_l)^2\right) - \varphi\left(2\eta_g|D(u_g)|^2 + \lambda_g(\mathrm{div} u_g)^2\right) \end{aligned}$$

and the energy balance reads

$$\begin{aligned} & \partial_t\left((1-\varphi)\rho_l\frac{|u_l|^2}{2} + \varphi\rho_g\frac{|u_g|^2}{2} - (g\cdot X)((1-\varphi)\rho_l + \varphi\rho_g) + \varphi\rho_g(c_{pg}-c_0)T_g + (1-\varphi)\rho_l(c_{p_l}T_l - p_l) + \varphi\frac{2\sigma}{R}\right) \\ & + \mathrm{div}\left((1-\varphi)\rho_l u_l\frac{|u_l|^2}{2} + \varphi\rho_g u_g\frac{|u_g|^2}{2} - (g\cdot X)((1-\varphi)\rho_l u_l + \varphi\rho_g u_g)\right) \\ & + \mathrm{div}\Big(\varphi c_{pg}\rho_g T_g u_g + (1-\varphi)c_{p_l}\rho_l T_l u_l\Big) \\ & - \mathrm{div}((1-\varphi)\kappa_l\nabla T_l + \varphi\kappa_g\nabla T_g) \\ & + \mathrm{div}\Big(-(1-\varphi)u_l(\lambda_l \mathrm{div}\, u_l + 2\eta_l D(u_l)) - \varphi u_g(\lambda_g \mathrm{div}\, u_g + 2\eta_g D(u_g))\Big) \\ & + K_d\varphi(1-\varphi)|u_g - u_l|^2 + \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right)^2 + \frac{1}{2}|u_g - u_l|^2|R^{H_2O}|\delta = 0. \end{aligned}$$

All the terms in the time derivative are positive except the liquid pressure term. □

Remark 2.9. *The presence of the term $2\varphi\sigma/R$ in (2.72) is interesting. [Kostin, Ilya et al., 2003] consider a system of two incompressible phases separated by a diffusive interface, which may be seen as a system with miscible phases. It is written as:*

$$\partial_t \varphi + \mathrm{div}(\varphi u) = 0, \tag{2.73a}$$

$$\partial_t u + u \cdot \nabla u + \nabla p - \varepsilon^2 \sigma \nabla \varphi \Delta \varphi = 0, \tag{2.73b}$$

$$\mathrm{div} u = 0 \tag{2.73c}$$

with initial conditions on φ and u that assume $\varphi_0 = 1$ inside the fluid 1, $\varphi_0 = 0$ inside the fluid 2 and $0 < \varphi < 1$ on a diffuse interface of size ε^2. The variable φ represents the volume fraction of fluid 1 in this non-overlapping continuum. The quantity $\sigma \nabla \varphi \Delta \varphi$ represents the surface tension force written for a diffusive interface of size ε where σ is the surface tension coefficient (assumed to be constant). The quantity $\nabla \varphi$ is an extension of the normal n on the whole space and $\Delta \varphi = \mathrm{div}(\nabla \varphi)$ is an extension of the curvature. When evaluating the energy balance, the term $-\varepsilon^2 \sigma \nabla \varphi \Delta \varphi$ tested against u and integrated in space yields

$$-\varepsilon^2 \int_\Omega \sigma \nabla \varphi \Delta \varphi \cdot u = -\varepsilon^2 \frac{d}{dt} \int_\Omega \sigma \varphi \Delta \varphi dx = \frac{\varepsilon^2}{2} \frac{d}{dt} \int_\Omega \sigma |\nabla \varphi|^2 dx.$$

This quantity encodes the effect of surface tension when using a diffuse interface PDE system with an order parameter φ. The limit $\varepsilon \to 0$ of such a system is the system for two incompressible phases separated by a sharp interface.

□

2.6 Resulting macroscopic system with energy balance

Here we present the final proposed system for the 8 unknowns $\varphi, \rho_g, C_l, p_l, u_l, u_g, T_g, T_l$, taking into account the specific closures deduced from the energy analysis:

$$\partial_t((1-\varphi)\rho_l) + \mathrm{div}((1-\varphi)\rho_l u_l) = -R^{H_2O} \tag{2.74a}$$

$$\partial_t(\varphi \rho_g) + \mathrm{div}(\varphi \rho_g u_g) = R^{H_2O} \tag{2.74b}$$

$$\partial_t((1-\varphi)\rho_l(1-C_l)) + \mathrm{div}((1-\varphi)\rho_l(1-C_l)u_l) = 0 \tag{2.74c}$$

$$\begin{aligned} &\partial_t((1-\varphi)\rho_l u_l) + \mathrm{div}((1-\varphi)\rho_l u_l \otimes u_l) + \nabla((1-\varphi)p_l) - \mathrm{div}((1-\varphi)\mathcal{D}_l) \\ &- \left(p_g - \frac{2\sigma}{R}\right) \nabla(1-\varphi) - K_d\, \varphi(1-\varphi)(u_g - u_l) - (1-\varphi)\rho_l g + \frac{u_g + u_l}{2} R^{H_2O} = 0 \end{aligned} \tag{2.74d}$$

$$\partial_t(\varphi \rho_g u_g) + \mathrm{div}(\varphi \rho_g u_g \otimes u_g) + \nabla(\varphi p_g) - \mathrm{div}(\varphi \mathcal{D}_g) - p_g \nabla \varphi + K_d\, \varphi(1-\varphi)(u_g - u_l) - \varphi \rho_g g - \frac{u_g + u_l}{2} R^{H_2O} = 0 \tag{2.74e}$$

$$\partial_t \varphi + u_l \cdot \nabla \varphi = \varphi(1-\varphi) \frac{3}{4\eta_l} \left(p_g - p_l - \frac{2\sigma}{R}\right), \tag{2.74f}$$

where we chose the averaged case for the exchange velocity ($\delta = 0$ in (2.67)) and the mean curvature $\bar{H} = 1/R$ in (2.42). Closures are given by the pressure gas law

$$p_g = c_0 \rho_g T_g, \tag{2.74g}$$

the bubble radius expression

$$R^3 = \frac{3}{4\pi N} \frac{\varphi}{1-\varphi}, \tag{2.74h}$$

the mass exchange expression

$$R^{H_2O} = 3^{1/3} \rho_l D \left(4\pi N \varphi (1-\varphi)\right)^{2/3} \frac{C_l - k_h \sqrt{p_g}}{1-\varphi^{1/3}}, \tag{2.74i}$$

equations (2.10)-(2.11) for the viscosity tensors $\mathcal{D}_g$ and $\mathcal{D}_l$, and the following equations for the temperatures:

$$\begin{aligned} \partial_t(\varphi c_{pg} \rho_g T_g) + \mathrm{div}(\varphi c_{pg} \rho_g T_g u_g) - \varphi \left(\partial_t \left(p_g - \frac{2\sigma}{R} \right) + u_g \cdot \nabla p_g \right) - \gamma(T_g - T_l) - \mathrm{div}(\varphi \kappa_g \nabla T_g) \\ - \varphi \left(2\eta_g D(u_g) : \nabla u_g + \lambda_g (\mathrm{div} u_g)^2 \right) = 0, \end{aligned} \tag{2.74j}$$

$$\begin{aligned} \partial_t((1-\varphi) c_{p_l} \rho_l T_l) + \mathrm{div}((1-\varphi) c_{p_l} \rho_l T_l u_l) - \frac{p_l}{\rho_l} R^{H_2O} + \gamma(T_g - T_l) - \mathrm{div}((1-\varphi)\kappa_l \nabla T_l) \\ - (1-\varphi) \left(2\eta_l D(u_l) : \nabla u_l + \eta_l (\mathrm{div} u_l)^2 \right) = 0. \end{aligned} \tag{2.74k}$$

The energy equation of this system is (2.58).

A few alternative equations highlighting various physical processes can be built. They present different sets of advantages. An alternative equation for (2.74f) is given by (2.45):

$$\mathrm{div}\, u_l = \varphi \frac{3}{4\eta_l} \left(p_g - p_l - \frac{2\sigma}{R} \right) - \frac{R^{H_2O}}{\rho_l (1-\varphi)}.$$

This expression allows to obtain the liquid pressure p_l in terms of the other variables of the system:

$$p_l = p_g - \frac{2\sigma}{R} - \frac{4\eta_l}{3\varphi} \left(\mathrm{div}\, u_l + \frac{R^{H_2O}}{\rho_l (1-\varphi)} \right). \tag{2.75}$$

Another equivalent equation for the gas temperature that can replace (2.74j) can be obtained in terms of ρ_g, p_g, T_g when using (2.74a) and the gas law (2.74g). First we write

$$\begin{aligned} \partial_t(\varphi c_0 \rho_g T_g) + \mathrm{div}(\varphi c_0 \rho_g T_g u_g) - \varphi(\partial_t p_g + u_g \cdot \nabla p_g) &= \partial_t(\varphi p_g) + \mathrm{div}(\varphi p_g u_g) - \varphi(\partial_t p_g + u_g \cdot \nabla p_g) \\ &= p_g(\partial_t \varphi + \mathrm{div}(\varphi u_g)) \\ &= -p_g \partial_t (1-\varphi) + p_g \mathrm{div}(\varphi u_g) \\ &= \frac{p_g}{\rho_l} R^{H_2O} + p_g \mathrm{div}(\varphi u_g + (1-\varphi) u_l). \end{aligned}$$

Thus,

$$c_0 \left(\partial_t(\varphi \rho_g T_g) + \mathrm{div}(\varphi \rho_g T_g u_g) \right) - \frac{p_g}{\rho_l} R^{H_2O} - p_g \mathrm{div}(\varphi u_g + (1-\varphi) u_l) = \varphi(\partial_t p_g + u_g \cdot \nabla p_g),$$

and embedding it in (2.74j) yields an alternative equation for the gas temperature:

$$\begin{aligned} \partial_t((c_{pg} - c_0)\varphi \rho_g T_g) + \mathrm{div}((c_{pg} - c_0)\varphi \rho_g T_g u_g) + \varphi \partial_t \left(\frac{2\sigma}{R} \right) - \mathrm{div}(\varphi \kappa_g \nabla T_g) + p_g \mathrm{div}\Big(\varphi u_g + (1-\varphi) u_l \Big) \\ = \gamma(T_g - T_l) - \frac{p_g}{\rho_l} R^{H_2O} + \varphi \left(2\eta_g D(u_g) : \nabla u_g + \lambda_g (\mathrm{div} u_g)^2 \right) \end{aligned} \tag{2.76}$$

3 Geometry, initial and boundary conditions

The geometry of volcanic conduits is complex and often poorly constrained. The simplest approximation is to consider the domain to be a cylinder with a very small ratio radius/length. Thus we consider that the total boundary of the domain is divided in three parts: the cylinder base at the bottom Γ_b, the cylinder top Γ_t, and the lateral walls Γ_w, which make $\partial\Omega = \Gamma_b \cup \Gamma_t \cup \Gamma_w$. Assuming an open volcanic conduit, we set an inflow boundary at the bottom Γ_b and an outflow boundary at the top Γ_t. The system has eight unknowns $(\varphi, \rho_g, C_l, p_l, u_l, u_g, T_g, T_l)$ for the eight equations in (2.74).

Establishing appropriate boundary conditions is not an easy task for viscous compressible flows [Strikwerda, 1977, Oliger and Sundstrom, 1978, Rudy and Strikwerda, 1981, Dutt, 1988, Poinsot and Lele, 1992]. While some theoretical results have been developed to prove the well-posed character of the Euler equations, it is not the case for the Navier-Stokes equations. This system can be viewed as an extension of the compressible Euler equations with viscous terms, so it seems reasonable to consider the boundary conditions for the inviscid case and to add the proper boundary conditions linked to the viscosity. These conditions are also differently addressed for subsonic and supersonic flows. Here we assume subsonic flow. Let us consider the 3D compressible system with unknowns: density ρ, velocity u, temperature T and pressure p (pressure satisfying the ideal gas relation), which is:

$$\begin{aligned} &\partial_t \rho + \mathrm{div}(\rho u) = 0 \\ &\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}\sigma = 0 \\ &(c_p - c_0)(\partial_t(\rho T) + \mathrm{div}(\rho T u)) - \mathrm{div}(\kappa \nabla T) - \sigma : \nabla u = 0 \\ &p = c_0 \rho T \end{aligned} \tag{3.1}$$

with the total stress tensor $\sigma = -p\mathrm{Id} + \tau$, τ being the viscous stress tensor, c_p the heat capacity and κ being the thermal conductivity. The initial conditions for this problem are given by $\rho(t=0) = \rho^0, \rho u(t=0) = \rho^0 u^0, \rho T(t=0) = \rho^0 T^0, p(t=0) = c_0 \rho^0 T^0$. In what follows, and according to the literature, the term "viscous" is used to describe both viscous and thermal diffusion effects related to the Navier-Stokes equations. The term "inviscid" refers then to the Euler equations where the viscosity of the fluid and the thermal diffusion are neglected (i.e. $\tau = \kappa = 0$ in the previous system).

The work developed in [Strikwerda, 1977] specifies the number of conditions to be imposed at inflow and outflow boundaries, which we summarize in Table 1. Table 2 summarizes boundary conditions for slip and no-slip walls following [Poinsot and Lele, 1992].

Boundary type	Euler (inviscid)	Navier-Stokes (inviscid & viscous)
inflow	4	5
outflow	1	4

Table 1: Inflow and outflow boundary conditions required for compressible flow problems as in (3.1)

Wall type	Euler (inviscid)	Navier-Stokes (inviscid & viscous)
isothermal no-slip	4	4
adiabatic no-slip	3	4
adiabatic slip	1	4

Table 2: Boundary conditions at walls required for compressible flow problems as in (3.1)

Many types of conditions can be found in the literature, such as the ones proposed in [Rudy and Strikwerda, 1981, Dutt, 1988, Poinsot and Lele, 1992], where they are also analyzed numerically. An interesting summary is presented in [Poinsot and Lele, 1992] (see Tables III and IV in that paper) with also a reasoning supporting the choice of such conditions. In particular, well-posed problems are found for the inviscid case when the pressure at the outlet boundary is imposed alongside, at the inlet boundary, either (i) the velocity and the density, or (ii) the velocity and the temperature. The pair velocity/pressure leads to an ill-posed problem (cf. [Oliger and Sundstrom, 1978]). As mentioned, several extensions to the viscous problem have been proposed. In particular, following [Rudy and Strikwerda, 1981, Dutt, 1988] and [Poinsot and Lele, 1992], stable solutions are found when we consider at the outlet boundary a fixed pressure together with zero normal viscous stress. In other words for the total tensor $\sigma = -p\mathrm{Id} + \tau$ we consider $(\tau n)_n = (\tau n)_{\tan} = 0$, where the sub-indices n and $\tan$ denote the normal and tangential components of the outlet normal, n, respectively.

At the inlet, different choices can be found in the literature. For example, in addition to conditions (i) (resp. (ii)) we may impose the temperature (resp. the density) compatible with the state equation, or alternatively $(\tau n)_n = 0$ or $\partial_n T = 0$ (resp. $\partial_n \rho = 0$). A different kind of inlet conditions consists in imposing the pressure $-$instead of the velocity$-$ together with the temperature, keeping also the pressure value at the outflow boundary. These conditions are usually found in engineering applications of gas dynamics for isentropic inflow in terms of the total pressure (static plus dynamic components) and total temperature, see for example [Briley and McDonald, 1977], or more recently [Kim et al., 2004, Carlson, 2011, Choudhary et al., 2016]. Notice that in this case, additional conditions are needed to ensure that the velocity at the inlet enters the domain by either imposing the flow directions, or by setting the pressures values ensuring that this condition is fulfilled. In [Benzoni-Gavage et al., 2003] it is proven that for the Euler equations, this additional condition may be given by prescribing the tangential velocity in 2D and the flow angles in 3D. In [Laurén and Nordström, 2018] the well-posedness of imposing total pressure, total temperature and vanishing tangent velocity at the inlet for the 2D Euler equations is proven. In [Odier et al., 2019], the entry angles of the flow are chosen instead to solve the problem.

Extensions to multi-component gas flows have been also developed, see for example [Baum et al., 1994, Okong'o and Bellan, 2002, Odier et al., 2019]. These components share the same temperature and velocity, so the system is defined as in (3.1) with additional equations for component mass fractions $\{\phi_j\}_{j=1}^s$ such that $\sum_{j=1}^s \phi_j = 1$. Such a system is presented in equation (3.2) for g_j being a source term:

$$\begin{aligned}
&\partial_t \rho + \mathrm{div}(\rho u) = 0 \\
&\partial_t (\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}\sigma = 0 \\
&(c_p - c_0)(\partial_t(\rho T) + \mathrm{div}(\rho T u)) - \mathrm{div}(\kappa \nabla T) - \sigma : \nabla u = 0 \\
&\partial_t(\rho \phi_j) + \partial_t(\rho \phi_j u) = g_j \\
&p = c_0 \rho T
\end{aligned} \tag{3.2}$$

together with the initial conditions $\rho(t=0) = \rho^0, \rho u(t=0) = \rho^0 u^0, \rho T(t=0) = \rho^0 T^0, \rho\phi_j(t=0) = \rho^0\phi_j^0, p(t=0) = c_0\rho^0 T^0$. In general, the same type of boundary conditions as for single component Euler or Navier-Stokes equations described before are considered, and one must additionally impose the component mass fractions at the inflow boundaries. So for s components, the number of conditions to be imposed at the inflow boundaries increases by $s-1$ those specified in Table 1.

To summarize, Tables 3 and 4 show a compilation of the boundary conditions proposed in the literature that we will consider to establish those suitable for our two-phase system. For convenience, we write the additional conditions to be considered for multi-component models with mass fraction $\{\phi_j\}_{j=1}^s$.

Remark 3.1. *In order to ensure a well-posed problem, the appropriate variables to be imposed at the boundaries are the "characteristic variables". This method is called the Characteristic Boundary Conditions and has been*

Boundary type	Euler (inviscid)	Navier-Stokes (inviscid & viscous)
inflow	$\{u, \rho\}$, $\{u, T\}$, $\{p, \rho, u_{\tan} = 0\}$, $\{p, T, u_{\tan} = 0\}$ additionally for multi-component: $\{\phi_j\}_{j=1}^{s-1}$	$\{u, \rho, T\}$, $\{p, T, u_{\tan} = 0, \partial_n(u_n) = 0\}$ additionally for multi-component: $\{\phi_j\}_{j=1}^{s-1}$
outflow	p	$p,\ (\tau n)_n = (\tau n)_{\tan} = 0$

Table 3: Inflow and outflow suitable boundary conditions for compressible flow problems of the form (3.1) and (3.2).

Boundary type	Euler (inviscid)	Navier-Stokes (inviscid & viscous)
isothermal no-slip	$u = 0,\ T = T^w$	$u = 0,\ T = T^w$
adiabatic no-slip	$u = 0$	$u = 0,\ \partial_n T = 0$
adiabatic slip	$(u)_n = 0$	$(u)_n = 0,\ \partial_n T = 0,\ (\tau n)_{\tan} = 0$

Table 4: Suitable wall conditions for compressible flow problems of the form (3.1) and (3.2).

largely studied and applied for Euler and Navier-Stokes equations, see for example [Thompson, 1987, Poinsot and Lele, 1992, Laney, 1998, Nicoud, 1999, Okong'o and Bellan, 2002, Benzoni-Gavage et al., 2003, Kim et al., 2004, Carlson, 2011, Odier et al., 2019]. To write an hyperbolic system in characteristic formulation, one must find the diagonalization of the Jacobian matrix of the flux function and to get a relationship between the characteristic and primitive variables. Notice that the primitive variables are the ones known in physics and thus easy to impose. The number of boundary conditions to be imposed at one boundary is given by the number of entering characteristic waves that moves with a characteristic velocity given by the eigenvalues of the Jacobian matrix. The boundary conditions are then imposed in such a way that all the characteristic variables associated to these entering waves are determined. The outgoing waves remain undetermined since the corresponding variables will be determined by the equations in the system. This is how one concludes in particular that the pair {velocity, pressure} provides an ill-posed problem for the Euler equations because it determines the characteristic variable associated to an outgoing wave. This kind of analysis is not simple and it is out of scope of the work presented here, although it would be useful to better know how boundary conditions must be set. Here, we simply use the related literature as guidelines to establish the boundary conditions for our model.

As discussed in [Poinsot and Lele, 1992, Choudhary et al., 2016], the mathematical/physical boundary conditions are not always enough to solve the problem numerically and additional numerical boundary conditions may be considered to avoid numerical instabilities. Here we focus on theoretical aspects and we only propose boundary conditions to close the problem based on the works cited above. It must nevertheless be kept in mind that our model (2.74) is a two-phase system for ten unknowns with strong couplings (mostly due to the term R^{H_2O}) that entails a more complex situation. The viability of a subset of the proposed boundary conditions, together with possible additional numerical boundary conditions to get stability in the numerical solutions, is analyzed in Part II, Section 5.2 [Burgisser et al., 2024] for the 1D case.

We frame the discussion on the boundary conditions by considering equation (2.75) as a closure equation for the liquid pressure p_l, which means that equation (2.74f) is no longer considered as a part of the system. At the outlet, we consider the conditions of viscous type presented in Table 3, by imposing the pressure value together with zero viscous tensor for both phases. The inlet is more difficult to handle. Our magmatic system is composed of two phases, a gas with volume fraction φ, and a liquid with volume fraction $1 - \varphi$. The liquid phase is in turn composed of two components, H_2O and other oxides with corresponding concentrations C_l and $(1 - C_l)$, their mass fractions in the system being determined by $\rho_l(1 - \varphi)C_l$ and $\rho_l(1 - \varphi)(1 - C_l)$, respectively. We can

thus interpret our system as a three-component model (gaseous H_2O, dissolved H_2O, and dry silicate liquid) with volume fractions φ, $(1-\varphi)C_l$, and $(1-\varphi)(1-C_l)$, respectively. The analogy with the works dealing with multi-component systems ([Baum et al., 1994, Okong'o and Bellan, 2002, Odier et al., 2019]) leads us to impose the values of the volume fraction φ and the concentration C_l at the inlet.

For clarity, we set aside for now the coupled nature of the system to focus on its two-phase nature with two components in the liquid phase. The equations of the system may be split in the following two groups corresponding to the gas (left) and to the liquid (right), respectively:

$$\begin{aligned}
&\partial_t \rho_{g\varphi} + \mathrm{div}(\rho_{g\varphi} u_g) = R^{H_2O} \\
&\partial_t (\rho_{g\varphi} u_g) + \mathrm{div}(\rho_{g\varphi} u_g \otimes u_g) - \mathrm{div}\sigma_g = \mathcal{F}_g \\
&\partial_t (c_{pg} \rho_{g\varphi} T_g) + \mathrm{div}(c_{pg} \rho_{g\varphi} T_g u_g) - \mathrm{div}(\kappa_g \nabla T_g) = \mathcal{G}_g \\
&p_{g\varphi} = c_0 \rho_{g\varphi} T_g
\end{aligned}
\qquad
\begin{aligned}
&\partial_t \rho_{l\varphi} + \mathrm{div}(\rho_{l\varphi} u_l) = -R^{H_2O} \\
&\partial_t (\rho_{l\varphi} u_l) + \mathrm{div}(\rho_{l\varphi} u_l \otimes u_l) - \mathrm{div}\sigma_l = \mathcal{F}_l \\
&\partial_t (c_{pl} \rho_{l\varphi} T_l) + \mathrm{div}(c_{pl} \rho_{l\varphi} T_l u_l) - \mathrm{div}(\kappa_l \nabla T_l) = \mathcal{G}_l \\
&\partial_t (\rho_{l\varphi}(1-C_l)) + \mathrm{div}(\rho_{l\varphi}(1-C_l) u_l) = 0 \\
&p_l = p_l(p_g, \varphi, \mathrm{div} u_l, R^{H_2O})
\end{aligned}$$

where $\rho_{g\varphi} = \rho_g \varphi$, $p_{g\varphi} = p_g \varphi$, $\rho_{l\varphi} = \rho_l(1-\varphi)$ and functions $\mathcal{F}_j, \mathcal{G}_j,\ j = g, l$, are defined appropriately to recover the system (2.74). The initial conditions are $\rho_{g\varphi}(t=0) = \rho^0_{g\varphi}, \rho_{g\varphi} u_g(t=0) = \rho^0_{g\varphi} u^0_g, \rho_{l\varphi}(t=0) = \rho^0_{l\varphi}, \rho_{l\varphi} u_l(t=0) = \rho^0_{l\varphi} u^0_l, \rho_{g\varphi} T_g(t=0) = \rho^0_{g\varphi} T^0_g, \rho_{l\varphi} T_l(t=0) = \rho^0_{l\varphi} T^0_l, \rho_{l\varphi} C_l(t=0) = \rho^0_{l\varphi} C^0_l, p_g(t=0) = c_0 \rho^0_g \varphi^0 T^0_g, p_l(t=0) = p_l(p^0_g, \varphi^0, \mathrm{div} u^0_l, R^{H_2O}(t=0))$. The similarity of the gas equations with the general system (3.1) is clear, so the same number and kind of conditions may be considered for both cases. Thus, from Table 1, four conditions are needed for the inviscid case and five for the viscous case at the inlet, following the same type of conditions presented in Table 3 for the gas variables $\rho_{g\varphi}, u_g, T_g, p_{g\varphi}$. The liquid equations have a two-component structure with an additional equation for the fraction C_l in affinity with the system (3.2). As argued before, in addition to the general conditions, the fraction of each component must be imposed. Thus five conditions are needed for the inviscid case and six for the viscous case at the inlet. The type of conditions are again those in Table 3 for the liquid variables $\rho_{l\varphi}, u_l, T_l, p_l, C_l$, with C_l being the additional variable to be prescribed.

To recover now the coupled nature of the two-phase system, the volume fraction φ has to be imposed at the inlet because otherwise the system loses consistency as a whole. The density ρ_l is constant, so by imposing a condition on $\rho_{l\varphi}$ we are prescribing the volume fraction φ as well and no additional measure is needed. If, however, $\rho_{l\varphi}$ is not chosen as a variable to be imposed, then the system requires an additional condition on φ. This is the case for example when the shortlist $\{p_l, T_l, (u_l)_{\tan} = 0\}$ is chosen for the inviscid problem. The situation is simpler when considering wall boundaries because the same number and type of boundary conditions as in Tables 2 and 4 are valid for both gas and liquid.

We summarize in Tables 5 and 6 the number of boundary conditions needed for our system to be closed. As in Table 1, one additional condition for the viscous case is needed at the inlet and three more at the outlet for both gas and liquid.

Boundary type	Proposed model (inviscid)	Proposed model (inviscid & viscous)
inflow	4 (gas) + 5 (liquid) + 2 (N, R) + 1*(φ)	5 (gas) + 6 (liquid) + 2 (N, R) + 1*(φ)
outflow	1 (gas) + 1 (liquid)	4 (gas) + 4 (liquid)

Table 5: Inflow and outflow. Number of boundary conditions required for the proposed compressible model (2.74). The last condition marked with an asterisk needs to be added iff φ is not already considered among the variables imposed for the liquid phase.

Boundary type	Proposed model (inviscid)	Proposed model (inviscid & viscous)
isothermal no-slip	4 (gas) + 4 (liquid)	4 (gas) + 4 (liquid)
adiabatic no-slip	3 (gas) + 3 (liquid)	4 (gas) + 4 (liquid)
adiabatic slip	1 (gas) + 1 (liquid)	4 (gas) + 4 (liquid)

Table 6: Walls. Number of boundary conditions required for the proposed compressible model (2.74).

Next, we propose suitable boundary conditions for our system (2.74) considering the two types of inflow conditions by imposing the discriminatory variables: velocity (denoted by BCU) or pressure (denoted by BCP). As further discussed in Part II, Section 5.2 [Burgisser et al., 2024], all the values have to be set to keep compatibility in the system because of its strong coupled structure.

Inlet boundary conditions.

