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# Mathematical topics in compressible flows from single-phase systems to two-phase averaged systems

Didier Bresch\*, Gladys Narbona-Reina†, Alain Burgisser‡, Marielle Collombet‡

August 24, 2024

## Abstract

We review the modeling and mathematical properties of compressible viscous flows, ranging from single-phase systems to two-phase systems, with a focus on the occurrence of oscillations and/or concentrations. We explain how establishing the existence of nonlinear weak stability ensures that no such instabilities occur in the solutions because of the system formulation. When oscillation/concentration are inherent to the nature of the physical situation modelled, we explain how the averaging procedure by homogenization helps to understand their effect on the averaged system. This review addresses systems of progressive complexity. We start by focusing on nonlinear weak stability — a crucial property for numerical simulations and well posedness — in single-phase viscous systems. We then show how a two-phase immiscible system may be rewritten as a single-phase system. Conversely, we show then how to derive a two-phase averaged system from a two-phase immiscible system by homogenization. As in many homogenization problems, this is an example where physical oscillation/concentration occur. We then focus on two-phase averaged viscous systems and present results on the nonlinear weak stability necessary for the convergence of numerical schemes. Finally, we review some singular limits frequently developed to obtain drift-flux systems. Additionally, the appendix provides a crash course on basic functional analysis tools for PDE and homogenization (averaging procedures) for readers unfamiliar with them. This review serves as the foundation for two subsequent papers (Part I and Part II in this same volume), which present averaged two-phase models with phase exchange applicable to magma flow during volcanic eruptions. Part I introduces the physical processes occurring in a volcanic conduit and establishes a two-phase transient conduit flow model ensuring: 1) mass and volatile species conservation, 2) disequilibrium degassing considering both viscous relaxation and volatile diffusion, and 3) dissipation of total energy. The relaxation limit of this model is then used to obtain a drift-flux system amenable to simplification. Part II revisits the model introduced in Part I and proposes a 1.5D simplification that addresses issues in its numerical implementation. Model outputs are compared to those of another well-established model under conditions typical of an effusive eruption at an andesitic volcano.

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# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
1.1	General motivations behind PDEs design . . . . .	4
1.2	Obstacles to finding stable solutions . . . . .	7
1.2.1	Examples of oscillations and concentration . . . . .	7
1.2.2	Oscillations-Concentration related to nonlinear stability and homogenization (averaging procedure) . . . . .	9
<b>2</b>	<b>Single-phase viscous systems (Nonlinear Stability – Well Posedness)</b>	<b>12</b>
2.1	Incompressible flows . . . . .	12
2.2	Compressible flows . . . . .	16
2.2.1	Constant viscosities . . . . .	18
2.2.2	Density dependent viscosities . . . . .	27
2.2.3	Porous media equation and pressureless Navier–Stokes equations . . . . .	32
2.3	Pressure dependent viscosity on granular material . . . . .	34
2.3.1	Local dependency . . . . .	34
2.3.2	Non-local dependency . . . . .	35
2.3.3	Granular material: A challenge in mathematical fluid mechanics . . . . .	37
2.4	Single-phase viscous model with heat conductivity . . . . .	38
2.5	Construction of approximate systems . . . . .	40
2.6	A two-phase immiscible system written as a single-phase system with an indicator function . . . . .	43
<b>3</b>	<b>From two-phase immiscible system to two-phase averaged system by homogenization</b>	<b>45</b>
3.1	Formal approach . . . . .	45
3.2	Mathematical derivation using homogenization techniques (averaging process) . . . . .	50
3.2.1	Two-phase system without temperature . . . . .	52
3.2.2	Two-phase system with temperature . . . . .	60
<b>4</b>	<b>Different two-phase averaged viscous systems (Nonlinear weak stability)</b>	<b>64</b>
4.1	Averaged two-phase systems with one velocity field . . . . .	64
4.2	Averaged two-phase systems with two velocity fields . . . . .	66
4.2.1	Two-phase system with pressure algebraic closure . . . . .	66
4.2.2	Two-phase system with pressure PDE closure . . . . .	69
<b>5</b>	<b>Simplified two-phase averaged systems (Singular limits)</b>	<b>69</b>
5.1	Relaxation limit . . . . .	69
5.2	The drift-flux system with no temperature equations . . . . .	71
5.2.1	Writing the two-phase system with drift variables . . . . .	72
5.2.2	Simplification and drift flux system . . . . .	76
5.3	Other generalized drift-flux systems considered in the literature . . . . .	76
<b>A</b>	<b>Crash course on functional analysis</b>	<b>79</b>
A.1	Mathematical concepts related to PDEs . . . . .	79
A.2	$L^p$ spaces and convergence in these spaces . . . . .	82
A.3	Distribution, Sobolev embedding . . . . .	84
A.4	Distribution with values in a Banach space . . . . .	85

---

<b>B</b>	<b>Crash course on compactness and homogenization</b>	<b>86</b>
B.1	Compactness tools for nonlinear weak stability . . . . .	86
B.1.1	Weak compactness in $L^1(\Omega)$ . . . . .	86
B.1.2	Space-time controls . . . . .	87
B.1.3	Commutation between weak limit and strictly convex function . . . . .	87
B.2	A non-local tool . . . . .	88
B.2.1	A property coming from monotony . . . . .	88
B.2.2	Minty trick for nonlinear functions depending on a sequence . . . . .	89
B.3	Some useful calculations to better understand the averaging of two-phase systems . . . . .	89
B.3.1	Homogenization (averaging procedure) for elliptic equation in one dimension . . . . .	89
B.3.2	Explicit multi-dimension averaging calculations . . . . .	91
	<b>References</b>	<b>94</b>

## List of Key insight

Key insight 1.	Weak solutions for incompressible NS with constant density . . . . .	13
Key insight 2.	Weak solutions for incompressible NS with non-constant density . . . . .	16
Key insight 3.	Effective flux property . . . . .	19
Key insight 4.	Weak solutions for compressible NS with non-decreasing pressure and constant viscosity	23
Key insight 5.	Weak solutions for compressible NS with non-monotone pressure or anisotropic viscosity	27
Key insight 6.	Weak solutions for compressible NS with density-dependent viscosity . . . . .	32
Key insight 7.	Continuous systems versus numerical schemes . . . . .	34
Key insight 8.	Local versus non-local dependency in term of the pressure . . . . .	37
Key insight 9.	Conclusion of sections 2.1-2.4 . . . . .	40
Key insight 10.	Small parameters and stability procedure . . . . .	43
Key insight 11.	Interfaces and weak regularity solutions . . . . .	45
Key insight 12.	Averaging procedure . . . . .	53
Key insight 13.	Averaging procedure with temperature equation . . . . .	63
Key insight 14.	Two-phase flows with a pure advective equation on the volume fraction . . . . .	66
Key insight 15.	Singular limits and two-phase averaged systems . . . . .	69

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## Glossary

**Two-phase mixture:** Physical system consisting of two phases with different physical properties.

**Two-phase averaged system:** An idealized representation of a two-phase mixture for modeling purpose where both phases are represented by interpenetrating continua regardless of their particulate or carrier nature.

**Two-phase immiscible system:** Physical system consisting of two phases that do not mix or dissolve to form a homogeneous mixture, thereby experiencing interfacial tension. Most gas–liquid mixtures are immiscible but mass transfer is possible.

**Two-phase system with interfaces:** An idealized representation of a two-phase mixture for modeling purpose where phases are represented by non-overlapping continua. Generally used to describe systems composed of at least one particulate phase such as an immiscible gas/liquid suspension or a mixture of solid-fluid phases.

## 1 Introduction

The aim of this review paper is to introduce fluid mechanics modeling and the mathematical properties of the related equations. We introduce progressively more complex models, from single phase flow to averaged two-phase systems, to reach the level of complexity necessary to describe the fluid mechanics of volcanic eruptions. The review can thus be used as an introduction to the two-phase flow systems describing magma flow in a volcanic conduit that we develop in Part I of our series of two papers focused on volcanology ( [Narbona-Reina et al., 2024]; [Burgisser et al., 2024]). It can also be read as a standalone compendium of derivation techniques and fundamental mathematical properties of two-phase flows. An essential tool to design complex models is the derivation of averaged two-phase systems from systems composed of two fluids separated by interfaces by using homogenization techniques (averaging procedures). An important mathematical properties is weak nonlinear stability, which can be obtained for both incompressible and compressible fluid systems by starting with single-phase viscous systems and then addressing different two-phase viscous systems. Once a given suitable two-phase model has been established, we discuss simplified formulations through relaxation limits or drift-flux systems. Such simplified systems are then adapted to encode the physical processes occurring in volcanic conduits in [Narbona-Reina et al., 2024]. For readers unfamiliar with the mathematical concepts we use herein, we have included a crash course in PDE and homogenization techniques in the appendix that provides mathematical definitions and key concepts for a more in-depth understanding. We also regularly provide short, plain language summaries labeled *Key Insight* throughout our explanations. Should a section prove too detailed, these summaries are here to help the reader to grasp the global flow of reasoning of the concerned section before moving on.

### 1.1 General motivations behind PDEs design

Modeling complex physical situations through systems of partial differential equations (PDEs) is a powerful tool to better understand the interactions involved in the phenomena of interest. Two main challenges of such modeling is being able to propose equation systems amenable to coherent numerical approaches by ensuring nonlinear weak stability and being able to propose simplified systems by implicitly encoding small-scale phenomena using homogenization techniques.

#### 1) Nonlinear weak stability (Approximation – Regularization – Discretization).

The structure of the Section has strongly been inspired by the introduction of Evans’ beautiful book [Evans, 1990], which explains the motivations to study the nonlinear weak stability of systems of equations occurring in various fields of applications. Suppose we wish to solve some nonlinear system, which we write symbolically as

$$A[V] = f \tag{1.1}$$

where  $A[\cdot]$  denotes a given nonlinear operator (which could be differential),  $f$  is a given function, and  $V$  is the unknown. The operator  $A$  may be differential, such as, for instance,

$$A(\partial^s V, \partial^{s-1} V, \dots, \partial V, V, t, x) = f(t, x)$$

where  $V := (v_1(t, x), \dots, v_p(t, x)) : (0, T) \times \Omega \mapsto \mathbb{R}^p$  and  $\partial^s$  denote any derivative of order  $s$  with respect to times  $t$  or coordinates  $x_j$  of space  $x$ . An instance of differential operator can be found in the transport equation:  $A(\partial_t V, \partial_x V, V, t, x) = \partial_t V + u \partial_x V$ , for a given scalar  $u$ . Here are some examples of how this operator  $A$  is defined for classical equations of fluid dynamics (with  $u \partial_x V$  which reads in a multi-dimension case  $u \cdot \nabla_x V = \sum_{j=1}^d u_j \partial_{x_j} V$  for  $u = (u_1, \dots, u_d)$  a vector field and  $V$  depending on  $(x_1, \dots, x_d)$ ).

- Transport equation:

$$A(\partial_t V, \partial_x V, V, t, x) = \partial_t V + u \cdot \nabla_x V, \text{ where } x \in \Omega \subset \mathbb{R}^2 \text{ and for a given vector field } u \text{ with } V : (0, T) \times \Omega \mapsto \mathbb{R}.$$

- Shallow-water equations:

$$A(\partial_t V, \partial_x V, V, t, x) = \partial_t V + v \cdot \nabla_x V + \begin{bmatrix} h \operatorname{div} v \\ g \nabla_x h \end{bmatrix}$$

where  $x \in \Omega \subset \mathbb{R}^2$  and  $V = (h, v)^t : (0, T) \times \Omega \mapsto \mathbb{R}^3$  with  $g$  the acceleration gravity.

- Porous media equations:

$$A(\partial_t V, \partial_x V, V, t, x) = \partial_t V - \Delta_x \varphi(V) \text{ for a given nonlinear function } s \mapsto \varphi(s) \text{ and } V : (0, T) \times \mathbb{R}.$$

Note that such systems, if they are differential, may be decomposed in the two main classes of boundary-value problems and time-dependent problems. The nonlinear PDEs systems introduced herein in such form are single-phase flow before transitioning to two-phase flows. To establish the existence or to find a numerical approximation of a solution  $V$  of this system, a powerful idea is to invent an appropriate collection of nicer approximate systems that can be solved exactly or numerically. We write abstractly these approximated systems as

$$A_n[V_n] = f_n \tag{1.2}$$

with  $n \in \mathbb{N}^*$  and where  $A_n[\cdot]$  represents a nonlinear operator which is somehow close to  $A[\cdot]$  for large  $n$ .  $f_n$  is close to  $f$  and  $V_n$  is a solution of (1.2). The hope is that the functions  $\{V_n\}_{n \in \mathbb{N}}$  will converge to a solution  $V$  of the initial system (1.1) in a sense to be defined. In practice, for instance in the stationary case, the operator  $A_n[\cdot]$  may represent finite-dimensional projections such as in the Galerkin method. Let  $P_n$  denote an appropriate projection into the computational space, which is a finite-dimensional space spanned by finite-element basis functions  $\{\varphi_j\}_{1 \leq j \leq n}$ . The Galerkin method seeks an approximate solution,  $V_n = \sum_{j=1}^n V_j \varphi_j(x)$  such that

$$P_n A(\partial^s V, \partial^{s-1} V, \dots, \partial V, V, t, x) = P_n f$$

and then calculates the limit  $n \rightarrow +\infty$  to solve the continuous equation

$$A(\partial^s V, \partial^{s-1} V, \dots, \partial V, V, t, x) = f(t, x).$$

Other approximated systems include singular regularizations (to enable one to use well-known results on the approximate system), discretization in space and/or time (to apply numerical methods such as finite-differences, finite-elements, finite-volumes, and spectral methods), gradients of approximate energy functionals if a minimization procedure is used (to perform calculus of variations), and systems collapsing at the limit  $n \rightarrow +\infty$  into a single equation (singular limits).

The main difficulty in defining these approximate systems is of course the nonlinear nature of the systems, knowing that certain uniform estimates available for the family  $\{V_n\}_{n \in \mathbb{N}}$ , for instance coming from physical properties, are not too strong. With such relatively poor estimates in hand, we can usually only show that the functions  $\{V_n\}_{n \in \mathbb{N}}$  (or a subsequence) converge weakly to a limit  $V$  in some function space (in a sense related to the bounds):

$$V_n \rightharpoonup V \text{ as } n \rightarrow +\infty.$$

Now that we have somehow constructed approximate operators  $A_n[\cdot]$  that tend to  $A[\cdot]$  in some sense, in applications it is not guaranteed that the weak convergence implies

$$A_n[V_n] \rightarrow A[V] \text{ as } n \rightarrow +\infty.$$

The obstacle we will address is that weak convergence behaves poorly with respect to nonlinearities. Nonetheless, such weak convergence appears to be the best outcome we can achieve, as seen in explicit well-known examples we provide later on. The key issue for many PDEs of interest lies in the explicit nonlinear structure of these systems, which may offer additional desirable properties when the equations are manipulated adequately because usually additional information is linked to hidden physical properties. Broadly speaking, while nonlinearity generally prohibits the derivation of good uniform estimates on the functions  $\{V_n\}_{k \in \mathbb{N}}$  and their derivatives, in certain settings the mere fact that  $V_n$  appropriately solves the designed approximation provides sufficient extra control to justify taking the limit. In fact, the first step in identifying the main ingredients allowing taking the limit is to assume we have a sequence of weak solutions  $\{V_n\}_{n \in \mathbb{N}}$  of (1.1) with uniform bounds given by the energy and to show that there exists a subsequence of  $\{V_n\}_{n \in \mathbb{N}}$  which converges weakly in some sense to a weak solution  $V$  of (1.1) satisfying the estimates. This will require obtaining extra bounds by manipulating the system. Establishing the global existence of weak solutions for nonlinear PDEs requires proving such nonlinear weak stability, which is based on compactness properties. More precisely, this requires showing that no oscillations and concentration phenomena occur when considering an *a priori* sequence of global weak solutions depending on a parameter  $n$  of the PDE system. In other words, the solutions have regularity given by the energy bound and they satisfy the system in a weak sense. For reader unfamiliar with these concepts, we provide examples of oscillations and concentration in the following section 1.2.1 that explain why they could prevent nonlinear weak stability. The next step after the nonlinear weak stability procedure is to construct the approximate problems (e.g., by singular regularizations, discretization in space/time, calculus of variations, or singular limits) while ensuring that the solutions of such a collection of systems preserve the estimates related to the uniform estimates deduced from the target system.

## 2) Small scales effects (Averaging process – Homogenization).

As described in [Allaire, Grégoire, 2012], averaging processes such as homogenization extract homogeneous effective parameters from disordered or heterogeneous media; it is a way to encode small scales into a continuum description. Homogenization was initially developed for periodic structures. Quite often, the period size is small compared to the size of a representative sample of the medium (often called a Representative Elementary Volume in physics literature). Denoting the size ratio by  $\varepsilon$ , an asymptotic analysis as  $\varepsilon$  tends to zero is required. Starting from a microscopic description of a problem, we aim to derive a macroscopic or effective description. This may be seen as follows: Let us start with the following problem to solve

$$A_{a(nx)}[V_n] = f \tag{1.3}$$

where  $A_{a(nx)}$  means a nonlinear operator depending on small scales through an oscillating coefficient  $a$  with respect to the variable  $y = nx$  with  $n \in \mathbb{N}^*$ . The main objective is to be able to find the operator  $B_{\bar{a}}$  such that  $V_n$  tends to  $V$  in some sense that solves

$$B_{\bar{a}}[V] = f. \tag{1.4}$$

The nonlinear operator  $B_{\bar{a}}$  may be different from  $A$ , and  $\bar{a}$  is the limit of  $a_n$  with  $a_n(x) = a(nx)$  when  $n$  tends to infinity in a sense to be defined. In the paper, we use sometimes the parameter  $\varepsilon = 1/n$  instead of  $n$ . In

Appendix B.3 we present a well-known example in one-dimensional space related to the Laplace operator, and we demonstrate the convergence related to highly oscillating functions in multidimensional settings through Young measures. We encourage people who are not aware on homogenization to consult the appendix to understand some phenomena with simple well-known examples. Important books related to homogenization that may be consulted include [Bakhvalov and Panasenko, 1990], [Bensoussan et al., 1978], [Jikov et al., 1995] and [Cioranescu and Donato, 1999]. As indicated previously, we refer the reader to [Allaire, Grégoire, 2012] for discussions regarding a model problem of diffusion for which the homogenized operator is of the same type (also a diffusion equation). In this context, homogenization involves defining and computing effective diffusion tensors. Our focus is on models that have different homogenized limits, meaning that the partial differential equations before and after the averaging procedure (homogenization) are of a different mathematical nature. For instance, Stokes equations for a viscous fluid in a porous medium yield either Darcy's law or Brinkman's law as a homogenized model. In this context of different limits, homogenization serves as a modeling tool that can justify new models arising as homogenized limits of complex microscopic equations. This will be the topic of Section 3, where we mathematically justify the derivation of two-phase averaged compressible systems starting from two-phase immiscible systems with interfaces on a periodic scale significantly smaller than the size of the domain. Note that even though periodicity is far from real applications, this procedure enables a better characterization of closures often presented a priori in the physics literature.

Note that the homogenization procedure can be developed without assuming periodicity conditions, using methods such as G or H convergence (see [Giorgi and Spagnolo, 1973, Murat and Tartar, 1997]). However, this approach does not seem to be appropriate for averaging purposes in multi-phase compressible systems. Recently, new insights have been gained, for instance, in the works of [Capdeville et al., 2010] and [Mizuno et al., 2020], which both develop homogenization methods to calculate an effective equivalent medium for wave propagation above a cut-off wavelength. This method, which is very powerful for real applications, could be the subject of mathematical analysis, including proofs of convergence and symmetry of the homogenized elastic tensor.

## 1.2 Obstacles to finding stable solutions

Let us first illustrate situations which may occur on sequences of solutions. Below are simple examples of oscillations and concentration that can be kept in mind.

### 1.2.1 Examples of oscillations and concentration

– For *oscillations*. Let us consider  $u \in L^\infty(0, 1)$  such that  $\int_0^1 u(x) dx = 0$ . We extend  $u$  to  $\mathbb{R}$  by 1-periodicity and we define the sequence

$$u_n(x) := u(nx) \text{ for almost all } x \in \mathbb{R}.$$

Then we can prove that  $u_n \rightarrow 0$  weakly in  $L^p(0, 1)$  for all  $1 < p < +\infty$  but it does not converge strongly to 0 in  $L^p(0, 1)$ . This implies that the weak limit of a nonlinear function of  $u_n$  is not necessarily the nonlinear function of the weak limit. Let us construct a sequence of functions  $\{u_n(x)\}_{n \in \mathbb{N}}$  such that  $\int_\Omega u_n(x) \rightarrow 0$  and  $\|u_n(x)\|_{L^2(\Omega)} \leq C < +\infty$  uniformly with respect to  $n$ . This implies  $u_n \rightarrow 0$  in  $L^2(\Omega)$  by proposition A.11 in Appendix A. The idea is to prove that we cannot get strong convergence to 0 in  $L^2(0, 1)$ , i.e. that  $\|u_n - 0\|_{L^2(0,1)}^2 \not\rightarrow 0$ , which shows that oscillations may hinder nonlinear stability. To construct the example, we start with a function  $u(x)$  such as the sinusoidal black curve in Figure 1. We then construct on this basis progressively more oscillating candidates:  $u(2x)$  (blue curve in Figure 1),  $u(3x)$ ,  $\dots$ ,  $u(nx)$  (red curve in Figure 1 for a given  $n$ ). The limit  $n \rightarrow +\infty$  will oscillate at high frequency, which prevents nonlinear strong convergence. Let us take for example the function  $u(x) = \sin(2\pi x)$  for  $0 \leq x \leq 1$  with  $p = 2$ . This function satisfies  $\int_0^1 u(x) dx = 0$ . Based on this function, let us introduce the sequence  $\{u_n\}_{n \in \mathbb{N}}$ , with

$$u_n(x) = u(nx) = \sin(2\pi nx) \text{ for } 0 \leq x \leq 1.$$



Figure 1 illustrates this sequence. When  $n$  increases, so does the number of oscillations of the function  $u_n$  in the span  $0 \leq x \leq 1$ .

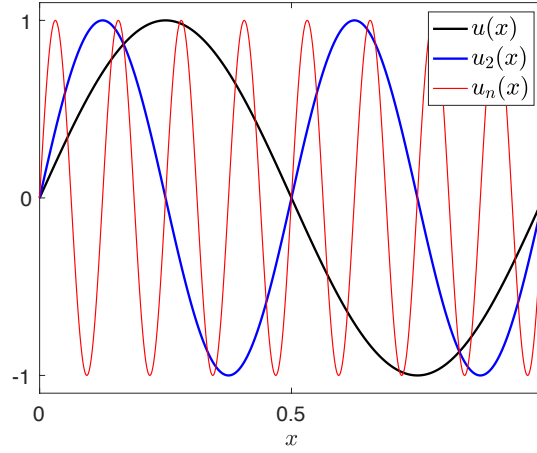


Figure 1: Example of oscillation as a property that does not ensure weak stability.

We can check that

$$\int_a^b u_n(x) dx \rightarrow 0 \quad \forall (a, b) \subset (0, 1)$$

and the norm of this sequence is bounded uniformly in  $n$ , indeed  $\|u_n\|_{L^2(0,1)}^2 = \int_0^1 (1 - \cos(4\pi nx))/2 = 1/2$ . So we have

$$\|u_n\|_{L^2(0,1)}^2 \rightarrow 1/2 \neq 0.$$

– For *concentration*. Let us suppose  $1 \leq p < +\infty$ . Let  $v \in \mathcal{C}_c^\infty(\mathbb{R})$  such that  $\int_{\mathbb{R}} v(x) dx = 1$  and let us define

$$v_n(x) := n^{1/p} v(nx) \text{ for all } x \in \mathbb{R}.$$

Then  $v_n \rightarrow 0$  weakly in  $L^p(\mathbb{R})$  for  $1 < p < +\infty$  but the convergence is not strong. Note that, for  $p = 1$ ,  $v_n$  does not converge weakly to 0 in  $L^1(\mathbb{R})$  but  $v_n \rightarrow \delta_0$  weakly in the sense of the measure  $\mathcal{M}(\mathbb{R})$ . Consider a sequence of functions  $\{u_n(x)\}_{n \in \mathbb{N}}$  such that  $\|u_n\|_{L^1(-1,1)} = 1$ . Its  $L^1$  norm is then constant and we observe that the sequence concentrates to a Dirac mass. Such Dirac mass is not  $L^1(-1,1)$ , which implies no weak nonlinear stability in  $L^1$ . To construct the example, we start with a function  $v(x)$  (black lines in Figure 2). On this basis we build progressively shorter and higher step functions:  $v(2x)$  (blue lines in Figure 2),  $v(3x)$ ,  $\dots$ ,  $v(nx)$  (red lines in Figure 2 for a given  $n$ ). The limit  $n \rightarrow +\infty$  will concentrate these step functions into a Dirac measure. Take for instance the function  $v(x) = 1$  for  $0 \leq x \leq 1$  and 0 elsewhere, with  $p = 1$ . We build from this function the sequence  $\{v_n\}_{n \in \mathbb{N}}$  with

$$v_n(x) = n v(nx) = \begin{cases} n & 0 \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

Figure 2 illustrates this sequence. When  $n$  increases, the function  $v_n(x)$  increases its value but over a decreasing range.

We can check that

$$\|v_n\|_{L^1(-1,1)} = 1$$

and that  $v_n$  does not converges weakly in  $L^1(-1,1)$  because

$$\langle v_n, \phi \rangle_{L^1 \times L^\infty} = n \int_0^{1/n} \phi(x) dx \rightarrow \langle \delta_0, \phi \rangle = \phi(0).$$

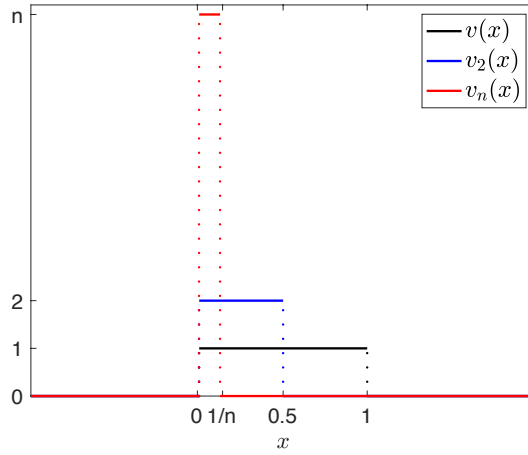


Figure 2: Example for concentration: A property which may prevent convergence in  $L^1$ : Dirac masses may appear.

### 1.2.2 Oscillations-Concentration related to nonlinear stability and homogenization (averaging procedure)

It is important to note that oscillations and concentration can be a consequence of mathematical phenomena (such as a poor choice of approximate PDEs) or physical phenomena (such as small-scale effects). In the former case, we need to find ways to demonstrate that these phenomena do not occur in the system of interest by using, for instance, equi-integrability, the Aubin-Lions-Simon Lemma, commutation between weak limits and strictly convex functions, or non-local compactness tools. This will prevent artificial behavior and could aid in designing appropriate numerical schemes that yield stable simulations. It is worth noting that understanding these mathematical properties and the chosen approximate system and iterative processes can also help to better understand numerical black boxes such as the COMSOL software we used in Part II [Burgisser et al., 2024]. In the latter case, where oscillations or concentrations arise from physical situations, we need to describe such phenomena and depict large-scale effects using appropriate tools to describe the averaged system. To do this, suitable tools such as Young measures, generalized Young measures, or more precisely directional oscillations, concentration, and compensated compactness via micro local compactness forms need to be introduced. This is necessary as we aim to derive averaged two-phase systems from two component systems with interfaces. In such situations, the density changes value at small scales, resulting in abrupt oscillations when attempting to obtain the averaged two-phase mixture at the macro scale.

Let us present an **example of discontinuous oscillations** in the density profiles which will reflect the scenario we will see to derive an averaged two-phase system. Let us define the following profile  $\rho$  on  $\mathbb{R}$ , see figure 3a:

$$\rho(x) = \begin{cases} \rho_l & x \in (0, \frac{1}{3}) \\ \rho_g & x \in (\frac{1}{3}, 1) \\ \text{extended on } \mathbb{R} & \text{by periodicity} \end{cases}$$

We construct the sequence  $\{\rho_n\}$  as  $\rho_n(x) = \rho(nx)$ . Then for example for  $n = 2$  we have:

$$\rho_2(x) = \rho(2x) = \begin{cases} \rho_l & x \in (0, \frac{1}{6}) \cup (\frac{1}{2}, \frac{2}{3}) \\ \rho_g & x \in (\frac{1}{6}, \frac{1}{2}) \cup (\frac{2}{3}, 1) \end{cases},$$

also extended in  $\mathbb{R}$  by periodicity, see figure 3b.

We can check (see Appendix Proposition B.2 for a proof in two dimensions) the following convergence

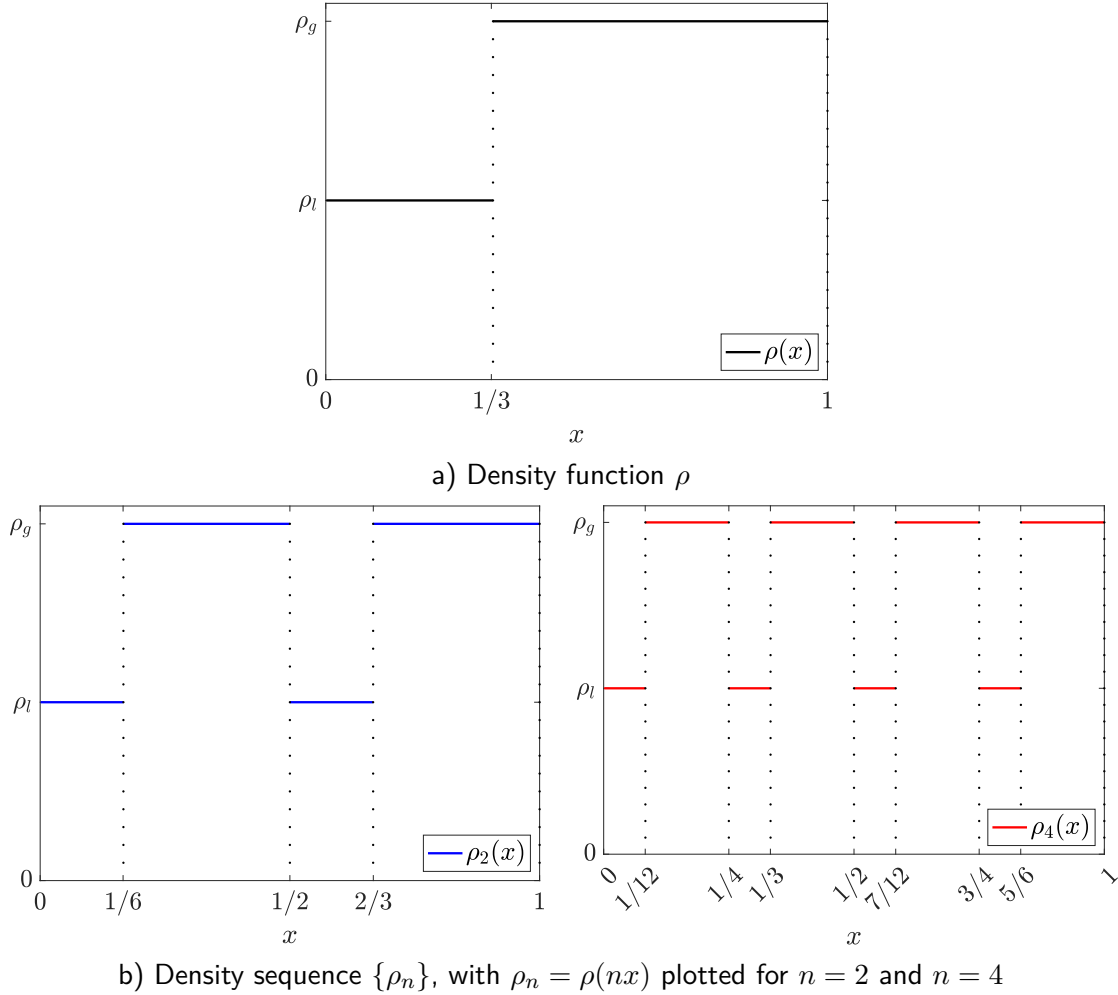


Figure 3: Example of discontinuous oscillations.

$$\begin{cases} \rho_n \rightharpoonup \bar{\rho} = \frac{1}{3}\rho_l + \frac{2}{3}\rho_g \\ \rho_n^2 \rightharpoonup \bar{\rho}^2 = \frac{1}{3}\rho_l^2 + \frac{2}{3}\rho_g^2 \\ \beta(\rho_n) \rightharpoonup \overline{\beta(\rho)} = \frac{1}{3}\beta(\rho_l) + \frac{2}{3}\beta(\rho_g) \end{cases}$$

This may be understood as

$$\rho_n \rightharpoonup \frac{1}{3}\delta_{\rho_l} + \frac{2}{3}\delta_{\rho_g} \text{ in the sense of measures.}$$

The averaged two-phase system, we will justify mathematically later-on, will reflect this kind of discontinuous oscillations and convergence related to two-dirac masses. Remark that the fraction associated to  $\rho_l$  is  $1/3$  and the one corresponding to  $\rho_g$  is  $1 - 1/3 = 2/3$ . Note that no compactness in space occurs reflected by the property that for all nonlinear function  $s \mapsto \beta(s)$ , we get

$$\overline{\beta(\rho)} \neq \beta(\bar{\rho}).$$

---

The weak limit related to a non-linear function is not necessarily the nonlinear function of the weak limit.

The two situations mentioned above (nonlinear weak stability and homogenization) have motivated us to present mathematical results that address each of these scenarios, including known results and open problems. The structure of our review follows that logic. Section 2 presents well-posedness results for single-phase viscous systems with interfaces. Section 3 presents the derivation of two-phase averaged systems from two-phase systems with interfaces and Section 4 assesses the well-posedness for two-phase averaged systems. The last Section 5 focuses on formal singular limits, which will be extensively studied in volcanic context in Part I and II of the series ([Narbona-Reina et al., 2024]; [Burgisser et al., 2024]).

We begin by establishing the **global existence of weak solutions "à la Leray"** (i.e. solutions with regularity deduced solely from energy). In his seminal paper, [Leray, 1934], J. Leray (1906-1998) introduces the concept of weak solutions (and also defines what is now called a Sobolev space), providing a precise definition of irregular solutions of the system. Leray shows that such weak solutions exist for the Navier–Stokes equations in their incompressible and homogeneous version. These solutions, characterized by minimal regularity (finite energy), are now commonly referred to as "solutions à la Leray." Although the global existence of weak solutions (turbulent solutions) contributes little to the well-posedness of the system, such analysis holds a significant practical interest. Apart from its physical significance, the minimal regularity of assumed initial data is strongly linked to well-identified physical quantities, and the stability properties of weak solutions in the continuous model aid in the construction of stable numerical schemes that often do not preserve strong regularity estimates.

To prove global existence of weak solutions, several steps are typically followed:

- a) Formally identify uniform estimates related to the energy and additional global information.
- b) Construct an approximate system compatible with the global uniform estimates.
- c) Demonstrate weak stability using the uniform estimates.

Usually, we start with step a), focusing on establishing uniform physical estimates. Subsequently, we investigate step c), assuming we have a sequence of weak solutions related to a parameter of the system with the uniform estimates from step a) satisfied uniformly with respect to this parameter. This parameter may be related to an approximate system or to a finite-dimensional approximation when numerical schemes are employed. The primary objective is to prove the existence of a subsequence converging in some sense to a global weak solution of the system, satisfying the uniform estimates; this is known as **nonlinear weak stability**. An essential aspect of analyzing nonlinear PDEs involves understanding oscillations and concentrations in weakly converging sequences of functions, as these phenomena differentiate weak from strong convergence in Lebesgue spaces. This observation is already expressed in Vitali's classical convergence theorem, where a norm-bounded, non-oscillating, non-concentrating sequence is seen to converge strongly.

The connection between the abstract concept of compactness and concrete oscillation and concentration effects underscores the importance of studying questions of compactness for their own sake. Starting with simple scalar defect measures, increasingly refined tools have been developed to study compactness. Here we focus on mathematical fluid mechanics, explaining the different tools used to ensure weak stability or, at least, to describe oscillations and concentration if they arise from physics. This is particularly relevant when deriving by homogenization two-phase averaged systems from two-phase systems with interfaces separating the fluids.

In the final step c), we construct approximate solutions, which are solutions of a modified system with approximate parameters that satisfy estimates uniformly with respect to these introduced parameters. We present several examples of approximate systems depending on the PDE system considered. We then discuss possible

extensions of well-posedness to two-phase systems for the global existence of weak solutions.

The final part of this review is devoted to the physical and mathematical derivations of some averaged two-phase flow systems from systems consisting of two phases separated by interfaces. To achieve such derivations, we introduce a small space scale for each phase in the initial data for density and let the corresponding parameter tend to 0. This process yields an averaged two-phase system starting from a single-phase system with initially high oscillating density profiles. We also discuss recent justifications of relaxation limits on two-phase systems, highlighting interesting open problems. In conclusion, we present some calculations related to the drift-flux system in multi-dimensions.

## 2 Single-phase viscous systems (Nonlinear Stability – Well Posedness)

### 2.1 Incompressible flows

The Navier–Stokes equations provide a basic mathematical model for describing the motion of a fluid. The classic paper of [Leray, 1934] shows the existence of a regular solution up to a time  $T$  and characterizes that solution for the incompressible Navier–Stokes equations with constant viscosity and constant density. It corresponds to the following system for  $t \in [0, T]$ ,  $x \in \Omega \subset \mathbb{R}^d$  with  $d = 2, 3$ ,

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p - \nu \Delta u = f, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \\ u|_{x \in \partial\Omega} = 0, \end{cases} \quad (2.1)$$

where  $u$  is the velocity vector field,  $p$  is the pressure and  $\nu$  is the dynamical shear viscosity. Note that  $u \otimes u$  is a  $d \times d$  tensor with  $(u \otimes u)_{i,j} = u_i u_j$ . For the sake of brevity, we only consider periodic boundary conditions for the systems presented here, i.e.  $\Omega = \mathbb{T}^d$  for  $d = 2$  or  $3$ . For the homogeneous incompressible Navier–Stokes equations (2.1), the energy balance is

$$\sup_{t \in (0, T)} \int_{\Omega} |u|^2(t) + \nu \int_0^T \int_{\Omega} |\nabla u|^2 = \int_{\Omega} |u_0|^2 + \int_0^T \int_{\Omega} f \cdot u.$$

To get such a balance, it suffices to take the scalar product of the momentum equation with  $u$  and integrate on the domain  $\Omega$ , which yields

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 + \frac{1}{2} \int_{\Omega} u \cdot \nabla |u|^2 + \int_{\Omega} \nabla p \cdot u + \nu \int_{\Omega} |\nabla u|^2 = \int_{\Omega} f \cdot u.$$

Observe that since  $\operatorname{div} u = 0$ ,

$$\operatorname{div}(u \otimes u) = u \cdot \nabla u + u \operatorname{div} u = u \cdot \nabla u.$$

Integrating by parts and using again that  $\operatorname{div} u = 0$ , we obtain

$$\int_{\Omega} u \cdot \nabla |u|^2 = 0, \quad \int_{\Omega} \nabla p \cdot u = 0.$$

Integrating in time and taking the supremum formally gives the result.

Thus one can assume there exists a sequence  $\{u_n\}_{n \in \mathbb{N}}$  that is a weak solution of the incompressible Navier–Stokes equations satisfying the energy estimates

$$\sup_{t \in (0, T)} \int_{\Omega} |u_n|^2(t) + \nu \int_0^T \int_{\Omega} |\nabla u_n|^2 < c < +\infty \quad (2.2)$$

where  $c$  does not depend on  $n$ . The main objective is to prove that we are able to take the limit with respect to  $n$  in the weak form of the incompressible Navier–Stokes equations, namely in the equation:

$$\langle \partial_t u_n + \operatorname{div}(u_n \otimes u_n) - \nu \Delta u_n - f, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = 0 \text{ in } \mathcal{D}'(0, T) \quad (2.3)$$

for all  $\varphi \in \{\varphi \in \mathcal{D}(\Omega) : \operatorname{div} \varphi = 0\}$ . See the crash course A for the introduction of  $\langle \cdot, \cdot \rangle$ ,  $\mathcal{D}$  and  $\mathcal{D}'$ . It is important to note that the pressure term does not appear in the weak form because we test the equation with divergence-free test functions. To get pressure after solving the weak form that involves quantities linked to  $u_n$ , the De Rham theorem is used [Simon, 1999].

The detailed reasoning is that, from the energy estimate control (2.2), we know that  $u_n \in L^2(0, T; V)$  uniformly with  $V = \{v : v \in H^1(\Omega) \text{ with } \operatorname{div} v = 0\}$  and from the equation itself we know that  $\partial_t u_n \in L^1(0, T; V')$  uniformly because  $u_n$  satisfies (2.3) and we use the bounds given by the energy. Then to take the limit in (2.3), the main difficulty is the quadratic term  $u_n \otimes u_n$ , that is the only nonlinear quantity in terms of the sequence  $\{u_n\}_{n \in \mathbb{N}}$ . To be able to take the limit in this term and then ensure the weak stability, we need to prove compactness in  $L^2((0, T) \times \Omega)$  for  $\{u_n\}_{n \in \mathbb{N}}$ . This comes from a control of the time and space derivative of  $u_n$  uniformly in complementary spaces: The Aubin-Lions-Simon compactness Lemma helps to check compatible properties (see Theorem B.2).

In our case, we have  $u_n \in L^2(0, T; V)$  and  $\partial_t u_n \in L^q(0, T; V')$  with some  $q > 1$  so considering  $H = \{v : v \in L^2(\Omega) \text{ with } \operatorname{div} v = 0\}$  we get compactness of  $u_n$  in  $L^2(0, T; H)$  (choosing  $X_0 = V$ ,  $X = H$  and  $X_1 = V'$  in Aubin-Lions-Simon Lemma B.2 and  $p = 2$ ,  $q > 1$ ), hence compactness of  $u_n$  in  $L^2(0, T; L^2(\Omega))$  writing  $\int_0^T |u_n|^2 = \int_0^T \langle u_n; u_n \rangle_{H \times H}$ . Note that it is also possible to show that  $u_n$  belongs uniformly in some Nikolskii space based on translation in time to prove compactness in time.

**Key insight 1.** For the incompressible Navier–Stokes equations with constant density, we have information on the space derivative of the velocity field from the energy identity and we have information on the time derivative of the velocity field from the momentum equation and the properties from the energy identity. The space+time control of a quantity implies compactness on this quantity by the Aubin-Lions-Simon Lemma.

In the 70-80's, the non-homogeneous version of the incompressible Navier–Stokes equations (i.e. with variable density) was investigated. It corresponds to the following system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - 2\operatorname{div}(\mu(\rho)D(u)) = \rho f, \\ \operatorname{div} u = 0, \\ \rho u|_{t=0} = m_0, \quad \rho|_{t=0} = \rho_0 \end{cases} \quad (2.4)$$

where  $D(u) = (\nabla u + (\nabla u)^t)/2$  is the strain tensor, which is symmetric. Note that the strain tensor  $2\mu(\rho)D(u)$  is traceless due to the incompressibility constraint when  $\mu(\rho_n) = \text{cst}$ . The initial velocity is defined as  $u_0 = m_0/\rho_0$  when  $\rho_0 \neq 0$  and  $u_0 = 0$  for a vanishing initial density.

This system has been studied for instance by [Kazhikhov and Smagulov, 1977] assuming  $0 < c \leq \rho_0 \leq C < +\infty$ , by [Simon, 1986] assuming  $0 \leq \rho_0 \leq C < +\infty$  before the renormalization concept introduced by [DiPerna and Lions, 1989], and then by P.-L. Lions in his book [Lions, 1998] considering  $\mu$  a continuous, positive function on  $[0, +\infty)$ . Note that for such system, assuming the same constraint on the initial density  $\rho_0$ , the uniform elements of information are the energy

$$\sup_t \int_{\Omega} \rho |u|^2(t) + \int_0^T \int_{\Omega} 2\mu(\rho) |D(u)|^2 = \int_{\Omega} \rho_0 |u_0|^2 + \int_0^T \int_{\Omega} f \cdot u$$

and the constraint

$$0 \leq \rho \leq C < +\infty$$

Since the velocity field satisfies the divergence-free constraint, the propagation in time of the initial bounds on the density is ensured. This is a general property of the transport equation

$$\partial_t \rho + u \cdot \nabla \rho + \rho \operatorname{div} u = 0$$

for a vector field for which we have a control of  $\operatorname{div} u$  in  $L_t^1 L_x^\infty$ . The case of divergence-free velocity field is thus straightforward. When, however, it is not a-priori divergence free and if we do not control  $\operatorname{div} u$  in  $L_t^1 L_x^\infty$ , concentration profiles of the density are implying possible blow-up phenomena ( $\rho$  tending to  $+\infty$ ).

