Homogenization of weakly damped Helmholtz resonators

Isabelle Gruais

Homogenization of weakly damped Helmholtz resonators

Isabelle Gruais
Univ Rennes, UR1, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France ;

ARTICLE HISTORY
Compiled August 23, 2024

ABSTRACT
We consider a material made of a periodic distribution of damped resonators embedded in a porous matrix. The homogenization method is used to grasp to which extend the resulting material behaves singularly with respect to the genuine system where seemingly vanishing resonators are weakly damped in contrast with the surrounding chamber and porous matrix. The resulting macroscopic material is gouverned by a complex coupled system involving two components from the porous and the resonating parts on one hand, and a third non local component arising from the wake inside the porous matrix.

Mathematical Subject Classification (2000). 35B27, 76M50, 76Rxx, 74F10, 74Q05.

Keywords. fractured porous media, two-scale homogenized system, Darcy law, Beavers-Joseph conditions, Helmholtz resonator.

1. Introduction

This paper deals with the modelisation of poro-acoustics media made of a rigid porous connected matrix with embedded inner resonators. Under the hypothesis that materials under consideration present a statistically invariant representative volume element and are conveniently represented by periodic media, they are eligible for mathematical homogenization methods and unconventional macroscopic acoustic properties leading to singularly behaving metamaterials.

More precisely, we consider highly heterogeneous materials where embedded Helmholtz resonators are characterized by small constricted ducts behaving differently from the surrounding chamber [9]. As the total volume of the ducts is chosen of unity order to balance their vanishing volume at local scale, the macroscopic asymptotic behaviour of the system may differ according to whether damping is weak inside the duct. Unlike [12] where damping is assumed to be of order one everywhere, the present study considers weak damping inside the thin ducts. To that aim, the control-zone method is revisited.

The paper is organized as follows. Section 2 is devoted to the description of the mathematical model and of the functional setting with an emphasis on the physical background [9]. In Section 3, the main results of [12] are recalled as long as they contribute to the present study. In particular, it requires results of two-scale homogenization [23] and of the so-called Control-zone method [6]. The first main convergence

*Corresponding author. Email: isabelle.gruais@univ-rennes.fr

result is established in Section 3. The solution of the heat equation in a cylindrical ring is explicited in Section 4 as part of the Control-zone method which needs profound changes to take into account weak damping in thin ducts. The resulting problem is derived in Section 5. It displays a complex coupling between two expected components arising from the porous and the resonating parts of the structure. More interestingly, a third non local component is identified with originating location in the wake of the resonator, inside the porous matrix.

2. The fluid problem

2.1. Geometric preliminaries

As is usual in homogenization theory, a small parameter ε is introduced to discriminate between microscopic and macroscopic scales. More precisely, denoting by L, resp. ℓ, the characteristic length of the long wavelength in the porous matrix, resp. of the distribution period size, we may define ε as the scale ration $\varepsilon = \frac{\ell}{L}$ which is small as soon as the size of the period is small compared to the characteristic length L.

We proceed to describe the geometry of the periodic structure. We refer to Figure 1 below for an illustrative example.

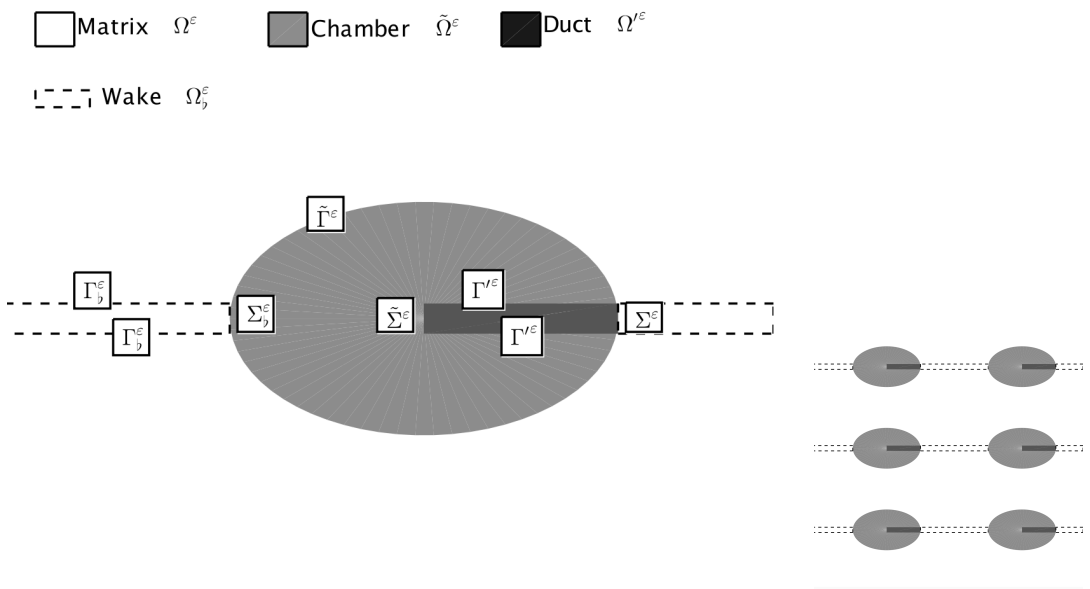

Figure 1. Helmholtz resonator in a cell (left) and the resulting periodic distribution (right)