◦ BCU. We consider inflow condition by fixing the values of the velocities

$$(u_l)_{|\Gamma_b} = u_l^b, \qquad (u_g)_{|\Gamma_b} = u_g^b, \qquad \text{such that} \quad u_l^b \cdot n_b < 0, \quad u_g^b \cdot n_b < 0, \quad \text{at} \quad \Gamma_b, \tag{3.3a}$$

for n_b being the normal vector to the boundary Γ_b pointing outwards. The presence of the thermal diffusion implies that the temperatures must be imposed, which is the additional condition related to the viscous contribution. To extend the stable conditions found in the literature for the gas phase, we chose to give the value for the conserved density $\varphi\rho_g$ that must be compatible with (2.74g). For the liquid phase we also impose the density $(1-\varphi)\rho_l$ (hence, φ) and the additional condition on C_l. These conditions thus are:

$$(\rho_g)_{|\Gamma_b} = (\rho_g)^b, \quad (\varphi)_{|\Gamma_b} = \varphi^b, \quad (C_l)_{|\Gamma_b} = C_l^b, \tag{3.3b}$$

and

$$(T_g)_{|\Gamma_b} = T_g^b, \qquad (T_l)_{|\Gamma_b} = T_l^b. \tag{3.3c}$$

◦ BCP. In this case the pressures are given at the inlet instead of the velocities. We must then impose the flow directions, for which we will consider a vanishing tangent component of the velocity as in [Laurén and Nordström, 2018] and a zero normal derivative of the normal velocity as in [Briley and McDonald, 1977]. Since the volume fraction is not chosen for the liquid phase, then it must be additionally prescribed. Note that the value of p_l at the boundary must be compatible with (2.75). These conditions thus are:

$$(p_g)_{|\Gamma_b} = p_g^b, \quad (T_g)_{|\Gamma_b} = T_g^b, \quad (p_l)_{|\Gamma_b} = p_l^b, \quad (T_l)_{|\Gamma_b} = T_l^b, \tag{3.4a}$$

$$((u_g)_{\tan})_{|\Gamma_b} = ((u_l)_{\tan})_{|\Gamma_b} = 0, \quad ((\mathcal{D}_g\, n_b)_n)_{|\Gamma_b} = ((\mathcal{D}_l\, n_b)_n)_{|\Gamma_b} = 0, \tag{3.4b}$$

$$(\varphi)_{|\Gamma_b} = \varphi^b, \quad (C_l)_{|\Gamma_b} = C_l^b. \tag{3.4c}$$

Outlet boundary conditions. We impose the value of the pressures and viscous, stress-free condition for both phases:

$$\begin{array}{llll} (p_l)_{|\Gamma_t} = p_l^t, & (\mathcal{D}_l\, n_t)_n = (\mathcal{D}_l\, n_t)_{\tan} = 0, & \text{where} \quad u_l^t \cdot n_t > 0, \quad \text{at} \quad \Gamma_t \\ (p_g)_{|\Gamma_t} = p_g^t, & (\mathcal{D}_g\, n_t)_n = (\mathcal{D}_g\, n_t)_{\tan} = 0, & \text{where} \quad u_g^t \cdot n_t > 0, \quad \text{at} \quad \Gamma_t \end{array} \tag{3.5a}$$

where the viscous tensors are given in equations (2.10)-(2.11) and n_t is the normal vector to the boundary Γ_t pointing outwards. In case the velocities are inbound at this boundary, that is, $u_l^t \cdot n_t < 0$, $u_g^t \cdot n_t < 0$, we should impose the values for the corresponding advected variables. In that case we introduce,

$$\begin{array}{llll} (\varphi)_{|\Gamma_t} = \varphi^t, \quad (C_l)_{|\Gamma_t} = C_l^t, \quad (T_l)_{|\Gamma_t} = T_l^t & \text{where} & u_l^t \cdot n_t < 0, & \text{at} \quad \Gamma_t \\ (\rho_g)_{|\Gamma_t} = (\rho_g)^t, \quad (T_g)_{|\Gamma_t} = T_g^t, & \text{where} & u_g^t \cdot n_t < 0, & \text{at} \quad \Gamma_t. \end{array} \tag{3.5b}$$

Wall boundary conditions. We consider impermeable walls by imposing no-slip or slip conditions for the velocity. If the wall is considered adiabatic then the normal derivatives of the temperature vanish and if it is considered as isothermal then fixed temperatures must be defined. We can also impose a temperature gradient at the wall by fixing the values of the derivatives. We denote n_w the normal vector to the boundary Γ_w pointing outwards.

$$\text{No-slip isothermal wall:} \quad (u_l)_{|\Gamma_w} = 0, \quad (u_g)_{|\Gamma_w} = 0, \quad (T_g)_{|\Gamma_w} = T_g^w, \quad (T_l)_{|\Gamma_w} = T_l^w \qquad \text{at} \quad \Gamma_w. \tag{3.6}$$

or

$$\text{No-slip adiabatic wall:} \quad (u_l)_{|\Gamma_w} = 0, \quad (u_g)_{|\Gamma_w} = 0, \quad (\partial_n T_g)_{|\Gamma_w} = 0, \quad (\partial_n T_l)_{|\Gamma_w} = 0 \qquad \text{at} \quad \Gamma_w. \tag{3.7}$$

or

$$\text{No-slip temperature gradient wall:} \quad (u_l)_{|\Gamma_w} = 0, \quad (u_g)_{|\Gamma_w} = 0, \quad (\partial_n T_g)_{|\Gamma_w} = q_g, \quad (\partial_n T_l)_{|\Gamma_w} = q_l \quad \text{at} \quad \Gamma_w. \tag{3.8}$$

or

$$\text{Slip adiabatic wall:} \quad \begin{array}{l} ((u_l)_n)_{|\Gamma_w} = ((u_g)_n)_{|\Gamma_w} = 0, \ (\partial_n T_g)_{|\Gamma_w} = (\partial_n T_l)_{|\Gamma_w} = 0, \\ (\mathcal{D}_l\, n_w)_{\tan} = (\mathcal{D}_g\, n_w)_{\tan} = 0 \end{array} \quad \text{at} \quad \Gamma_w. \tag{3.9}$$

Remark 3.2 (Simplified case used in Part II, Section 2 [Burgisser et al., 2024])**.** *If we neglect the thermal conductivity (i.e. $\kappa_l = \kappa_g = 0$), the temperature no longer follows a parabolic equation but follows a transport equation. Thus, the values of the temperatures may only be imposed at the inlet and so the corresponding conditions in* (3.6)-(3.9) *must be eliminated. We then impose inflow conditions as in* (3.3) *or* (3.4)*, and outflow conditions in* (3.5)*. The boundary conditions at the walls are instead*

$$\begin{array}{ll} \textit{Wall:} & \textit{(No-slip)}\ (u_l)_{|\Gamma_w} = 0, \quad (u_g)_{|\Gamma_w} = 0 \\ & \textit{or (Slip)}\ ((u_l)_n)_{|\Gamma_w} = ((u_g)_n)_{|\Gamma_w} = 0, \quad (\mathcal{D}_l \cdot n_w)_{\tan} = (\mathcal{D}_g \cdot n_w)_{\tan} = 0. \end{array} \tag{3.10}$$

If we also neglect the gas and liquid viscosities in the stress tensor, we obtain an Euler-type system. For the outlet condition (3.5a)*, only the pressure should be prescribed. The number of inflow conditions must be reduced by one and the walls conditions must be adapted by eliminating the conditions related to viscosity. To summarize, the conditions are then*

$$\begin{array}{ll} \textit{Outlet:} & (p_l)_{|\Gamma_t} = p_l^t, \quad (p_g)_{|\Gamma_t} = p_g^t, \\ & (\varphi)_{|\Gamma_t} = \varphi^t, \quad (C_l)_{|\Gamma_t} = C_l^t, \quad (T_l)_{|\Gamma_t} = T_l^t \quad \textit{where} \quad u_l^t \cdot n_t < 0, \\ & (\rho_g)_{|\Gamma_t} = (\rho_g)^t, \quad (T_g)_{|\Gamma_t} = T_g^t, \quad \textit{where} \quad u_g^t \cdot n_t < 0. \end{array} \tag{3.11a}$$

$$\begin{array}{ll} \textit{Wall:} & \textit{(No-slip)}\ (u_l)_{|\Gamma_w} = 0, \quad (u_g)_{|\Gamma_w} = 0 \\ & \textit{or (Slip)}\ ((u_l)_n)_{|\Gamma_w} = ((u_g)_n)_{|\Gamma_w} = 0. \end{array}$$

together with

$$\begin{array}{ll} \textit{Inlet BCU:} & \Big\{(\rho_g)_{|\Gamma_b} = (\rho_g)^b \quad \textit{or} \quad (T_g)_{|\Gamma_b} = T_g^b\Big\}, \quad (\varphi)_{|\Gamma_b} = \varphi^b, \quad (T_l)_{|\Gamma_b} = T_l^b, \\ & (u_l)_{|\Gamma_b} = u_l^b, \quad (u_g)_{|\Gamma_b} = u_g^b, \quad (C_l)_{|\Gamma_b} = C_l^b, \end{array} \tag{3.11b}$$

or

$$\begin{array}{ll} \textit{Inlet BCP:} & (p_g)_{|\Gamma_b} = p_g^b, \quad (T_g)_{|\Gamma_b} = T_g^b, \quad (p_l)_{|\Gamma_b} = p_g^b, \quad (T_l)_{|\Gamma_b} = T_l^b, \\ & (\varphi)_{|\Gamma_b} = \varphi^b, \quad (C_l)_{|\Gamma_b} = C_l^b, \quad ((u_g)_{\tan})_{|\Gamma_b} = ((u_l)_{\tan})_{|\Gamma_b} = 0. \end{array}$$

In the simplified case used in Part II, Section 2 [Burgisser et al., 2024] we neglect the thermal conductivity and the gas viscosity and we keep the liquid viscosity. The boundary conditions then differ for the gas and the liquid. Based on the development above, these conditions are:

$$\begin{aligned}
&\textit{Outlet:}\quad (p_l)_{|\Gamma_t} = p_l^t, \quad (p_g)_{|\Gamma_t} = p_g^t, \quad (\mathcal{D}_l\, n_t)_n = (\mathcal{D}_l\, n_t)_{\tan} = 0,\\
&\qquad (\varphi)_{|\Gamma_t} = \varphi^t, \quad (C_l)_{|\Gamma_t} = C_l^t, \quad (T_l)_{|\Gamma_t} = T_l^t \quad \textit{where} \quad u_l^t \cdot n_t < 0,\\
&\qquad (\rho_g)_{|\Gamma_t} = (\rho_g)^t, \quad (T_g)_{|\Gamma_t} = T_g^t, \quad \textit{where} \quad u_g^t \cdot n_t < 0.\\
&\textit{Wall:}\quad \textit{(No-slip)}\ (u_l)_{|\Gamma_w} = 0, \quad (u_g)_{|\Gamma_w} = 0\\
&\qquad \textit{or (Slip)}\ ((u_l)_n)_{|\Gamma_w} = ((u_g)_n)_{|\Gamma_w} = 0, \quad (\mathcal{D}_l \cdot n_w)_{\tan} = 0.
\end{aligned} \tag{3.12a}$$

together with

$$\begin{aligned}
&\textit{Inlet BCU:}\quad (T_g)_{|\Gamma_b} = T_g^b \quad (\varphi)_{|\Gamma_b} = \varphi^b, \quad (T_l)_{|\Gamma_b} = T_l^b,\\
&\qquad (u_l)_{|\Gamma_b} = u_l^b, \quad (u_g)_{|\Gamma_b} = u_g^b, \quad (C_l)_{|\Gamma_b} = C_l^b,\\
&\textit{or}\\
&\textit{Inlet BCP:}\quad (p_g)_{|\Gamma_b} = p_g^b, \quad (T_g)_{|\Gamma_b} = T_g^b, \quad (p_l)_{|\Gamma_b} = p_l^b, \quad (T_l)_{|\Gamma_b} = T_l^b,\\
&\qquad ((u_g)_{\tan})_{|\Gamma_b} = ((u_l)_{\tan})_{|\Gamma_b} = 0, \quad ((\mathcal{D}_l\, n_b)_n)_{|\Gamma_b} = 0,\\
&\qquad (\varphi)_{|\Gamma_b} = \varphi^b, \quad (C_l)_{|\Gamma_b} = C_l^b.
\end{aligned} \tag{3.12b}$$

where we chose to keep the gas temperature as the imposed variable at the inlet.

4 Drift flux system

A drift flux formulation characterizes two-phase systems in which one velocity (pressure, temperature) is an average of that of the two phases and the other is a relative velocity (pressure, temperature) between the two phases [Ishii, 1977]. The objective of rewriting our two-phase system (2.74) into a drift flux form is to propose a system that is amenable to simplification by neglecting terms that have vanishing control on the the relative behavior of the two phases involved, gas and liquid. We first write the system as a drift flux model before formulating it in a dimensionless form in Section 4.2. We then assess in Section 4.3 the relative importance of physically meaningful terms under ranges of parameter values that are relevant in magmatic systems. Section 4.3 ends with a simplified model in dimensionless and dimensional drift-flux form and in the original two-phase form. This is a generalization of the approach detailed in Section 5 of [Bresch et al., 2024] to a temperature-dependent case involving unknowns specific to the situation of volcanic conduit flow.

Remark 4.1. *For clarity, we neglect in this Section 4 the effects of surface tension that are included in* (2.74)*. This can be justified a priori by considering surface tension effect as a modification of the gas pressure (i.e. surface tension is systematically additive to pressure in* (2.74)*):*

$$p_g - \frac{2\sigma}{R} = p_g - 2\sigma \left(\frac{4\pi N(1-\varphi)}{3\varphi}\right)^{1/3} \tag{4.1}$$

In Section 4.3 we present a simplified model by neglecting terms $< 10^{-3}$ in the dimensionless drift flux system using characteristic ranges for all variables that are set in Table 7. As it will be developed in Section 4.2, we use the characteristic variables denoted by subscript 0 and the diacritical mark tilde for the dimensionless variables, hence $\phi = \phi_0\tilde{\phi}$ for any variable ϕ. Following this procedure, let us write (4.1) *in non-dimensional form*

$$p_0\tilde{p}_g - 2\sigma_0 N_0^{1/3}\tilde{\sigma}\tilde{N}^{1/3}\left(\frac{4\pi(1-\varphi)}{3\varphi}\right)^{1/3}.$$

Thus surface tension can be neglected if

$$2\frac{\sigma_0 N_0^{1/3}}{p_0}\left(\frac{4\pi(1-\varphi)}{3\varphi}\right)^{1/3} < 10^{-3} \Longleftrightarrow p_0 > 2\sigma_0 N_0^{1/3}\left(\frac{4\pi(1-\varphi)}{3\varphi}\right)^{1/3} 10^3.$$

We assume that a typical value of the surface tension between water vapor and silicate melt is $\sigma_0 = 10^{-2}$ and we use the range values in Table 7, that is $N_0 = 10^{12}$, $10^6 \leq p_0 \leq 10^8$ Pa. For the most unfavorable case, $p_0 = 10^6$ Pa, we get that surface tension can be neglected if $\varphi > 3 \times 10^{-2}$, which corresponds to nearly the whole range of φ values.

We define the bulk variables Y, ρ, u, w, p, q, T and δT, which we link to the primary unknowns with:

$$\rho Y = \varphi\rho_g \tag{4.2}$$

$$\rho = \varphi\rho_g + (1-\varphi)\rho_l \tag{4.3}$$

$$u = Yu_g + (1-Y)u_l \tag{4.4}$$

$$w = u_g - u_l \tag{4.5}$$

$$p = \varphi p_g + (1-\varphi)p_l \tag{4.6}$$

$$q = p_g - p_l \tag{4.7}$$

$$T = Y_T T_g + (1-Y_T)T_l \tag{4.8}$$

$$\delta T = T_g - T_l \tag{4.9}$$

where the coefficients for the temperature variables are given by

$$c_{pm}Y_T = \varphi\rho_g c_{vg}, \quad c_{pm} = \varphi\rho_g c_{vg} + (1-\varphi)\rho_l c_{pl}, \tag{4.10}$$

where, for clarity, we used the shorthand $c_{vg} = c_{pg} - c_0$ (see Remark 2.7). To ease the readability of the variable change, the expressions of the original variables in terms of the new ones are:

$$p_l = p - \varphi q, \quad p_g = p + (1-\varphi)q, \quad u_l = u - Yw, \quad u_g = u + (1-Y)w, \quad T_l = T - Y_T\delta T, \quad T_g = T + (1-Y_T)\delta T \tag{4.11a}$$

and we can also notice that

$$\rho_l(1-\varphi) = \rho(1-Y), \quad \rho u = \varphi\rho_g u_g + (1-\varphi)\rho_l u_l, \quad c_{pm}(1-Y_T) = c_{pl}\rho_l(1-\varphi). \tag{4.11b}$$

The constitutive equation for the ideal gas (2.13) reads:

$$c_0\rho Y T_g = \varphi(p + (1-\varphi)q). \tag{4.12}$$

To transform the system (2.74) into drift flux variables, we choose convenient combinations of the equations. The first one is the total mass as the sum of (2.74a) and (2.74b). It is followed by the equations for the gas mass (2.74b), the concentration (2.74c) and the volume fraction (2.74f). For the velocity equations, we consider the sum of (2.74d) and (2.74e) that yields (4.15e) and we keep the momentum equation for the gas phase (2.74e). Finally, for the temperature equations, we consider the alternate equation (2.76) instead of (2.74j) and we keep (2.74k) for the liquid temperature. These primary temperature equations in drift flux form are:

$$\begin{aligned} &c_{vg}\rho Y(\partial_t T_g + (u + (1-Y)w)\cdot\nabla T_g) + (p + (1-\varphi)q)\mathrm{div}\Big(u + (\varphi - Y)w\Big) \\ &\quad = \gamma(T_g - T_l) + \mathrm{div}(\varphi\kappa_g\nabla T_g) + \eta_g\varphi\Big(2D(u_g) : \nabla u_g + 3(\mathrm{div}u_g)^2\Big) - \frac{p + (1-\varphi)q}{\rho_l}R^{H_2O} - c_{vg}T_g R^{H_2O} \end{aligned} \tag{4.13}$$

$$c_{p_l}\rho(1-Y)(\partial_t T_l + (u-Yw)\cdot\nabla T_l) = -\gamma(T_g - T_l) + \mathrm{div}((1-\varphi)\kappa_l \nabla T_l) \\ + \eta_l(1-\varphi)\Big(2D(u_l) : \nabla u_l + \frac{1}{3}(\mathrm{div} u_l)^2\Big) + \frac{p-\varphi q}{\rho_l} R^{H_2O} + c_{pl} T_l R^{H_2O}. \tag{4.14}$$

Appendix D presents a detailed calculation of the drift-flux conversion. Briefly, we chose the sum of (2.76) and (2.74k) to get (4.15g) and we calculated $(1-Y_T)\times$(4.13)$-Y_T\times$(4.14) to get (4.15h). The resulting converted system (2.74) is:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0 \tag{4.15a}$$

$$\partial_t(\rho Y) + \mathrm{div}(\rho Y(u + (1-Y)w)) = R^{H_2O} \tag{4.15b}$$

$$\partial_t(\rho(1-Y)(1-C_l)) + \mathrm{div}(\rho(1-Y)(1-C_l)(u-Yw)) = 0 \tag{4.15c}$$

$$\partial_t \varphi + (u-Yw)\cdot\nabla\varphi = \varphi(1-\varphi)\chi q \tag{4.15d}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) + \mathrm{div}(\rho Y(1-Y) w\otimes w) + \nabla p - \mathrm{div}\mathcal{D} - \rho g = 0 \tag{4.15e}$$

$$\begin{aligned} &\partial_t(\rho Y(u+(1-Y)w)) + \mathrm{div}(\rho Y u\otimes u) + \mathrm{div}(\rho Y(1-Y)^2 w\otimes w) \\ &\quad + \mathrm{div}(\rho Y(1-Y)u\otimes w) + \mathrm{div}(\rho Y(1-Y) w\otimes u) \\ &\quad + \varphi\nabla(p + (1-\varphi)q) - \mathrm{div}(\varphi\mathcal{D}_g) + K_d\,\varphi(1-\varphi)w - \rho Y g - \Big(u + \Big(\frac{1}{2} - Y\Big)w\Big) R^{H_2O} = 0 \end{aligned} \tag{4.15f}$$

$$\begin{aligned} &\partial_t(c_{pm}T) + \mathrm{div}(c_{pm}Tu) + \mathrm{div}\Big(c_{pm}((Y_T - Y)T + Y_T(1-Y_T)\delta T)w\Big) + (p + (1-\varphi)q)\mathrm{div}(u + (\varphi - Y)w) \\ &\quad = -\frac{q}{\rho_l} R^{H_2O} + \mathrm{div}(\varphi\kappa_g\nabla(T + (1-Y_T)\delta T) + (1-\varphi)\kappa_l\nabla(T - Y_T\delta T)) \\ &\qquad + \eta_g\varphi\Big(2D(u + (1-Y)w) : \nabla(u + (1-Y)w) + 3(\mathrm{div}(u + (1-Y)w))^2\Big) \\ &\qquad + \eta_l(1-\varphi)\Big(2D(u - Yw) : \nabla(u - Yw) + \frac{1}{3}(\mathrm{div}(u - Yw))^2\Big) \end{aligned} \tag{4.15g}$$

$$\begin{aligned} &c_{pm}Y_T(1-Y_T)\Big(\partial_t(\delta T) + (u-Yw)\cdot\nabla(\delta T) + w\cdot\nabla(T + (1-Y_T)\delta T)\Big) + (1-Y_T)(p + (1-\varphi)q)\mathrm{div}(u + (\varphi - Y)w) \\ &\quad = \gamma\delta T - (p + (1-\varphi-Y_T)q)\frac{R^{H_2O}}{\rho_l} - \frac{c_{pm}Y_T(1-Y_T)}{\rho Y(1-Y)}(T + (1-Y-Y_T)\delta T)R^{H_2O} \\ &\qquad + (1-Y_T)\mathrm{div}(\varphi\kappa_g\nabla(T + (1-Y_T)\delta T) - Y_T\mathrm{div}((1-\varphi)\kappa_l\nabla(T - Y_T\delta T)) \\ &\qquad\quad + (1-Y_T)\eta_g\varphi\Big(2D(u + (1-Y)w) : \nabla(u + (1-Y)w) + 3(\mathrm{div}(u + (1-Y)w))^2\Big) \\ &\qquad\quad - Y_T\eta_l(1-\varphi)\Big(2D(u - Yw) : \nabla(u - Yw) + \frac{1}{3}(\mathrm{div}(u - Yw))^2\Big) \end{aligned} \tag{4.15h}$$

where $\chi = \dfrac{3}{4\eta_l}$ and $\mathcal{D} = (1-\varphi)\mathcal{D}_l + \varphi\mathcal{D}_g$ (see (2.10)-(2.11)).

The unknowns of the system are: $p, q, C_l, u, w, \varphi, T, \delta T$. The density ρ is obtained from the constitutive equation (4.12). From the original system, we also have this relation as an alternative to equation (4.15d):

$$\mathrm{div}(u - Yw) = \varphi\chi q - \frac{R^{H_2O}}{\rho(1-Y)}. \tag{4.16}$$

Thus, considering (4.16) instead of (4.15d), the unknowns become $p, \rho, C_l, u, w, q, T_g, T_l$ and φ is obtained from the constitutive equation (4.12).

The alternative equations (2.54)–(2.55) for the total water content, C_T, give the following equivalent conservation and transport equations:

$$\partial_t(\rho(1-C_T)) + \mathrm{div}(\rho(1-C_T)(u - Yw)) = 0, \tag{4.17}$$

$$\partial_t C_T + (u - Yw)\cdot\nabla C_T = -\frac{1-C_T}{\rho}\mathrm{div}(\rho Y w). \tag{4.18}$$

4.1 Alternative momentum equations

In the momentum equations, (4.15f) needs to be replaced by an alternate equation on the relative velocity w (i.e. a equation with only w in the time derivative). This calculation is similar to those performed in the review paper [Bresch et al., 2024] in Section 5.2 with the following equivalence in the notation: $\Gamma = R^{H_2O}$, $\Lambda|w| = K_d\varphi(1-\varphi)$, $\Theta = \frac{3\varphi(1-\varphi)}{4\eta_l}$, $f_g = f_l = g$, $\beta_u = 1$, $\beta_p = 0$. Notice that this calculation is made for the model in [Ambroso et al., 2008] where the viscous tensor is not considered and no exchange appears in the momentum equation due to the term R^{H_2O} (Γ in their paper). Nevertheless we can follow the same calculations to obtain the following equation for w:

$$\begin{aligned}
&\partial_t w + \mathrm{div}((1-Y)w\otimes w) + u\cdot\nabla w + w\cdot\nabla u - Yw\cdot\nabla w - (1-Y)w\,\mathrm{div}w \\
&\quad = \frac{1}{\rho Y(1-Y)}\Big(-K_d\varphi(1-\varphi)w + (1-Y)\mathrm{div}(\varphi\mathcal{D}_g) - Y\mathrm{div}((1-\varphi)\mathcal{D}_l) + (Y-\varphi)\nabla p - \varphi\nabla((1-\varphi)q) \\
&\qquad\qquad - R^{H_2O}\left(\frac{1}{2} - Y\right)w\Big).
\end{aligned} \tag{4.19}$$

This formulation is useful when the asymptotic equilibrium $w = 0$ is satisfied in the physical system considered. It is not the case for magmas because the gas phase can accelerate significantly compared to the liquid owing to the large viscosity contrast between the two phases. The large density contrast between the two phases, however, causes $|u - u_l| \ll 1$ (Part II, Section 6 [Burgisser et al., 2024]). That $u - u_l = Yw$ makes an equation based on this quantity a desirable goal. We thus follow the reasoning used to obtain (4.19) to find an equation on Yw instead of w. From (4.15b) and (4.15e), we have:

$$\partial_t(\rho Y u) + \mathrm{div}(\rho Y u\otimes u) + \mathrm{div}(\rho Y(1-Y)w)u = uR^{H_2O} + Y\Big(-\mathrm{div}(\rho Y(1-Y)w\otimes w) - \nabla p + \mathrm{div}\mathcal{D} + \rho g\Big).$$

Then we write (4.15f) as

$$\begin{aligned}
&\partial_t(\rho Y(1-Y)w) + \mathrm{div}(\rho Y(1-Y)^2 w\otimes w) \\
&\quad + \mathrm{div}(\rho Y(1-Y)u\otimes w) + \rho Y(1-Y)w\cdot\nabla u - (\frac{1}{2} - Y)wR^{H_2O} \\
&\quad - Y\mathrm{div}(\rho Y(1-Y)w\otimes w) + Y(-\nabla p + \mathrm{div}\mathcal{D} + \rho g) \\
&\quad + \varphi\nabla(p + (1-\varphi)q) - \mathrm{div}(\varphi\mathcal{D}_g) + K_d\varphi(1-\varphi)w - \rho Y g = 0.
\end{aligned} \tag{4.20}$$

Using mass equations (4.15a), (4.15b) we can write

$$Yw\Big(\partial_t(\rho(1-Y)) + \mathrm{div}(\rho(1-Y)u) - \mathrm{div}(\rho Y(1-Y)w)\Big) = -R^{H_2O}Yw. \tag{4.21}$$

We now rewrite (4.20) as follows:

$$\begin{aligned}
&\rho(1-Y)\partial_t(Yw) + Yw\partial_t(\rho(1-Y)) + \mathrm{div}(\rho(1-Y)^2w)Yw \\
&\quad + \rho(1-Y)^2w\cdot\nabla(Yw) + \mathrm{div}(\rho(1-Y)u)Yw + \rho(1-Y)u\cdot\nabla(Yw) \\
&\quad + \rho Yw(1-Y)\cdot\nabla u - \left(\frac{1}{2} - Y\right)wR^{H_2O} - Y^2\mathrm{div}(\rho(1-Y)w)w \\
&\quad - \rho Y(1-Y)w\cdot\nabla(Yw) + Y(-\nabla p + \mathrm{div}\mathcal{D} + \rho g) \\
&\quad + \varphi\nabla(p + (1-\varphi)q) - \mathrm{div}(\varphi\mathcal{D}_g) + K_d\varphi(1-\varphi)w - \rho Yg = 0.
\end{aligned} \tag{4.22}$$

This last identity can be simplified using (4.21):

$$\begin{aligned}
&\rho(1-Y)\partial_t(Yw) + \mathrm{div}(\rho(1-Y)w)(1-Y)Yw + \rho(1-2Y)(1-Y)w \\
&\cdot\nabla(Yw) + \rho(1-Y)u\cdot\nabla(Yw) + \rho(1-Y)Yw\cdot\nabla u - \frac{1}{2}wR^{H_2O} \\
&+ Y(-\nabla p + \mathrm{div}\mathcal{D} + \rho g) + \varphi\nabla(p + (1-\varphi)q) - \mathrm{div}(\varphi\mathcal{D}_g) + K_d\varphi(1-\varphi)w = 0.
\end{aligned} \tag{4.23}$$

Dividing by $\rho(1-Y)$, collecting similar terms together, and using $\varphi w = \frac{\rho}{\rho_g}Yw$ in the drag term yields:

$$\begin{aligned}
&\partial_t(Yw) + \mathrm{div}(\rho(1-Y)w)\frac{1}{\rho}Yw + (1-2Y)w\cdot\nabla(Yw) + u\cdot\nabla(Yw) + Yw\cdot\nabla u \\
&= \frac{1}{\rho(1-Y)}\Big((1-Y)\mathrm{div}(\varphi\mathcal{D}_g) - Y\mathrm{div}((1-\varphi)\mathcal{D}_l) + (Y-\varphi)\nabla p - \varphi\nabla((1-\varphi)q) \\
&\qquad\qquad - K_d(1-\varphi)\frac{\rho}{\rho_g}Yw + \frac{1}{2}wR^{H_2O}\Big).
\end{aligned} \tag{4.24}$$

Equations (4.19) or (4.24) may replace (4.15f).

Remark 4.2. *Note that the term in* (4.24)

$$\mathrm{div}(\rho(1-Y)w)\frac{1}{\rho}Yw$$

may be written as

$$\mathrm{div}(\rho(1-Y)w)\frac{1}{\rho}Yw = \mathrm{div}((1-Y)w)Yw + Y(1-Y)w\cdot\frac{\nabla\rho}{\rho}w. \tag{4.25}$$

4.2 Dimensionless drift flux and ranges of physical parameters

The goal of this section is to transform the drift flux model into a dimensionless form that contain classical dimensionless numbers. The resulting model features dimensionless factors that are easily compared to each other to decide which terms have the most weight or on the contrary which ones can be neglected. The natural application of the (full) model in Part II [Burgisser et al., 2024] suggests three relevant limits. As mentioned in Section 4.1, $|u-u_l| \ll 1$. For the temperatures it is observed that $|T_g - T_l| \ll 1$. For the water concentration, the steady state solution is close to the equilibrium state $|C_l - k_h\sqrt{p_g}| \ll 1$, which in turn suggests that $|R^{H_2O}| \ll 1$. We analyze here these cases to understand why these limits are achieved by bracketing the ranges of the physical quantities involved.

