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Study of symmetries through the action on torsors of the Janus Symplectic Group

Jean-Pierre Petit
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Keywords : Dynamic groups, Symplectic groups, Lorentz group, Poincaré group, Janus group, torsors of a Lie group, Action on the torsors.

Abstract

In this article, we analyze the Poincaré group as the isometry group of Minkowski space, approached here within the framework of dynamic groups. Using the work of J.M. Souriau, we investigate how the action of the group on its coadjoint orbit structure reveals physical quantities such as energy, momentum, and spin as emergent geometric entities. Extending this analysis, we incorporate antichronous transformations, thereby generalizing the group to a five-dimensional context, which geometrically encodes the presence of electric charge and the symmetry between matter and antimatter. This extension culminates in the formulation of the Janus group, which encompasses **CPT** symmetry. Our findings suggest that these developments initiate a geometric interpretation of Andrei Sakharov's model, wherein primordial antimatter resides within an antichronous sector of the universe, providing a novel perspective on the baryon asymmetry problem in cosmology.

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1 Preliminaries

1.1. The vector space \mathbb{R}^4 has also an affine structure: its elements can be seen as vectors or points. We denote by $(O; e_0, e_1, e_2, e_3)$ its usual canonical frame where O is the origin and $e_0 = (1, 0, 0, 0)$ $e_1 = (0, 1, 0, 0)$ $e_2 = (0, 0, 1, 0)$ $e_3 = (0, 0, 0, 1)$. The coordinates of a point x are given by a quadruplet of real numbers (x_0, x_1, x_2, x_3) . The first one x_0 , which physicists call the *time coordinate* is sometimes replaced by $x_0 = ct$ where c is the speed of light in a vacuum.

For a positive integer n , we denote by I_n the identity matrix of order n and we set:

$$I_{1,3} = \begin{pmatrix} -1 & 0 \\ 0 & I_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1.2. A *bilinear form* on \mathbb{R}^4 is a map $g : \mathbb{R}^4 \times \mathbb{R}^4 \mapsto \mathbb{R}$ such that, for any fixed $x \in \mathbb{R}^4$, the partial map $g(x, \cdot)$ is linear and for any fixed $y \in \mathbb{R}^4$, the partial map $g(\cdot, y)$ is linear. We will say that g is *symmetric* if, for all $x, y \in \mathbb{R}^4$, we have $g(y, x) = g(x, y)$.

Let g be a symmetric bilinear form on \mathbb{R}^4 . We say that g is *non-degenerate* if: $\{g(x, y) = 0 \text{ for all } y\} \implies x = 0$; *positive definite* if, for all non-zero $x \in \mathbb{R}^4$, we have $g(x, x) > 0$. In this case we say that g defines a *scalar product* on \mathbb{R}^4 .

The vectors e_0, e_1, e_2, e_3 also form a basis of the vector space \mathbb{R}^4 . If g is a bilinear form on \mathbb{R}^4 , we set $g_{\lambda\mu} = g(e_\lambda, e_\mu)$ for $\lambda, \mu = 0, 1, 2, 3$. If g is symmetric we have of course $g_{\lambda\mu} = g_{\mu\lambda}$ for all $\lambda, \mu \in \{0, 1, 2, 3\}$. The symmetric bilinear form g is therefore determined by the symmetric matrix:

$$g = (g_{\lambda\mu}) \quad \text{where } \lambda \text{ and } \mu \text{ vary in } \{0, 1, 2, 3\}.$$

There exists a basis of \mathbb{R}^4 in which g is equivalent to a diagonal matrix. If in addition g is non-degenerate, all terms in the diagonal (the eigenvalues) are non-zero.

1.3. The *Minkowski space* is the space \mathbb{R}^4 equipped with the symmetric and non-degenerate bilinear form g whose matrix in the canonical basis (e_0, e_1, e_2, e_3) is:

$$I_{1,3} = \begin{pmatrix} -1 & 0 \\ 0 & I_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We say that g is the *Lorentz metric* on \mathbb{R}^4 .

The couple (\mathbb{R}^4, g) will be denoted simply by $\mathbb{R}^{1,3}$. (Do not confuse with Euclidean space $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ is the usual scalar product $\langle x, y \rangle = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$ whose matrix is the identity I_4 .)

For infinitesimal variation (x_0, x_1, x_2, x_3) of the coordinates, we have a dual basis (dx_0, dx_1, dx_2, dx_3) whose vectors are defined by $dx_\lambda(e_\mu) = \delta_\lambda^\mu$ where δ_λ^μ is the Kronecker symbol (equal to

1 if $\lambda = \mu$ and 0 otherwise). We can therefore write the metric g in the form $dx^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$ or, if we put $x_0 = ct$, $dx^2 = -c^2 dt^2 + dx_1^2 + dx_2^2 + dx_3^2$.