Let $\varepsilon > 0$ denote the period and let $\varepsilon > 0$ denote the size of the small period designed to tend to zero. Set

$$I = \left]-\frac{1}{2}, \frac{1}{2}\right[, \quad I_k^\varepsilon = \varepsilon k + \varepsilon I, \quad k \in \mathbb{Z}, \quad \varepsilon > 0.$$

and let $Y = I^3$ denote the rescaled period of the microstructure. In what follows, we

introduce:

$$\mathbb{Z}_\varepsilon = \{k \in \mathbb{Z}^3,\ \overline{Y}_\varepsilon^k \cap \Omega \neq \emptyset\}.$$

Without loss of generality, we may assume that the problem under consideration is set in a domain Ω of the form:

$$\Omega = \left]-\frac{L}{2}, \frac{L}{2}\right[^3 \quad \text{for some } L > 0. \tag{1}$$

Then, setting: $\varepsilon_n = \frac{L}{(2n+1)}$, $\quad \forall n \geq 0$ we immediately find that $\mathbb{Z}_{\varepsilon_n} \neq \emptyset$, $\forall n \geq 0$.

Let $\tilde{Y} \subset\subset Y$ denote the rescaled embedded resonator. Denoting by $R > 0$ its typical length, we assume that $R \in \left]0, \frac{1}{2}\right[$ and set:

$$Y = \underbrace{I^2}_{=:Y'} \times I, \quad Y_\varepsilon^k = \varepsilon k + \underbrace{\varepsilon Y}_{=:Y^\varepsilon} =: I_{k_1}^\varepsilon \times I_{k_2}^\varepsilon \times I_{k_3}^\varepsilon \quad \forall k \in \mathbb{Z}^3.$$

The set of resonators $\tilde{\Omega}_\sharp^\varepsilon$ is the periodic repetition $\cup_{k\in\mathbb{Z}_\varepsilon} \tilde{Y}_\varepsilon^k$ of individual resonators $\tilde{Y}_\varepsilon^k$ defined as the homothetic images of the unscaled resonator defined by:

$$\tilde{Y} = \{(x', x_3) \in Y, \quad |x_3| < R,\ |x'| < \varrho(x_3)\}, \quad \tilde{I} :=]-R, R[, \quad \tilde{I}_\varepsilon^k = \varepsilon k + \varepsilon \tilde{I}$$

for some $2R$-periodic $\mathcal{C}^1$-function $\varrho : \mathbb{R} \to]0, +\infty[$, namely:

$$\tilde{Y}_\varepsilon^k = \varepsilon k + \{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R}, \quad |x_3 - \varepsilon k_3| < \varepsilon R,\ |x' - \varepsilon k'| < \varepsilon \varrho\left(\frac{x_3 - \varepsilon k_3}{\varepsilon}\right) =: \varepsilon \varrho_\varepsilon^k(x_3)\}$$

$$\tilde{\Omega}_\sharp^\varepsilon = \cup_{k\in\mathbb{Z}_\varepsilon} \underbrace{\{x \in \Omega,\ |x' - \varepsilon k'| \leq \varepsilon \varrho\left(\frac{x_3 - \varepsilon k_3}{\varepsilon}\right) =: \varepsilon \varrho_\varepsilon^k(x_3) \quad \text{and} \quad |x_3 - \varepsilon k_3| < \varepsilon R\}}_{=:\tilde{Y}_\varepsilon^k}$$

We assume that resonators are embedded in a porous matrix Ω^ε, which amounts to define Ω^ε as the complementary part of the resonators $\tilde{\Omega}_\sharp^\varepsilon$ in the domain Ω, namely:

$$\Omega^\varepsilon := \Omega \setminus \tilde{\Omega}_\sharp^\varepsilon.$$

The set of ducts Ω'^ε at the core of the resonators are defined by the periodic repetition of individual cylindrical ducts $T_\varepsilon'^k$ of axis $I_\varepsilon'^k$ and straight circular sections of very small radius $r_\varepsilon << \varepsilon$, namely:

$$\Omega'^\varepsilon = \cup_{k\in\mathbb{Z}^\varepsilon} T_\varepsilon'^k \quad \text{where:} \quad T_\varepsilon'^k = \varepsilon k + D_{r_\varepsilon} \times I_\varepsilon' = \varepsilon k + D_{r_\varepsilon} \times]0, \varepsilon R[, \quad \forall k \in \mathbb{Z}^3,$$

$$D_r = \{x' \in \mathbb{R}^2, \quad |x'| < r\}, \quad \mathrm{C}_r = \partial D_r, \quad D_r^{k,\varepsilon} := \varepsilon k' + \varepsilon D_r, \quad \mathrm{C}_r^{k,\varepsilon} := \varepsilon k' + \varepsilon \mathrm{C}_r, \quad r > 0.$$

and:

$$I'^{k}_{\varepsilon} = \varepsilon k + \underbrace{\varepsilon I'}_{:=I'_{\varepsilon}} = \varepsilon k +]0, \varepsilon R[\quad \text{where} \quad I' =]0, R[.$$

Our main assumption is that the radius r_ε of the section of small ducts is very small, compared with the period ε, that is:

$$\lim_{\varepsilon \to} \frac{r_\varepsilon}{\varepsilon} = 0.$$

and that:

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon = \gamma \in]0, \infty[. \tag{2}$$