To isolate the three observed limits in the dimensionless system, we replace R^{H_2O} in (4.15g), (4.15h) and (4.24) with its expression in (4.15b). The same substitution is not possible for the other quantities, $Yw = u - u_l$, $\delta T = T_g - T_l$, because they are affected by continuity equations, (4.24) and (4.15h), that act as relaxed equations for the relaxation parameters K_d and γ, respectively. These equations must thus be analyzed first to find the respective orders of Yw and δT. We then use in Section 4.3 the orders of magnitude of these two quantities to perform the scaling of the rest of the equations. The system to be analyzed for unknowns $\rho, \varphi, C_l, q, u, w, T, \delta T$ is thus:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{4.26a}$$

$$\partial_t(\rho Y) + \mathrm{div}(\rho Y(u + (1-Y)w)) = R^{H_2O} \tag{4.26b}$$

$$\partial_t(\rho(1-Y)(1-C_l)) + \mathrm{div}(\rho(1-Y)(1-C_l)(u - Yw)) = 0 \tag{4.26c}$$

$$\partial_t \varphi + (u - Yw)\cdot\nabla\varphi = \frac{3\varphi}{4\eta_l}(1-\varphi)q \tag{4.26d}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \mathrm{div}(\rho Y(1-Y) w \otimes w) + \nabla p - \mathrm{div}\Big((1-\varphi)\mathcal{D}_l + \varphi \mathcal{D}_g\Big) = \rho g \tag{4.26e}$$

$$\begin{aligned}
&\partial_t(Yw) + \mathrm{div}(\rho(1-Y)w)\frac{1}{\rho} Yw + (1-2Y) w\cdot\nabla(Yw) + u\cdot\nabla(Yw) + Yw\cdot\nabla u \\
&\quad = \frac{1}{\rho(1-Y)}\Big((1-Y)\mathrm{div}(\varphi\mathcal{D}_g) - Y\mathrm{div}((1-\varphi)\mathcal{D}_l) + (Y-\varphi)\nabla p - \varphi\nabla((1-\varphi)q) - K_d(1-\varphi)\frac{\rho}{\rho_g} Yw\Big) \\
&\qquad + \frac{1}{2}\frac{1}{\rho(1-Y)} w(\partial_t(\rho Y) + \mathrm{div}(\rho Y(u + (1-Y)w))).
\end{aligned} \tag{4.26f}$$

$$\begin{aligned}
&\partial_t(c_{pm}T) + \mathrm{div}(c_{pm}Tu) + \mathrm{div}\Big(c_{pm}((Y_T - Y)T + Y_T(1-Y_T)\delta T)w\Big) + (p + (1-\varphi)q)\mathrm{div}(u + (\varphi - Y)w) \\
&\quad = \mathrm{div}(\varphi\kappa_g\nabla(T + (1-Y_T)\delta T) + (1-\varphi)\kappa_l\nabla(T - Y_T\delta T)) \\
&\qquad + \eta_g\varphi\Big(2D(u + (1-Y)w) : \nabla(u + (1-Y)w) + 3(\mathrm{div}(u + (1-Y)w))^2\Big) \\
&\qquad + \eta_l(1-\varphi)\Big(2D(u - Yw) : \nabla(u - Yw) + \frac{1}{3}(\mathrm{div}(u - Yw))^2\Big) - \frac{1}{\rho_l} q(\partial_t(\rho Y) + \mathrm{div}(\rho Y(u + (1-Y)w)))
\end{aligned} \tag{4.26g}$$

$$\begin{aligned}
&c_{pm}Y_T(1-Y_T)\Big(\partial_t(\delta T) + (u - Yw)\cdot\nabla(\delta T) + w\cdot\nabla(T + (1-Y_T)\delta T)\Big) + (1-Y_T)(p + (1-\varphi)q)\mathrm{div}(u + (\varphi - Y)w) \\
&\quad = \gamma\delta T - \left((p + (1-\varphi-Y_T)q)\frac{1}{\rho_l} + \frac{c_{pm}Y_T(1-Y_T)}{\rho Y(1-Y)}(T + (1-Y-Y_T)\delta T)\right)(\partial_t(\rho Y) + \mathrm{div}(\rho Y(u + (1-Y)w))) \\
&\qquad + (1-Y_T)\mathrm{div}\Big(\varphi\kappa_g\nabla(T + (1-Y_T)\delta T)\Big) - Y_T\mathrm{div}\Big((1-\varphi)\kappa_l\nabla(T - Y_T\delta T)\Big) \\
&\qquad + (1-Y_T)\eta_g\varphi\Big(2D(u + (1-Y)w) : \nabla(u + (1-Y)w) + 3(\mathrm{div}(u + (1-Y)w))^2\Big) \\
&\qquad - Y_T\eta_l(1-\varphi)\Big(2D(u - Yw) : \nabla(u - Yw) + \frac{1}{3}(\mathrm{div}(u - Yw))^2\Big)
\end{aligned} \tag{4.26h}$$

where

$$\mathcal{D}_l = \eta_l\Big(2D(u - Yw) + \frac{1}{3}\mathrm{div}(u - Yw)\Big), \qquad \mathcal{D}_g = \eta_g\Big(2D(u + (1-Y)w) + 3\mathrm{div}(u + (1-Y)w)\Big).$$

The calculated variables are Y, Y_T, p from

$$Y = \frac{\varphi p_g}{c_0 \rho T_g}, \qquad Y_T = \frac{\rho Y c_{vg}}{c_{pm}}, \qquad c_0 \rho Y (T + (1 - Y_T)\delta T) = \varphi(p + (1-\varphi)q) \tag{4.27}$$

We introduce the diacritical mark tilde for the dimensionless variables, that is $\phi = \phi_0 \tilde{\phi}$ where ϕ_0 is some reference value. As we deal with a three dimensional problem, the space variable (x, y, z) has a different characteristic length for each component. In the present application, the domain is the volcano conduit that can be depicted as a cylinder where the z component, representing the conduit length, spans a distance much larger than the horizontal component representing the cylinder diameter. Here we are interested in analyzing a most unfavorable case, so we consider only one characteristic length L_0 for the three dimensions, which we equate to the cylinder radius. For simplicity, some dimensional constants are kept unchanged in the calculations $(D, k_h, c_{pl}, c_{pg}, c_0, \kappa_l, \kappa_g)$. The dimensionless variables are thus defined as:

$$t = t_0\tilde{t}, \quad u = u_0\tilde{u}, \quad w = w_0\tilde{w}, \quad \rho_l = \rho_{l0}\tilde{\rho}_l, \quad \rho_g = \rho_{g0}\tilde{\rho}_g, \quad p = p_0\tilde{p}, \quad q = q_0\tilde{q},$$

$$C_l = C_{l0}\tilde{C}_l, \quad T = T_0\tilde{T}, \quad \delta T = \delta_{T_0}\tilde{\delta T},$$

$$\eta_l = \eta_{l0}\tilde{\eta}_l, \quad \eta_g = \eta_{g0}\tilde{\eta}_g, \quad g = |g|\tilde{g},$$

$$K_d = K_{d0}\tilde{K}_d, \quad \gamma = \gamma_0\tilde{\gamma}, \quad R^{H_2O} = R_0\tilde{R}^{H_2O},$$

and the space derivative operators are denoted as

$$\mathrm{div}(\cdot) = \frac{1}{L_0}\tilde{\mathrm{div}}(\cdot), \quad \nabla(\cdot) = \frac{1}{L_0}\tilde{\nabla}(\cdot).$$

The characteristic values C_{l0}, K_{d0}, γ_0 and R_0 correspond to the particular expressions considered for these terms. In the exchange term R^{H_2O} given by (2.38d), we consider $N = N_0\tilde{N}, p_g = p_{g0}\tilde{p}_g$ and we identify

$$R_0 = 3^{1/3}(4\pi N_0)^{2/3} D\rho_{l0} C_{l0}, \qquad \tilde{R}^{H_2O} = \tilde{\rho}_l(\tilde{N})^{2/3}\frac{\varphi^{2/3}(1-\varphi)^{2/3}}{1-\varphi^{1/3}}\left(\tilde{C}_l - \sqrt{\tilde{p}_g}\right), \quad \text{with} \quad C_{l0} = k_h\sqrt{p_{g0}}, \tag{4.28a}$$

To avoid confusion with the operator $D(u) = \frac{\nabla u + \nabla^t u}{2}$ appearing in the velocity and temperature equations, we will use the notation $D_u(u)$ for this operator in this Section, hence $\tilde{D}_u(u)$ for those without dimension. In Part II, Section 2 [Burgisser et al., 2024], we will use the following expression for the heat transfer coefficient γ:

$$\gamma = \frac{24\kappa_l\varphi\mathrm{Nu}}{R_b^2}, \qquad \mathrm{Nu} = 7 - 10(1-\varphi) + 5(1-\varphi)^2, \qquad R_b^3 = \frac{3\varphi}{4\pi N(1-\varphi)}.$$

Then, considering that $\mathrm{Nu} = \mathrm{Nu}_0\tilde{\mathrm{Nu}}$, we identify

$$\gamma_0 = 24\left(\frac{4\pi}{3}\right)^{2/3} N_0^{2/3}\kappa_l \mathrm{Nu}_0, \qquad \tilde{\gamma} = (1-\varphi)^{2/3}\varphi^{1/3}\tilde{\mathrm{Nu}}\tilde{N}^{2/3}. \tag{4.28b}$$

The drag coefficient K_d is given in equation (2.12) for bubbly and permeable flow. As our application in Part II [Burgisser et al., 2024] is based on permeable flows, we choose $K_d = \eta_g/k$, where k is the permeability coefficient of the magma. Then we set

$$K_{d0} = \frac{\eta_{g0}}{k_0}, \qquad \tilde{K}_d = \frac{\tilde{\eta}_g}{\tilde{k}}. \tag{4.28c}$$

The bulk density variable Y is written as

$$Y = \frac{\rho_{g0}}{\rho_{l0}} \frac{1}{1 + \frac{\rho_{g0}}{\rho_{l0}} \tilde{Y}} \tilde{Y}, \qquad \text{so} \qquad 1 - Y = \frac{1}{1 + \frac{\rho_{g0}}{\rho_{l0}} \tilde{Y}},$$

for $\tilde{Y} = \frac{\varphi \tilde{\rho}_g}{(1-\varphi)\tilde{\rho}_l} = \frac{\rho_{l0} Y}{\rho_{g0}(1-Y)}$. Two-way variable conversion can be done with:

$$\rho = \rho_{l0} \left(1 + \frac{\rho_{g0}}{\rho_{l0}} \tilde{Y}\right) (1 - \varphi)\tilde{\rho}_l \quad \text{and} \quad \rho Y = \rho_{g0}(1 - \varphi)\tilde{\rho}_l \tilde{Y}.$$

To keep the temperature equations concise, we introduce

$$C_p = \frac{c_{vg}}{c_{pl}}$$

and we write

$$Y_T = \frac{\rho_{g0}}{\rho_{l0}} C_p \frac{1}{1 + \frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y}} \tilde{Y}, \quad \text{and} \quad c_{pm} = \rho_{l0} \left(1 + \frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y}\right) (1 - \varphi)\tilde{\rho}_l c_{pl}.$$

Two-way variable conversion can be done with:

$$1 - Y_T = \frac{1}{1 + \frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y}} \quad \text{and} \quad c_{pm} Y_T = \rho_{g0} c_{vg} (1 - \varphi)\tilde{\rho}_l \tilde{Y}.$$

Several classical dimensionless numbers are found in the equations: St is the Strouhal number, Ma is the Mach number, Re is the Reynolds number, Fr is the Froude number, Pe is the thermal Peclet number, and Br is the Brinkman number based on a reference temperature (see [Costa et al., 2007]):

$$\mathrm{St} = \frac{L_0}{u_0 t_0}, \quad \mathrm{Fr}^2 = \frac{u_0^2}{L_0 |g|}, \quad \mathrm{Ma}^2 = \frac{\rho_{l0} u_0^2}{p_0}, \quad \mathrm{Re}_l = \frac{\rho_{l0} u_0 L_0}{\eta_{l0}}, \quad \mathrm{Re}_g = \frac{\rho_{g0} u_0 L_0}{\eta_{g0}}$$

$$\mathrm{Pe}_l = \frac{\rho_{l0} u_0 L_0 c_{pl}}{\kappa_l}, \quad \mathrm{Pe}_g = \frac{\rho_{g0} u_0 L_0 c_{vg}}{\kappa_g}, \quad \mathrm{Br}_l = \frac{\eta_{l0} u_0^2}{\kappa_l \delta_{T_0}}, \quad \mathrm{Br}_g = \frac{\eta_{g0} u_0^2}{\kappa_g \delta_{T_0}}$$

The resulting dimensionless, drift flux system is lengthy, which led us to present it in Appendix E. The simplified version of the system omitting lower-order terms is presented in the next Section.

4.3 Drift flux simplification to the main order

In this section we write the dimensionless drift flux system up to order 10^{-3}. The complete system is written in Appendix E. Tables 7, 8, and 9 contain characteristic value ranges of the physical quantities involved. In a volcanic conduit, we expect different ranges of values between the bottom and the top of the conduit because of the different gas volume fractions and pressures at each end (generally p_g and ρ_g are high at the base and low at the top and φ has the opposite trend). For completeness, we assess both situations separately to then consider the more unfavorable case between them to write the asymptotic system.

For simplicity, we assume $\mathrm{St} \sim 1$, which implies a characteristic time of $t_0 \sim 10^3$ s. This assumption is revisited when discussing the final simplified system in two-phase form (see (4.37) below). The particular limits we hope to find are related to R^{H_2O}, Yw and δT. As R^{H_2O} has been already replaced in the system, we must first study the equations of Yw and δT to find their orders and then replace them as well in the other equations of the system.

Description	Conduit bottom	Conduit top	Remarks
L_0	10	10	
t_0	?	?	to be determined (see main text)
u_0	10^{-2}	1	
w_0	0 if $u_g = u_l$	0 if $u_g = u_l$	maximum value 100
ρ_{l0}	2200	2200	
ρ_{g0}	200	$0,2$	
p_0	10^8	10^6	
q_0	0 if $p_g = p_l$	0 if $p_g = p_l$	maximum value 10^6
p_{g0}	10^8	10^6	
T_0	10^3	10^3	
δ_{T_0}	1	1	
η_{l0}	10^6	10^6	
η_{g0}	10^{-5}	10^{-5}	
k_h	4×10^{-6}	4×10^{-6}	
c_{pl}	10^3	10^3	
c_{pg}	2×10^3	2×10^3	
c_0	5×10^2	5×10^2	
C_p	1.5	1.5	
κ_l	2.3	2.3	
κ_g	2×10^{-2}	2×10^{-2}	
N_0	10^{12}	10^{12}	
Nu_0	2	2	

Table 7: Dimensionless quantities I: elementary values

Description	Conduit bottom	Conduit top	Remarks
$\mathrm{St} = L_0/(u_0 t_0)$	$10^3/t_0$	$10/t_0$	
$\mathrm{Fr}^2 = u_0^2/(L_0\lvert g\rvert)$	10^{-6}	10^{-2}	
$\mathrm{Ma}^2 = \rho_{l0} u_0^2/p_0$	2×10^{-7}	2×10^{-3}	
$\mathrm{Re}_l = \rho_{l0} u_0 L_0/\eta_{l0}$	2×10^{-4}	2×10^{-2}	
$\mathrm{Re}_g = \rho_{g0} u_0 L_0/\eta_{g0}$	2×10^6	2×10^5	
$\mathrm{Pe}_l = \rho_{l0} u_0 L_0 c_{pl}/\kappa_l$	10^5	10^7	
$\mathrm{Pe}_g = \rho_{g0} u_0 L_0 c_{vg}/\kappa_g$	10^6	10^5	undefined if $\varphi = 0$
$\mathrm{Br}_l = \eta_{l0} u_0^2/(\kappa_l \delta_{T_0})$	5×10^1	5×10^5	
$\mathrm{Br}_g = \eta_{g0} u_0^2/(\kappa_g \delta_{T_0})$	5×10^{-8}	5×10^{-4}	undefined if $\varphi = 0$

Table 8: Dimensionless quantities II: classical dimensionless numbers

Description	Conduit bottom	Conduit top	Remarks
ρ_{g0}/ρ_{l0}	10^{-1}	10^{-4}	
q_0/p_0	0 if $p_g = p_l$	0 if $p_g = p_l$	maximum value 1
w_0/u_0	0 if $u_g = u_l$	0 if $u_g = u_l$	maximum value 100
T_0/δ_{T_0}	$10^3 - 10^8$	$10^3 - 10^8$	
$p_0/(\rho_{g0} T_0 c_{pl})$	0.5	0.5	
$C_{l0} = k_h\sqrt{p_{g0}}$	4×10^{-2}	4×10^{-3}	
L_0^2/k_0	10^{21}	10^{14}	
R_0	3×10^4	3×10^2	for $D = 10^{-10}$, or 0 if $D = 0$
$L_0 R_0/(\rho_{l0} u_0) = 3^{1/3}(4\pi N_0)^{2/3} D C_{l0} L_0/u_0$	10^4	1	for $D = 10^{-10}$, or 0 if $D = 0$
$\gamma_0 L_0/(\rho_{l0} u_0 c_{pl}) = 24\,(4\pi/3)^{2/3}\,\mathrm{Pe}_l^{-1}\mathrm{Nu}_0\,N_0^{2/3} L_0^2$	10^5	10^7 if $\varphi = 0$	

Table 9: Dimensionless quantities III: particular relations

The drift velocity Yw is involved in a relaxation equation for the drag coefficient K_d. In the dimensionless equation (E.6) , the coefficient becomes L_0^2/k_0, varying from 10^{14} to 10^{21} (see Table 9). The term giving the main order is the same at the bottom and at the top of the conduit, the order being however different:

$$\frac{\tilde{K}_d(1-\varphi)}{\tilde{\rho}_g}\tilde{Y}\tilde{w} = O\left(\left(\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{1}{\mathrm{Re}_g}\frac{L_0^2}{k_0}\right)^{-1}\frac{1}{\mathrm{Ma}^2}\frac{-\varphi}{\tilde{\rho}_l(1-\varphi)}\tilde{\nabla}\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)\right) = O(10^{m_1}),\quad m_1 = \begin{cases} -9 & \text{bottom} \\ -4 & \text{top} \end{cases}$$

Similarly, the temperature difference (E.8) is involved in a relaxation equation for the coefficient γ. The corresponding dimensionless coefficient is $\gamma_0 L_0/(\rho_{l0} u_0 c_{pl})$, which is on the order of $10^5 - 10^7$ (see Table 9):

$$\tilde{\gamma}\delta\tilde{T} = O\left(\left(\frac{\gamma_0 L_0}{\rho_{l0} u_0 c_{pl}}\right)^{-1}\frac{p_0}{\delta_{T_0}\rho_{l0} c_{pl}}\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)\tilde{\mathrm{div}}\,(\tilde{u}+\varphi\tilde{w})\right) = O(10^{m_2}),\ m_2 = \begin{cases} -3 & \text{bottom} \\ -7 & \text{top} \end{cases}$$

We can now study the rest of equations in the system where we embed

$$\tilde{Y}\tilde{w} = \left(\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{1}{\mathrm{Re}_g}\frac{L_0^2}{k_0}\right)^{-1}\frac{1}{\mathrm{Ma}^2}\frac{\tilde{\rho}_g}{\tilde{K}_d(1-\varphi)}\frac{-\varphi}{\tilde{\rho}_l(1-\varphi)}\tilde{\nabla}\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right), \tag{4.29}$$

$$\delta\tilde{T} = \left(\frac{\gamma_0 L_0}{\rho_{l0} u_0 c_{pl}}\right)^{-1}\frac{p_0}{\delta_{T_0}\rho_{l0} c_{pl}}\frac{1}{\tilde{\gamma}}\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)\tilde{\mathrm{div}}\,(\tilde{u}+\varphi\tilde{w})\,. \tag{4.30}$$

The mass equations.
The total mass equation (E.1) is:

- at the bottom of the conduit:

$$\mathrm{St}\,\partial_{\tilde{t}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\right)+\tilde{\mathrm{div}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{u}\right)=0,$$

- at the top of the conduit:

$$\mathrm{St}\,\partial_{\tilde{t}}\left((1-\varphi)\tilde{\rho}_l\right)+\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\rho}_l\tilde{u}\right)=O(10^{-4}).$$

The gas mass equation (E.2) gives the limit for R^{H_2O}:

$$\tilde{R}^{H_2O}=O\left(\left(\frac{\rho_{l0}}{\rho_{g0}}\frac{L_0R_0}{\rho_{l0}u_0}\right)^{-1}\left(\mathrm{St}\,\partial_{\tilde{t}}\left((1-\varphi)\tilde{\rho}_l\tilde{Y}\right)+\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\rho}_l\tilde{Y}\tilde{u}\right)\right)\right)=O(10^{m_3}),\qquad m_3=\begin{cases}-5 & \text{bottom}\\ -4 & \text{top}\end{cases}$$

The water concentration (E.3) is:

- at the bottom of the conduit:

$$\mathrm{St}\,\partial_{\tilde{t}}\left((1-\varphi)\tilde{\rho}_l(1-C_{l0}\tilde{C}_l)\right)+\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\rho}_l(1-C_{l0}\tilde{C}_l)\tilde{u}\right)=O(10^{-8}),\tag{4.31}$$

- at the top of the conduit:

$$\mathrm{St}\,\partial_{\tilde{t}}\left((1-\varphi)\tilde{\rho}_l\right)+\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\rho}_l\tilde{u}\right)=O(10^{-3}).$$

  This equation does not give any information about $\tilde{C}_l$ because the characteristic concentration is $C_{l0}\sim 10^{-3}$ at the top of the conduit (see Table 9).

Finally, the gas volume fraction (E.4) is:

$$\varphi(1-\varphi)\frac{3}{4\tilde{\eta}_l}\tilde{q}=\left(\frac{\mathrm{Re}_l}{\mathrm{Ma}^2}\frac{q_0}{p_0}\right)^{-1}\left(\mathrm{St}\,\partial_{\tilde{t}}\varphi+\tilde{u}\cdot\tilde{\nabla}\varphi\right)+O(10^{m_4}),\qquad m_4=\begin{cases}-8 & \text{bottom}\\ -6 & \text{top}\end{cases}$$

with $O\left(\left(\frac{\mathrm{Re}_l}{\mathrm{Ma}^2}\frac{q_0}{p_0}\right)^{-1}\right)=O(10^{-1})$.

The mixture velocity equation (E.5)

- at the bottom of the conduit:

$$\tilde{\nabla}\tilde{p}=\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l+O(10^{-3})\tag{4.32}$$

  where $O\left(\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\right)=O(10^{-1})$.

- at the top of the conduit:

$$\tilde{\nabla}\tilde{p}=\frac{\mathrm{Ma}^2}{\mathrm{Re}_l}\left(\tilde{\nabla}\left((1-\varphi)\tilde{\eta}_l\frac{1}{3}\tilde{\mathrm{div}}\,\tilde{u}\right)+2\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\eta}_l\tilde{D}_u(\tilde{u})\right)\right)+\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}(1-\varphi)\tilde{\rho}_l\tilde{g}+O(10^{-3})$$

  where $O(\mathrm{Ma}^2/\mathrm{Re}_l)=O(\mathrm{Ma}^2/\mathrm{Fr}^2)=O(10^{-1})$.

The mixture temperature equation (E.7)

- at the bottom of the conduit:

$$\begin{aligned}\mathrm{St}\,\partial_{\tilde{t}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}C_p\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{T}\right)+\tilde{\mathrm{div}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}C_p\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{T}\tilde{u}\right)\\ =-\frac{\rho_{g0}}{\rho_{l0}}\frac{p_0}{\rho_{g0}T_0c_{pl}}\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)\tilde{\mathrm{div}}\tilde{u}+O(10^{-3})\end{aligned} \tag{4.33}$$

with $O\left(\frac{\rho_{g0}}{\rho_{l0}}\frac{p_0}{\rho_{g0}T_0c_{pl}}\right)=O(10^{-2})$ and $O(q_0/p_0)=O(1)$.

- at the top of the conduit:

$$\mathrm{St}\,\partial_{\tilde{t}}\left((1-\varphi)\tilde{\rho}_l\tilde{T}\right)+\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\rho}_l\tilde{T}\tilde{u}\right)=O(10^{-4}). \tag{4.34}$$

Pressure equation (E.9)

$$C_p(1-\varphi)\tilde{\rho}_l\tilde{Y}\ \tilde{T}=\frac{c_{vg}}{c_0}\frac{p_0}{\rho_{g0}T_0c_{pl}}\varphi\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)+O(10^{m_5}),\qquad m_5=\begin{cases}-6 & \text{bottom}\\ -10 & \text{top}\end{cases}$$

where $O\left(\frac{c_{vg}}{c_0}\frac{p_0}{\rho_{g0}T_0c_{pl}}\right)=O(C_p)=O(q_0/p_0)=O(1)$. Using the expression of C_p it can also be written as

$$(1-\varphi)\tilde{\rho}_l\tilde{Y}\ \tilde{T}=\frac{p_0}{\rho_{g0}T_0c_0}\varphi\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)+O(10^{m_5}).$$

We can now write the asymptotic system where only the most unfavorable cases remain:

$$\tilde{R}^{H_2O}=O\left(\left(\frac{\rho_{l0}}{\rho_{g0}}\frac{L_0R_0}{\rho_{l0}u_0}\right)^{-1}\left(\mathrm{St}\,\partial_{\tilde{t}}\left((1-\varphi)\tilde{\rho}_l\tilde{Y}\right)+\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\rho}_l\tilde{Y}\tilde{u}\right)\right)\right)=O(10^{-4}) \tag{4.35a}$$

$$\tilde{Y}\tilde{w}=O\left(\left(\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{1}{\mathrm{Re}_g}\frac{L_0^2}{k_0}\right)^{-1}\frac{1}{\mathrm{Ma}^2}\frac{\tilde{\rho}_g}{\tilde{K}_d(1-\varphi)}\frac{-\varphi}{\tilde{\rho}_l(1-\varphi)}\tilde{\nabla}\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)\right)=O(10^{-4}) \tag{4.35b}$$

$$\delta\tilde{T}=O\left(\left(\frac{\gamma_0L_0}{\rho_{l0}u_0c_{pl}}\right)^{-1}\frac{p_0}{\delta_{T_0}\rho_{l0}c_{pl}}\frac{1}{\tilde{\gamma}}\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)\tilde{\mathrm{div}}\left(\tilde{u}+\varphi\tilde{w}\right)\right)=O(10^{-7}) \tag{4.35c}$$

$$\mathrm{St}\,\partial_{\tilde{t}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\right)+\tilde{\mathrm{div}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{u}\right)=0, \tag{4.35d}$$

$$\mathrm{St}\,\partial_{\tilde{t}}\left((1-\varphi)\tilde{\rho}_l(1-C_{l0}\tilde{C}_l)\right)+\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\rho}_l(1-C_{l0}\tilde{C}_l)\tilde{u}\right)=O(10^{-8}), \tag{4.35e}$$

$$\mathrm{St}\,\partial_{\tilde{t}}\varphi+\tilde{u}\cdot\tilde{\nabla}\varphi=\frac{\mathrm{Re}_l}{\mathrm{Ma}^2}\frac{q_0}{p_0}\frac{3}{4\tilde{\eta}_l}\varphi(1-\varphi)\tilde{q}+O(10^{-6}) \tag{4.35f}$$

$$\begin{aligned}\tilde{\nabla}\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)=\frac{\mathrm{Ma}^2}{\mathrm{Re}_l}\left(\tilde{\nabla}\left(\frac{1}{3}(1-\varphi)\tilde{\eta}_l\tilde{\mathrm{div}}\,\tilde{u}\right)+2\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\eta}_l\tilde{D}_u(\tilde{u})\right)\right)\\ +\frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{g}+O(10^{-3})\end{aligned} \tag{4.35g}$$

$$\begin{aligned}
&\mathrm{St}\,\partial_{\tilde{t}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}C_p\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{T}\right)+\tilde{\mathrm{div}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}C_p\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{T}\tilde{u}\right)\\
&\quad=-\frac{\rho_{g0}}{\rho_{l0}}\frac{p_0}{\rho_{g0}T_0c_{pl}}\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)\tilde{\mathrm{div}}\tilde{u}+O(10^{-3})
\end{aligned}\tag{4.35h}$$

$$(1-\varphi)\tilde{\rho}_l\tilde{Y}\ \tilde{T}=\frac{p_0}{\rho_{g0}T_0c_0}\varphi\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)+O(10^{-13}).\tag{4.35i}$$

Convection effects are kept in the velocity equation because it is driven by the pressure and viscosity forces. In dimensional form, the system (4.35) and the three limits for R^{H_2O}, Yw and δT are:

$$\begin{aligned}
&R^{H_2O}=\partial_t(\rho Y)+\mathrm{div}(\rho Yu)\\
&Yw=-\frac{Y}{(1-\varphi)K_d}\nabla\left(p+(1-\varphi)q\right)\\
&\delta T=\frac{1}{\gamma}\left(p+(1-\varphi)q\right)\mathrm{div}\left(u+\varphi w\right)\\
&\partial_t\rho+\mathrm{div}(\rho u)=0\\
&\partial_t(\rho(1-Y)(1-C_l))+\mathrm{div}(\rho(1-Y)(1-C_l)u)=0\\
&\partial_t\varphi+u\cdot\nabla\varphi=\varphi(1-\varphi)\frac{3}{4\eta_l}q\\
&\nabla\left(p+(1-\varphi)q\right)=\frac{1}{3}\nabla\left((1-\varphi)\eta_l\mathrm{div}u\right)+2\mathrm{div}\left((1-\varphi)\eta_lD_u(u)\right)+\rho g\\
&\partial_t(c_{pm}T)+\mathrm{div}(c_{pm}Tu)=-\left(p+(1-\varphi)q\right)\mathrm{div}u
\end{aligned}\tag{4.36}$$

In this system, not all drift-flux variables have an immediate physical meaning, so we convert some of them back into their two-phase counterparts. Mass conservation involves the drift variables ρ and Y, the latter of which we convert into a combination of ρ_g, φ, and ρ_l. The pressures are also converted because p_l can be written explicitly. We keep u and w because the bulk velocity is obtained from a single Stokes equation and the drift velocity is obtained from a single Darcy equation. Finally, δT is neglected owing to its very small order. The strict drift-flux system (4.36) can thus be rewritten in a system depending on, respectively, ρ, φ, C_l, u, and T :

$$\begin{aligned}
&\partial_t((1-\varphi)\rho_l)+\mathrm{div}((1-\varphi)\rho_lu)=-R^{H_2O}\\
&\partial_t\rho+\mathrm{div}(\rho u)=0\\
&\partial_t((1-\varphi)\rho_l(1-C_l))+\mathrm{div}((1-\varphi)\rho_l(1-C_l)u)=0\\
&\nabla p_g=\frac{1}{3}\nabla\left((1-\varphi)\eta_l\mathrm{div}u\right)+2\mathrm{div}\left((1-\varphi)\eta_lD_u(u)\right)+\rho g\\
&\partial_t(c_{pm}T)+\mathrm{div}(c_{pm}Tu)=-p_g\mathrm{div}u
\end{aligned}\tag{4.37a}$$

where the variables w and p_l are respectively given by:

$$\begin{aligned}
&w=-\frac{1}{(1-\varphi)K_d}\nabla p_g\\
&p_l=p_g-\frac{4\eta_l}{3\varphi}\left(\mathrm{div}u+\frac{R^{H_2O}}{(1-\varphi)\rho_l}\right)
\end{aligned}\tag{4.37b}$$

and the variables p_g, ρ_g, and c_{pm} are respectively given by:

$$\begin{aligned} p_g &= \rho_g c_0 T \\ \varphi\rho_g &= \rho - (1-\varphi)\rho_l \\ c_{pm} &= \rho_g \varphi c_{vg} + (1-\varphi)\rho_l c_{pl} \end{aligned} \tag{4.37c}$$

This system (4.37) highlights the essential physical processes governing volcanic conduit flows. The mass exchange is a major control of flow dynamics and cannot be neglected. At the time scale assumed here (10^3 s), the main transient terms are those related to mass exchange and conservation. This drove us to propose in Section 5 a limit case of mass exchange by establishing a relaxed system in which $C_l = C_l^{eq}$. The bulk magma flow can be represented by a viscous, steady-state Stokes flow. We expect that transient terms in the momentum play an important role at time scales $< 10^3$ s, such as those involved in pulsatory eruptive dynamics (e.g., Strombolian and Vulcanian regimes). Conversely, the largest length scale is the conduit length, which is on the order of several kilometers (say 10 km). A Strouhal number, $\mathrm{St} = L_0/(u_0 t_0)$, of 1 thus implies that the transit time of a parcel of magma from the base to the top of the conduit occurs over $t_0 \sim 10^6$ s, and that the transient term of the liquid momentum equation can no longer be neglected. Part II, Section 6 [Burgisser et al., 2024], illustrates transient behaviors lasting $10^3 - 10^5$ s and the conditions to obtain steady-state solutions in which all transient terms can be neglected.