1.4. Let $x \in \mathbb{R}^4$. We will say that x is a:

- (i) *space vector* if $g(x, x) > 0$;
- (ii) *light vector* if $g(x, x) = 0$;
- (iii) *time vector* if $g(x, x) < 0$.

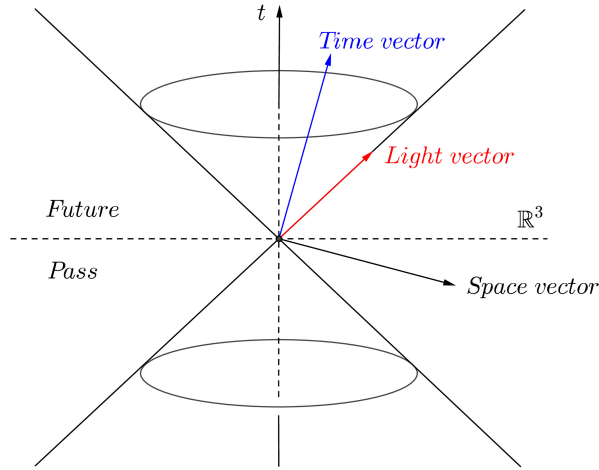


Figure 1: Visualization of Minkowski spacetime with light cones

2 The Lorentz group

2.1. Let $L : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be a linear bijection. We will say that L is an *automorphism* of the Minkowski space $\mathbb{R}^{1,3}$ or *Lorentz automorphism* if it leaves the metric g invariant, that is to say:

$$g(L(x), L(y)) = g(x, y) \quad \text{for all } x, y \in \mathbb{R}^{1,3}.$$

The identity is a Lorentz automorphism; the product of two Lorentz automorphisms is a Lorentz automorphism, as is the inverse of a Lorentz automorphism. The set of Lorentz automorphisms is therefore a group called the *Lorentz group* and denoted by \mathcal{Lor} . It is the subgroup $O(1, 3)$ of the linear group $GL(\mathbb{R}, 4)$. As a Lie group, its dimension is 6.

Let L be an element of \mathcal{Lor} with matrix $L = (\ell_{\lambda\mu})$ ($\lambda, \mu \in \{0, 1, 2, 3\}$) in the canonical basis and let L^t denote its transpose. Then the invariance of g by L requires the latter to satisfy $L^t I_{1,3} L = I_{1,3}$. From which we deduce:

$$(\det L)^2 = 1 \quad \text{and} \quad -(\ell_{00})^2 + (\ell_{01})^2 + (\ell_{02})^2 + (\ell_{03})^2 = -1.$$

The first relation implies $\det L = 1$ or $\det L = -1$; the second implies $(\ell_{00})^2 \geq 1$, that is, $\ell_{00} \geq 1$ or $\ell_{00} \leq -1$. The four "combinations":

$$\{\det L = 1 \text{ and } \ell_{00} \geq 1\}, \quad \{\det L = -1 \text{ and } \ell_{00} \geq 1\}$$

$$\{\det L = 1 \text{ and } \ell_{00} \leq -1\}, \quad \{\det L = -1 \text{ and } \ell_{00} \leq -1\}$$

show that the group $\mathcal{Lor} = O(1,3)$ has four connected components:

- (i) $\mathcal{Lor}_n = \{L \in \mathcal{Lor} : \det L = 1 \text{ and } \ell_{00} \geq 1\}$. The *neutral component* whose elements do not reverse space and do not reverse time.
- (ii) $\mathcal{Lor}_s = \{L \in \mathcal{Lor} : \det L = -1 \text{ and } \ell_{00} \geq 1\}$. Its elements reverse space but not time.
- (iii) $\mathcal{Lor}_t = \{L \in \mathcal{Lor} : \det L = 1 \text{ and } \ell_{00} \leq -1\}$. Its elements reverse time but not space.
- (iv) $\mathcal{Lor}_{st} = \{L \in \mathcal{Lor} : \det L = -1 \text{ and } \ell_{00} \leq -1\}$. Its elements reverse time and space.

3 The Poincaré group

3.1. The Poincaré group is the group of all affine transformations of \mathbb{R}^4 that leave the Lorentz metric g invariant. These transformations are of the form $P(x) = Lx + D$, where $L \in \mathcal{Lor}$ and D is a vector in \mathbb{R}^4 . We denote it by \mathcal{Poin} .

3.2. As we have seen, the group \mathcal{Lor} acts linearly on the vector space \mathbb{R}^4 , preserving the Lorentz metric g . We can therefore construct the semi-direct product $\mathcal{Lor} \ltimes \mathbb{R}^4$. This is precisely the Poincaré group of Minkowski space $\mathbb{R}^{1,3}$. It is a Lie group of dimension 10 with four connected components.