Thus, since the density remains bounded in this case and we consider  $\operatorname{div} u = 0$ , the step of demonstrating weak stability amounts to the following problem:

Assume that a sequence of weak solution of (2.4) satisfies

$$\sup_t \int_{\Omega} \rho_n |u_n|^2(t) + \int_0^T \int_{\Omega} 2\mu(\rho_n) |D(u_n)|^2 < c < +\infty \quad (2.5)$$

and the constraint

$$0 \leq \rho_n \leq C < +\infty \quad (2.6)$$

uniformly with respect to  $n$  with  $C > 0$ . More precisely, assume that the sequence  $\{(\rho_n, u_n)\}_{n \in \mathbb{N}}$  satisfies in a weak sense the approximated system mass, momentum and divergence free constraint namely the mass equation in the distribution sense  $\mathcal{D}'((0, T) \times \Omega)$ :

$$\partial_t \rho_n + \operatorname{div}(\rho_n u_n) = 0, \quad \operatorname{div} u_n = 0, \quad (2.7)$$

and the momentum equation

$$\langle \partial_t(\rho_n u_n) + \operatorname{div}(\rho_n u_n \otimes u_n) - 2\operatorname{div}(\mu(\rho_n) D(u_n)); \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = 0 \text{ in } \mathcal{D}'(0, T) \quad (2.8)$$

for all  $\varphi \in \mathcal{D}(\Omega)$  such that  $\operatorname{div} \varphi = 0$ .

**Then let us prove a nonlinear weak stability property** by proving that there exists a subsequence of  $\{(\rho_n, u_n)\}_{n \in \mathbb{N}}$  that converges weakly in a sense to be defined to  $(\rho, u)$  and that also satisfies (2.8)–(2.7) in a weak sense as well as the initial data  $\rho|_{t=0} = \rho_0$  and  $(\rho u)|_{t=0} = m_0$ . To prove this property, we first focus on the proof of the uniform estimates (2.5) and (2.6), and in a second step we take the limit.

Let us begin with the proof of (2.5). Firstly, from the energy estimate and assuming  $\mu(\rho_n) > c > 0$  uniformly in  $n$ , we have a bound on  $D(u_n)$  in  $L^2((0, T) \times \Omega)$ . Next, to conclude a uniform bound of  $u_n$  in  $L^2(0, T; H^1(\Omega))$  we need an information on  $\frac{1}{|\Omega|} \int_{\Omega} u_n$  in  $L^2(0, T)$  to be able to apply the Poincaré-Wirtinger estimate, namely the inequality

$$\left\| V - \frac{1}{|\Omega|} \int_{\Omega} V \right\|_{L^2(\Omega)}^2 \leq C \|\nabla V\|_{L^2(\Omega)}^2.$$

By integrating in space the mass equation, we have the conservation of the mass ( $\int_{\Omega} \rho_n = \text{cst} > 0$ ). To conclude on the control of  $\frac{1}{|\Omega|} \int_{\Omega} |u_n|^2$ , using that  $a^2 \leq 2((a-b)^2 + b^2)$ , we observe that

$$\left| \frac{1}{|\Omega|} \int_{\Omega} |u_n|^2 \right|^2 \leq 2 \left| u_n - \frac{1}{|\Omega|} \int_{\Omega} |u_n|^2 \right|^2 + 2 |u_n|^2$$

and therefore

$$\left( \int_{\Omega} \rho_n \right) \frac{1}{|\Omega|} \int_{\Omega} |u_n|^2 \leq 2 \int_{\Omega} \rho_n |u_n|^2 + 2 \int_{\Omega} \rho_n \left| u_n - \frac{1}{|\Omega|} \int_{\Omega} |u_n|^2 \right|^2 \leq C + C \int_{\Omega} \left| u_n - \frac{1}{|\Omega|} \int_{\Omega} |u_n|^2 \right|^2 \leq C + C \|\nabla u\|_{L^2}^2$$

for all  $t \in (0, T)$ . We conclude with the estimate (2.5).

Let us now prove (2.6). To get the first bound on  $\rho_n$ , it suffices to multiply the mass equation by  $\rho_n^{p-1}$  to get, using the divergence free property of  $u_n$ :

$$\partial_t \rho_n^p + \operatorname{div}(\rho_n^p u_n) = 0$$

and therefore, integrating in space

$$\frac{d}{dt} \int_{\Omega} \rho_n^p dx = 0$$

which gives

$$\int_{\Omega} \rho_n^p(t, \cdot) dx = \int_{\Omega} \rho_n^p(0, \cdot) dx.$$

Letting  $p$  tend to  $+\infty$  and using that  $\rho_n^0$  is uniformly bounded by  $C$ , we get  $\rho_n < C$ . To prove the lower bound, it suffices to write the equation satisfied by  $\beta(\rho_n) = \min(0, \rho_n)$  and show that it is equal to 0 if initially it is the case. Remark that multiplying the mass equation by  $\beta'(\rho_n)$  and using that  $\operatorname{div} u_n = 0$ , we have

$$\partial_t \beta(\rho_n) + \operatorname{div}(\beta(\rho_n) u_n) = 0$$

and therefore integrating in space

$$\int_{\Omega} \beta(\rho_n)(t) = \int_{\Omega} \beta(\rho_0).$$

Observe now that  $\int_{\Omega} \beta(\rho_0) = 0$  because initially  $\rho_0 \geq 0$  and therefore  $\beta(\rho_0) = 0$ . This implies that for all  $t \in [0, T]$ ,

$$\int_{\Omega} \beta(\rho_n)(t) = 0$$

which implies  $\beta(\rho_n(t)) = 0$  a.e. because  $s \mapsto \beta(s)$  is a positive function and therefore  $\rho_n(t) \geq 0$  a.e.. To justify mathematically the calculation that provides the equation on  $\beta(\rho_n)$  with low regularity, one needs to use the renormalization technique introduced in [DiPerna and Lions, 1989] that we will not detail here.

Thus we state the bounds recovered from the energy estimate and the mass equation:

$$\rho_n \in L^\infty((0, T) \times \Omega), \quad u_n \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \text{ uniformly with respect to } n.$$

Let us now study the limit to complete the proof of nonlinear weak stability. This limit concerns the nonlinear quantities  $\rho_n u_n$  and  $\rho_n u_n \otimes u_n$  and then the diffusive term  $\mu(\rho_n) D(u_n)$ . Let us first focus on the first two. We will use the fact that if we have a product to be controlled, we will be able to prove the weak stability if we control time derivative of one of the quantity and control the space derivative of the other in appropriate spaces. Note that time derivatives are present in  $\partial_t \rho_n$  and in  $\partial_t(\rho_n u_n)$ , which occur in the mass and the momentum equations, respectively.

To take the limit in  $\rho_n u_n$ , it suffices to have strong convergence in some space for  $\rho_n$  in duality with the space where we have the weak convergence for  $u_n$ . More precisely, let us work with the following writing

$$\int_0^T \int_{\Omega} \rho_n u_n \cdot \nabla \varphi = \int_0^T \langle \rho_n; u_n \cdot \nabla \varphi \rangle_{H^{-1}(\Omega) \times H^1(\Omega)}$$

for all  $\varphi \in \mathcal{D}((0, T) \times \Omega) = \mathcal{C}_0^\infty((0, T) \times \Omega)$ .

Let us first recall that  $\partial_t \rho_n \in L^2(0, T; H^{-1}(\Omega))$  with  $\rho_n \in L^\infty((0, T) \times \Omega)$  and thus is compact in  $L^2(0, T; H^{-1}(\Omega))$  (by using the Aubin-Lions-Simon Lemma B.2 with  $X_0 = L^2(\Omega)$ ,  $X = X_0 = H^{-1}(\Omega)$  and



$p = q = 2$ ). Knowing that  $u_n$  is uniformly bounded in  $L^2(0, T; H^1(\Omega))$  converging (up to a subsequence) to  $u$  weakly in  $L^2(0, T; H^1(\Omega))$ , we can take the limit in the duality product.

Concerning the term  $\rho_n u_n \otimes u_n$ , it is interesting to note that we have a control on  $\partial_t(P(\rho_n u_n)) \in L^1(0, T; W^{-1,1}(\Omega))$  where  $P$  is the Leray-Hopf projector (see [Galdi, 2011]) on the divergence-free space using the momentum equation and the information coming from the energy (using the Aubin-Lions-Simon Lemma B.2 with  $X_0 = L^2(\Omega)$ ,  $X = H^{-1}(\Omega)$ ,  $X_1 = W^{-1,1}(\Omega)$  and  $p = q = 2$ ). Now we write

$$\int_0^T \int_{\Omega} \rho_n |u_n|^2 = \int_0^T \langle P(\rho_n u_n); u_n \rangle_{H^{-1}(\Omega) \times H^1(\Omega)}$$

where  $P$  is the Leray-Hopf projector on the divergence-free space. We take the limit in this quantity to conclude the convergence

$$\int_0^T \int_{\Omega} \rho_n |u_n|^2 \rightarrow \int_0^T \int_{\Omega} \rho |u|^2.$$

To do so, let us first see how to get the strong convergence of  $\rho_n$  to  $\rho$ . We know that  $\rho_n \rightarrow \rho$  in  $\mathcal{C}([0, T]; L^p(\Omega))$  weakly for all  $0 < p < +\infty$ . We introduce the sequence  $\rho_n^0$  such that  $\rho_n^0 \rightarrow \rho_0$  in  $L^p(\Omega)$  for all  $0 < p < +\infty$  and  $\rho_n|_{t=0} = \rho_n^0$ . We remark that using the conservation of the  $L^p$  norm then

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\rho_n|^p(t) = \lim_{n \rightarrow +\infty} \int_{\Omega} |\rho_n^0|^p = \int_{\Omega} |\rho_0|^p = \int_{\Omega} |\rho|^p(t).$$

This concludes the strong convergence of  $\rho_n$  to  $\rho$  in  $\mathcal{C}([0, T]; L^p(\Omega))$  for all  $p < +\infty$ . Thus  $\sqrt{\rho_n} u_n$  weakly converges in  $L^2((0, T) \times \Omega)$  and, using the convergence of  $\sqrt{\rho_n} u_n$  in norm  $L^2((0, T) \times \Omega)$ , this provides the strong convergence (weak convergence + convergence in norm  $\implies$  strong convergence).

The last limit to take involves the viscous quantity  $\mu(\rho_n)D(u_n)$ . The strong convergence on  $\rho_n$  is enough. Using the property of  $s \mapsto \mu(s)$  to get the convergence of  $\mu(\rho_n)$  to  $\mu(\rho)$  in  $L^2((0, T) \times \Omega)$ , combined with the weak limit of  $D(u_n)$  in  $L^2((0, T) \times \Omega)$ , we get the convergence of  $\mu(\rho_n)D(u_n)$  to  $\mu(\rho)D(u)$ .

**Key insight 2.** For the incompressible Navier–Stokes equations with non-constant density, we have a uniform bound of the space derivative of the velocity field in  $L^2(0, T; H^1(\Omega))$  from the energy identity and we have uniform bounds of the time derivative of the density from the mass equation using the bounds given by the energy and  $L^\infty$  bound on the density. We conclude the stability of the product  $\rho_n u_n$  by combining these properties and writing the product as a dual product  $\langle \rho_n, u_n \rangle_{H^{-1}(\Omega) \times H^1(\Omega)}$  and using Aubin-Lions-Simon Lemma to deduce some compactness of  $\rho_n$  in  $L^2(0, T; H^{-1}(\Omega))$ . Importantly, if one sequence does not oscillate in time and the other does not oscillate in space, we can then hope to have stability on the product. This is also the case if we want to take the limit in  $\rho_n u_n \otimes u_n$ . On the one hand, we have a bound of the space derivative of  $u_n$  and on the other hand we have a bound of the time derivative of  $\rho_n u_n$  by using the momentum equation and the bounds deduced from the energy estimates and from the mass equation with  $\operatorname{div} u_n = 0$ .

## 2.2 Compressible flows

Note that usually incompressible and compressible flows are treated separately in the literature. It is however interesting to realize that (2.4) is a particular case of the more general system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - 2\operatorname{div} \left( \mu(\rho) \left( D(u) - \frac{1}{d} \operatorname{div} u \operatorname{Id} \right) \right) = \rho f, \\ \rho u|_{t=0} = m_0, \quad \rho|_{t=0} = \rho_0 \end{cases} \quad (2.9)$$

where  $d$  is the space dimension and  $p$  is given by solving a nonlinear constraint

$$\mathcal{F}(\operatorname{div}u, |D(u)|, p, \rho) = 0$$

with an implicit function theorem. Choosing the relation

$$\mathcal{F}(\operatorname{div}u, |D(u)|, p, \rho) = \operatorname{div}u$$

yields the non-homogeneous incompressible Navier–Stokes equations (2.4). Alternatively, choosing the following simple relation

$$\mathcal{F}(\operatorname{div}u, |D(u)|, p, \rho) = p - p(\rho) + \left( \lambda(\rho) + \frac{2\mu(\rho)}{d} \right) \operatorname{div}u \quad (2.10)$$

where the pressure state law  $s \mapsto p(s)$  is given, yields the barotropic compressible Navier–Stokes equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - 2\operatorname{div} \left( \mu(\rho) \left( D(u) - \frac{1}{d} \operatorname{div}u \operatorname{Id} \right) \right) - \nabla \left( \left( \lambda(\rho) + \frac{2\mu(\rho)}{d} \right) \operatorname{div}u \right) = \rho f, \\ \rho u|_{t=0} = m_0, \quad \rho|_{t=0} = \rho_0. \end{cases} \quad (2.11)$$

Here we do not discuss general boundary conditions but consider for simplicity the periodic boundary conditions  $\Omega = \mathbb{T}^d$  with  $d = 2$  or  $3$ . The existence of global weak solutions of such system depends entirely on the assumptions taken for the shear and volume viscosities  $s \mapsto \mu(s)$ ,  $s \mapsto \lambda(s)$  and for the pressure  $s \mapsto p(s)$ . In the volcanological application presented in Part I [Narbona-Reina et al., 2024] for a gas-liquid mixture, we use a relation which is similar to (2.10) with  $\mathcal{F} = 0$  that links the liquid pressure to the gas pressure and to the divergence of the liquid velocity (see Subsection 2.3 of that paper).

*Barotropic compressible Navier–Stokes equations.* We consider the compressible Navier–Stokes equations (2.9) with the nonlinear constraint  $p = p(\rho) + (\lambda(\rho) + 2\mu(\rho)/d)\operatorname{div}u$  that yields the system (2.11). As in the incompressible case, we seek to establish the differential equation of energy that is coming from that system (2.11):

$$\frac{d}{dt} \int_{\Omega} \left( \frac{\rho}{2} |u|^2 + \rho e(\rho) \right) + \int_{\Omega} (2\mu(\rho) |D(u)|^2 + \lambda(\rho) |\operatorname{div}u|^2) = \int_{\Omega} \rho f \cdot u. \quad (2.12)$$

where  $\frac{1}{2}|u|^2$  is the kinetic energy and the internal energy  $e(\rho)$  is

$$e(\rho) = \int_0^\rho p(s)/s^2 ds. \quad (2.13)$$

Equation (2.12) is obtained by taking the scalar product with  $u$  of the momentum equation and using the mass equation to obtain after integrating on the domain  $\Omega$

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u|^2 + \int_{\Omega} (2\mu(\rho) |D(u)|^2 + \lambda(\rho) |\operatorname{div}u|^2) + \int_{\Omega} \nabla(p(\rho)) \cdot u = \int_{\Omega} \rho f \cdot u. \quad (2.14)$$

As we do not have a divergence-free equation, we use the mass equation to write the convective part as:

$$\begin{aligned} (\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u)) \cdot u &= \rho u \cdot (\partial_t u + u \cdot \nabla u) \\ &= \frac{1}{2} \rho (\partial_t |u|^2 + u \cdot \nabla |u|^2) \\ &= \frac{1}{2} (\partial_t(\rho |u|^2) + \operatorname{div}(\rho u |u|^2)) \end{aligned}$$

that we integrate to find the first term in (2.12). The main novelty compared to the incompressible flow is that the term involving the pressure  $p(\rho)$  does not vanish in the weak formulation; it yields the term  $\int_{\Omega} u \cdot \nabla p(\rho)$  in the energy calculation. To treat such a term, we use the mass equation to deduce that

$$\partial_t(\rho e(\rho)) + \operatorname{div}(\rho e(\rho)u) + p(\rho)\operatorname{div}u = 0$$

with  $e$  given by (2.13). Integrating this equality in space and adding the result to the previous relation yields the energy balance (2.12).

### 2.2.1 Constant viscosities

We focus again here on the multi-dimensional case ( $d = 2$  or  $3$ ) with periodic boundary conditions for simplicity. In the case where  $\mu$  and  $\lambda$  are constant with  $\lambda + 2\mu/d > 0$ , the existence of global weak solutions of (2.11) has been proved first by [Lions, 1993] and [Lions, 1998] with a pressure law  $p(s) = as^\gamma$  for  $\gamma \geq 3d/(d+2)$ , then by [Feireisl et al., 2001] for  $\gamma > d/2$ . By constructing approximate system and limit and by using the stability process, [Feireisl, 2002] proved the existence of global weak solutions more generally with a pressure law satisfying  $p \in \mathcal{C}^2([0, +\infty))$  with  $p(0) = 0$  and

$$a\rho^{\gamma-1} - b \leq p'(\rho) \leq a^{-1}\rho^{\gamma-1} + b$$

where  $a, b > 0$  and  $\gamma > d/2$ . More recently, an important result concerning the existence of global solutions has been obtained by [Plotnikov and Weigant, 2015] when  $d = 2$  with a pressure state law  $p(\rho) = a\rho^{d/2} = a\rho$ .

**Remark 2.1.** *The lower value  $d/2$  is linked to the equi-integrability property that guarantees the weak convergence of  $\rho_n u_n \otimes u_n$  in  $L^1((0, T) \times \Omega)$ , which means no concentration phenomena. This can be seen by first recalling that*

$$\rho_n u_n \otimes u_n \in L^\infty(0, T; L^1(\Omega))$$

*by using the energy estimate. In addition, we also know from the energy estimates that  $\rho_n \in L^\infty(0, T; L^\gamma(\Omega))$  from the pressure information and  $u_n \otimes u_n \in L^1(0, T; L^{d/(d-2)}(\Omega))$  from the viscous term. Therefore*

$$\rho_n u_n \otimes u_n \in L^1(0, T; L^\varepsilon(\Omega))$$

*for some  $\varepsilon > 0$ . By interpolation we get the equi-integrability by using that the quantity is  $s$ -integrable for  $s > 1$ . Note that similar equi-integrability estimates in the constant viscosities case if  $\gamma < d/2$  have been found by [Plotnikov, 2023] and by [Mellet and Vasseur, 2007] in the density dependent case (see later on) when considering the quantity  $\rho_n(1 + |u_n|^2) \log(1 + |u_n|^2)$ .*

Note that in all these results, the pressure law is at least an increasing function after a fixed value. Recently this monotony assumption after a fixed value has been removed by [Bresch and Jabin, 2018a], who proved the global existence of weak solutions for non monotone pressure  $p \in \mathcal{C}([0, +\infty))$  with  $p$  locally Lipschitz on  $(0, +\infty)$  with  $p(0) = 0$  satisfying

$$C^{-1}\rho^\gamma - C \leq p(\rho) \leq C\rho^\gamma + C \tag{2.15a}$$

and

$$|p'(s)| \leq \bar{p}s^{\gamma-1} \tag{2.15b}$$

with  $\bar{p} > 0$  and  $\gamma \geq 2d/(d+2)$ . This result has been improved by [Bresch et al., 2021] to take into account pressure terms of the form  $p(\rho, t, x)$  (i.e. a pressure law depending on  $\rho$  but also on space and time), which is a prerequisite to treat the case of heat-conducting, compressible Navier–Stokes equations where the pressure law  $p$  depends on the density  $\rho$  and the temperature  $\theta$ , such as the truncated virial pressure law. Note that such pressure  $p(\rho, t, x)$  is obtained when assuming that the temperature is given by an iterative constructing scheme.

This case of compressible Navier–Stokes system with heat conductivity and a truncated virial pressure law has been recently considered in [Bresch et al., 2023b] to prove global existence of weak solutions solving by the way a mathematical problem which remained open for a long time.

Let us focus on the case of **a non monotone pressure satisfying** (2.15) to prove the existence of global weak solutions of the system (2.11). The following quantity, that we will call *effective flux*,

$$F = (\lambda + 2\mu)\operatorname{div}u - p(\rho)$$

plays a crucial role in all these proofs. There is a kind of weak compactness property on this quantity, which provides some regularization in space on it. If we look at the momentum equation

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \mu\Delta u - (\lambda + \mu)\nabla\operatorname{div}u = \rho f$$

and express the divergence of it, we can write the *effective flux property*

$$-\Delta F = \operatorname{div}G, \quad G = -\partial_t(\rho u) - \operatorname{div}(\rho u \otimes u) + \rho f \quad (2.16)$$

This property is essential to obtain compactness properties on the density and we will explain it later on a simpler system.

**Key insight 3.** When the shear and bulk viscosities are constant, the effective flux  $F = (\lambda + 2\mu)\operatorname{div}u - p(\rho)$  satisfies the elliptic equation (2.16), which allows one to replace  $\operatorname{div}u$  by  $p(\rho)/(\lambda + 2\mu)$  in some sense when looking for compactness in space on  $\{\rho_n\}_{n \in \mathbb{N}}$ .

As in the incompressible case, we want to take the limit in the weak form of the Navier–Stokes equations. To do so, we assume that the pressure satisfies (2.15) and we introduce a sequence  $\{(\rho_n, u_n)\}_{n \in \mathbb{N}}$  such that

$$\langle \partial_t \rho_n + \operatorname{div}(\rho_n u_n), \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} = 0$$

and

$$\langle \partial_t(\rho_n u_n) + \operatorname{div}(\rho_n u_n \otimes u_n) - \mu\Delta u_n - (\lambda + \mu)\nabla\operatorname{div}u_n + \nabla p(\rho_n) - \rho_n f, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} = 0,$$

for all  $\varphi \in \mathcal{D}$  where  $\mathcal{D} = \{\varphi : \varphi \in C^\infty(\Omega)\}$ . The equivalent property on the effective flux may be written for

$$F_n = (\lambda + 2\mu)\operatorname{div}u_n - p(\rho_n)$$

as

$$-\Delta F_n = \operatorname{div}G_n, \quad G_n = -\partial_t(\rho_n u_n) - \operatorname{div}(\rho_n u_n \otimes u_n) + \rho_n f. \quad (2.17)$$

Using the same reasoning than in the incompressible case, we can use the Aubin-Lions-Simon Lemma (Lemma B.2) to take the limit in the nonlinear quantities  $\rho_n u_n$  and  $\rho_n u_n \otimes u_n$ . We only have to look to what space  $\partial_t \rho_n$  and  $\partial_t(\rho_n u_n)$  belong to use the bounds deduced from the energy estimates. More precisely, we have  $\partial_t \rho_n \in L^1(0, T; W^{-1,1}(\Omega))$  and  $\partial_t(\rho_n u_n) \in L^1(0, T; W^{-1,1}(\Omega))$ . For the compactness related to  $\rho_n$ , it suffices to choose  $B = H^{-1}(\Omega)$ ,  $X = L^\gamma$  and  $Y = W^{-1,1}(\Omega)$  in Aubin-Lions-Simon Lemma (note that  $X$  is compactly embedded in  $B$  since  $\gamma > 6/5$ ). This is obtained by duality since  $H^1(\Omega)$  is compactly embedded in  $L^r(\Omega)$  for  $r < 6$  in space dimension less or equal to 3. For the compactness related to  $u_n$ , it suffices to choose  $B = H^{-1}(\Omega)$ ,  $X = L^r(\Omega)$  with  $r = 6\gamma/(6 + \gamma)$  and  $Y = W^{-1,1}(\Omega)$  (note that  $X$  is compactly embedded in  $B$  since  $r > 6/5$ ). To take the limit, we start by writing

$$\int_0^T \int_\Omega \rho_n u_n \cdot \psi = \langle \rho_n; u_n \cdot \psi \rangle_{L^2(0, T; H^{-1}(\Omega)) \times L^2(0, T; H^1(\Omega))}$$

for all  $\psi \in \mathcal{D}((0, T) \times \Omega)$  and

$$\int_0^T \int_{\Omega} \rho_n v_n w_n \psi = \langle \rho_n v_n; w_n \psi \rangle_{L^2(0, T; H^{-1}(\Omega)) \times L^2(0, T; H^1(\Omega))}$$

for  $v_n$  and  $w_n$  components of  $u_n$  and  $\psi \in \mathcal{D}((0, T) \times \Omega)$ . Then we can take the limit in the system to obtain

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla(\operatorname{div} u) + \nabla \overline{p(\rho)} = \rho f$$

where  $\overline{p(\rho)}$  is the limit of  $p(\rho_n)$  in a sense to be precised. If we are looking at nonlinear weak stability, **the main difficulty is to prove that**

$$\overline{p(\rho)} = p(\rho). \quad (2.18)$$

If we want to describe a physical situation where possible density oscillations appear, then we have to characterize the limit. This is the subject of Section 3, where the derivation of two-phase averaged systems is explained.

Back to demonstrating nonlinear weak stability, we note that the pressure state law  $s \mapsto p(s)$  is a nonlinear function and we only have Lebesgue information on  $\{\rho_n\}_{n \in \mathbb{N}}$ . We have no control on the space derivative at all but we have a control on the time derivative  $\partial_t \rho_n$  thanks to the mass equation. We need thus to use a more complex tool to deduce compactness in space for the density sequence. The proof of (2.18) is made of two steps. The first one [S1] is to obtain better integrability on the density, and the second one [S2] is to really identify  $\overline{p(\rho)}$  with  $p(\rho)$ .

Proof of [S1]. The extra integrability on the density provides equi-integrability of the pressure and therefore the absence of concentration phenomena. It has been proved by [Lions, 1998] and [Feireisl et al., 2001] that it is possible to get the following extra integrability on the density:

$$\int_0^T \int_{\Omega} \rho_n^{\gamma + \omega} dx dy \leq C < +\infty$$

where

$$\omega = \frac{2\gamma}{d} - 1,$$

and therefore, since  $p(\rho_n) \in L^\infty(0, T; L^1(\Omega))$  and thanks to (2.15),

$$\overline{p(\rho)} \in L^{1+\varepsilon}((0, T) \times \Omega) \text{ for some } \varepsilon > 0 \text{ if } \gamma > d/2.$$

Proof of [S2]. Let us identify that  $\overline{p(\rho)} = p(\rho)$ . We also have the property on the effective flux for this limit equation because we can write

$$-\Delta \bar{F} = \operatorname{div} G \quad \text{with} \quad \bar{F} = (\lambda + 2\mu) \operatorname{div} u - \overline{p(\rho)} \quad (2.19)$$

and  $G$  is the same as in (2.16). Since  $p$  is a nonlinear function of  $\rho$ , we need to prove the strong convergence  $\rho_n \rightarrow \rho$  in  $L^1((0, T) \times \Omega)$ . To do so, we will use Lemma B.1 that helps to get compactness if we are able to compute the weak convergence with strictly convex functions. To apply this lemma, we must prove its hypothesis for  $f(s) = s \log s$ . As this is not an easy development, we refer the reader to [Bresch and Jabin, 2018a] for its details.

However, for completeness, we explain the procedure of steps [S1] and [S2] for a simpler system with a monotone pressure. We then give some insights of the more complex case of non-monotone pressure following [Bresch and Jabin, 2018a].

We consider the quasi-static Stokes equations with the pressure law  $p(\rho) = \rho^\gamma$ :

$$\begin{cases} \partial_t \rho_n + \operatorname{div}(\rho_n u_n) = 0, \\ -\mu \Delta u_n - (\lambda + \mu) \nabla \operatorname{div} u_n + \nabla \rho_n^\gamma = g, \\ \rho_n|_{t=0} = \rho_n^0 \end{cases} \quad (2.20)$$

with  $\gamma > 1$  and for a given function  $g \in L^{2\gamma}((0, T) \times \Omega)$  and an initial condition  $\rho_n^0$  which is assumed to be compact in space. Note that (2.15) is exactly satisfied.

Proof of [S1] with (2.20). This is how to get an extra estimate on the density, and hence on the pressure. The equivalent energy estimate of (2.12) is for such a system:

$$\frac{d}{dt} \int_{\Omega} \rho_n e(\rho_n) + \int_{\Omega} (\mu |\nabla u_n|^2 + (\lambda + \mu) |\operatorname{div} u_n|^2) = \int_{\Omega} g \cdot u$$

where  $e(\rho_n) = \rho_n^{\gamma-1}/(\gamma-1)$ . As we assumed  $g$  in  $L^{2\gamma}((0, T) \times \Omega)$ , we have an energy inequality control by Cauchy-Schwarz. Taking the divergence of the momentum equation, we get again the property on the effective flux, which is now written as

$$-\Delta F_n = \operatorname{div} g \quad \text{with} \quad F_n = (\lambda + 2\mu) \operatorname{div} u_n - \rho_n^\gamma, \quad (2.21)$$

and which may also be written as

$$-(\lambda + 2\mu) \operatorname{div} u_n + \rho_n^\gamma = \Delta^{-1} \operatorname{div} g.$$

The advantage of the relation above compared to the more complex case (2.19) is that here  $g$  is a given function and it does not depend on  $\rho$ , unlike  $u_n$  and  $G$  in (2.19).

We multiply the result by  $\rho_n^\gamma$  and integrate in space to get

$$\int_{\Omega} \rho_n^{2\gamma} = (\lambda + 2\mu) \int_{\Omega} \rho_n^\gamma \operatorname{div} u_n + \int_{\Omega} \rho_n^\gamma \Delta^{-1} \operatorname{div} g.$$

By Cauchy-Schwarz, we conclude that  $\rho_n \in L^{2\gamma}((0, T) \times \Omega)$  uniformly in  $n$  if  $\Delta^{-1} \operatorname{div} g \in L^{2\gamma}((0, T) \times \Omega)$  because due to the energy we know that  $\operatorname{div} u_n \in L^{2\gamma}((0, T) \times \Omega)$  uniformly in  $n$ . Note that we obtain a gain of integrability for the quasi-static compressible Stokes system.

The gain of integrability for the full compressible Navier–Stokes equations follows the same lines but the result is of lower order because of the quantity coming from the total time derivative  $\partial_t(\rho_n u_n) + \operatorname{div}(\rho_n u_n \otimes u_n)$  in  $G$  has to be treated. This calculation has been done by [Lions, 1998] and [Feireisl et al., 2001]. Furthermore, recalling that the compressible Navier–Stokes system is an hyperbolic/parabolic system, new findings are needed to treat the density because one cannot hope to use the Aubin-Lions-Simon Lemma for a nonlinear function of the density. In the work by [Lions, 1998] and [Feireisl et al., 2001] the authors present a tool to show this strong convergence, that is, the property of commutation between strictly convex function and weak convergence. For that, a monotonicity at least after a fixed value is needed for the pressure state law.

Proof of [S2]-(2.20). This second step consists in proving the hypothesis of Lemma B.1. We need to prove the weak convergence of the function  $f(\rho_n) = \rho_n \log \rho_n$  in  $L^1((0, T) \times \Omega)$ , that is, we must prove

$$\int_{\Omega} \rho \log \rho = \lim_n \int_{\Omega} \rho_n \log \rho_n = \int_{\Omega} \overline{\rho \log \rho}. \quad (2.22)$$

Recall that the following mass equations for  $\rho$  and  $\rho_n$  are, respectively:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\partial_t \rho_n + \operatorname{div}(\rho_n u_n) = 0$$

Applying the renormalization procedure –we divide each equation by its variable ( $\rho$  and  $\rho_n$ , respectively) and then combine it with the resulting equation– yields the following equations

$$\partial_t(\rho \log \rho) + \operatorname{div}(\rho u \log \rho) + \rho \operatorname{div} u = 0 \quad (2.23)$$

and

$$\partial_t(\rho_n \log \rho_n) + \operatorname{div}(\rho_n u_n \log \rho_n) + \rho_n \operatorname{div} u_n = 0$$

and, taking the limit,

$$\partial_t(\overline{\rho \log \rho}) + \operatorname{div}(\overline{\rho \log \rho} u) + \overline{\rho \operatorname{div} u} = 0 \quad (2.24)$$

where again  $\overline{f(v)}$  denotes the weak limit of  $f(v_n)$ . Subtracting (2.23) and (2.24) and integrating in space and time yields

$$\int_{\Omega} (\rho \log \rho - \overline{\rho \log \rho})(t) + \int_0^T \int_{\Omega} (\rho \operatorname{div} u - \overline{\rho \operatorname{div} u}) = \int_{\Omega} (\rho_0 \log \rho_0 - \overline{\rho_0 \log \rho_0}).$$

As for the incompressible case, we introduce the sequence  $\{\rho_n^0\}_{n \in \mathbb{N}}$  such that  $\rho_n^0 \rightarrow \rho_0$  in  $L^p(\Omega)$  for all  $0 < p < +\infty$  and  $\rho_n|_{t=0} = \rho_n^0$ . Let us assume compactness on the sequence  $\rho_n^0$  (see Remark 2.2 below), then

$$\int_{\Omega} (\rho_0 \log \rho_0 - \overline{\rho_0 \log \rho_0}) = 0$$

and consequently

$$\int_{\Omega} (\rho \log \rho - \overline{\rho \log \rho})(t) = - \int_0^T \int_{\Omega} (\rho \operatorname{div} u - \overline{\rho \operatorname{div} u}). \quad (2.25)$$

Applying Lemma B.2 to our case for the weak lower semi-continuity of the convex function  $\Phi(\rho) = \rho \log \rho$ , we know that

$$\int_{\Omega} \rho \log \rho(t) \leq \int_{\Omega} \overline{\rho \log \rho}(t). \quad (2.26)$$

Then, in order to get (2.22), it suffices to prove

$$\int_0^T \int_{\Omega} (\rho \operatorname{div} u - \overline{\rho \operatorname{div} u}) \leq 0 \quad (2.27)$$

because in this case, thanks to (2.25), we get

$$\int_{\Omega} \rho \log \rho(t) \geq \int_{\Omega} \overline{\rho \log \rho}(t),$$

which, together with (2.26), implies that the equality  $\int_{\Omega} \rho \log \rho(t) = \int_{\Omega} \overline{\rho \log \rho}(t)$  is ensured for all time  $t$ , and hence we have (2.22). Therefore, under (2.27), we have that (2.22) is proven and we are able to apply Lemma B.1 to get the compactness in space in  $L^1((0, T) \times \Omega)$  of the density, which gives us the convergence on the pressure law, thus ending the proof of [S2]-(2.20).

The remaining task is to prove (2.27). To do so, we use the property of the effective flux (2.21), which implies for the limit  $F = (\lambda + 2\mu) \operatorname{div} u - \overline{p(\rho)}$  the following equation

$$- \Delta F = \operatorname{div} g \quad (2.28)$$

Therefore for all  $\tau \in (0, T)$  and  $\phi \in \mathcal{D}(0, \tau)$ , we get

$$\lim_{n \rightarrow +\infty} \int_0^\tau \int_\Omega \phi(\rho_n p(\rho_n) - (\lambda + 2\mu)\rho_n \operatorname{div} u_n) = \int_0^\tau \int_\Omega \phi(\rho \overline{p(\rho)} - (\lambda + 2\mu)\rho \operatorname{div} u). \quad (2.29)$$

This property is obtained by testing the equation (2.21) by  $\varphi_n \phi$  with a test function  $\varphi_n = \Delta^{-1}(1_\Omega \rho_n)$  and testing (2.28) by  $\varphi \phi$  with a test function  $\varphi = \Delta^{-1}(1_\Omega \rho)$ , and subtracting the results. We use the relationship (2.29), which in some sense says that we can exchange  $\operatorname{div} u$  by  $\overline{p(\rho)}/(\lambda + 2\mu)$ . Then, proving (2.27) is equivalent to prove that we have the right sign on

$$\int_0^T \int_\Omega (\rho \overline{p(\rho)} - \overline{\rho p(\rho)}) \leq 0. \quad (2.30)$$

This property is ensured by Lemma B.4 for a pressure state law  $p$  that is a non-decreasing function, such as in our example  $p(\rho) = \rho^\gamma$ . This completes the proof for the quasi-static Stokes (2.20) equations with the pressure law  $p(\rho) = \rho^\gamma$ .

**Remark 2.2.** *Importantly, we assume here compactness on the initial density sequence  $\{\rho_n^0\}_{n \in \mathbb{N}}$ . There are physical situations in which this is not the case, such as if we want an averaged system made of several compressible phases. We will discuss this issue when deriving a multiphase (averaged) system starting from a compressible system and high-oscillating data. If oscillations occur on the initial density, then  $\overline{p(\rho)}$  is not  $p(\rho)$  and the result will depend on the measure family associated to the initial density (see Section 4 for a concrete example).*

**Key insight 4.** For the compressible Navier–Stokes equations with constant viscosities and a non-decreasing pressure state law that depends on the density, we conclude on the stability of  $\rho_n u_n$  and  $\rho_n u_n \otimes u_n$  in a similar way than for the density-dependent incompressible Navier–Stokes equations. The main remaining difficulty is to show that  $p(\rho_n)$  converges in some sense to  $p(\rho)$ . To do so, we know that

$$\partial_t \rho_n + \operatorname{div}(\rho_n u_n) = 0, \quad \partial_t \rho + \operatorname{div}(\rho u) = 0.$$

We also have

$$\partial_t(\rho_n u_n) + \operatorname{div}(\rho_n u_n \otimes u_n) - \mu \Delta u_n - (\lambda + 2\mu) \nabla \operatorname{div} u_n + \nabla p(\rho_n) = 0,$$

and

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + 2\mu) \nabla \operatorname{div} u + \nabla \overline{p(\rho)} = 0,$$

with  $\overline{p(\rho)}$  the weak limit of  $p(\rho_n)$ . We can thus show first that  $p(\rho_n) \in L^s((0, T) \times \Omega)$  uniformly with  $s > 1$ . The main difficulty lies in proving that  $\overline{p(\rho)} = p(\rho)$ . This is done by using a compactness Lemma involving commutation between a weak limit and a strictly convex function that is defined by working with the equations written above. We then use an important property on the effective flux showing that we control the space derivatives of  $(\lambda + 2\mu) \operatorname{div} u_n - p(\rho_n)$ , which, formally, means that  $\operatorname{div} u_n$  behaves as  $p(\rho_n)/(\lambda + 2\mu)$ . The proof can be concluded only if the pressure state law  $s \mapsto p(s)$  is non decreasing. To design appropriate numerical schemes, it is important that the approximate system satisfies at the discrete level properties such as that on the effective flux. For compressible Navier–Stokes equations, this has been for instance studied by [Gallouët, 2018].

**Insights into the non-monotone pressure case.** When the pressure law is non monotone and satisfies (2.15), we are not able to prove (2.30) because Lemma B.4 cannot be used, which in turn causes us to fail proving (2.27) and, consequently, [S2]. To address this complex case, a new tool has been introduced by [Bresch and Jabin, 2018a]. It is based on a non-local tool with the use of appropriate dynamical weights: a semi-norm that includes critical regularity issues for the compressible Navier–Stokes system in bounded domains with values



of the semi-norm in different places of the domain.

The non-monotone pressure case is an open problem in general but it is possible to address it using this non-local tool when the diffusion terms are defined as follows

$$\mathcal{D} = -\operatorname{div}(A(t)\nabla u) - (\mu + \lambda)\nabla\operatorname{div}u \quad (2.31)$$

where

$$A(t) = \mu\operatorname{Id} + (\delta A)(t)$$

with  $\mu > 0$  and  $2\mu/d + \lambda - \|\delta A\|_{L^\infty} > 0$ . Assuming

$$\gamma > d/2 \left( (1 + 1/d) + \sqrt{1 + 1/d^2} \right),$$

there exists  $C_\star$  such that if

$$\|\delta A\|_{L^\infty} \leq C_\star(2\mu + \lambda)$$

there exists global weak solution of the system for the compressible Navier–Stokes equations with the diffusion given by (2.31) and  $p(\rho) = a\rho^\gamma$  with  $\gamma \geq 3d/(d + 2)$ .

The study of [Bresch and Jabin, 2018a] also treats the case where the viscosity may be not the same in different directions, that is, for **anisotropic materials**. This extension is possible because this case entails the same problem as that of non monotone pressure, so it can be addressed with the same tools. To highlight the key points of the [Bresch and Jabin, 2018a] result, let us illustrate the anisotropic case by considering the following compressible momentum equation

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu_x \Delta_x u - \mu_z \partial_z^2 u + \nabla p(\rho) = 0,$$

where  $\mu_x, \mu_z$  are the viscosities in the respective directions. If we apply the divergence on this equation, we get again the effective flux property

$$-\Delta_\mu F_\mu = \operatorname{div}(\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u)) \quad \text{with} \quad F_\mu = \operatorname{div}u + (\Delta_\mu)^{-1}\Delta p(\rho),$$

where we denoted the operator  $\Delta_\mu f = \mu_x \Delta_x f + \mu_z \partial_z^2 f$ . Therefore,  $\operatorname{div} u$  is linked to the pressure quantity in a non-local fashion. If we follow the development presented before, we find an issue in step [S2]; we are not able to provide a sign to the quantity

$$\int_0^T \int_\Omega \left( \rho \overline{\mathcal{A}p(\rho)} - \overline{\rho \mathcal{A}p(\rho)} \right)$$

with  $\mathcal{A} = (\Delta_\mu)^{-1}\Delta$  which is a zero order operator. Note that the sign of this quantity coincides with the sign of  $\int_0^T \int_\Omega (\rho \operatorname{div}u - \overline{\rho \operatorname{div}u})$  in (2.27) thanks to the effective flux property. Even if the pressure is monotone, this information is lost for  $\mathcal{A}p(\rho)$ , leaving us with the same problem as in the case a non-monotone pressure. This is a non-local quantity and this explains in some sense why the non-local method introduced by [Bresch and Jabin, 2018a] has also been able to treat viscous anisotropy in the system.

To treat this viscous anisotropy, the authors adapted the non-local compactness tool given by Lemma B.3. The line of reasoning is that if we were able to apply Lemma B.3, we would get the compactness of  $\rho_n$  in  $L^1((0, T) \times \Omega)$  and we could conclude [S2]. For this to work, we must prove the hypothesis in Lemma B.3 for  $p = 1$ . We already have  $\partial_t \rho_n \in L^1(0, T; W^{-1,1}(\Omega))$ , which is one of the required hypothesis. We thus only have to prove that

$$R_{h_0} = \limsup_n \left( \frac{1}{\|\mathcal{K}_{h_0}\|_{L^1}} \int_{\Omega^2} \mathcal{K}_{h_0}(x - y) |\rho_n(t, x) - \rho_n(t, y)| dx dy \right) \rightarrow 0 \quad \text{as} \quad h \rightarrow 0 \quad (2.32)$$

for  $\mathcal{K}_{h_0}$  a Kernel satisfying some particular hypothesis.