We introduce the wake $\Omega^\varepsilon_\flat$ of the resonator as a parallel extension of the ducts Ω'^ε contained in the global wake T^ε, namely:

$$T^\varepsilon = \cup_{k \in \mathbb{Z}^\varepsilon} T^k_\varepsilon \quad \text{where} \quad T^k_\varepsilon = \varepsilon k + D_{r_\varepsilon} \times \Big]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big[, \quad \forall k \in \mathbb{Z}^3, \tag{3}$$

but reduced to the porous matrix, namely:

$$\Omega^\varepsilon_\flat := \Omega^\varepsilon \cap T^\varepsilon.$$

To discriminate between the resonators and the chambers inside the resonators, we defined:

$$\tilde{\Omega}^\varepsilon := \tilde{\Omega}^\varepsilon_\sharp \setminus T^\varepsilon = \tilde{\Omega}^\varepsilon_\sharp \setminus \Omega'^\varepsilon$$

so that resonators $\tilde{\Omega}^\varepsilon_\sharp$ may be seen as an extension of the chambers $\tilde{\Omega}^\varepsilon$ by the ducts Ω'^ε.

2.2. A priori estimates

Resonators $\tilde{\Omega}^\varepsilon_\sharp$ are made of chambers $\tilde{\Omega}^\varepsilon$ and constricted ducts Ω'^ε delimited by thin impervious rigid surfaces $\tilde{\Gamma}^\varepsilon$ and Γ'^ε. Each duct T'^k_ε, $k \in \mathbb{Z}_\varepsilon$, is of length εR and its constant section Σ^ε (aperture on the matrix) and $\tilde{\Sigma}^\varepsilon$ (aperture on the chamber) is much smaller than that of the period, i.e. $\lim_{\varepsilon \to 0} r_\varepsilon \varepsilon^{-1} = 0$, so that the duct volume $|\Omega'^\varepsilon|$ is negligeable, compared to $|\Omega|$, namely:

$$|\Omega'^\varepsilon| \underset{\varepsilon \to 0}{\sim} 2\pi R |\Omega| \left(\frac{r_\varepsilon}{\varepsilon}\right)^2 \underset{\varepsilon \to 0}{\to} 0,$$

and negligeable, compared to the resonators volume $|\tilde{\Omega}^\varepsilon_\sharp|$, namely:

$$|\tilde{\Omega}^\varepsilon_\sharp| = \pi \varepsilon^2 \sum_{k \in \mathbb{Z}\varepsilon} \int_{\varepsilon k_3 - \varepsilon R}^{\varepsilon k_3 + \varepsilon R} \varrho^k_\varepsilon(x_3) dx_3 \underset{\varepsilon \to 0}{\sim} \pi |\Omega| \int_{-R}^{R} \varrho(y_3) dy_3 > 0.$$

Remark 1. In the present study, we consider weakly damped resonators in the sense of the following model heat equation (to be compared with Section 4)

$$\rho_\varepsilon \frac{y^\varepsilon}{\partial t} - k_\varepsilon \Delta y^\varepsilon = 0 \quad \text{in} \quad \Omega^T$$

where coefficients ρ^ε and k^ε are defined by

$$(\rho_\varepsilon, k_\varepsilon) = \begin{cases} \left(\frac{a}{|T^\varepsilon|}, b\right) & \text{in} \quad T^\varepsilon, \\ (1,1) & \text{in} \quad \Omega \setminus T^\varepsilon, \end{cases}$$

and where Ω, resp. T^ε, is defined by (1), resp. (3). Direct computation yields that a system of fundamental solutions is given by:

$$y^\varepsilon(x,t) = e^{-\frac{k_\varepsilon}{\rho_\varepsilon}\omega^2 t} y_\omega(x), \quad \omega > 0.$$

where the damping term satisfies:

$$e^{-\frac{k_\varepsilon}{\rho_\varepsilon}\omega^2 t} = 1_{T^\varepsilon} e^{-\frac{b}{a}|T^\varepsilon|\omega^2 t} + 1_{\Omega\setminus T^\varepsilon} e^{-\omega^2 t} \underset{\varepsilon\to 0}{\sim} 1_{T^\varepsilon} + 1_{\Omega\setminus T^\varepsilon} e^{-\omega^2 t}$$

that is, damping asymptotically vanishes in the thin duct T^ε.

Let $f \in \mathcal{C}_c(\Omega)$, $g \in \mathcal{C}_c(\Omega)$ and let $T, a, b > 0$. Having in mind the homogenization methods developped in [6], [8], [13], we consider the following problem where poro-acoustic equations in the matrix are coupled with a viscous compressible fluid in resonators.

$$v^\varepsilon = K_\varepsilon(-\nabla q^\varepsilon + f) \quad \text{in} \quad \Omega^\varepsilon \times]0,T[, \tag{4}$$

$$\frac{1}{E_m}\frac{\partial q^\varepsilon}{\partial t} + \operatorname{div} v^\varepsilon = 0 \quad \text{in} \quad \Omega^\varepsilon \setminus T^\varepsilon \times]0,T[, \quad \frac{a}{|T^\varepsilon|}\frac{\partial q^\varepsilon}{\partial t} + b \operatorname{div} v^\varepsilon = 0 \quad \text{in} \quad \Omega_\flat^\varepsilon \times]0,T[,$$

$$\frac{\partial \tilde{u}^\varepsilon}{\partial t} - \Delta \tilde{u}^\varepsilon + \nabla \tilde{p}^\varepsilon = f, \quad \frac{1}{E_r}\frac{\partial \tilde{p}^\varepsilon}{\partial t} + \operatorname{div} \tilde{u}^\varepsilon = 0 \quad \text{in} \quad \tilde{\Omega}^\varepsilon \times]0,T[,$$