3.3. We can embed the Poincaré group into the matrix group $GL(\mathbb{R}, 5)$ as follows. To any $(L, D) \in \mathcal{Lor} \ltimes \mathbb{R}^4 = \mathcal{Poin}$, we associate the following 5×5 matrix:

$$\begin{pmatrix} L & D \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \ell_{00} & \ell_{01} & \ell_{02} & \ell_{03} & d_0 \\ \ell_{10} & \ell_{11} & \ell_{12} & \ell_{13} & d_1 \\ \ell_{20} & \ell_{21} & \ell_{22} & \ell_{23} & d_2 \\ \ell_{30} & \ell_{31} & \ell_{32} & \ell_{33} & d_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

That is:

$$\mathcal{Poin} = \left\{ \begin{pmatrix} L & D \\ 0 & 1 \end{pmatrix} : L \in \mathcal{Lor}, \quad D \in \mathbb{R}^4 \right\}.$$

4 Geometrization of physical quantities through Symplectic Group actions

The application of the coadjoint action of a symplectic group on the dual of its Lie algebra, initiated by the mathematician Jean-Marie Souriau, has shed light on specific aspects of the approach followed by physics. The orbit method is due to Kirillov ([4], [7], [5], [6], [12], [13], [15], [21], [24] and [25]).

Thus, the restricted Lorentz symplectic group, limited to its two orthochrone components, translates, through the invariance properties that result from it, the aspects of special relativity. In 1970, J-M Souriau established that the analysis of the components of its moment makes it possible to shed light on the geometric nature of a spin (not quantized): see [20] and [19]. He uses for this purpose symplectic methods ([10], [8], [22] and [23]). In the theory of symplectic groups, we show a classification in terms of movements.

By operating the product of this group by that of the spatio-temporal translations, we obtain the restricted Poincaré symplectic group, always limited to its two orthochrone components. In its moment, we first find the energy related to the subgroup of temporal translations. Then the momentum, linked to the spatial translations, the two being linked by the invariance of the modulus of the energy-momentum four-vector under the action of the Lorentz group.

By adding a translation along a fifth dimension to the restricted Poincaré group, we form a Lie group to which we will give the name *Restricted Kaluza Group* ([1], [2], [3], [11], [14]). This group is not the 15-dimensional Kaluza group associated with a 5-dimensional Lorentzian manifold but a new 11-dimensional group, including 5-dimensional space-time translation. This new dimension endows the momentum with an additional scalar that can be identified with the electric charge q , which may be positive, negative, or zero, and is still not quantized. We then bring out the geometric translation according to a scalar ϕ due to endowing the masses with an invariant electric charge. Then, by bringing in a new symmetry reflecting the inversion of the fifth dimension, synonymous with an inversion of the scalar from q to $-q$, we double the number of its connected components from 2 to 4. The action on the moment then links this new symmetry to the inversion of the electric charge q . We thus deduce the geometric modeling of charge conjugation or *C-Symmetry*¹. It's then logical to name this new extension, the *Restricted Janus Group*.

By introducing a new symmetry to the previous group, which we describe as *T-Symmetry*², we build the *Janus Symplectic Group*. Thus, we double the number of connected components from four to eight, grouped into two subsets: "*Orthochronous*", conserving time and energy properties, and "*Antichronous*", reversing time and energy. Therefore, we bring forth the geometric translation of endowing masses with an invariant electric charge. As the Jean-Marie Souriau demonstrated as early as 1970, a pioneer in the theory of symplectic groups

¹which translates the matter-antimatter symmetry introduced by Dirac.

²which converts matter into antimatter with negative mass, a concept we could name *antimatter in the Feynman sense*.

([20], [9], [19]), this approach has allowed key elements, which have marked the progress of relativistic physics, to be given a purely geometric nature.

In relation to the world of physics, wouldn't the role of mathematics be to illuminate the path traveled? Conversely, could it be possible that the exploration of new symmetries, accompanying this decoding using symplectic groups, contains more than what we thought we put into it? That it could designate new paths to follow?

This is what we will consider with the Janus Symplectic Group with charge symmetry, by integrating the antichronous components of the Lorentz group, resulting from its simple axiomatic definition, with the obvious repercussions on the Poincaré group and its extensions.

5 Janus Symplectic Group

Let:

$$I_{1,3} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

be the Minkowski metric tensor in four dimensions, and define the following matrices:

$$\tilde{\mathbf{T}} := I_{1,3}, \quad \tilde{\mathbf{P}} := -\tilde{\mathbf{T}}.$$

Thus:

$$\forall \lambda, \nu \in \{0, 1\}, \quad \mathcal{L}or \left(\tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda \right) := \left\{ L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda, L_n \in \mathcal{L}or_n \right\}.$$

Then, there are 4 connected components of $\mathcal{L}or$, given by³

$$\begin{aligned} \mathcal{L}or_n &= \mathcal{L}or \left(\tilde{\mathbf{P}}^0 \tilde{\mathbf{T}}^0 \right) & \mathcal{L}or_s &= \mathcal{L}or \left(\tilde{\mathbf{P}}^1 \tilde{\mathbf{T}}^0 \right) \\ \mathcal{L}or_t &= \mathcal{L}or \left(\tilde{\mathbf{P}}^0 \tilde{\mathbf{T}}^1 \right) & \mathcal{L}or_{st} &= \mathcal{L}or \left(\tilde{\mathbf{P}}^1 \tilde{\mathbf{T}}^1 \right) \end{aligned}$$

and we have the decomposition:

$$\mathcal{L}or = \bigsqcup_{\nu, \lambda \in \{0, 1\}} \mathcal{L}or \left(\tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda \right) \quad (1)$$

Then, we define the Janus symplectic group.