To prove (2.32), let us first show that the propagation of the intermediate regularity (namely non regularity on  $\rho$  but regularity on  $u$ ) given in Lemma B.3 is satisfied when the velocity field is regular and when we have a control on its derivative, that is, if  $\operatorname{div} u \in L^1(0, T; L^\infty(\Omega))$  then the density is propagated regularly. However, from the energy of the system we only have  $u \in L^2(0, T; H^1(\Omega))$ , so  $\operatorname{div} u$  may blow up (i.e.  $\operatorname{div} u$  may tend to  $+\infty$ , thus creating a concentration), which leaves us unable to prove this requirement of Lemma B.3. This obstacle can be lifted by looking at how the quantity  $\int_{\Omega^2} \mathcal{K}_{h_0}(x-y) |\rho_k(t, x) - \rho_k(t, y)| dx dy$  is propagated in time, where  $\mathcal{K}_{h_0}$  is a Kernel defined later on. For simplicity, we use the shorthand  $g_k^x = g_k(t, x)$  and  $g_k^y = g_k(t, y)$  for any function  $g_k$ . We start with the equations

$$\partial_t \rho_k^x + \operatorname{div}_x(\rho_k^x u_k^x) = 0$$

and

$$\partial_t \rho_k^y + \operatorname{div}_y(\rho_k^y u_k^y) = 0.$$

Subtracting these two equations and using the renormalization technique, we can write

$$\partial_t |\rho_k^x - \rho_k^y| + \operatorname{div}_x(u_k^x |\rho_k^x - \rho_k^y|) + \operatorname{div}_y(u_k^y |\rho_k^x - \rho_k^y|) = \frac{1}{2} (\operatorname{div}_x u_k^x + \operatorname{div}_y u_k^y) |\rho_k^x - \rho_k^y| - \frac{1}{2} (\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y) (\rho_k^x + \rho_k^y) s_k$$

where  $s_k = \operatorname{sign}(\rho_k^x - \rho_k^y)$ . We now multiply this equation by  $\mathcal{K}_{h_0}(x-y)$  and integrate it with respect to  $x$  and  $y$  over  $\Omega^2$ . Defining  $R_{h_0} = \int_{\Omega^2} (\mathcal{K}_{h_0}(x-y) |\rho_k^x - \rho_k^y|) dx dy$ , we get

$$\begin{aligned} \partial_t R_{h_0} &= \int_{\Omega^2} \nabla \mathcal{K}_{h_0}(x-y) (u_k^x - u_k^y) |\rho_k^x - \rho_k^y| \\ &\quad - \int_{\Omega^2} \mathcal{K}_{h_0}(x-y) (\operatorname{div}_x u_k^x - \operatorname{div}_y u_k^y) \rho_k^x s_k + \int_{\Omega^2} \mathcal{K}_{h_0}(x-y) |\rho_k^x - \rho_k^y| \operatorname{div}_x u_k^x. \end{aligned}$$

This gives the extra quantities we have to bound to be able to use a Grönwall Lemma. Such expression shows that the right-hand side is not simply comparable to  $R_{h_0}$  without assuming some  $L^\infty$  bounds in space on  $\operatorname{div} u_k$  (for the last two terms) and at least (for the first term) on  $M(|\nabla u_k|)$ , which is the maximal function of  $|\nabla u_k|$ . The maximal function is defined as follows

$$M(f)(x) = \sup_{r \leq 1} \frac{1}{|B(0, r)|} \int_{B(0, r)} f(x+z) dz.$$

Note that

$$\nabla \mathcal{K}_{h_0}(x-y) |u_k^x - u_k^y| \leq C \mathcal{K}_{h_0}(x-y) (D_{|x-y|} u(x) + D_{|x-y|} u(y))$$

where the square function is defined by

$$D_h u(x) = \frac{1}{h} \int_{|z| \leq h} \frac{\nabla u(x+z)}{|z|^{d-1}} dz.$$

Then, since we do not have enough regularity on  $\operatorname{div} u$ , we must find other way to demonstrate that the hypothesis (2.32) is valid to apply Lemma B.3. In next lines we show how this problem has been tackled in [Bresch and Jabin, 2018a]. The main idea is to introduce dynamical weights  $w_n$  in  $R_{h_0}$  considering

$$R_{h_0, w} = \limsup_n \left( \frac{1}{\|\mathcal{K}_{h_0}\|_{L^1}} \int_{\Omega^2} \mathcal{K}_{h_0}(x-y) |\rho_n^x - \rho_n^y| (w_n^x + w_n^y) dx dy \right)$$

where the kernel  $\mathcal{K}_{h_0}$  is defined by

$$\mathcal{K}_{h_0} = \int_{h_0}^{+\infty} \mathcal{K}_h \frac{dh}{h}$$

with  $\mathcal{K}_h$  as in Lemma B.3. Let us see how this quantity help us to prove that  $R_{h_0} \rightarrow 0$  as  $h_0 \rightarrow 0$ .

The dynamical weight  $w_n$  is assumed to satisfy the following equation

$$\partial_t w_n + u_k \cdot \nabla w_n + \lambda P_n w_n = 0, \quad w_n|_{t=0} = 1 \quad (2.33)$$

with  $\lambda$  a parameter which is chosen sufficiently large and  $P_n \geq 0$  an appropriate damping quantity linked to the unknown  $\rho_n$  and  $u_n$ . The dynamical weights can be viewed as passive tracers in the flow that track the regularity on  $\operatorname{div} u_n$ ,  $\nabla u_n$  and  $\rho_n^\gamma$ . To ensure this behavior for compressible Navier–Stokes equations, the damping term  $P_n$  is chosen as:

$$P_n = |\operatorname{div} u_n| + M |\nabla u_n| + \rho_n^\gamma.$$

Note that a comparison between the square function and the maximal function is used namely we know that there exists  $C > 0$ , for any  $u \in W^{1,p}(\mathbb{T}^d)$  with  $p \leq 1$  such that

$$D_h u(x) \leq CM(|\nabla u|)(x).$$

First, as  $P_n$  is a damping term, since we start with  $w_n|_{t=0} = 1$ , this ensures that

$$0 \leq w_n \leq 1.$$

Assuming  $P_n \rho_n \in L^1((0, T) \times \Omega)$  uniformly, then dividing the equation (2.33) by  $w_n$ , multiplying by  $\rho_n$  and integrating we conclude that

$$\int_{\Omega} \rho_n |\log w_n| \leq C < +\infty \quad (2.34)$$

uniformly with respect to  $n$ . These properties will be used to get rid of the weights. Note that it is possible to consider less rigid assumptions on  $P_n$  by relaxing the property on  $w_n$  but we choose the stronger assumption to simplify the calculations in this sketch of the proof. This is how to get rid of the weights and stay convergent to zero as  $h_0 \rightarrow 0$ . We decompose

$$\Omega_\eta^2 = \{(x, y) : w_n^x \geq \eta \quad \text{or} \quad w_n^y \geq \eta\}$$

and

$$(\Omega_\eta^2)^c = \{(x, y) : w_n^x \leq \eta \quad \text{and} \quad w_n^y \leq \eta\}.$$

The integral of  $R_{h_0}$  in (2.32) can thus be decomposed in  $\Omega_\eta^2$  and  $(\Omega_\eta^2)^c$ . We then use the previous properties related to  $w_n$ . We look first at the integral corresponding to  $\Omega_\eta^2$ ,

$$\begin{aligned} (R_{h_0})_1 &= \frac{1}{\|\mathcal{K}_{h_0}\|_{L^1}} \int_{\Omega_\eta^2} \mathcal{K}_{h_0}(x-y) |\rho_n^x - \rho_n^y| dx dy \\ &\leq \frac{1}{\|\mathcal{K}_{h_0}\|_{L^1}} \int_{\Omega_\eta^2} \mathcal{K}_{h_0}(x-y) |\rho_n^x - \rho_n^y| \frac{w_n^x}{\eta} dx dy \\ &\leq \frac{1}{\|\mathcal{K}_{h_0}\|_{L^1}} \int_{\Omega_\eta^2} \mathcal{K}_{h_0}(x-y) |\rho_n^x - \rho_n^y| \frac{w_n^x + w_n^y}{\eta} dx dy \\ &\leq \frac{R_{h_0, w}}{\eta} \end{aligned}$$

The integral corresponding to  $(\Omega_\eta^2)^c$  yields

$$\begin{aligned}
(R_{h_0})_2 &= \frac{1}{\|\mathcal{K}_{h_0}\|_{L^1}} \int_{(\Omega_\eta^2)^c} \mathcal{K}_{h_0}(x-y) |\rho_n^x - \rho_n^y| dx dy \\
&\leq \frac{1}{\|\mathcal{K}_{h_0}\|_{L^1}} \int_{(\Omega_\eta^2)^c} \mathcal{K}_{h_0}(x-y) \rho_n^x \frac{|\log w_n^x|}{|\log \eta|} dx dy \\
&\leq \frac{1}{|\log \eta|} \int_{(\Omega_\eta^2)^c} \rho_n^x |\log w_n^x| dx dy \\
&\leq \frac{C}{|\log \eta|}
\end{aligned}$$

where in the last inequality we used (2.34). Thus  $R_{h_0} = (R_{h_0})_1 + (R_{h_0})_2$  satisfies the following relation

$$R_{h_0} \leq C \left( \frac{R_{h_0,w}}{\eta} + \frac{1}{|\log \eta|} \right)$$

where the weights are only present in  $R_{h_0,w}$ . To get the convergence of  $R_{h_0}$  to zero when  $h_0 \rightarrow 0$ , it suffices to choose  $\eta$  in terms of  $h_0$  such that  $\eta$  converges to zero slower than  $R_{h_0,w}$  does.

The flexibility of this compactness tool is that we have to prove a convergence to 0 when  $h_0 \rightarrow 0$ . Thus we can start from the quantity  $R_{h_0,w}$  with the appropriate weights to prove convergence to zero and thus enabling us to apply Lemma B.3. We then prove some properties on the weight to show that we can get rid of them while conserving the convergence to zero.

**Remark 2.3.** *It seems impossible to address the case of density dependent viscosities with such mathematical tools. It necessitates another set of tools, which is the subject of the next section.*

**Key insight 5.** If the pressure state law is not monotone or anisotropic viscosities are involved, the tool related to a strictly convex function to prove compactness on  $\rho_n$  seems to be inappropriate. To conclude stability, the idea is to introduce a quantitative non-local tool composed of dynamical weights that act as passive tracer monitoring the behavior of quantities of interest such as  $\operatorname{div} u$ . To find such weights we have to solve a PDE with two parts: a transport quantity (in some sense they follow the flow) and a damping quantity to be chosen depending on the problem (in some sense this quantity will cause the weights to vanish if the quantity of interest blows up into a concentration) because the final step will be to get rid of the weights to conclude compactness. With this tool it is possible that divergent behavior of  $\rho^\gamma$ ,  $M(|\nabla u_n|)$ , and  $|\operatorname{div} u_n|$  occurs, but such concentrations are restricted to small sets. Such a compactness tool is helpful when dealing with PDEs with non-local quantities.

## 2.2.2 Density dependent viscosities

If one assumes that the viscosity depends on the density, the property on the effective flux we described previous Section cannot be obtained. This is because taking the divergence of the momentum equation yields a commutator related to the density. The density is not *a priori* regular and even the energy estimate does not provide bounds on  $\nabla \rho$  so we cannot bound the commutator. The reader interested in the compressible Navier–Stokes equations with density-dependent viscosities may consult F. Rousset’s presentation (in French) at the Bourbaki seminar [Rousset, 2017] as well as [Bresch et al., 2018] and [Bresch and Jabin, 2020].

We consider the following system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - 2\operatorname{div}(\mu(\rho)D(u)) - \nabla(\lambda(\rho)\operatorname{div}u) + \nabla p(\rho) = 0, \\ \rho u|_{t=0} = m_0, \quad \rho|_{t=0} = \rho_0 \end{cases} \quad (2.35)$$

with  $s \mapsto p(s) = as^\gamma$  where  $a > 0$  and  $\gamma > 1$  are given constants. The energy conservation reads

$$\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \rho e(\rho) \right) + 2 \int_{\Omega} \mu(\rho) \left| D(u) - \frac{1}{d} \operatorname{div}u \operatorname{Id} \right|^2 + \int_{\Omega} \left( \lambda(\rho) + \frac{2\mu(\rho)}{d} \right) |\operatorname{div}u|^2 = 0 \quad (2.36)$$

where  $e(\rho) = \int_0^\rho p(s)/s^2 ds$ . The case of density-dependent viscosities has attracted attention since the discovery by D. Bresch and B. Desjardins in [Bresch and Desjardins, 2004] of a new mathematical entropy (now called BD entropy) in dimension 2 or 3 that provides an a priori estimate on the gradient of a density function. Let us first assume a relationship between shear viscosity  $\mu(\rho)$  and bulk viscosity  $\lambda(\rho)$ :

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)). \quad (2.37)$$

This is how the BD entropy is formulated. To get such an extra estimate on  $\nabla \rho$ , one needs to recognize that if  $\rho$  satisfies the mass equation

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

then  $\mu(\rho)$  satisfies the equation

$$2\partial_t \mu(\rho) + 2\operatorname{div}(\mu(\rho)u) + \lambda(\rho)\operatorname{div}u = 0$$

and therefore taking the gradient of this relation yields

$$2\partial_t(\nabla(\mu(\rho))) + 2\operatorname{div}((\nabla\mu(\rho)) \otimes u) + 2\operatorname{div}(\mu(\rho)\nabla^t u) + \nabla(\lambda(\rho)\operatorname{div}u) = 0.$$

Denoting  $\varphi'(\rho) = \mu'(\rho)/\rho$  and adding the result to the momentum equation in (2.35) we get

$$\partial_t(\rho(u + 2\nabla\varphi(\rho))) + \operatorname{div}(\rho(u + 2\nabla\varphi(\rho)) \otimes u) - 2\operatorname{div}(\mu(\rho)A(u)) + \nabla p(\rho) = 0 \quad (2.38)$$

where  $A(u) = (\nabla u - \nabla^t u)/2$ . Taking the scalar product of this identity with  $u + 2\nabla\varphi(\rho)$  and integrating with respect to the space variable, we get

$$\frac{d}{dt} \int_{\Omega} (\rho|u + 2\nabla\varphi|^2 + \rho e(\rho)) + 2 \int_{\Omega} \mu(\rho)|A(u)|^2 + 2 \int_{\Omega} \rho'(\rho)\varphi'(\rho) \frac{|\nabla\rho|^2}{\rho} = 0. \quad (2.39)$$

This is the new conserved quantity that has been identified by [Bresch and Desjardins, 2004] and which is now called the BD entropy. This identity is complementary to the energy identity (2.36). It will provide uniform bounds on the gradient of the density sequence  $\{\rho_n\}_{n \in \mathbb{N}}$  and a bound on  $\sqrt{\mu(\rho_n)}A(u_n)$  uniformly in  $L^2((0, T) \times \Omega)$ .

**Remark 2.4** (a degenerate Parabolic/Hyperbolic system). *It is important to specify which types of shear viscosity  $\mu(\rho)$  and bulk viscosity  $\lambda(\rho)$  are appropriate because to get the uniform bounds we have to ensure that*

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)), \quad \mu(\rho) \geq 0, \quad \lambda(\rho) + 2\mu(\rho)/d \geq \varepsilon\mu(\rho) \quad \text{for some } \varepsilon > 0.$$

*The second and third conditions ensure the control of the derivatives in the energy relation (2.36). Assuming  $\mu(\rho) = \rho^\alpha$ , for  $\alpha$  a constant, this implies that  $\lambda(\rho) = 2(\alpha - 1)\rho^\alpha$  and therefore we need that*

$$2(\alpha - 1) + 2/d > \varepsilon > 0.$$

This implies that  $\alpha > 1 - 1/d > 0$  and therefore the viscosity  $\mu(\rho)$  vanishes when  $\rho$  vanishes. Note that a more general calculus can be made by replacing  $\lambda(\rho)$  by  $2(\mu'(\rho)\rho - \mu(\rho))$  in the inequality

$$\lambda(\rho) + 2\mu(\rho)/d \geq \varepsilon\mu(\rho)$$

to obtain

$$\frac{\mu'(\rho)}{\mu(\rho)} \geq \frac{d-1+\varepsilon}{\rho d} \text{ for all } \rho > 0$$

and therefore for  $\rho \leq 1$ , we get

$$\mu(\rho) \leq C\rho^{(d-1+\varepsilon)/d}.$$

This indicates that we have to treat a degenerate hyperbolic/parabolic system (i.e. a system where the shear viscosity vanishes when  $\rho$  vanishes). There is also an additional constraint in such a system because the velocity field is not necessarily defined when the density vanishes. This requires to understand what are the appropriate quantities to involve into the calculation. Note that the dual formulation to take the limit in  $\rho_n u_n$  and  $\rho_n u_n \otimes u_n$  cannot be used here (unlike in the case of constant viscosities) because we loose the estimates  $u_n \in L^2(0, T; H^1(\Omega))$ .

Now equipped with the concept of BD entropy, let us focus on the **nonlinear weak stability**. For a sequence  $\{\rho_n, u_n\}_{n \in \mathbb{N}}$  satisfying (2.35) in a distributional sense, we start with the two uniform bounds from the energy and the BD entropy,

$$\begin{aligned} \sup_{t \in (0, T)} \frac{1}{2} \int_{\Omega} (\rho_n |u_n|^2 + \rho_n e(\rho_n))(t) + 2 \int_0^T \int_{\Omega} \mu(\rho_n) |D(u_n) - \frac{1}{d} \operatorname{div} u_n \operatorname{Id}|^2 \\ + \int_0^T \int_{\Omega} (\lambda(\rho_n) + \frac{\mu(\rho_n)}{\rho_n}) |\operatorname{div} u_n|^2 \leq c < +\infty \end{aligned} \quad (2.40)$$

and

$$\begin{aligned} \sup_{t \in (0, T)} \int_{\Omega} (\rho_n |u_n + 2\nabla \varphi(\rho_n)|^2 + \rho_n e(\rho_n))(t) + 2 \int_0^T \int_{\Omega} \mu(\rho_n) |A(u_n)|^2 \\ + 2 \int_0^T \int_{\Omega} p'(\rho_n) \varphi'(\rho_n) \frac{|\nabla \rho_n|^2}{\rho_n} \leq c < +\infty. \end{aligned} \quad (2.41)$$

The reasoning step is to prove the nonlinear weak stability by knowing that we obtain the control of space derivatives for the density. This allows to take the limit in a simple way for the pressure  $p(\rho_n)$  using the Aubin-Lions-Simon Lemma: as we did in the previous sections, we have the control on the space derivative of a function of the density and the control on the time derivative of the density using the mass equation. The difficulty lies as usual on the terms involving the velocity, namely  $\rho_n u_n$ ,  $\rho_n u_n \otimes u_n$  and  $\mu(\rho_n) D(u_n)$ ,  $\lambda(\rho_n) \operatorname{div} u_n$ . Since this is a complex task, we focus on a **didactic example with given viscosities** by considering the simple case  $\mu(\rho_n) = \rho_n$  that gives  $\lambda(\rho_n) = 2(\mu'(\rho_n)\rho_n - \mu(\rho_n)) = 0$ . Our task is to prove the controls of  $\rho_n u_n$ ,  $\rho_n u_n \otimes u_n$  and  $\rho_n D(u_n)$ . Taking first the term  $\rho_n u_n$ , we note that

$$\nabla(\rho_n u_n) = \sqrt{\rho_n} \nabla(u_n \sqrt{\rho_n}) + \sqrt{\rho_n} u_n \cdot \nabla \sqrt{\rho_n}$$

and therefore it belongs to  $L^2(0, T; L^2(\Omega))$  uniformly. Since  $\rho_n u_n \in L^\infty(0, T; L^1(\Omega))$ , we get the following control:  $\rho_n u_n \in L^2(0, T; W^{1,1}(\Omega))$ . Looking at the mass balance and using the estimates, we can prove that  $\partial_t(\rho_n u_n)$  is bounded in  $L^2(0, T; W^{-2,4/3}(\Omega))$  uniformly. Therefore, we get from the Aubin-Lions-Simon Lemma the compactness of  $\rho_n u_n$  in  $L^2(0, T; L^p(\Omega))$  for all  $p \in [1, 3/2]$ . We will see below how to get the convergence on  $\rho_n u_n \otimes u_n$  by showing that  $\sqrt{\rho_n} u_n \rightarrow \sqrt{\rho} u$  in  $L^2((0, T) \times \Omega)$ .

Focusing now on the term  $\rho_n D(u_n)$ , since the velocity field alone is not so easily defined, it is better to write the momentum equation in a more convenient form. Noting that

$$\rho_n \partial_i u_n = \partial_i(\rho_n u_n) - 2\sqrt{\rho_n} u_n \partial_i(\sqrt{\rho_n}),$$

we use it in the term  $\operatorname{div}(\rho_n D(u_n))$  to write the momentum equation as:

$$\partial_t(\rho_n u_n) + \operatorname{div}(\rho_n u_n \otimes u_n) - \Delta(\rho_n u_n) - \nabla \operatorname{div}(\rho_n u_n) + 2\operatorname{div}(\nabla \sqrt{\rho_n} \otimes \sqrt{\rho_n} u_n) + 2\operatorname{div}(\sqrt{\rho_n} u_n \otimes \nabla \sqrt{\rho_n}) + \nabla p(\rho_n) = 0.$$

Due to the nonlinearity, we see that the most important issue is to be able to show that  $\sqrt{\rho_n} u_n \rightarrow \sqrt{\rho} u$  in  $L^2((0, T) \times \Omega)$  strongly because of the weak convergence in  $L^2((0, T) \times \Omega)$  for the complementary quantity  $\sqrt{\rho_n}$ . In other words, one finds that  $\sqrt{\rho_n} \in L^2((0, T); H^1(\Omega))$  by using the estimates given by (2.40) and (2.41) with  $\varphi(\rho_n) = \log \rho_n$  and the mass conservation. The simplest way to prove such a convergence is to be able to prove the equi-integrability of  $|u_n|^2 \in L^1_{\rho_n}((0, T) \times \Omega)$ , for instance by proving the existence of extra information on  $\rho(1 + |u|^2) \ln(1 + |u|^2)$  as proposed in [Mellet and Vasseur, 2007]. This can be done by using the bounds given by (2.40) and (2.41), which allows us to prove that

$$\sup_{t \in (0, T)} \int \rho_n (1 + |u_n|^2) \ln(1 + |u_n|^2) \leq C < +\infty$$

if it is initially the case for  $\rho_n^0$ . Such a control is enough to prove the strong convergence of  $\sqrt{\rho_n} u_n$  in  $L^2((0, T) \times \Omega)$  by using the strong convergence of  $\rho_n u_n$  in  $L^2(0, T; L^p(\Omega))$  for all  $p \in [1, 3/2]$  and the strong convergence of  $\rho_n$ . Let us show now this convergence of  $\sqrt{\rho_n} u_n$  in  $L^2((0, T) \times \Omega)$ .

We know that  $m_n = \rho_n u_n$  and  $\rho_n$  converge almost everywhere. It is readily seen that in  $\{\rho(t, x) \neq 0\}$  we have  $\sqrt{\rho_n} u_n = m_n / \sqrt{\rho_n}$  converging almost everywhere to  $\sqrt{\rho} u = m / \sqrt{\rho}$ . We also have that

$$\sqrt{\rho_n} u_n 1_{|u_n| \leq M} \rightarrow \sqrt{\rho} u 1_{|u| \leq M} \text{ almost everywhere for } M > 0.$$

So we have the convergence almost everywhere in  $\{\rho(t, x) \neq 0\}$ , and in  $\{\rho(t, x) = 0\}$  we have  $\sqrt{\rho_n} u_n 1_{|u_n| \leq M} \leq M \sqrt{\rho_n} \rightarrow 0$ . To complete the proof, for  $M > 0$ , we write

$$\int |\sqrt{\rho_n} u_n - \sqrt{\rho} u|^2 \leq \int |\sqrt{\rho_n} u_n 1_{|u_n| \leq M} - \sqrt{\rho} u 1_{|u| \leq M}|^2 + 2 \int |\sqrt{\rho_n} u_n 1_{|u_n| \geq M}|^2 + 2 \int |\sqrt{\rho} u 1_{|u| \geq M}|^2.$$

It is obvious that  $\sqrt{\rho_n} u_n 1_{|u_n| \leq M}$  is bounded uniformly in  $L^\infty(0, T; L^3(\Omega))$  and the "almost everywhere" convergence provides that

$$\int |\sqrt{\rho_n} u_n 1_{|u_n| \leq M} - \sqrt{\rho} u 1_{|u| \leq M}|^2 \rightarrow 0.$$

Finally we write

$$\int |\sqrt{\rho_n} u_n|^2 1_{|u_n| \geq M} \leq \frac{1}{\ln(1 + M^2)} \int \rho_n u_n^2 \ln(1 + |u_n|^2) \leq \frac{C}{\ln(1 + M^2)}$$

and the same for the last term  $\int |\sqrt{\rho} u|^2 1_{|u| \geq M}$  for all  $M > 0$ . The strong convergence of  $\sqrt{\rho_n} u_n$  in  $L^2((0, T) \times \Omega)$  is found by taking  $M \rightarrow +\infty$ .

**Remark 2.5.** *This extra information is not really needed if one uses a more general approach introduced first by [Vasseur and Yu, 2016] and then by [Lacroix-Violet and Vasseur, 2018] that may be seen as a renormalized method for the velocity. Usually, the renormalized technique has been used for the density in the mass equation. We do not give details on this method because it is beyond the scope of this review but we refer readers to [Bresch and Jabin, 2018b] and references therein.*

**Remark 2.6.** Several remarks can be made at this stage for mathematicians interested in theoretical investigations:

— The genesis of this result stems from the work of [Bresch et al., 2003] on Navier–Stokes–Korteweg equations (where the shear viscosity was taken as being equal to the density, which implies zero bulk viscosity).

— Such a gain on the gradient of a density function was highlighted in dimension 1 by [Kanel, 1979]. This gain in regularity in density is at the cost of a loss in regularity on the speed close to vacuum conditions. One then switches to degenerate hyperbolic-parabolic systems where a nonlinear hypocoercivity structure exists. Note that the hypocoercivity at the center of studies such as those of [Matsumura and Nishida, 1980] or [Danchin, 2001] corresponds in a way to the linearization of the nonlinear hypocoercivity property that we have highlighted.

— A relationship similar to the one we introduced in [Bresch and Desjardins, 2004] between the shear viscosity and the bulk viscosity is found when considering the Euler–Korteweg system and rewriting the dispersive term as a diffusion operator applied to the gradient of a rho-function. This rho-function has a link with the surface tension coefficient chosen.

After more than ten years of international research, it has been found in [Bresch et al., 2015] that the compressible Navier–Stokes system with density-dependent viscosities satisfying the relation (2.37) can be reformulated as an augmented system (via what they called a  $\kappa$  entropy where  $\kappa$  is a parameter between 0 and 1). This reformulation explains how the additional control on the density is related to a two-phase averaged structure of the compressible Navier–Stokes system when there is a strong density gradient, which veers towards questions related to the presence of two velocities [Brenner, 2005, Brenner, 2006]. Interestingly, we find a concept of effective and drift velocities due to the strong density gradient that are in line with the works of [Einstein and Fürth, 1956, Nelson, 1967]. In one of the most recent paper written on the subject [Bresch et al., 2022b], the following hypotheses are used to get the **existence of global weak solutions of the compressible Navier–Stokes equations**. For the shear and volume viscosities satisfying (2.37), we consider

$$\mu \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R}_+) \cap \mathcal{C}^2(\mathbb{R}_+^*; \mathbb{R})$$

where  $\mathbb{R}_+ = [0, +\infty)$  and  $\mathbb{R}_+^* = (0, +\infty)$ . It is also assumed that there exists two positive numbers  $\alpha_1, \alpha_2$  such that

$$2/3 < \alpha_1 < \alpha_2 < 4$$

$$\text{and that, for any } \rho > 0, \quad 0 < \rho\mu'(\rho)/\alpha_2 \leq \mu(\rho) \leq \rho\mu'(\rho)/\alpha_1$$

and there exists a constant  $C > 0$  such that

$$|\rho\mu''(\rho)/\mu'(\rho)| \leq C < +\infty.$$

Note that the upper bound  $\alpha_2 < 4$  may be relaxed as indicated in [Alazard and Bresch, 2020]. The study of [Bresch et al., 2022b] is based on the following interesting inequality. There exists  $C > 0$  independent of  $\alpha$  and  $\varepsilon > 0$  such that

$$\begin{aligned} +\infty > \frac{C}{\varepsilon} \int_{\mathbb{T}^d} \rho^\alpha |\nabla^2 \rho^{\alpha-1}|^2 dx &\geq \frac{4}{(3\alpha-2)^2} \int_{\mathbb{T}^d} |\nabla^2 \rho^{(3\alpha-2)/2}|^2 dx \\ &+ \left( \frac{1}{\alpha} - \frac{1}{4} - \varepsilon \right) \frac{4^4}{(3\alpha-2)^4} \int_{\mathbb{T}^d} |\nabla \rho^{(3\alpha-2)/4}|^4 dx \end{aligned}$$

The authors assume  $\alpha < 4$  to have two positive quantities in the right-hand side. An idea to cover a larger range of power is based on the following uniform inequality: for any  $d \geq 1$  and any positive function  $\theta \in H^2(\mathbb{T}^d)$ ,

$$\int_{\mathbb{T}^d} |\nabla \theta^{1/2}|^4 dx \leq \frac{9}{16} \int_{\mathbb{T}^d} (\Delta \theta)^2 dx.$$



**Remark 2.7** (Important applications from the density dependent case introduced in the multi-dimensional space case by D. Bresch and B. Desjardins). *Being able to treat density-dependent viscosities in the compressible Navier–Stokes equations have had many important physical applications. This could motivate researchers to try opening the way for other more complicated dependencies.*

**Key insight 6.** When the shear and bulk viscosities are constant, the main difficulty is to ensure the weak stability in the pressure term  $p(\rho_n)$ . The stability for  $\rho_n u_n$  and  $\rho_n u_n \otimes u_n$  is obtained using that  $u_n \in L^2(0, T; H^1(\Omega))$  and the bound of  $\partial_t \rho_n$  given by the mass equation. When the shear and bulk viscosities depend on the density, the main difficulty lies in the weak stability in the terms  $\rho_n u_n$  and  $\rho_n u_n \otimes u_n$  because we loose any information on  $u_n$  close to vacuum namely  $u_n \notin L^2(0, T; H^1(\Omega))$  a priori. The stability in the pressure term does not cause convergence issues due to the BD entropy tool which provides bounds on the space derivative of a function of  $\rho_n$ . The bound on the time derivative of  $\rho_n$  is given by the mass equations and bounds related to the energy and the BD entropy. For compressible, barotropic Navier–Stokes equations, considering density-dependent viscosities is in some sense the same as considering a pressure-dependent viscosity if the pressure law is strictly monotonous with respect to the density.

### 2.2.3 Porous media equation and pressureless Navier–Stokes equations

We use the case of the pressureless version of the compressible Navier–Stokes equations with density dependent viscosities to discuss the non-linear weak stability for a simple physical system. This system allows us also to comment on the associated discrete setting that mimics the continuous stability properties. This system is:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - 2\operatorname{div}(\mu(\rho)D(u)) - \nabla(\lambda(\rho)\operatorname{div}u) = 0, \\ \rho u|_{t=0} = m_0, \quad \rho|_{t=0} = \rho_0 \end{cases} \quad (2.42)$$

with

$$\lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)).$$

*Continuous setting.* When the pressure state law is not taken into account, equation (2.38) reads

$$\partial_t(\rho(u + 2\nabla\varphi(\rho))) + \operatorname{div}(\rho(u + 2\nabla\varphi(\rho)) \otimes u) - 2\operatorname{div}(\mu(\rho)A(u)) = 0$$

with  $A(u) = (\nabla u - \nabla^t u)/2$  and  $\varphi'(\rho) = \mu'(\rho)/\rho$ . Note that if  $u_0 = -2\nabla\varphi(\rho_0)$  then  $u = -2\nabla\varphi(\rho)$  is solution of this equation. From the mass equation  $\partial_t \rho + \operatorname{div}(\rho u) = 0$  we deduce that

$$\partial_t \rho - 2\Delta\mu(\rho) = 0, \quad \rho|_{t=0} = \rho_0 \quad (2.43)$$

with  $\rho_0 \in L^2(\Omega)$ . This is the Stefan problem discussed in [Gallouët, 2018] and studied in [Eymard and Gallouët, 2023] in a bounded domain. We reproduce here parts of the content of [Gallouët, 2018] that illustrate how the proof of the global existence of weak solutions for such systems (i.e. demonstrating the nonlinear weak stability) may help to design numerical schemes that mimic at the discrete level these continuous properties of stability. We consider implicitly periodic boundary conditions. At the continuous level, the approximate equation is:

$$\partial_t \rho_n - 2\Delta(\mu_n) - \frac{1}{n}\Delta\rho_n = 0, \quad \text{with } \mu_n = \mu(\rho_n). \quad (2.44)$$

This formulation helps one to obtain the global existence of weak solutions at fixed  $n$  by using classical results on parabolic equations. The existence of weak solutions to (2.44) and the appropriate initial conditions can be

proven using Schauder's Theorem and the resolution of linear parabolic equations by the Faedo-Galerkin method. The proof of the existence of weak solution to the Stefan system on  $\mu$  is based on the compactness in a negative exponent Sobolev space ( $L^2(0, T; H^{-1}(\Omega))$ ) on  $\{\rho_n\}_{n \in \mathbb{N}}$  and on the weak convergence of the sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  in  $L^2(0, T; H^1(\Omega))$  to  $\overline{\mu(\rho)}$ . The problem being to prove that  $\overline{\mu(\rho)} = \mu(\rho)$ . The following theorem is the result without external forces obtained by [Eymard and Gallouët, 2023].

**Theorem 2.1.** *Let  $T > 0$  and let  $\mu$  be a nondecreasing Lipschitz continuous function and  $\rho_0 \in L^2(\Omega)$  be given. Let  $\{\rho_n\}_{n \in \mathbb{N}}$  be the solution of (2.44) for all  $n \in \mathbb{N}^*$ . Then there exists  $\rho$ , solution of (2.43), such that, as  $n \rightarrow +\infty$  up to some subsequence,  $\rho_n$  converges to  $\rho$  in  $\mathcal{C}([0, T]; H^{-1}(\Omega))$  and weakly  $\star$  in  $L^\infty(0, T; L^2(\Omega))$  and  $\mu_n$  converges weakly to  $\mu(\rho)$  in  $L^2(0, T; H^1(\Omega))$ .*

Hints of the proof. The estimates obtained multiplying equation (2.44) by  $\mu_n$  provides the uniform bounds

$$\mu_n \in L^2(0, T; H^1(\Omega)), \quad \frac{1}{\sqrt{n}} \rho_n \in L^2(0, T; H^1(\Omega))$$

and the estimate obtained multiplying equation (2.44) by  $\rho_n$  provides the uniform bound

$$\rho_n \in L^\infty(0, T; L^2(\Omega)).$$

Using the bound on  $\mu_n$  and the equation (2.44), we can deduce that  $\partial_t \rho_n$  is bounded uniformly in  $L^2(0, T; H^{-1}(\Omega))$  and we know that  $\rho_n$  is bounded uniformly in  $L^2(0, T; L^2(\Omega))$ . Thus  $\{\rho_n\}_{n \in \mathbb{N}}$  is compact in  $L^2(0, T; H^{-1}(\Omega))$ . Using that  $\mu_n = \mu(\rho_n)$  converges weakly in  $L^2(0, T; H^1(\Omega))$  to  $\overline{\mu(\rho)}$  we have

$$\int_0^T \int_\Omega \rho_n \mu_n = \langle \rho_n, \mu_n \rangle_{L^2(0, T; H^{-1}(\Omega)) \times L^2(0, T; H^1(\Omega))} \rightarrow \langle \rho, \overline{\mu(\rho)} \rangle_{L^2(0, T; H^{-1}(\Omega)) \times L^2(0, T; H^1(\Omega))} = \int_0^T \int_\Omega \rho \overline{\mu(\rho)}.$$

It remains to prove that  $\overline{\mu(\rho)} = \mu(\rho)$  which involves what is called the Minty trick, written in Lemma B.5 that we give in the appendix, assuming for simplicity that  $|\mu(s)| \leq C|s| + C$  for all  $s \in \mathbb{R}$  as done in Lemma 1 by [Gallouët, 2018].

*Discrete setting.* Without providing detailed calculations, we here illustrate why theoretical studies on continuous PDEs may help designing stable numerical schemes. The objective is to adapt the continuous methods to a discrete setting to prove the convergence of numerical schemes. The set  $\Omega$  is now a bounded open set of  $\mathbb{R}^d$  adapted to a space discretization. The time interval is  $[0, T]$  with  $T > 0$ . With  $n \in \mathbb{N}$ , a time step  $k_n$  is defined such that  $T = k_n N_n$  with some positive integer  $N_n$ . The space discretization is chosen to yield a finite-dimensional space  $H_n$ , where  $H_n$  is a space of functions constant on each control volume defined by some mesh. We assume that  $\rho_n$  is a function constant in time on each interval  $((l-1)k_n, lk_n)$  for  $l = 1, \dots, N_n$ . For all  $t \in [(l-1)k_n, lk_n)$ ,  $l = 1, \dots, N_n$ , one has

$$\rho_n(\cdot, t) = \rho_n^{(l)} \in H_n.$$

As we consider constant functions in time on each interval  $((l-1)k_n, lk_n)$  with  $l = 1, \dots, N_n$ , we also have to define discrete derivatives  $\partial_{t,n}$  as follows: for  $l \in \{2, \dots, N_n\}$ ,

$$\partial_{t,n} \rho_n(\cdot, t) = \partial_{t,k_n}^{(l)} \rho_n(\cdot, t) = \frac{1}{k_n} (\rho_n^{(l)} - \rho_n^{(l-1)}) \text{ for } t \in ((l-1)k_n, lk_n)$$

and for  $l = 1$

$$\partial_{t,n} \rho_n(\cdot, t) = \partial_{t,n} \rho_n(\cdot, t) = 0 \text{ for } t \in (0, k_n).$$

The discretization of the Stefan problem yields the couple  $(\rho_n, \mu_n)$  satisfying

$$\partial_{t,k_n} \rho_n - \Delta_n \mu_n = 0, \quad \mu_n = \mu(\rho_n)$$

where  $\Delta_n$  is a convenient discrete operator from  $H_n$  to  $H_n$  (see for example section 2.2.1 in [Gallouët, 2018]). We then assume  $k_n \rightarrow 0$  when  $n \rightarrow +\infty$  and  $\lim_{n \rightarrow +\infty} h_n = 0$  where  $h_n$  is the maximum diameter of the control volumes of the mesh defining  $H_N$ . We are now able to study the weak non-linear stability of the discrete setting, which will yield the convergence properties of the numerical scheme. To do so, we need discrete versions of compactness Lemmas such as the Aubin-Lions-Simon Lemma. Such discrete Lemmas can be found in [Gallouët, 2018] and references cited therein.

**Key insight 7.** Studying systems of partial differential equations in a theoretical manner can help understand the key elements that must be kept at the discrete level to ensure the stability and the convergence of the associated numerical scheme. The interested reader can consult the works [Gallouët et al., 2018, Gallouët et al., 2017, Perrin and Saleh, 2022, Herbin et al., 2021, Feireisl et al., 2019] and references therein on this subject.

## 2.3 Pressure dependent viscosity on granular material

### 2.3.1 Local dependency

A crucial insight is that the compressible Navier–Stokes equations with pressure dependent viscosities in the barotropic case are nothing else than the compressible Navier–Stokes equations with density dependent viscosities in the framework of global well-posedness properties exposed in the previous Section. In the incompressible case, a similar line of study remains an open (and interesting) problem. This is because we are not able to prove, for instance, the global existence of weak solutions for the following system in the full generality:

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) - 2\operatorname{div}(\nu(p)D(u)) + \nabla p = f, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0 \end{cases} \quad (2.45)$$

The situation is more complicated than the classical case studied by [Leray, 1934] with  $\nu(p) = \nu = \text{cst}$  where the pressure is only present as a gradient and does not appear in the weak form of the system because its contribution tested with any divergence-free test function is zero. Usually, when assuming the velocity to be known as a weak solution for the incompressible Navier–Stokes equations, the pressure is obtained by solving the elliptic equation

$$-\Delta p = \operatorname{div}(\operatorname{div}(u \otimes u) - f).$$

In the pressure-dependent viscosity case, we cannot solve in  $u$  and  $p$  separately but we have a coupled system

$$\begin{cases} \langle \partial_t u + \operatorname{div}(u \otimes u) - 2\operatorname{div}(\nu(p)D(u)) - f, \phi \rangle_{\mathcal{D}' \times \mathcal{D}} = 0 \text{ for all } \phi \text{ such that } \operatorname{div} \phi = 0, \\ -\operatorname{div}((\operatorname{Id} - 2\nu'(p)D(u))\nabla p) - 2\operatorname{div}(\nu(p)\operatorname{div}(D(u))) = \operatorname{div}(\operatorname{div}(u \otimes u) - f). \end{cases} \quad (2.46)$$

Note that the second-order operator in  $p$ ,

$$\operatorname{div}((\operatorname{Id} - 2\nu'(p)D(u))\nabla p),$$

may degenerate when  $D(u) = \operatorname{Id}/(2\nu'(p))$  and may change sign, thereby loosing an elliptic behavior. As a result, the system may be generally ill-posed if no real assumptions are done about the form of the viscosity. This is something that we will revisit later when addressing granular media: the real difference between incompressible

and compressible systems with a better behavior for the compressible ones. To keep the ellipticity property, one needs that

$$\text{Id} - 2\nu'(p)D(u) > 0.$$

The first result related to incompressible Navier–Stokes equations with pressure-dependent viscosity is due to [Renardy, 1986] and centers on the local existence of strong solutions. The main idea is to ensure the elliptic property of the nonlinear problem satisfied by the pressure by pushing the eigenvalues of  $D(u)$  to be strictly bounded from above by  $1/(2\nu'(p))$ . This leads to assume that the nonlinear function  $\nu$  is at most linear in the sense

$$1/(\lim_{p \rightarrow +\infty} \nu'(p)) > 0.$$

The only result concerning the global existence for the incompressible system can be found in [Málek et al., 2002]. Their hypothesis is to consider a shear-thinning rheology, which means to assume a viscosity  $\nu(p, |D(u)|^2)$  such that

$$\nu(p, |D(u)|^2) = (1 + \gamma(p) + |D(u)|^2)^{(r-2)/2} \text{ with } r \in (1, 2)$$

with for instance

$$\gamma(p) = \frac{1}{\sqrt{1 + \alpha^2 p^2}}.$$

With this definition we may have

$$\max \left| 2 \frac{\partial \nu}{\partial p}(p, |D(u)|^2) D(u) \right| < 1$$

for all the vector field  $u$ , so the elliptic behavior is ensured.

### 2.3.2 Non-local dependency

We present here some models from the literature where the pressure has a non-local dependency. For that, a new variable  $\gamma$  is introduced as an "activation" factor for the pressure. Let us first mention a 1-d macroscopic model of suspensions proposed in [Lefebvre-Lepot and Maury, 2012]:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \nabla p = 0, \\ \partial_t \gamma + u \partial_x \gamma = -p, \\ 0 \leq \rho \leq \rho^*, \quad \text{supp } \gamma \subset \{\rho = \rho^*\}, \quad \gamma \geq 0, \end{cases} \quad (2.47)$$

where  $\rho^*$  is the maximal density and  $\text{supp } g$  design the support of a function  $g$ , that is, the values  $(t, x)$  where  $g(t, x) \neq 0$ . We have the following behaviors on such a system:

- when  $\rho < \rho^*$ :  $\gamma = 0$ , so we have  $p = 0$  (free evolution)
- when  $\rho = \rho^*$ :  $\partial_x u = 0$  (incompressible flow) and  $\gamma \geq 0$ , so we have the activation of  $p$ .

Interestingly, such a system may be obtained formally as the limit  $\varepsilon \rightarrow 0$  of the following viscous system:

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) - \partial_x(\lambda_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon) = 0, \end{cases}$$

with

$$\lambda_\varepsilon(\rho_\varepsilon) = \frac{\varepsilon}{\rho^* - \rho_\varepsilon}.$$

Let us explain formally how to obtain such a conclusion. From the mass equation given in (2.47), we can write

$$\partial_t \mu_\varepsilon(\rho_\varepsilon) + \partial_x(u_\varepsilon \mu_\varepsilon(\rho_\varepsilon)) - \lambda_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon = 0$$

where  $\mu_\varepsilon \geq 0$  is defined such that  $\mu_\varepsilon(\rho_\varepsilon) - \mu'_\varepsilon(\rho_\varepsilon) \rho_\varepsilon = -\lambda_\varepsilon(\rho_\varepsilon)$ . We first understand that if  $\rho_\varepsilon$  is asymptotically far from  $\rho^*$  then  $\lambda_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon$  and  $\mu_\varepsilon(\rho_\varepsilon)$  tends to zero. If  $\rho_\varepsilon$  is asymptotically closed to  $\rho^*$  then  $\partial_x u = 0$ . We also conclude that  $\lambda_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon$  tends to some  $p$  and that  $\mu_\varepsilon$ , which is positive, tends to some  $\gamma \geq 0$ . We finally obtain the equation

$$\partial_t \gamma + u \partial_x \gamma + p = 0$$

since  $\partial_x u = 0$  in the congestion zone.