$$\tilde{u}^\varepsilon(0) = \tilde{u}_0^\varepsilon \quad \text{in} \quad \tilde{\Omega}^\varepsilon,$$

$$\frac{a}{|T^\varepsilon|}\frac{\partial u'^\varepsilon}{\partial t} - b\Delta u'^\varepsilon + \nabla p'^\varepsilon = \frac{1}{|T^\varepsilon|} g, \quad \frac{1}{E_r}\frac{\partial p'^\varepsilon}{\partial t} + \operatorname{div} u'^\varepsilon = 0 \quad \text{in} \quad \Omega'^\varepsilon \times]0,T[,$$

$$u'^\varepsilon(0) = u_0'^\varepsilon \quad \text{in} \quad \Omega'^\varepsilon,$$

$$\tilde{u}^\varepsilon = u'^\varepsilon, \quad \tilde{u}^\varepsilon \cdot \tilde{n}^\varepsilon = u'^\varepsilon \cdot n'^\varepsilon = 0 \quad \text{on} \quad \Gamma'^\varepsilon \times]0,T[$$

$$bv^\varepsilon \cdot n_\flat^\varepsilon + v^\varepsilon \cdot n^\varepsilon = 0 \quad \text{on} \quad \Gamma_\flat^\varepsilon \times]0,T[$$

$$\tilde{u}^\varepsilon \cdot \tilde{n}^\varepsilon + v^\varepsilon \cdot n^\varepsilon = 0 \quad \text{on} \quad \tilde{\Gamma}^\varepsilon \times]0,T[, \quad \tilde{u}^\varepsilon \cdot \tilde{n}^\varepsilon + bv_\flat^\varepsilon \cdot n_\flat^\varepsilon = 0 \quad \text{on} \quad \Sigma_\flat^\varepsilon \times]0,T[$$

$$\frac{\partial \tilde{u}_i^\varepsilon}{\partial \tilde{n}^\varepsilon} - \tilde{p}^\varepsilon \tilde{n}_i^\varepsilon - q^\varepsilon n_i^\varepsilon = \alpha_\varepsilon \mathrm{tr}(K_\varepsilon)^{-\frac{1}{2}} (\tilde{u}_i^\varepsilon - (\tilde{u}^\varepsilon \cdot \tilde{n}^\varepsilon)\tilde{n}_i^\varepsilon), \text{ on } \tilde{\Gamma}^\varepsilon \times]0,T[\tag{5}$$

$$b\,\frac{\partial u_i'^\varepsilon}{\partial n'^\varepsilon} - p'^\varepsilon n_i'^\varepsilon - q^\varepsilon n_i^\varepsilon = 0, \quad u'^\varepsilon \cdot n'^\varepsilon + v^\varepsilon \cdot n^\varepsilon = 0 \quad \text{on} \quad \Sigma^\varepsilon \times]0,T[$$

$$-b\frac{\partial u_i'^\varepsilon}{\partial n'^\varepsilon} + p'^\varepsilon n_i'^\varepsilon - \frac{\partial \tilde{u}_i^\varepsilon}{\partial \tilde{n}^\varepsilon} + \tilde{p}^\varepsilon \tilde{n}_i^\varepsilon = 0, \quad u_i'^\varepsilon = \tilde{u}_i^\varepsilon \quad \text{on} \quad \tilde{\Sigma}^\varepsilon \times]0,T[, \quad i \in \{1,2,3\}.$$

$$v^\varepsilon \cdot n = 0 \quad \text{on} \quad \partial\Omega \times]0,T[, \tag{6}$$

where n^ε, resp. $n_\flat^\varepsilon$, $\tilde{n}^\varepsilon$, n'^ε, n, denotes the outward unit normal on $\partial\Omega^\varepsilon$, resp. $\partial\Omega_\flat^\varepsilon$, $\partial\tilde{\Omega}^\varepsilon$, ∂T^ε, $\partial\Omega$, $K_\varepsilon > 0$ is the constant (positive) permeability tensor, $\alpha_\varepsilon \in L^\infty(\Omega)$ is the non-dimensional Beavers-Joseph number, $g \in L^2(\Omega)$ is the exterior force.

The variational formulation of the boundary value problem (4)–(6) will be based on the following spaces. Set:

$$L_0^2(\Omega) = \{v \in L^2(\Omega), \int_\Omega v dx = 0\}$$

$$H(\mathrm{div},\Omega) = \{v \in L^2(\Omega), \quad \mathrm{div}(v) \in L^2(\Omega)\}, \quad H_0(\mathrm{div},\Omega) = \{v \in H(\mathrm{div},\Omega), \quad v{\cdot}n|_{\partial\Omega} = 0\},$$

$$\tilde{V}_\varepsilon := \{v \in H^1(\tilde{\Omega}^\varepsilon), \quad v \cdot \tilde{n}^\varepsilon|_{\Gamma'^\varepsilon} = 0\}, \quad V_\varepsilon' := \{v \in H^1(\Omega'^\varepsilon), \quad v \cdot n'^\varepsilon|_{\Gamma'^\varepsilon} = 0\}$$

$$U_\varepsilon := \{u \in H_0(\mathrm{div},\Omega), \quad u|_{\Omega\setminus\Omega^\varepsilon} \in H^1(\Omega \setminus \Omega^\varepsilon), \quad u|_{\tilde{\Omega}^\varepsilon} \in \tilde{V}_\varepsilon, \quad u|_{\Omega'^\varepsilon} \in V_\varepsilon'\}.$$

$$H_\varepsilon = \{v \in H_0(\mathrm{div},\Omega),\ v|_{\Omega\setminus\Omega^\varepsilon} \in H^1(\Omega \setminus \Omega^\varepsilon)\}. \tag{7}$$