³Equalities are shown by double inclusion. For example, let's demonstrate that $\mathcal{L}or_s = \mathcal{L}or \left(\tilde{\mathbf{P}}^1 \tilde{\mathbf{T}}^0 \right)$. Take $L \in \mathcal{L}or_s$ ($\det(L) = -1$ et $[L]_{00} \geq 1$). Then we have $\det(L\tilde{\mathbf{P}}) = -1$ and $[L\tilde{\mathbf{P}}]_{00} \geq 1$ i.e., we have $L_n := L\tilde{\mathbf{P}} \in \mathcal{L}or_n$. Since $\tilde{\mathbf{P}}^{-1} = \tilde{\mathbf{P}}$, we can conclude that $L = L_n \tilde{\mathbf{P}} \in \mathcal{L}or \left(\tilde{\mathbf{P}}^1 \tilde{\mathbf{T}}^0 \right)$. The inclusion in the other direction is trivial.

Definition 5.1. The *Janus symplectic group* is defined as the subgroup of $GL(6, \mathbb{R})$:

$$\mathcal{J}an := \left\{ \begin{pmatrix} L & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix}, \eta \in \{0, 1\} \wedge \phi \in \mathbb{R} \wedge L \in \mathcal{L}or \wedge D \in \mathbb{R}^4 \right\}$$

The Janus symplectic group is therefore a subgroup of the group of isometries in dimension 5 given by⁴:

$$\text{Aff}(\mathcal{O}(1, 4)) := \left\{ \begin{pmatrix} L & D' \\ 0 & 1 \end{pmatrix}, L \in \mathcal{O}(1, 4) \wedge D' \in \mathbb{R}^5 \right\}$$

with $\tau_{1,4}(L) := I_{1,4}L^T I_{1,4}$ and $\mathcal{O}(1, 4) := \{L \in GL(5, \mathbb{R}), \tau_{1,4}(L)L = I_5\}$ ⁵.

The elements of $\text{Aff}(\mathcal{O}(1, 4))$ are the elements which preserve the distance between two events (*pentavectors*) $X := (t, x, y, z, \xi)$ and $X' := (t', x', y', z', \xi')$ given by:

$$d(X, X') := c^2(t - t')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2 - (\xi - \xi')^2$$

This fifth dimension is of space type (we note the variable ξ). Each dimension is therefore associated with a symmetry, there are three types of symmetries:

- the **T**-*symmetry*;
- the **P**_x-*symmetry*, **P**_y-*symmetry*, **P**_z-*symmetry* grouped together what we call the **P**-*symmetry*;
- the ξ -*symmetry* corresponding to the **C**-*symmetry* (the charge conjugation).

This space of dimension 5 is a Minkowski metric space to which we have added one dimension, it has the metric $I_{1,4}$.

We also define the *restricted Janus group* as the subgroup of $\mathcal{J}an$ given by:

$$\mathcal{J}an_n := \left\{ \begin{pmatrix} L_n & 0 & D \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix}, \phi \in \mathbb{R} \wedge L_n \in \mathcal{L}or_n \wedge D \in \mathbb{R}^4 \right\}$$

⁴ $\text{Aff}(\mathcal{O}(1, 4))$ is the affine group associated with $\mathcal{O}(1, 4)$, it is also defined by the semi-direct product $\text{Aff}(\mathcal{O}(1, 4)) := \mathcal{O}(1, 4) \ltimes \mathbb{R}^5$. We can also define the symplectic Janus group as being the affine group associated with the subgroup of $\mathcal{O}(1, 4)$ given by:

$$\mathcal{E}lec := \left\{ \begin{pmatrix} L & 0 \\ 0 & (-1)^\eta \end{pmatrix}, \eta \in \{0, 1\} \wedge L \in \mathcal{L}or \right\}$$

called the *symplectic electric group* and we have $\mathcal{J}an := \text{Aff}(\mathcal{E}lec)$.