Unlike the vanishing small temperature differences, bulk magma temperature is expected to vary because of gas expansion (quantified by $p_g \mathrm{div} u$). This is noteworthy as the non isothermal character of the flow has often been neglected in conduit flow studies. One notable exception is the series of works by La Spina and collaborators [La Spina and de' Michieli Vitturi, 2012, La Spina et al., 2014, La Spina et al., 2017], in which temperature changes are explicitly taken into account. This drove us to carry out a detailed comparison between our model and that of [La Spina and de' Michieli Vitturi, 2012] in Section 6.

In Part II [Burgisser et al., 2024], Section 2, we neglect surface tension effects, temperature diffusion, and shear effects in the gas phase. Other effects absent from the first-order system (4.37) are kept and quantified. Scaling argument for neglecting surface tension were stated at the beginning of Section 4. Our scaling choice in Table 7 is not the best to estimate whether gas viscosity can be neglected because we chose L_0 as the conduit diameter. The gas flow, however, occurs within the interconnected bubble network, which has a characteristic cross-section on the order of $(4\pi N_0/3)^{1/3}$ (see equation (4.1)), which is $O(10^{-6} L_0)$. Taking this correction factor into account, the η_g term in the mixture velocity equation is at most $O(10^{-14})$ whereas all the other terms are $> O(1)$. In the drift flux velocity equation, the η_g term is at most $O(10^{-4})$ whereas all the other terms are at least one order of magnitude larger. Shear effects can thus be neglected in the gas momentum equation. In the temperature equation, the viscous dissipation due to the gas is at most $O(10^{-15})$ and heat diffusion terms are at most $O(10^{-5})$, whereas the leading order terms are $O(1)$. In the relative temperature equation, the viscous dissipation due to the gas is at most $O(10^{-12})$ and heat diffusion terms are at most $O(10^{-4})$, whereas the leading order terms are $> O(1)$. Thus, neglecting temperature diffusion does not affect the two-phase flow dynamics. Finally, the system (4.37) suggests that relative velocities can be captured to the main order with a Darcy equation (4.37b). This simplification is only accurate within limits that are explored in Part II, Section 6 [Burgisser et al., 2024].

5 Relaxed system

In this section we analyze a particular case of the proposed system (2.74) that we assume to be at the chemical equilibrium, that is, $C_l = C_l^{\text{eq}}$ (see equation (2.39)):

$$C_l = k_h \sqrt{p_g}.$$

This limit is achieved for small Peclet number when the diffusive effect is more important that the viscous one, which corresponds to a infinite diffusion coefficient D. Since R^{H_2O} is defined in terms of the D (see equation (2.38d)), this limit implies that R^{H_2O} is not defined, so it can no longer be part of the new, relaxed system. In other words, one equation must be eliminated from the system (2.74) to establish the relaxed system since now we know C_l in terms of p_g. So we sum up the equations of mass and momentum conservation, respectively to eliminate the term in R^{H_2O}. The relaxed system at the chemical equilibrium for unknowns $\varphi, \rho_g, u_l, u_g, p_l, T_g, T_l$ then reads:

$$\partial_t \rho + \operatorname{div}((1-\varphi)\rho_l u_l + \varphi \rho_g u_g) = 0, \tag{5.1a}$$

$$\partial_t((1-\varphi)\rho_l(1-C_l)) + \operatorname{div}((1-\varphi)\rho_l(1-C_l)u_l) = 0 \tag{5.1b}$$

$$\begin{aligned} &\partial_t((1-\varphi)\rho_l u_l + \varphi\rho_g u_g) + \operatorname{div}((1-\varphi)\rho_l u_l \otimes u_l + \varphi\rho_g u_g \otimes u_g) \\ &\quad + \nabla((1-\varphi)p_l + \varphi p_g) + \frac{2\sigma}{R}\nabla(1-\varphi) - \operatorname{div}((1-\varphi)\mathcal{D}_l + \varphi\mathcal{D}_g) - \rho g = 0 \end{aligned} \tag{5.1c}$$

$$\partial_t(\varphi\rho_g u_g) + \operatorname{div}(\varphi\rho_g u_g \otimes u_g) + \nabla(\varphi p_g) - \operatorname{div}(\varphi\mathcal{D}_g) - p_g\nabla\varphi + K_d\,\varphi(1-\varphi)(u_g-u_l) - \varphi\rho_g g - \frac{u_g+u_l}{2}(\partial_t(\varphi\rho_g) + \operatorname{div}(\varphi\rho_g u_g)) = 0 \tag{5.1d}$$

$$\partial_t\varphi + u_l\cdot\nabla\varphi = \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right) \tag{5.1e}$$

$$\begin{aligned} \partial_t(\varphi c_{pg}\rho_g T_g) + \operatorname{div}(\varphi c_{pg}\rho_g T_g u_g) - \varphi\left(\partial_t\left(p_g - \frac{2\sigma}{R}\right) + u_g\cdot\nabla p_g\right) - \gamma(T_g - T_l) \\ -2\eta_g D(u_g) : \nabla u_g - \lambda_g(\operatorname{div} u_g)^2 - \operatorname{div}(\varphi\kappa_g\nabla T_g) = 0, \end{aligned} \tag{5.1f}$$

$$\begin{aligned} \partial_t((1-\varphi)c_{p_l}\rho_l T_l) + \operatorname{div}((1-\varphi)c_{p_l}\rho_l T_l u_l) + p_l(\partial_t((1-\varphi)) + \operatorname{div}((1-\varphi)u_l)) + \gamma(T_g - T_l) \\ -2\eta_l D(u_l) : \nabla u_l - \lambda_l(\operatorname{div} u_l)^2 - \operatorname{div}((1-\varphi)\kappa_l\nabla T_l) = 0. \end{aligned} \tag{5.1g}$$

where $\rho = (1-\varphi)\rho_l + \varphi\rho_g$ and $p_g = c_0\rho_g T_g$. Note that equation (5.1b) is no longer used to solve C_l but that it can be used now to solve ρ_g and φ together with (5.1a).

The system being still complicated, we consider two additional relaxation limits. The first one is to assume an infinite drag coefficient K_d, which leads to a system with a unique velocity $u_g = u_l = u$. This is a restrictive assumption according to our dimensional analysis in Section 4.3. The second limit assumes an infinite heat transfer coefficient γ, which leads to a unique temperature $T_g = T_l = T$. Our dimensional analysis shows that this is a generally valid assumption. The unknowns of the system are reduced to φ, ρ_g, u, p_l and T and we have to build equations without R^{H_2O}, K_d and γ. We thus add the equations of mass, momentum and temperatures

of the two phases to eliminate one equation for each eliminated unknown, respectively. The system is now given by the following equations:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{5.2a}$$

$$\partial_t((1-\varphi)\rho_l(1-C_l)) + \mathrm{div}((1-\varphi)\rho_l(1-C_l)u) = 0 \tag{5.2b}$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla((1-\varphi)p_l + \varphi p_g) + \frac{2\sigma}{R}\nabla(1-\varphi) - \mathrm{div}\mathcal{D} - \rho g = 0 \tag{5.2c}$$

$$\partial_t \varphi + u \cdot \nabla\varphi = \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right) \tag{5.2d}$$

$$\begin{aligned} &\Big(\varphi c_{vg}\rho_g + (1-\varphi)c_{p_l}\rho_l\Big)(\partial_t T + u\cdot\nabla T) + (c_{p_g}\rho_g T - p_l - c_{p_l}\rho_l T)(\partial_t\varphi + u\cdot\nabla\varphi) \\ &+ (\varphi c_{pg}\rho_g T + (1-\varphi)p_l + (1-\varphi)c_{p_l}\rho_l T)\mathrm{div}u + \varphi c_{vg}T(\partial_t\rho_g + u\cdot\nabla\rho_g) + \varphi\partial_t\left(\frac{2\sigma}{R}\right) \\ &- \mathcal{D} : \nabla u - \mathrm{div}(((1-\varphi)\kappa_l + \varphi\kappa_g)\nabla T) = 0, \end{aligned} \tag{5.2e}$$

where now $p_g = c_0\rho_g T$, $c_{vg} = c_{pg} - c_0$, and $\mathcal{D} = (1-\varphi)\mathcal{D}_l + \varphi\mathcal{D}_g = -\nabla(\varphi\lambda\mathrm{div}\,u) - 2\mathrm{div}(\varphi\eta D(u))$ with $\lambda = (1-\varphi)\lambda_l + \varphi\lambda_g$, $\eta = (1-\varphi)\eta_l + \varphi\eta_g$ (see (2.10)-(2.11)). We now neglect the temperature diffusion term, $\kappa_l = \kappa_g = 0$, which is a valid assumption according to our dimensional analysis.

We now prove that $\partial_t\varphi + u\cdot\nabla\varphi$ may be expressed with respect to $\mathrm{div}u$. In such case, thanks to (5.2d) there is a dilatancy effect (as introduced in Remark 2.5). [Fowler and Robinson, 2018] present a similar relaxation analysis, which will be shown to be a particular case of the relaxed system developed herein. In [Fowler and Robinson, 2018], the equation for the gas volume fraction is $\partial_t(1-\varphi) + \mathrm{div}((1-\varphi)u) = 0$ for their unique velocity u, so in their system $\partial_t\varphi + u\cdot\nabla\varphi = (1-\varphi)\mathrm{div}u$.

We introduce the variable $Q_g = 1 - C_l = 1 - k_h p_g^\nu$, which uses a more general, power-law form of Henry's law. This generalization is motivated by the fact that we have hither to used $\nu = 0.5$, whereas the Fowler-Robinson work is based on $\nu = 1$. Keeping ν unspecified for now will become handy to relate our work to that of [Fowler and Robinson, 2018].

From equation (5.2b), Q_g satisfies

$$-Q_g(\partial_t\varphi + u\cdot\nabla\varphi) + (1-\varphi)(\partial_t Q_g + u\cdot\nabla Q_g) + (1-\varphi)Q_g\mathrm{div}u = 0. \tag{5.3}$$

From equation (5.2a) we write

$$(\rho_g - \rho_l)(\partial_t\varphi + u\cdot\nabla\varphi) + \varphi(\partial_t\rho_g + u\cdot\nabla\rho_g) + \rho\mathrm{div}u = 0. \tag{5.4}$$

Since our objective is to find an equivalent expression for $\partial_t\varphi + u\cdot\nabla\varphi$ in terms of $\mathrm{div}u$, we want to cancel the terms on ρ_g and Q_g. First we use that $Q_g = 1 - k_h(c_0\rho_g T)^\nu$, which yields the following material derivative:

$$\partial_t Q_g + u\cdot\nabla Q_g = Q_g'(p_g)c_0 T(\partial_t\rho_g + u\cdot\nabla\rho_g) + Q_g'(p_g)c_0\rho_g(\partial_t T + u\cdot\nabla T)$$

and then we use (5.2e) to get

$$\begin{aligned} c_{pm}(\partial_t T + u\cdot\nabla T) = &-c_{pr}(\partial_t\varphi + u\cdot\nabla\varphi) \\ &- P_m\mathrm{div}u - \varphi(c_{pg} - c_0)T(\partial_t\rho_g + u\cdot\nabla\rho_g) - \varphi\partial_t\left(\frac{2\sigma}{R}\right) + \mathcal{D} : \nabla u \end{aligned} \tag{5.5}$$

where

$$
\begin{aligned}
&c_{pm} = \varphi c_{vg}\rho_g + (1-\varphi)c_{pl}\rho_l \\
&c_{pr} = c_{pg}\rho_g T - (p_l + c_{pl}\rho_l T) \\
&P_m = \varphi c_{pg}\rho_g T + (1-\varphi)(p_l + c_{pl}\rho_l T) = c_{pg}\rho_g T - (1-\varphi)c_{pr}.
\end{aligned}
\tag{5.6}
$$

Embedding these expressions into (5.3) yields:

$$
\begin{aligned}
&-\Big(Q_g(p_g) + (1-\varphi)\frac{Q_g'(p_g)c_0\rho_g c_{pr}}{c_{pm}}\Big)(\partial_t\varphi + u\cdot\nabla\varphi) \\
&+(1-\varphi)\Big(Q_g(p_g) - P_m\frac{Q_g'(p_g)c_0\rho_g}{c_{pm}}\Big)\mathrm{div}u \\
&+(1-\varphi)\Big(Q_g'(p_g)c_0 T - \frac{\varphi Q_g'(p_g)c_0\rho_g}{c_{pm}}c_{vg}T\Big)(\partial_t\rho_g + u\cdot\nabla\rho_g) \\
&+(1-\varphi)\frac{Q_g'(p_g)c_0\rho_g}{c_{pm}}\mathcal{D}:\nabla u - (1-\varphi)\frac{Q_g'(p_g)c_0\rho_g}{c_{pm}}\varphi\partial_t\left(\frac{2\sigma}{R}\right) = 0
\end{aligned}
\tag{5.7}
$$

Now we compute the combination (5.4) $\times\,(1-\varphi)\Big(Q_g'(p_g)c_0T - \frac{\varphi Q_g'(p_g)c_0\rho_g}{c_{pm}}c_{vg}T\Big)$ + (5.7) $\times\,\varphi$ that gives

$$
\begin{aligned}
&(\partial_t\varphi + u\cdot\nabla\varphi)\mathcal{N} + (1-\varphi)\mathcal{M}\mathrm{div}u \\
&-\varphi(1-\varphi)\frac{Q_g'(p_g)c_0\rho_g}{c_{pm}}\left(\mathcal{D}:\nabla u - \varphi\partial_t\left(\frac{2\sigma}{R}\right)\right) = 0
\end{aligned}
\tag{5.8}
$$

where

$$
\mathcal{M} := \varphi\Big(Q_g(p_g) + P_m\frac{Q_g'(p_g)c_0\rho_g}{c_{pm}}\Big) + \rho Q_g'(p_g)c_0T(1 + \frac{\varphi\rho_g}{c_{pm}}c_{vg}) \tag{5.9}
$$

$$
\mathcal{N} := -\varphi\Big(-Q_g(p_g) + (1-\varphi)\frac{Q'(p_g)c_0\rho_g c_{pr}}{c_{pm}}\Big) + (\rho_g - \rho_l)(1-\varphi)Q_g'(p_g)c_0T(1 + \frac{\varphi\rho_g}{c_{pm}}c_{vg}) \tag{5.10}
$$

Then we find

$$
\partial_t\varphi + u\cdot\nabla\varphi = (1-\varphi)\mathcal{A}\mathrm{div}u + \varphi(1-\varphi)\frac{1}{\mathcal{N}}\frac{Q_g'(p_g)c_0\rho_g}{c_{pm}}\left(\mathcal{D}:\nabla u - \varphi\partial_t\left(\frac{2\sigma}{R}\right)\right) \qquad \text{where } \mathcal{A} = -\frac{\mathcal{M}}{\mathcal{N}}. \tag{5.11}
$$

Notice that when $\frac{\rho_g}{\rho_l}\to 0$, $\mathcal{N}\approx_{\rho_l\to\infty}\rho_l(1-\varphi)k_hc_0T$ and $\mathcal{M}\approx_{\rho_l\to\infty} -\rho_l(1-\varphi)k_hc_0T$ then $-\mathcal{N}/\mathcal{M}\approx 1$, so in that case we find (notice that however $\mathcal{M}\to\infty$)

$$
\partial_t\varphi + u\cdot\nabla\varphi = (1-\varphi)\mathrm{div}u, \qquad \text{when} \quad \frac{\rho_g}{\rho_l}\to 0,
$$

which coincides with the expression in [Fowler and Robinson, 2018]. In the general form, using that $Q'(p_g) = -\nu C_l/p_g$, (5.11) can be written as:

$$
\partial_t\varphi + u\cdot\nabla\varphi = (1-\varphi)\mathcal{A}\,\mathrm{div}u + \varphi(1-\varphi)\frac{1}{\mathcal{N}_s}\frac{\nu}{Tc_{pm}}\left(\mathcal{D}:\nabla u - \varphi\partial_t\left(\frac{2\sigma}{R}\right)\right) \tag{5.12}
$$

where

$$
\mathcal{A} = -\frac{\mathcal{M}}{\mathcal{N}} = -\frac{\mathcal{M}_s}{\mathcal{N}_s}. \tag{5.13}
$$

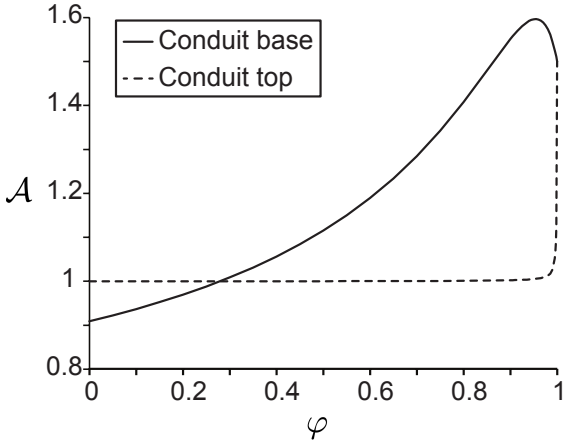

Figure 4: Rayleigh-Plesset correction factor $\mathcal{A}$ (5.13) as a function of φ for two extreme values of ρ_g (conduit base and top).

and

$$\begin{aligned}\mathcal{M}_s &= -\nu\frac{\rho}{\rho_g}(1+f_e)+\varphi+(1-\varphi)\frac{\nu\varphi c_{pr}}{Tc_{pm}}-\frac{\nu\varphi c_{pg}\rho_g}{c_{pm}},\\ \mathcal{N}_s &= -\nu(1-\frac{\rho}{\rho_g})(1+f_e)+\varphi-(1-\varphi)\frac{\nu\varphi c_{pr}}{Tc_{pm}},\end{aligned} \tag{5.14}$$

where $f_e=\varphi c_{vg}\rho_g/c_{pm}$ is the fraction of total enthalpy carried by the gas phase. The terms multiplied by ν in the factors $\mathcal{M}_s$ and $\mathcal{N}_s$ are ratios of energy densities (units of $\mathrm{J/m^3}$). Thus, the factor $\mathcal{A}$ is a correction factor to the Rayleigh-Plesset dilatancy (2.47) that accounts for the work done by changing phases when degassing occurs under a fully relaxed fashion to maintain chemical equilibrium. $\mathcal{A}$ is a function of thermodynamical constants and of four variables (φ,p_l,ρ_g, and T). Using the same numerical values as in the assessment of the main order approximation of the drift flux system (Section 4.3, Table 7), we find that $\mathcal{A}$ depends very weakly on p_l. Table 7 also suggests that using only one value of T is sufficient. The behavior of $\mathcal{A}$ can thus be captured by varying φ from 0 to 1 and ρ_g within the bounds of Table 7. Figure 4 shows the evolution of $\mathcal{A}$ as a function of φ at the conduit base and top. $\mathcal{A}$ is close to 1 except at high gas volume fraction. At the limit $\varphi\to 1$, $\mathcal{A}=c_{vg}/((3\nu-1)c_{pg}-(2\nu-1)c_0)$, which is equal to 1.5 with the values of Table 7.

Note that along the lines of Remark 2.5 we can also relate (5.12) with (5.2d) to write

$$\mathrm{div}u=\frac{1}{\mathcal{A}}\frac{3\varphi}{4\eta_l}(p_g-p_l)+\varphi\frac{1}{\mathcal{N}_s}\frac{\nu}{Tc_{pm}}\mathcal{D}:\nabla u. \tag{5.15}$$

finding also a dilatancy-like relation that can be compared with (4.16) without drift velocity w:

$$\mathrm{div}u=\frac{3\varphi}{4\eta_l}(p_g-p_l)-\frac{R^{H_2O}}{\rho_l(1-\varphi)}.$$

The relaxation thus converts the gas exsolution rate R^{H_2O} into a modified viscous heating regardless of the expression chosen for R^{H_2O} (see Remark A.1).

One can build a conservation equation on the quantity $\Theta=(1-\varphi)Q_g(p_g)$:

$$\partial_t\Theta+\mathrm{div}(\Theta u)=0.$$

We prefer to use the alternative equation for the total water content, C_T, to change this conservation equation on Θ into a perhaps more intuitive transport equation:

$$\partial_t C_T + u \cdot \nabla C_T = 0.$$

All the calculations made before allows us to write the following system (neglecting viscosity and surface tension):

$$\begin{aligned} &\partial_t \rho + \mathrm{div}(\rho u) = 0, \\ &\partial_t C_T + u \cdot \nabla C_T = 0, \\ &\partial_t (\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla\Big((1-\varphi)(p_l - p_g)\Big) + \nabla p_g = \rho g, \end{aligned} \tag{5.16a}$$

where p_g and φ are given in terms of ρ and C_T solving the algebraic constraint

$$\begin{aligned} &(\varphi\rho_g + (1-\varphi)\rho_l Q_g(p_g))/\rho = C_T, \\ &(1-\varphi)\rho_l + \varphi\rho_g = \rho, \\ &p_g = c_0 \rho_g T, \qquad Q_g(p_g) = 1 - k_h p_g^{\nu}. \end{aligned} \tag{5.16b}$$

where p_l is solution of the nonlinear relation

$$\mathrm{div} u = \frac{1}{\mathcal{A}} \frac{3\varphi}{4\eta_l}(p_g - p_l). \tag{5.16c}$$

and the gas volume fraction φ is the solution of the following equation

$$(\partial_t \varphi + u \cdot \nabla \varphi)\mathcal{N}_s + (1-\varphi)\mathcal{M}_s \mathrm{div} u = 0 \tag{5.16d}$$

Note that (5.16) is a closed system with the unknowns $(\rho, C_T, u, T, \varphi, p_l, p_g, \rho_g)$ where $\mathcal{A}$, $\mathcal{M}_s$, and $\mathcal{N}_s$ are given by (5.13)–(5.14). The system obtained by [Fowler and Robinson, 2018] can be retrieved exactly by choosing $\nu = 1$ and letting ρ_g/ρ_l tend to zero.

6 Incompressible vs. compressible liquid phase

One assumption of our conduit flow model is that the liquid is incompressible and thus that ρ_l is constant. In a series of papers [La Spina and de' Michieli Vitturi, 2012, La Spina et al., 2014, La Spina et al., 2017], a one-dimensional conduit flow model of two compressible phases is analyzed and applied, the model being originally presented in [Romenski et al., 2010]. In this model a unique temperature is considered and the system is written with a drift-flux structure (see Section 4). Depending on the application, different relaxation terms are chosen in [La Spina and de' Michieli Vitturi, 2012, La Spina et al., 2014, La Spina et al., 2017]. Here we chose the mass conservation equations of the bulk density, $\rho = \varphi\rho_g + (1-\varphi)\rho_l$, given in [La Spina et al., 2017], which is a version of the model close to the one we propose herein:

$$\partial_t \rho + \partial_x(\rho u) = 0 \tag{6.1}$$

with $u = Y u_g + (1-Y)u_l$ where $Y = \rho_g \varphi/\rho$ (see (4.2) sqq. in Section 4). The equations on φ and ρ_l are given by:

$$\partial_t(\rho\varphi) + \partial_x(\rho\varphi u) = \frac{1}{\tau_p}(p_g - p_l) \tag{6.2}$$

and

$$\partial_t(\rho_l(1-\varphi)) + \partial_x(\rho_l(1-\varphi)u_l) = -\frac{1}{\tau_d}\rho_l(1-\varphi)(C_l - C_l^{eq}) \tag{6.3}$$

where C_l satisfies:

$$\partial_t(\rho_l(1-\varphi)C_l) + \partial_x(\rho_l(1-\varphi)C_l u_l) = -\frac{1}{\tau_d}\rho_l(1-\varphi)(C_l - C_l^{eq}). \tag{6.4}$$

For the momentum equations, we chose the form given in [La Spina et al., 2014] p. 286-287 in the case when $\tau^{(c)} \to +\infty$ because they use the two phase velocities:

$$\begin{aligned}\partial_t(\varphi\rho_g u_g) + \partial_x(\varphi\rho_g u_g^2 + \varphi p_g) = &\left(\frac{(1-\varphi)\rho_l p_g + \varphi\rho_g p_l}{\rho}\right)\partial_x\varphi - \frac{\varphi(1-\varphi)\rho_l\rho_g}{\rho}(s_l - s_g)\partial_x T \\ &- \frac{\varphi(1-\varphi)\rho_l\rho_g}{\rho^2}\frac{1}{\tau_f}Y(1-Y)(u_g - u_l) + \varphi\rho_g g\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}\partial_t((1-\varphi)\rho_l u_l) + \partial_x((1-\varphi)\rho_l u_l^2 + (1-\varphi)p_l) = &-\left(\frac{(1-\varphi)\rho_l p_g + \varphi\rho_g p_l}{\rho}\right)\partial_x\varphi + \frac{\varphi(1-\varphi)\rho_l\rho_g}{\rho}(s_l - s_g)\partial_x T \\ &+ \frac{\varphi(1-\varphi)\rho_l\rho_g}{\rho^2}\frac{1}{\tau_f}Y(1-Y)(u_g - u_l) + (1-\varphi)\rho_l g - \tilde{\beta}u_l\end{aligned} \tag{6.6}$$

with $\tau_f, \tilde{\beta}$ constants and where $\mu_i = e_i + p_i/\rho_i - s_i T$ for $i = g, l$ (precised later on). Note that the authors consider two compressible components with pressure state laws in the form [La Spina and de' Michieli Vitturi, 2012]

$$p_i = (c_{pi} - c_{vi})\rho_i T - \bar{p}_i$$

where T is the single temperature, c_{vi} is the heat capacity at constant volume and $\bar{p}_i$ quantifies the effects of the molecular attraction in phase i at a reference pressure, density, and sound speed. This is in contrast with our work, where the liquid phase is incompressible with a constant liquid density and the gas phase is compressible with $\bar{p}_g = 0$ and $c_0 = c_{pg} - c_{vg}$. Our pressure, p_l, is an unknown given (without surface tension effect) by the relation

$$\mathrm{div} u_l = \varphi\frac{3}{4\eta_l}(p_g - p_l) - \frac{R^{H_2O}}{\rho_l(1-\varphi)}.$$

The total energy conservation to characterize the temperature is chosen by considering that the mixture is mainly governed by the total density ρ and the averaged velocity u. They thus start from the quantity

$$\int_\Omega \rho(E + \frac{|u|^2}{2})$$

where $E = e + Y(1-Y)w^2/2$ with the internal energy $e = Ye_g(\rho_g, s_g) + (1-Y)e_l(\rho_l, s_l)$ recalling that $p_i = \rho_i^2\partial e_i/\partial\rho_i$, $T = \partial e_i/\partial s_i = T$ where $i = g, l$. They then derive the momentum equations on u and w and the corresponding equation on the total energy from the Lagrangian associated to such variables. Their conservation energy reads

$$\begin{aligned}\partial_t(\rho(E + \frac{u^2}{2})) + \partial_x\Big(&\rho_l(1-\varphi)u_l(e_l + \frac{p_l}{\rho_l} + \frac{u_l^2}{2}) + \rho_g\varphi u_g(e_g + \frac{p_g}{\rho_g} + \frac{u_g^2}{2}) \\ &-\frac{\rho_l\rho_g\varphi(1-\varphi)}{\rho}(s_g - s_l)wT\Big) = -\rho g u - \tilde{\beta}u_l^2.\end{aligned} \tag{6.7}$$

Remark 6.1. *One difference with our system is that the La Spina model has the following pressure at the interface*

$$(p_I)_{LS} = \frac{(1-\varphi)\rho_l p_g + \varphi\rho_g p_l}{\rho} = (1-Y)p_g + Y p_l$$

instead of our interface pressure

$$\langle p_{int,g}\rangle = p_g.$$

As a result, the difference between interface pressures reads

$$(p_I)_{LS} - \langle p_{int,g}\rangle = \frac{\varphi\rho_g(p_g - p_l)}{\rho} = Y(p_l - p_g).$$

Remark 6.2. *Another difference is that the La Spina model includes in the momentum equations the term*

$$-\frac{\varphi(1-\varphi)\rho_l\rho_g}{\rho}(s_l - s_g)\partial_x T = -\rho Y(1-Y)(s_l - s_g)\partial_x T$$

that is absent from our model.