An important mathematical paper by [Perrin, 2016] addresses what non-local dependency means for granular media. This work considers the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \gamma - \nabla p - 2 \operatorname{div}(c_\mu(\rho + \gamma/c_\pi) D(u)) + r \rho |u| u = 0, \\ 0 \leq \rho \leq \rho^*, \\ \partial_t \gamma + \operatorname{div}(u \gamma) = -\frac{c_\pi}{2c_\mu} p, \\ (\rho^* - \rho) \gamma = 0 \text{ with } \gamma \geq 0, \end{cases} \quad (2.48)$$

where  $c_\mu, c_\pi$  are positive constants and  $r$  is a coefficient. The last expression encodes the activation effect. It is possible to justify this system by letting  $\varepsilon$  tend to zero in the following singular, degenerate system:

$$\begin{cases} \partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \operatorname{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla \pi_\varepsilon(\rho_\varepsilon) - \nabla(\lambda_\varepsilon(\rho_\varepsilon) \operatorname{div} u_\varepsilon) - \operatorname{div}(\mu_\varepsilon(\rho_\varepsilon) D(u_\varepsilon)) + r \rho_\varepsilon |u_\varepsilon| u_\varepsilon = 0. \end{cases} \quad (2.49)$$

where

$$\mu_\varepsilon(\rho_\varepsilon) = c_\mu(\rho_\varepsilon + \mu_\varepsilon^1(\rho_\varepsilon))$$

with the degenerate behavior

$$\mu_\varepsilon^1(\rho_\varepsilon) = \varepsilon \frac{\rho_\varepsilon}{(1 - \rho_\varepsilon)^\beta}$$

and the relationships

$$\lambda_\varepsilon(\rho_\varepsilon) = 2(\mu'_\varepsilon(\rho_\varepsilon) \rho_\varepsilon - \mu_\varepsilon(\rho_\varepsilon)), \quad \pi_\varepsilon(\rho_\varepsilon) = c_\pi \mu_\varepsilon^1(\rho_\varepsilon).$$

The main insight on non-local dependency is that taking  $\rho = \rho^*$  in (2.48) means, assuming an incompressible material everywhere, that we obtain the following system:

$$\begin{cases} \operatorname{div} u = 0, \\ \rho^*(\partial_t u + \operatorname{div}(u \otimes u)) + \nabla Q - 2 \operatorname{div}(c_\mu(\rho^* + \gamma/c_\pi) D(u)) + r \rho^* |u| u = 0, \\ \partial_t \gamma + \operatorname{div}(u \gamma) + \frac{c_\pi}{2c_\mu} \gamma = \frac{c_\pi}{2c_\mu} Q. \end{cases} \quad (2.50)$$

where  $Q = \gamma - p$ . This is the incompressible Navier–Stokes equations with viscosity depending on the pressure  $Q$  but in a non-local manner through the activation parameter  $\gamma$ . This system admits global weak solutions à la Leray. The important insights are:

- The compressible Navier–Stokes equations may be well posed for density dependent viscosities if a relationship between the shear and the bulk viscosities is assumed.
- The incompressible Navier–Stokes equations may be well posed for a pressure-dependent shear viscosity if this dependency is non-local and related to an activation parameter.

All these results have been obtained assuming the relationship  $\lambda_\varepsilon(\rho_\varepsilon) = 2(\mu'_\varepsilon(\rho_\varepsilon)\rho_\varepsilon - \mu_\varepsilon(\rho_\varepsilon))$  (that is linked to the BD entropy, see (2.37)) for compressible flows, which allows one to write the equation

$$2(\partial_t \mu_\varepsilon(\rho_\varepsilon) + \operatorname{div}(\mu_\varepsilon(\rho_\varepsilon)u_\varepsilon)) + \lambda_\varepsilon(\rho_\varepsilon)\operatorname{div}u_\varepsilon = 0. \quad (2.51)$$

Note that  $\lambda_\varepsilon(\rho_\varepsilon)$  is more singular than  $\mu_\varepsilon(\rho_\varepsilon)$  when  $\rho_\varepsilon$  is close to  $\rho^*$ . In the limit  $\varepsilon \rightarrow 0$ , using the extra estimates mentioned in Section 2.2.2 for the density-dependent viscosities in the Navier–Stokes system, it is possible to prove the convergence of  $\gamma_\varepsilon(\rho_\varepsilon) = c_\pi \mu_\varepsilon^1(\rho_\varepsilon)$  to  $\gamma$ ,  $\mu_\varepsilon(\rho_\varepsilon)$  to  $c_\mu(\rho + \gamma/c_\pi)$  and  $\lambda_\varepsilon(\rho_\varepsilon)\operatorname{div}u_\varepsilon$  to  $\lambda$ . The equation on  $\gamma$  is then deduced from the limit in (2.51).

One way forward could be to try to generalize such a dependency with an equation related to a tensor matrix (texture), which is currently investigated using DEM simulations aimed at deducing appropriate forms in the macroscopic system.

**Key insight 8.** This part shows that there is no point in trying to complicate a model in hope to obtain a better one because a non-local dependence on pressure can have much better properties to establish the existence of global weak solutions. In a certain sense, this non-local dependence encodes a memory effect in time or in space.

### 2.3.3 Granular material: A challenge in mathematical fluid mechanics

In dense granular systems, where the particle volume fraction  $\varphi$  is close to its maximum value, the viscosity has been proposed to depend on the pressure  $p$  and on the inertial number

$$I = \frac{2d_s|D(u)|}{\sqrt{p/\rho_s}} \quad (2.52)$$

through the  $\mu(I)$  rheology [Jop et al., 2006] with  $\mu$  the function given by

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + I_0/I}. \quad (2.53)$$

In the expressions (2.52) and (2.53),  $d_s$  is the mean diameter of the grain particles,  $\rho_s$  is the grain density and  $I_0, \mu_1, \mu_2$  are constants related to the material and the setup used to measure them. This rheology is local and the constants are based on steady-state measurements. As a result, limitations occur when applied to the quasi-static regime transitioning towards the rest state (i.e. when  $I \rightarrow 0$ ) or from dense to dilute conditions when  $\varphi$  decreases away from maximum packing [Andreotti et al., 2013]. The viscous stress tensor

$$\tau = \mu(I)p \frac{D(u)}{|D(u)|} \quad (2.54)$$

may be written as

$$\tau = 2\eta D(u) + \mu_1 p \frac{D(u)}{|D(u)|}$$

with

$$\eta = \frac{(\mu(I) - \mu_1)p}{2|D(u)|}.$$

The definitions (2.52) and (2.53) imply that the function  $\eta$  is not singular when  $|D(u)|$  vanishes and that the other part  $\mu_1 p D(u)/|D(u)|$  behaves as a plastic material with threshold  $\mu_1 p$ . In [Barker et al., 2015], the authors first prove the ill-posedness for incompressible  $\mu(I)$  flows (using linearized systems and spectral analysis) and

then argue that the compressibility will provide better properties. This is exactly what we have seen above for the incompressible or compressible systems depending on the pressure.

Importantly, a relationship is missing to close the system if the model is compressible. This relationship should be an evolution equation on the volume fraction  $\varphi$  (related to the effective density  $\varphi\rho$ ). Some relationships have been proposed in the literature:

- [Roux and Radjai, 1998]:  $\operatorname{div}u = 2|D(u)|(\varphi - \varphi_{\text{eq}}(I))$  with (2.54).
- [Iverson and George, 2014]:  $p = p(\varphi)$  with (2.54).

Note that it is possible to propose a more general constitutive law  $\tau$  with a more general closure state law:

- [Barker et al., 2017]:  $\operatorname{div}u = 2|D(u)|f(\varphi, p, I)$  and  $\tau = 2\mu(I, \varphi, p)\frac{D(u)}{|D(u)|}$  with some compatibility conditions between  $f$  and  $\mu$ .

To be able to provide even partial results related to weakly nonlinear stability on such topics would be really helpful for numerical and physical applications.

## 2.4 Single-phase viscous model with heat conductivity

There are not many results on global weak solutions for heat-conducting Navier–Stokes equations. Such compressible, Navier–Stokes–Fourier system reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}S + \nabla p = 0, \\ \partial_t(\rho E) + \operatorname{div}(\rho u E) + \operatorname{div}(Pu) = \operatorname{div}(Su) + \operatorname{div}(\kappa \nabla \theta) \end{cases} \quad (2.55)$$

where  $E = |u|^2/2 + e$  is the total energy,  $\theta$  is the temperature, and  $\kappa$  is the heat conductivity.  $P = P(\rho, \theta)$  and  $e = e(\rho, \theta)$  respectively stand for the pressure and the internal energy. We take here the isotropic stress tensor

$$S = \mu(\nabla u + \nabla^t u) + \lambda \operatorname{div}u \operatorname{Id}$$

where  $\mu$  and  $\lambda$  are two constants satisfying  $\mu > 0$  and  $\lambda + 2\mu/d > 0$  with  $d$  the dimension considered. The pressure and the internal energy are related thermodynamically:

$$P = \rho^2 \frac{\partial e}{\partial \rho} + \theta \frac{\partial P}{\partial \theta}.$$

If  $(\rho, \theta)$  are smooth and bounded from below away from zero and if the velocity field is smooth, then the total energy balance can be replaced by the thermal energy balance

$$c_v \rho (\partial_t \theta + u \cdot \nabla \theta) - \operatorname{div}(\kappa(\theta) \nabla \theta) = S : \nabla u - \theta \frac{\partial P(\rho, \theta)}{\partial \theta} \operatorname{div}u,$$

where  $c_v$  is the specific heat at constant volume,  $c_v = (\frac{\partial e}{\partial \theta})_v$ . The initial conditions are given by

$$\rho|_{t=0} = \rho_0, \quad (\rho u)|_{t=0} = m_0, \quad (\rho E)|_{t=0} = \rho_0 E_0.$$

Furthermore, defining the entropy  $s = s(\rho, \theta)$  up to an additive constant by

$$\left. \frac{\partial s}{\partial \theta} \right|_{\rho} = \frac{1}{\theta} \left. \frac{\partial e}{\partial \theta} \right|_{\rho}, \quad \text{and} \quad \left. \frac{\partial s}{\partial \rho} \right|_{\theta} = -\frac{1}{\rho^2} \left. \frac{\partial P}{\partial \theta} \right|_{\theta}$$

we can write the entropy equation

$$\partial_t(\rho s) + \operatorname{div}(\rho s u) - \operatorname{div}\left(\frac{\kappa(\theta)\nabla\theta}{\theta}\right) = \frac{1}{\theta}\left(S : \nabla u + \frac{\kappa(\theta)|\nabla\theta|^2}{\theta}\right).$$

Importantly, the total mass and total energy are formally conserved. That is, no dissipation occurs in the energy estimates coming, for instance, from the viscous terms in the momentum equations. This is a major issue when looking for global weak solutions of the compressible Navier–Stokes–Fourier equations. Finding such solutions entirely depends on the pressure law chosen because this law enables one to get extra bounds, which in turn allows one to obtain compactness. Finding global weak solutions à la Leray by considering a perfect gas law ( $P = c_0\rho\theta$ ) is an open problem.

Let us briefly list the pressure state laws that have been considered in the literature to obtain the global existence of weak solution. For constant viscosities, there are only three cases:

- 1) The pressure law as a monotone perturbation of the barotropic case [Feireisl, 2003].
- 2) Self-similar pressure laws with a large radiative contribution [Feireisl and Novotný, 2009].
- 3) The thermodynamically unstable truncated virial pressure law [Bresch et al., 2023b].

More precisely, these laws are

Case 1) The pressure state law

$$P(\rho, \theta) = P_c(\rho) + \theta P_\theta(\rho)$$

with

$$\begin{cases} P_c(0) = 0, & P'_c(\rho) \geq a_1\rho^{\gamma-1} - b \text{ for } \rho > 0, \\ P_c(\rho) \leq a_2\rho^\gamma + b \text{ for all } \rho \geq 0, \\ P_\theta(0) = 0, & P'_\theta(\rho) \geq 0 \text{ for all } \rho \geq 0, \\ P_\theta(\rho) \leq c(1 + \rho^\beta) \end{cases} \quad (2.56)$$

with

$$\gamma > d/2, \quad \beta < \gamma/2 \text{ for } d = 2, \quad \beta = \gamma/3 \text{ for } d = 3$$

with constant  $a_1 > 0$ ,  $a_2$ ,  $b$  and  $P_c$ ,  $P_\theta$  in  $\mathcal{C}([0, +\infty)) \cap \mathcal{C}^1(0, +\infty)$ .

Case 2) The pressure state law

$$P(\rho, \theta) = \theta^{\gamma/(\gamma-1)} Q\left(\frac{\rho}{\theta^{1/(\gamma-1)}}\right) + \frac{a}{3}\theta \text{ with } a > 0 \text{ and } \gamma > 3/2,$$

$$Q \in \mathcal{C}^1([0, +\infty)), \quad Q(0) = 0, \quad Q'(Z) > 0 \text{ for all } Z \geq 0,$$

$$\lim_{Z \rightarrow +\infty} \frac{Q(Z)}{Z^\gamma} = Q_\infty > 0.$$

The authors impose

$$0 < S'(Z) = \frac{1}{\gamma-1} \frac{\gamma Q(Z) - Q'(Z)Z}{Z} < c < +\infty \text{ for all } Z > 0$$

with  $\lim_{Z \rightarrow +\infty} S(Z) = 0$  so that thermodynamical stability holds.

Case 3) The pressure state law

$$P(\rho, \theta) = \rho^\gamma + \theta \sum_{n=0}^N B_n(\theta) \rho^\varepsilon$$



where  $\gamma > \max(4, 2N, d)$  and with assumptions on the coefficient  $B_j$  that does not a priori imply  $\partial e / \partial \rho > 0$ . This law provides the first example of the global existence (on  $(0, T)$ ) of weak solutions for the compressible Navier–Stokes–Fourier equations with a pressure state law that is thermodynamically unstable.

Importantly, the pressure behaves as  $\rho^\gamma$  for large density in all the above examples. Taking the barotropic case, we can write

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u|^2 + \frac{d}{dt} \int_{\Omega} \phi(\rho) + \int_{\Omega} S : \nabla u = \int_{\Omega} (P(\rho, \theta) - \phi'(\rho)\rho + \phi(\rho)) \operatorname{div} u$$

for some  $\phi(\rho)$ . To get the diffusion control related to the velocity, we need to be able to make a link between  $P(\rho, \theta) - \phi'(\rho)\rho + \phi(\rho)$  and  $\sqrt{\rho e}$  plus some quantities that will be controlled using the bounds given by the entropy. Instead of providing details for the Navier–Stokes–Fourier equations, we refer the interested readers to the recent paper [Bresch et al., 2023b] for further explanations and discussions.

There is also an interesting concept of solutions for density-dependent viscous systems such as the non-homogenous, incompressible Navier–Stokes equations or the barotropic, compressible Navier–Stokes equations. These solutions have intermediate regularity, which means that they have regularity on the velocity but only an integrability constraint on the density. The interested reader may consult [Bresch et al., 2022a] on the anisotropic compressible Navier–Stokes system. It contains a justification of averaged two-phase systems stemming from two-phase systems with interfaces that follows an homogenization procedure assuming high-oscillating initial density.

**Key insight 9.** To conclude Sections 2.1–2.4, there are many possibilities to try enriching the stress tensor so that it covers realistic physical situations and a lot of work remains to be done from a mathematical standpoint. We expect that it will not only allow a better understanding of the physical properties involved but also help to design appropriate numerical schemes. We focused in this Section on global existence à la Leray. For studies related to strong solution with regularity control, we refer the reader to [Danchin and Tolksdorf, 2022] for Navier–Stokes compressible with critical regularity, to [Beauchard and Zuazua, 2011] for partially dissipative or diffusive hyperbolic systems (extending the results of [Kawashima and Shizuta, 1988]), and to the seminal work of [Matsumura and Nishida, 1980] on compressible Navier–Stokes and solution with strong Sobolev regularity.

## 2.5 Construction of approximate systems

As summarized in the Introduction (Section 1.1), to study the global existence of weak solutions in the previous Sections 2.1–2.4, we have used approximated systems of the original problem in order to demonstrate that a sequence of solutions of this approximated system converges to the solution of the original one. For example the equation (2.3) approximates the homogeneous incompressible Navier–Stokes system (2.1). These previous Sections have also presented the first and third step of establishing the global existence of weak solutions, that is,

- Formal identification of uniform estimates.
- Weak nonlinear stability using the uniform estimates.

For completeness, we now give an overview of different constructions of approximate systems that correspond to the physical systems discussed before and that keep intact the uniform estimates presented alongside. Although quite technical, this Section may help to design appropriate iterative numerical schemes. It also could be helpful for physicists interested in understanding how the numerical iterations are done while keeping the property of nonlinear weak stability. Of course other construction may be done just keeping in mind to conserve

the estimations uniformly with respect to introduced parameters and analyzing the structure of the equations to get weak nonlinear stability.

Let us first address the incompressible systems studied in Section 2.1. Since we know that the velocity field is divergence free, we have important properties at hand that do not exist in compressible systems.

*Incompressible Navier–Stokes system* (2.1). The approximate system (2.3) is based on the following Galerkin approximation:

$$\partial_t u_n + P_n(P_n u_n \cdot \nabla P_n u_n) + \nabla p_n = \nu P_n \Delta(P_n u_n)$$

with

$$u_n|_{t=0} = P_n u_0$$

where the projection operator  $P_n$  is characterized by

$$P_n \left( \sum_{k \in \mathbb{Z}^d} c_k \exp(ik \cdot x) \right) = \sum_{-n \leq k \leq n} c_k \exp(ik \cdot x).$$

Approximating  $u = \sum_{k \in \mathbb{Z}^d} c_k \exp(ik \cdot x)$  by a sum consisting of a finite number of modes yields a velocity field  $u_n$  that is a solution of a differential equation in finite dimension that satisfies the energy estimate uniformly with respect to  $n$ . To solve this system is not complicated. One can for instance use a fixed point procedure  $w_n \mapsto u_n$  where  $u_n$  is solution of the system

$$\partial_t u_n + P_n(P_n w_n \cdot \nabla P_n u_n) + \nabla p_n = \nu P_n \Delta(P_n u_n).$$

The important property here is that  $u_n$  satisfies the energy estimates independently with respect to  $w_n$ .

*Non-homogeneous incompressible Navier–Stokes equations* (2.4). The study of such system is based on a Galerkin approximation combined with a fixed point procedure that is based on:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho w) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho w \otimes u) - \operatorname{div}(\mu(\rho)D(u)) + \nabla p = 0, \\ \operatorname{div} u = 0. \end{cases} \quad (2.57)$$

After proving the existence of a fixed point  $w \mapsto u$ , one can take the limit with respect to the dimension  $n$ .

*Barotropic compressible Navier–Stokes equation with constant viscosities* (2.9). Since we have no control *a priori* on  $\operatorname{div} u$  that could allow us to prove that the density remains far from vacuum if the initial density does as well, we add an artificial diffusion  $\varepsilon \Delta \rho$  in the mass equation. The sequence of approximate solutions will be obtained by solving a two-index system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho w) = \varepsilon \Delta \rho, \\ \partial_t(\rho u) + \operatorname{div}(\rho w \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla(p(\rho)) + \delta \nabla \rho^\beta + \varepsilon(\nabla \rho \cdot \nabla) u = 0. \end{cases} \quad (2.58)$$

after a Galerkin procedure. Recently, another approximate system has been proposed by [Chaudhuri et al., 2023]. It is compatible with the non-local compactness procedure proposed by [Bresch and Jabin, 2018a]. We refer to the case of compressible Navier–Stokes equations with an heterogeneous pressure law for a new approximate system [Bresch et al., 2021] which has been helpful to treat the heat conducting Navier–Stokes equations with the truncated virial pressure law.

*Barotropic compressible Navier–Stokes equation* (2.11) *with density-dependent viscosities*. The construction of approximate system is not straightforward in this case. We need to write an approximate system that preserves

the estimates used for nonlinear weak stability. As noted by [Rousset, 2017], for  $\mu(\rho) = \rho$  and  $\lambda(\rho) = 0$  the construction of approximate solutions by [Vasseur and Yu, 2016] starts from an idea close to that of [Bresch and Desjardins, 2006] with drag terms, a cold pressure, and a capillarity quantity but with a different capillarity coefficient linked to the quantum compressible Euler system. For more general shear and bulk viscosities satisfying the BD relationship, the idea proposed in [Bresch et al., 2022b] is based on the identification made in [Bresch et al., 2015] that a two-velocity property is included in such systems. Let us consider the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho[w]_\delta) = 2\kappa \operatorname{div}([\mu'(\rho)]_\alpha \nabla \rho) = 0, \\ \partial_t(\rho w) + \operatorname{div}(\rho u_{\delta,\alpha} \otimes w) - 2(1 - \kappa) \operatorname{div}(\mu(\rho) \nabla w) - 2\kappa \operatorname{div}(\mu(\rho) A(w)) \\ \quad + 4(1 - \kappa) \kappa \operatorname{div}(\mu(\rho) \nabla^2 \varphi(\rho)) - \nabla((\lambda(\rho) - 2\kappa(\mu'(\rho)\rho - \mu(\rho))) \operatorname{div} u) + \nabla p(\rho) \\ \quad + \varepsilon \Delta^{2s} w - \varepsilon \operatorname{div}((1 + |\nabla w|^2) \nabla w) = 0, \\ \partial_t(\rho \nabla \phi(\rho)) + \operatorname{div}(\rho u_{\delta,\alpha} \otimes \nabla \phi(\rho)) - 2\kappa \operatorname{div}(\mu(\rho) \nabla^2 \phi(\rho)) + \operatorname{div}(\mu(\rho) \nabla^t w) + \nabla((\mu'(\rho)\rho - \mu(\rho)) \operatorname{div} u) = 0, \\ \rho[w]_\delta = \rho u_{\delta,\alpha} + 2\kappa[\mu'(\rho)]_\alpha \nabla \rho. \end{cases} \quad (2.59)$$

Note that two smoothing parameters  $\alpha > 0$  and  $\delta > 0$  (associated with a standard mollification with respect to  $t$  and  $x$ ) are introduced in all the transport velocities and  $s \in \mathbb{N}^*, s > 2$ . To build such a solution if  $\varepsilon$  is given, we need to go through several levels of approximations. For example, building a solution of the nonlinear parabolic equation for  $\rho$  requires some assumptions on the coefficients. When  $\varepsilon, \alpha, \delta$  are fixed, then the construction is done by using a Faedo-Galerkin approximation combined with a fixed point procedure. Then, we let  $\alpha$  tend to zero before letting  $n$  tend to infinity. Next, we let  $\delta$  tend to zero to prove that  $v = 2\nabla \phi(\rho)$ . At the end, we combine the equations satisfied by  $v$  and  $w$  to get an equation satisfied by  $u$ . Finally, we let  $\varepsilon$  tend to zero.

*Compressible Navier–Stokes equations (2.11) with an heterogeneous pressure law.* The corresponding system is:

$$\begin{cases} \partial_t \rho_{\varepsilon,\eta} + \operatorname{div}(\rho_{\varepsilon,\eta} u_{\varepsilon,\eta}) = 0, \\ \partial_t(\rho_{\varepsilon,\eta} u_{\varepsilon,\eta}) + \operatorname{div}(\rho_{\varepsilon,\eta} u_{\varepsilon,\eta} \otimes u_{\varepsilon,\eta}) - \mu \Delta u_{\varepsilon,\eta} + \nabla(p_{\text{art},\eta}(\rho_{\varepsilon,\eta}) + \mathcal{L}_\varepsilon \star_x p(\rho_{\varepsilon,\eta}, t, x)) = 0 \end{cases} \quad (2.60)$$

where

$$p_{\text{art},\eta}(\rho_{\varepsilon,\eta}) = \eta_1 \rho_{\varepsilon,\eta}^{\gamma_{\text{art},1}} + \dots + \eta_m \rho_{\varepsilon,\eta}^{\gamma_{\text{art},m}}$$

for some fixed parameters  $\gamma_{\text{art}} = \gamma_{\text{art},1} > \gamma_{\text{art},2} > \dots > \gamma_{\text{art},m}$  with

$$\gamma_{\text{art},1} > 2\gamma, \quad \gamma_{\text{art},i+1} + 2\frac{\gamma_{\text{art},i+1}}{d} - 1 > \gamma_{\text{art},i}, \quad \gamma + 2\frac{\gamma}{d} - 1 > \gamma_{\text{art},m}.$$

The symbol  $\star_x$  means the convolution operator in space where the mollifying operator  $\mathcal{L}_\varepsilon$  is defined as:

$$\mathcal{L}_\varepsilon(x) = \frac{1}{\log 2} \int_\varepsilon^{2\varepsilon} L_{\varepsilon'}(x) \frac{d\varepsilon'}{\varepsilon'}$$

where  $L_\varepsilon$  is a standard mollifier given by

$$L_\varepsilon(x) = \frac{1}{\varepsilon^d} L\left(\frac{x}{\varepsilon}\right)$$

with  $L$  a non-negative smooth function such that  $L \in \mathcal{C}_0^\infty(\mathbb{T}^d)$  and  $\int_{\mathbb{T}^d} L(x) dx = 1$ . Note that  $\mathcal{L}_\varepsilon \star_x g$  means a convolution between  $\mathcal{L}_\varepsilon$  and  $g$  in the space variable. The approximate pressure is obtained by regularizing the pressure law. The a priori energy estimates are tricky because of the spatial and time dependence in  $p$ . Obtaining the existence for such a system is straightforward by using a fixed point argument. Limits are taken one after the other: first by letting  $\varepsilon \rightarrow 0$  and then  $\eta_1 \rightarrow 0, \dots, \eta_m \rightarrow 0$ . The main difficulty is to handle the limit  $\varepsilon \rightarrow 0$ , which strongly uses the non-local method introduced in [Bresch and Jabin, 2018a]. The limit  $\varepsilon \rightarrow 0$  requires a large integrability on  $\rho$  that imposes the presence of some large  $\rho_{\varepsilon,\eta}^{\gamma_{\text{art},1}}$ . This coefficient is too large to

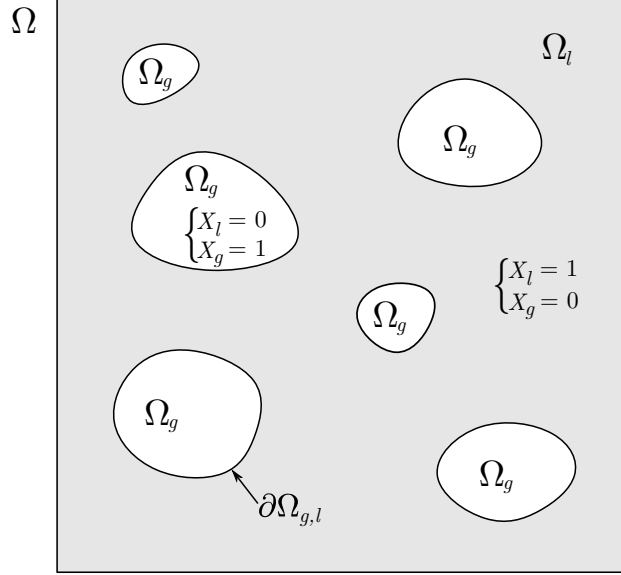


Figure 4: Two-phase domain  $\Omega = \Omega_g \cup \Omega_l \cup \partial\Omega_{g,l}$  and notation. Phase  $g$  occupies white zone and phase  $l$  occupies grey zone. For reference, the indicator function  $X_i$  and the interface  $\partial\Omega_{g,l}$  have been marked only for one subdomain  $\Omega_g$ .

directly take the limit  $\eta_1 \rightarrow 0$  without another artificial pressure, hence we need a kind of hierarchy  $\gamma_{\text{art},i}$ . Such result has been used in [Bresch et al., 2023b] to prove the global existence of weak solution of the compressible Navier–Stokes equations with heat-conductivity where the state law is the truncated virial pressure law. This work is based on a new construction of approximate solutions that uses an iterative scheme and a fixed point procedure that could be very helpful to design efficient numerical schemes.

**Key insight 10.** Systems of partial differential equations are not characterized by a single parameter but rather by several parameters of approximations and regularizations that must converge towards 0 or infinity in a very precise sequence. Such sequence guarantees the uniformity of the limits in these parameters by demonstrating the absence of artificial oscillations and concentrations.

## 2.6 A two-phase immiscible system written as a single-phase system with an indicator function

In this section we review theoretical results in the literature regarding the existence of weak solutions for a system with two different phases that are separated by interfaces. The respective phase variables (density, velocity, viscosity, etc.) are denoted with the subscripts  $g, l$ . Each phase occupies a domain  $\Omega_{g,l}$ , respectively, with interfaces  $\partial\Omega_{g,l} = \overline{\Omega_g} \cap \overline{\Omega_l}$ . The whole domain is thus  $\Omega = \Omega_g \cup \Omega_l \cup \partial\Omega_{g,l}$  (see Figure 4). Let us introduce indicator functions  $X_i$  with  $i = g$  or  $i = l$  that has the value 1 if  $x \in \Omega_i$  and 0 otherwise, which implies that  $X_l = 1 - X_g$ . These functions satisfy the following transport equation

$$\partial_t X_i + u_I \cdot \nabla X_i = 0$$

where  $u_I$  is the interface velocity. The phase  $g$  is governed in the domain  $\Omega_g$  by the following conservation equations:

$$\begin{cases} \partial_t \rho_g + \operatorname{div}(\rho_g u_g) = 0, \\ \partial_t(\rho_g u_g) + \operatorname{div}(\rho_g u_g \otimes u_g) - \operatorname{div}(\mu_g D(u_g)) - \nabla(\lambda_g \operatorname{div} u_g) + \nabla(p_g(\rho_g)) = \rho_g f, \\ \rho_g u_g|_{t=0} = m_{0,g}, \quad \rho_g|_{t=0} = \rho_{0,g}. \end{cases} \quad (2.61)$$

In the domain  $\Omega_l$  the phase  $l$  follows the conservation equations,

$$\begin{cases} \partial_t \rho_l + \operatorname{div}(\rho_l u_l) = 0, \\ \partial_t(\rho_l u_l) + \operatorname{div}(\rho_l u_l \otimes u_l) - \operatorname{div}(\mu_l D(u_l)) - \nabla(\lambda_l \operatorname{div} u_l) + \nabla(p_l(\rho_l)) = \rho_l f, \\ \rho_l u_l|_{t=0} = m_{0,l}, \quad \rho_l|_{t=0} = \rho_{0,l}. \end{cases} \quad (2.62)$$

Defining

$$\begin{aligned} \rho &= X_g \rho_g + X_l \rho_l, & p(\rho, X_g) &= X_g p_g(\rho) + X_l p_l(\rho) \\ \lambda(\rho, X_g) &= X_g \lambda_g(\rho) + X_l \lambda_l(\rho), & \mu(\rho, X_g) &= X_g \mu_g(\rho) + X_l \mu_l(\rho) \end{aligned}$$

and choosing one velocity field  $u_g = u_l = u_I = u$ , we can write the following system on  $(X_g, \rho, u)$ :

$$\begin{cases} \partial_t X_g + u \cdot \nabla X_g = 0 & \text{with } X_g X_l = 0 & \text{and } X_l = 1 - X_g, \\ \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho, X_g) D(u)) - \nabla(\lambda(\rho, X_g) \operatorname{div} u) + \nabla(p(\rho, X_g)) = \rho f, \\ \rho u|_{t=0} = m_0, \quad \rho|_{t=0} = \rho_0, \quad X_g|_{t=0} = X_{g,0} & \text{with } X_{g,0} X_{l,0} = 0. \end{cases} \quad (2.63)$$

Remark that since  $X_g X_l = 0$ , we can find the equation on each phase from the equation on  $X_l$  and the equation on  $\rho$ . Indeed

$$\partial_t \rho + \operatorname{div}(\rho u) = (\partial_t X_g + u \cdot \nabla X_g)(\rho_g - \rho_l) + (\partial_t(\rho_g) + \operatorname{div}(\rho_g u))X_g + (\partial_t(\rho_l) + \operatorname{div}(\rho_l u))(1 - X_g) = 0.$$

The condition that  $X_g X_l = 0$  means that we are able to track the interface.

**Global weak solutions à la Leray versus solutions à la Hoff.** The global weak solutions framework is not appropriate for such a system; it cannot ensure that  $\operatorname{div} u \in L^1(0, T; L^\infty(\Omega))$ , which is needed to follow interfaces with possible discontinuous jump in density across these interfaces. The appropriate framework to handle such discontinuities is that of Hoff's solution and we refer the reader to [Bresch et al., 2022a] for a discussion on this topic. For such systems with an indicator function, the effective flux property is required to have a relationship between  $\operatorname{div} u$  and the pressure state law. This is the main reason why the existence of weak solutions à la Hoff for such system has only been recently proved in the simple, one-dimensional case.

In one dimension, the momentum equation reads

$$\partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu(\rho, X_g) \partial_x u) + \partial_x(p(\rho, X_g)) = \rho f.$$

Applying  $\partial_x$  on such equation yields

$$-\partial_x^2(\mu(\rho, X_g) \partial_x u - p(\rho, X_g)) = \partial_x(-\partial_t(\rho u) - \partial_x(\rho u^2) + \rho f).$$

This provides an elliptic property on  $\mu(\rho, X_g) \partial_x u - p(\rho, X_g)$ , corresponding to the effective flux property in (2.16) for  $F = \mu(\rho, X_g) \partial_x u - p(\rho, X_g)$  and  $G = -\partial_t(\rho u) - \partial_x(\rho u^2) + \rho f$ , helping to find compactness on the density. In the multidimensional case, applying the divergence operator on the momentum equation in (2.63),

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho, X_g) D(u)) - \nabla(\lambda(\rho, X_g) \operatorname{div} u) + \nabla(p(\rho, X_g)) = \rho f,$$

---

yields

$$-\Delta\left((2\mu(\rho, X_g) + \lambda)\operatorname{div}u - p(\rho, X_g)\right) = \operatorname{div}\left(-\partial_t(\rho u) - \operatorname{div}(\rho u \otimes u) + \rho f\right) + \left[\operatorname{div}, \operatorname{div}(\mu(\rho, X_g)D(u))\right]$$

where  $\left[\operatorname{div}, \operatorname{div}(\mu(\rho, X_g)D(u))\right]$  is the commutator, which involves derivatives of  $\rho$  or  $X_g$  with respect to  $x$ . This commutator hinders the possibility to define a usable property to find the compacity on the density because we do not have bounds on the derivative of the density, which explains why only the one-dimensional case has been considered.

*Dissipative solutions with appropriate viscous potential.* The recent work of [Feireisl and Novotný, 2022] introduces a new concept of dissipative varifold solution to address weak regularity in two-phase compressible viscous fluids with interfaces. (Varifolds are used to deal with quantities coming from interface motions when weak regularity occurs.) This new formulation is variational, combining the energy and momentum balance in a single inequality. This work shows the existence of dissipative varifold solutions for a large class of general viscous fluids featuring a nonlinear dependence of their viscous stress on the symmetric velocity gradient. The problem admits global solutions depending on time as long as the hypothesis on the dissipative potential yields the bound  $|\operatorname{div}u| \leq d$  almost everywhere in  $(0, T) \times \Omega$ . It is noteworthy that in this case, the physical viscous law (rheology) ensures an appropriate bound on  $\operatorname{div}u$ .

**Key insight 11.** Few results exist in the presence of interfaces when one wishes to study stability results with weak regularity. Exceptions include the one-dimensional case and when the rheology relationship ensures appropriate bounds on  $\operatorname{div}u$ .

## 3 From two-phase immiscible system to two-phase averaged system by homogenization

### 3.1 Formal approach

This subsection reviews well-known two-phase flow modeling procedures. There is a large amount of work in the literature dedicated to modeling two-phase flows. The interested reader may consult the popular books of [Gidaspow, 1994, Drew and Passman, 1999, Brennen, 2005, Kolev, 2007, Ishii and Hibiki, 2011] but we also refer to the book of [Dobran, 2001] for an application of multiphase flow theory to volcanology.

The accuracy of two-phase modeling hinges on the effects that small scales have on the large scales. Focusing on the critical level of mesoscales, mixtures made of two phases may be understood as two single-phase regions separated by an evolving interface. Such an interface evolution involving multiple locations requires an appropriate averaging to properly represent the macroscopic behavior, i.e. assuming a certain amount of homogeneity at the mesoscale to enable the description of the flow at the macroscopic level.

An alternative and more popular approach to such an homogenization is based on the theory of multiphase flows [Gidaspow, 1994, Drew and Passman, 1999, Dobran, 2001, Ishii and Hibiki, 2011]. This approach assumes each phase to be distinct and described by its own set of equations, typically the Euler equations or the Navier–Stokes equations [Bresch et al., 2018, Bresch and Hillaire, 2015, Bresch and Hillaire, 2019, Bresch et al., 2022a, Bresch et al., 2023a]. Pioneering works in this direction include those of [Bruce Stewart and Wendroff, 1984, Baer and Nunziato, 1986, Saurel and Abgrall, 1999]. This approach has now been extended into a wide variety of possible models. Following the observation that different phases interact through the interface until

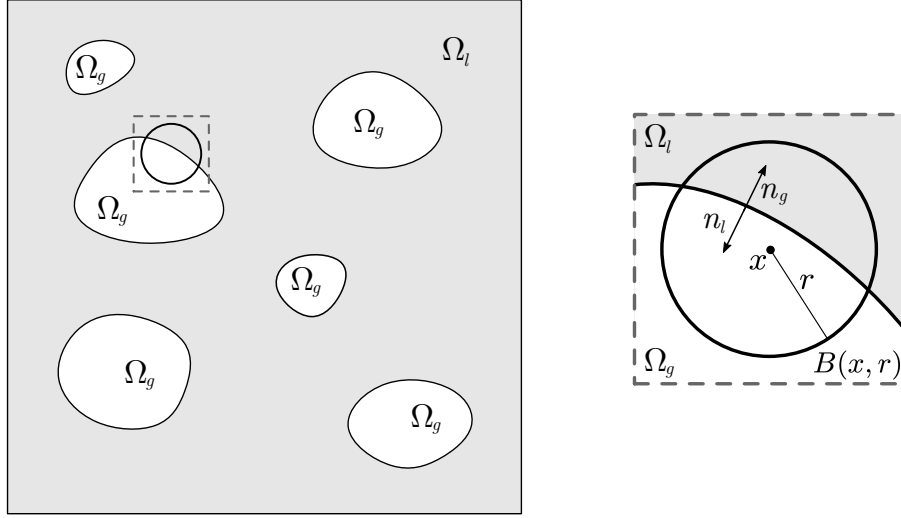


Figure 5: Left: Two-phase domain  $\Omega = \Omega_g \cup \Omega_l$ . The phase  $g$  occupies the white zone and the phase  $l$  occupies the gray zone. Right: Detail of the nomenclature for the averaging process with a characteristic sphere  $B(x, r)$  for any  $x \in \Omega$ .

reaching uniform conditions [Bdzil et al., 1999] (i.e. they move with approximately the same pressure and velocity), the resulting set of equations is classified according to the set of independent variables considered.