⁵with

$$I_{1,4} := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let:

$$\mathbf{C} := \begin{pmatrix} I_4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P} := \begin{pmatrix} \tilde{\mathbf{P}} & 0 \\ 0 & I_2 \end{pmatrix}, \quad \mathbf{T} := \begin{pmatrix} \tilde{\mathbf{T}} & 0 \\ 0 & I_2 \end{pmatrix}.$$

We have:

$$\forall \lambda, \eta, \nu \in \{0, 1\}, \quad \begin{pmatrix} L_n & 0 & D \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix} \mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda = \begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix}$$

and therefore by equation (1):

$$\mathcal{J}an = \left\{ \left(\begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix}, \lambda, \eta, \nu \in \{0, 1\} \wedge \phi \in \mathbb{R} \wedge L_n \in \mathcal{L}or_n \wedge D \in \mathbb{R}^4 \right) \right\}.$$

Definition 5.2. (i) The *CPT*-group is the subgroup \mathcal{K} of $\mathcal{J}an$ of order 8 generated by \mathbf{C} , \mathbf{P} and \mathbf{T} ie:

$$\mathcal{K} := \left\{ \mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda, \eta, \nu, \lambda \in \{0, 1\} \right\} = \{I_6, \mathbf{T}, \mathbf{P}, \mathbf{PT}, \mathbf{C}, \mathbf{CT}, \mathbf{CP}, \mathbf{CPT}\}.$$

(ii) For all $\mathbf{X} \in \mathcal{K}$, the \mathbf{X} -component of $\mathcal{J}an$ is:

$$\mathcal{J}an(\mathbf{X}) := \{J\mathbf{X}, J \in \mathcal{J}an_n\}.$$

Thus, we have:

$$\mathcal{J}an(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) = \left\{ \left(\begin{pmatrix} L_n \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix}, \phi \in \mathbb{R} \wedge L_n \in \mathcal{L}or_n \wedge D \in \mathbb{R}^4 \right) \right\}.$$

These 8 components are the 8 connected components of $\mathcal{J}an$, we have the decomposition:

$$\mathcal{J}an = \bigsqcup_{\mathbf{X} \in \mathcal{K}} \mathcal{J}an(\mathbf{X}) = \bigsqcup_{\eta, \nu, \lambda \in \{0, 1\}} \mathcal{J}an(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda).$$

The group $\mathcal{L}or$ is the Lie group of dimension 6 and its Lie algebra is:

$$\mathfrak{lor} := \mathcal{A}(1, 3) := \{\Lambda \in \mathcal{M}(4, \mathbb{R}), \tau_{1,3}(\Lambda) = -\Lambda\}$$

Then, the group $\mathcal{J}an$ is a Lie group of dimension 11 and its Lie algebra is:

$$\mathfrak{j}an = \left\{ \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}, \Lambda \in \mathcal{A}(1, 3) \wedge \Gamma \in \mathbb{R}^4 \wedge \varepsilon \in \mathbb{R} \right\}.$$

We have two characterisations⁶:

$$\begin{aligned} (\mathbb{R}^5)^* &= \left\{ \begin{pmatrix} \Gamma \\ \varepsilon \end{pmatrix} \mapsto - \begin{pmatrix} P^T & q \end{pmatrix} I_{1,4} \begin{pmatrix} \Gamma \\ \varepsilon \end{pmatrix} = -\tau(P)\Gamma - q\varepsilon, \begin{pmatrix} P \\ q \end{pmatrix} \in \mathbb{R}^5 \right\} \\ \mathcal{A}(1,3)^* &= \left\{ \Lambda \mapsto -\frac{1}{2}\text{Tr}(M\Lambda), M \in \mathcal{A}(1,3) \right\} \end{aligned}$$

Then, we have:

$$\mathbf{jan}^* = \left\{ \left\{ M \mid P \mid q \right\} : \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \mapsto -\frac{1}{2}\text{Tr}(M\Lambda) - \tau(P)\Gamma - q\varepsilon, M \in \mathcal{A}(1,3) \wedge P \in \mathbb{R}^4 \wedge q \in \mathbb{R} \right\}^7.$$

The *action of the group \mathcal{Jan} on \mathbf{jan}^** is defined by the coadjoint representation *i.e.*, for any $a \in \mathcal{Jan}$ and any $\mu \in \mathbf{jan}^*$, we denote this action by:

$$a \bullet \mu := \text{Ad}_a^*(\mu).$$

with

$$\begin{aligned} \text{Ad}^* : \mathcal{Jan} &\longrightarrow \text{Aut}(\mathbf{jan}^*) \\ a &\longmapsto \text{Ad}_a^* : \mu \longmapsto (Z \longmapsto \mu(a^{-1}Za)) \end{aligned}$$

Proposition 5.1. Let:

$$a := \begin{pmatrix} L & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{Jan}, \quad \left\{ M \mid P \mid q \right\} \in \mathbf{jan}^*.$$

We have:

$$\begin{aligned} a \bullet \left\{ M \mid P \mid q \right\} \\ = \left\{ LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D) \mid LP \mid (-1)^\eta q \right\}. \end{aligned}$$

⁶For all $\beta \in \mathbb{R}^*$, the application Φ_β which to $M \in \mathcal{A}(1,3)$ associates the linear form $\Lambda \mapsto \beta\text{Tr}(M\Lambda)$ is an isomorphism of $\mathcal{A}(1,3)$ to $\mathcal{A}(1,3)^*$. Taking $\{A_{kl} := -E_{kl} + [I_{1,3}]_{ll}[I_{1,3}]_{kk}E_{lk}, k, l \in \{1, \dots, 4\}, k < l\}$ the canonical basis of $\mathcal{A}(1,3)$, we have $\Phi_{-1/2}(A_{kl})(A_{kl}) = 1$, hence the choice of $\beta := -1/2$.