Remark 6.3. *A similar approach of deriving two-phase systems has been introduced by Shugrin [Shugrin, 1994]. In this approach, one starts from thermodynamics, derives the equations on u and w and then deduces the equation on u_g and u_l. [Gavrilyuk, 2020] proved that different two-phases equations may be obtained depending on the governing unknowns chosen. As indicated in his abstract, "a very surprising fact is that one can obtain different governing equations from the same Lagrangian. Different types of the governing equations are due to the choice of independent variables and the corresponding virtual motions. Even if the total momentum and total energy equations are the same, the equations for individual components differ from each other by the presence or absence of gyroscopic forces (also called 'lift' forces). These forces have no influence on the hyperbolicity of the governing equations, but can drastically change the distribution of density and velocity of components."*

Let us now explain how the equations on the mean velocity u and the drift velocity w are obtained in the La Spina model. The authors start from a Lagrangian with the unknowns u and w and deduce the following equations:

$$\partial_t(\rho u) + \partial_x(\rho_l(1-\varphi)u_l^2 + \rho_g\varphi u_g^2 + (1-\varphi)p_l + \varphi p_g) = -\rho g - \tilde{\beta}u_l \tag{6.8}$$

and

$$\partial_t w + \partial_x\left(\frac{u_g^2}{2} - \frac{u_l^2}{2} + e_g + \frac{p_g}{\rho_g} - e_l - \frac{p_l}{\rho_l} - (s_g - s_l)T\right) = -\frac{1}{\tau_f}\varphi(1-\varphi)\frac{\rho_l\rho_g}{\rho^3}w + \frac{1}{\rho_l(1-\varphi)}\tilde{\beta}u_l. \tag{6.9}$$

The equations on the variables u_g and u_l are then deduced from these u and w equations. For the readers convenience, we present the reverse (i.e. how to deduce equations on u and w from equations on u_g and u_l). We follow the procedure described in the review paper, Section 5.2 [Bresch et al., 2024]. Table 10 lists useful relationships to convert phase variables into drift-flux variables.

The equation on u (6.8) is easily obtained adding the momentum equation on u_g to the momentum equation on u_l and using relationship 1 from Table 10. To obtain the equation on w, we use the relationship 2 of Table 10 and the equation on $\varphi\rho_g u_g$ from which we subtract the equation on $\rho Y u$ obtained from the equations on u multiplied by Y, and use the equation on Y. After calculations, we get

$$\begin{aligned}&\partial_t w + \partial_x\left(\frac{u_g^2}{2} - \frac{u_l^2}{2}\right) + \frac{1}{\rho Y(1-Y)}\left(\partial_x(\varphi p_g) - Y\partial_x p - \Big(\frac{(1-\varphi)\rho_l p_g + \varphi\rho_g p_l}{\rho}\Big)\partial_x\varphi\right) - (s_g - s_l)\partial_x T \\ &\quad = -\frac{1}{\tau_f}\varphi(1-\varphi)\frac{\rho_l\rho_g}{\rho^3}w + \frac{1}{\rho_l(1-\varphi)}\tilde{\beta}u_l.\end{aligned} \tag{6.10}$$

Relationship	Formula
1	$\rho_l(1-\varphi)u_l^2 + \rho_g\varphi u_g^2 = \rho u^2 + \rho Y(1-Y)w^2$
2	$\rho_g\varphi u_g = \rho Y(u+(1-Y)w) = \rho Y u + \rho Y(1-Y)$
3	$\frac{u_g^2}{2} - \frac{u_l^2}{2} = uw + (\frac{1}{2} - Y)w^2$

Table 10: Relationships to convert phase variables into drift-flux variables (see also the review paper, Section 5.2 [Bresch et al., 2024]).

To obtain equation (6.9) on w, we first note relationship 3 in Table 10 and that

$$\partial_x(\varphi p_g) - Y\partial_x p - \left(\frac{(1-\varphi)\rho_l p_g + \varphi\rho_g p_l}{\rho}\right)\partial_x\varphi = \rho Y(1-Y)\left(\frac{\partial_x p_g}{\rho_g} - \frac{\partial_x p_l}{\rho_l}\right)$$

Then we use the definitions $\mu_i = e_i + p_i/\rho_i - s_i T$ with $p_i = \rho_i^2 \partial e_i/\partial\rho_i$ and $\partial e_i/\partial s_i = T$ coming from the identity

$$T ds_i = de_i - \frac{p_i}{\rho_i^2} d\rho_i$$

to establish that

$$\partial_x(\mu_g - \mu_l) = \frac{1}{\rho_g}\partial_x p_g - \frac{1}{\rho_l}\partial_x p_l - (s_g - s_l)\partial_x T.$$

Finally, we remark that

$$T\partial_x s_i = \partial_x e_i - p_i \partial_x \rho_i/\rho_i^2.$$

Remark 6.4. *In the equation on w from our model, the terms coming from the pressure and volume fraction φ are*

$$\frac{1}{\rho_g}\partial_x p_g - \frac{1}{\rho_l}\partial_x p_l + \frac{1}{(1-\varphi)\rho_l}(p_g - p_l)\partial_x\varphi.$$

The difference between these terms and the equivalent terms in the La Spina equation on w is

$$-\frac{1}{(1-\varphi)\rho_l}(p_g - p_l)\partial_x\varphi - (s_g - s_l)\partial_x T.$$

The first quantity comes from the difference in the expressions of the interface pressure in the two systems (see Remark 6.1). The second quantity comes from the derivation of the two-phase system from a Lagrangian quantity when choosing u and ρ as the reference velocity and density.

The relaxation parameters τ_p, τ_d, τ_f are linked respectively to the pressure, the dissolved gas, and the velocity. The mixture energy E depends on $(\rho, Y, \varphi, w, s_g, s_l)$. So when each relaxation parameter tends to zero, we find the following respective equilibrium limits: $p_g = p_l$, $C_l = C_l^{\text{eq}}$, and $u_l = u_g$.

One can easily check the equivalence of equations (6.1), (6.3), (6.4), (6.8) with our equations (4.15a), (2.74a), (2.74c), (4.15e) for the 1D case, neglecting the tension surface effect and taking $\mathcal{D}_g = 0$. By fitting these equations, we find the relaxation parameter for the dissolved gas, denoted as τ_d, to be

$$\tau_d = \frac{\rho_l(1-\varphi)(C_l - C_l^{\text{eq}})}{R^{H_2O}} = \frac{S-R}{4\pi R^2 N D}$$

since from (2.34) and (2.38d), we have $R^{H_2O} = 4\pi R^2 N(1-\varphi)\rho_l D\frac{C_l - k_h\sqrt{p_g}}{S-R}$, where R is the local bubble radius and $S-R$ is half the interbubble distance. Hence the relaxation exsolution coefficient in our model would be a

$1/D$ function of the ratio between the interbubble distance and the surface area of the bubbles. Conversely, we could give an expression for the exchange term for the system given by La Spina (along the lines of Remark **??**) as

$$R_{LS}^{H_2O} = \mathcal{A}_{LS} R^{H_2O} \quad \text{with} \quad A_{LS} = \frac{1}{\tau_d} \frac{(1-\varphi)^{1/3}(1-\varphi^{1/3})}{3^{1/3} D \left(4\pi N \varphi\right)^{2/3}}.$$

Focusing now on the continuity equations for φ, (6.2) and (2.74f), our model and that of La Spina differ on the velocity transport. In the La Spina model, we found that the carrier velocity is the mixture velocity, u, while in our model it is instead the liquid velocity. Our φ equation comes from the microscopic analysis related to the Rayleigh-Plesset equation in Section 2.3. Following [Leighton, 2007], we assumed that the macroscopic velocity related to the transport of the bubbles in the melt is the velocity field of the fluid phase, u_l. The La Spina model uses a different choice; the transport is ensured by the mixture velocity u. Assuming for simplicity that transport velocities are identical, an equivalence between the two models would be achieved for the following pressure relaxation parameter:

$$\tau_p = \frac{4\eta_l}{3\rho\varphi(1-\varphi)} = \frac{\eta_l}{\rho_l} \frac{4}{3\varphi(1-\varphi)} \frac{\rho_l}{\rho}$$

where η_l/ρ_l is the kinematic liquid viscosity and we can write $\frac{\rho_l}{\rho} = \left(\varphi \frac{\rho_g}{\rho_l} + (1-\varphi)\right)^{-1}$.

We here recall that the La Spina equation (6.9) on w reads

$$\partial_t w + \partial_x \left(\frac{u_g^2}{2} - \frac{u_l^2}{2} + \mu_g - \mu_l \right) = -\frac{1}{\tau_f} \varphi(1-\varphi) \frac{\rho_l \rho_g}{\rho^3} w + \frac{1}{\rho_l (1-\varphi)} \tilde{\beta} u_l$$

where $\partial E/\partial Y = \mu_g - \mu_l + (1-2Y)|w|^2/2$ with $\mu_g - \mu_l = (e_g + p_g/\rho_g - s_g T) - (e_l + p_l/\rho_l - s_l T)$. The term $\tilde{\beta} u_l$ with $\tilde{\beta} = \frac{8\eta_l}{r_c^2}$, where r_c is the radius of the cylindrical conduit, replaces the diffusion term $-\partial_x((1-\varphi)\mathcal{D}_l)$ as it is usually done in one-dimensional problems to keep wall effects (see Part II, Section 3 [Burgisser et al., 2024]). The equation for the drift velocity for our system is given in (4.19). As before we neglect $\mathcal{D}_g$ and replace the diffusion liquid term by $\tilde{\beta} u_l$, which yields:

$$\begin{aligned} &\partial_t w + \partial_x \left(uw + (1-Y)w^2 - \frac{1}{2} w^2 \right) - \frac{1}{\rho Y(1-Y)} \left((Y-\varphi)\partial_x p - \varphi \partial_x ((1-\varphi) q) \right) \\ &= -\frac{1}{\rho Y(1-Y)} \left(K_d \varphi(1-\varphi) + R^{H_2O} \left(\frac{1}{2} - Y \right) \right) w + \frac{\tilde{\beta} u_l}{\rho(1-Y)} \end{aligned}$$

Relationship 3 in Table 10 yields $uw + (1-Y)w^2 - \frac{1}{2}w^2 = \frac{1}{2}(u_g^2 - u_l^2)$. The last term coincides with those in the La Spina equation since $\rho(1-Y) = \rho_l(1-\varphi)$. The first term on the right-hand side of the La Spina equation may be compared with the drag term in our model with coefficient K_d providing the following definition of the velocity relaxation parameter:

$$\tau_f = \frac{1}{K_d} \rho_l \rho_g Y(1-Y) = \frac{1}{K_d} \frac{Y(1-Y)}{\varphi(1-\varphi)}.$$

To compare the energy balances of the two models, we first rewrite the La Spina energy equation (6.7) as

$$\begin{aligned} &\partial_t (\rho \frac{u^2}{2} + \rho E) + \partial_x \Big(\rho_g \varphi u_g \frac{u_g^2}{2} + \rho_l (1-\varphi) u_l \frac{u_l^2}{2} \Big) \\ &\quad + \partial_x \Big(\rho_g \varphi u_g (e_g + \frac{p_g}{\rho_g}) + \rho_l (1-\varphi) u_l (e_l + \frac{p_l}{\rho_l}) - \rho Y(1-Y)(s_g T - s_l T) w \Big) \\ &\quad = -\rho g u - \widetilde{\beta} u_l^2 \end{aligned} \tag{6.11}$$

Note that this equation could be fully recast as a function of u, w, and Y by using :

$$\begin{aligned}&\rho_g\varphi u_g(e_g+\frac{p_g}{\rho_g})+\rho_l(1-\varphi)u_l(e_l+\frac{p_l}{\rho_l})-\rho Y(1-Y)(s_gT-s_lT)w\\&=\rho u\Big(Y(e_g+\frac{p_g}{\rho_g})+(1-Y)(e_l+\frac{p_l}{\rho_l})\Big)+\rho Y(1-Y)w\Big((e_g+\frac{p_g}{\rho_g}-s_gT)-(e_l+\frac{p_l}{\rho_l}-s_lT)\Big)\end{aligned}\tag{6.12}$$

and

$$\rho_g\varphi u_g\frac{u_g^2}{2}+\rho_l(1-\varphi)u_l\frac{u_l^2}{2}=\rho u(\frac{u^2}{2}+Y(1-Y)\frac{w^2}{2})+\rho Y(1-Y)uw^2+\rho Y(1-Y)w(\frac{1}{2}-Y)w^2\tag{6.13}$$

Using the relationships 1 and 2 of Table 10, we can recast our energy equation (2.58) with drift-flux variables to match the form of (6.11):

$$\begin{aligned}&\partial_t\Big(\rho\frac{u^2}{2}+\rho Y(1-Y)\frac{w^2}{2}+\rho(Yc_{vg}T+(1-Y)c_{pl}T)\Big)+\partial_x\Big(\rho_g\varphi u_g\frac{u_g^2}{2}+\rho_l(1-\varphi)u_l\frac{u_l^2}{2}\Big)\\&+\partial_x\Big(\rho_g\varphi u_g(c_{vg}T+\frac{p_g}{\rho_g})+\rho_l(1-\varphi)u_l(c_{pl}T+\frac{p_l}{\rho_l})\Big)\\&=-\rho gu-\frac{3+\varphi}{4}\widetilde{\beta}u_l^2\\&\quad-K_d\varphi(1-\varphi)|u_g-u_l|^2-\varphi(1-\varphi)\frac{3}{4\eta_l}(p_g-p_l)^2-\delta R^{H_2O}\frac{|u_g-u_l|^2}{2}\end{aligned}\tag{6.14}$$

In both model we can define $E=Y(1-Y)w^2/2+Ye_g+(1-Y)e_l$, but the definitions of the internal energies differ:

$$\begin{aligned}e_g(\rho_g,s_g)_{LS}\quad&\text{is found using}\quad p_g=\rho_g^2\partial e_g/\partial\rho_g\quad\text{and}\quad T=\partial e_g/\partial s_g,\\e_l(\rho_l,s_l)_{LS}\quad&\text{is found using}\quad p_l=\rho_l^2\partial e_l/\partial\rho_l\quad\text{and}\quad T=\partial e_l/\partial s_l,\end{aligned}$$

whereas for our model these quantities are:

$$e_g=c_{vg}T,\qquad e_l=c_{pl}T.$$

The main difference between our energy equation and that of La Spina thus comes from the addition in the latter of the entropy quantity $\partial_x(\rho Y(1-Y)(s_g-s_l)Tw)$, which explains the concurrent addition of the term $\rho Y(1-Y)(s_g-s_l)\partial_xT$ in the momentum equation (see Remark 6.2). A more minor difference is that the La Spina model does not feature the term $3\varphi(1-\varphi)(p_g-p_l)^2/(4\eta_l)$, which quantifies the work done by growing or shrinking bubbles against the surrounding viscous liquid.

Conclusions

There are several ways to create two-phase flow models, mostly because of the diversity of ways to average microscopic processes to the macroscopic scale of the transport equations. One goal of this Part I of our three-part series of works dedicated to two-phase flow modeling is to detail the steps needed to obtain a self-consistent set of equations adapted to the modeling of magma flow in a volcanic conduit. We adopted a series of assumptions to reduce the physical processes to sets that are compatible with the two-phase flow framework.

The three-phase nature of the magma was simplified by considering a gas–liquid mixture where the gas is pure $\mathrm{H_2O}$. The incompressible liquid phase ignores the presence of crystals but is composed of dissolved $\mathrm{H_2O}$ and dry silicate liquid. Fragmentation is not considered explicitly (although doing so does not change the structure of the proposed transport equations). Under these assumptions, the mass and momentum conservation

yield 5 transport equations with 7 unknowns: gas volume fraction and density (φ, ρ_g), dissolved water content (C_l), liquid pressure and velocity (p_l, u_l), and gas velocity and temperature (u_g, T_g). Closing this partial system involves microscopic processes. The physics of gas bubble growth adds one equation for the gas volume fraction (φ) and a novel closure for the mass exchange term (R^{H_2O}). Neglecting nucleation and coalescence yields a constant bubble number density with respect to the liquid (N). Scaling these processes up to the macroscopic scale yields a system without energy balance with 7 unknowns for 6 transport equations. Finally, considering the energy balance adds liquid temperature (T_l) and the equations for both temperatures. The final system (2.74) has 8 transport equations on 8 unknowns $(\varphi, \rho_g, C_l, p_l, u_l, u_g, T_g, T_l)$ as well as algebraic closures for the gas pressure (p_g) and bubble radius (R). A 1.5D implementation and application of this model is presented in Part II, Sections 5-6 [Burgisser et al., 2024].

Establishing appropriate boundary conditions is not an easy task for viscous compressible flows. The two-phase, multi-component nature of our system adds complexity to that task. We started from a literature review of the number and type of conditions to be imposed on single-phase, multi-component Euler or Navier-Stokes equations. We then extended the underlying reasoning to establish sets of boundary conditions suitable to our system. Our analysis suggests that maintaining compatibility between conditions at the boundaries and within the system restricts the possible sets of conditions. We listed valid sets of boundary conditions linked to imposing pressures and stress-free conditions at the outlet and either velocity or pressure at the inlet.

Using principles from the review paper [Bresch et al., 2024], Section 5.2, we rewrote our two-phase system following a drift flux structure so that velocities, pressures, and temperatures are separated into a bulk and a differential component. Dimensional arguments led us to focus on the mass-averaged drift velocity, Yw, instead of the classical drift velocity w. A dimensionless version of this novel drift flux system was used to assess the relative importance of physically meaningful terms. Considering ranges of parameter values relevant to magmatic systems led us to propose a simplified model (4.37) valid up to order 10^{-3}. This simplified system suggests that mass exchange is a major control of flow dynamics. Bulk magma temperature is expected to vary because of gas expansion but temperature differences can be neglected. Relative velocities can be captured to the main order with a Darcy equation and pressures differences partly control magma acceleration. The accuracy and limitations of these simplifications are assessed in Part II, Section 6 [Burgisser et al., 2024].

The simplified system highlights the importance of mass exchange. This drove us to propose a limit case of mass exchange of the proposed system (2.74) by establishing a relaxed system at chemical equilibrium (i.e. $C_l = C_l^{\mathrm{eq}}$). This system (5.16) has a single velocity and a single temperature without diffusive effects. We demonstrate that it is a generalization of the system obtained by [Fowler and Robinson, 2018].

Finally, we explored the consequences of choosing different sets of initial assumptions to create a two-phase flow model by comparing our new system (2.74) to that of La Spina and co-workers [La Spina and de' Michieli Vitturi, 2012, La Spina et al., 2014, La Spina et al., 2017]. Unlike our compressible/incompressible mixture, these authors assume energy conservation from thermodynamics in a system where both phases are compressible. Their model construction starts from a Lagrangian on the bulk and drift velocities u and w. The equations on u_g and u_l and the energy conservation then deduced from these initial mass and momentum equations. The main differences arising from these different constructions are liquid pressure relationships, which differ because of the contrasted definitions of the internal energies of both phases, the carrier velocity of φ (u in the La Spina model and u_l in our model), the interface pressures, and the mass exchange (based on a constant time scale in the La Spina model and based on diffusion in our model). This comparison illustrates that different two-phase equations may be obtained depending on the governing unknowns chosen. Numerical evaluations of these differences is carried out in Part II, Section 6 [Burgisser et al., 2024].

A Details on the microscopic analysis

This section contains details on the microscopic analysis developed in Section 2.2.

A.1 Rayleigh-Plesset equation

In this section we present the derivation of the Rayleigh-Plesset equation. We consider Lagrangian coordinates as it has been introduced in Section 2.2.1 (see Figure 5 for sketch and notation).

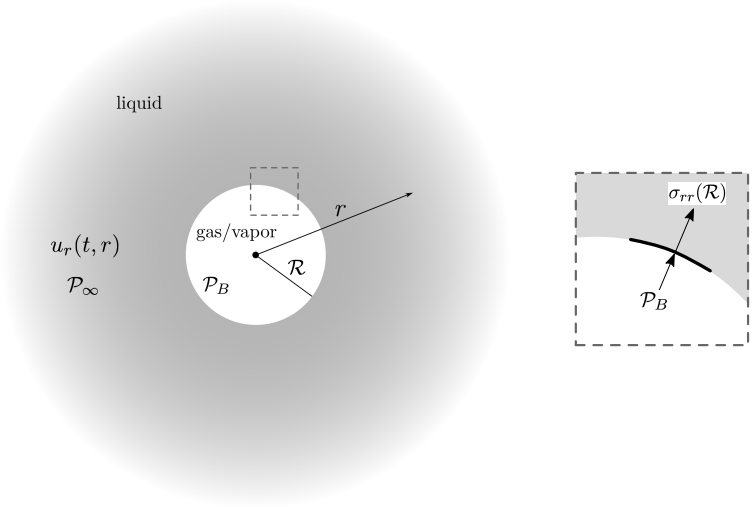

Figure 5: Configuration of a spherical bubble in an infinite liquid (adapted from [Brennen, 1995])

Rayleigh-Plesset with a bubble in an infinite medium. The Rayleigh-Plesset equation originally presented in [Scriven, 1959] is based on the analysis of the behavior of a bubble of radius $\mathcal{R}$, the center of which is assumed to be at rest. The gas inside the bubble is at the vapour pressure, $\mathcal{P}_v$, and density, ρ_v. Both are assumed constant.

The integration of the incompressibility equation, $\frac{1}{r^2}\partial_r(r^2 u_r) = 0$ in $r \in [\mathcal{R}, \infty)$ gives the radial velocity of the liquid adjacent to the bubble wall (i.e. the velocity orthonormal to the wall). As boundary condition we use the Lagrangian coordinates to introduce the velocity of the bubble surface,

$$u_r(t,\mathcal{R}) = (1 - \frac{\rho_v}{\rho_l})\frac{d\mathcal{R}}{dt} := (1 - \frac{\rho_v}{\rho_l})\dot{\mathcal{R}},$$

which yields

$$u_r(t,r) = (1 - \frac{\rho_v}{\rho_l})\frac{\mathcal{R}^2}{r^2}\dot{\mathcal{R}}, \qquad r \geq \mathcal{R}. \tag{A.1}$$

The momentum equation in radial coordinates reads,

$$\rho_l(\partial_t u_r + u_r\partial_r u_r) = -\partial_r\mathcal{P} + 2\left(\frac{1}{r^2}\partial_r(\eta_l r^2\partial_r u_r) - \eta_l\frac{2u_r}{r^2}\right) \tag{A.2}$$

where $\mathcal{P}$ is the pressure in the fluid adjacent to the bubble. If η_l is assumed constant, the viscosity term on the right hand side vanishes for the radial velocity (A.1), leading to

$$\rho_l(\partial_t u_r + u_r\partial_r u_r) = -\partial_r\mathcal{P}. \tag{A.3}$$

This equation is integrated in $r \in [\mathcal{R}, \infty)$ using the dynamic mechanical equilibrium (cf. chapter 4 of [Toramaru, 2022]) given by the Young-Laplace equation as the boundary condition at the bubble interface, that is,

$$\mathcal{P}_B - (-\sigma_{rr}(\mathcal{R})) = \frac{2\tilde{\sigma}}{\mathcal{R}}, \qquad \text{with} \quad \sigma_{rr} = -\mathcal{P} + 2\eta_l \partial_r u_r \quad \text{the stress orthonormal to the bubble wall (see Figure 5)},$$

where $\mathcal{P}_B$ is the pressure in the bubble (the sum of the pressure in the bubble gas, $\mathcal{P}_g$, and the vapour pressure due to the temperature changes, $\mathcal{P}_v$) and $\tilde{\sigma}$ is the surface tension of the bubble. So we obtain the value of the pressure $\mathcal{P}$ at the bubble surface:

$$\mathcal{P}(\mathcal{R}) = \mathcal{P}_B - 2\frac{\tilde{\sigma}}{\mathcal{R}} - 4\eta_l \frac{\dot{\mathcal{R}}}{\mathcal{R}}. \tag{A.4}$$

We integrate equation (A.3) in r from $\mathcal{R}$ to ∞, remarking that

$$\rho_l(\partial_t u_r + u_r \partial_r u_r) = (\rho_l - \rho_v)\left(\frac{\mathcal{R}^2}{r^2}\ddot{\mathcal{R}} + \frac{2\mathcal{R}\ddot{\mathcal{R}}^2}{r^2} + \frac{1}{2}\partial_r(\frac{\mathcal{R}^4}{r^4})\ddot{\mathcal{R}}^2\right)$$

using expression (A.1). This integration using the boundary condition on $\mathcal{P}(\mathcal{R})$ then leads to the Rayleigh-Plesset equation:

$$\frac{1}{\rho_l - \rho_v}\left(\mathcal{P}_B - \mathcal{P}_\infty - 2\frac{\tilde{\sigma}}{\mathcal{R}}\right) = \mathcal{R}\ddot{\mathcal{R}} + \frac{3}{2}\dot{\mathcal{R}}^2 + 4\frac{\eta_l}{\rho_l}\frac{\dot{\mathcal{R}}}{\mathcal{R}} \tag{A.5}$$

where $\mathcal{P}_\infty$ is the pressure of the liquid far from the bubble. This derivation can be found in several references, such as [Leighton, 1994, Brennen, 1995].

A usual simplification is found when the evaporation or condensation effect is neglected ($\mathcal{P}_v = 0$ under a constant temperature) and the vapour density is considered small in front of the liquid density giving

$$\frac{1}{\rho_l}\left(\mathcal{P}_g - \mathcal{P}_\infty - 2\frac{\tilde{\sigma}}{\mathcal{R}}\right) = \mathcal{R}\ddot{\mathcal{R}} + \frac{3}{2}\dot{\mathcal{R}}^2 + 4\frac{\eta_l}{\rho_l}\frac{\dot{\mathcal{R}}}{\mathcal{R}}.$$

An even more simplified equation is found when the inertial terms are neglected, which is the case for magma with high viscosity liquid and small particle Reynolds number [Arefmanesh and Advani, 1991, Proussevitch et al., 1993b, Proussevitch and Sahagian, 1998]:

$$\frac{\dot{\mathcal{R}}}{\mathcal{R}} = \frac{1}{4\eta_l}\left(\mathcal{P}_g - \mathcal{P}_\infty - 2\frac{\tilde{\sigma}}{\mathcal{R}}\right). \tag{A.6}$$

Rayleigh-Plesset with a bubble population. To extend this work to a magma containing many bubbles, the magma is considered as a set of identical spherical cells composed of a spherical gas bubble of radius $\mathcal{R}$ in a surrounding liquid melt of radius $\mathcal{S} > \mathcal{R}$, so the bubbles do not interact with each other. The packing arrangement considers that these cells slightly overlap uniformly in the suspension in a way such that the gas volume fraction of magma is (see Figure 2 for a sketch and notation):

$$\tilde{\varphi} = \frac{\mathcal{R}^3}{\mathcal{S}^3}, \tag{A.7}$$

The reasoning to deduce the Rayleight-Plesset equation may be adopted to this case as it has been developed in [Arefmanesh and Advani, 1991, Lensky et al., 2001] for a general non-constant viscosity. Here we consider that η_l is constant in the melt cell as in [Proussevitch et al., 1993b] and we also neglect the inertial terms (that means

the total time derivative) and set $\mathcal{P}_v = 0$. The radial velocity is still given by (A.1) but now for $r \in [\mathcal{R}, \mathcal{S}]$ and we integrate the momentum equation (A.2) in $r \in [\mathcal{R}, \mathcal{S}]$, which yields, after integration in space:

$$\mathcal{P}(\mathcal{S}) - \mathcal{P}(\mathcal{R}) = 0. \tag{A.8}$$

It remains to express $\mathcal{P}(\mathcal{S})$ and $\mathcal{P}(\mathcal{R})$. At the bubble surface ($r = \mathcal{R}$) we use the boundary condition (A.4) so we have

$$\mathcal{P}(\mathcal{R}) = \mathcal{P}_g - 2\frac{\tilde{\sigma}}{\mathcal{R}} - 4\eta_l \frac{\dot{\mathcal{R}}}{\mathcal{R}}. \tag{A.9}$$

At the limit of the melt cell ($r = \mathcal{S}$), it is assumed (cf. [Arefmanesh and Advani, 1991, Proussevitch et al., 1993b, Lensky et al., 2001]) that

$$\sigma_{rr}(\mathcal{S}) = (-\mathcal{P} + 2\eta_l \partial_r u_r)_{|r=\mathcal{S}} = -\mathcal{P}_{\text{amb}} \tag{A.10}$$

where $\mathcal{P}_{\text{amb}}$ is the external ambient pressure to the cell. Plugging expressions (A.9) and (A.10) in (A.8) yields

$$\mathcal{P}_g - \mathcal{P}_{\text{amb}} - 2\frac{\tilde{\sigma}}{\mathcal{R}} = 4\eta_l \dot{\mathcal{R}} \left(\frac{1}{\mathcal{R}} - \frac{\mathcal{R}^2}{\mathcal{S}^3} \right) \quad \Rightarrow \quad \frac{\dot{\mathcal{R}}}{\mathcal{R}} = \frac{1}{4\eta_l(1-\tilde{\varphi})} \left(\mathcal{P}_g - \mathcal{P}_{\text{amb}} - \frac{2\tilde{\sigma}}{\mathcal{R}} \right). \tag{A.11}$$

where we used (2.18). Note that there is a modification of the left hand side compared to (A.6) because the factor $1 - \tilde{\varphi}$ appears in the denominator. [Toramaru, 2022] interprets $\eta_l(1-\tilde{\varphi})$ as the effective viscosity of the liquid that is modified by the presence of the bubbles.