The same arguments as in [16] yield that when equipped with the scalar product

$$(u,v)_{H_\varepsilon} := \int_{\Omega^\varepsilon\setminus\Omega_\flat^\varepsilon} u{\cdot}v dx + b\int_{\Omega_\flat^\varepsilon} u{\cdot}v dx + \int_{\Omega^\varepsilon} \mathrm{div}(u)\mathrm{div}(v) dx + \int_{\Omega\setminus\Omega^\varepsilon} \nabla u{\cdot}\nabla v dx + b\int_{\Omega'^\varepsilon} \nabla u \nabla v dx +$$

$$+ \int_{\tilde{\Gamma}^\varepsilon} \alpha_\varepsilon (u - (u \cdot \tilde{n}^\varepsilon)\tilde{n}^\varepsilon) v d\sigma \tag{8}$$

H_ε is a Hilbert space for the norm: $u \mapsto \|u\|_{H_\varepsilon} = \sqrt{(u,u)_{H_\varepsilon}}$.

In the functional framework (7)–(8), existence and unicity of the problem (4)–(6) will result from Cauchy-Lipschitz's theorem once we have introduced the following rescaled bilinear forms. Set

$$A^\varepsilon = \mathrm{tr}(K_\varepsilon) K_\varepsilon^{-1}$$

so that there exists $a_0 > 0$ such that:

$$\beta_\varepsilon = \sqrt{\mathrm{tr}(K_\varepsilon)} > 0 \quad \text{and} \quad A^\varepsilon \xi \cdot \xi \geq a_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^N.$$

Consider the rescaled velocity:

$$\mathrm{tr}(K_\varepsilon) u^\varepsilon = \begin{cases} u'^\varepsilon & \text{in} \quad \Omega'^\varepsilon, \\ \tilde{u}^\varepsilon & \text{in} \quad \tilde{\Omega}^\varepsilon, \\ v^\varepsilon & \text{in} \quad \Omega^\varepsilon. \end{cases} \qquad p^\varepsilon = \begin{cases} p'^\varepsilon & \text{in} \quad \Omega'^\varepsilon, \\ \tilde{p}^\varepsilon & \text{in} \quad \tilde{\Omega}^\varepsilon, \\ q^\varepsilon & \text{in} \quad \Omega^\varepsilon, \end{cases}$$

and define: $\forall u, v \in H_\varepsilon$, $\forall q \in L^2_0(\Omega)$,

$$a_\varepsilon(u,v) = {\beta_\varepsilon}^2 \left(\int_{\tilde{\Omega}^\varepsilon} \nabla u \nabla v dx + b \int_{\Omega'^\varepsilon} \nabla u \nabla v dx \right) + \varepsilon^{-1} \beta_\varepsilon \int_{\tilde{\Gamma}^\varepsilon} \alpha_\varepsilon (u - (u \cdot \tilde{n}^\varepsilon)\tilde{n}^\varepsilon) v d\sigma,$$

$$+ \int_{\Omega^\varepsilon \setminus \Omega_\flat^\varepsilon} A^\varepsilon u \cdot v dx + b \int_{\Omega_\flat^\varepsilon} A^\varepsilon u \cdot v dx$$

$$b_\varepsilon(q,v) = - \int_{\Omega \setminus \Omega_\flat^\varepsilon} q \mathrm{div} v dx - b \int_{\Omega_\flat^\varepsilon} q \mathrm{div} v dx$$

$$(u,v)_\varepsilon = Ra \fint_{\Omega'^\varepsilon} u^\varepsilon v dx + \int_{\Omega \setminus \Omega'^\varepsilon} u^\varepsilon v dx,$$

$$(p,q)_\varepsilon^\sharp = \left(\frac{1}{2} - R \right) a \fint_{\Omega_\flat^\varepsilon} p^\varepsilon q dx + \frac{1}{E_m} \int_{\Omega^\varepsilon \setminus T^\varepsilon} p^\varepsilon q dx + \frac{1}{E_r} \int_{\Omega \setminus \Omega^\varepsilon} p^\varepsilon q dx$$

With these notations in mind, $(u^\varepsilon, p^\varepsilon)$ is solution of: $(u^\varepsilon, p^\varepsilon) \in \mathcal{C}^1(0,T; U_\varepsilon) \times \mathcal{C}^1(0,T; L^2_0(\Omega))$ and

$$\frac{d}{dt}(u^\varepsilon, v)_\varepsilon + a_\varepsilon(u^\varepsilon, v) + b_\varepsilon(p^\varepsilon, v) = \int_{\Omega \setminus \Omega'^\varepsilon} f v dx + \fint_{\Omega'^\varepsilon} g v dx, \quad \forall v \in U_\varepsilon \tag{9}$$

$$\frac{d}{dt}(p^\varepsilon, q)_\varepsilon^\sharp - b_\varepsilon(q, u^\varepsilon) = 0, \quad \forall q \in L^2_0(\Omega) \tag{10}$$

Proposition 2.1. *Problem (9)–(10) admits a unique solution* $(u^\varepsilon, p^\varepsilon) \in \mathcal{C}^1(0,T;U_\varepsilon) \times \mathcal{C}^1(0,T;L^2_0(\Omega))$.