⁷The elements of \mathbf{jan}^* are called **torsors**.

Proof. We have:

$$\begin{aligned}
& \left(a \bullet \left\{ M \mid P \mid q \right\} \right) \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \\
&= \left\{ M \mid P \mid q \right\} \left(a^{-1} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} a \right) \\
&= \left\{ M \mid P \mid q \right\} \left(\begin{pmatrix} \tau(L) & 0 & -\tau(L)D \\ 0 & (-1)^\eta & (-1)^{\eta+1}\phi \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L & 0 & D \\ 0 & (-1)^\eta & \phi \\ 0 & 0 & 1 \end{pmatrix} \right) \\
&= \left\{ M \mid P \mid q \right\} \begin{pmatrix} \tau(L)\Lambda L & 0 & \tau(L)(\Lambda D + \Gamma) \\ 0 & 0 & (-1)^\eta \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \\
&= -\frac{1}{2} \text{Tr} (M\tau(L)\Lambda L) - \tau(P)\tau(L)(\Lambda D + \Gamma) - (-1)^\eta q\varepsilon \\
&= -\frac{1}{2} \text{Tr} [(LM\tau(L) + 2D\tau(P)\tau(L))\Lambda] - \tau(LP)\Gamma - (-1)^\eta q\varepsilon \\
&= -\frac{1}{2} \text{Tr} [(LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D))\Lambda] - \tau(LP)\Gamma - (-1)^\eta q\varepsilon \\
&= \left\{ LM\tau(L) + D\tau(P)\tau(L) - LP\tau(D) \mid LP \mid (-1)^\eta q \right\} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

□

To describe the Lie algebra of $\mathcal{J}an$, we can also use the isomorphism of Lie algebras⁸:

$$\begin{aligned}
j: (\mathbb{R}^3, \wedge) &\longrightarrow (\mathcal{A}(3), [\cdot, \cdot]) \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &\longmapsto \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}
\end{aligned}$$

with \wedge the cross product on \mathbb{R}^3 and $\mathcal{A}(3)$ the vector space of antisymmetric matrices of size

3. Then, we have:

$$\mathbf{jan} = \left\{ \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}, \Lambda \in \mathcal{A}(1, 3) \wedge \Gamma \in \mathbb{R}^4 \wedge \varepsilon \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}, \beta, w, \gamma \in \mathbb{R}^3 \wedge v, \varepsilon \in \mathbb{R} \right\}.$$

⁸We have for all $u, v \in \mathbb{R}^3$: $u \wedge v = j(u)(v)$ and $j(u \wedge v) = [j(u), j(v)] = j(u)j(v) - j(v)j(u)$.

Therefore, for all $\left\{ M \mid P \mid q \right\} \in \mathbf{jan}^*$, there are unique $\ell, g, p \in \mathbb{R}^3$ and $E, q \in \mathbb{R}$ such as:

$$\begin{aligned} \left\{ M \mid P \mid q \right\} \begin{pmatrix} \Lambda & 0 & \Gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} &= \left\{ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \mid \begin{pmatrix} E \\ p \end{pmatrix} \mid q \right\} \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= -\frac{1}{2} \text{Tr} \left(\begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix} \right) - (E \ p^T) I_{1,3} \begin{pmatrix} v \\ \gamma \end{pmatrix} - q\varepsilon \\ &= \ell^T w - g^T \beta + p^T \gamma - Ev - q\varepsilon \end{aligned}$$

We denote this last equality as:

$$\left\{ \ell \mid g \mid p \mid E \mid q \right\} \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The dual \mathbf{jan}^* has the following descriptions:

$$\left\{ \left\{ \ell \mid g \mid p \mid E \mid q \right\} : \begin{pmatrix} 0 & \beta^T & 0 & v \\ \beta & j(w) & 0 & \gamma \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \end{pmatrix} \mapsto \ell^T w - g^T \beta + p^T \gamma - Ev - q\varepsilon, \ell, g, p \in \mathbb{R}^3 \wedge E, q \in \mathbb{R} \right\}.$$

Definition 5.3. Let

$$\mu := \left\{ M \mid P \mid q \right\} := \left\{ l \mid g \mid p \mid E \mid q \right\} \in \mathbf{jan}^*$$

with relations:

$$M = \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix}, \quad P = \begin{pmatrix} E \\ p \end{pmatrix}.$$