As explained in [Gidaspow, 1994, Drew and Passman, 1999, Collombet, 2009, Gavriluk, 2020], there is no consensus on what constitutes a suitable, general modeling framework for two-phase flows. In particular, a uniform description of the vast range of flow regimes seems yet out of reach. The main art in modeling such flows is to filter unnecessary details of the flow and retain all the properties that really characterize the flow and confer emerging properties at the macroscale such as different dynamical regimes. It is then sometimes necessary to design specific constitutive relations for closure. Consider the following conservation equations where  $i = g, l$  marks the two phases:

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i u_i) = 0, \\ \partial_t(\rho_i u_i) + \operatorname{div}(\rho_i u_i \otimes u_i) + \nabla p_i = \operatorname{div} \mathcal{D}_i + \rho_i f_i, \\ \partial_t(\rho_i E_i) + \operatorname{div}((\rho_i E_i + p_i) u_i) = \operatorname{div}(\mathcal{D}_i \cdot u_i - q_i) + \rho_i (u_i \cdot f_i) \end{cases} \quad (3.1)$$

where  $\rho_i$ ,  $u_i$ ,  $p_i$  are respectively the density, the velocity and the pressure of phase  $i$ ,  $\mathcal{D}_i$  is the viscous tensor,  $f_i$  is any body force acting upon phase  $i$ ,  $q_i$  is the heat flux for phase  $i$ , and the total energy is  $E_i = |u_i|^2 + e_i$  with  $e_i$  the internal energy for each phase  $i$ . We close this system by choosing the form of the viscous term, the body forces and the energetic constitutive law through the heat flux and heat source terms, and finally the constitutive law that relates the system of equations to thermodynamics. We also need to add jump conditions across the interfaces where mass, momentum and energy exchanges between each sub-regions happen. To establish a macroscopic description of a two-phase flow with an averaging procedure, we introduce the same domains and characteristic functions to each phase as in the previous section, that is, 1) the global domain  $\Omega = \Omega_g \cup \Omega_l \cup \partial\Omega_{g,l}$  where  $\Omega_i$  is the domain for phase  $i$  and 2) a characteristic function  $X_i(x)$  with value 1 if  $x \in \Omega_i$  and 0 otherwise (Figure 4). These characteristic functions are transported by the interfacial velocity field  $u_I$ :

$$\partial_t X_i + u_I \cdot \nabla X_i = 0 \quad (3.2)$$

where  $\nabla X_i = -n_i \delta_i$  with  $\delta_i$  the Dirac distribution of the interface associated to phase  $i$  and  $n_i$  the normal vectors pointing outwards  $\Omega_i$ . For any point  $x \in \Omega$ , we consider a sphere with center in  $x$  and radius  $r$ ,  $B(x, r)$  (see Figure 5). Then, for  $i = g, l$ ,

$$\int_{B(x,r)} X_i(y) dy = |B(x, r) \cap \Omega_i|.$$

We introduce the space average for any function  $\phi$  as

$$\langle \phi \rangle = \lim_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x,r)} \phi(y) dy.$$

The volume fraction of each phase  $\varphi_i$  for  $i = g$  and  $i = l$  as a function of  $x$  is then given by

$$\varphi_i = \langle X_i \rangle = \lim_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x,r)} X_i(y) dy = \lim_{r \rightarrow 0} \frac{|B(x, r) \cap \Omega_i|}{|B(x, r)|} \text{ for } i = g, l.$$

We can also introduce the phase average and the Favre average of any variable  $\phi_i$  as, respectively,

$$\bar{\phi}_i = \frac{\langle X_i \phi_i \rangle}{\varphi_i}, \quad \tilde{\phi}_i = \frac{\langle X_i \rho_i \phi_i \rangle}{\varphi_i \bar{\rho}_i} \text{ for } i = g, l. \quad (3.3)$$

We then multiply the conservative equations in (3.1) by  $X_i$  and use this averaging operator, which influences the closure of interfacial source terms. We get the following averaged system:

$$\begin{cases} \partial_t(\varphi_i \bar{\rho}_i) + \text{div}(\varphi_i \bar{\rho}_i \tilde{u}_i) = \Gamma_i, \\ \partial_t(\varphi_i \bar{\rho}_i \tilde{u}_i) + \text{div}(\varphi_i \bar{\rho}_i \tilde{u}_i \otimes \tilde{u}_i) + \text{div}(\varphi_i (\tilde{p}_i \text{Id} - \tilde{\mathcal{D}}_i)) = \varphi_i \bar{\rho}_i \tilde{f}_i + M_i, \\ \partial_t(\varphi_i \bar{\rho}_i \tilde{E}_i) + \text{div}(\varphi_i ((\bar{\rho}_i \tilde{E}_i + \tilde{p}_i) \text{Id} - \tilde{\mathcal{D}}_i) \tilde{u}_i + \varphi_i \tilde{q}_i) = \varphi_i \bar{\rho}_i \widetilde{(u_i \cdot f_i)} + \mathcal{E}_i \end{cases} \quad (3.4)$$

for  $i = g, l$  where the quantities  $\Gamma_i$ ,  $M_i$ , and  $\mathcal{E}_i$  are the interfacial source terms defined by

$$\begin{cases} \Gamma_i = \langle \rho_i (u_i - u_I) \cdot \nabla X_i \rangle, \\ M_i = \langle \rho_i (u_i - u_I) \otimes u_i \nabla X_i \rangle + \langle p_i \nabla X_i \rangle - \langle \mathcal{D}_i \nabla X_i \rangle, \\ \mathcal{E}_i = \langle \rho_i E_i (u_i - u_I) \cdot \nabla X_i \rangle + \langle p_i u_i \cdot \nabla X_i \rangle - \langle (\mathcal{D}_i u_i) \cdot \nabla X_i \rangle + \langle q_i \cdot \nabla X_i \rangle \end{cases} \quad (3.5)$$

with the interface jump assumptions

$$\begin{aligned} \Gamma_g + \Gamma_l &= 0 \\ M_g + M_l &= 2\sigma \langle H \nabla X_g \rangle \\ \mathcal{E}_g + \mathcal{E}_l &= \langle \sigma H u_I \cdot \nabla (X_g + X_l) \rangle \end{aligned}$$

where  $\Gamma_i$  corresponds to the mass transfer to phase  $i$ ,  $M_i$  is the momentum transfer due to mass transfer, and  $\mathcal{E}_i$  is the energy transfer due to mass transfer. The quantity  $H$  denotes the mean curvature and  $\sigma$  is the surface tension. Several closures may be introduced for these terms. We will follow [Drew and Passman, 1999] to calculate the interfacial terms in (3.5). This leads us to define the averaged interfacial velocity of phase  $i$ , denoted by  $u_{int,i}$ , as

$$\Gamma_i \langle u_{int,i} \rangle = \langle (\rho_i (u_i - u_I)) \otimes u_i \nabla X_i \rangle$$

to obtain a simple closure of the momentum transfer due to mass transfer. The interfacial pressure and the heat flux of phase  $i = g, l$  are decomposed into their average and fluctuating parts:

$$p_{int,i} = \langle p_{int,i} \rangle + p'_{int,i}, \quad q_{int,i} = \langle q_{int,i} \rangle + q'_{int,i}.$$





with the jump conditions

$$\Gamma_g + \Gamma_l = 0,$$

$$\Gamma_g(\langle u_{int,g} - u_{int,l} \rangle) + (\langle p_{int,g} \rangle - \langle p_{int,l} \rangle) \nabla \varphi_g + F_{d,g} + F_{d,l} = 2\sigma(\langle H \rangle \nabla \varphi_g + \langle H' \nabla X_g \rangle),$$

$$\begin{aligned} & \Gamma_g(\langle E_{int,g} - E_{int,l} \rangle) + (\langle p_{int,g} \rangle \langle u_{int,g} \rangle - \langle p_{int,l} \rangle \langle u_{int,l} \rangle) \cdot \nabla \varphi_g + (\langle q_{int,g} \rangle - \langle q_{int,l} \rangle) \cdot \nabla \varphi_g + E_{d,g} + E_{d,l} \\ & = \sigma(\langle H \rangle (\langle u_{int,g} \rangle - \langle u_{int,l} \rangle) \cdot \nabla \varphi_g + \langle H' \nabla X_g \rangle \cdot (\langle u_{int,g} \rangle - \langle u_{int,l} \rangle)). \end{aligned}$$

**Remark 3.1.** In the companion paper Part I [Narbona-Reina et al., 2024], this averaging is used to establish the two-phase (gas–liquid) model of volcanic conduit flow. In the momentum equation, the term  $F_{d,i}$  has to be written explicitly:

$$F_{d,i} = \langle (p'_{int,i} \text{Id} - \mathcal{D}_i) \nabla X_i \rangle = \langle p'_{int,i} \nabla X_i \rangle - \langle \mathcal{D}_i \nabla X_i \rangle.$$

In [Narbona-Reina et al., 2024], we neglect the fluctuation of the interfacial pressure  $p'_{int,i}$ . We do not know, however, how to calculate the second term. We have instead the property (3.7) for the scalar variables  $\phi_i$  indicating that  $\langle \phi_i \nabla X_i \rangle = \langle \phi_{int,i} \rangle \nabla \varphi_i + \langle \phi'_{int,i} \nabla X_i \rangle$ . In (3.8), this property is applied to each component of the vector  $\phi_i u_i$  to write equivalently  $\langle \phi_i u_i \cdot \nabla X_i \rangle = \langle \phi_{int,i} u_{int,i} \rangle \cdot \nabla \varphi_i + \langle (\phi_{int,i} u_{int,i})' \cdot \nabla X_i \rangle$ . The idea to move forward is to extend this property to a tensor  $\mathcal{D}$ , considering it as “an array of an array”. We denote by  $(R_{\mathcal{D}})_k$  the  $k$ -th row of  $\mathcal{D}$  so that the tensor  $\mathcal{D}$  is written in vectorial form as

$$\mathcal{D} = ((R_{\mathcal{D}})_1, (R_{\mathcal{D}})_2, \dots, (R_{\mathcal{D}})_d)^t, \quad \text{with} \quad (R_{\mathcal{D}})_k = (\mathcal{D}_{k,1}, \mathcal{D}_{k,2}, \dots, \mathcal{D}_{k,d}), \quad k = 1 \dots d.$$

The  $k$ -th component of the vector  $\mathcal{D} \nabla X$ , denoted by  $(\mathcal{D} \nabla X)_k$ , is

$$(\mathcal{D} \nabla X)_k = (R_{\mathcal{D}})_k \cdot \nabla X.$$

The property (3.8) can be then applied to this product as

$$\langle (R_{\mathcal{D}})_k \cdot \nabla X \rangle = \langle (R_{\mathcal{D},int})_k \rangle \cdot \nabla \varphi + \langle (R_{\mathcal{D},int})'_k \cdot \nabla X \rangle$$

The average  $\langle \mathcal{D} \nabla X \rangle$  in vectorial form is

$$\begin{aligned} \langle \mathcal{D} \nabla X \rangle & = (\langle (\mathcal{D} \nabla X)_1 \rangle, \langle (\mathcal{D} \nabla X)_2 \rangle, \dots, \langle (\mathcal{D} \nabla X)_d \rangle)^t \\ & = (\langle (R_{\mathcal{D}})_1 \cdot \nabla X \rangle, \langle (R_{\mathcal{D}})_2 \cdot \nabla X \rangle, \dots, \langle (R_{\mathcal{D}})_d \cdot \nabla X \rangle)^t \end{aligned}$$

Applying the property to each component and neglecting fluctuations yields the approximation

$$\langle \mathcal{D} \nabla X \rangle \simeq (\langle (R_{\mathcal{D},int})_1 \rangle \cdot \nabla \varphi, \langle (R_{\mathcal{D},int})_2 \rangle \cdot \nabla \varphi, \dots, \langle (R_{\mathcal{D},int})_d \rangle \cdot \nabla \varphi)^t = \langle \mathcal{D}_{int} \rangle \nabla \varphi$$

This approximation allows us to write the momentum equation in (3.9) in terms of the total stress tensor as:

$$\partial_t(\varphi_i \bar{\rho}_i \tilde{u}_i) + \text{div}(\varphi_i \bar{\rho}_i \tilde{u}_i \otimes \tilde{u}_i) + \text{div}(\varphi_i (\tilde{p}_i \text{Id} - \tilde{\mathcal{D}}_i)) = \varphi_i \bar{\rho}_i \tilde{f}_i + (\langle p_{int,i} \text{Id} - \mathcal{D}_{int,i} \rangle) \nabla \varphi_i + \Gamma_i \langle u_{int,i} \rangle$$

This formulation of the momentum equation is used in [Narbona-Reina et al., 2024] to establish the gas–liquid system.

We now have to choose interfacial quantity closures  $\langle u_{int,g} \rangle$ ,  $\langle u_{int,l} \rangle$ ,  $\langle p_{int,g} \rangle$ ,  $\langle p_{int,l} \rangle$ ,  $\langle q_{int,g} \rangle$  and  $\langle q_{int,l} \rangle$  and to define the averaged interfacial force density and the averaged interfacial energy. This choice depends on the physical characteristics of the application considered.

Here we assume a vanishing heat flux at the interface,  $\langle q_{int,g} \rangle = \langle q_{int,l} \rangle = 0$ . Usually the interface is assumed to have no intrinsic mass and it seems reasonable to assume

$$\langle u_{int,g} \rangle = \langle u_{int,l} \rangle = \langle u_I \rangle$$

as done by [Saurel and Abgrall, 1999]. Many closures are possible for the interfacial pressures. When no surface tension occurs

$$\langle p_{int,g} \rangle = \langle p_{int,l} \rangle = \langle p_I \rangle$$

where many closures for  $\langle p_I \rangle$  can be found in the literature. Our choice will be guided by the requirement to ensure energy dissipation. When involving the concept of surface tension, this yields

$$\langle p_{int,g} \rangle - \langle p_{int,l} \rangle = 2\sigma \langle H \rangle.$$

If we want to take into account surface tension and packing in a dispersed flow where the dispersed phase is the phase  $l$

$$\langle p_{int,g} \rangle - \langle p_{int,l} \rangle = 2\sigma \langle H \rangle - p_c(\varphi_l).$$

The contact pressure  $p_c$  is null for  $\varphi_l$  under a certain  $\varphi_c$  called the random close packing and then increases drastically when  $\varphi_l$  approaches the critical packing fraction  $\varphi_c$ .

**Remark 3.2.** *The approach considered in Part I [Narbona-Reina et al., 2024] to model gas–liquid flows in volcanic conduits involves the viscous tensor in the choice of the interfacial pressure to ensure the balance force at the gas/liquid interface by using the approximation mentioned in Remark 3.1:*

$$\langle p_{int,g} \text{Id} - \mathcal{D}_{int,g} \rangle - \langle p_{int,l} \text{Id} - \mathcal{D}_{int,l} \rangle = 2\sigma \langle H \rangle \quad (3.10)$$

*The interfacial velocity is considered identical for both phases:  $\langle u_{int,g} \rangle = \langle u_{int,l} \rangle = \langle u_I \rangle$ .*

Hereafter, we drop the averaging symbols  $\tilde{\cdot}$ ,  $\bar{\cdot}$  and  $\langle \cdot \rangle$  to increase readability so that two-phase immiscible systems and two-phase averaged systems share the same variables.

### 3.2 Mathematical derivation using homogenization techniques (averaging process)

This section focuses on how averaged two-phase models may be obtained rigorously with no closure assumptions. Note that this mathematical development may only be justified in the viscous case because it requires that the velocity is the same for the two phases. Another limitation is the need to assume only one spatial dimension because, as explained previously, of the lack of the possibility of commutation to obtain information of the effective flux.

A series of recent mathematical results justify the construction of two-phase averaged systems thanks to an homogenization process starting from the compressible Navier–Stokes equations with rapidly oscillating distributions of density. The homogenized effective model is derived as the frequency of the oscillations tends to  $\infty$ . A kinetic equation is obtained to track the evolution of oscillations coupled with the velocity field. The effect of oscillations in density has been studied in the pioneer papers of [Serre, 1991], [Weinan, 1992], [Amosov and Zlotnik, 1998]. Results on global weak solutions can be found in [Plotnikov and Sokolowski, 2012], [Bresch and Hillairet, 2015]. Some authors have started to consider solutions *à la Hoff* to perform the homogenization (i.e. the averaging process), which allow one to better characterize the defect measures associated to the kinetic equation. This enables interfaces tracking and shows that if the Young measure family is initially restricted to a finite convex combination of Dirac measures, this is also the case as a function of time. Such results allows one to characterize the kinetic equations and to obtain a two-phase system if the number of Dirac masses is restricted to two (e.g., [Bresch and Huang, 2011], [Bresch and Hillairet, 2015], [Bresch and Hillairet, 2019], and recently [Bresch et al., 2022a]). The effects of temperature were considered in [Hillairet, 2019] and those of surface tension were taken into account in [Hillairet et al., 2023a] and [Hillairet et al., 2023b]. The convergence of numerical schemes introduced to approximate averaged two-phase system was considered in [Bresch et al., 2023a].

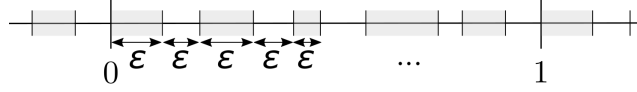


Figure 6: Sketch of the cell division for the one dimensional case. Cell  $[0, 1]$  is assumed periodic. Phase  $g$  is in white and phase  $l$  is in gray.

After this brief overview, we now describe the starting system of equations, the process and the result mathematically justified in [Bresch et al., 2022a]. All these calculations are done in one dimension in space and with a unique velocity  $u$  for the two phases. The application we consider is a bubbly flow where  $\Omega_g$  is the union of a large number of small gas bubbles suspended in a liquid occupying the domain  $\Omega_l$  (Figure 5). The homogenization procedure uses the concepts and tools described in Section B.3. As in [Bresch et al., 2022a], we consider two compressible fluids separated by interfaces in a domain  $\Omega$  such that any infinitesimal volume contains both phases. This situation is described in general in Appendix B.3 (Figure 8) and it is illustrated in the 1D case by Figure 6. The domain  $\Omega_g(t)$  represents the volume occupied at time  $t$  by fluid  $g$  (white in Figure 6) and  $\Omega_l(t)$  the volume occupied at time  $t$  by fluid  $l$  (gray in Figure 6). We assume that initially  $\Omega_g(0) \cap \Omega_l(0) = \emptyset$  and  $\Omega_g(0) \cup \Omega_l(0) = \Omega$ . As before,  $X_g(t, x)$  is the characteristic function of  $\Omega_g(t)$  transported by the unique velocity  $u$ :

$$\partial_t X_g + u \partial_x X_g = 0, \quad \text{and} \quad X_l(t, x) = 1 - X_g(t, x).$$

The densities of the two phases are  $\rho_g$  and  $\rho_l$  and they are initially defined in  $\Omega_g(0)$  and  $\Omega_l(0)$ , respectively. We extend  $\rho_g$  and  $\rho_l$  by 0 in  $\Omega$  and, slightly abusing the notation, we still call them  $\rho_i$  with  $i = g$  or  $i = l$  depending on the phase. Because of the separation hypothesis and the definition of  $X_i$ , the bulk density is

$$\rho = X_g \rho_g + (1 - X_g) \rho_l.$$

We assume that both fluids have positive constant viscosities  $\mu_i$  for  $i = g, l$ , respectively and define the viscosity field of the two-phase system as

$$\mu(X_g) = X_g \mu_g + (1 - X_g) \mu_l.$$

We further assume that both fluids are barotropic fluids with pressure laws  $p_g(\rho)$  and  $p_l(\rho)$ , and define the pressure field as

$$p(\rho, X_g) = X_g p_g(\rho) + (1 - X_g) p_l(\rho).$$

The general system for two immiscible phases is (2.61)-(2.62), which we now write for the single-velocity, one-dimensional case with  $i = g, l$ :

$$\begin{cases} \partial_t \rho_i + \partial_x(\rho_i u) = 0, \\ \partial_t(\rho_i u) + \partial_x(\rho_i u^2) - \partial_x(\mu_i \partial_x u) + \partial_x(p_i(\rho_i)) = \rho_i f, \\ \rho_i u|_{t=0} = \rho_{i,0} u_0, \quad \rho_i|_{t=0} = \rho_{i,0}. \end{cases} \quad (3.11)$$

The velocity is assumed regular in the whole domain  $\Omega$ . As detailed in Appendix B.3, the homogenization procedure is to consider a sequence of solutions generated by a sequence of initial data that are widely oscillating in space and to analyze its limiting behavior. A typical example of such initial data is:

$$X_{g,0}^\varepsilon = X_{g,0}(x/\varepsilon), \quad X_{l,0}^\varepsilon = 1 - X_{g,0}^\varepsilon, \quad \rho_0^\varepsilon = X_{g,0}^\varepsilon \rho_{g,0}(x) + (1 - X_{g,0}^\varepsilon) \rho_{l,0}(x)$$

where  $X_{g,0} : \Omega \mapsto \{0, 1\}$  is a fixed profile and  $\rho_{g,0}, \rho_{l,0}$  are bounded initial data far from vacuum (i.e. a density far from 0) and where the initial velocity  $u_0^\varepsilon = u_0(x) \in H^1(\Omega)$ . We now show how to obtain the averaged two-phase system with a relaxation quantity in the equation related to the volume fraction for two cases. The first case is without temperature as proposed by [Bresch et al., 2022a], and the second case is with temperature as proposed by [Hillairet, 2019].

### 3.2.1 Two-phase system without temperature

As indicated in Section 2.6, a one-dimensional, two phase immiscible system with interfaces but without the source term  $f$  may be defined as:

$$\begin{cases} \partial_t X_g + u \partial_x X_g = 0 & \text{with } X_g X_l = 0 \text{ where } X_l = 1 - X_g, \\ \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu(X_g) \partial_x u) + \partial_x p(\rho, X_g) = 0 \end{cases} \quad (3.12)$$

with (as above)

$$\rho = \rho_g X_g + \rho_l(1 - X_g), \quad \mu(X_g) = X_g \mu_g + (1 - X_g) \mu_l, \quad p(\rho, X_g) = X_g p_g(\rho) + (1 - X_g) p_l(\rho).$$

The initial conditions are

$$X_i|_{t=0} = X_{i,0} \quad \text{with } X_{g,0} X_{l,0} = 0 \text{ and } X_{l,0} = 1 - X_{g,0}.$$

and

$$u|_{t=0} = u_0, \quad \rho|_{t=0} = \rho_{g,0} X_{g,0} + \rho_{l,0} (1 - X_{g,0}).$$

We assume that the stress tensor  $-p(\rho, X_g) + \mu(X_g) \partial_x u$  has no discontinuity across the interfaces.

We first show the final, averaged two-phase system to facilitate the understanding of the reasoning steps involved. We then explain three way to conduct the homogenization procedure to obtain this final averaged system from the immiscible two-phase system described above. Defining  $\varphi_i$  as the volume fraction of phase  $i$ , this final homogenized system is:

$$\begin{cases} \partial_t \varphi_g + u \partial_x \varphi_g = \frac{\varphi_g \varphi_l}{\varphi_g \mu_l + \varphi_l \mu_g} (\sigma_g - \sigma_l) \\ \partial_t \rho_i + \partial_x(\rho_i u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(\mu_{\text{eff}} \partial_x u) + \partial_x p_{\text{eff}} = 0 \\ \rho = \varphi_g \rho_g + \varphi_l \rho_l, \end{cases} \quad (3.13)$$

with

$$\sigma_i = -p_i(\rho_i) + \mu_i \partial_x u \quad \text{for } i = g, l. \quad (3.14)$$

and

$$\mu_{\text{eff}} = \frac{\mu_g \mu_l}{\varphi_g \mu_l + \varphi_l \mu_g}, \quad p_{\text{eff}} = \frac{\varphi_g \mu_l p_g(\rho_g) + \varphi_l \mu_g p_l(\rho_l)}{\varphi_g \mu_l + \varphi_l \mu_g}$$

**Remark 3.3.** Averaged, compressible two-phases systems are frequently derived formally (e.g., the Baer Nunziato systems). Interestingly, the first equation of (3.13),

$$\partial_t \varphi_g + u \partial_x \varphi_g = \frac{\varphi_g \varphi_l}{\varphi_g \mu_l + \varphi_l \mu_g} (\sigma_g - \sigma_l)$$

with  $\sigma_g$  and  $\sigma_l$  given by (3.14), is similar to equation (3.6) that we derived formally in Section 3.1. In other words, it can be viewed as a viscous generalization of equation (3.6). Also, one can recover the averaged two-phase system (3.11) because  $\rho_g = X_g \rho$  and  $\rho_l = X_l \rho$  almost everywhere in  $\Omega$ .

We now explain three different approaches to derive the averaged two-phase system (3.13). The first one is related to the process of the limit and the measure concept, the second one uses a discrete approximation as in [Bresch et al., 2023a], and the third one uses WKB techniques as in [Bresch et al., 2022a].

**Key insight 12.** It is not mandatory to understand all the mathematical aspects of the different homogenization approaches to obtain an averaged two-phase system from a two-phase model with interface. The following 3-step procedure is sufficient. A) Bring the study back to a system of partial differential equations in one spatial dimension that is valid over the entire space thanks to a quantity that allows you to know in which part of the domain you are located. B) Create a small parameter that gives the scale of the mixture and place yourself in a periodic configuration by simplifying the geometry considered. C) Carry out the asymptotic analysis when this small parameter tends towards 0. To do this, you can choose between an approach involving a family of measurements that encode the mixture, a discrete approach closer to what would be encoded numerically to solve the system, or an approach with multi-scale development that require an intuition of the scales involved. In all cases, the calculations are carried out in an idealized situation and the results will be used to better understand the phenomena involved, creating a mutually beneficial interaction between mathematicians and physicists.

**Approach 1. Process of the limit and the measure concept.** This approach starts from a transport equation and the mass and momentum equations for the sequences  $\{X_g^\varepsilon\}_{\varepsilon>0}$ ,  $\{\rho^\varepsilon\}_{\varepsilon>0}$ ,  $\{u^\varepsilon\}_{\varepsilon>0}$ , respectively:

$$\begin{aligned}\partial_t X_g^\varepsilon + u^\varepsilon \partial_x X_g^\varepsilon &= 0 \\ \partial_t \rho^\varepsilon + \partial_x(\rho^\varepsilon u^\varepsilon) &= 0 \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \partial_x(\rho^\varepsilon (u^\varepsilon)^2) - \partial_x \sigma^\varepsilon &= 0\end{aligned}$$

where

$$\sigma^\varepsilon = -p(\rho^\varepsilon, X_g^\varepsilon) + \mu(X_g^\varepsilon) \partial_x u^\varepsilon, \quad (3.15)$$

where

$$p(\rho^\varepsilon, X_g^\varepsilon) = p_g(\rho^\varepsilon) X_g^\varepsilon + p_l(\rho^\varepsilon)(1 - X_g^\varepsilon), \quad \mu(X_g^\varepsilon) = \mu_g X_g^\varepsilon + \mu_l(1 - X_g^\varepsilon). \quad (3.16)$$

with the initial data

$$X_g^\varepsilon|_{t=0} = X_{g,0}(x/\varepsilon), \quad u^\varepsilon|_{t=0} = u_0, \quad \rho^\varepsilon|_{t=0} = \rho_{g,0} X_{g,0}^\varepsilon + \rho_{l,0}(1 - X_{g,0}^\varepsilon)$$

where  $\rho_{i,0}$  and  $u_0$  depend on  $x$  and not on  $x/\varepsilon$ . Recall that  $\sigma^\varepsilon$  is regular with no discontinuity across the interfaces due to the boundary condition. Letting  $\varepsilon$  tend to zero, as explained in Section 2.2 where we treated the stability of compressible Navier–Stokes, we can perform this limit in the density and velocity equations to obtain

$$\partial_t \rho + \partial_x(\rho u) = 0$$

and

$$\partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x \sigma = 0,$$

where  $\sigma$  is the strong limit of  $\sigma^\varepsilon$ . Applying the method explained in the Appendix B.3 that is related to elliptic systems, we use the expression (3.15) to write

$$\partial_x u^\varepsilon = \frac{\sigma^\varepsilon + p(\rho^\varepsilon, X_g^\varepsilon)}{\mu(X_g^\varepsilon)}. \quad (3.17)$$

We calculate the limit in this expression using the concept of measures as in proposition B.2. To find the appropriate measure, we use the renormalization method by writing an equation on  $b(\rho^\varepsilon, X_g^\varepsilon)$  for all functions  $b$  such that  $b \in \mathcal{C}_c(\mathbb{R}_\xi \times \mathbb{R}_\eta)$ . To do so, we use that

$$\partial_t X_g^\varepsilon + u^\varepsilon \partial_x X_g^\varepsilon = 0 \quad (3.18)$$

and

$$\partial_t \rho^\varepsilon + u^\varepsilon \partial_x \rho^\varepsilon + \rho^\varepsilon \partial_x u^\varepsilon = 0. \quad (3.19)$$

We define  $\partial_k b$  as the derivative of  $b$  with respect to its argument  $k$ . We then multiply equation (3.18) by  $\partial_2 b(\rho^\varepsilon, X_g^\varepsilon)$  and equation (3.19) by  $\partial_1 b(\rho^\varepsilon, X_g^\varepsilon)$ . Adding the two yields

$$\partial_t b(\rho^\varepsilon, X_g^\varepsilon) + \partial_x (u^\varepsilon b(\rho^\varepsilon, X_g^\varepsilon)) + \left( \rho^\varepsilon \partial_1 b(\rho^\varepsilon, X_g^\varepsilon) - b(\rho^\varepsilon, X_g^\varepsilon) \right) \partial_x u^\varepsilon = 0$$

for any  $b \in \mathcal{C}_c(\mathbb{R}_\xi \times \mathbb{R}_\eta)$ . Now we use relation (3.17) to write

$$\partial_t b(\rho^\varepsilon, X_g^\varepsilon) + \partial_x (u^\varepsilon b(\rho^\varepsilon, X_g^\varepsilon)) + \left( \rho^\varepsilon \partial_1 b(\rho^\varepsilon, X_g^\varepsilon) - b(\rho^\varepsilon, X_g^\varepsilon) \right) \frac{\sigma^\varepsilon + p(\rho^\varepsilon, X_g^\varepsilon)}{\mu(X_g^\varepsilon)} = 0. \quad (3.20)$$

We now introduce a sequence of measures  $\nu_{t,x}^\varepsilon$  on  $\mathbb{R}_\xi \times \mathbb{R}_\eta$  as

$$\langle \nu_{t,x}^\varepsilon, b \rangle := b(\rho^\varepsilon(t, x), X_g^\varepsilon(t, x)) \quad \text{for all } b \in \mathcal{C}_c(\mathbb{R}_\xi \times \mathbb{R}_\eta).$$

The information on  $\nu_{t,x}^\varepsilon$  (from (3.20)) provides its strong convergence to  $\bar{\nu}_{t,x}$  in  $C_w([0, +\infty) : \mathcal{M}_+(\mathbb{R}_\xi \times \mathbb{R}_\eta))$  with the relation

$$\langle \bar{\nu}_{t,x}, \beta \rangle = \overline{b(\rho^\varepsilon, X_g^\varepsilon)}$$

where  $\overline{b(\rho^\varepsilon, X_g^\varepsilon)}$  is the weak limit of  $b(\rho^\varepsilon, X_g^\varepsilon)$  in  $L^\infty((0, T) \times \mathbb{T})$  weak-star. If we couple this convergence with the strong convergence on  $u^\varepsilon \rightarrow u$  in  $L^2(0, T; \mathcal{C}(\mathbb{T}^1))$ , we can calculate the limit in (3.20) and prove that the limit of the measure  $\bar{\nu}_{t,x}^\varepsilon$  is the limit quantity  $\bar{\nu}_{t,x}$  that satisfies the following kinetic equation

$$\partial_t \bar{\nu}_{t,x} + \partial_x (u \bar{\nu}_{t,x}) - \partial_\xi \left( \xi \frac{\sigma + p(\xi, \eta)}{\mu(\eta)} \bar{\nu}_{t,x} \right) - \frac{\sigma + p(\xi, \eta)}{\mu(\eta)} \bar{\nu}_{t,x} = 0. \quad (3.21)$$

Using the following property of the limit of  $\bar{\nu}_{t,x}$ ,

$$\bar{\nu}_{0,x} = \varphi_{g,0}(x) \delta_{(\rho_g, 0(x), 1)} + \varphi_{l,0}(x) \delta_{(\rho_l, 0(x), 0)}$$

we deduce that

$$\bar{\nu}_{t,x} = \varphi_g(t, x) \delta_{(\rho_g(t,x), 1)} + \varphi_l(t, x) \delta_{(\rho_l(t,x), 0)}. \quad (3.22)$$

This means that we obtained the propagation in time of the decomposition of the defect measure into a convex combination of two Dirac masses. A complete demonstration of (3.22) that is based on a uniform bound of  $\partial_x u$  in  $L^1(0, T; L^\infty(\Omega))$  can be found in [Bresch et al., 2022a]. This is the key point in the mathematical proof but we can assume this property to deduce the two-phase system.

Let us adopt this expression (3.22) for the limit quantity  $\bar{\nu}_x(t)$ . Proposition B.2 implies that, for any function  $b \in \mathcal{C}_c(\mathbb{R}_\xi \times \mathbb{R}_\eta)$ ,

$$\langle \bar{\nu}_{t,x}, b \rangle = \varphi_g b(\rho_g, 1) + \varphi_l b(\rho_l, 0).$$

If we consider the function  $b$  such that  $b(\xi, \eta) = 1$  in a neighborhood of  $(\rho_g, 1)$  and  $b(x, \xi, \eta) = 0$  in a neighborhood of  $(\rho_l, 0)$ , we can recover the volume fraction  $\varphi_g$  equation from (3.21):

$$\langle \partial_t \varphi_g + \partial_x (\varphi_g u), \phi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = \left\langle \frac{\sigma + p_g}{\mu_g} \varphi_g, \phi \right\rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)}$$

It can also be written as

$$\partial_t \varphi_g + u \partial_x \varphi_g = \left( \frac{\sigma + p_g}{\mu_g} - \partial_x u \right) \varphi_g \text{ in the sense of distributions.} \quad (3.23)$$

What remains now is to characterize  $\sigma$ . Taking the limit of (3.17) and using the strong convergence on  $\sigma^\varepsilon$  yields

$$\partial_x u = \sigma \langle \bar{v}_x(t), \frac{1}{\mu(\eta)} \rangle + \langle \bar{v}_x(t), \frac{p(\xi, \eta)}{\mu(\eta)} \rangle.$$

We use (3.22) and that  $u, \sigma$  have no discontinuities to write

$$\partial_x u = \left( \frac{\varphi_g}{\mu_g} + \frac{\varphi_l}{\mu_l} \right) \sigma + \left( \varphi_g \frac{p_g}{\mu_g} + \varphi_l \frac{p_l}{\mu_l} \right)$$

which implies the following expression for  $\sigma$ :

$$\sigma = \frac{\partial_x u}{\frac{\varphi_g}{\mu_g} + \frac{\varphi_l}{\mu_l}} - \frac{\varphi_g \frac{p_g}{\mu_g} + \varphi_l \frac{p_l}{\mu_l}}{\frac{\varphi_g}{\mu_g} + \frac{\varphi_l}{\mu_l}}.$$

We can define this limit as  $\sigma = -p_{\text{eff}} + \mu_{\text{eff}} \partial_x u$  with the following coefficients deduced from the previous relationship:

$$\mu_{\text{eff}} = \left( \frac{\varphi_g}{\mu_g} + \frac{\varphi_l}{\mu_l} \right)^{-1} = \frac{\mu_g \mu_l}{\varphi_g \mu_l + \varphi_l \mu_g}, \quad p_{\text{eff}} = \frac{\varphi_g \frac{p_g}{\mu_g} + \varphi_l \frac{p_l}{\mu_l}}{\frac{\varphi_g}{\mu_g} + \frac{\varphi_l}{\mu_l}} = \frac{\varphi_g \mu_l p_g + \varphi_l \mu_g p_l}{\varphi_g \mu_l + \varphi_l \mu_g}.$$

We can finally use this formulation to write explicitly the right-hand side of (3.23). Noting that

$$\sigma + p_g - \mu_g \partial_x u = (\mu_{\text{eff}} - \mu_g) \partial_x u - (p_{\text{eff}} - p_g)$$

and

$$\mu_{\text{eff}} - \mu_g = \frac{\varphi_l \mu_g (\mu_l - \mu_g)}{\varphi_g \mu_l + \varphi_l \mu_g}, \quad p_{\text{eff}} - p_g = \frac{\varphi_l \mu_g (p_g - p_l)}{\varphi_g \mu_l + \varphi_l \mu_g}$$

allows us to obtain the equation on  $\varphi_g$  as presented in (3.13):

$$\partial_t \varphi_g + u \partial_x \varphi_g = \frac{\varphi_g \varphi_l}{\varphi_l \mu_g + \varphi_g \mu_l} \left( - (p_g - p_l) + (\mu_l - \mu_g) \partial_x u \right).$$

To obtain the equation on  $\varphi_g \rho_g$ , one needs to choose  $b$  for  $\langle \bar{v}_x(t), b \rangle$  with  $b(x, \xi, \eta) = \xi$  in a neighborhood of  $(\rho_g, 1)$  and 0 in a neighborhood of  $(\rho_l, 0)$  and use again equation (3.21) to obtain, after integration by part with respect to  $\xi$ :

$$\partial_t (\varphi_g \rho_g) + \partial_x (\varphi_g \rho_g u) = 0.$$

Considering  $b$  for  $\langle \bar{v}_x(t), b \rangle$  with  $b(x, \xi, \eta) = \xi$  in a neighborhood of  $(\rho_l, 0)$  and 0 in a neighborhood of  $(\rho_g, 1)$  yields the equivalent equation for phase  $l$ . Summing them yields the equation on  $\rho = \varphi_g \rho_g + \varphi_l \rho_l$ :

$$\partial_t \rho + \partial_x (\rho u) = 0.$$



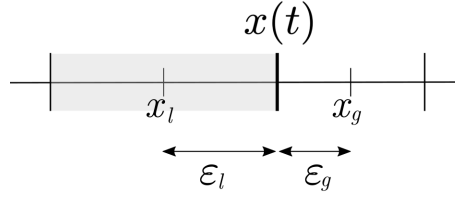


Figure 7: Sketch for the discrete approach. Phase  $g$  is white and phase  $l$  is gray.

**Approach 2. Discrete approach following [Bresch et al., 2023a].** This approach provides a formal discrete procedure to derive the relationship between the volume fraction  $\varphi_g$  and the expressions of the effective viscosity  $\mu_{\text{eff}}$  and effective pressure  $p_{\text{eff}}$  in (3.13)-(3.14).

In a situation where the fluids are separated (say, at a small scale  $\varepsilon$ ), consider a point  $x(t) \in \mathbb{T}$  at an interface between fluid  $g$  on its right and fluid  $l$  on its left, for any time  $t$  (see Figure 7). We define  $x_g(t)$  as the center of the zone of pure fluid  $g$  on the right of  $x(t)$  and  $x_l(t)$  as the center of the zone of pure fluid  $l$  on the left of  $x(t)$ . We define the small scales

$$\varepsilon_g(t) = x_g(t) - x(t), \quad \varepsilon_l(t) = x(t) - x_l(t)$$

The variable  $\varphi_g(t)$  is the local volume fraction of fluid  $g$  at the point  $x(t)$ :

$$\varphi_g(t) = \frac{\varepsilon_g(t)}{\varepsilon_g(t) + \varepsilon_l(t)},$$

and  $\varphi_l = 1 - \varphi_g$ . Using the Lagrangian derivative respect to the velocity  $u$  (i.e.  $D_t\phi = \partial_t\phi + u\partial_x\phi$  for any function  $\phi$ ), we have

$$D_t\varepsilon_g(t) = u(t, x_g(t)) - u(t, x(t)), \quad D_t\varepsilon_l(t) = u(t, x(t)) - u(t, x_l(t))$$

and therefore

$$D_t(\varepsilon_g + \varepsilon_l)(t) = u(t, x_g(t)) - u(t, x_l(t)).$$

This allows us to write

$$\begin{aligned} D_t\varphi_g(t) &= \frac{(\varepsilon_g + \varepsilon_l)D_t\varepsilon_g - \varepsilon_g D_t(\varepsilon_g + \varepsilon_l)}{(\varepsilon_g + \varepsilon_l)^2} \\ &= \frac{\varepsilon_l(u(t, x_g(t)) - u(t, x(t))) - \varepsilon_g(u(t, x(t)) - u(t, x_l(t)))}{(\varepsilon_g + \varepsilon_l)^2} \end{aligned} \quad (3.24)$$

The regularity of the solution is expected to be the following: at any time  $t$ , the pressure and the space derivative of the velocity should be continuous in space in each pure region (namely, in  $(x_l - \varepsilon_l, x_l + \varepsilon_l)$  and in  $(x_g - \varepsilon_g, x_g + \varepsilon_g)$ ), but not at the point  $x(t)$ . At this point, what it is expected is that the constraint  $p - \mu\partial_x u$  is continuous (this continuity in space stems from the law of reciprocal forces of Newton). It is convenient to define the approximate space derivatives of the velocity  $d_l(t) \approx (\partial_x u)_l$  and  $d_g(t) \approx (\partial_x u)_g$  as

$$d_l(t) = \frac{u(t, x(t)) - u(t, x_l(t))}{\varepsilon_l(t)}, \quad d_g(t) = \frac{u(t, x_g(t)) - u(t, x(t))}{\varepsilon_g(t)}.$$

As  $u$  is intended to converge strongly but  $\partial_x u$  only weakly,  $d_g(t)$  and  $d_l(t)$  are not approximations of  $\partial_x u(t, x(t))$ . On the other hand,  $\frac{\varepsilon_l}{\varepsilon_l + \varepsilon_g}d_l + \frac{\varepsilon_g}{\varepsilon_l + \varepsilon_g}d_g$  is intended to converge toward  $\partial_x u$ . The formal computation to obtain

the equation on  $\varphi_g$  in (3.13) is straightforward when the two viscosity coefficients are equal. We thus first assume this is the case before obtaining the general law for  $\varphi_g$ .

- Case where  $\mu_g = \mu_l = \mu$

The continuity of the effective flux combined with the regularity on pure zones is expressed as

$$p_l(t) - \mu d_l(t) = p_g(t) - \mu d_g(t) + r(\varepsilon_l + \varepsilon_g),$$

where  $p_i(t)$  (for  $i = g, l$ ) is a shorthand for  $p_i(\rho(t, x_i(t)))$  and  $r$  is a function measuring the error of the derivative approximation such that  $r(x) \rightarrow 0$  as  $x \rightarrow 0^+$ . This can be rewritten

$$p_g(t) - p_l(t) = \mu \frac{\varepsilon_l(u(t, x_g(t)) - u(t, x(t))) - \varepsilon_g(u(t, x(t)) - u(t, x_l(t)))}{\varepsilon_g \varepsilon_l} + r(\varepsilon_l + \varepsilon_g),$$

and, thanks to (3.24) and letting  $\varepsilon_i \rightarrow 0$  for  $i = g, l$ ,

$$p_g(t) - p_l(t) \approx \mu \frac{(\varepsilon_g + \varepsilon_l)^2}{\varepsilon_g \varepsilon_l} D_t \varphi_g = \frac{\mu}{\varphi_g \varphi_l} D_t \varphi_g,$$

which is exactly what is stated in (3.13).

- Case where  $\mu_g \neq \mu_l$

Using  $d_l(t)$  and  $d_g(t)$ , we can rewrite (3.24) as

$$D_t \varphi_g(t) = \frac{\varepsilon_l \varepsilon_g}{(\varepsilon_l + \varepsilon_g)^2} (d_g(t) - d_l(t)).$$

We would like to express the limit, as  $\varepsilon_l + \varepsilon_g$  tends to 0, of the right-hand side term as a function of the limit quantities. In other words, the limit of the right-hand side should be expressed as a function of the limit unknowns  $\varphi_g, \varphi_l, p_g, p_l, \partial_x u \dots$ . We already know that  $\frac{\varepsilon_l \varepsilon_g}{(\varepsilon_l + \varepsilon_g)^2}$  converges to  $\varphi_g \varphi_l$ . Only the term  $d_g - d_l$  remains to be processed. As  $\mu_g d_g - \mu_l d_l$  is intended to converge to  $p_g - p_l$ , it is quite natural to try writing

$$d_g - d_l = a(\mu_g d_g - \mu_l d_l) + (1 - a\mu_g)d_g - (1 - a\mu_l)d_l$$

with  $a \in \mathbb{R}$  such that there exists  $b \in \mathbb{R}$  satisfying

$$1 - a\mu_g = b\varphi_g \quad \text{and} \quad 1 - a\mu_l = -b\varphi_l.$$

In such a case one would have

$$\lim_{\varepsilon_l + \varepsilon_g \rightarrow 0} (d_g - d_l) = a(p_g - p_l) + b\partial_x u.$$

The linear system in  $a$  and  $b$  has a unique solution,  $a = \frac{1}{\varphi_l \mu_g + \varphi_g \mu_l}$  and  $b = \frac{\mu_l - \mu_g}{\varphi_l \mu_g + \varphi_g \mu_l}$ , which finally gives

$$D_t \varphi_g = \frac{\varphi_g \varphi_l}{\varphi_l \mu_g + \varphi_g \mu_l} (p_g - p_l - (\mu_g - \mu_l) \partial_x u),$$

which is exactly the equation on  $\varphi_g$ .

We do not discuss here the derivation of the homogenized equation on  $\rho = \varphi_g \rho_g + \varphi_l \rho_l$  and  $u$ . We instead focus on how to recover the effective viscosity  $\mu_{\text{eff}}$  and pressure  $p_{\text{eff}}$ . To that end, we use that the Cauchy stresses defined by  $\sigma_l = -p_l + \mu_l d_l$  and  $\sigma_g = -p_g + \mu_g d_g$  converge strongly to a same function  $\sigma$  because of the continuity of the stress tensor. To obtain this stress tensor  $\sigma$ , we write

$$\frac{\varepsilon_l}{\varepsilon_l + \varepsilon_g} d_l + \frac{\varepsilon_g}{\varepsilon_l + \varepsilon_g} d_g = \frac{\varepsilon_l}{\varepsilon_l + \varepsilon_g} \frac{p_l + \sigma_l}{\mu_l} + \frac{\varepsilon_g}{\varepsilon_l + \varepsilon_g} \frac{p_g + \sigma_g}{\mu_g}$$

As the left hand-side converges to  $\partial_x u$ , the strong convergence of the Cauchy stress yields that

$$\partial_x u = \left( \varphi_l \frac{p_l}{\mu_l} + \varphi_g \frac{p_g}{\mu_g} \right) - \left( \frac{\varphi_l}{\mu_l} + \frac{\varphi_g}{\mu_g} \right) \sigma.$$

and therefore the Cauchy stress limit is

$$\sigma = -\frac{\varphi_l \mu_g p_l + \varphi_g \mu_l p_g}{\varphi_l \mu_g + \varphi_g \mu_l} + \frac{\mu_l \mu_g}{\varphi_l \mu_g + \varphi_g \mu_l} \partial_x u.$$

Assuming as before the form  $\sigma = -p_{\text{eff}} + \mu_{\text{eff}} \partial_x u$ , the identification of the coefficients in the above formula yields the coefficient  $\mu_{\text{eff}}$  and  $p_{\text{eff}}$  of (3.14).