Corollary 2.2. *There exists some constant* $C > 0$ *independent of* $\varepsilon > 0$ *such that*

$$\|v\|_{H^1(\Omega\setminus\Omega^\varepsilon)} \leq C\|v\|_{H_\varepsilon}, \quad \forall v \in H_\varepsilon.$$

Proof. This is a direct consequence of Korn's inequality. See [16]. □

In the following, we assume that $A_\varepsilon = A$ is a positive constant operator and then, we may set $A = I$ without loss of generality. Moreover, we assume that $\beta_\varepsilon = \beta > 0$ is constant and that α_ε behaves in a such a way that:

$$\varepsilon^{-1}\beta\alpha_\varepsilon = \alpha\left(\frac{x}{\varepsilon}\right) \quad \text{for some positive } \alpha \in \mathcal{C}^1_{\text{per}}(Y).$$

Now, we are in a position to establish the main preparatory estimate towards compacity results of Theorems 3.1 and Proposition 3.8 resp.

Proposition 2.3. *Assume:*

$$|u_0^\varepsilon|^2_\varepsilon + |p_0^\varepsilon|^{\sharp 2}_\varepsilon \leq C, \quad \forall \varepsilon > 0. \tag{11}$$

Then:

$$\sup_{0\leq t\leq T}(|u^\varepsilon(t)|^2_\varepsilon + |p^\varepsilon(t)|^{\sharp 2}_\varepsilon) + \int_0^T |u^\varepsilon(t)|^2_{H_\varepsilon} dt \leq C \tag{12}$$

Proof. Setting $v = u^\varepsilon$ and $q = p^\varepsilon$ in (9)–(10) yields: $\forall t \in [0,T]$,

$$|u^\varepsilon(t)|^2_\varepsilon + |p^\varepsilon(t)|^{\sharp 2}_\varepsilon \leq |u^\varepsilon(t)|^2_\varepsilon + |p^\varepsilon(t)|^{\sharp 2}_\varepsilon + \int_0^t a_\varepsilon(u^\varepsilon(s), u^\varepsilon(s))ds =$$

$$= |u_0^\varepsilon|^2_\varepsilon + |p_0^\varepsilon|^{\sharp 2}_\varepsilon + \int_0^t (g^\varepsilon, u^\varepsilon(s))_\varepsilon ds \leq C + (\int_\Omega |g^\varepsilon|_\Omega + |g^\varepsilon|_\infty) \sup_{0\leq t\leq T} |u^\varepsilon(t)|_\varepsilon \leq$$

$$\leq C(1 + \sup_{0\leq t\leq T} |u^\varepsilon(t)|_\varepsilon)$$

from which we deduce that

$$\sup_{0\leq t\leq T}(|u^\varepsilon(t)|^2_\varepsilon + |p^\varepsilon(t)|^{\sharp 2}_\varepsilon) \leq C.$$

Moreover, the ellipticity of a_ε leads to:

$$C\int_0^t \|u^\varepsilon(s)\|^2_{H_\varepsilon} ds \leq C\int_0^t a_\varepsilon(u^\varepsilon(s), u^\varepsilon(s))ds \leq |u_0^\varepsilon|^2_\varepsilon + |p_0^\varepsilon|^{\sharp 2}_\varepsilon + \int_0^t (g^\varepsilon, u^\varepsilon(s))_\varepsilon ds \leq C$$

which ends the proof. □

3. Preliminary convergence results

3.1. Two-scale convergence limits

We pass to the limit in the variational formulation (9)–(10) is two steps, first with two-scale convergence arguments in Theorem 3.1, second with tools of the Control-zone method [8] in Proposition 3.8.

To that aim, we introduce classical notations of two-scale convergence theory [1], [16], [23]. In the following, $\Omega^T = (0,T)\times\Omega$ will denote the time-dependent domain. Let $\tilde{Y} \subset Y$ denote the rescaled resonator inside the period cell Y, namely:

$$\tilde{Y} = \{y = (y', y_3) \in Y,\ |y_3| < R,\ |y'| < \varrho(y_3)\}, \quad Y_m := Y \setminus \tilde{Y}$$

To avoid technicalities, we have assumed that the duct is contained into the chamber. Then, Y_m coincides with the porous matrix inside Y.

Following [16], [17], [18], [19], we introduce the periodic distributions of resonators $\tilde{\mathbb{R}}^3$ and the periodic porous matrix $\mathbb{R}^3_m$ defined by:

$$\tilde{\mathbb{R}}^3 = \cup_{k\in\mathbb{Z}^3}(k+\tilde{Y}), \quad \mathbb{R}^3_m = \cup_{k\in\mathbb{Z}^3}(k+Y_m)$$

and associated characteristic functions of $\tilde{\Omega}^\varepsilon_\sharp$ and Ω^ε resp. in Ω:

$$1_{\tilde{\Omega}^\varepsilon_\sharp}(x) := \tilde{\chi}^\varepsilon(x) = \chi_{\tilde{Y}}\left(\frac{x}{\varepsilon}\right), \quad 1_{\Omega^\varepsilon}(x) := \chi^\varepsilon_m(x) = \chi_{Y_m}\left(\frac{x}{\varepsilon}\right), \quad \forall x \in \mathbb{R}^3.$$

In the following theorem we introduce H^1 periodic functions of period $\tilde{Y}$, namely:

$$\tilde{H}^1_{\rm per}(\tilde{Y}) = \{\varphi \in H^1_{\rm loc}(\tilde{\mathbb{R}}^3),\ Y\text{-periodic},\ \int_{\tilde{Y}} \varphi dy = 0\}$$

Theorem 3.1 below yields the main two-scale convergence result about our structure.