- (i) The matrix $M := M(\mu) \in \mathcal{A}(1, 3)$ is called the *moment matrix associated with μ* . The vector $\ell := \ell(\mu) \in \mathbb{R}^3$ is called the *angular momentum of M* , and the vector $g := g(\mu) \in \mathbb{R}^3$ is the *relativist barycenter of M* .
- (ii) (a) The vector $P := P(\mu) \in \mathbb{R}^4$ is called the *stress-energy vector associated with μ* . The vector $p := p(\mu) \in \mathbb{R}^3$ is called the *linear momentum of P* , and the scalar $E := E(\mu) \in \mathbb{R}$ is called the *energy of P* .
 (b) The *first Casimir number associated with μ* is defined by:

$$C_1 := C_1(\mu) := P^T I_{1,3} P = E^2 - p^2.$$

- (c) The *mass associated to μ* is defined by :

$$m := m(\mu) := \text{sign}(E) \sqrt{C_1} = \text{sign}(E) \sqrt{E^2 - p^2}.$$

- (iii) The scalar $q := q(\mu) \in \mathbb{R}$ is called the *electric charge associated with μ* .

We deduce a simple expression of the action of the **CPT**-group \mathcal{K} on the torsors of \mathbf{jan}^* .

Corollary 5.2. Let $\{ l \mid g \mid p \mid E \mid q \} \in \mathbf{jan}^*$. For all $\lambda, \eta, \nu \in \{0, 1\}$, we have:

$$(\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \{ l \mid g \mid p \mid E \mid q \} = \{ l \mid (-1)^{\lambda+\nu} g \mid (-1)^\nu p \mid (-1)^\lambda E \mid (-1)^\eta q \}.$$

Proof. We apply the Proposition 5.1 with $a := \mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda$:

$$\begin{aligned} (\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \{ l \mid g \mid p \mid E \mid q \} &= (\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \left\{ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \middle| \begin{pmatrix} E \\ p \end{pmatrix} \middle| q \right\} \\ &= \left\{ \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu \middle| I_{1,3} \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu I_{1,3} \begin{pmatrix} E \\ p \end{pmatrix} \middle| (-1)^\eta q \right\} \\ &= \left\{ \begin{pmatrix} 0 & (-1)^{\lambda+\nu} g^T \\ (-1)^{\lambda+\nu} g & j(\ell) \end{pmatrix} \middle| \begin{pmatrix} (-1)^\lambda E \\ (-1)^\nu p \end{pmatrix} \middle| (-1)^\eta q \right\} \\ &= \{ l \mid (-1)^{\lambda+\nu} g \mid (-1)^\nu p \mid (-1)^\lambda E \mid (-1)^\eta q \} \end{aligned}$$

□

So we have:

$$\begin{aligned} \mathbf{C} \bullet \{ l \mid g \mid p \mid E \mid q \} &= \{ l \mid g \mid p \mid E \mid -q \} \\ \mathbf{P} \bullet \{ l \mid g \mid p \mid E \mid q \} &= \{ l \mid -g \mid -p \mid E \mid q \} \\ \mathbf{T} \bullet \{ l \mid g \mid p \mid E \mid q \} &= \{ l \mid -g \mid p \mid -E \mid q \} \end{aligned}$$

Corollary 5.3. Let $\mu \in \mathbf{jan}^*$. For all $\lambda, \eta, \nu \in \{0, 1\}$, we have:

$$\begin{aligned} P \left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu \right) &= \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda P(\mu) \\ C_1 \left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu \right) &= C_1(\mu) \\ m \left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu \right) &= (-1)^\lambda m(\mu) \end{aligned}$$

Proof. Let $\mu := \{ l \mid g \mid p \mid E \mid q \} \in \mathbf{jan}^*$. We have for the stress-energy tensor:

$$\begin{aligned} P(\mathbf{P} \bullet \mu) &= P \left(\{ l \mid -g \mid -p \mid E \mid q \} \right) = \begin{pmatrix} E \\ -p \end{pmatrix} = \tilde{\mathbf{P}} P(\mu) \\ P(\mathbf{T} \bullet \mu) &= P \left(\{ l \mid -g \mid p \mid -E \mid q \} \right) = \begin{pmatrix} -E \\ p \end{pmatrix} = \tilde{\mathbf{T}} P(\mu) \\ P(\mathbf{C} \bullet \mu) &= P \left(\{ l \mid g \mid p \mid E \mid -q \} \right) = \begin{pmatrix} E \\ p \end{pmatrix} = P(\mu) \end{aligned}$$

for the first Casimir number:

$$C_1 \left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu \right) = P(\mu)^T \tilde{\mathbf{T}}^\lambda \tilde{\mathbf{P}}^\nu I_{1,3} \tilde{\mathbf{P}}^\nu \tilde{\mathbf{T}}^\lambda P(\mu) = P(\mu)^T I_{1,3} P(\mu) = C_1(\mu)$$

for the mass:

$$m \left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu \right) = \text{sign} \left(E \left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu \right) \right) \sqrt{C_1 \left((\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu \right)} = \text{sign}((-1)^\lambda E) \sqrt{C_1(\mu)} = (-1)^\lambda m(\mu)$$