In our case, we consider that including a bubble population instead of a single bubble means that there is a continuity of the liquid phase that implies a continuity of the pressure and the viscosity component of the normal stress. As a result, the boundary of the melt cell (see equation (A.10)) follows:

$$\mathcal{P}_{\text{amb}} = \mathcal{P}_l - 2\eta_l(\partial_r u_r)_{|r=\mathcal{S}} \quad \text{so} \quad \mathcal{P}(\mathcal{S}) = \mathcal{P}_l, \tag{A.12}$$

and the resulting equation reads:

$$\dot{\mathcal{R}} = \frac{\mathcal{R}}{4\eta_l} \left(\mathcal{P}_g - \mathcal{P}_l - \frac{2\tilde{\sigma}}{\mathcal{R}} \right). \tag{A.13}$$

Equation (A.13) coincides with the original Rayleigh-Plesset equation (A.6) for $\mathcal{P}_\infty = \mathcal{P}_l$. Following the [Toramaru, 2022] interpretation, this assumption is equivalent to considering that the effective viscosity equals the liquid viscosity.

A.2 Mass exchange term R^{H_2O} in the literature.

As mentioned in Section 2.2.2, we analyzed two propositions from the literature ([Lyakhovsky et al., 1996]; [Mancini et al., 2016]), to define the water mass balance and its impact on the exchange mass term R^{H_2O}.

Approaches to find $\partial_r \mathcal{C}$ at the interface. The first approach is from [Lyakhovsky et al., 1996]. Since the boundary condition at the bubble interface $r = \mathcal{R}$ in (2.27) is taken to consider the equilibrium state at that place, it seems suitable (or at least compatible) to search for the value of the concentration gradient at the interface with the same assumption. In [Lyakhovsky et al., 1996], an explicit expression is found for the concentration gradient at the bubble interface for the quasi-static case (small Peclet number) given in terms of the cell size (see equation 13 in that paper):

$$(\partial_r \mathcal{C})_{|r=\mathcal{R}} = \frac{\mathcal{C}_0 - \mathcal{C}_{|r=\mathcal{R}} - \frac{\tilde{\rho}_g}{\rho_l}\frac{\mathcal{R}^3}{\mathcal{S}_0^3}}{\mathcal{R} - \frac{3}{2}(\mathcal{S}^2 - \mathcal{R}^2)\frac{\mathcal{R}^2}{\mathcal{S}_0^3}} \sim \frac{\mathcal{C}_0 - k_h\sqrt{\mathcal{P}_g}}{\mathcal{R} - \frac{3}{2}(\mathcal{S}^2 - \mathcal{R}^2)\frac{\mathcal{R}^2}{\mathcal{S}_0^3}} \qquad \text{with} \qquad \mathcal{S}_0^3 = \mathcal{S}^3 - \mathcal{R}^3 \tag{A.14}$$

where for the last approximation the authors assumed $\tilde{\rho}_g \ll \rho_l$. For the case of a tiny bubble, $\mathcal{S} \gg \mathcal{R}$, equation (A.14) yields $(\partial_r \mathcal{C})_{|r=\mathcal{R}} = \frac{\mathcal{C}_0 - k_h\sqrt{\mathcal{P}_g}}{\mathcal{R}}$.

The second approach is given by [Mancini et al., 2016]. The water mass balance (2.30) is written in terms of the water mass at the equilibrium, $\mathcal{M}_g^{\text{eq}}$:

$$\frac{d\mathcal{M}_g}{dt} = \frac{\mathcal{A}}{\mathcal{R}_0^2} D(\mathcal{M}_g^{\text{eq}} - \mathcal{M}_g) \tag{A.15}$$

for a dimensionless coefficient $\mathcal{A}$ and a characteristic radius $\mathcal{R}_0$, both of them to be defined. To find the equilibrium water mass, $\mathcal{M}_g^{\text{eq}}$, the authors use the conservation equation (2.29) for $\mathcal{C} = \mathcal{C}^{\text{eq}}$ that reads:

$$\mathcal{M}_g^{\text{eq}} + 4\pi\rho_l \mathcal{C}^{\text{eq}} \frac{\mathcal{S}^3 - \mathcal{R}^3}{3} = \frac{4\pi}{3}\mathcal{S}_0^3 \mathcal{C}_0 \rho_l,$$

leading to

$$\mathcal{M}_g^{\text{eq}} = \frac{4\pi}{3}\rho_l \mathcal{S}_0^3 (\mathcal{C}_0 - \mathcal{C}^{\text{eq}}). \tag{A.16}$$

Identification of (A.15) with (2.30) using (A.16) and the definition of $\mathcal{M}_g$ gives

$$(\partial_r \mathcal{C})_{|r=\mathcal{R}} = \frac{1}{3}\frac{\mathcal{A}}{\mathcal{R}_0^2}\frac{\mathcal{S}_0^3}{\mathcal{R}^2}\left(\left(\mathcal{C}_0 - \frac{\tilde{\rho}_g \mathcal{R}^3}{\rho_l \mathcal{S}_0^3}\right) - \mathcal{C}^{\text{eq}}\right) \tag{A.17}$$

or, using that $\mathcal{S}_0^3 = \mathcal{S}^3 - \mathcal{R}^3$,

$$(\partial_r \mathcal{C})_{|r=\mathcal{R}} = \frac{1}{3}\frac{\mathcal{A}}{\mathcal{R}_0^2}\frac{\mathcal{S}^3 - \mathcal{R}^3}{\mathcal{R}^2}\left(\mathcal{C}^* - k_h\sqrt{\mathcal{P}_g}\right), \quad \text{with} \quad \mathcal{C}^* = \mathcal{C}_0 - \frac{\tilde{\rho}_g \mathcal{R}^3}{\rho_l(\mathcal{S}^3 - \mathcal{R}^3)}. \tag{A.18}$$

Embedding (A.14) or (A.18) in equation (2.30) yields the water mass balance at the bubble interface.

Limit behavior of the rate of mass exchange. We analyze the behavior of the rate of mass exchange $\frac{d\mathcal{M}_g}{dt}$ as made in Remark 2.3 for the previous approaches. In particular we check if the following limits to be satisfied:

$$\lim_{\mathcal{R}\to 0}\frac{d\mathcal{M}_g}{dt} = 0 \qquad \text{and} \qquad \lim_{\mathcal{R}\to \mathcal{S}}\frac{d\mathcal{M}_g}{dt} = \infty. \tag{A.19}$$

When the first approach (A.14) is considered, (2.30) reads

$$\frac{d\mathcal{M}_g}{dt} = 4\pi\rho_l D \frac{\mathcal{R}}{1 - \frac{3}{2}\frac{\mathcal{R}(\mathcal{S}^2 - \mathcal{R}^2)}{\mathcal{S}^3 - \mathcal{R}^3}}(\mathcal{C}_0 - k_h\sqrt{\mathcal{P}_g})$$

As $\mathcal{S}^3 - \mathcal{R}^3 = (\mathcal{S} - \mathcal{R})(\mathcal{S}^2 + \mathcal{S}\mathcal{R} + \mathcal{R}^2)$, then $\frac{\mathcal{S}^2 - \mathcal{R}^2}{\mathcal{S}^3 - \mathcal{R}^3} = \frac{\mathcal{S} + \mathcal{R}}{\mathcal{S}^2 + \mathcal{S}\mathcal{R} + \mathcal{R}^2}$ and we also obtain both desired limits. When the second approach (A.18) is considered, (2.30) reads

$$\frac{d\mathcal{M}_g}{dt} = 4\pi\rho_l D \frac{1}{3}\frac{\mathcal{A}}{\mathcal{R}_0^2}(\mathcal{S}^3 - \mathcal{R}^3)\left(\left(\mathcal{C}_0 - \frac{\tilde{\rho}_g \mathcal{R}^3}{\rho_l(\mathcal{S}^3 - \mathcal{R}^3)}\right) - k_h\sqrt{\mathcal{P}_g}\right)$$

and we are only able to write

$$\lim_{\mathcal{R}\to 0}\frac{d\mathcal{M}_g}{dt} = \frac{4\pi}{3}\rho_l D \mathcal{S}^3(\mathcal{C}_0 - k_h\sqrt{\mathcal{P}_g}) \lim_{\mathcal{R}\to 0}\frac{\mathcal{A}}{\mathcal{R}_0^2},$$

$$\lim_{\mathcal{R}\to\mathcal{S}} \frac{d\mathcal{M}_g}{dt} = \frac{4\pi}{3}\rho_l D \lim_{\mathcal{R}\to 0}\left(\frac{\mathcal{A}}{\mathcal{R}_0^2}(\mathcal{S}^3-\mathcal{R}^3)\left(\left(\mathcal{C}_0 - \frac{\tilde{\rho}_g\mathcal{R}^3}{\rho_l(\mathcal{S}^3-\mathcal{R}^3)}\right) - k_h\sqrt{\mathcal{P}_g}\right)\right)$$

The limits depend on the behavior of $\frac{\mathcal{A}}{\mathcal{R}_0^2}$ and so we cannot be conclusive at this point. The choice of [Mancini et al., 2016] of $\mathcal{A}=1$ and $\mathcal{R}_0=\mathcal{R}$ yields the undesirable limits of ∞ for $\mathcal{R}\to 0$ and $-4\pi D\rho_g\mathcal{S}/3$ for $\mathcal{R}\to\mathcal{S}$. Other values for $\mathcal{A}$ and $\mathcal{R}_0$ are discussed below in Remark A.1.

Remark A.1. *The previous propositions of [Lyakhovsky et al., 1996] and [Mancini et al., 2016] can also be written under the structure of our proposed approach in* (2.31)*. Thus, in the definition given in equation* (A.14)*, we write the denominator as follows*

$$\mathcal{R} - \frac{3}{2}(\mathcal{S}^2-\mathcal{R}^2)\frac{\mathcal{R}^2}{\mathcal{S}_0^3} = \frac{\mathcal{R}\mathcal{S}^3}{\mathcal{S}^3-\mathcal{R}^3}\left(\frac{1}{2}\left(\frac{\mathcal{R}}{\mathcal{S}}\right)^3 - \frac{3}{2}\frac{\mathcal{R}}{\mathcal{S}}+1\right) = \frac{1}{2}\frac{\mathcal{R}\mathcal{S}^3}{\mathcal{S}^3-\mathcal{R}^3}\left(\frac{\mathcal{R}}{\mathcal{S}}-1\right)\left(\left(\frac{\mathcal{R}}{\mathcal{S}}\right)^2+\frac{\mathcal{R}}{\mathcal{S}}-2\right),$$

that using (2.18) *it becomes,*

$$\mathcal{R} - \frac{3}{2}(\mathcal{S}^2-\mathcal{R}^2)\frac{\mathcal{R}^2}{\mathcal{S}_0^3} = \frac{1}{2}\frac{\tilde{\varphi}^{1/3}(2-\tilde{\varphi}^{2/3}-\tilde{\varphi}^{1/3})}{1-\tilde{\varphi}}(\mathcal{S}-\mathcal{R}).$$

Then using the definition of φ*,* (A.14) *also reads*

$$(\partial_r\mathcal{C})_{|r=\mathcal{R}} \sim \mathcal{A}_L\frac{\mathcal{C}_0 - k_h\sqrt{\mathcal{P}_g}}{\mathcal{S}-\mathcal{R}} \qquad \textit{with} \quad \mathcal{A}_L = \frac{1-\tilde{\varphi}}{\tilde{\varphi}^{1/3}(1-\frac{1}{2}\tilde{\varphi}^{2/3}-\frac{1}{2}\tilde{\varphi}^{1/3})}. \tag{A.20}$$

The proposition in (A.18) *presented in [Mancini et al., 2016] reads as*

$$(\partial_r\mathcal{C})_{|r=\mathcal{R}} = \mathcal{A}_M\frac{\mathcal{C}^* - k_h\sqrt{\mathcal{P}_g}}{\mathcal{S}-\mathcal{R}}, \qquad \textit{with} \quad \mathcal{A}_M = \frac{1}{3}\frac{\mathcal{A}}{\mathcal{R}_0^2}(\mathcal{S}-\mathcal{R})R\frac{1-\tilde{\varphi}}{\tilde{\varphi}}, \quad \mathcal{C}^* = \mathcal{C}_0 - \frac{\tilde{\rho}_g\tilde{\varphi}}{\rho_l(1-\tilde{\varphi})}. \tag{A.21}$$

Note that $\mathcal{A}_M$ *can also be written as*

$$\mathcal{A}_M = \frac{1}{3}\frac{\mathcal{A}}{\mathcal{R}_0^2}\mathcal{R}^2\frac{(1-\tilde{\varphi})(1-\tilde{\varphi}^{1/3})}{\tilde{\varphi}^{4/3}}. \tag{A.22}$$

The values of $\mathcal{A}$ *and* $\mathcal{R}_0$ *need to be set. One way to do so is to use equation* (2.31) *as a reference. In other words, to fit* (A.21) *with* (2.31) *we may set* $(\mathcal{C})_{|r=\mathcal{S}} = \mathcal{C}^*$ *and* $\mathcal{A}_M = 1$*, so that* $\frac{\mathcal{A}}{\mathcal{R}_0^2} = 3\frac{\tilde{\varphi}}{1-\tilde{\varphi}}\frac{1}{\mathcal{R}(\mathcal{S}-\mathcal{R})}$*. Then, if we define for instance* $\mathcal{R}_0 = \mathcal{S}-\mathcal{R}$*, then* $\mathcal{A} = 3\frac{\tilde{\varphi}}{1-\tilde{\varphi}}(\tilde{\varphi}^{-1/3}-1)$*. Another way was chosen in Part II, Section 6.1.2 [Burgisser et al., 2024] for consistency with [Mancini et al., 2016] ;* $\mathcal{A}$ *was set to 1 and* $\mathcal{R}_0$ *was set to* $\mathcal{R}$*.* □

Macroscopic mass exchange rate. We present the mass exchange term R^{H_2O} for the two proposals. If with the first approach (A.20) we identify $(\mathcal{C})_{|r=\mathcal{S}}$ with $\mathcal{C}_0$, we can write

$$\mathcal{R}_L^{H_2O} = \mathcal{A}_L\mathcal{R}^{H_2O} \qquad \text{with} \quad \mathcal{A}_L = \frac{1-\tilde{\varphi}}{\tilde{\varphi}^{1/3}(1-\frac{1}{2}\tilde{\varphi}^{2/3}-\frac{1}{2}\tilde{\varphi}^{1/3})}, \tag{A.23}$$

and then

$$\begin{aligned}\lim_{\tilde{\varphi}\to 0}\mathcal{R}_L^{H_2O} &= 3^{1/3}\rho_l D(4\pi\mathcal{N})^{2/3}(\mathcal{C}_0 - k_h\sqrt{\mathcal{P}_g})\lim_{\tilde{\varphi}\to 0}\frac{\tilde{\varphi}^{2/3}}{1-\tilde{\varphi}^{1/3}}\mathcal{A}_L \\ &= 3^{1/3}\rho_l D(4\pi\mathcal{N})^{2/3}(\mathcal{C}_0 - k_h\sqrt{\mathcal{P}_g})\lim_{\tilde{\varphi}\to 0}\frac{\tilde{\varphi}^{1/3}}{1-\tilde{\varphi}^{1/3}}\frac{1-\tilde{\varphi}}{(1-\frac{1}{2}\tilde{\varphi}^{2/3}-\frac{1}{2}\tilde{\varphi}^{1/3})} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lim_{\tilde{\varphi}\to 1}\mathcal{R}_L^{H_2O} &= 3^{1/3}\rho_l D(4\pi\mathcal{N})^{2/3}(\mathcal{C}_0 - k_h\sqrt{\mathcal{P}_g})\lim_{\tilde{\varphi}\to 1}\frac{(1-\tilde{\varphi})^{2/3}}{1-\tilde{\varphi}^{1/3}}\mathcal{A}_L \\ &= 3^{1/3}\rho_l D(4\pi\mathcal{N})^{2/3}(\mathcal{C}_0 - k_h\sqrt{\mathcal{P}_g})\lim_{\tilde{\varphi}\to 1}\frac{(1-\tilde{\varphi})^{5/3}}{(1-\frac{1}{2}\tilde{\varphi}^{2/3}-\frac{1}{2}\tilde{\varphi}^{1/3})(1-\tilde{\varphi}^{1/3})} \\ &= 3^{1/3}\rho_l D(4\pi\mathcal{N})^{2/3}(\mathcal{C}_0 - k_h\sqrt{\mathcal{P}_g})\lim_{\tilde{\varphi}\to 1}\frac{(1-\tilde{\varphi})^{2/3}(1+\tilde{\varphi}^{1/3}+\tilde{\varphi}^{2/3})^{5/3}}{1-\frac{1}{2}\tilde{\varphi}^{2/3}-\frac{1}{2}\tilde{\varphi}^{1/3}} \\ &= \infty\end{aligned}$$

where in the last equality we used again $1-\tilde{\varphi} = (1-\tilde{\varphi}^{1/3})(1+\tilde{\varphi}^{1/3}+\tilde{\varphi}^{2/3})$. In order to consider the second approach (A.21), we must identify $(\mathcal{C})_{|r=\mathcal{S}}$ with $\mathcal{C}^* = \mathcal{C}_0 - \frac{\tilde{\rho}_g\tilde{\varphi}}{\rho_l(1-\tilde{\varphi})}$. Note that for this definition, when $\tilde{\varphi}$ tends to zero we find $\mathcal{C}^* = \mathcal{C}_0$ but, on the contrary, when $\tilde{\varphi}$ tends to one, $\mathcal{C}^*$ is not bounded. This last limit is consistent with the fact that when there is only gas, the concentration tends to infinity. Focusing on $\mathcal{R}^{H_2O}$, we can similarly write

$$\mathcal{R}_M^{H_2O} = \mathcal{A}_M\mathcal{R}^{H_2O} \qquad \text{with} \quad \mathcal{A}_M = \frac{1}{3}\frac{\mathcal{A}}{\mathcal{R}_0^2}\left(\frac{3}{4\pi\mathcal{N}}\right)^{2/3}\frac{(1-\tilde{\varphi})^{1/3}(1-\tilde{\varphi}^{1/3})}{\tilde{\varphi}^{2/3}}. \tag{A.24}$$

where we used (A.22) together with the definition of $\tilde{n}$. Then

$$\lim_{\tilde{\varphi}\to 0}\mathcal{R}_M^{H_2O} = \rho_l D(\mathcal{C}^* - k_h\sqrt{\mathcal{P}_g})\lim_{\tilde{\varphi}\to 0}(1-\tilde{\varphi})\frac{\mathcal{A}}{\mathcal{R}_0^2} = \rho_l D(\mathcal{C}^* - k_h\sqrt{\mathcal{P}_g})\lim_{\tilde{\varphi}\to 0}\frac{\mathcal{A}}{\mathcal{R}_0^2}$$

and

$$\lim_{\tilde{\varphi}\to 1}\mathcal{R}_M^{H_2O} = \rho_l D(\mathcal{C}^* - k_h\sqrt{\mathcal{P}_g})\lim_{\tilde{\varphi}\to 1}(1-\tilde{\varphi})\frac{\mathcal{A}}{\mathcal{R}_0^2}.$$

B Relating the proposed temperature equations to thermodynamics

We followed the approach of deducing the temperature equations, (2.56) and (2.57), a posteriori to obtain a dissipative energy balance. Classically, in single-phase, compressible flows models, the mass and momentum equations are completed with the energy equation from the thermodynamic theory. As we show below, the temperature equation can be obtained from the energy equation through the thermal equations of state (e.g., [Kolev, 2007, Müller and Müller, 2009]).

From the thermodynamics of compressible fluids [Müller and Müller, 2009], the general, single-phase balance of the total energy E for a fluid with density ρ, velocity u and internal energy e, $E = \frac{1}{2}|u|^2 + e$, reads

$$\partial_t(\rho E) + \operatorname{div}(\rho E u) = \operatorname{div}(\Sigma u) + \operatorname{div}(\kappa\nabla T) + \rho(f\cdot u) + r \tag{B.1}$$

where $\Sigma = -p\mathrm{Id} + \mathcal{D}$ is the total stress tensor and $\mathcal{D}$ is the viscous tensor. The term $\kappa\nabla T$ is the flux of internal energy according to Fourier's law with thermal conductivity coefficient κ, f represents the potential energy and r is the external absorption/emission energy contribution (in this work, we ignore the contributions from chemical reaction and nuclear, radiative, or electromagnatic processes).

Building two-phase flow systems involves homogenization procedures (see Section 2.3) that rest on various sets of assumptions. Because of this variety, two-phase systems may share the same total energy balance, but feature different partitions between the phases (see Remark 6.3). Unlike single-phase flows, two-phase energy balances involve the material derivative of the phase volume fraction φ, which is the work done by the expansion of volume fraction [Gidaspow, 1994]. In our case, we first established the mass and momentum conservation

equations of our two-phase system (2.52) by leaving some quantities undefined: the interface quantities $\langle p_{int,g}\rangle$ and $\langle u_I\rangle$, and an equation for the gas temperature T_g. We then used the calculation of the total energy balance obtained explicitly from the mass and momentum equations to find suitable relationships for these quantities. The equation for the gas temperature needing to be consistent with thermodynamics, we based ourselves on the standard structure coming from the thermodynamic theory to propose relevant adjustments or extensions that match the specificity of our two-phase model. One of these extensions is the inclusion of the heat transfer between the two phases, which is quantified by the term $\gamma(T_g - T_l)$. This term, involving the additional variable T_l, drove us to prescribe the closure equation (2.57) for the liquid temperature. In other words, our approach aims at finding a minimal model that ensures energy dissipation.

The total energy balance (B.1), which is constructed from first law of thermodynamics (e.g., [Brennen, 2005]), contains the contribution from the momentum equation, which can be seen by splitting it into a balance for the internal energy e and one for the kinetic energy $\frac{1}{2}|u|^2$:

$$\partial_t(\rho e) + \mathrm{div}(\rho e u) = \Sigma : \nabla u + \mathrm{div}(\kappa \nabla T) + r, \tag{B.2}$$

and

$$\partial_t\left(\frac{1}{2}\rho|u|^2\right) + \mathrm{div}\left(\frac{1}{2}\rho|u|^2 u\right) = -\Sigma : \nabla u + \mathrm{div}(\Sigma u) + \rho(f \cdot u).$$

As the kinetic energy balance is constructed from the momentum and mass transport equation, it can be viewed as a transport equation of energy [Gidaspow, 1994, Bird et al., 2007, Ishii and Hibiki, 2011]. To find the corresponding temperature equations, we use the equation of state of the fluid, which in particular provides the following rule for derivatives,

$$\rho D_t e = \rho c_p D_t T + \frac{p}{\rho} D_t \rho + \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p D_t p,$$

where $D_t\xi = \partial_t \xi + u \cdot \nabla \xi$ is the material derivative for any variable ξ. Then we get

$$\begin{aligned}\partial_t(c_p\rho T) + \mathrm{div}(c_p\rho T u) + \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p D_t p - \mathcal{D} : \nabla u - \mathrm{div}(\kappa \nabla T) - r + \frac{p}{\rho}(\partial_t \rho + \mathrm{div}(\rho u)) \\ = (c_p T - e)(\partial_t \rho + \mathrm{div}(\rho u)).\end{aligned} \tag{B.3}$$

For an ideal gas, $p_1 = c_0 \rho_1 T_1$. Using $\Sigma_1 : \nabla u_1 = -p_1 \,\mathrm{div} u_1 + \mathcal{D}_1 : \nabla u_1$ and that c_{p1} is constant, we find the temperature equation

$$\begin{aligned}\partial_t(c_{p1}\rho_1 T_1) + \mathrm{div}(c_{p1}\rho_1 T_1 u_1) - D_t p_1 - \mathcal{D}_1 : \nabla u_1 - \mathrm{div}(\kappa_1 \nabla T_1) - r_1 \\ = ((c_{p1} - c_0)T_1 - e_1)(\partial_t \rho_1 + \mathrm{div}(\rho_1 u_1))\end{aligned} \tag{B.4}$$

or equivalently

$$\begin{aligned}\partial_t((c_{p1} - c_0)\rho_1 T_1) + \mathrm{div}((c_{p1} - c_0)\rho_1 T_1 u_1) - \Sigma_1 : \nabla u_1 - \mathrm{div}(\kappa_1 \nabla T_1) - r_1 \\ = ((c_{p1} - c_0)T_1 - e_1)(\partial_t \rho_1 + \mathrm{div}(\rho_1 u_1)).\end{aligned}$$

As the internal energy is $e_1 = (c_{p1} - c_0)T_1$, we obtain (B.2) whether the mass is conserved or not.

For a compressible fluid with constant density ρ_2, the same procedure yields the temperature equation obtained from (B.3):

$$\partial_t(c_{p2}\rho_2 T_2) + \mathrm{div}(c_{p2}\rho_2 T_2 u_2) + p_2 \mathrm{div} u_2 - \mathcal{D}_2 : \nabla u_2 - \mathrm{div}(\kappa_2 \nabla T_2) - r_2 = (c_{p2} T_2 - e_2)\rho_2 \mathrm{div} u_2, \tag{B.5}$$

where the term $p_2 \mathrm{div} u_2$ depends on the compressibility law chosen. Again, if the internal energy is $e_2 = c_{p2} T_2$, we get (B.2).

As mentioned above, the internal energy balance for multiphase flows involves an additional term that takes into account the work done by the expansion of the volume fraction φ_i [Gidaspow, 1994]. For each phase i, the equivalent equation to (B.2) is thus

$$\partial_t(\varphi_i\rho_i e_i) + \mathrm{div}(\varphi_i\rho_i e_i u_i) = -p_i D_t\varphi_i + \varphi_i\Sigma_i : \nabla u_i + \mathrm{div}(\varphi_i\kappa_i\nabla T_i) + r_i, \tag{B.6}$$

Following the same procedure as before yields the alternate equation to (B.3) for each phase,

$$\begin{aligned}\partial_t(c_{pi}\varphi_i\rho_i T_i) + \mathrm{div}(c_{pi}\varphi_i\rho_i T_i u_i) + \varphi_i\frac{T_i}{\rho_i}\left(\frac{\partial\rho_i}{\partial T_i}\right)_{p_i} D_t p_i - \varphi_i\mathcal{D}_i : \nabla u_i - \mathrm{div}(\varphi_i\kappa_i\nabla T_i) - r_i \\ + p_i D_t\varphi_i + \varphi_i\frac{p_i}{\rho_i}(\partial_t\rho_i + \mathrm{div}(\rho_i u_i)) = (c_{pi}T_i - e_i)(\partial_t(\varphi_i\rho_i) + \mathrm{div}(\varphi_i\rho_i u_i)).\end{aligned} \tag{B.7}$$

For an ideal gas, with the fractional density $\varphi_1\rho_1$ and pressure $p_1 = c_0\rho_1 T_1$, we get the equivalent equation to (B.4),

$$\begin{aligned}\partial_t(c_{p1}\varphi_1\rho_1 T_1) + \mathrm{div}(c_{p1}\varphi_1\rho_1 T_1 u_1) - \varphi_1 D_t p_1 - \varphi_1\mathcal{D}_1 : \nabla u_1 - \mathrm{div}(\varphi_1\kappa_1\nabla T_1) - r_1 \\ = ((c_{p1} - c_0)T_1 - e_1)(\partial_t(\varphi_1\rho_1) + \mathrm{div}(\varphi_1\rho_1 u_1))\end{aligned}$$

where the term on the right hand side vanishes for $e_1 = (c_{p1} - c_0)T_1$. Our proposed equation for the gas temperature (2.56) follows this structure, considering surface tension plus heat exchange as the external contribution r_1. From the energy equation (2.58) we deduce that the associated internal energy would be $e_g = (c_{pg} - c_0)T_g$. For a compressible fluid with constant density ρ_2 and fractional density $\varphi_2\rho_2$, (B.7) yields:

$$\begin{aligned}\partial_t(c_{p2}\varphi_2\rho_2 T_2) + \mathrm{div}(c_{p2}\varphi_2\rho_2 T_2 u_2) - \varphi_2\mathcal{D}_2 : \nabla u_2 - \mathrm{div}(\varphi_2\kappa_2\nabla T_2) - r_2 + p_2(\partial_t\varphi_2 + \mathrm{div}(\varphi_2 u_2)) \\ = (c_{p2}T_2 - e_2)(\partial_t(\varphi_2\rho_2) + \mathrm{div}(\varphi_2\rho_2 u_2)),\end{aligned}$$

that is the equivalent equation to (B.5). Again the term on the right-hand side vanishes for $e_2 = c_{p2}T_2$. Our proposed equation for the liquid temperature (2.57) also follows this structure, r_2 containing again the heat exchange. Note that from (2.52a) we have the value of $\partial_t\varphi_2 + \mathrm{div}(\varphi_2 u_2)$, providing the term $\frac{p_l}{\rho_l}R^{H_2O}$ appearing in (2.57). The corresponding internal energy would be $e_l = c_{pl}T_l$.