Theorem 3.1. *There exist* $u \in L^2(\Omega^T; H_{\rm loc}({\rm div}_y, \mathbb{R}^3))$ *and* $p \in L^\infty(0,T; L^2_0(\Omega))$ *such that*

$$⨍_Y u dy \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0({\rm div}, \Omega))$$

$$⨍_Y p dy \in L^\infty(0,T; L^2_0(\Omega))$$

and:

$$u^\varepsilon \rightharpoonup u \quad \textit{and} \quad p^\varepsilon \rightharpoonup p \quad \textit{in} \quad L^2(\Omega^T \times Y).$$

$$u^\varepsilon \rightharpoonup ⨍_Y u dy \quad \textit{in} \quad L^2(\Omega^T) \quad \textit{and} \quad u^\varepsilon \overset{\star}{\rightharpoonup} ⨍_Y u dy \quad \textit{in} \quad L^\infty(0,T; L^2(\Omega)).$$

$$p^\varepsilon \rightharpoonup \fint_Y p dy \quad in \quad L^2(\Omega^T) \quad and \quad p^\varepsilon \overset{\star}{\rightharpoonup} \fint_Y p dy \quad in \quad L^\infty(0,T;L^2\Omega))$$

Set:

$$u^r = \chi_{\tilde{Y}} u, \quad u^m = \chi_{Y_m} u, \quad \tilde{u}^r = \fint_{\tilde{Y}} u dy, \quad \tilde{u}^m = \fint_{Y_m} u dy,$$

resp.:

$$p^r = \chi_{\tilde{Y}} p, \quad p^m = \chi_{Y_m} p, \quad \tilde{p}^r = \fint_{\tilde{Y}} p dy, \quad \tilde{p}^m = \fint_{Y_m} p dy.$$

Then:

$$\tilde{\chi}^\varepsilon u^\varepsilon \rightharpoonup\!\!\!\rightharpoonup u^r \quad in \quad L^2(\Omega^T \times Y) \tag{13}$$

$$\chi_m^\varepsilon p^\varepsilon \rightharpoonup\!\!\!\rightharpoonup p^m \quad in \quad L^2(\Omega^T \times Y)) \tag{14}$$

$$\chi_m^\varepsilon u^\varepsilon \rightharpoonup\!\!\!\rightharpoonup u^m \quad and \quad \tilde{\chi}^\varepsilon p^\varepsilon \rightharpoonup\!\!\!\rightharpoonup p^r \quad in \quad L^2(\Omega^T \times Y))$$

where:

$$p^m = \tilde{p}^m \in L^2(0,T;H^1(\Omega)) \cap L^2(0,T;L_0^2(\Omega))$$

and

$$u^r = \tilde{u}^r \in L^2(0,T;H^1(\Omega)) \cap L^2(0,T;H_0(\mathrm{div},\Omega)) \quad \textit{are independent of } y$$

and there exist $w \in L^2(\Omega^T, \tilde{H}^1_{\mathrm{per}}(\tilde{Y}))$, $p^\sharp \in L^2(\Omega^T, \tilde{L}^2_{\mathrm{per}}(Y_m))$ *such that*

$$\tilde{\chi}^\varepsilon \nabla u_i^\varepsilon \rightharpoonup\!\!\!\rightharpoonup \chi_{\tilde{Y}}(\nabla u_i^r + \nabla_y w_i) \quad in \quad L^2(\Omega^T \times Y) \tag{15}$$

$$\chi_m^\varepsilon \nabla p^\varepsilon \rightharpoonup\!\!\!\rightharpoonup \chi_m(\nabla p^m + \nabla_y p^\sharp) \quad in \quad L^2(\Omega^T \times Y) \tag{16}$$

$$\mathrm{div}_y u = 0 \quad in \quad \Omega^T \times Y. \tag{17}$$

$$\tilde{u}^m \cdot n = 0 \quad on \quad \partial\Omega. \tag{18}$$

Proof. We refer to [12]. See also Lemma 3.1, Theorem 3.2 and Theorem 3.3 of [16] for the proof of (13), (15) and (18).

□

3.2. The Control-zone method (see [8])

The analysis of the ducts deserves a special treatment to take into account the main feature of its geometry characterized by its global vanishing volume:

$$|\Omega'^{\varepsilon}| \underset{\varepsilon\to 0}{\sim} 2\pi|\Omega|R\left(\frac{r_\varepsilon}{\varepsilon}\right)^2 \underset{\varepsilon\to 0}{\to} 0$$

contrasting with its non negligeable influence on the macroscopic behaviour of the structure due to the rescaled coefficient $|\Omega'^{\varepsilon}|^{-1}$ in front of its contribution in the energy. Similar problems have been studied in [6], [7], [8], [13], [14], [15], by means of the so-called control-zone method introduced in [6] in the case of a distribution of thin spherical particules, and inspired by the pioneering article [5] dealing with a distribution of thin parallel rods.