Therefore the elements variable by these actions are:

$$P(\mathbf{P} \bullet \mu) = \tilde{\mathbf{P}}P(\mu) \quad P(\mathbf{T} \bullet \mu) = \tilde{\mathbf{T}}P(\mu) \quad m(\mathbf{T} \bullet \mu) = -m(\mu) \quad (2)$$

and we have the following table:

		$\lambda = 0$	$\lambda = 1$
$\eta = 0$	$\nu = 0$	<ul style="list-style-type: none"> • neutral symmetry • $a = I_6$ • $\mu' = \{\ell, g, p, E, q\}$ 	<ul style="list-style-type: none"> • \mathbf{T} – symmetry • $a = \mathbf{T}$ • $\mu' = \{\ell, -g, p, -E, q\}$
	$\nu = 1$	<ul style="list-style-type: none"> • \mathbf{P} – symmetry • $a = \mathbf{P}$ • $\mu' = \{\ell, -g, -p, E, q\}$ 	<ul style="list-style-type: none"> • \mathbf{PT} – symmetry • $a = \mathbf{PT}$ • $\mu' = \{\ell, g, -p, -E, q\}$
$\eta = 1$	$\nu = 0$	<ul style="list-style-type: none"> • \mathbf{C} – symmetry • $a = \mathbf{C}$ • $\mu' = \{\ell, g, p, E, -q\}$ 	<ul style="list-style-type: none"> • \mathbf{CT} – symmetry • $a = \mathbf{CT}$ • $\mu' = \{\ell, -g, p, -E, -q\}$
	$\nu = 1$	<ul style="list-style-type: none"> • \mathbf{CP} – symmetry • $a = \mathbf{CP}$ • $\mu' = \{\ell, -g, -p, E, -q\}$ 	<ul style="list-style-type: none"> • \mathbf{CPT} – symmetry • $a = \mathbf{CPT}$ • $\mu' = \{\ell, g, -p, -E, -q\}$

Figure 2: This table lists the 8 values of $\mu' := (\mathbf{C}^\eta \mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \{ l \mid g \mid p \mid E \mid q \}$ for all $\lambda, \eta, \nu \in \{0, 1\}$.

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6 Discussion & Conclusion

In this paper, we have performed a double extension of the restricted Poincaré group limited to its orthochronous components, which are classically used in physics. This extension also includes the transition from the four-dimensional Minkowski spacetime to a new space of the same dimension, to which we have added a translation along an additional fifth dimension to form a new Lie group. The existence of this additional subgroup results in the invariance of a scalar, identified as the electric charge. A symmetry is introduced along this fifth dimension, and we have shown that this leads to the inversion of the electric charge. This provides a geometric representation of the symmetry between matter and antimatter.

In 1905, Albert Einstein revolutionized physics by introducing the theory of special relativity, integrating time, via the constant c , as a coordinate comparable to spatial dimensions in the geometry of Minkowski space. Ten years later, with his field equation, he explained phenomena such as the precession of Mercury's perihelion and the deflection of light, while laying the foundations of modern cosmology. The Big Bang theory, supported by Edwin Hubble's observations and Friedmann's work, revealed that the early universe was characterized by extreme density and temperature conditions.

Simultaneously, quantum mechanics emerged with the discovery of antimatter by Paul Dirac, an essential postulate in the cosmological model where, a fraction of a second after the Big Bang, matter and antimatter coexisted in equilibrium with gamma photons. However, the progressive disappearance of antimatter, implied by annihilations, remains a mystery, reinforced by the discovery of the cosmic microwave background in 1965.

The paradox of baryonic asymmetry found a theoretical response in 1967 with Sakharov, who postulated the existence of a twin universe symmetric to ours according to **CPT** symmetry, containing antimatter and evolving with an opposite arrow of time [16, 17, 18]. This model is based on a natural extension of fundamental symmetries, although the complete separation of the two universes leaves unresolved questions.

Within the framework of a geometric approach via dynamical groups, and building on the theory of symplectic groups by J.-M. Souriau [20], we consider here an extension of Sakharov's model, the Janus model. The latter proposes an alternative vision where the two universes, far from being disjoint, are connected by a covering structure. This configuration allows gravitational interaction between particles of opposite masses, thereby redefining **CPT** symmetry in a larger geometric framework, including the antichronous components of the Poincaré group.

In conclusion, this approach not only sheds light on the path physics has taken but also suggests new avenues, offering a possible resolution to contemporary crises in cosmology, as evidenced by the data from the Hubble and James Webb telescopes. A more detailed study will be presented, including a system of coupled field equations modeling the gravitational interactions induced by these new symmetries.

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