As for our system the internal energies are $e_g = (c_{pg} - c_0)T_g, e_l = c_{pl}T_l$, the total energy balance obtained in (2.58) becomes:

$$\begin{aligned}&\partial_t\Big(\varphi\rho_g(\tfrac{1}{2}|u_g|^2 + e_g) + (1-\varphi)\rho_l(\tfrac{1}{2}|u_l|^2 + e_l) - (\varphi\rho_g + (1-\varphi)\rho_l)(g\cdot X) + \varphi\tfrac{2\sigma}{R}\Big) \\ &+\mathrm{div}\Big(\varphi\rho_g(\tfrac{1}{2}|u_g|^2 + e_g)u_g + (1-\varphi)\rho_l(\tfrac{1}{2}|u_l|^2 + e_l)u_l - (\varphi\rho_g u_g + (1-\varphi)\rho_l u_l)(g\cdot X)\Big) \\ &-\mathrm{div}(\varphi\Sigma_g u_g + (1-\varphi)\Sigma_l u_l) - \mathrm{div}\left(\varphi\kappa_g\nabla T_g + (1-\varphi)\kappa_l\nabla T_l\right) \\ &+K_d\varphi(1-\varphi)|u_g - u_l|^2 + \varphi(1-\varphi)\tfrac{3}{4\eta_l}\left(p_g - p_l - \tfrac{2\sigma}{R}\right)^2 = 0.\end{aligned} \tag{B.8}$$

where we added and subtracted $\mathrm{div}(p_g u_g)$ to (2.58) using $p_g = c_0\rho_g T_g$ and we denoted $\Sigma_g = -p_g\mathrm{Id} + \mathcal{D}_g$, $\Sigma_l = -p_l\mathrm{Id} + \mathcal{D}_l$, with $\mathcal{D}_g, \mathcal{D}_l$ the viscosity tensors defined in (2.10)-(2.11). This equation is identical to (B.1) if we neglect surface tension and we consider the potential energy as the gravitational potential, $f = \nabla(g\cdot X)$. The terms on the last line represent the loss of energy due to the drag terms on the relative velocities and pressures. Finally, the internal energy equations for gas and liquid phases can be written from the proposed temperature equations (2.56) and (2.57) using (2.13), (2.52a) and (2.52b):

$$\partial_t(\varphi\rho_g e_g) + \mathrm{div}(\varphi\rho_g e_g u_g) = -p_g\left(\partial_t\varphi + u_g\cdot\nabla\varphi\right) + \varphi\Sigma_g : \nabla u_g + \mathrm{div}(\varphi\kappa_g\nabla T_g) + \gamma(T_g - T_l) - \varphi\partial_t\left(\frac{2\sigma}{R}\right) \tag{B.9}$$

$$\partial_t((1-\varphi)\rho_l e_l) + \mathrm{div}((1-\varphi)\rho_l e_l u_l) = -p_l\left(\partial_t(1-\varphi) + u_l\cdot\nabla(1-\varphi)\right) + (1-\varphi)\Sigma_l : \nabla u_l \\ +\mathrm{div}((1-\varphi)\kappa_l\nabla T_l) - \gamma(T_g - T_l) \tag{B.10}$$

These equations follow the same structure as equation (B.6) where the external contributions are $r_g = \gamma(T_g - T_l) - \varphi\partial_t\left(\frac{2\sigma}{R}\right)$ and $r_l = -\gamma(T_g - T_l)$. By subtracting the internal energy equations (B.9), (B.10) from the total energy (B.8), we can also get the kinematic energy balance:

$$\begin{aligned}
&\partial_t\left((1-\varphi)\rho_l\frac{|u_l|^2}{2} + \varphi\rho_g\frac{|u_g|^2}{2} - (g\cdot X)((1-\varphi)\rho_l + \varphi\rho_g)\right) \\
&+ \mathrm{div}\left((1-\varphi)\rho_l u_l\frac{|u_l|^2}{2} + \varphi\rho_g u_g\frac{|u_g|^2}{2} - (g\cdot X)((1-\varphi)\rho_l u_l + \varphi\rho_g u_g)\right) \\
&= -\Big(\varphi\Sigma_g : \nabla u_g + (1-\varphi)\Sigma_l : \nabla u_l\Big) + \mathrm{div}(\varphi\Sigma_g u_g + (1-\varphi)\Sigma_l u_l) - K_d\varphi(1-\varphi)|u_g - u_l|^2 + p_g(u_g - u_l)\nabla\varphi.
\end{aligned} \tag{B.11}$$

The last term comes from the remaining terms

$$\begin{aligned}
&-\frac{2\sigma}{R}\partial_t\varphi + p_g\left(\partial_t\varphi + u_g\cdot\nabla\varphi\right) + p_l\left(\partial_t(1-\varphi) + u_l\cdot\nabla(1-\varphi)\right) - \varphi(1-\varphi)\frac{3}{4\eta_l}\left(p_g - p_l - \frac{2\sigma}{R}\right)^2 \\
&= -\frac{2\sigma}{R}\partial_t\varphi + p_g\left(\partial_t\varphi + u_g\cdot\nabla\varphi\right) - p_l\left(\partial_t\varphi + u_l\cdot\nabla\varphi\right) - \left(p_g - p_l - \frac{2\sigma}{R}\right)\left(\partial_t\varphi + u_l\cdot\nabla\varphi\right) \\
&= p_g(u_g - u_l)\nabla\varphi,
\end{aligned}$$

where we used (2.52f).

Our proposed temperature equations are thus compatible with thermodynamics. They keep the structure of classical equations and they provide a dissipative energy balance for the proposed system. We also refer the reader to Section 6 where we conduct a comparison with the thermodynamics-derived model proposed in [Romenski et al., 2010, La Spina and de' Michieli Vitturi, 2012].

C From compressible-compressible to incompressible-compressible at low Mach

Connecting seamlessly compressible-compressible formulations to incompressible-compressible formulations is not trivial. [Varsakelis and Papalexandris, 2011] generalize the concept of low-Mach-number approximation to multiphase flows and apply it to a two-phase flow model by performing a joint, low-Mach-number limit for the two phases. Here we show that it is possible to obtain our incompressible-compressible system representing liquid–gas flows with respective indices g and l from a compressible-compressible system governing two gas phases with respective indices $(g,1)$ and $(g,2)$ by performing a low-Mach-number limit $\mathrm{Ma}\to 0$ with respect to the phase $(g,2)$.

For simplicity, we replace the Mach number Ma by a parameter ε. We also assume that all quantities are dimensionless. To perform the asymptotic calculation, we choose a perfect gas law for the phase $(g,1)$,

$$p^\varepsilon_{g,1} = r_0\,\rho^\varepsilon_{g,1}T^\varepsilon_{g,1},$$

and use a simplified stiff gas pressure state law for the other phase $(g,2)$:

$$\rho^\varepsilon_{g,2} = \widetilde{\rho} + \varepsilon\Big(p^\varepsilon_{g,2} - \alpha T^\varepsilon_{g,2}\Big)$$

where $\alpha > 0$ is a fixed constant, ε is the Mach number associated to the gas $(g,2)$, and $\widetilde{\rho}$ is a given positive constant that is chosen to be equal to our constant fluid density ρ_l. The asymptotic is obtained by looking at solutions that are expanded with respect to ε as follows:

$$\begin{aligned}
\rho_{g,i}^{\varepsilon} &= \rho_{g,i}^{0} + \varepsilon\rho_{g,i}^{1} + \varepsilon^2\rho_{g,i}^{2} + \cdots \\
T_{g,i}^{\varepsilon} &= T_{g,i}^{0} + \varepsilon T_{g,i}^{1} + \varepsilon^2 T_{g,i}^{2} + \cdots \\
\varphi_{g,i}^{\varepsilon} &= \varphi_{g,i}^{0} + \varepsilon\varphi_{g,i}^{1} + \varepsilon^2\varphi_{g,i}^{2} + \cdots \\
u_{g,i}^{\varepsilon} &= u_{g,i}^{0} + \varepsilon u_{g,i}^{1} + \varepsilon^2 u_{g,i}^{2} + \cdots \\
C_{l}^{\varepsilon} &= C_{l}^{0} + \varepsilon C_{l}^{1} + \varepsilon^2 C_{l}^{2} + \cdots
\end{aligned}$$

where ε is assumed to be sufficiently small. The equations involving the phase $(g,2)$ are the most relevant ones because the $(g,2)$ pressure law depends critically on ε. These equations are

$$\begin{aligned}
&\partial_t(\varphi_{g,2}^{\varepsilon}\rho_{g,2}^{\varepsilon}) + \operatorname{div}(\varphi_{g,2}^{\varepsilon}\rho_{g,2}^{\varepsilon}u_{g,2}^{\varepsilon}) = -R_{H_2O}^{\varepsilon}, \\
&\partial_t\varphi_{g,2}^{\varepsilon} + u_{g,2}^{\varepsilon}\cdot\nabla\varphi_{g,2}^{\varepsilon} = \varphi_{g,2}^{\varepsilon}(1-\varphi_{g,2}^{\varepsilon})\frac{3}{4\eta_l}(p_{g,1}^{\varepsilon} - p_{g,2}^{\varepsilon}), \\
&\partial_t(\varphi_{g,2}^{\varepsilon}\rho_{g,2}^{\varepsilon}u_{g,2}^{\varepsilon}) + \operatorname{div}(\varphi_{g,2}^{\varepsilon}\rho_{g,2}^{\varepsilon}u_{g,2}^{\varepsilon}\otimes u_{g,2}^{\varepsilon}) + \nabla\Big(\varphi_{g,2}^{\varepsilon}p_{g,2}^{\varepsilon}\Big) - \operatorname{div}(\varphi_{g,2}^{\varepsilon}\mathcal{D}_{g,2}^{\varepsilon}) - p_{g,1}^{\varepsilon}\nabla\varphi_{g,2}^{\varepsilon} \\
&\qquad - K_d\varphi_{g,2}^{\varepsilon}(1-\varphi_{g,2}^{\varepsilon})(u_{g,1}^{\varepsilon} - u_{g,2}^{\varepsilon}) - \varphi_{g,2}^{\varepsilon}\rho_{g,2}^{\varepsilon}g + \frac{u_{g,1}^{\varepsilon}+u_{g,2}^{\varepsilon}}{2}R_{H_2O}^{\varepsilon} = 0, \\
&\partial_t(\varphi_{g,2}^{\varepsilon}c_{p,2}\rho_{g,2}^{\varepsilon}T_{g,2}^{\varepsilon}) + \operatorname{div}(\varphi_{g,2}^{\varepsilon}c_{p,2}\rho_{g,2}^{\varepsilon}T_{g,2}^{\varepsilon}u_{g,2}^{\varepsilon}) - \varepsilon\frac{T_{g,2}^{\varepsilon}}{\rho_{g,2}^{\varepsilon}}\ \varphi_{g,2}^{\varepsilon}(\partial_t p_{g,2}^{\varepsilon} + u_{g,2}^{\varepsilon}\cdot\nabla p_{g,2}^{\varepsilon}) \\
&\qquad - \frac{p_{g,2}^{\varepsilon}}{\rho_{g,2}^{\varepsilon}}R_{H_2O}^{\varepsilon} + \gamma(T_{g,1}^{\varepsilon} - T_{g,2}^{\varepsilon}) \\
&\qquad - \operatorname{div}(\varphi_{g,2}^{\varepsilon}k_2\nabla T_{g,2}^{\varepsilon}) - \varphi_2\Big(2\eta_2 D(u_{g,2}^{\varepsilon}) : \nabla u_{g,2}^{\varepsilon} + \lambda_2(\operatorname{div}u_{g,2}^{\varepsilon})^2\Big) = 0.
\end{aligned} \tag{C.1a}$$

Note that the stiff gas pressure state law and the expansions of the unknowns we consider imply the following pressure expansion with respect to the Mach number:

$$p_{g,2}^{\varepsilon} = \frac{1}{\varepsilon}p_{g,2}^{-1} + p_{g,2}^{0} + \varepsilon p_{g,2}^{1} + \cdots = \frac{\rho_{g,2}^{0} - \widetilde{\rho}}{\varepsilon} + (\rho_{g,2}^{1} + \alpha T_{g,2}^{0}) + \varepsilon^2(\rho_{g,2}^{2} + \alpha T_{g,2}^{1}) + \cdots.$$

In the gas fraction equation on $\varphi_{g,2}^{\varepsilon}$, the main contribution at order ε^{-1} only involves the pressure $p_{g,2}^{-1}$. This contribution is

$$\varphi_{g,2}^{0}(1-\varphi_{g,2}^{0})p_{g,2}^{-1} = 0,$$

which can be written as

$$\varphi_{g,2}^{0}(1-\varphi_{g,2}^{0})(\rho_{2,g}^{0} - \widetilde{\rho}) = 0. \tag{C.2}$$

Using the momentum equation related to $u_{g,2}^{\varepsilon}$, we get at order ε^{-1} that

$$\nabla(\varphi_{g,2}^{0}(\rho_{2,g}^{0} - \widetilde{\rho})) = 0$$

and therefore

$$\varphi_{g,2}^{0}(\rho_{g,2}^{0} - \widetilde{\rho}) = C(t).$$

Coupling this result with the constraint (C.2) yields

$$(1-\varphi_{g,2}^{0})C(t) = 0,$$

which implies that $C(t) = 0$ or $\varphi^0_{g,2} = 1$ identically because $C(t)$ does not depend on space. The second choice is not possible because it implies a single phase system. Thus,

$$\varphi^0_{g,2}(\rho^0_{2,g} - \widetilde{\rho}) = 0.$$

Note that if $\varphi^0_{g,2} = 0$ at some point in space, it means that phase $(g, 2)$ is zero at that point and we therefore can choose $\rho^0_{g,2} = \widetilde{\rho}$. Therefore the conclusion is that $\rho^0_{g,2} = \widetilde{\rho}$ everywhere. Note that this implies that $p^{-1}_{g,2} = 0$, which is be used later on when looking at the temperature equation on $T^0_{g,2}$.

We address now the mass and gas fraction equations at order ε^0. Using that $\rho^0_{g,2} = \widetilde{\rho} = \text{constant}$, we obtain that

$$\partial_t(\varphi^0_{g,2}\rho^0_{g,2}) + \text{div}(\varphi^0_{g,2}\rho^0_{g,2}u^0_{g,2}) = -R^0_{H_2O}$$

and therefore

$$\partial_t\varphi^0_{g,2} + \text{div}(\varphi^0_{g,2}u^0_{g,2}) = -\frac{1}{\widetilde{\rho}}R^0_{H_2O}. \tag{C.3}$$

The volume fraction equation yields

$$\partial_t\varphi^0_{g,2} + u^0_{g,2}\cdot\nabla\varphi^0_{g,2} = \varphi^0_{g,2}(1-\varphi^0_{g,2})\frac{3}{4\eta_l}(p^0_{g,1} - p^0_{g,2}) \tag{C.4}$$

with $p^0_{g,2} = \rho^1_{g,2} + \alpha T^0_{g,2}$. Combining (C.3) and (C.4) yields:

$$\varphi^0_{g,2}\text{div}u^0_{g,2} = -\frac{1}{\widetilde{\rho}}R^0_{H_2O} + \varphi^0_{g,2}(1-\varphi^0_{g,2})\frac{3}{4\eta_l}(p^0_{g,1} - p^0_{g,2})$$

where $p^0_{g,1} = r_0\rho^0_{g,1}T^0_{g,1}$. Note that $p^0_{g,2}$ corresponds to p_l in our incompressible-compressible system. Importantly, $p^0_{g,2} = \rho^1_{g,2} + \alpha T^0_{g,2}$, where $\rho^1_{g,2}$ is an unknown. The salient unknown to consider is therefore $p^0_{g,2}$ because it appears everywhere. In the temperature equation at the main order, $(\text{C.1a})_4$, the quantity with ε in front yields the quantity

$$\frac{T^0_{g,2}}{\rho^0_{g,2}}\ \varphi^0_{g,2}\Big(\partial_t p^{-1}_{g,2} + u_{g,2}\cdot\nabla p^{-1}_{g,2}\Big)$$

which is zero because $p^{-1}_{g,2} = 0$. Also, the quantity related to R_{H_2O} in $(\text{C.1a})_4$ simplify to

$$\frac{p^0_{g,2}}{\rho^0_{g,2}}R^0_{H_2O}.$$

Therefore we obtain that

$$\begin{aligned}\partial_t(\varphi^0_{g,2}c_{p,2}\rho^0_{g,2}T^0_{g,2}) &+ \text{div}(\varphi^0_{g,2}c_{p,2}\rho^0_{g,2}T^0_{g,2}u^0_{g,2})\\ &- \frac{p^0_{g,2}}{\rho^0_{g,2}}R^0_{H_2O} + \gamma(T^0_{g,1} - T^0_{g,2})\\ &- \text{div}(\varphi^0_{g,2}k_2\nabla T^0_{g,2}) - \varphi_2\Big(2\eta_2 D(u^0_{g,2}) : \nabla u^0_{g,2} + \lambda_2(\text{div}u^0_{g,2})^2\Big) = 0\end{aligned} \tag{C.5a}$$

which is the equation on T_l that we have considered. The main order of the other equations (e.g., that of C_l and those related to the phase $(g, 1)$) easily yield equations that are identical to the ones we use in our incompressible-compressible system.

D Detailed calculations of the drift flux system

The drift flux system (4.15) is obtained from the original system (2.74) by performing the change of variables detailed in Section 4 (see in particular the relationships (4.11)). Obtaining the equations for mass, concentration and volume fraction, (4.15a)-(4.15d) is straightforward. Here we detail the calculations for the remaining equations (4.15e)-(4.15h), which are less obvious.

Equation (4.15e) is the sum of (2.74d) and (2.74e). From the definition of the drift flux variables we can easily achieve the conversion into drift-flux variables. We only detail the conversion of the divergence term, where we use relationships (4.11) to write

$$\begin{aligned}\mathrm{div}((1-\varphi)\rho_l u_l\otimes u_l+\varphi\rho_g u_g\otimes u_g) &= \mathrm{div}((1-Y)\rho(u-Yw)\otimes(u-Yw))\\ &\quad+\mathrm{div}(Y\rho(u+(1-Y)w)\otimes(u+(1-Y)w))\\ &= \mathrm{div}(\rho(1-Y)(u\otimes u)-\rho Y(1-Y)(u\otimes w)-\rho Y(1-Y)(w\otimes u)\\ &\qquad+\rho Y^2(1-Y)(w\otimes w))+\mathrm{div}(\rho Y(u\otimes u)+\rho Y(1-Y)(u\otimes w)\\ &\qquad+\rho Y(1-Y)(w\otimes u)+\rho Y(1-Y)^2(w\otimes w))\\ &= \mathrm{div}(\rho u\otimes u+\rho Y(1-Y)w\otimes w)\end{aligned}$$

Equation (4.15f) is the momentum equation for the gas phase (2.74e). We also only detail the divergence term, the other terms being easily converted:

$$\begin{aligned}\mathrm{div}(\rho_g\varphi u_g\otimes u_g) &= \mathrm{div}(\rho Y(u+(1-Y)w)\otimes(u+(1-Y)w))\\ &= \mathrm{div}(\rho Y(u\otimes u)+\rho Y(1-Y)(u\otimes w)+\rho Y(1-Y)(w\otimes u)+\rho Y(1-Y)^2(w\otimes w))\end{aligned}$$

The coefficient of R^{H_2O} is calculated using the definitions of the velocities,

$$\frac{1}{2}(u_g+u_l)=\frac{1}{2}(u+(1-Y)w+u-Yw)=u+(\frac{1}{2}-Y)w.$$

Equation (4.15g) is the sum of (2.76) and (2.74k). In the primary variables, that sum reads

$$\begin{aligned}&\partial_t((c_{pg}-c_0)\varphi\rho_g T_g+(1-\varphi)c_{p_l}\rho_l T_l)+\mathrm{div}((c_{pg}-c_0)\varphi\rho_g T_g u_g+(1-\varphi)c_{p_l}\rho_l T_l u_l)\\ &\quad-\mathrm{div}(\varphi\kappa_g\nabla T_g+(1-\varphi)\kappa_l\nabla T_l)+p_g\mathrm{div}\Big(\varphi u_g+(1-\varphi)u_l\Big)\\ &\quad+\frac{(p_g-p_l)}{\rho_l}R^{H_2O}-\varphi\left(2\eta_g D(u_g):\nabla u_g+\lambda_g(\mathrm{div}u_g)^2\right)-(1-\varphi)\left(2\eta_l D(u_l):\nabla u_l+\eta_l(\mathrm{div}u_l)^2\right)=0.\end{aligned}$$

Using the definition of c_{pm} in (4.10) and T in (4.8), the time derivative term reads

$$\partial_t((c_{pg}-c_0)\varphi\rho_g T_g+(1-\varphi)c_{p_l}\rho_l T_l)=\partial_t(c_{pm}Y_T T_g+c_{pm}(1-Y_T)T_l)=\partial_t(c_{pm}T)$$

We develop the divergence term where we replace all variables to get

$$\begin{aligned}\mathrm{div}((c_{pg}-c_0)\varphi\rho_g T_g u_g+(1-\varphi)c_{p_l}\rho_l T_l u_l) &= \mathrm{div}(c_{pm}Y_T(T+(1-Y_T)\delta T)(u+(1-Y)w)\\ &\qquad+c_{pm}(1-Y_T)(T-Y_T\delta T)(u-Yw))\\ &= \mathrm{div}(c_{pm}u(Y_T(T+(1-Y_T)\delta T)+(1-Y_T)(T-Y_T\delta T))\\ &\quad+c_{pm}w(Y_T(T+(1-Y_T)\delta T)(1-Y)-(1-Y_T)(T-Y_T\delta T)Y))\\ &= \mathrm{div}(c_{pm}uT+c_{pm}wT(Y_T-Y)\\ &\qquad+c_{pm}w\delta T(1-Y_T)(Y_T(1-Y)+YY_T))\\ &= \mathrm{div}(c_{pm}uT+c_{pm}w((Y_T-Y)T+Y_T(1-Y_T)\delta T))\end{aligned}$$

The remaining terms in the temperature equations are directly obtained by using (4.11) to convert the original variables.

Equation (4.15h) is calculated as $(1-Y_T)\times$ (4.13) $-\,Y_T\times$ (4.14). This combination yields:

$$\begin{aligned}
&(1-Y_T)c_{vg}\rho Y(\partial_t T_g + (u+(1-Y)w)\cdot\nabla T_g) - Y_T c_{p_l}\rho(1-Y)(\partial_t T_l + (u-Yw)\cdot\nabla T_l)\\
&\quad + (1-Y_T)(p+(1-\varphi)q)\mathrm{div}\Big(u+(\varphi-Y)w\Big)\\
&\quad = \gamma(T_g-T_l) + (1-Y_T)\mathrm{div}(\varphi\kappa_g\nabla T_g) - Y_T\mathrm{div}((1-\varphi)\kappa_l\nabla T_l)\\
&\qquad + (1-Y_T)\eta_g\varphi\Big(2D(u_g):\nabla u_g + 3(\mathrm{div}u_g)^2\Big) - Y_T\eta_l(1-\varphi)\left(2D(u_l):\nabla u_l + \frac{1}{3}(\mathrm{div}u_l)^2\right)\\
&\qquad - ((1-Y_T)(p+(1-\varphi)q) + Y_T(p-\varphi q))\frac{R^{H_2O}}{\rho_l} - ((1-Y_T)c_{vg}T_g + Y_T c_{pl}T_l)R^{H_2O}.
\end{aligned}$$

The time derivative terms are

$$(1-Y_T)c_{vg}\rho Y\partial_t T_g - Y_T c_{p_l}\rho(1-Y)\partial_t T_l = (1-Y_T)c_{pm}Y_T\partial_t T_g - Y_T c_{pm}(1-Y_T)\partial_t T_l = c_{pm}Y_T(1-Y_T)\partial_t(\delta T)$$

The velocity terms are

$$\begin{aligned}
&(1-Y_T)c_{vg}\rho Y(u+(1-Y)w)\cdot\nabla T_g - Y_T c_{p_l}\rho(1-Y)(u-Yw)\cdot\nabla T_l\\
&\quad = c_{pm}Y_T(1-Y_T)((u-Yw)\cdot\nabla(T_g-T_l) + w\cdot\nabla T_g)\\
&\quad = c_{pm}Y_T(1-Y_T)((u-Yw)\cdot\nabla(\delta T) + w\cdot\nabla(T+(1-Y_T)\delta T))
\end{aligned}$$

Finally, we develop the term in R^{H_2O},

$$\begin{aligned}
&-((1-Y_T)(p+(1-\varphi)q) + Y_T(p-\varphi q))\tfrac{R^{H_2O}}{\rho_l} - ((1-Y_T)c_{vg}T_g + Y_T c_{pl}T_l)R^{H_2O}\\
&\quad = -(p+(1-\varphi)q - Y_T q))\tfrac{R^{H_2O}}{\rho_l} - \left((1-Y_T)\tfrac{c_{pm}Y_T}{\rho Y}(T+(1-Y_T)\delta T) + Y_T\tfrac{c_{pm}(1-Y_T)}{\rho(1-Y)}(T-Y_T\delta T)\right)R^{H_2O}\\
&\quad = -(p+(1-\varphi-Y_T)q))\tfrac{R^{H_2O}}{\rho_l} - \tfrac{c_{pm}Y_T(1-Y_T)}{\rho Y(1-Y)}\left((1-Y)(T+(1-Y_T)\delta T) + Y(T-Y_T\delta T)\right)R^{H_2O}\\
&\quad = -(p+(1-\varphi-Y_T)q))\tfrac{R^{H_2O}}{\rho_l} - \tfrac{c_{pm}Y_T(1-Y_T)}{\rho Y(1-Y)}\left(T+(1-Y-Y_T)\delta T\right)R^{H_2O}
\end{aligned}$$

Like before, the rest of the terms are obtained by using (4.11).

E Dimensionless drift flux system

Mass equations

The mass equations for ρ, C_l, and φ, (4.26a), (4.26b), (4.26c), and (4.15d) are:

$$\mathrm{St}\,\partial_{\tilde t}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde Y\right)(1-\varphi)\tilde\rho_l\right) + \tilde{\mathrm{div}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde Y\right)(1-\varphi)\tilde\rho_l\tilde u\right) = 0, \tag{E.1}$$

$$\mathrm{St}\,\partial_{\tilde t}\left((1-\varphi)\tilde\rho_l\tilde Y\right) + \tilde{\mathrm{div}}\left((1-\varphi)\tilde\rho_l\tilde Y\left(\tilde u + \frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde Y}\tilde w\right)\right) = \frac{\rho_{l0}}{\rho_{g0}}\frac{L_0R_0}{\rho_{l0}u_0}\tilde R^{H_2O}, \tag{E.2}$$

$$\mathrm{St}\,\partial_{\tilde t}\left((1-\varphi)\tilde\rho_l(1-C_{l0}\tilde C_l)\right) + \tilde{\mathrm{div}}\left((1-\varphi)\tilde\rho_l(1-C_{l0}\tilde C_l)\left(\tilde u - \frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde Y}\tilde Y\tilde w\right)\right) = 0, \tag{E.3}$$

$$\mathrm{St}\,\partial_{\tilde t}\varphi + \left(\tilde u - \frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde Y}\tilde Y\tilde w\right)\cdot\tilde\nabla\varphi = \frac{\mathrm{Re}_l}{\mathrm{Ma}^2}\frac{q_0}{p_0}\,\varphi(1-\varphi)\frac{3}{4\tilde\eta_l}\tilde q \tag{E.4}$$

The exchange coefficient on the right-hand side of the second equation, $\frac{L_0R_0}{\rho_{l0}u_0}$, also appears in other equations of the system. Using (4.28a), it reads

$$\frac{L_0R_0}{\rho_{l0}u_0} = \frac{3^{1/3}(4\pi N_0)^{2/3}DC_{l0}L_0}{u_0}.$$

Mixture velocity equation

Equation (4.26e) in dimensionless form is:

$$
\begin{aligned}
&\mathrm{St}\,\partial_{\tilde{t}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{u}\right)+\tilde{\mathrm{div}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{u}\otimes\tilde{u}\right)+\tilde{\mathrm{div}}\left(\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0^2}{u_0^2}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}(1-\varphi)\tilde{\rho}_l\tilde{Y}\tilde{w}\otimes\tilde{w}\right)\\
&\quad+\frac{1}{\mathrm{Ma}^2}\tilde{\nabla}\tilde{p}-\frac{1}{\mathrm{Re}_l}\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\eta}_l\left(\frac{1}{3}\tilde{\mathrm{div}}\left(\tilde{u}-\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{\tilde{Y}\tilde{w}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\right)\mathrm{Id}+2\tilde{D}_u\Big(\tilde{u}-\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{\tilde{Y}\tilde{w}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\Big)\right)\right)\\
&\quad-\frac{\rho_{g0}}{\rho_{l0}}\frac{1}{\mathrm{Re}_g}\tilde{\mathrm{div}}\left(\varphi\tilde{\eta}_g\left(3\tilde{\mathrm{div}}\left(\tilde{u}+\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{w}\right)\mathrm{Id}+2\tilde{D}_u\left(\tilde{u}+\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{w}\right)\right)\right)\\
&\quad=\frac{1}{\mathrm{Fr}^2}\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{g}
\end{aligned}
\tag{E.5}
$$

Velocity equation on Yw

Equation (4.26f) in dimensionless form is:

$$
\begin{aligned}
&\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\mathrm{St}\,\partial_{\tilde{t}}\left(\frac{\tilde{Y}\tilde{w}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\right)+\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0^2}{u_0^2}\frac{1}{\tilde{\rho}_l(1-\varphi)}\frac{\tilde{Y}\tilde{w}}{\left(1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}\right)^2}\ \tilde{\mathrm{div}}(\tilde{\rho}_l(1-\varphi)\tilde{w})\\
&+\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0^2}{u_0^2}\left(\tilde{w}-2\frac{\rho_{g0}}{\rho_{l0}}\frac{\tilde{Y}\tilde{w}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\right)\cdot\tilde{\nabla}\left(\frac{\tilde{Y}\tilde{w}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\right)+\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\tilde{u}\cdot\tilde{\nabla}\left(\frac{\tilde{Y}\tilde{w}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\right)+\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{\tilde{Y}\tilde{w}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\cdot\tilde{\nabla}\tilde{u}\\
&=\frac{1}{\tilde{\rho}_l(1-\varphi)}\left(\frac{\rho_{g0}}{\rho_{l0}}\frac{1}{\mathrm{Re}_g}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{\mathrm{div}}\left(\varphi\tilde{\eta}_g\left(3\tilde{\mathrm{div}}\left(\tilde{u}+\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{w}\right)\mathrm{Id}+2\tilde{D}_u\left(\tilde{u}+\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{w}\right)\right)\right)\right)\\
&\quad-\frac{1}{\tilde{\rho}_l(1-\varphi)}\left(\frac{\rho_{g0}}{\rho_{l0}}\frac{1}{\mathrm{Re}_l}\frac{\tilde{Y}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\eta}_l\left(\frac{1}{3}\tilde{\mathrm{div}}\left(\tilde{u}-\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{\tilde{Y}\tilde{w}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\right)\mathrm{Id}+2\tilde{D}_u\left(\tilde{u}-\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{\tilde{Y}\tilde{w}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\right)\right)\right)\right)\\
&\quad+\frac{1}{\tilde{\rho}_l(1-\varphi)}\frac{1}{\mathrm{Ma}^2}\left(\left(\frac{\rho_{g0}}{\rho_{l0}}\frac{\tilde{Y}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}-\varphi\right)\tilde{\nabla}\tilde{p}-\frac{q_0}{p_0}\varphi\tilde{\nabla}((1-\varphi)\tilde{q})\right)-\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{1}{\mathrm{Re}_g}\frac{K_{d0}L_0^2}{\eta_{g0}}\frac{\tilde{K}_d(1-\varphi)}{\tilde{\rho}_g}\tilde{Y}\tilde{w}\\
&\quad+\frac{1}{2}\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\frac{1}{\tilde{\rho}_l(1-\varphi)}\left(\mathrm{St}\,\partial_{\tilde{t}}\left((1-\varphi)\tilde{\rho}_l\tilde{Y}\right)+\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\rho}_l\tilde{Y}\left(\tilde{u}+\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{w}\right)\right)\right)\tilde{w}
\end{aligned}
\tag{E.6}
$$