**Approach 3. WKB technique as in [Bresch et al., 2022a]** (see also [Bresch and Hillairet, 2015]). The two-scale asymptotic technique is an intuitive averaging tool to perform an homogenization. Its key idea is to assume that the unknowns depend on different scales of time and of space. We focus on its formal application to the two-phase, compressible setting by using some compactness properties of the effective flux and using the re-normalization approach for the transport equation. We follow the formal calculation performed in [Bresch et al., 2022a]. It starts by assuming two additional scale dependencies at long time,  $\tau = \frac{t}{\varepsilon}$ , and distance,  $y = \frac{x}{\varepsilon}$ . The asymptotic system is written by assuming formally the following ansatz

$$\begin{aligned} X_g^\varepsilon(t, x) &= X_{g,0} \left( t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon} \right) + \varepsilon X_{g,1} \left( t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon} \right) + O(\varepsilon^2) \\ u^\varepsilon(t, x) &= u_0 \left( t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon} \right) + \varepsilon u_1 \left( t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon} \right) + O(\varepsilon^2) \\ \rho^\varepsilon(t, x) &= X_g \left( t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon} \right) \rho_g^\varepsilon(t, x) + \left( 1 - X_g \left( t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon} \right) \right) \rho_l^\varepsilon(t, x) \\ \mu^\varepsilon(t, x) &= X_g^\varepsilon(t, x) \mu_g + (1 - X_g^\varepsilon(t, x)) \mu_l \\ p^\varepsilon(t, x) &= X_g \left( t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon} \right) p_g(\rho) + \left( 1 - X_g \left( t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon} \right) \right) p_l(\rho) \end{aligned}$$

assuming, for  $i = g, l$ , that

$$\rho_i^\varepsilon(t, x) = \rho_i^0(t, x) + O(\varepsilon), \quad X_g(t, \tau, x, y) \in \{0, 1\} \text{ periodic in variables } \tau, y.$$

In essence, the reasoning consists in plugging these ansatz into the two-phase system (3.12) and identify the equations related to different powers of  $\varepsilon$ . Starting with the mass equation for  $X_g$ ,  $\partial_t X_g + u^\varepsilon \partial_x X_g = 0$  is written as:

$$\partial_t X_{g,0} + \frac{1}{\varepsilon} \partial_\tau X_{g,0} + \partial_\tau X_{g,1} + u_0 \partial_x X_{g,0} + \frac{1}{\varepsilon} u_0 \partial_y X_{g,0} + u_0 \partial_y X_{g,1} + u_1 \partial_y X_{g,0} = O(\varepsilon)$$

Then, we get at order  $\varepsilon^{-1}$ ,

$$\partial_\tau X_{g,0} + u_0 \partial_y X_{g,0} = 0 \tag{3.25}$$

and at order 1

$$\partial_t X_{g,0} + \partial_\tau X_{g,1} + u_0 \partial_x X_{g,0} + u_1 \partial_y X_{g,0} + u_0 \partial_y X_{g,1} = 0. \quad (3.26)$$

Equation (3.25) yields the behavior of  $X_g$  on a cell; this equation is compatible with the assumption that  $X_g$  is an indicator function on the fast variables  $\tau$  and  $y$ . Averaging the second equation (3.26) with respect to the fast variables  $y$  yields:

$$\partial_t \varphi_{g,0} + \partial_\tau \overline{X_{g,1}} + \overline{u_0 \partial_x X_{g,0}} + \overline{u_0 \partial_y X_{g,1}} = -\overline{u_1 \partial_y X_{g,0}} \quad (3.27)$$

where we denote for now the average with respect to  $y$  on a cell with a bar. We deduce next that

$$\varphi_g = \overline{X_g} = \overline{X_{g,0}} + \varepsilon \overline{X_{g,1}} + \mathcal{O}(\varepsilon^2) = \varphi_{g,0} + \varepsilon \varphi_{g,1} + \mathcal{O}(\varepsilon^2).$$

The main objective is now to characterize all the averaging quantities  $\overline{u_0 \partial_x X_{g,0}}$ ,  $\overline{u_1 \partial_y X_{g,0}}$  and  $\overline{u_0 \partial_y X_{g,1}}$  to deduce the equation on  $\varphi_{g,0}$ . We start by searching what is the behavior of the velocity  $u$  by plugging the ansatz in the momentum equation. We obtain at order  $\varepsilon^{-2}$  that

$$\partial_y \left( (X_{g,0} \mu_g + (1 - X_{g,0}) \mu_l) \partial_y u_0 \right) = 0.$$

Multiplying this equation by  $u_0$ , integrating the result with respect to  $(t, \tau, x, y)$ , and using that

$$X_{g,0} \mu_g + (1 - X_{g,0}) \mu_l \geq \min(\mu_g, \mu_l) > 0$$

we obtain that

$$\partial_y u_0 = 0,$$

which means that  $u_0$  does not depend on  $y$ . Using that lack of dependency and the periodical boundary conditions, the averaged terms on the left-hand side of (3.27) can be written as

$$\overline{u_0 \partial_x X_{g,0}} + \overline{u_0 \partial_y X_{g,1}} = u_0 \overline{\partial_x X_{g,0}} = u_0 \partial_x \varphi_{g,0}.$$

Then (3.27) becomes

$$\partial_t \varphi_{g,0} + \partial_\tau \varphi_{g,1} + u_0 \partial_x \varphi_{g,0} = -\overline{u_1 \partial_y X_{g,0}} \quad (3.28)$$

Coming back to the equation (3.25) and averaging it with respect to  $y$ , we deduce that  $\varphi_{g,0}$  does not depend on  $\tau$  because  $u_0$  does not depend on  $y$ . Focusing on the main part of the momentum equation, using the ansatz yields at order  $\varepsilon^{-1}$ :

$$\rho_0 (\partial_\tau u_0 + u_0 \partial_y u_0) + \partial_y p_0 = \partial_y (\mu_0 \partial_y u_1) + \partial_x (\mu_0 \partial_y u_0) + \partial_y (\mu_0 \partial_x u_0),$$

where we used the shorthand  $p_0 = X_{g,0} p_g(\rho_{g,0}) + (1 - X_{g,0}) p_l(\rho_{l,0})$  and  $\mu_0 = X_{g,0} \mu_g + (1 - X_{g,0}) \mu_l$ . Multiplying this equation by  $\partial_\tau u_0$  and integrating with respect to  $(t, \tau, x, y)$  yields that

$$\partial_\tau u_0 = 0.$$

So  $u_0$  is independent of the fast variables  $\tau$  and  $y$ . Thus this equation simplifies to

$$\partial_y (\mu_0 \partial_y u_1) - \partial_y p_0 = -\partial_y (\mu_0 \partial_x u_0).$$

Considering that  $\partial_y \phi = 0$  implies  $\phi - \overline{\phi} = 0$  for any variable  $\phi$ , we obtain for  $\phi = \mu_0 \partial_y u_1 - p_0 + \mu_0 \partial_x u_0$  that

$$\mu_0 \partial_y u_1 - \overline{\mu_0 \partial_y u_1} - (p_0 - \overline{p_0}) = -(\mu_0 - \overline{\mu_0}) \partial_x u_0.$$

By writing  $\mu_0 = \mu_l + (\mu_g - \mu_l)X_{g,0}$  and  $\overline{\mu_0 \partial_y u_1} = (\mu_g - \mu_l)\overline{X_{g,0} \partial_y u_1}$ , we can obtain

$$\mu_0 \partial_y u_1 = (p_0 - \overline{p_0}) - (\mu_0 - \overline{\mu_0}) \partial_x u_0 + (\mu_g - \mu_l) \overline{X_{g,0} \partial_y u_1},$$

which may be rewritten as

$$X_{g,0} \partial_y u_1 = \frac{X_{g,0}}{\mu_0} \left( (p_0 - \overline{p_0}) - (\mu_0 - \overline{\mu_0}) \partial_x u_0 + (\mu_g - \mu_l) \overline{X_{g,0} \partial_y u_1} \right).$$

Therefore, averaging with respect to  $y$  and using that some quantities do not depend on  $y$  yields

$$\overline{X_{g,0} \partial_y u_1} = \frac{\varphi_{g,0}(1 - \varphi_{g,0})}{\mu_l} \left( (p_g(\rho_{g,0}) - p_l(\rho_{l,0})) - (\mu_g - \mu_l) \partial_x u_0 \right) + \varphi_{g,0} \left( 1 - \frac{\mu_g}{\mu_l} \right) \overline{X_{g,0} \partial_y u_1}.$$

This relationship allows us to find that

$$\overline{X_{g,0} \partial_y u_1} = -\frac{\varphi_{g,0}(1 - \varphi_{g,0})}{\mu_0} (\sigma_{g,0} - \sigma_{l,0}) \quad \text{with} \quad \sigma_{i,0} = -p_i(\rho_{i,0}) + \mu_i \partial_x u_0.$$

Using that

$$\overline{u_1 \partial_y X_{g,0}} = -\overline{X_{g,0} \partial_y u_1},$$

we obtain the formula of  $-\overline{u_1 \partial_y X_{g,0}}$  that can be plugged into (3.28). Averaging equation (3.25) in  $y$  and using that  $u_0$  does not depend on  $y$  yields

$$\partial_\tau \varphi_{g,0} = 0.$$

This result allows us to conclude also that  $-\overline{X_{g,0} \partial_y u_1}$  does not depend on  $\tau$  because  $\varphi_{g,0}$  does not depend on  $\tau$ . Finally, differentiating (3.28) with respect to  $\tau$  proves that  $\partial_\tau^2 \varphi_{g,1} = 0$  and therefore that  $\partial_\tau \varphi_{g,1} = 0$  if it holds initially. This yields the equation on  $\varphi_g$  in (3.13).

### 3.2.2 Two-phase system with temperature

The system equivalent to (3.11) with varying temperature  $\theta$  is the heat-conducting, viscous compressible Navier–Stokes system with  $i = g, l$ :

$$\begin{cases} \partial_t \rho_i + \partial_x(\rho_i u) = 0, \\ \partial_t(\rho_i u) + \partial_x(\rho_i u^2) = \partial_x \sigma_i, \\ \partial_t(\rho_i e_i) + \partial_x(\rho_i e_i u) = \partial_x Q_i + \sigma_i \partial_x u \end{cases} \quad (3.29)$$

where

$$\sigma_i = \mu_i \partial_x u - p_i, \quad Q_i = \kappa_i \partial_x \theta$$

where  $\mu_i$  and  $\kappa_i$  (heat conductivity) are constant. The constitutive equations for the pressure  $p_i$  and the internal energy  $e_i$  are

$$p_i = c_{p,i} \rho_i \theta, \quad e_i = c_{v,i} \theta,$$

where the thermal capacities  $c_{v,i}$  and  $c_{p,i}$  are also constant. The initial conditions are

$$\rho_i|_{t=0} = \rho_{i,0}, \quad u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0.$$

The homogenization (or averaging) of this system has been carried out in [Hillairet, 2019] by introducing the same indicator function  $X_g$  as before such that

$$\partial_t X_g + u \partial_x X_g = 0, \quad \text{with} \quad X_g X_l = 0 \quad \text{where} \quad X_l = 1 - X_g.$$

As in the case without temperature, this quantity indicates if we are in the zone  $g$  or in the zone  $l$ . As a result, the thermal capacities  $c_{v,i}, c_{p,i}$  and the heat conductivity  $\kappa_i$  may be different for each phase. All these parameters are assumed to be constant and strictly positive. We may rewrite the last equation of (3.29) in terms of temperature, by using that  $q_i = c_{v,i}\rho_i$ :

$$\partial_t(q_i\theta) + \partial_x(q_i\theta u) = \partial_x Q_i + \sigma_i \partial_x u.$$

Using the  $X_g$  variable, the full system now reads

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t q + \partial_x(qu) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x \sigma, \\ \partial_t(q\theta) + \partial_x(q\theta u) = \partial_x Q + \sigma \partial_x u, \end{cases} \quad (3.30)$$

where

$$\rho = X_g \rho_g + (1 - X_g) \rho_l, \quad \sigma = \mu \partial_x u - p, \quad Q = \kappa \partial_x \theta$$

with the constitutive equations

$$\mu = \mu_g X_g + \mu_l (1 - X_g), \quad q = (c_{v,g} X_g + c_{v,l} (1 - X_g)) \rho, \quad \kappa = \kappa_g X_g + \kappa_l (1 - X_g), \quad p = (c_{p,g} X_g + c_{p,l} (1 - X_g)) \rho \theta.$$

To extend the validity of this system over the whole domain, [Hillairet, 2019] uses the continuity of the normal stress and the thermal fluxes at the interfaces. Then, assuming that the initial data for  $X_g$  oscillates in space as

$$X_g^\varepsilon|_{t=0} = X_{g,0}(x/\varepsilon)$$

and using homogenization techniques aimed at obtaining the limit when  $\varepsilon \rightarrow 0$ , [Hillairet, 2019] derives the following homogenized system:

$$\begin{cases} \partial_t \varphi_g + u \partial_x \varphi_g = \frac{\varphi_g \varphi_l}{\varphi_g \mu_l + \varphi_l \mu_g} \left( (c_{p,g} \rho_g - c_{p,l} \rho_l) \theta + (\mu_l - \mu_g) \partial_x u \right) \\ \partial_t(\varphi_g \rho_g) + \partial_x(\varphi_g \rho_g u) = 0 \\ \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x \sigma \\ \partial_t(q\theta) + \partial_x(q\theta u) = \partial_x Q + \sigma \partial_x u, \end{cases} \quad (3.31)$$

with the constitutive equations

$$\begin{cases} \rho = \varphi_g \rho_g + \varphi_l \rho_l, \\ q = \varphi_g c_{v,g} \rho_g + \varphi_l c_{v,l} \rho_l, \\ \sigma = \frac{\mu_l \mu_g}{\varphi_l \mu_g + \varphi_g \mu_l} \partial_x u - \frac{\theta}{\varphi_g \mu_l + \varphi_l \mu_g} (\varphi_g c_{p,g} \mu_l \rho_g + \varphi_l c_{p,l} \mu_g \rho_l) \\ Q = \frac{\kappa_g \kappa_l}{\varphi_g \kappa_l + \varphi_l \kappa_g} \partial_x \theta \end{cases} \quad (3.32)$$

The work of [Hillairet, 2019] rigorously develops the derivation of this two-phase system with one velocity for viscous compressible fluids with temperature, thus generalizing what was done in the barotropic case in [Bresch and Huang, 2011, Bresch and Hillairet, 2015, Bresch et al., 2022a, Bresch et al., 2023a]. The key point of this analysis is to derive a family of estimates for the system considered (here the compressible Navier–Stokes equations with temperature) that is compatible with discontinuous densities and smooth velocities. This analysis also yields the family of solutions on which the homogenization process can be performed, highlighting which terms oscillate

and which ones do not when obtaining the homogenized system. In the final step, [Hillairet, 2019] obtained the multiphase system by plugging an explicit form on the measure families into the homogenized system. It is not clear, however, if such an ansatz is preserved even at short time scales. In the case of isentropic systems, [Bresch and Hillairet, 2015] and [Bresch and Hillairet, 2019] proposed to address this issue with a weak-strong uniqueness argument. We note that when calculating the limit, the velocity field of the fluid has the additional property  $\partial_x u \in L^1(0, T; L^\infty(\Omega))$ . This property opens the possibility to construct solutions as convex combinations of Dirac measures – with smooth weights  $\varphi_j$  and centers  $\rho_j$  – and then to show that any solution is necessarily equal to these solutions at a short time scale.

*How does such derivation works, formally speaking?* Let us try to justify formally the derivation performed in [Hillairet, 2019]. Note first that we have temperature conduction and diffusion of the velocity field. In other words, there is no step discontinuity for  $\sigma^\varepsilon = \mu(X_g^\varepsilon) \partial_x u^\varepsilon - p(\rho^\varepsilon, X_g^\varepsilon, \theta^\varepsilon)$  with  $p(\rho^\varepsilon, X_g^\varepsilon, \theta^\varepsilon) = c_p(X_g^\varepsilon) \rho^\varepsilon \theta^\varepsilon$  and no discontinuity on the temperature flux  $Q^\varepsilon = \kappa(X_g^\varepsilon) \partial_x \theta^\varepsilon$ , which ensures compactness in space. We define the two respective limits by  $\sigma$  and  $Q$ . As in the case without temperature, we use this strong convergence in space to rewrite the equations on  $\sigma^\varepsilon$  and  $Q^\varepsilon$  as:

$$\partial_x u^\varepsilon = \frac{1}{\mu(X_g^\varepsilon)} \sigma^\varepsilon + \frac{p(\rho^\varepsilon, X_g^\varepsilon, \theta^\varepsilon)}{\mu(X_g^\varepsilon)}$$

and

$$\partial_x \theta^\varepsilon = \frac{1}{\kappa(X_g^\varepsilon)} Q^\varepsilon.$$

This allows us to take the limit in products on both right-hand sides by using the time derivative control of  $X_g^\varepsilon$  and the bound on the space derivative of  $Q^\varepsilon$  and  $\sigma^\varepsilon$  to obtain that

$$\partial_x u = \left\langle \frac{1}{\mu(X_g)} \right\rangle \sigma + \left\langle \frac{p(\rho, X_g, \theta)}{\mu(X_g)} \right\rangle$$

and

$$\partial_x \theta = \left\langle \frac{1}{\kappa(X_g)} \right\rangle Q$$

where  $\langle g \rangle = \langle \bar{v}_x, g \rangle$  is the weak limit of any function  $g^\varepsilon$ . We assume, as in the case without temperature, that the measures associated to the weak limits are characterized by a convex combination of two Dirac masses (see equation (3.22)):

$$\bar{v}_x(t) = \varphi_g(t, x) \delta_{(\rho_g(t, x), 1)} + \varphi_l(t, x) \delta_{(\rho_l(t, x), 0)}.$$

This provides the following equations of  $\sigma$  and  $Q$ :

$$\sigma = \frac{1}{\left\langle \frac{1}{\mu(X_g)} \right\rangle} \partial_x u - \frac{1}{\left\langle \frac{1}{\mu(X_g)} \right\rangle} \left\langle \frac{p(\rho, X_g, \theta)}{\mu(X_g)} \right\rangle = \frac{1}{\frac{\varphi_g}{\mu_g} + \frac{\varphi_l}{\mu_l}} \partial_x u - \frac{1}{\frac{\varphi_g}{\mu_g} + \frac{\varphi_l}{\mu_l}} \left( \frac{c_{p,g} \varphi_g \rho_g}{\mu_g} + \frac{c_{p,l} \varphi_l \rho_l}{\mu_l} \right) \theta$$

and

$$Q = \frac{1}{\left\langle \frac{1}{\kappa(X_g)} \right\rangle} \partial_x \theta = \frac{1}{\frac{\varphi_g}{\kappa_g} + \frac{\varphi_l}{\kappa_l}} \partial_x \theta$$

We now calculate the limit in the momentum equation and in the temperature equation by using a similar convergence as described in Approach 1 in Section 3.2.1, which yields

$$\partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x \sigma$$

and

$$\partial_t(q\theta) + \partial_x(q\theta u) = \partial_x Q + \sigma \partial_x u$$

where  $\rho$  and  $q$  are the weak limits of  $\rho^\varepsilon$  and  $q^\varepsilon$  (i.e.  $\rho = \varphi_g \rho_g + \varphi_l \rho_l$  and  $q = \varphi_g c_{p,g} \rho_g + \varphi_l c_{p,l} \rho_l$ ). For the equation on  $\varphi_g$ , we obtain a relationship that is similar to that of the barotropic case:

$$\partial_t \varphi_g + u \partial_x \varphi_g = \frac{\varphi_g \varphi_l}{(\varphi_l \mu_g + \varphi_g \mu_l)} \left( (p_g(\rho_g, \theta) - p_l(\rho_l, \theta)) + (\mu_l - \mu_g) \partial_x u \right)$$

and therefore we obtain the final equation on  $\varphi_g$  by recalling that  $p_g(\rho_g, \theta) = c_{p,g} \rho_g \theta$  and  $p_l(\rho_l, \theta) = c_{p,l} \rho_l \theta$ .

**Key insight 13.** This Section focused on the case with thermal conductivity (i.e. no oscillations on the temperature), which greatly simplifies the calculations and the limit model that is obtained if the thermal conductivity is negligible. Such a study is possible with more quantities that oscillate than this case. This is the subject of a forthcoming paper: the relaxed system is much richer physically and mathematically.

**Some insights into the multidimensional case.** The derivation of a two-phase system in more than one spatial dimension using the homogenization procedure is an open problem when two different pressure laws and different viscosities are required. With identical constant viscosities, the homogenization with fast-oscillating initial density (averaging procedure) has been investigated in [Hillairet, 2007] and [Plotnikov and Sokolowski, 2012] starting with global weak solutions that justify a kinetic equation on the defect measures. More specifically, one can justify the following coupled macroscopic-kinetic system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla \langle \bar{v}_x, p(\xi) \rangle = 0, \\ \partial_t \bar{v}_x + \operatorname{div}(u \bar{v}_x) - \frac{1}{\lambda + 2\mu} \left( \partial_\xi(\xi \bar{v}_x) G - \partial_\xi(\xi p(\xi) \bar{v}_x) \right) = 0 \end{cases} \quad (3.33)$$

where  $G = (2\mu + \lambda) \operatorname{div} u - \langle \bar{v}_x, p(\xi) \rangle$ . To characterize the limit and show that it may be written as a two-phase system, we need more information on the unknowns. Specifically, more information on  $\operatorname{div} u$  is important because having appropriate bounds on  $\operatorname{div} u$  allows one to follow the interfaces and therefore opens the possibility to characterize the macroscopic limit. This can be done by using the Hoff's solutions framework as first mentioned in [Bresch and Huang, 2011] and then fully developed in [Bresch and Hillairet, 2015]. Such solution framework starts with

**Theorem 3.1.** *Assume  $\Omega = \mathbb{T}^3$ . Given a positive  $\rho^0 \in L^\infty(\Omega)$  and  $u^0 \in H^1(\mathbb{T}^3)$ , there exists  $T_0 > 0$  and a finite energy, weak solution  $(\rho, u)$  to the compressible Navier–Stokes equations that satisfy the initial conditions such that*

- $\rho \in L^\infty((0, T_0) \times \mathbb{T}^3)$  and  $\nabla u \in L^\infty((0, T); L^2(\mathbb{T}^3))$
- $\sqrt{\rho} \partial_t u \in L^2((0, T_0) \times \mathbb{T}^3)$  and  $Pu \in L^2((0, T_0); H^2(\mathbb{T}^3))$
- $G := (\lambda + 2\mu) \operatorname{div} u - p(\rho) \in L^2(0, T_0; H^1(\mathbb{T}^3))$

where  $T_0$  depends only on  $\|\rho_0\|_{L^\infty(\mathbb{T}^3)}$  and  $\|u_0\|_{H^1(\mathbb{T}^3)}$ .

Then the following result is proven:

**Theorem 3.2.** *Let initial data  $(\rho_n^0, u_n^0) \in L^\infty(\mathbb{T}^3) \times H^1(\mathbb{T}^3)$  satisfy*

- $\|\rho_n^0\|_{L^\infty(\mathbb{T}^3)} + \|u_n^0\|_{H^1(\mathbb{T}^3)} \leq C$
- $0 < 1/C \leq \rho_n^0(x)$



- The Young measures  $\nu_n^0$  associated with  $\rho_n^0$  converge weakly to

$$\nu^0 = \varphi_g^0 \delta_{\rho_g^0(x)} + (1 - \varphi_g^0) \delta_{\rho_l^0(x)} \text{ on } \Omega.$$

Then, given  $p(z) = az^\gamma$  with  $\gamma > 1$ ,

- There exists  $T > 0$  which does not depend on  $n$  and a solution  $(\rho_n, u_n)$  of the compressible Navier–Stokes equation in the sense given above.
- Up to the extraction of a subsequence

$$\nu_n \rightharpoonup \nu = \varphi_g \delta_{\rho_g} + (1 - \varphi_g) \delta_{\rho_l}, \quad u_n \rightharpoonup u, \quad p_n \rightharpoonup p$$

- $(\varphi_g, \rho_g, u, p)$  is solution of the homogenized two-phase system.

**Remark 3.4.** Interestingly, this result holds for the same pressure state law  $p(z) = az^\gamma$  for the two phases because of the absence of the function  $X_g$  in the mathematical analysis. This result has been extended recently in [Bresch et al., 2022a] in one spatial dimension. The multidimensional case is an open problem if the phases do not have the same viscosities.

## 4 Different two-phase averaged viscous systems (Nonlinear weak stability)

This Section reviews different formulations of two-phase averaged systems with viscous effects. We discuss well-posedness with a focus on weakly nonlinear stability in the Leray sense, listing the important quantities and properties that allow one to conclude such stability.

### 4.1 Averaged two-phase systems with one velocity field

*Averaged two-phase flows with heterogeneous pressure.* Assume that, instead of the indicator function  $X_i$  introduced in the previous Section, which solves an advection equation for  $\varphi_g$ , we consider an advected function  $c$  satisfying the equation

$$\partial_t c + u \cdot \nabla c = 0 \text{ with the constraint } 0 \leq c \leq 1$$

that is coupled with the mass and momentum equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho, c) D(u)) + \nabla p(\rho, c) = 0, \\ \rho u|_{t=0} = m_0, \quad \rho|_{t=0} = \rho_0, \quad c|_{t=0} = c_0. \end{cases} \quad (4.1)$$

Physically,  $c$  may be for instance the entropy  $S$  for the compressible Navier–Stokes equations with a pressure  $p(\rho, S) = \rho S$  and  $\mu(\rho, S) = \mu = \text{cst}$ ,  $\lambda(\rho, S) = \lambda = \text{cst}$ , such as mentioned in [Bresch et al., 2002]. It could also be the chemical potential  $\mu_p$ , as considered by [Michoski and Vasseur, 2008] with a pressure

$$p(\rho, \mu_p) = C_1 \rho^{\gamma_1(\mu_p)} - C_2 \rho^{\gamma_2(\mu_p)} + C_3 \rho^{\gamma_3(\mu_p)}$$

and an appropriate viscous term  $\mu(\rho, \mu_p)$  in one dimension. In all these cases the pressure depends on two quantities, one that satisfies a transport equation ( $\rho$ ) and the other that satisfies an advection equation ( $c$ ). The proof of the global existence of weak solutions depends on whether the spatial dimension is one or higher. In one dimension, [Michoski and Vasseur, 2008] have proved global existence of strong solutions for a pressure state law that increases as a function of density with a viscosity  $\mu(\rho, \mu_p)$  that depends on  $\mu_p$  and  $\rho$  and satisfies the

following constraint:  $\mu(\rho, \mu_p) = \rho \partial_\rho p(\rho, \mu_p) \psi'(p(\rho, \mu_p))$ , where  $\psi(p)$  is a function of pressure restricted by the form of its derivative in  $p$ . The proof is based on the standard energy estimates and on a generalization of the BD entropy (see (2.39)) related to the velocity  $u + \rho^{-1} \psi'(p)$ . We refer the readers interested in learning more about this proof to [Michoski and Vasseur, 2008].

*Averaged two-phase flows with algebraic closure.* This system with only one velocity field is:

$$\begin{cases} \partial_t(\varphi_g \rho_g) + \operatorname{div}(\varphi_g \rho_g u) = 0, \\ \partial_t(\varphi_l \rho_l) + \operatorname{div}(\varphi_l \rho_l u) = 0, \\ \partial_t((\varphi_g \rho_g + \varphi_l \rho_l)u) + \operatorname{div}((\varphi_g \rho_g + \varphi_l \rho_l)u \otimes u) - \operatorname{div}((\varphi_g \mu_g(\rho_g) + \varphi_l \mu_l(\rho_l))D(u)) \\ \quad - \nabla((\varphi_g \lambda_g + \varphi_l \lambda_l) \operatorname{div} u) + \nabla(p_g(\rho_g)) = (\varphi_g \rho_g + \varphi_l \rho_l) f, \\ p_g(\rho_g) = p_l(\rho_l). \end{cases} \quad (4.2)$$

This system can be written in a more appropriate way for our purpose by defining  $\rho = \varphi_g \rho_g$  and  $Z = \varphi_l \rho_l$  and solving the algebraic constraint

$$\frac{\rho}{\rho_g} + \frac{Z}{\rho_l} = 1, \quad p_g(\rho_g) = p_l(\rho_l), \quad (4.3)$$

to write  $p_g(\rho_g) = p(\rho, Z)$ . The rewritten system becomes

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t Z + \operatorname{div}(Z u) = 0, \\ \partial_t((\rho + Z)u) + \operatorname{div}((\rho + Z)u \otimes u) - \operatorname{div}((\alpha_g \mu_g(\rho_g) + \alpha_l \mu_l(\rho_l))D(u)) \\ \quad - \nabla((\alpha_g \lambda_g + \alpha_l \lambda_l) \operatorname{div} u) + \nabla(P(\rho, Z)) = (\rho + Z) f, \end{cases} \quad (4.4)$$

Different authors have studied such a system when assuming  $\mu_g = \mu_l = \text{cst}$  and  $\lambda_g = \lambda_l = \text{cst}$ . We refer the readers interested in learning more about the global existence of weak solutions (and therefore nonlinear weak stability) to [Novotný and Pokorný, 2020] and references therein, such as those of [Maltese et al., 2016, Vasseur et al., 2019].

*Averaged two-phase flows with PDE closure.* Recently, [Novotný, 2020] has studied a single velocity system that mimics an averaged mixture with no relaxation term in the equation related to the fraction:

$$\begin{cases} \partial_t \varphi_g + u \cdot \nabla \varphi_g = 0, \quad \varphi_l = 1 - \varphi_g \text{ with } 0 \leq \varphi_g \leq 1 \\ \partial_t(\varphi_g \rho_g) + \operatorname{div}(\varphi_g \rho_g u) = 0, \\ \partial_t(\varphi_l \rho_l) + \operatorname{div}(\varphi_l \rho_l u) = 0, \\ \partial_t((\varphi_g \rho_g + \varphi_l \rho_l)u) + \operatorname{div}((\varphi_g \rho_g + \varphi_l \rho_l)u \otimes u) - \operatorname{div}((\varphi_g \mu_g(\rho_g) + \varphi_l \mu_l(\rho_l))D(u)) \\ \quad - \nabla((\varphi_g \lambda_g(\rho_g) + \varphi_l \lambda_l(\rho_l)) \operatorname{div} u) + \nabla(\varphi_g p_g(\rho_g) + \varphi_l p_l(\rho_l)) = (\varphi_g \rho_g + \varphi_l \rho_l) f, \end{cases} \quad (4.5)$$

where  $\varphi_i$  is the volume fraction of each phase  $i = g, l$ , respectively. The author has succeeded to establish the global existence of weak solutions when the viscosities are assumed constant. To do so, the system is written as before in terms of  $\rho = \varphi_g \rho_g$ ,  $Z = \varphi_l \rho_l$  and choosing the same viscosities for all phases,  $\mu_g = \mu_l = \mu = \text{cst}$  and  $\lambda_g = \lambda_l = \lambda = \text{cst}$ :

$$\begin{cases} \partial_t \varphi_g + u \cdot \nabla \varphi_g = 0 \text{ with } 0 \leq \varphi_g \leq 1, \\ \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t Z + \operatorname{div}(Z u) = 0, \\ \partial_t((\rho + Z)u) + \operatorname{div}((\rho + Z)u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla p(f(\varphi_g) \rho, g(\varphi_g) Z) = (\rho + Z) f, \\ \rho u|_{t=0} = m_0, \quad \rho|_{t=0} = \rho_0, \quad Z|_{t=0} = Z_0, \quad \varphi_g|_{t=0} = \varphi_{g,0}. \end{cases} \quad (4.6)$$



**Theorem 4.1.** Assume  $F_{int,g} = F_{int,l} = 0$ ,  $p_{int,g} = p_g$  and  $p_{int,l} = p_l$  where  $p = p_g = \rho_g^{\gamma_g} = \rho_l^{\gamma_l} = p_l$  and  $f = 0$ . Let  $\gamma_g > 1$  and  $\gamma_l > 1$  and the initial data  $(R_{g,0}, m_{g,0})$  and  $(R_{l,0}, m_{l,0})$  satisfy initial data assumptions related to the energy and to the BD entropy and assume  $|m_{g,0}|^{2+\alpha_0}/R_{g,0}^{1+\alpha_0}$  and  $|m_{l,0}|^{2+\alpha_0}/R_{l,0}^{1+\alpha_0}$  in  $L^1(\Omega)$  for some constant  $\alpha_0$ . Then for any  $T > 0$ , there exists a global weak solution  $(\varphi_g, \varphi_l, \rho_l, \rho_g, u_l, u_g)$  to the two-phase system.

The proof relies on an original approximate system with the viscosities  $\mu(\rho_g) = \rho_g$  and  $\mu(\rho_l) = \rho_l$ .  $R_g = \varphi_g \rho_g$  and  $R_l = \varphi_l \rho_l$  are respectively replaced by  $R_g + \varepsilon R_g^\beta$ ,  $R_l + \varepsilon R_l^\beta$  with  $\beta$  depending on  $\gamma_g$  and  $\gamma_l$  in  $[1/4, 1/2)$ . An extra pressure term is added for each phase, respectively  $\varepsilon^{1/2}(R_g^{1/2} + R_g^\Lambda)$  and  $\varepsilon^{1/2}(R_l^{1/2} + R_l^\Lambda)$  with  $\Lambda = 2 \max(\gamma_g, \gamma_l)$ . This perturbation allows one to construct a global strong solution with  $R_i$  far from the vacuum and satisfying uniformly the energy, the BD entropy and an extra estimate on  $R_g |u_g|^{2+\delta_0}$  and  $R_l |u_l|^{2+\delta_0}$ . The proof also needs a Lemma that allows one to deduce useful information on  $\rho_g$ ,  $\rho_l$  when some information is known on  $R_g$  and  $R_l$ . When these elements are in place, the line of reasoning follows that used for single-phase systems.

We provide for the reader's convenience the energy estimate associated to the system and the BD entropy as described in [Bresch et al., 2012]. The framework of Theorem 4.1 above allows us to rewrite the system in terms of  $R_g = \varphi_g \rho_g$  and  $R_l = \varphi_l \rho_l$ :

$$\begin{cases} \partial_t R_g + \partial_x(R_g u_g) = 0, \\ \partial_t(R_g u_g) + \operatorname{div}(R_g u_g^2) - a_g \partial_x(R_g \partial_x u_g) = -\frac{\gamma_g}{\gamma_g - 1} R_g \partial_x(\rho_g^{\gamma_g - 1}) \\ \partial_t R_l + \partial_x(R_l u_l) = 0, \\ \partial_t(R_l u_l) + \operatorname{div}(R_l u_l^2) - a_l \partial_x(R_l \partial_x u_l) = -\frac{\gamma_l}{\gamma_l - 1} R_l \partial_x(\rho_l^{\gamma_l - 1}) \end{cases} \quad (4.8)$$

coupled with an algebraic system on  $\rho_g$ ,  $\rho_l$  that is obtained using the relationships  $\varphi_g = R_g/\rho_g$ ,  $\varphi_l = R_l/\rho_l$ ,  $\varphi_g + \varphi_l = 1$ , and  $p_g = \rho_g^{\gamma_g} = \rho_l^{\gamma_l} = p_l$ . This algebraic system reads

$$\begin{cases} R_g \rho_l + R_l \rho_g = \rho_g \rho_l, \\ \rho_g^{\gamma_g} = \rho_l^{\gamma_l}. \end{cases} \quad (4.9)$$

*Energy estimates.* To obtain the energy inequality for each phase, we start by multiplying the corresponding momentum equation by  $u_g$ ,  $u_l$ , which yields:

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} R_g |u_g|^2 + a_g \int_{\Omega} R_g |\partial_x u_g|^2 + \frac{\gamma_g}{\gamma_g - 1} \int_{\Omega} R_g u_g \partial_x(\rho_g^{\gamma_g - 1}) = 0$$

and

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} R_l |u_l|^2 + a_l \int_{\Omega} R_l |\partial_x u_l|^2 + \frac{\gamma_l}{\gamma_l - 1} \int_{\Omega} R_l u_l \partial_x(\rho_l^{\gamma_l - 1}) = 0.$$

Integrating by parts and using the equations on  $R_g$  and  $R_l$ , we can write

$$\begin{aligned} \frac{\gamma_g}{\gamma_g - 1} \int_{\Omega} R_g u_g \partial_x(\rho_g^{\gamma_g - 1}) &= \frac{\gamma_g}{\gamma_g - 1} \int_{\Omega} \rho_g^{\gamma_g - 1} \partial_t R_g \\ &= \frac{d}{dt} \int_{\Omega} R_g e_g(\rho_g) + \int_{\Omega} p \partial_t \left( \frac{R_g}{\rho_g} \right) \\ \frac{\gamma_l}{\gamma_l - 1} \int_{\Omega} R_l u_l \partial_x(\rho_l^{\gamma_l - 1}) &= \frac{\gamma_l}{\gamma_l - 1} \int_{\Omega} \rho_l^{\gamma_l - 1} \partial_t R_l \\ &= \frac{d}{dt} \int_{\Omega} R_l e_l(\rho_l) + \int_{\Omega} p \partial_t \left( \frac{R_l}{\rho_l} \right) \end{aligned} \quad (4.10)$$

with  $e_g = \int_0^{\rho_g} (p_g(\tau)/\tau^2)d\tau$  and  $e_l = \int_0^{\rho_l} (p_l(\tau)/\tau^2)d\tau$ . Therefore adding the  $g$  and  $l$  contributions together yields:

$$\sum_{i=g,l} \left( \frac{1}{2} \frac{d}{dt} \int_{\Omega} R_i |u_i|^2 + \frac{d}{dt} \int_{\Omega} R_i e_i(\rho_i) \right) + a_i \int_{\Omega} R_i |\partial_x u_i|^2 = - \sum_i \int_{\Omega} p \partial_t \varphi_i = 0 \quad (4.11)$$

Then the quantity

$$\sup_{t \in (0,T)} \left( \frac{1}{2} \int_{\Omega} R_i |u_i|^2 + \int_{\Omega} R_i e_i(\rho_i) \right) + a_i \int_0^T \int_{\Omega} R_i |\partial_x u_i|^2$$

is bounded for each phase  $i = g, l$ , and in consequence we find for each phase the bound

$$\left| \int_0^T \int_{\Omega} p \partial_t \varphi_i \right|,$$

which controls the energy exchange. Note that this result is no longer valid when distinct pressures are considered.

*BD entropy.* For the BD entropy, the new unknowns are  $a_i \log R_i$  for  $i = g, l$ . As in the single-phase case (see Section 2.2.2 for a proof of the relationship (2.39)), the equation satisfied by  $w_i = u_i + a_i \partial_x \log R_i$  for  $i = g, l$  is

$$\partial_t (R_i w_i) + \partial_x (R_i u_i w_i) + \frac{R_i}{\rho_i} \partial_x p = 0.$$

Multiplying this equation by  $w_i = u_i + a_i \partial_x \log R_i$  and using the mass equation yields

$$\frac{d}{dt} \int_{\Omega} R_i \frac{|w_i|^2}{2} + \int_{\Omega} \frac{R_i}{\rho_i} w_i \partial_x p = 0.$$

Observing that

$$\int_{\Omega} \frac{R_i}{\rho_i} w_i \partial_x p = \frac{d}{dt} \int_{\Omega} R_i e_i(\rho_i) + \int_{\Omega} p \partial_t \varphi_i + a_i \int_{\Omega} \frac{1}{\rho_i} \partial_x p \partial_x R_i$$

for  $i = g, l$ , we can write the above equality as

$$\frac{d}{dt} \int_{\Omega} R_i \frac{|w_i|^2}{2} + \frac{d}{dt} \int_{\Omega} R_i e_i(\rho_i) + \int_{\Omega} p \partial_t \varphi_i + a_i \int_{\Omega} \frac{1}{\rho_i} \partial_x p \partial_x R_i = 0.$$

We use the following equality

$$\int_{\Omega} \frac{1}{\rho_i} \partial_x p \partial_x R_i = \int_{\Omega} \partial_x p \partial_x \left( \frac{R_i}{\rho_i} \right) + \int_{\Omega} R_i p'_i(\rho_i) \frac{|\partial_x \rho_i|^2}{|\rho_i|^2}$$

to multiply the  $i = g$  instance by  $a_l$  and the  $i = l$  instance by  $a_g$ . Integrating them in time and adding them yields the following BD entropy inequality:

$$\begin{aligned} & \sup_{t \in [0,T]} \left( a_l \left( \int_{\Omega} R_g \frac{|w_g|^2}{2} + \int_{\Omega} R_g e_g(\rho_g) \right) + a_g \left( \int_{\Omega} R_l \frac{|w_l|^2}{2} + \int_{\Omega} R_l e_l(\rho_l) \right) \right) (t) \\ & + a_g a_l \left( \int_0^T \int_{\Omega} R_g p'_g(\rho_g) \frac{|\partial_x \rho_g|^2}{|\rho_g|^2} + \int_0^T \int_{\Omega} R_l p'_l(\rho_l) \frac{|\partial_x \rho_l|^2}{|\rho_l|^2} \right) \leq C < +\infty \end{aligned} \quad (4.12)$$

if the initial energy is bounded and if the initial data satisfy

$$\left( a_l \left( \int_{\Omega} R_g \frac{|w_g|^2}{2} + \int_{\Omega} R_g e_g(\rho_g) \right) + a_g \left( \int_{\Omega} R_l \frac{|w_l|^2}{2} + \int_{\Omega} R_l e_l(\rho_l) \right) \right) (0) \leq C.$$

The proof is concluded by using the energy estimate to deduce that the right-hand side is bounded. Similar relationships may be obtained in the multi-dimensional case.

**Multi-dimensional case.** The global existence of weak solution has been shown in a similar way by [Bresch et al., 2010] when capillarity quantities are included into the equation system. It would be useful to obtain similar results without capillarity by the recent studies done on single-phase system with density-dependent viscosities.

## 4.2.2 Two-phase system with pressure PDE closure

The two-phase system with a PDE closure for the pressures reads

$$\left\{ \begin{array}{l} \varphi_g + \varphi_l = 1, \\ \partial_t \varphi_g + u_I \cdot \nabla \varphi_g = \frac{\varphi_g \varphi_l}{\varphi_g \mu_l + \varphi_l \mu_g} (p_g(\rho_g) - p_l(\rho_l)), \\ \partial_t(\varphi_g \rho_g) + \operatorname{div}(\varphi_g \rho_g u_g) = 0, \\ \partial_t(\varphi_g \rho_g u_g) + \operatorname{div}(\varphi_g \rho_g u_g \otimes u_g) \\ \quad - 2 \operatorname{div}(\varphi_g \mu_g(\rho_g) D(u_g)) - \nabla(\varphi_g \lambda_g(\rho_g) \operatorname{div} u_g) + \varphi_g \nabla p_g(\rho_g) = \varphi_g \rho_g f \\ \partial_t(\varphi_l \rho_l) + \operatorname{div}(\varphi_l \rho_l u_l) = 0, \\ \partial_t(\varphi_l \rho_l u_l) + \operatorname{div}(\varphi_l \rho_l u_l \otimes u_l) \\ \quad - 2 \operatorname{div}(\varphi_l \mu_l(\rho_l) D(u_l)) - \nabla(\varphi_l \lambda_l(\rho_l) \operatorname{div} u_l) + \varphi_l \nabla p_l(\rho_l) = \varphi_l \rho_l f \\ (\varphi_g)_{t=0} = \varphi_{g,0}, \quad (\varphi_g \rho_g)|_{t=0} = R_{g,0}, \quad (\varphi_g \rho_g u_g)|_{t=0} = m_{g,0}, \\ (\varphi_l \rho_l)|_{t=0} = R_{l,0}, \quad (\varphi_l \rho_l u_l)|_{t=0} = m_{l,0}, \end{array} \right. \quad (4.13)$$

where  $u_I$  is the interface velocity, which needs to be specified according to the physical situation considered. There are no real results on the global existence of weak solutions for this system. The main difficulties stems from the non-conservative quantities  $\varphi_g \nabla p_g(\rho_g)$  and  $\varphi_l \nabla p_l(\rho_l)$ , from the relaxation quantity, and from the shear and bulk viscosities that do not have a priori relationships. We surmise that this limits severely the approach used to treat density-dependent viscosities. To our knowledge, the well-posedness of the two-phase system with a PDE closure for the pressures is an open problem without preliminary results.

## 5 Simplified two-phase averaged systems (Singular limits)

**Key insight 15.** The averaged multiphase models can be rewritten in different forms depending on the physical system considered. This reformulation can be as a system with algebraic relationships by using relaxation limits, as a drift-flux system coupling a reference velocity to a relative, drift velocity, or as dimensionless system. This last reformation allows one to simplify the model formally while keeping as much of the physics of the analyzed system as possible. This section aims at justifying these approximations mathematically and explaining the stability procedure that yields the different asymptotic formula obtained when specific small parameters vanish.

### 5.1 Relaxation limit

The relaxation limit allows one to go from the single-velocity, two-phase system with PDE closure (see (3.13)-(3.14)) to the single-velocity, two-phase system with algebraic constraint (see equation (4.7) for  $u_g = u_l = u$ ) when the viscosities tend to zero. This limit has been studied by [Burtea et al., 2023]. We focus on the single-velocity case with constant viscosities  $\mu$  and  $\lambda$ . We consider the following 2D system with PDE closure equivalent to (3.13) with  $\mu_g = \mu_l = 2\mu + \lambda$ :

$$\left\{ \begin{array}{l} \varphi_g + \varphi_l = 1, \\ \partial_t \varphi_g + u \cdot \nabla \varphi_g = \frac{\varphi_g \varphi_l}{2\mu + \lambda} (p_g(\rho_g) - p_l(\rho_l)), \\ \partial_t(\varphi_i \rho_i) + \operatorname{div}(\varphi_i \rho_i u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla p + \beta \rho u = 0 \\ p = \varphi_g p_g(\rho_g) + \varphi_l p_l(\rho_l), \quad \rho = \varphi_g \rho_g + \varphi_l \rho_l \end{array} \right. \quad (5.1)$$

for  $i = g, l$  with

$$p_i(s) = A_i s^{\gamma_i} \quad \text{for all } s \geq 0, \quad \text{and } \gamma_g > \gamma_l,$$

where  $A_g$  and  $A_l$  are two constants and  $\beta > 0$  is a fixed parameter.

The relaxation parameter is the viscosity  $2\mu + \lambda$ . The last term in the momentum equation represents a drag term with constant coefficient  $\beta$ , which is essential to obtain the sought convergence. An interesting mathematical open problem would be to allow such parameter  $\beta$  to vanish. The limit of this system (5.1) is the system with algebraic constraint (4.7) with  $u_g = u_l = u$ ,  $p_{int,i} = p_g = p_l$ ,  $F_{int,i} = \beta \rho u$  and  $f = 0$ :

$$\begin{cases} \varphi_g + \varphi_l = 1, \\ \partial_t(\varphi_i \rho_i) + \operatorname{div}(\varphi_i \rho_i u) = 0, \\ \partial_t(\varphi_i \rho_i u) + \operatorname{div}(\varphi_i \rho_i u \otimes u) + \nabla p + \beta \rho u = 0 \\ p = p_g(\rho_g) = p_l(\rho_l), \quad \rho = \varphi_g \rho_g + \varphi_l \rho_l. \end{cases} \quad (5.2)$$

This is an important result that indicates that a system with algebraic constraint has to be inviscid and, conversely, that the two-phase compressible system with PDE closure has a parameter depending on the viscosities even if it is usually neglected because it is considered small.

The main result in [Burtea et al., 2023] provides a rate of convergence from the system with relaxation to the system with identical pressures assuming an initial rate of convergence of some quantities related to the initial data at the order  $\sqrt{2\mu + \lambda}$ . This result is obtained with techniques linked to partially dissipative, first-order, quasilinear systems with the following general form:

$$\partial_t w + \sum_{j=1}^d \partial_j F_j(w) - \varepsilon \sum_{i,j=1}^d \partial_j (B_{ij}(w) \partial_j w) = \frac{Q(w)}{\varepsilon}$$

where

$$Q(w) = \begin{pmatrix} 0 \\ q(w) \end{pmatrix},$$

$B$  is a partially degenerate matrix,  $F_j$  are smooth functions from  $\mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $q(w) \in \mathbb{R}^{d_2}$  with  $d_2 + d_1 = d$  where  $0$  in  $Q(w)$  is  $0 \in \mathbb{R}^{d_1}$ ,  $w \in \mathbb{R}^d$ , and  $\varepsilon > 0$  is a given parameter. [Burtea et al., 2023] indicate how to recast the system (5.1) in this form with  $\varepsilon = \lambda + 2\mu$ . Partially dissipative means that the system may have second derivatives on some unknowns and no second-order derivatives on others; this is for instance the case of the compressible Navier-Stokes system and the system (5.1).

These types of quasilinear models govern the dynamics of physical system out of thermodynamic equilibrium. The constant  $\varepsilon$  may be seen as a relaxation time that is in practice small. A first approach has been proposed to obtain results on the global-in-time existence with initial data around a constant equilibrium. This approach is restricted to systems verifying the Shizuta Kawashima (SK) condition, which is an explicit linear stability criterion ensuring that all the components of  $w$  decay as  $t \rightarrow +\infty$ .

[Burtea et al., 2023] contains an overview of an extension of these results to more general quasilinear systems done by [Yong, 2004]. This extension involves a series of advanced techniques that illustrates the complexity of obtaining relaxation limits for such systems. The study of [Yong, 2004] proposes a second condition that consists of the existence of an entropy that provides a symmetrization compatible with the dissipation  $Q$ .

Inspired by the work of [Beauchard and Zuazua, 2011], [Crin-Barat and R.Danchin, 2022, Danchin, 2001] consider non-conservative partially dissipative hyperbolic systems that are Friedrichs-symmetrizable and construct a

Lyapunov functional implying that, close to equilibrium, the low frequencies of the solutions of the nonlinear system behave, qualitatively, like a heat equation. They also highlight a damped mode that has better decay properties in low frequencies than in high frequencies and deduce from it a crucial regularity enhancement to close the a priori estimates.

The method in [Crin-Barat and R.Danchin, 2022] is flexible enough to be adapted to the two-phase system for well-posedness purposes but issues arise when we want to have uniform estimates to justify the limit from the system with relaxation to the system with algebraic constraint. The first obvious reason is that the starting PDEs with relaxation are not all conservation laws because the equations for the volume fractions cannot be written in conservative form. The second reason is that the entropy naturally associated with this system is not positive definite since it is linear with respect to the volume fractions. Another issue concerns the results on the global existence of solutions because the associated quasilinear system does not satisfy the SK condition as it admits the eigenvalue 0. This situation, however, is not too degenerate because the eigen-space associated to the eigenvalue 0 is of dimension 1 and because, roughly speaking, the non-degenerate part (i.e. the part associated to non-zero eigenvalues) fulfills the SK condition. This allows [Burtea et al., 2023] to isolate the undamped mode and rewrite the remaining system as a partially dissipative quasilinear system satisfying the SK condition. The undamped mode is then seen as a parameter and always appears in nonlinear terms as a prefactor of a function of the damped variable. Note, finally, that the unknowns chosen by [Burtea et al., 2023] are related to the drift-flux system presented in the next section.

## 5.2 The drift-flux system with no temperature equations

In a drift-flux systems, the velocities of the phases are recast as a reference velocity and a relative (drift) velocity. We focus on how to derive a drift-flux system from a two-phase system with two compressible phases by using the work of [Ambroso et al., 2008]. We also highlight aspects of this derivation that have similarities with the drift flux model we propose in Part I, Section 4 [Narbona-Reina et al., 2024] so as to comment on the assumptions that may or may not be done when modeling volcanic conduit flows.

We first consider the following system for unknowns  $\mathbf{u} = (\varphi_g, \varphi_g \rho_g, \varphi_l \rho_l, \varphi_g \rho_g u_g, \varphi_l \rho_l u_l)$ ,

$$\left\{ \begin{array}{l} \varphi_g + \varphi_l = 1, \\ \partial_t(\varphi_g \rho_g) + \operatorname{div}(\varphi_g \rho_g u_g) = \Gamma(\mathbf{u}), \\ \partial_t(\varphi_g \rho_g u_g) + \operatorname{div}(\varphi_g \rho_g u_g \otimes u_g) + \nabla(\varphi_g p_g) = p_I(\mathbf{u}) \nabla \varphi_g \\ \quad + \varphi_g \rho_g f_g(\mathbf{u}) + \Lambda(\mathbf{u}) |u_l - u_g| (u_l - u_g), \\ \partial_t(\varphi_l \rho_l) + \operatorname{div}(\varphi_l \rho_l u_l) = -\Gamma(\mathbf{u}), \\ \partial_t(\varphi_l \rho_l u_l) + \operatorname{div}(\varphi_l \rho_l u_l \otimes u_l) + \nabla(\varphi_l p_l) = p_I(\mathbf{u}) \nabla \varphi_l \\ \quad + \varphi_l \rho_l f_l(\mathbf{u}) + \Lambda(\mathbf{u}) |u_g - u_l| (u_g - u_l), \\ \partial_t \varphi_g + u_I(\mathbf{u}) \cdot \nabla \varphi_g = \Theta(\mathbf{u}) (p_g - p_l) \end{array} \right. \quad (5.3)$$

where  $\varphi_i$ ,  $\rho_i$  and  $u_i$  are the void fraction, the density and the velocity of the phase  $i$  (for  $i = g, l$ ), respectively. External forces are denoted by  $f_i$ ,  $\Lambda$  and  $\Theta$  are positive relaxation functions, and the source term  $\Gamma$  corresponds to the mass transfer from phase  $g$  to phase  $l$ . The two functions  $u_I$  and  $p_I$  are the so-called interfacial velocity and interfacial pressure, which are convex combinations of  $u_g$  and  $u_l$ , and  $p_g$  and  $p_l$ , respectively:

$$\begin{aligned} u_I(\mathbf{u}) &= \beta_u(\mathbf{u}) u_l + (1 - \beta_u(\mathbf{u})) u_g \\ p_I(\mathbf{u}) &= \beta_p(\mathbf{u}) p_l + (1 - \beta_p(\mathbf{u})) p_g \end{aligned}$$



with  $\beta_u, \beta_p \in [0, 1]$ . The study of [Ambroso et al., 2008], assuming

$$\Lambda(\mathbf{u}) = \frac{\lambda(\mathbf{u})}{\varepsilon^2}, \quad \Theta(\mathbf{u}) = \frac{\theta(\mathbf{u})}{\varepsilon^2}, \quad \Gamma(\mathbf{u}) = \varepsilon\gamma(\mathbf{u}), \quad (5.4)$$

and defining the following drift variables

$$\begin{aligned} \rho &= \varphi_g \rho_g + \varphi_l \rho_l, & \rho Y &= \varphi \rho_g, & u &= Y u_g + (1 - Y) u_l, \\ w &= u_g - u_l, & p &= \varphi_g p_g + \varphi_l p_l, & q &= p_g - p_l, \end{aligned} \quad (5.5)$$

proves the following proposition:

**Proposition 5.1** (Proposition 3.1 in the numbering of [Ambroso et al., 2008]). *Let us denote  $\mathbf{v} = (\rho, \rho Y, \rho u)^T$  where  $\rho > 0$  and  $\rho Y / \varphi_g \in [0, 1]$ , and assume that  $\Gamma, \lambda$ , and  $f_i$  for  $i = g, l$  only depend on  $\mathbf{v}$ . Consider the system*

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho Y) + \operatorname{div}(\rho u Y + \rho Y(1 - Y)w) = \Gamma(\mathbf{v}), \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + p \operatorname{Id} + \rho Y(1 - Y)w \otimes w) = \rho Y f_g(\mathbf{v}) + \rho(1 - Y) f_l(\mathbf{v}), \end{cases} \quad (5.6)$$

where

$$|w|w = \frac{\rho Y(1 - Y)}{\Lambda(\mathbf{v})} \left( f_g(\mathbf{v}) - f_l(\mathbf{v}) + \left( \frac{1}{\rho_l(\mathbf{v})} - \frac{1}{\rho_g(\mathbf{v})} \right) \nabla p \right)$$

and with the pressure law  $p = \mathcal{P}(\mathbf{v})$  given by the solution  $(p, \varphi_g)$  of the  $2 \times 2$  nonlinear system

$$p = \mathcal{P}_l(\rho(1 - Y)/(1 - \varphi_g)),$$

$$\mathcal{P}_l(\rho(1 - Y)/(1 - \varphi_g)) = \mathcal{P}_g(\rho Y / \varphi_g).$$

This system (5.6) is closed and its solutions correspond to equilibrium solutions of the two-phase pressure model (5.3) up to the first order in  $\varepsilon$ .

We detail the derivation of this proposition in the two following sections. The first is dedicated to writing the system with the drift variables  $(\rho, Y, u, w, p, q)$  and the second makes the ansatz to derive the drift-flux system assuming an appropriate relaxation effect that drives the closure of the two velocity fields.

### 5.2.1 Writing the two-phase system with drift variables

Many algebraic steps are involved when writing the mixture system from (5.3) using the new variables (5.5). To simplify the explanation of this conversion, we remove the dependencies of functions  $\Gamma, \Lambda, f_g, f_l$  on  $\mathbf{v}$  (see Proposition 5.1) in the primitive system (5.3). The conversion, which we use extensively in Part I [Narbona-Reina et al., 2024], starts by adding the equations for phase  $g$  and  $l$  while keeping the mass and momentum conservation for the phase  $g$ . This yields an alternate, equivalent set of 4 equations. Using that the primitive variables can be written in terms of the drift ones as

$$u_g = u + (1 - Y)w, \quad u_l = u - Yw, \quad p_g = p + (1 - \varphi_g)q, \quad p_l = p - \varphi_g q,$$

obtaining the three first equations is straightforward. The mass equations read

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \quad (5.7)$$

$$\partial_t(\rho Y) + \operatorname{div}(\rho Y u + \rho Y(1 - Y)w) = \Gamma. \quad (5.8)$$

The momentum equation for the mean velocity  $u$  reads

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + \rho Y(1 - Y)w \otimes w + p\operatorname{Id}) = \rho Y f_g + \rho(1 - Y)f_l. \quad (5.9)$$

The momentum equation for the phase  $g$  reads

$$\begin{aligned} & \partial_t(\rho Y u + \rho Y(1 - Y)w) + \operatorname{div}(\rho Y u \otimes u) + \operatorname{div}(\rho Y(1 - Y)^2 w \otimes w) \\ & + \operatorname{div}(\rho Y(1 - Y)u \otimes w) + \operatorname{div}(\rho Y(1 - Y)w \otimes u) \\ & + \nabla(\varphi_g(p + (1 - \varphi_g)q)) - (p + (1 - \beta_p - \varphi_g)q)\nabla\varphi_g = \rho Y f_g - \Lambda|w|w \end{aligned} \quad (5.10)$$

since  $p_I = p + (1 - \beta_p - \varphi_g)q$ . This fourth equation is not the final converted equation; working on it will help us to find the equation satisfied by  $w$ . To do so, we use the three relationships (5.7), (5.8) and (5.9) to simplify (5.10). Let us split equation (5.10) in the following form

$$\begin{aligned} & \partial_t(\rho Y u) + \rho Y(1 - Y)\partial_t w + \partial_t(\rho Y(1 - Y))w + \operatorname{div}(\rho Y u \otimes u) + \rho Y(1 - Y)\operatorname{div}((1 - Y)w \otimes w) \\ & + (1 - Y)(w \cdot \nabla(\rho Y(1 - Y)))w + \rho Y(1 - Y)u \cdot \nabla w + \operatorname{div}(\rho Y(1 - Y)u)w \\ & + \rho Y(1 - Y)w \cdot \nabla u + \operatorname{div}(\rho Y(1 - Y)w)u \\ & + \varphi_g \nabla p + \nabla(\varphi_g(1 - \varphi_g)q) - (1 - \beta_p - \varphi_g)q \nabla \varphi_g = \rho Y f_g(\mathbf{v}) - \Lambda(\mathbf{v})|w|w. \end{aligned} \quad (5.11)$$

Let us now rewrite (5.11) by regrouping similar terms as follows

$$\begin{aligned} & I_1 + I_2 + I_3 + (1 - Y)w \cdot \nabla(\rho Y(1 - Y))w \\ & + \varphi_g \nabla p + \nabla(\varphi_g(1 - \varphi_g)q) - (1 - \beta_p - \varphi_g)q \nabla \varphi_g = \rho Y f_g - \Lambda|w|w. \end{aligned} \quad (5.12)$$

with

$$\begin{aligned} I_1 &= \rho Y(1 - Y)\left(\partial_t w + \operatorname{div}((1 - Y)w \otimes w) + u \cdot \nabla w + w \cdot \nabla u\right) \\ I_2 &= \partial_t(\rho Y u) + \operatorname{div}(\rho Y u \otimes u) + \operatorname{div}(\rho Y(1 - Y)w)u \\ I_3 &= \left(\partial_t(\rho Y(1 - Y)) + \operatorname{div}(\rho Y(1 - Y)u)\right)w. \end{aligned} \quad (5.13)$$

The expressions  $I_1$ ,  $I_2$  and  $I_3$  can be simplified significantly.

*New expression for  $I_3$ .* Using (5.7) and (5.8), we can write

$$\rho(\partial_t Y + u \cdot \nabla Y) + \operatorname{div}(\rho Y(1 - Y)w) = \Gamma.$$

Multiplying this relation by  $(1 - 2Y)$  and using the mass equation (5.7) yields

$$\partial_t(\rho Y(1 - Y)) + \operatorname{div}(\rho Y(1 - Y)u) = -(1 - 2Y)\left(\operatorname{div}(\rho Y(1 - Y)w) - \Gamma\right).$$

and therefore

$$I_3 = -(1 - 2Y)\left(\operatorname{div}(\rho Y(1 - Y)w)w - \Gamma w\right). \quad (5.14)$$

*New expression for  $I_2$ .* This expression can be simplified by using (5.8), (5.9), and the mass equation (5.7). We write

$$\begin{aligned} \partial_t(\rho Y u) + \operatorname{div}(\rho Y u \otimes u) &= (\partial_t(\rho Y) + \operatorname{div}(\rho Y u))u + \rho Y(\partial_t u + u \cdot \nabla u) \\ &= \left(-\operatorname{div}(\rho Y(1 - Y)w) + \Gamma\right)u + \rho Y(\partial_t u + u \cdot \nabla u) \end{aligned} \quad (5.15)$$

The momentum equation (5.9) may be written

$$\rho(\partial_t u + u \cdot \nabla u) = -\operatorname{div}(\rho Y(1-Y)w \otimes w) - \nabla p + \rho Y f_g + \rho(1-Y)f_l.$$

Therefore

$$I_2 = u\Gamma + Y \left( -\operatorname{div}(\rho Y(1-Y)w \otimes w) - \nabla p + \rho Y f_g + \rho(1-Y)f_l \right). \quad (5.16)$$

Using expressions (5.14) and (5.16), we can rewrite (5.12) as:

$$I_1 + I_4 - (Y - \varphi_g)\nabla p + \nabla(\varphi_g(1 - \varphi_g)q) - (1 - \beta_p - \varphi_g)q\nabla\varphi_g + (u + (1 - 2Y)w)\Gamma + \Lambda|w|w + \rho Y(1 - Y)(f_g - f_l) = 0 \quad (5.17)$$

where

$$I_4 = -Y\operatorname{div}(\rho Y(1-Y)w \otimes w) - (1 - 2Y)\operatorname{div}(\rho Y(1-Y)w)w + (1 - Y)w\nabla(\rho Y(1-Y))w.$$

Let us now simplify the expression of  $I_4$ . Splitting all the quantities, we find that

$$I_4 = -\rho Y(1-Y) \left( Yw \cdot \nabla w + (1-Y)w \operatorname{div} w \right).$$

Substituting  $I_1$  and the simplified  $I_4$  into (5.17) allows us to write the equation on  $w$ :

$$\begin{aligned} & \partial_t w + \operatorname{div}((1-Y)w \otimes w) + u \cdot \nabla w + w \cdot \nabla u - Yw \cdot \nabla w - (1-Y)w \operatorname{div} w \\ & + \frac{1}{\rho Y(1-Y)} \left( -(Y - \varphi_g)\nabla p + \varphi_g(1 - \varphi_g)\nabla q - (1 - \beta_p - \varphi_g)q\nabla\varphi_g + q\nabla(\varphi_g(1 - \varphi_g)) \right) \\ & = -\frac{1}{\rho Y(1-Y)} \left( (u + (1 - 2Y)w)\Gamma + \Lambda|w|w \right) + f_g - f_l. \end{aligned} \quad (5.18)$$

Finally, to obtain the equation on  $q = p_g - p_l$  (the last equation in (5.3)), [Ambroso et al., 2008] use the two mass equations and the density equation (the equivalent deduction for the one dimensional case can be found in p. 149-150 of [Galié, 2009]). Gathering the equations, the drift-flux system for unknowns  $(\rho, Y, u, w, p, q)$  corresponding to (5.3) is thus:

$$\left\{ \begin{array}{l} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho Y) + \operatorname{div}(\rho Y u + \rho Y(1-Y)w) = \Gamma, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + \rho Y(1-Y)w \otimes w + p\operatorname{Id}) = \rho Y f_g + \rho(1-Y)f_l, \\ \partial_t w + \operatorname{div}((1-Y)w \otimes w) + u \cdot \nabla w + w \cdot \nabla u - Yw \cdot \nabla w - (1-Y)w \operatorname{div} w \\ + \frac{1}{\rho Y(1-Y)} \left( -(Y - \varphi_g)\nabla p + \varphi_g(1 - \varphi_g)\nabla q - (1 - \beta_p - \varphi_g)q\nabla\varphi_g + q\nabla(\varphi_g(1 - \varphi_g)) \right) \\ = -\frac{1}{\rho Y(1-Y)} \left( (u + (1 - 2Y)w)\Gamma + \Lambda|w|w \right) + f_g - f_l \\ \partial_t q + w \cdot \nabla p + \left( \rho_g \mathcal{P}'_g(\rho_g) - \rho_l \mathcal{P}'_l(\rho_l) \right) (\operatorname{div} u - w \cdot \nabla Y) + \left( (1-Y)\rho_g \mathcal{P}'_g(\rho_g) + Y\rho_l \mathcal{P}'_l(\rho_l) \right) \operatorname{div} w \\ + (\varphi_l u_g - \varphi_g u_l) \cdot \nabla q + \left( \frac{\beta_u \rho_g \mathcal{P}'_g(\rho_g)}{\varphi_g} - \frac{(1 - \beta_u)\rho_l \mathcal{P}'_l(\rho_l)}{\varphi_l} - q \right) w \cdot \nabla \varphi_g = - \left( \frac{\rho_g \mathcal{P}'_g(\rho_g)}{\varphi_g} + \frac{\rho_l \mathcal{P}'_l(\rho_l)}{\varphi_l} \right) \Theta q \end{array} \right. \quad (5.19)$$

As we show in the next section, to find the relaxed form of the system (5.6), it is enough to replace the  $q$  equation above by the following volumic fraction equation:

$$\partial_t \varphi_g + (u + (\beta_u - Y)w) \cdot \nabla \varphi_g = \Theta q \quad (5.20)$$

**Remark 5.1** (Checking the drift system in [Ambroso et al., 2008]). *The equation for  $w$  appearing in [Ambroso et al., 2008] is the following*

$$\begin{aligned} \partial_t w + \operatorname{div} \left( \left( \frac{1}{2} - Y \right) w \otimes w \right) + u \cdot \nabla w + w \cdot \nabla u + \left( \frac{1}{\rho_g} - \frac{1}{\rho_l} \right) \nabla p \\ + \left( \frac{\beta_p}{\rho Y} - \frac{1 - \beta_p}{\rho(1 - Y)} + \frac{1}{\rho_l} - \frac{1}{\rho_g} \right) q \nabla \varphi_g + \frac{\rho}{\rho_g \rho_l} \nabla q = f_g - f_l - \frac{\Lambda}{\rho Y(1 - Y)} |w| w \end{aligned}$$

and the equation we have obtained here is (5.18). The coefficient of  $\nabla p$  and  $\nabla q$  coincide with the ones in (5.18) because

$$\rho Y(1 - Y) = \frac{\rho_l \rho_g \varphi_g (1 - \varphi_g)}{\rho}, \quad Y - \varphi_g = \frac{\varphi_g (1 - \varphi_g) (\rho_g - \rho_l)}{\rho},$$

and then

$$\frac{Y - \varphi_g}{\rho Y(1 - Y)} = \frac{\rho_g - \rho_l}{\rho_g \rho_l} = \frac{1}{\rho_l} - \frac{1}{\rho_g} \quad \text{and} \quad \frac{\varphi_g (1 - \varphi_g)}{\rho Y(1 - Y)} = \frac{\rho}{\rho_g \rho_l}.$$

Similarly, the terms in front of  $q \nabla \varphi_g$  are also equivalent because the one in [Ambroso et al., 2008] is

$$\left( \frac{\beta_p}{\rho Y} - \frac{1 - \beta_p}{\rho(1 - Y)} + \frac{1}{\rho_l} - \frac{1}{\rho_g} \right) q \nabla \varphi_g = \left( \frac{\beta_p - Y}{\rho Y(1 - Y)} + \frac{Y - \varphi_g}{\rho Y(1 - Y)} \right) q \nabla \varphi_g = \frac{\beta_p - \varphi_g}{\rho Y(1 - Y)} q \nabla \varphi_g$$

while our version of the term in (5.18) reads

$$\frac{1}{\rho Y(1 - Y)} (-(1 - \beta_p - \varphi_g) q \nabla \varphi_g + q \nabla (\varphi_g (1 - \varphi_g))) = \frac{\beta_p - \varphi_g}{\rho Y(1 - Y)} q \nabla \varphi_g.$$

A first discrepancy is that the term corresponding to  $\Gamma$  is missing from the [Ambroso et al., 2008] equation on  $w$ . In other words, [Ambroso et al., 2008] consider mass exchange but the corresponding contribution is missing in the momentum equations. Another discrepancy lies in the velocity terms, which are in [Ambroso et al., 2008]:

$$\operatorname{div} \left( \left( \frac{1}{2} - Y \right) w \otimes w \right) + u \cdot \nabla w + w \cdot \nabla u,$$

while our version of these terms in (5.18) is:

$$\operatorname{div}((1 - Y)w \otimes w) - Yw \cdot \nabla w - (1 - Y)w \operatorname{div} w + u \cdot \nabla w + w \cdot \nabla u.$$

Note that in one dimension

$$I_4 = -\frac{1}{2} \rho Y(1 - Y) \partial_x (w^2)$$

and therefore

$$I_1 + I_4 = \rho Y(1 - Y) \left( \partial_t w + \partial_x \left( \left( \frac{1}{2} - Y \right) w^2 \right) + u \partial_x w + w \partial_x u \right)$$

which are exactly the quantities that can be found in the PhD thesis of [Galié, 2009]. The drift system for the one dimensional case is presented on p. 148 of this work. The multi-dimensional version is then proposed on p. 285, and was then included in [Ambroso et al., 2008]. The calculations above prove that this multi-dimensional version of the  $w$  equation is incorrect. It thus seems that the inexact  $w$  equation of [Ambroso et al., 2008] stems from an inappropriate extrapolation of the result in the one-dimension; the expression of  $I_4$  is not so straightforward in the multidimensional setting.

**Remark 5.2.** In Section 4 of the Part I companion paper, [Narbona-Reina et al., 2024], we discuss drift-flux formulations for volcanic conduit flow. To do so, the quantities corresponding to the system (5.3) are:

$$\Gamma = R^{H_2O}, \quad \Lambda|w| = K_d\varphi(1 - \varphi), \quad \Theta = \frac{3\varphi(1 - \varphi)}{4\eta_l}, \quad f_g = f_l = g, \quad \beta_u = 1, \quad \beta_p = 0.$$

The phase  $g$  represents the gas and the phase  $l$  is the liquid. There are some differences with the model studied in [Ambroso et al., 2008] that we would like to point out. First, the system of [Ambroso et al., 2008] does not take viscosity into account and therefore the viscous tensor (denoted  $\mathcal{D}$  in [Narbona-Reina et al., 2024]) does not appear in the calculations presented above. This term is added in the development of [Narbona-Reina et al., 2024]. Another difference is that the system of [Ambroso et al., 2008] does not consider the momentum exchange due to  $\Gamma$ . The development of [Narbona-Reina et al., 2024] has thus different coefficients in their drift-flux system, which are related to terms involved into the equation satisfied by  $u_g - u_l$  (see Remark 5.1).

### 5.2.2 Simplification and drift flux system

Using the ansatz assumption related to  $\Lambda, \Theta, \Gamma$  in (5.4), the system (5.19) with the  $q$  equation replaced by (5.20) is:

$$\left\{ \begin{array}{l} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho Y) + \operatorname{div}(\rho u Y + \rho Y(1 - Y)w) = \varepsilon \gamma, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + \rho Y(1 - Y)w \otimes w + p \operatorname{Id}) = \rho Y f_g + \rho(1 - Y) f_l \\ \partial_t w + \operatorname{div}((1 - Y)w \otimes w) + u \cdot \nabla w + w \cdot \nabla u - Y w \cdot \nabla w - (1 - Y)w \operatorname{div} w \\ \quad + \frac{1}{\rho Y(1 - Y)} \left( -(Y - \varphi_g) \nabla p + \varphi_g(1 - \varphi_g) \nabla q - (\beta_p - \varphi_g) q \nabla \varphi_g + q \nabla(\varphi_g(1 - \varphi_g)) \right) \\ \quad = -\frac{1}{\rho Y(1 - Y)} \left( (u + (1 - 2Y)w) \varepsilon \gamma + \frac{\lambda}{\varepsilon^2} |w| w \right) + f_g - f_l \\ \partial_t \varphi_g + (u + (\beta_u - Y)w) \cdot \nabla \varphi_g = \frac{\theta}{\varepsilon^2} q \end{array} \right. \quad (5.21)$$

These asymptotic hypotheses are taken with the goal of finding solutions near equilibrium defined by

$$w = 0, \quad q = 0,$$

that is, when velocities and pressures are the same for the two phases. We consider the ansatz

$$w = w_0 + \varepsilon w_1 + \mathcal{O}(\varepsilon^2), \quad q = q_0 + \varepsilon q_1 + \mathcal{O}(\varepsilon^2).$$

Multiplying the two last equations in (5.21) by  $\varepsilon^2$  yields  $w_0 = q_0 = \mathcal{O}(\varepsilon^2)$ . Writing again these equations to get  $w_1$  and  $q_1$  at first order in  $\varepsilon$  yields

$$|w_1| w_1 = \frac{\rho Y(1 - Y)}{\lambda} \left( f_g - f_l + \frac{Y - \varphi_g}{\rho Y(1 - Y)} \nabla p \right) + \mathcal{O}(\varepsilon), \quad q_1 = \mathcal{O}(\varepsilon).$$

Recalling that  $\frac{Y - \varphi_g}{\rho Y(1 - Y)} = \frac{1}{\rho_l} - \frac{1}{\rho_g}$ , we find the result of Proposition 5.1.

### 5.3 Other generalized drift-flux systems considered in the literature

Over the years, many drift-flux systems have been used to represent two-phase mixtures composed of liquid and gas (e.g., [Ishii and Hibiki, 2011]). This Section reviews several liquid–gas, drift-flux systems without temperature.

*A simplified model to discuss numerical schemes.* There is a wide variety of two-phase flows. The studies of [Gastaldo et al., 2010, Gastaldo et al., 2011] focus on dispersed flows, which are characterized by one phase present in the form of inclusions (dispersed phase, here the gaseous phase) and the other phase being continuous (here the liquid phase). As seen in Section 2.6, at the microscopic scale, two-phase flows can be seen as a set of single-phase regions separated by interfaces through which there are transfers of mass, momentum and energy. To develop a macroscopic model (Section 3), we write the balance equations valid inside each pure phase region, and then, by taking the average, we extend these equations to the entire domain and we model the transfers at the interfaces. The studies of [Gastaldo et al., 2010, Gastaldo et al., 2011] show how to solve numerically a drift-flux system by taking advantage of the parabolic behavior with respect to  $p$ . These studies consider the following system, which models a two-phase flow composed of an incompressible liquid (denoted with subscript  $l$ ) and a compressible gas phase (denoted with subscript  $g$ ). This system is written in terms of  $\rho, Y, u$ , where, as before,

$$\rho = \varphi_l \rho_l + \varphi_g \rho_g, \quad \rho u = \varphi_l \rho_l u_l + \varphi_g \rho_g u_g, \quad \rho Y = \varphi_g \rho_g, \quad \varphi_l + \varphi_g = 1.$$

It reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho Y) + \operatorname{div}(\rho u Y + \rho Y(1 - Y)w) = \operatorname{div}(d_g \nabla Y), \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p - \operatorname{div}(\tau(u)) = f \end{cases} \quad (5.22)$$

The term  $\operatorname{div}(d_g \nabla Y)$  accounts for the dispersive effect of the gas phase by using a positive coefficient  $d_g$ . The stress tensor reads

$$\tau(u) = \mu(\nabla u + \nabla^t u) - \frac{2}{3} \mu \operatorname{div} u \operatorname{Id}$$

and the pressure law is

$$p = a^2 \rho_g.$$

The liquid density  $\rho_l$  is assumed constant and  $\rho_g = \rho_g(p)$ . Finally,  $f$  represents the forcing term such as that caused by the gravity force. The relative velocity  $w$  is given by

$$w = \frac{1}{\lambda} (1 - \varphi_g) \varphi_g \frac{\rho_g(p) - \rho_l}{\rho} \nabla p.$$

To get such an expression for  $w$ , the studies start with a two-phase system with a linear drag term. The  $w$  equation is obtained by multiplying the momentum equation of the gas by  $\varphi_l \rho_l$  and the momentum equation of the liquid by  $\varphi_g \rho_g$  and suppressing the terms with  $\rho_g$  in front of  $\rho_l$  because the gas density is assumed small compared to that of the liquid.

*A gas-liquid drift-flux model with zero order slip law.* The study of [Evje and Wen, 2013] considers a 1D compressible gas-liquid model that is highly relevant for wellbore operations linked to rock drilling. The drift-flux model is composed of two continuity equations coupled with a mixture momentum equation. The model allows unequal gas and liquid velocities, the differential velocity of which is dictated by a so-called slip law, which is used to model flow scenarios involving counter currents. The model is written in Lagrangian coordinates. The difference in fluid velocities gives rise to new terms in the mixture momentum equation that are challenging to deal with. The study starts with the following system

$$\begin{cases} \varphi_g + \varphi_l = 1, \\ \partial_t(\varphi_g \rho_g) + \partial_x(\varphi_g \rho_g u_g) = 0, \\ \partial_t(\varphi_l \rho_l) + \partial_x(\varphi_l \rho_l u_l) = 0, \\ \partial_t(\varphi_l \rho_l u_l + \varphi_g \rho_g u_g) + \partial_x(\varphi_l \rho_l u_l^2 + \varphi_g \rho_g u_g^2) + \partial_x p = \partial_x(\varepsilon \partial_x u_M) \end{cases} \quad (5.23)$$

where  $\varepsilon$  is a viscosity depending on the different phase fractions with  $u_M = \varphi_g u_g + \varphi_l u_l$  and the pressure  $p$  depends on the gas density  $\rho_g$  following

$$p = C \rho_g^\gamma \quad \text{with} \quad \gamma > 1 \quad \text{and} \quad C > 0.$$

As indicated in [Evje and Wen, 2013], since the momentum balance is only specified for the mixture, we need an additional hydrodynamic closure law that connects the two phase velocities. This law should be able to take into account different flow regimes. A commonly used relationship is the Zuber-Findlay slip (see [Evje and Flatten, 2007]):

$$u_g = c_0 u_M + c_1$$

where  $c_0$  and  $c_1$  are flow-dependent coefficients (see more details below). These two coefficients are referred to as the distribution parameter and the drift velocity, respectively. The following drift variables are introduced:

$$\begin{aligned} m &= \varphi_l \rho_l, & n &= \varphi_g \rho_g \\ \rho &= m + n - k^* \\ c &= (m - k^*) / \rho \\ u &= u_g \end{aligned}$$

where  $k^* = \rho_l(1 - 1/c_0)$  represents a minimal mass of liquid that must be present in order to make the slip law well-defined. The viscosity  $\varepsilon$  is chosen as follows

$$\varepsilon(n, m) = \frac{(m - k^*)^{\beta+1}}{(n + m - k^*)(\rho_l - m)^{\beta+1}} \quad \text{with} \quad \beta > 0.$$

The system (5.23) can be rewritten in Lagrangian coordinates in terms of  $(c, \rho, u)$ , which yields:

$$\begin{cases} D_t c = 0, \\ D_t \rho + \rho^2 \partial_x u = 0, \\ D_t u + \partial_x (p(c, \rho) - u^2 g(c\rho) - u h(c\rho) + j(c\rho)) = \partial_x (E(c, \rho) \partial_x u) \end{cases} \quad (5.24)$$

where  $D_t \phi = \partial_t \phi + u \partial_x \phi$  for any function  $\phi$ . The different functions appearing in the momentum equation are:

$$p(c, \rho) = \left( \frac{(1-c)\rho}{a^* - c\rho} \right)^\gamma, \quad a^* = \rho_l / c_0,$$

$$g(c\rho) = k^* \frac{c\rho}{k^* + c\rho}, \quad h(c\rho) = 2\rho_l \left( \frac{c_1}{c_0} \right) \frac{c\rho}{k^* + c\rho}, \quad j(c\rho) = \rho_l^2 \left( \frac{c_1}{c_0} \right)^2 \frac{1}{k^* + c\rho},$$

and

$$E(c, \rho) = \frac{(c\rho)^{\beta+1}}{(a^* - c\rho)^{\beta+1}}$$

where  $\beta > 0$ .

*More general slip laws.* In some physical applications, a slip law on relative velocity is given as:

$$u_g - u_l = \Phi(p, \varphi_g, u_g)$$

where  $\Phi$  is a prescribed function (see [Evje and Flatten, 2007]). The Zuber-Findlay slip relationship presented before is a special case where

$$\Phi(p, \varphi_g, u_g) = \frac{(c_0 - 1)u_g + c_1}{c_0 \varphi_l}.$$

The coefficient  $c_0$  is defined as a shape factor taking into account nonuniform velocity and concentration profiles. The coefficient  $c_1$  is defined as the terminal velocity of a single gas bubble rising through the liquid. This particular drift-flux model has been extensively studied in [Théron, 1989] and [Benzoni-Gavage, 1991], where the eigenvalues have been characterized following the work of [Boure, 1997]. Provided that the liquid phase is incompressible, [Gavrilyuk and Fabre, 1996] have demonstrated that with a suitable variable transformation, the drift-flux model with Zuber-Findlay slip is mathematically similar to the Euler equations describing gas dynamics. There are also generalized drift-flux models where  $\Phi$  may also contain differential operators [Boure, 1997]. This is the case for our volcanic conduit flow model (Section 4 of Part I [Narbona-Reina et al., 2024]), where the drift velocity is given by a Darcy law in a simplified version of the drift-flux model. This kind of drift-flux model are quite useful because one can incorporate many physical processes into the slip law. This is especially the case when the slip law also includes derivatives.

## A Crash course on functional analysis

This section aims at introducing some essential mathematical concepts related to functional analysis that we use in this paper. We do not provide proofs but we rely heavily on the content given in the book of [Cioranescu and Donato, 1999], where readers interested by an introduction to functional analysis will find the related proofs. The notion of duality is at the heart of modern methods of analysis, and it plays a fundamental role in this course because it is at the origin of the notion of distributions named  $\mathcal{D}'(\Omega)$ . One of the most important objectives of this crash course is to use this new point of view to study functional spaces. The space of distributions is, in a proper sense, the “smallest space” containing continuous functions, and where the derivation is defined everywhere. The theory of distributions, developed by Schwartz at the end of the 1940s, is the culmination of a process spanning more than half a century and including, among others, the symbolic calculation of the engineer Heaviside (1893), the formalism introduced by the physicist Dirac (1926), the “finite parts” of divergent integrals of Hadamard (1932), and the generalized derivatives of Sobolev (1936). The initial idea consists in shifting the point of view to describe the functions; rather than considering a function  $f$  of a real variable as the collection of its values  $f(x)$  where  $x \in \Omega$ , we describe  $f$  by the collection of its weighted averages  $\int_{\Omega} f \varphi(x) dx$  where  $\varphi \in \mathcal{D}(\Omega) = \mathcal{C}_c^{\infty}(\Omega)$ . We write this average as  $\langle f, \varphi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)}$ . The integration formula by parts  $\int_{\Omega} f' \varphi(x) dx = \int_{\Omega} f \varphi'(x) dx$  then allows us to define the object  $f'$ . The crash course also defines the Lebesgue and Sobolev spaces as well as the weak and strong convergences and the distributions with values in a Banach space, which is important when one considers PDEs with time and space variables.

### A.1 Mathematical concepts related to PDEs

#### Weak and weak\* convergences in Banach spaces

Let us first define the notion of dual space. From now on,  $E$  and  $F$  denote two Banach spaces.  $E$  being a Banach space, the set of the linear and continuous maps from  $E$  to  $\mathbb{R}$  is called the dual space of  $E$  and is denoted  $E'$ . If  $x' \in E'$ , the image of  $x'(x)$  of  $x \in E$  is denoted  $\langle x', x \rangle_{E' \times E}$ . The bracket  $\langle \cdot, \cdot \rangle_{E' \times E}$  is called the duality pairing between  $E'$  and  $E$ . Assuming that  $E$  is a (real) Banach space equipped with the norm  $\|\cdot\|_E$ , we can define the weak limit as follows

**Definition A.1.** A sequence  $\{x_n\}$  in  $E$  is said to converge weakly to  $x$  iff

$$\text{for all } x' \in E', \quad \langle x', x_n \rangle_{E' \times E} \rightarrow \langle x', x \rangle_{E' \times E}.$$

This weak convergence is denoted

$$x_n \rightharpoonup x \text{ weakly in } E'.$$

Here are two important properties:



- Note that the strong convergence implies weak convergence.
- If  $\dim E = N < +\infty$ , the strong and weak convergences are equivalent.

Let us now write a particular case of the Banach-Steinhaus theorem.

**Proposition A.1.** *Let  $\{x_n\}$  be a sequence weakly convergent to  $x$  in  $E$ . Then*

- $\{x_n\}$  is a bounded sequence in  $E$ , which means that there exists a constant  $C$  independent of  $n$  such that

$$\text{for all } n \in \mathbb{N}, \quad \|x_n\|_E \leq C.$$

- The norm on  $E$  is lower semi-continuous with respect to the weak convergence, which means

$$\|x\|_E \leq \liminf_{n \rightarrow +\infty} \|x_n\|_E.$$

Recall the following definition of uniformly convex Banach space:

**Definition A.2.** We say that the Banach space  $E$  is uniformly convex if for  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\left(x, y \in E, \quad \|x\|_E \leq 1, \quad \|y\|_E \leq 1, \quad \|x - y\|_E > \varepsilon\right) \implies \left(\left\|\frac{x + y}{2}\right\|_E < 1 - \delta\right).$$

**Proposition A.2.** *Let  $\{x_n\}$  be a sequence in a uniformly convex Banach space  $E$ . One has the equivalence between the two properties (a) and (b) where*

$$\begin{aligned} & \text{(a) } x_n \rightarrow x \text{ strongly in } E \\ & \text{and} \\ & \text{(b) } x_n \rightharpoonup x \text{ weakly in } E \quad \text{and} \quad \|x_n\|_E \rightarrow \|x\|_E. \end{aligned}$$

The following theorem is useful:

**Theorem A.1.** *Assume that  $E$  is a reflexive Banach space and let  $\{x_n\}$  be a bounded sequence in  $E$ . Then*

- There exists a subsequence of  $\{x_n\}$  of  $\{x_n\}$  and  $x \in E$  such that, as  $k \rightarrow +\infty$

$$x_{n_k} \rightharpoonup x \text{ weakly in } E.$$

- If each weakly convergence subsequence of  $\{x_n\}$  has the same limit  $x$ , then the whole sequence  $\{x_n\}$  weakly converges to  $x$  that means

$$x_n \rightharpoonup x \text{ weakly in } E.$$

The next result is frequently used in this work. It enables us to take the limit in products of weak-strong convergence sequences.

**Proposition A.3.** *Let  $\{x_n\} \subset E$  and  $\{y_n\} \subset E'$  such that*

$$x_n \rightharpoonup x \text{ weakly in } E$$

and

$$y_n \rightarrow y \text{ strongly in } E'.$$

Then

$$\lim_{n \rightarrow +\infty} \langle y_n, x_n \rangle_{E' \times E} = \langle y, x \rangle_{E' \times E}.$$

We have seen previously that to check weak convergence for a sequence of  $E$ , one needs to know what is the space  $E'$  and it may happen that  $E'$  is too complicated to handle. Therefore another notion is needed. It is called weak $\star$  convergence and it is defined as:

**Definition A.3.** Assume that  $F$  is a Banach space and the set  $E = F'$ . A sequence  $\{x_n\}$  in  $E$  is said to converge weakly $\star$  to  $x$  iff

$$\langle x_n, x' \rangle_{F' \times F} \rightarrow \langle x, x' \rangle_{F' \times F} \quad \text{for all } x' \in F.$$

This weak $\star$  convergence is denoted

$$x_n \rightharpoonup x \text{ weakly } \star \text{ in } E.$$

Then we have the following properties:

**Proposition A.4.** Let  $F$  be a Banach space and  $E = F'$ . Then any weakly convergent sequence in  $E$  is also weakly $\star$  convergent.

**Proposition A.5.** Let  $\{x_n\}$  be a sequence weakly $\star$  convergent to  $x$  in  $E = F'$ , where  $F$  is a Banach space. Then

- $\{x_n\}$  is a bounded sequence in  $E$ , which means that there exists a constant  $C$  independent of  $n$  such that

$$\text{for all } n \in \mathbb{N}, \quad \|x_n\|_E \leq C.$$

- The norm is lower semi-continuous with respect to the weak $\star$  convergence, which means

$$\|x\|_E \leq \liminf_{n \rightarrow +\infty} \|x_n\|_E.$$

Concerning the convergence properties for bounded sequence in  $E = F'$ , we need a separability property of the space in the following sense:

**Definition A.4.** We say that the Banach space  $F$  is separable if there exists a set, at most countable, which is dense in  $F$ .

Then the following result holds:

**Theorem A.2.** Let  $F$  be a separable Banach space and let  $E = F'$ . If  $\{x_n\}$  is a bounded sequence in  $E$ , then

- i) There exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  and  $x \in E$  such that, as  $k \rightarrow +\infty$

$$x_{n_k} \rightharpoonup x \text{ weakly } \star \text{ in } E.$$

- ii) If each weakly $\star$  convergent subsequence of  $\{x_n\}$  has the same limit  $x$ , then the whole sequence  $\{x_n\}$  weakly $\star$  converges to  $x$  which means

$$x_n \rightharpoonup x \text{ weakly } \star \text{ in } E.$$

We also get the following weak $\star$ -strong convergence property:

**Proposition A.6.** Let  $\{x_n\}$  be a sequence in  $E = F'$  where  $F$  is a Banach space and  $\{y_n\}$  a sequence in  $E'$  such that

$$y_n \rightarrow y \text{ strongly in } F$$

and

$$x_n \rightharpoonup x \text{ weakly } \star \text{ in } E.$$

Then

$$\lim_{n \rightarrow +\infty} \langle x_n, y_n \rangle_{F' \times F} = \langle x, y \rangle_{F' \times F}.$$

## A.2 $L^p$ spaces and convergence in these spaces

We first give the definition of a space usually named  $\mathcal{D}(\mathcal{O})$  that will be used later on.

**Definition A.5.** For any function  $\phi : \mathcal{O} \mapsto \mathbb{R}$ , the support of  $\phi$  denoted  $\text{supp } \phi$  is defined as the following closed set of  $\mathcal{O}$ :

$$\text{supp } \phi = \overline{\{x \in \mathcal{O} : \phi(x) \neq 0\}} \cap \mathcal{O}.$$

We denote by  $\mathcal{D}(\mathcal{O})$  the set of infinitely differentiable functions, the support of which is a compact set of  $\mathbb{R}^N$  contained in  $\mathcal{O}$ . We denote by  $\mathcal{C}_c^0(\mathcal{O})$  the set of continuous functions, the support of which is a compact set of  $\mathbb{R}^N$  contained in  $\mathcal{O}$ .