Using (4.28c), the drag term coefficient reads:

$$\frac{K_{d0}L_0^2}{\eta_{g0}}=\frac{L_0^2}{k_0}.$$

Temperature equations

We write equations (4.26g) and (4.26h) in dimensionless form by using again $\rho_g\varphi=\rho Y$ to write $(\varphi-Y)w=-\left(1-\frac{\rho_l}{\rho_g}\right)(1-\varphi)Yw$. Equation (4.26g) becomes:

$$
\begin{aligned}
&\mathrm{St}\,\partial_{\tilde{t}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}C_p\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{T}\right)+\tilde{\mathrm{div}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}C_p\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\tilde{T}\tilde{u}\right)\\
&\quad+\frac{\rho_{g0}}{\rho_{l0}}\frac{w_0}{u_0}\tilde{\mathrm{div}}\left(\left(1+\frac{\rho_{g0}}{\rho_{l0}}C_p\tilde{Y}\right)(1-\varphi)\tilde{\rho}_l\left(\left(\frac{C_p}{1+\frac{\rho_{g0}}{\rho_{l0}}C_p}-\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\right)\tilde{T}+\frac{\delta_{T_0}}{T_0}\frac{C_p}{1+\frac{\rho_{g0}}{\rho_{l0}}C_p\tilde{Y}}\delta\tilde{T}\right)\tilde{Y}\tilde{w}\right)\\
&\quad+\frac{\rho_{g0}}{\rho_{l0}}\frac{p_0}{\rho_{g0}T_0c_{pl}}\left(\tilde{p}+\frac{q_0}{p_0}(1-\varphi)\tilde{q}\right)\tilde{\mathrm{div}}\left(\tilde{u}-\frac{w_0}{u_0}\left(\frac{\rho_{g0}}{\rho_{l0}}-\frac{\tilde{\rho}_l}{\tilde{\rho}_g}\right)\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}(1-\varphi)\tilde{Y}\tilde{w}\right)\\
&\quad=-\left(\frac{\rho_{g0}}{\rho_{l0}}\right)^2\frac{q_0}{p_0}\frac{p_0}{\rho_{g0}T_0c_{pl}}\left(\mathrm{St}\,\partial_{\tilde{t}}\left((1-\varphi)\tilde{\rho}_l\tilde{Y}\right)+\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\rho}_l\tilde{Y}\left(\tilde{u}+\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{w}\right)\right)\right)\frac{\tilde{q}}{\tilde{\rho}_l}\\
&\qquad+\frac{\rho_{g0}}{\rho_{l0}}C_p\frac{1}{\mathrm{Pe}_g}\tilde{\mathrm{div}}\left(\varphi\tilde{\nabla}\left(\tilde{T}+\frac{\delta_{T_0}}{T_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}C_p\tilde{Y}}\delta\tilde{T}\right)\right)+\frac{1}{\mathrm{Pe}_l}\tilde{\mathrm{div}}\left((1-\varphi)\tilde{\nabla}\left(\tilde{T}-\frac{\rho_{g0}}{\rho_{l0}}\frac{\delta_{T_0}}{T_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}C_p\tilde{Y}}\tilde{Y}\delta\tilde{T}\right)\right)\\
&\qquad+2\frac{\delta_{T_0}}{T_0}\frac{\rho_{g0}}{\rho_{l0}}C_p\frac{\mathrm{Br}_g}{\mathrm{Pe}_g}\varphi\tilde{\eta}_g\;\tilde{D}_u\left(\tilde{u}+\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{w}\right):\tilde{\nabla}\left(\tilde{u}+\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{w}\right)\\
&\qquad+3\frac{\delta_{T_0}}{T_0}\frac{\rho_{g0}}{\rho_{l0}}C_p\frac{\mathrm{Br}_g}{\mathrm{Pe}_g}\varphi\tilde{\eta}_g\;\left(\tilde{\mathrm{div}}\left(\tilde{u}+\frac{w_0}{u_0}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{w}\right)\right)^2\\
&\qquad+2\frac{\delta_{T_0}}{T_0}\frac{\mathrm{Br}_l}{\mathrm{Pe}_l}(1-\varphi)\tilde{\eta}_l\;D_u\left(\tilde{u}-\frac{w_0}{u_0}\frac{\rho_{g0}}{\rho_{l0}}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{Y}\tilde{w}\right):\tilde{\nabla}\left(\tilde{u}-\frac{w_0}{u_0}\frac{\rho_{g0}}{\rho_{l0}}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{Y}\tilde{w}\right)\\
&\qquad+\frac{1}{3}\frac{\delta_{T_0}}{T_0}\frac{\mathrm{Br}_l}{\mathrm{Pe}_l}(1-\varphi)\tilde{\eta}_l\;\left(\tilde{\mathrm{div}}\left(\tilde{u}-\frac{w_0}{u_0}\frac{\rho_{g0}}{\rho_{l0}}\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}}\tilde{Y}\tilde{w}\right)\right)^2
\end{aligned}
\tag{E.7}
$$

The relative temperature equation (4.26h) becomes:

$$
\begin{aligned}
&\frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y}(1-\varphi)\tilde{\rho}_l \left(\mathrm{St}\, \partial_{\tilde{t}}(\delta\tilde{T}) + \left(\tilde{u} - \frac{\rho_{g0}}{\rho_{l0}} \frac{w_0}{u_0} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} \tilde{Y}\tilde{w} \right) \cdot \tilde{\nabla}(\delta\tilde{T}) + \frac{w_0}{u_0}\tilde{w} \cdot \tilde{\nabla} \left(\frac{T_0}{\delta_{T_0}}\tilde{T} + \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y}} \delta\tilde{T} \right) \right) \\
&\quad + \frac{\rho_{g0}}{\rho_{l0}} \frac{p_0}{\rho_{g0}\delta_{T_0} c_{pl}} \left(\tilde{p} + \frac{q_0}{p_0}(1-\varphi)\tilde{q} \right) \tilde{\mathrm{div}} \left(\tilde{u} + \frac{w_0}{u_0} \Big(\varphi\tilde{w} - \frac{\rho_{g0}}{\rho_{l0}} \frac{\tilde{Y}\tilde{w}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} \Big) \right) \\
&\quad = \frac{\gamma_0 L_0}{\rho_{l0} u_0 c_{pl}} \left(1 + \frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y} \right) \tilde{\gamma} \delta\tilde{T} \\
&\qquad - \frac{\rho_{g0}}{\rho_{l0}} \frac{T_0}{\delta_{T_0}} \frac{p_0}{\rho_{l0} T_0 c_{pl}} \left(1 + \frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y} \right) \frac{1}{\tilde{\rho}_l} \left(\tilde{p} + \frac{q_0}{p_0} \left(\frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y}} - \varphi \right) \tilde{q} \right) \left(\mathrm{St}\, \partial_{\tilde{t}} \left((1-\varphi)\tilde{\rho}_l \tilde{Y} \right) \right. \\
&\qquad\qquad \left. + \tilde{\mathrm{div}} \left((1-\varphi)\tilde{\rho}_l \tilde{Y} \left(\tilde{u} + \frac{w_0}{u_0} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} \tilde{w} \right) \right) \right) \\
&\qquad - \frac{\rho_{g0}}{\rho_{l0}} \frac{C_p}{\rho_{g0}} \left(1 + \frac{\rho_{g0}}{\rho_{l0}} \tilde{Y} \right) \left(\frac{T_0}{\delta_{T_0}} \tilde{T} + \Big(1 - \frac{\rho_{g0}}{\rho_{l0}} \Big(\frac{\tilde{Y}}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} - \frac{C_p \tilde{Y}}{1+\frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y}} \Big) \Big) \delta\tilde{T} \right) \left(\mathrm{St}\, \partial_{\tilde{t}} \left((1-\varphi)\tilde{\rho}_l \tilde{Y} \right) \right. \\
&\qquad\qquad \left. + \tilde{\mathrm{div}} \left((1-\varphi)\tilde{\rho}_l \tilde{Y} \left(\tilde{u} + \frac{w_0}{u_0} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} \tilde{w} \right) \right) \right) \\
&\qquad + \frac{\rho_{g0}}{\rho_{l0}} C_p \frac{T_0}{\delta_{T_0}} \frac{1}{\mathrm{Pe}_g} \tilde{\mathrm{div}} \left(\varphi\tilde{\nabla} \left(\tilde{T} + \frac{\delta_{T_0}}{T_0} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y}} \delta\tilde{T} \right) \right) \\
&\qquad - \frac{\rho_{g0}}{\rho_{l0}} C_p \frac{T_0}{\delta_{T_0}} \frac{1}{\mathrm{Pe}_l} \tilde{Y} \tilde{\mathrm{div}} \left((1-\varphi)\tilde{\nabla} \left(\tilde{T} - \frac{\rho_{g0}}{\rho_{l0}} \frac{\delta_{T_0}}{T_0} \frac{C_p}{1+\frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y}} \tilde{Y}\delta\tilde{T} \right) \right) \\
&\qquad + 2 \frac{\rho_{g0}}{\rho_{l0}} C_p \frac{\mathrm{Br}_g}{\mathrm{Pe}_g} \varphi\tilde{\eta}_g \; \tilde{D}_u \left(\tilde{u} + \frac{w_0}{u_0} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} \tilde{w} \right) : \tilde{\nabla} \left(\tilde{u} + \frac{w_0}{u_0} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} \tilde{w} \right) \\
&\qquad + 3 \frac{\rho_{g0}}{\rho_{l0}} C_p \frac{\mathrm{Br}_g}{\mathrm{Pe}_g} \varphi\tilde{\eta}_g \; \left(\tilde{\mathrm{div}} \left(\tilde{u} + \frac{w_0}{u_0} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} \tilde{w} \right) \right)^2 \\
&\qquad - 2 \frac{\rho_{g0}}{\rho_{l0}} c_{vg} \frac{\mathrm{Br}_l}{\mathrm{Pe}_l} (1-\varphi)\tilde{\eta}_l \; \tilde{D}_u \left(\tilde{u} - \frac{w_0}{u_0} \frac{\rho_{g0}}{\rho_{l0}} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} \tilde{Y}\tilde{w} \right) : \tilde{\nabla} \left(\tilde{u} - \frac{w_0}{u_0} \frac{\rho_{g0}}{\rho_{l0}} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} \tilde{Y}\tilde{w} \right) \\
&\qquad - \frac{1}{3} \frac{\rho_{g0}}{\rho_{l0}} c_{vg} \frac{\mathrm{Br}_l}{\mathrm{Pe}_l} (1-\varphi)\tilde{\eta}_l \; \left(\tilde{\mathrm{div}} \left(\tilde{u} - \frac{w_0}{u_0} \frac{\rho_{g0}}{\rho_{l0}} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}}\tilde{Y}} \tilde{Y}\tilde{w} \right) \right)^2
\end{aligned}
\tag{E.8}
$$

Using (4.28b), the heat transfer coefficient, $\frac{\gamma_0 L_0}{\rho_{l0} u_0 c_{pl}}$ reads:

$$
\frac{\gamma_0 L_0}{\rho_{l0} u_0 c_{pl}} = 24 \left(\frac{4\pi}{3} \right)^{2/3} \frac{\mathrm{Nu}_0}{Pe_l} N_0^{2/3} L_0^2
$$

Pressure

Finally we write the calculated mixture pressure p from (4.27)

$$
C_p(1-\varphi)\tilde{\rho}_l \tilde{Y} \left(\tilde{T} + \frac{\delta_{T_0}}{T_0} \left(1 - \frac{\rho_{g0}}{\rho_{l0}} \frac{1}{1+\frac{\rho_{g0}}{\rho_{l0}} C_p \tilde{Y}} C_p \tilde{Y} \right) \delta\tilde{T} \right) = \frac{c_{vg}}{c_0} \frac{p_0}{\rho_{g0} T_0 c_{pl}} \varphi \left(\tilde{p} + \frac{q_0}{p_0}(1-\varphi)\tilde{q} \right). \tag{E.9}
$$

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References

[Ambroso et al., 2008] Ambroso, A., Chalons, C., Coquel, F., Galié, T., Godlewski, E., Raviart, P. A., and Seguin, N. (2008). The drift-flux asymptotic limit of barotropic two-phase two-pressure models. *Communications in Mathematical Sciences*, 6(2):521–529.

[Arefmanesh and Advani, 1991] Arefmanesh, A. and Advani, S. G. (1991). Diffusion-induced growth of a gas bubble in a viscoelastic fluid. *Rheologica Acta*, 30(3):274–283.

[Arzilli et al., 2019] Arzilli, F., La Spina, G., Burton, M. R., Polacci, M., Le Gall, N., Hartley, M. E., Di Genova, D., Cai, B., Vo, N. T., Bamber, E. C., Nonni, S., Atwood, R., Llewellin, E. W., Brooker, R. A., Mader, H. M., and Lee, P. D. (2019). Magma fragmentation in highly explosive basaltic eruptions induced by rapid crystallization. *Nature Geoscience*, 12(12):1023–1028.

[Bamber et al., 2022] Bamber, E. C., La Spina, G., Arzilli, F., de' Michieli Vitturi, M., Polacci, M., Hartley, M. E., Petrelli, M., Fellowes, J., and Burton, M. (2022). Basaltic Plinian eruptions at Las Sierras-Masaya volcano driven by cool storage of crystal-rich magmas. *Communications Earth & Environment*, 3(1):1–17. Number: 1 Publisher: Nature Publishing Group.

[Baum et al., 1994] Baum, M., Poinsot, T., and Thévenin, D. (1994). Accurate boundary conditions for multicomponent reactive flows. *Journal of Computational Physics*, 116(2):247–261.

[Benzoni-Gavage et al., 2003] Benzoni-Gavage, S., Coulombel, J.-F., and Aubert, S. (2003). Boundary conditions for euler equations. *AIAA Journal*, 41(1):56–63.

[Bird et al., 2007] Bird, R. B., Stewart, W. E., and Lightfoot, E. N. (2007). *Transport Phenomena*. John Wiley & Sons Inc, New York, revised 2nd édition edition.

[Brennen, 1995] Brennen, C. E. (1995). *Cavitation and Bubble Dynamics*. Oxford University Press.

[Brennen, 2005] Brennen, C. E. (2005). *Fundamentals of Multiphase Flow*. Cambridge University Press.

[Bresch et al., 2024] Bresch, D., Narbona-Reina, G., Burgisser, A., and Collombet, M. (2024). Mathematical topics in compressible flows from mono-phase systems to two-phase systems. Submitted.

[Briley and McDonald, 1977] Briley, W. and McDonald, H. (1977). Solution of the multidimensional compressible navier-stokes equations by a generalized implicit method. *Journal of Computational Physics*, 24(4):372–397.

[Burgisser et al., 2024] Burgisser, A., Collombet, M., Narbona-Reina, G., and Bresch, D. (2024). 1d numerical simulations of two-phase magma flows with phase exchanges – part ii – physical modeling for the evolution of magma in a volcanic conduit. Submitted.

[Burgisser and Degruyter, 2015] Burgisser, A. and Degruyter, W. (2015). Magma Ascent and Degassing at Shallow Levels. In *The Encyclopedia of Volcanoes*, pages 225–236. Elsevier.

[Burnham, 1975] Burnham, C. (1975). Water and magmas; a mixing model. *Geochimica et Cosmochimica Acta*, 39(8):1077–1084.

[Burtea et al., 2023] Burtea, C., Crin-Barat, T., and Tan, J. (2023). Pressure-relaxation limit for a one-velocity baer–nunziato model to a kapila model. *Mathematical Models and Methods in Applied Sciences*, 33(04):687–753.

[Carlson, 2011] Carlson, J. R. (2011). Inflow/outflow boundary conditions with application to fun3d. In *NASA/TM-2011-217181*.

[Castro et al., 2012] Castro, J. M., Burgisser, A., Schipper, C. I., and Mancini, S. (2012). Mechanisms of bubble coalescence in silicic magmas. *Bulletin of Volcanology*, 74(10):2339–2352.

[Chernov et al., 2014] Chernov, A. A., Kedrinsky, V. K., and Pil'nik, A. A. (2014). Kinetics of gas bubble nucleation and growth in magmatic melt at its rapid decompression. *Physics of Fluids*, 26(11):1–19.

[Choudhary et al., 2016] Choudhary, A., Roy, C. J., Luke, E. A., and Veluri, S. P. (2016). Code verification of boundary conditions for compressible and incompressible computational fluid dynamics codes. *Computers and Fluids*, 126:153–169.

[Costa et al., 2007] Costa, A., Melnik, O., and Vedeneeva, E. (2007). Thermal effects during magma ascent in conduits. *Journal of Geophysical Research: Solid Earth*, 112(B12).

[Costa et al., 2013] Costa, A., Wadge, G., Stewart, R., and Odbert, H. (2013). Coupled subdaily and multiweek cycles during the lava dome eruption of Soufrière Hills Volcano, Montserrat. *Journal of Geophysical Research: Solid Earth*, 118(5):1895–1903.

[Degruyter et al., 2012] Degruyter, W., Bachmann, O., Burgisser, A., and Manga, M. (2012). The effects of outgassing on the transition between effusive and explosive silicic eruptions. *Earth and Planetary Science Letters*, 349-350:161–170.

[Dutt, 1988] Dutt, P. (1988). Stable boundary conditions and difference schemes for navier-stokes equations. *SIAM Journal on Numerical Analysis*, 25(2):245–267.

[Forestier-Coste et al., 2012] Forestier-Coste, L., Mancini, S., Burgisser, A., and James, F. (2012). Numerical resolution of a mono-disperse model of bubble growth in magmas. *Applied Mathematical Modelling*, 36(12):5936 – 5951.

[Fowler and Robinson, 2018] Fowler, A. and Robinson, M. (2018). Counter-current convection in a volcanic conduit. *Journal of Volcanology and Geothermal Research*, 356:141–162.

[Gaskell and Laughlin, 2017] Gaskell, D. R. and Laughlin, D. E. (2017). *Introduction to the Thermodynamics of Materials*. CRC Press.

[Gavrilyuk, 2020] Gavrilyuk, S. (2020). 'uncertainty' principle in two fluid-mechanics. *ESAIM: ProcS*, 69:47–55.

[Gavrilyuk and Saurel, 2002] Gavrilyuk, S. and Saurel, R. (2002). Mathematical and numerical modeling of two-phase compressible flows with micro-inertia. *Journal of Computational Physics*, 175(1):326–360.

[Gidaspow, 1994] Gidaspow, D. (1994). *Multiphase Flow and Fluidization: Continuum and Kinetic Theory Descriptions*. Academic Press.

[Gonnermann and Manga, 2007] Gonnermann, H. and Manga, M. (2007). The fluid mechanics inside a volcano. *Annual Review of Fluid Mechanics*, 39:321–356.

[Guillemaud, 2007] Guillemaud, V. (2007). *Modélisation et simulation numérique des écoulements diphasiques par une approche bifluide à deux pressions*. PhD thesis, Univ. Provence-Aix-Marseille I.

[Hajimirza et al., 2019] Hajimirza, S., Gonnermann, H. M., Gardner, J. E., and Giachetti, T. (2019). Predicting Homogeneous Bubble Nucleation in Rhyolite. *Journal of Geophysical Research: Solid Earth*, 124(3):2395–2416.

[Hess and Dingwell, 1996] Hess, K.-U. and Dingwell, D. (1996). Viscosities of hydrous leucogranitic melts: A non-Arrhenian model. *American Mineralogist*, 81:1297–1300.

[Ishii, 1975] Ishii, M. (1975). *Thermo-fluid dynamic theory of two-phase flow*. Collection de la Direction des Etudes et recherches d'Electricité de France. Eyrolles, Paris.

[Ishii, 1977] Ishii, M. (1977). One-dimensional drift-flux model and constitutive equations for relative motion between phases in various two-phase flow regimes. Technical report, IAEA, United States.

[Ishii and Hibiki, 2011] Ishii, M. and Hibiki, T. (2011). *Thermo-fluid dynamics of two-phase flow*. Springer Verlag, New York.

[Kim et al., 2004] Kim, S., Alonso, J., Schluter, J., Wu, X., and Pitsch, H. (2004). *Integrated Simulations for Multi-Component Analysis of Gas Turbines:RANS Boundary Conditions*, chapter 3415, pages 1–13. AIAA.

[Kolev, 2007] Kolev, N. I. (2007). *Multiphase Flow Dynamics*. Springer, Berlin, Heidelberg.

[Kostin, Ilya et al., 2003] Kostin, Ilya, Marion, Martine, Texier-Picard, Rozenn, and Volpert, Vitaly A. (2003). Modelling of miscible liquids with the korteweg stress. *ESAIM: M2AN*, 37(5):741–753.

[Kozono and Koyaguchi, 2009] Kozono, T. and Koyaguchi, T. (2009). Effects of relative motion between gas and liquid on 1-dimensional steady flow in silicic volcanic conduits: 1. an analytical method. *Journal of Volcanology and Geothermal Research*, 180(1):21–36.

[Kozono and Koyaguchi, 2010] Kozono, T. and Koyaguchi, T. (2010). A simple formula for calculating porosity of magma in volcanic conduits during dome-forming eruptions. *Earth, Planets and Space*, 62(5):483–488.

[La Spina et al., 2016] La Spina, G., Burton, M., de' Michieli Vitturi, M., and Arzilli, F. (2016). Role of syn-eruptive plagioclase disequilibrium crystallization in basaltic magma ascent dynamics. *Nature Communications*, 7(1):13402.

[La Spina and de' Michieli Vitturi, 2012] La Spina, G. and de' Michieli Vitturi, M. (2012). High-resolution finite volume central schemes for a compressible two-phase model. *SIAM Journal on Scientific Computing*, 34(6):B861–B880.

[La Spina et al., 2017] La Spina, G., de' Michieli Vitturi, M., and Clarke, A. (2017). Transient numerical model of magma ascent dynamics: application to the explosive eruptions at the soufrière hills volcano. *Journal of Volcanology and Geothermal Research*, 336:118–139.

[La Spina et al., 2014] La Spina, G., de' Michieli Vitturi, M., and Romenski, E. (2014). A compressible single-temperature conservative two-phase model with phase transitions. *International Journal for Numerical Methods in Fluids*, 76(5):282–311.

[Laney, 1998] Laney, C. B. (1998). *Computational Gasdynamics*. Cambridge University Press.

[Laurén and Nordström, 2018] Laurén, F. and Nordström, J. (2018). Practical inlet boundary conditions for internal flow calculations. *Computers and Fluids*, 175:159–166.

[Leighton, 1994] Leighton, T. G. (1994). *The Acoustic Bubble*. Academic Press.

[Leighton, 2007] Leighton, T. G. (2007). Derivation of the rayleigh-plesset equation in terms of volume. Technical report, Institute of Sound and Vibration Research.

[Lensky et al., 2004] Lensky, N., Navon, O., and Lyakhovsky, V. (2004). Bubble growth during decompression of magma: experimental and theoretical investigation. *Journal of Volcanology and Geothermal Research*, 129(1):7 – 22. The role of laboratory experiments in volcanology.

[Lensky et al., 2001] Lensky, N. G., Lyakhovsky, V., and Navon, O. (2001). Radial variations of melt viscosity around growing bubbles and gas overpressure in vesiculating magmas. *Earth and Planetary Science Letters*, 186(1):1–6.

[Lyakhovsky et al., 1996] Lyakhovsky, V., Hurwitz, S., and Navon, O. (1996). Bubble growth in rhyolitic melts: experimental and numerical investigation. *Bulletin of Volcanology*, 58(1):19–32.

[Mancini et al., 2016] Mancini, S., Forestier-Coste, L., Burgisser, A., James, F., and Castro, J. (2016). An expansion–coalescence model to track gas bubble populations in magmas. *Journal of Volcanology and Geothermal Research*, 313:44–58.

[Mangan and Cashman, 1996] Mangan, M. T. and Cashman, K. V. (1996). The structure of basaltic scoria and reticulite and inferences for vesiculation, foam formation, and fragmentation in lava fountains. *Journal of Volcanology and Geothermal Research*, 73:1–18.

[Mason et al., 2006] Mason, R., Starostin, A., Melnik, O., and Sparks, R. (2006). From vulcanian explosions to sustained explosive eruptions: The role of diffusive mass transfer in conduit flow dynamics. *Journal of Volcanology and Geothermal Research*, 153 (1-2):148 – 165. Publisher: Elsevier.

[Müller and Müller, 2009] Müller, I. and Müller, W. H. (2009). *Fundamentals of Thermodynamics and Applications*. Springer Berlin, Heidelberg.

[Melnik, 2000] Melnik, O. (2000). Dynamics of two-phase conduit flow of high-viscosity gas-saturated magma: large variations of sustained explosive eruption intensity. *Bulletin of Volcanology*, 62(3):153–170.

[Melnik and Sparks, 1999] Melnik, O. and Sparks, R. (1999). Nonlinear dynamics of lava dome extrusion. *Nature*, 402:37–41.

[Michaut et al., 2009] Michaut, C., Bercovici, D., and Sparks, R. (2009). Ascent and compaction of gas rich magma and the effects of hysteretic permeability. *Earth and Planetary Science Letters*, 282:258–267.

[Michaut et al., 2013] Michaut, C., Ricard, Y., Bercovici, D., and Sparks, R. S. J. (2013). Eruption cyclicity at silicic volcanoes potentially caused by magmatic gas waves. *Nature Geoscience*, 6(10):856–860.

[Nicoud, 1999] Nicoud, F. (1999). Defining wave amplitude in characteristic boundary conditions. *Journal of Computational Physics*, 149(2):418–422.

[Odier et al., 2019] Odier, N., Sanjosé, M., Gicquel, L., Poinsot, T., Moreau, S., and Duchaine, F. (2019). A characteristic inlet boundary condition for compressible, turbulent, multispecies turbomachinery flows. *Computers and Fluids*, 178:41–55.

[Okong'o and Bellan, 2002] Okong'o, N. and Bellan, J. (2002). Consistent boundary conditions for multicomponent real gas mixtures based on characteristic waves. *Journal of Computational Physics*, 176(2):330–344.

[Oliger and Sundstrom, 1978] Oliger, J. and Sundstrom, A. (1978). Theoretical and practical aspects of some initial boundary value problems in fluid dynamics. *SIAM Journal on Applied Mathematics*, 35(3):419–446.

[Plesset and Prosperetti, 1977] Plesset, M. S. and Prosperetti, A. (1977). Bubble dynamics and cavitation. *Annual Review of Fluid Mechanics*, 9(1):145–185.

[Poinsot and Lele, 1992] Poinsot, T. and Lele, S. (1992). Boundary conditions for direct simulations of compressible viscous flows. *Journal of Computational Physics*, 101(1):104–129.

[Proussevitch et al., 1993a] Proussevitch, A., Sahagian, D., and Kutolin, V. (1993a). Stability of foams in silicate melts. *Journal of Volcanology and Geothermal Research*, 59:161–178.

[Proussevitch and Sahagian, 1998] Proussevitch, A. A. and Sahagian, D. L. (1998). Dynamics and energetics of bubble growth in magmas: Analytical formulation and numerical modeling. *Journal of Geophysical Research: Solid Earth*, 103(B8):18223–18251.

[Proussevitch et al., 1993b] Proussevitch, A. A., Sahagian, D. L., and Anderson, A. T. (1993b). Dynamics of diffusive bubble growth in magmas: Isothermal case. *Journal of Geophysical Research: Solid Earth*, 98(B12):22283–22307.

[Romenski et al., 2010] Romenski, E., Drikakis, D., and Toro, E. (2010). Conservative models and numerical methods for compressible two-phase flow. *Journal of Scientific Computing*, 42(1):68.

[Rudy and Strikwerda, 1981] Rudy, D. H. and Strikwerda, J. C. (1981). Boundary conditions for subsonic compressible navier-stokes calculations. *Computers and Fluids*, 9(3):327–338.

[Scriven, 1959] Scriven, L. (1959). On the dynamics of phase growth. *Chemical Engineering Science*, 10(1):1–13.

[Shugrin, 1994] Shugrin, S. M. (1994). Two-velocity hydrodynamics and thermodynamics. *Journal of Applied Mechanics and Technical Physics*, 35(4):522–537.

[Strikwerda, 1977] Strikwerda, J. C. (1977). Initial boundary value problems for incompletely parabolic systems. *Communications on Pure and Applied Mathematics*, 30(6):797–822.

[Su and Huber, 2017] Su, Y. and Huber, C. (2017). The effect of nonlinear decompression history on H2O/CO2 vesiculation in rhyolitic magmas. *Journal of Geophysical Research: Solid Earth*, 122(4):2712–2723.

[Thompson, 1987] Thompson, K. W. (1987). Time dependent boundary conditions for hyperbolic systems. *Journal of Computational Physics*, 68(1):1–24.

[Toramaru, 1995] Toramaru, A. (1995). Numerical study of nucleation and growth of bubbles in viscous magmas. *Journal of Geophysical Research*, 100:1913–1931.

[Toramaru, 2022] Toramaru, A. (2022). *Vesiculation and Crystallization of Magma. Fundamentals of the Volcanic Eruption Process*. Springer Singapore.

[Varsakelis and Papalexandris, 2011] Varsakelis, C. and Papalexandris, M. V. (2011). Low-mach-number asymptotics for two-phase flows of granular materials. *Journal of Fluid Mechanics*, 669:472–497.

[Webb and Dingwell, 1990] Webb, S. and Dingwell, D. (1990). The Onset of Non-Newtonian Rheology of Silicate Melts: a fiber elongation study. *Physics and Chemistry of Minerals*, 17:125–132.