All our calculations are henceforth related to Lebesgue  $L^p$  spaces defined as:

**Definition A.6.** Let  $p \in \mathbb{R}$  with  $1 \leq p < +\infty$ . Define

$$L^p(\mathcal{O}) = \{f \text{ with } f : \mathcal{O} \mapsto \mathbb{R}, \quad f \text{ measurable and such that } \int_{\mathcal{O}} |f(x)|^p dx < +\infty\}$$

and

$$L^\infty(\mathcal{O}) = \{f \text{ with } f : \mathcal{O} \mapsto \mathbb{R}, \quad f \text{ measurable and such that there exists } C \in \mathbb{R} \text{ with } |f| \leq C \text{ a.e. on } \mathcal{O}\}.$$

The next two propositions give the main properties of  $L^p$  spaces that are frequently used to prove convergence in these spaces.

**Proposition A.7.** Let  $p \in \mathbb{R}$  with  $1 \leq p \leq +\infty$ . The set  $L^p(\mathcal{O})$  is a Banach space for the norm

$$\|f\|_{L^p(\mathcal{O})} = \left( \int_{\mathcal{O}} |f(x)|^p dx \right)^{1/p} \quad \text{if } p < \infty$$

and

$$\|f\|_{L^\infty(\mathcal{O})} = \inf\{C, |f| \leq C \text{ a.e. on } \mathcal{O}\}.$$

**Proposition A.8.** The space  $L^p(\mathcal{O})$  is separable for  $1 \leq p < +\infty$  and is uniformly convex for  $1 < p < +\infty$ .

An important inequality is the Hölder inequality:

**Proposition A.9.** Let  $1 \leq p \leq +\infty$  and  $p'$  be its conjugate which means

- $\frac{1}{p'} = 1 - \frac{1}{p}$  if  $1 < p < +\infty$
- $p' = 1$  if  $p = +\infty$
- $p' = +\infty$  if  $p = 1$ .

Then

$$\int_{\mathcal{O}} |f(x)g(x)| dx \leq \|f\|_{L^p(\mathcal{O})} \|g\|_{L^{p'}(\mathcal{O})}.$$

Note that if  $p = 2$ , this inequality is known as the Cauchy-Schwarz inequality.

The following theorem is important:

**Theorem A.3.** Let  $1 \leq p < +\infty$  and  $p'$  be its conjugate. Let  $f \in [L^p(\mathcal{O})]'$ . Then there exists a unique  $g \in L^{p'}(\mathcal{O})$  such that

$$\langle f, \varphi \rangle_{[L^p(\mathcal{O})]' \times L^p(\mathcal{O})} = \int_{\mathcal{O}} g(x)\varphi(x) dx \quad \text{for all } \varphi \in L^p(\mathcal{O}).$$

Moreover

$$\|g\|_{L^{p'}(\mathcal{O})} = \|f\|_{[L^p(\mathcal{O})]'}$$

Many sets useful in fluid dynamics are dense in  $L^p$  spaces. This is the case of the set given by the following theorem:

**Theorem A.4.**

$$\mathcal{D}(\mathcal{O}) \text{ is dense in } L^p(\mathcal{O}) \text{ for } 1 \leq p < +\infty.$$

The next result on sets is related to the step functions that are defined as:

**Definition A.7.** Let  $A$  be a measurable set of  $\mathbb{R}^N$ . The characteristic function of  $A$  is the function  $\chi_A$  defined by

$$\chi_A(x) = 1 \text{ if } x \in A, \quad \chi_A(x) = 0 \text{ elsewhere.}$$

**Definition A.8.** A function  $f : \mathbb{R}^N \mapsto \mathbb{R}$  is called a step function if

$$f(x) = \sum_{k=1}^m \alpha_k \chi_{I_k}$$

with  $m \in \mathbb{N}$ ,  $\alpha_k \in \mathbb{R}^N$  and where  $I_k$  is an interval in  $\mathbb{R}^N$  for any  $k \in \{1, \dots, m\}$ .

If  $\Omega \subset \mathbb{R}^N$  is a bounded open set of  $\mathbb{R}^N$ , we denote by  $\mathcal{S}(\Omega)$  the set of step functions as defined above. Obviously we have  $\mathcal{S}(\Omega) \subset L^p(\Omega)$  for  $1 \leq p < +\infty$ . Another more precise property is:

**Theorem A.5.** If  $1 \leq p < +\infty$  then  $\mathcal{S}(\Omega)$  is dense in  $L^p(\Omega)$ .

In this Lebesgue space, we have the following weak and strong convergences propositions that use the definitions of weak and strong convergences:

**Proposition A.10.** Let  $\{u_n\}$  be a sequence in  $L^p(\Omega)$  with  $1 < p < \infty$ . Then there exists  $u$  in  $L^p(\Omega)$  such that

$$u_n \rightharpoonup u \text{ weakly in } L^p(\Omega)$$

which means that

$$\int_{\Omega} u_n \phi \, dx \rightarrow \int_{\Omega} u \phi \, dx \text{ for all } \phi \in L^{p'}(\Omega) \text{ with } 1/p + 1/p' = 1.$$

For  $1 < p < +\infty$ ,  $L^p(\Omega) = (L^{p'}(\Omega))'$ , and the weak convergence is equivalent to the weak\* convergence.

The next proposition is important because it helps to characterize the weak convergence in the  $L^p(\Omega)$  spaces.

**Proposition A.11.** Let  $1 < p < +\infty$  and  $\{u_n\}$  be a sequence in  $L^p(\Omega)$ . Then, the following equivalence holds true

$$u_n \rightharpoonup u \text{ weakly in } L^p(\Omega)$$

and

$$\|u_n\|_{L^p(\Omega)} \leq C \text{ (independently of } n) \quad \int_I u_n \, dx \rightarrow \int_I u \, dx \text{ for any interval } I \subset \Omega.$$

The weak convergence in  $L^1(\Omega)$  namely

$$u_n \rightharpoonup u \text{ weakly in } L^1(\Omega)$$

means that

$$\int_{\Omega} u_n \phi \, dx \rightarrow \int_{\Omega} u \phi \, dx \text{ for all } \phi \in L^{\infty}(\Omega).$$

Remark that because  $L^1(\Omega)$  is not reflexive, the weak compactness theorem does not apply. This makes the study of bounded sequences in  $L^1(\Omega)$  much more difficult.

### A.3 Distribution, Sobolev embedding

Let us first discuss convergences of sequences defined in this space  $\mathcal{D}(\mathcal{O})$  and the definition of the set of distributions on  $\mathcal{O}$ . More precisely,

**Definition A.9.** Let  $\{\phi_n\}$  be a sequence in  $\mathcal{D}(\mathcal{O})$ . We say that  $\phi_n$  converges to an element  $\phi \in \mathcal{D}(\mathcal{O})$ , iff

- i) There exists a compact set  $K \subset \mathcal{O}$  such that, for any  $n \in \mathbb{N}$ ,  $\text{supp } \phi_n \subset K$ ,
- ii) For any  $\alpha \in \mathbb{N}^N$ ,  $\partial^\alpha \phi_n$  converges uniformly to  $\partial^\alpha \phi \in K$ .

This is enough to define a distribution on  $\mathcal{O}$ . More precisely, we have the following definition

**Definition A.10.** A map  $T : \mathcal{D}(\mathcal{O}) \mapsto \mathbb{R}$  is called a distribution on  $\mathcal{O}$ , iff

- i) The map  $T$  is linear, which means

$$\forall \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } \forall \phi_1, \phi_2 \in \mathcal{D}(\mathcal{O}), \quad T(\lambda_1 \phi_1 + \lambda_2 \phi_2) = \lambda_1 T(\phi_1) + \lambda_2 T(\phi_2)$$

- ii) The map  $T$  is sequentially continuous, which means

$$(\phi_n \rightarrow \phi \text{ in } \mathcal{D}(\mathcal{O})) \implies (T(\phi_n) \rightarrow T(\phi)).$$

We denote by  $\mathcal{D}'(\mathcal{O})$  the set of distributions on  $\mathcal{O}$ .

We are now ready to give a notion of convergence for sequences in  $\mathcal{D}'(\mathcal{O})$ . This is a weak- $\ast$  convergence but not in a Banach space.

**Definition A.11.** A sequence  $\{T_n\}$  in  $\mathcal{D}'(\mathcal{O})$  is said to converge (in the sense of distribution) to an element  $T \in \mathcal{D}'(\mathcal{O})$  iff

$$\langle T_n, \phi \rangle_{\mathcal{D}'(\mathcal{O}) \times \mathcal{D}(\mathcal{O})} \rightarrow \langle T, \phi \rangle_{\mathcal{D}'(\mathcal{O}) \times \mathcal{D}(\mathcal{O})} \quad \text{for all } \phi \in \mathcal{D}(\mathcal{O}).$$

We also define the derivative of a distribution as follows:

**Definition A.12.** Let  $T \in \mathcal{D}'(\mathcal{O})$ . For any  $i = 1, \dots, N$  the derivative  $\partial T / \partial x_i$  of  $T$  with respect to  $x_i$  is defined by

$$\left\langle \frac{\partial T}{\partial x_i}, \phi \right\rangle = - \left\langle T, \frac{\partial \phi}{\partial x_i} \right\rangle \text{ for all } \phi \in \mathcal{D}(\mathcal{O}).$$

We are now ready to define the Sobolev space  $W^{1,p}$  as follows:

**Definition A.13.** Let  $1 \leq p \leq +\infty$ . The Sobolev space  $W^{1,p}(\mathcal{O})$  is defined by

$$W^{1,p}(\mathcal{O}) = \{u : u \in L^p(\mathcal{O}) \text{ and } \partial u / \partial x_i \in L^p(\mathcal{O}) \text{ for all } i = 1, \dots, N\}$$

where the derivatives are taken in the sense of distribution.

We next provide Sobolev embeddings that are listed in the following theorem.

**Theorem A.6.** Suppose  $\Omega$  is an open set in  $\mathbb{R}^N$  such that  $\partial\Omega$  is Lipschitz continuous. Then

- i) If  $1 \leq p < N$  then  $W^{1,p}(\Omega) \subset L^q(\Omega)$  with
  - Compact embedding for  $q \in [1, p^\star)$  where

$$\frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N}$$

– Continuous injection for  $q = p^*$ .

ii) If  $p = N$ ,  $W^{1,N}(\Omega) \subset L^q(\Omega)$  with compact embedding if  $q \in [1, +\infty)$ .

iii) If  $p > N$ ,  $W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$  with compact embedding.

An other important space is the dual space of  $H_0^1(\Omega)$  where

$$H_0^1(\Omega) = W_0^{1,2}(\Omega)$$

with, for any  $1 \leq p < +\infty$ ,  $W_0^{1,p}(\Omega)$  is defined as the closure of  $\mathcal{D}(\Omega)$  with respect to the norm of  $W^{1,p}(\Omega)$ . The definition of the dual space is:

**Definition A.14.** We denote by  $H^{-1}(\Omega)$  the Banach space defined by

$$H^{-1}(\Omega) = (H_0^1(\Omega))'$$

equipped with the norm

$$\|F\|_{H^{-1}(\Omega)} = \sup_{H_0^1(\Omega) \setminus \{0\}} \frac{|\langle F, u \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}|}{\|u\|_{H_0^1(\Omega)}}.$$

The following proposition gives an important characterization of  $H^{-1}(\Omega)$ .

**Proposition A.12.** Let  $F$  be in  $H^{-1}(\Omega)$ , where  $\Omega$  is an open set in  $\mathbb{R}^N$ . Then there exist  $N + 1$  functions  $f_0, f_1, \dots, f_N$  in  $L^2(\Omega)$  such that

$$F = f_0 + \sum_{i=1}^N \frac{\partial f_i}{\partial x_i} \quad (\text{A.1})$$

in the sense of distributions. Moreover

$$\|F\|_{H^{-1}(\Omega)}^2 = \inf \sum_{i=0}^N \|f_i\|_{L^2(\Omega)}^2$$

where the infimum is taken over all the vectors  $(f_0, f_1, \dots, f_N) \in (L^2(\Omega))^{N+1}$  such that (A.1) holds. Conversely if  $(f_0, f_1, \dots, f_N)$  is a vector in  $(L^2(\Omega))^{N+1}$  then (A.1) defines an element  $F$  of  $H^{-1}(\Omega)$  that satisfies

$$\|F\|_{H^{-1}(\Omega)}^2 \leq \sum_{i=1}^N \|f_i\|_{L^2(\Omega)}^2$$

We have also the following embedding:

**Proposition A.13.**

$$L^2(\Omega) \subset H^{-1}(\Omega) \text{ with compact embedding.}$$

#### A.4 Distribution with values in a Banach space

This part is important when we consider PDEs with time and space variables. The notion of distribution can be generalized to vector-valued functions as follows:

**Definition A.15.** Let  $X$  be a Banach space and  $\Omega \subset \mathbb{R}^N$ . A map  $T : \mathcal{D}(\Omega) \mapsto X$  is called a distribution on  $\Omega$  with values in  $X$  iff

- i) The map  $T$  is linear, which means

$$\forall \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } \forall \phi_1, \phi_2 \in \mathcal{D}(\Omega) \quad T(\lambda_1 \phi_1 + \lambda_2 \phi_2) = \lambda_1 T(\phi_1) + \lambda_2 T(\phi_2).$$

- The map  $T$  is sequentially continuous, which means

$$(\phi_n \rightarrow \phi \text{ in } \mathcal{D}(\Omega)) \implies (T(\phi_n) \rightarrow T(\phi) \text{ in } X).$$

We denote by  $\mathcal{D}'(\Omega; X)$  the set of distributions on  $\Omega$  with values in  $X$ .

Similarly, one can also define the  $L^p$ -spaces for vector-valued functions.

**Definition A.16.** Let  $X$  be a Banach space,  $\Omega \subset \mathbb{R}^N$  and  $p$  such that  $1 \leq p \leq \infty$ . We denote by  $L^p(\Omega; X)$  the set of measurable functions  $u : x \in \Omega \rightarrow u(x) \in X$  such that  $\|u(x)\|_X \in L^p(\Omega)$ .

In the vector-valued case, we choose that a measurable function is an almost everywhere limit of step functions. Concerning the  $L^p$  function, we have the following proposition:

**Proposition A.14.** *The following quantity*

$$\|u\|_{L^p(\Omega; X)} = \left( \int_{\Omega} \|u\|_X^p dx \right)^{1/p}$$

defines a norm on  $L^p(\Omega; X)$  that is a Banach space. If  $X$  is reflexive and  $1 < p < +\infty$ , the space  $L^p(\Omega; X)$  is reflexive too. Moreover, if  $X$  is separable and  $1 \leq p < \infty$ , then  $L^p(\Omega; X)$  is separable.

We can also define the derivative in time for time-space dependent quantities as:

**Proposition A.15.** *Let  $u$  be in  $L^p(a, b; X)$ . The derivative  $\partial u / \partial t$  is the distribution in  $\mathcal{D}'(a, b; X)$  defined by*

$$\frac{\partial u}{\partial t}(\phi) = - \int_a^b u \frac{\partial \phi}{\partial t} dt \text{ for all } \phi \in \mathcal{D}(a, b).$$

## B Crash course on compactness and homogenization

This section introduces some essential mathematical concepts related to compactness (non-linear stability) and to homogenization (characterization of weak limits – averaging procedure).

### B.1 Compactness tools for nonlinear weak stability

#### B.1.1 Weak compactness in $L^1(\Omega)$

Let us first give the definition of equi-integrable functions.

**Definition B.1.** Let  $\Omega$  be a bounded set in  $\mathbb{R}^N$  and  $\{u_n\}$  a sequence in  $L^1(\Omega)$ . The functions  $u_n$  are equi-integrable if for any  $\eta > 0$ , there exists  $\delta > 0$  such that

$$\text{For all } n \in \mathbb{N} \quad \int_E |u_n(x)| dx < \eta \text{ for any } E \subset \Omega \text{ with } |E| < \delta.$$

Then we have the following important theorem due to Dunford-Pettis.

**Theorem B.1.** *Let  $\{u_n\}$  be a sequence in  $L^1(\Omega)$ . Then we have the equivalence of*

$$\{u_n\} \text{ is weakly compact in } L^1(\Omega)$$

and

$$\{u_n\} \text{ is bounded in } L^1(\Omega) \text{ with the functions } u_n \text{ equi-integrable}$$

recalling that  $\{u_n\}_{n \in \mathbb{N}}$  weakly converges to  $u$  in  $L^1(\Omega)$  means

$$\int_{\Omega} u_n \varphi \rightarrow \int_{\Omega} u \varphi \text{ for all } \varphi \in L^\infty(\Omega).$$

### B.1.2 Space-time controls

An important compactness theorem that is widely used in this work that states:

**Theorem B.2.** (Aubin-Lions-Simon) *Let  $X_0$ ,  $X$  and  $X_1$  three Banach spaces with  $X_0 \subset X \subset X_1$ . Suppose that  $X_0$  is compactly embedded in  $X$  and  $X$  is continuously embedded in  $X_1$ . For  $1 \leq p, q \leq +\infty$ , let*

$$W = \{u \in L^p(0, T; X_0) : \partial_t u \in L^q(0, T; X_1)\}.$$

Then we have

- i) *If  $p < +\infty$  then the embedding of  $W$  into  $L^p(0, T; X)$  is compact.*
- ii) *If  $p = +\infty$  and  $q > 1$  then the embedding of  $W$  into  $\mathcal{C}([0, T]; X)$  is compact.*

Another useful result is:

**Theorem B.3.** *Let us define the Banach spaces*

$$W = \{v : v \in L^2(a, b; H_0^1(\Omega)) \text{ and } \partial_t v \in L^2(a, b; H^{-1}(\Omega))\},$$

and

$$W_1 = \{v : v \in L^2(a, b; L^2(\Omega)) \text{ and } \partial_t v \in L^2(a, b; H^{-1}(\Omega))\}$$

*equipped with the norm of the graph. Then the following properties hold*

- i) *The embeddings*

$$W \subset L^2(a, b; L^2(\Omega)) \text{ and } W_1 \subset L^2(a, b; H^{-1}(\Omega))$$

*are compact.*

- ii) *One has the continuous embeddings*

$$\begin{aligned} W &\subset \mathcal{C}([a, b]; L^2(\Omega)) \\ W_1 &\subset \mathcal{C}([a, b]; H^{-1}(\Omega)). \end{aligned}$$

- iii) *For any  $u, v \in W$ , one has*

$$\frac{d}{dt} \int_{\Omega} u(x, t)v(x, t) dx = \langle \partial_t u'(\cdot, t), v \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \langle \partial_t v(\cdot, t), u(\cdot, t) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}.$$

### B.1.3 Commutation between weak limit and strictly convex function

The following lemma is used for the compressible Navier-Stokes equations to take the limit in the pressure quantity.

**Lemma B.1.** *Let  $G$  be a bounded domain in  $\mathbb{R}^N$ ,  $I$  an interval in  $\mathbb{R}$ ,  $1 < p < +\infty$  and  $f : I \rightarrow \mathbb{R}$  a strictly convex function. Let  $\rho_n$  be a sequence of functions in  $L^p(G)$  with values in  $I$ . If  $\rho_n \rightarrow \rho$  weakly in  $L^p(G)$  and  $f(\rho_n) \rightarrow f(\rho)$  weakly in  $L^1(G)$  then*

$$\rho_n \rightarrow \rho \text{ strongly in } L^1(G).$$

Another useful lemma is:

**Lemma B.2.** *Let  $G$  be a bounded domain in  $\mathbb{R}^N$  and  $\{\rho_n\}_{n \in \mathbb{N}}$  a sequence of functions such that*

$$\rho_n \rightarrow \rho \text{ weakly in } L^1(G).$$

*Let  $\Phi : \mathbb{R} \rightarrow (-\infty, +\infty]$  be a convex lower semi-continuous function. Then  $\Phi(\rho) : G \rightarrow \mathbb{R}$  is integrable and*

$$\int_G \Phi(\rho) dy \leq \liminf_{n \rightarrow +\infty} \int_G \Phi(\rho_n) dy.$$



## B.2 A non-local tool

The following non-local compactness tool is useful.

**Lemma B.3.** *Let  $\{\rho_n\}$  be a sequence uniformly bounded in some  $L^p((0, T) \times \Pi^d)$  with  $1 < p < +\infty$ . Assume that  $\mathcal{K}_h$  be a sequence of positive, bounded functions such that*

- i) *For all  $\eta > 0$ ,  $\sup_h \int_{\Pi^d} \mathcal{K}_h(x) 1_{\{|x| \geq \eta\}} dx < +\infty$ ,*
- ii)  *$\|\mathcal{K}_h\|_{L^1(\Pi^d)} \rightarrow \infty$  as  $h \rightarrow +\infty$ .*

*If  $\partial_t \rho_n \in L^q(0, T; W^{-1, q}(\Pi^d))$  (with  $q \geq 1$ ) uniformly in  $n$  and*

$$R_h = \limsup_n \left( \frac{1}{\|\mathcal{K}_h\|_{L^1}} \int_0^T \int_{\Pi^d} \mathcal{K}_h(x-y) |\rho_n(t, x) - \rho_n(t, y)|^p dx dy, dt \right) \rightarrow 0 \text{ as } h \rightarrow 0 \quad (\text{B.1})$$

*then  $\rho_n$  is compact in  $L^p((0, T) \times \Pi^d)$ . Conversely if  $\rho_n$  is compact in  $L^p((0, T) \times \Pi^d)$  then the above quantity converges to 0 with  $h$ .*

Let us show how this property is obtained for a special particular kernel defined as

$$\mathcal{K}_h(x) = \frac{K_h}{\|K_h\|_{L^1}} \quad \text{where } K_h = 1/(h + |x|)^d.$$

Defining the following semi-norm

$$\|f\|_{p, \theta} = \sup_{h \leq 1/2} |\log h|^{-\theta} \int_{\Omega^2} \frac{|f(x) - f(y)|^p}{(h + |x - y|)^d} dx dy$$

we use that for all  $s > 0$ ,  $0 < \theta < 1$  and  $p \in [1, +\infty)$  we have

$$W^{s, p} \subset W_{\log, \theta} \subset L^p \text{ compact}$$

where

$$W_{\log, \theta} = \{f \text{ such that } f \in L^p(\Omega) \text{ with } \|f\|_{p, \theta} < +\infty\}.$$

To prove the right-hand compactness, it suffices to remark that  $\|K_h\|_{L^1} = |\log h|$

$$\|f_n - \mathcal{K}_h * f_n\|_{L^p}^p \leq \frac{1}{\|\mathcal{K}_h\|_{L^1}^p} \int_{\Omega} \left( \int_{\Omega} K_h(x-y) |f_n(x) - f_n(y)| dx \right)^p dy \leq \frac{1}{\|\mathcal{K}_h\|_{L^1}} \int_{\Omega} K_h(x-y) |f_n(x) - f_n(y)|^p dx dy$$

which converges to zero as  $h \rightarrow 0$  uniformly in  $n$  by assumption. On the other hand, for a fixed  $h$ , the sequence  $\mathcal{K}_h * f_n$  in  $n$  is compact. This completes the proof.

### B.2.1 A property coming from monotony

The following result is related to non-decreasing functions.

**Lemma B.4.** *Let  $G$  be a domain of  $\mathbb{R}^N$ ,  $I$  an interval of  $\mathbb{R}$ , and let  $P : I \rightarrow \mathbb{R}$  be a non-decreasing function (defined on  $I$ ). Let  $\rho_n$  be a sequence of functions from  $L^1(G)$  with values in  $I$  such that*

- $\rho_n \rightarrow \rho$  weakly in  $L^1(G)$ ,
- $P(\rho_n) \rightarrow \overline{P(\rho)}$  weakly in  $L^1(G)$ ,

- $P(\rho_n)\rho \rightarrow \overline{P(\rho)}\rho$  weakly in  $L^1(G)$ ,
- $P(\rho_n)\rho_n \rightarrow \overline{P(\rho)}\rho$  weakly in  $L^1(G)$ ,
- $P(\rho)(\rho_n - \rho) \rightarrow 0$  weakly in  $L^1(G)$ .

Then

$$\overline{P(\rho)}\rho \geq \overline{P(\rho)}\rho \quad \text{a.e. in } G.$$

**Remark B.1.** Note that the proof of the previous Lemma is not complicated when taking the limit in the following inequality

$$(P(\rho_n) - P(\rho))(\rho_n - \rho) \geq 0$$

which is satisfied using that  $P$  is a non-decreasing function.

## B.2.2 Minty trick for nonlinear functions depending on a sequence

In this part, we recall a Lemma from [Gallouët, 2018] that allows one to characterize the weak limit of nonlinear function (see this paper for the proof).

**Lemma B.5.** Let  $Q$  be a bounded open set of  $\mathbb{R}^N$ ,  $N \geq 1$ , and let  $\mu$  be a continuous nondecreasing function from  $\mathbb{R}$  to  $\mathbb{R}$ . We assume that there exists  $C \in \mathbb{R}_+$  such that

$$|\mu(s)| \leq C|s| + C \quad \text{for all } s \in \mathbb{R}.$$

Let  $\{\rho_n\}_{n \in \mathbb{N}}$  and  $\{\mu_n\}_{n \in \mathbb{N}}$  be sequences such that  $\rho_n \rightarrow \rho$  weakly in  $L^2(Q)$  and  $\mu_n \rightarrow \bar{\mu}$  weakly in  $L^2(Q)$ . We assume that  $\mu_n = \mu(\rho_n)$  a.e. for all  $n$  and that

$$\lim_{n \rightarrow +\infty} \int_Q \rho_n(y) \mu_n(y) dy = \int_Q \rho(y) \overline{\mu(y)} dy.$$

Then  $\bar{\mu} = \mu(\rho)$  a.e. □

## B.3 Some useful calculations to better understand the averaging of two-phase systems

### B.3.1 Homogenization (averaging procedure) for elliptic equation in one dimension

We use here a well-known one-dimensional system with oscillating coefficient to obtain an homogenized system. This helps to understand how we get  $\mu_{\text{eff}}$  and  $p_{\text{eff}}$  in Section 3.2. Let us consider the following ODE on  $(0, 1)$

$$\begin{cases} -\frac{d}{dx} \left( a^\varepsilon \frac{d}{dx} u^\varepsilon \right) = f \\ u^\varepsilon|_{x=0} = u^\varepsilon|_{x=1} = 0 \end{cases} \quad (\text{B.2})$$

where  $f \in L^2(0, 1)$  is prescribed and with  $a^\varepsilon$  given by

$$a^\varepsilon = a\left(\frac{x}{\varepsilon}\right)$$

with  $a$  a positive function in  $L^\infty(0, \ell_1)$  such that

$$\begin{cases} a \text{ is } \ell_1\text{-periodic} \\ 0 < m \leq a(x) \leq M < +\infty \end{cases} \quad (\text{B.3})$$

where  $m$  and  $M$  are constant. The objective is to show the following result

**Theorem B.4.** Let  $u^\varepsilon \in H_0^1(0, 1)$  be the solution of the problem (B.2). Then

$$u^\varepsilon \rightharpoonup u^0 \text{ weakly in } H_0^1(0, 1)$$

where  $u^0$  is the unique solution in  $H_0^1(0, 1)$  of the problem

$$\begin{cases} -\frac{d}{dx} \left( \frac{1}{\mathcal{M}_{(0,\ell_1)}\left(\frac{1}{a}\right)} \frac{d}{dx} u^0 \right) = f \in (0, 1) \\ u^0|_{x=0} = u^0|_{x=1} = 0. \end{cases} \quad (\text{B.4})$$

where

$$\mathcal{M}_{(0,\ell_1)}\left(\frac{1}{a}\right) = \frac{1}{\ell_1} \int_0^{\ell_1} \frac{1}{a} dx.$$

**Proof of Theorem B.4.** We first multiply formally the equation in (B.2) by  $u^\varepsilon$  and integrate by parts to get

$$\int_0^1 a^\varepsilon \left| \frac{d}{dx} u^\varepsilon \right|^2 = \int_0^1 f u^\varepsilon.$$

Then we use the Cauchy-Schwarz inequality,

$$\int_a^b gh \leq \|g\|_{L^2(a,b)} \|h\|_{L^2(a,b)},$$

and the Poincaré inequality

$$\int_a^b |g|^2 \leq (b-a) \int_a^b |\partial_x g|^2 \quad \text{for all } g \in H_0^1(a, b),$$

to write

$$\int_0^1 a^\varepsilon \left| \frac{d}{dx} u^\varepsilon \right|^2 \leq \|f\|_{L^2(0,1)} \|\partial_x u^\varepsilon\|_{L^2(0,1)}.$$

Next, using the lower bound  $m$  on  $a^\varepsilon$  and the fact that  $cd \leq c^2/2 + d^2/2$ , we can write

$$\int_0^1 a^\varepsilon \left| \frac{d}{dx} u^\varepsilon \right|^2 \leq \frac{1}{m^2} \|f\|_{L^2(0,1)}^2.$$

This implies that

$$\sqrt{a^\varepsilon} \frac{d}{dx} u^\varepsilon \in L^2(0, 1)$$

and therefore

$$\tau^\varepsilon = a^\varepsilon \frac{du^\varepsilon}{dx} \in L^2(0, 1)$$

because  $a^\varepsilon$  is bounded by  $M$  in  $L^\infty(0, 1)$ . Thus  $\tau^\varepsilon \in L^2(0, 1)$  but we also remark that

$$\frac{d}{dx} \tau^\varepsilon \in L^2(0, 1)$$

uniformly because this quantity is equal to  $f$ , which is in  $L^2(0, 1)$ . This indicates that  $\tau^\varepsilon$  will have no oscillation and will strongly converge to  $\tau^0$  in  $L^2(0, 1)$  for instance. To take advantage of this information, we observe that we can write

$$\partial_x u^\varepsilon = \frac{1}{a^\varepsilon} \tau^\varepsilon \quad (\text{B.5})$$

which is a better formulation to take the limit than this one:

$$\tau^\varepsilon = a^\varepsilon \frac{du^\varepsilon}{dx}.$$

Indeed, this second expression has a trivial convergence on its left-hand side because it is linear. Its right-hand side has a  $L^\infty$  weak\* convergence information on  $a^\varepsilon$  with a  $L^2$ -weak convergence of  $du^\varepsilon/dx$  that is a priori not useful to take the limit in this quadratic form. On the contrary, the first expression (B.5) is better suited because its left-hand side is linear and then a weak convergence is enough. To treat its right-hand side, we take advantage of the strong convergence  $\tau^\varepsilon$  in any  $L^p$  space ( $1 \leq p < \infty$ ) combined with the  $L^p$  weak convergence of  $1/a^\varepsilon$  for all  $p < +\infty$ . More precisely, we use the following lemma:

**Lemma B.6.** *Let  $f$  be  $\ell_1$ -periodic in  $L^p(0, \ell_1)$  where  $1 \leq p \leq +\infty$ . Let us consider  $f^\varepsilon$  defined by*

$$f^\varepsilon = f\left(\frac{x}{\varepsilon}\right) \text{ a.e. on } \mathbb{R}.$$

*Then, if  $p < +\infty$ , as  $\varepsilon \rightarrow 0$*

$$f_\varepsilon \rightharpoonup \mathcal{M}_{(0, \ell_1)} = \frac{1}{\ell_1} \int_0^{\ell_1} f(y) dy \text{ weakly in } L^p(\omega)$$

*for any bounded open subset  $\omega$  of  $\mathbb{R}$ .*

This lemma indicates that it is possible to take the limit in (B.5) to get

$$\partial_x u = \tau \mathcal{M}_{(0, \ell_1)}\left(\frac{1}{a}\right). \quad (\text{B.6})$$

Now let us observe that

$$-\partial_x \tau^\varepsilon = f$$

and therefore at the limit  $\varepsilon \rightarrow 0$ , we get

$$-\partial_x \tau = f \quad (\text{B.7})$$

which gives (B.4) when using the expression (B.6). The boundary condition is obtained because we have that  $u^\varepsilon \in H_0^1(0, 1)$  uniformly in  $\varepsilon$  and therefore we get the continuity on the boundary.

### B.3.2 Explicit multi-dimension averaging calculations

An application we have in mind is a bubbly flow where  $\Omega_g$  is the union of a large number of small gas bubbles suspended into a liquid occupying the domain  $\Omega_l$ . To describe the homogenization procedure for this particular case, we introduce a parameter  $\varepsilon > 0$  (small) and denote  $(T_k^\varepsilon)_{k \in K^\varepsilon}$  a covering of  $[0, 1]^d$  with cubes of width  $\varepsilon$  (see Figure 8). Each cell  $T_k^\varepsilon$  contains a cube  $D_k^\varepsilon$  of volume  $\varphi_g \varepsilon^d$ , say  $D_k^\varepsilon$  is the cube with center  $x_k^\varepsilon$  and width  $\varphi_g^{1/d} \varepsilon$  where  $x_k^\varepsilon \in T_k^\varepsilon$  and  $\varphi_g \in [0, 1]$  represents the volume fraction of phase  $g$ . Using the previous notation of the indicator function  $X_i$ , we introduce the bulk density

$$\rho(x) = \rho_g(x)X_g(x) + \rho_l(x)X_l(x).$$

Since we assume  $\varepsilon \ll 1$ , we want to compute a description of the mixture by finding a limit for a density sequence  $\{(\rho^\varepsilon, X_g^\varepsilon)\}_{\varepsilon > 0}$  when  $\varepsilon \rightarrow 0$ , for  $\rho^\varepsilon$  being

$$\rho^\varepsilon(x) = \rho_g(x)X_g^\varepsilon(x) + \rho_l(x)X_l^\varepsilon(x) \quad \text{where} \quad X_g^\varepsilon = X_g\left(\frac{x}{\varepsilon}\right) \text{ and } X_l^\varepsilon(x) = (1 - X_g^\varepsilon)(x).$$

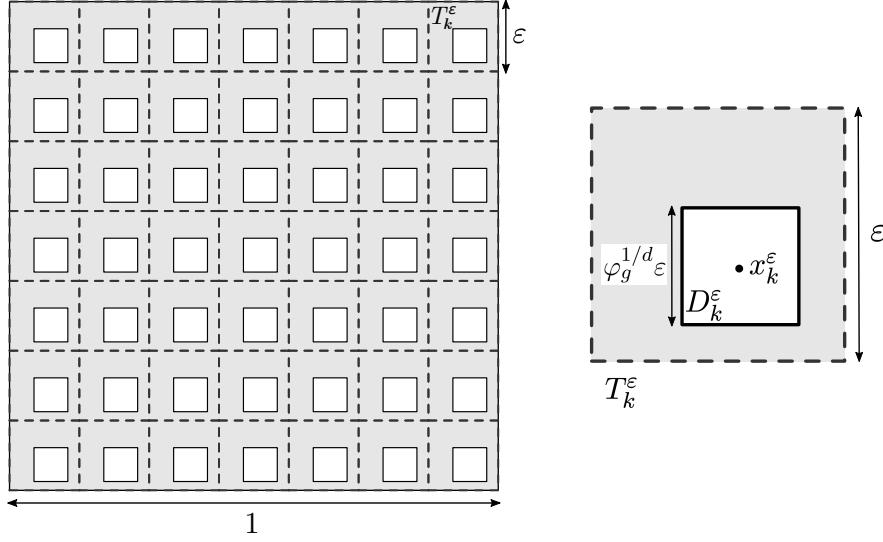


Figure 8: Left: Sketch of the two-phase domain  $[0, 1]^d$  covered by cubes  $T_k^\epsilon$  of volume  $\epsilon^d$ . Phase  $g$  is white and phase  $l$  is gray. Right: Detail of the elementary cell  $T_k^\epsilon$  that includes the cube  $D_k^\epsilon$  of volume  $\varphi_g \epsilon^d$ .

**Proposition B.1.** *The limit is*

$$(\rho^\epsilon, X_g^\epsilon) \rightharpoonup (\varphi_g \rho_g + \varphi_l \rho_l, \varphi_g) \text{ in } (L^p(\Omega))^2 \text{ weak for all } p \in [1, +\infty).$$

Moreover, we have

$$\rho^\epsilon \rightarrow \bar{\rho} \text{ in } L^p(\Omega) \text{ for all } p \in [1, +\infty) \text{ if and only if } \rho_g = \rho_l.$$

If  $\rho_g \neq \rho_l$ , the lack of strong convergence is due to the strong oscillations of  $\rho^\epsilon$ . These oscillations are related to the multiphase nature of the flow under consideration (see example shown in Section 1.2.1). Introducing the Young measures – which enables us to describe the density oscillations – we expect to be able to represent the different phases in the flow. Before a brief description of these Young measures, several remarks are needed. First, we emphasize that we need a tool able to split the information on the partial densities  $\rho_g$  and  $\rho_l$  and volume fractions  $\varphi_g$  and  $\varphi_l = 1 - \varphi_g$ . Although these quantities are known in the above construction, in an applied case, one may only have access to bulk quantities such as  $\rho$ .

**Proposition B.2.** *For all  $\beta \in \mathcal{C}_b(\mathbb{R}^+ \times \mathbb{R})$  and  $\phi \in \mathcal{C}_c^\infty(\Omega)$ , we have the following limit*

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} \beta(\rho^\epsilon, X_g^\epsilon) \phi(x) dx = \int_{\Omega} (\varphi_g(x) \beta(\rho_g(x), 1) + \varphi_l(x) \beta(\rho_l(x), 0)) \phi(x) dx.$$

Introducing the measures

$$(x \mapsto \bar{\nu}_x) = \varphi_g(x) \delta_{(\rho_g(x), 1)} + \varphi_l(x) \delta_{(\rho_l(x), 0)},$$

where we denote  $\delta_{\xi_0}$  the Dirac measure centered in  $\xi_0$ , the limit quantity may be written

$$\langle \bar{\nu}_x, \beta \rangle = \varphi_g(x) \beta(\rho_g(x), 1) + \varphi_l(x) \beta(\rho_l(x), 0).$$

**Proof.** For clarity, we assume that  $\rho_g, \rho_l$  and  $\varphi_l, \varphi_g$  are constant. When they are not, the following calculations can be reproduced under some restrictions of the continuous functions defined on  $\bar{\Omega}$ . Given  $\beta \in \mathcal{C}_b(\mathbb{R}^+ \times \mathbb{R})$  and  $\phi \in \mathcal{C}_c^\infty(\Omega)$ ,

$$\int_{\Omega} \beta(\rho^\epsilon, X_g^\epsilon) \phi(x) dx = \beta(\rho_g, 1) \sum_{k \in K^\epsilon} \int_{T_k^\epsilon} \phi(x) X_g dx + \beta(\rho_l, 0) \sum_{k \in K^\epsilon} \int_{T_k^\epsilon} \phi(x) X_l dx.$$

Let us calculate the quantity

$$\int_{\Omega} \phi(x) X_g$$

taking into account the domain decomposition described above. The quantity involving  $X_l^\varepsilon$  is calculated using  $X_l^\varepsilon = 1 - X_g^\varepsilon$ . We assume then that for any function  $\phi \in C_c^\infty$ ,

$$\phi(x) = \phi(x_k^\varepsilon) + O(\varepsilon) \text{ in } T_k^\varepsilon.$$

$$\begin{aligned} \int_{\Omega} \phi(x) X_g dx &= \sum_{k \in K^\varepsilon} \int_{T_k^\varepsilon} \phi(x) X_g dx = \sum_{k \in K^\varepsilon} \int_{D_k^\varepsilon} \phi(x) dx = \sum_{k \in K^\varepsilon} \int_{D_k^\varepsilon} \phi(x_k^\varepsilon) dx + O(\varepsilon) = \sum_{k \in K^\varepsilon} |D_k^\varepsilon| \phi(x_k^\varepsilon) + O(\varepsilon) \\ &= \sum_{k \in K^\varepsilon} \varphi_g \varepsilon^d \phi(x_k^\varepsilon) + O(\varepsilon) = \varphi_g \sum_{k \in K^\varepsilon} \varepsilon^d \phi(x_k^\varepsilon) + O(\varepsilon) = \varphi_g \sum_{k \in K^\varepsilon} \int_{T_k^\varepsilon} \phi(x_k^\varepsilon) dx + O(\varepsilon) \\ &= \varphi_g \sum_{k \in K^\varepsilon} \int_{T_k^\varepsilon} \phi(x) dx + O(\varepsilon) = \varphi_g \int_{\Omega} \phi(x) dx + O(\varepsilon) \end{aligned}$$

and similarly

$$\int_{\Omega} \phi(x) X_l = \varphi_l \int_{\Omega} \phi(x) dx + O(\varepsilon).$$

Then the limit  $\varepsilon \rightarrow 0$  yields

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \beta(\rho^\varepsilon(x), X_g^\varepsilon(x)) \phi(x) dx = \left( \varphi_g \beta(\rho_g, 1) + \varphi_l \beta(\rho_l, 0) \right) \int_{\Omega} \phi(x) dx.$$

If  $\rho_g$  and  $\rho_l$  depend on  $x$  then slightly more complicated calculations yield that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \beta(\rho^\varepsilon(x), X_g^\varepsilon(x)) \phi(x) dx = \int_{\Omega} \left( \varphi_g \beta(\rho_g(x), 1) + \varphi_l \beta(\rho_l(x), 0) \right) \phi(x) dx.$$

So if we define the measures  $(x \mapsto \bar{\nu}_x) = \varphi_g(x) \delta_{(\rho_g(x), 1)} + \varphi_l(x) \delta_{(\rho_l(x), 0)}$  then

$$\langle \bar{\nu}_x, \beta \rangle = \varphi_g \beta(\rho_g, 1) + \varphi_l \beta(\rho_l, 0) \text{ in } \mathcal{D}'(\Omega).$$

**A mathematical tool to describe oscillations.** The lack of strong compactness of the density sequence  $\{(\rho^\varepsilon, X_g^\varepsilon)\}_{\varepsilon > 0}$  is due to the oscillations of the indicator functions  $X_g^\varepsilon$  and  $X_l^\varepsilon$ , the convergence of which is calculated more easily in the space of measures (see the example in the introduction: Figure 3). As seen in the explicit calculations above, to describe oscillations one proposes that a couple  $(\rho, X) \in L^1(\Omega) \times L^\infty(\Omega)$  with  $\rho \geq 0$  is identified with a measure  $\nu_x$  on  $\Omega \times \mathbb{R}^+ \times \mathbb{R}$  as follows

$$\langle \nu_x, b \rangle = b(\rho(x), X(x)) \quad \text{for all } b \in \mathcal{C}_b(\mathbb{R}^+ \times \mathbb{R}).$$

Thus we can write

$$\langle \nu_x, \phi \otimes b \rangle = \int_{\Omega} b(\rho(x), X(x)) \phi(x) dx \quad \text{for all } b \in \mathcal{C}_b(\mathbb{R}^+ \times \mathbb{R}) \text{ and } \phi \in \mathcal{D}(\Omega).$$

This identification yields that  $\nu_x$  is positive and that its mass is  $|\Omega|$ . The above identification can be interpreted as seeing the mapping  $x \mapsto (x, \rho(x), X(x))$  as a random variable on  $\Omega$  and constructing the associated law  $\nu_x$ . One advantage of this identification is that we may apply directly the results on random variables, which implies the following lemma.

**Lemma B.7.** *Let  $\{\rho^\varepsilon\}_{\varepsilon > 0}$  be a bounded sequence in  $L^1(\Omega)$  and  $X_g^\varepsilon$  be a bounded sequence in  $L^\infty(\Omega)$ . Let  $\{\nu^\varepsilon\}_{\varepsilon > 0}$  be the associated sequence of measures defined by  $\langle \nu^\varepsilon, \phi \otimes b \rangle = \int_{\Omega} b(\rho^\varepsilon(x)) \phi(x) dx$  for all  $b \in \mathcal{C}_b(\mathbb{R}^+ \times \mathbb{R})$  and  $\phi \in \mathcal{D}(\Omega)$ . Then, up to the extraction of a subsequence, we have that  $\{\nu^\varepsilon\}_{\varepsilon > 0}$  converges to a positive measure  $\bar{\nu}$  in the sense*

$$\lim_{\varepsilon \rightarrow 0} \langle \nu^\varepsilon, \phi \otimes b \rangle = \langle \bar{\nu}, \phi \otimes b \rangle \quad \text{for all } b \in \mathcal{C}_b(\mathbb{R}^+) \text{ and } \phi \in \mathcal{D}(\Omega).$$

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