We begin with a priori estimates involving specific quantities (19), (20) and (21).

Consider : $\forall\varphi\in\mathcal{C}^1(\Omega)$, $\forall x\in\Omega$,

$$G_r^\varepsilon(\varphi)(x_3)=\sum_{k\in\mathbb{Z}_\varepsilon}\underbrace{\fint_{\mathrm{C_r}}\varphi(r,\theta,x_3)d\theta}_{=:G_r^{k,\varepsilon}(\varphi)(x_3)}1_{Y_\varepsilon^k}(x) \tag{19}$$

and set: $\forall(p,u)\in\mathcal{C}^1(\Omega)\times\mathcal{C}^1(\Omega)^3$, $\forall x\in\Omega$,

$$\vec{u}^\varepsilon=\begin{pmatrix} p(x)1_{\Omega^\varepsilon}(x)\\ \\ u(x)1_{\tilde{\Omega}^\varepsilon_\sharp}(x)\end{pmatrix},\quad \nabla\vec{u}^\varepsilon=\begin{pmatrix} \nabla p(x)1_{\Omega^\varepsilon}(x)\\ \\ \nabla u(x)1_{\tilde{\Omega}^\varepsilon_\sharp}(x)\end{pmatrix},\quad \vec{u}=\begin{pmatrix} p\\ \\ u\end{pmatrix},\quad \nabla\vec{u}=\begin{pmatrix} \nabla p\\ \\ \nabla u(x)\end{pmatrix}$$

$$G_r^{\varepsilon,k}(\vec{u})(x_3)=\begin{pmatrix} G_r^{k,\varepsilon}(p)1_{I_\varepsilon^k\setminus\tilde{I}_\varepsilon^k}(x_3)+G^{k,\varepsilon}_{\varepsilon\varrho_\varepsilon^k(x_3)}(p)1_{\tilde{I}_\varepsilon^k}(x_3)\\ \\ G_r^{k,\varepsilon}(u)1_{\tilde{I}_\varepsilon^k}(x_3)\end{pmatrix},\quad \forall k\in\mathbb{Z}_\varepsilon \tag{20}$$

$$G_r^\varepsilon(\vec{u})(x)=\sum_{k\in\mathbb{Z}_\varepsilon}G_r^{\varepsilon,k}(\vec{u})(x_3)1_{Y_\varepsilon^k}(x) \tag{21}$$

Below, we set:

$$\mathcal{C}_r^\varepsilon=\cup_{k\in\mathbb{Z}_\varepsilon}\underbrace{\{x\in\tilde{\Omega}^\varepsilon,\ |x'-\varepsilon k'|<r\}}_{=:\mathcal{C}_r^{k,\varepsilon}},\quad \forall r>0. \tag{22}$$

Proposition 3.2. *Let $(p,u)\in H^1(\Omega)\times H^1(\Omega)^3$. Then:*

$$|\vec{u}^\varepsilon-G^\varepsilon_{R_\varepsilon}(\vec{u})|^2_\Omega\le C\varepsilon^2\ln\frac{1}{R_\varepsilon}\left(|\nabla u|^2_{\tilde{\Omega}^\varepsilon_\sharp}+|\nabla p|^2_{\Omega^\varepsilon}\right), \tag{23}$$

where we set (see (22):

$$\mathcal{C}_\varepsilon=\mathcal{C}^\varepsilon_{R_\varepsilon}\setminus\mathcal{C}^\varepsilon_{r_\varepsilon} \tag{24}$$

Moreover:

$$\fint_{T^\varepsilon} |G^\varepsilon_{R_\varepsilon}(\vec{u})|^2 dx = \fint_{\Omega} |G^\varepsilon_{R_\varepsilon}(\vec{u})|^2 dx, \tag{25}$$

Proof. See [12]. See also [5], [6] and [14]. □

Proposition 3.3. *Let $(p,u) \in H^1(\Omega) \times H^1(\Omega)^3$. There holds:*

$$\fint_{T^\varepsilon} |\vec{u^\varepsilon}|^2 dx \leq C \max\left(\varepsilon^2 \ln \frac{1}{r_\varepsilon}, 1\right) \left(\int_{\tilde{\Omega}^\varepsilon_\sharp} |\nabla u|^2 dx + \int_{\Omega^\varepsilon} |\nabla p|^2 dx + \int_{\Omega} |\vec{u^\varepsilon}|^2 dx \right) \tag{26}$$

Proof. See [12]. See also [6] and [14]. □

Proposition 3.4. *The solution $(u^\varepsilon, p^\varepsilon) \in \mathcal{C}^1(0,T;U_\varepsilon) \times \mathcal{C}^1(0,T;L^2_0(\Omega))$ of Problem (9)–(10) satisfies:*

$$\fint_{\Omega^\varepsilon_\flat} |p^\varepsilon|^2 dx + \int_{\Omega'^\varepsilon} |u^\varepsilon|^2 dx \leq C \max\left(\varepsilon^2 \ln \frac{1}{r_\varepsilon}, 1\right) \left(|p^\varepsilon|^2_{H^1(\Omega^\varepsilon)} + |u^\varepsilon|^2_{H^1(\tilde{\Omega}^\varepsilon_\sharp)} \right). \tag{27}$$

Proof. See [12]. □

Proposition 3.5. *Let $\varphi \in \mathcal{C}_c(\Omega)$ and set:*

$$\varphi^\varepsilon := \sum_{k \in \mathbb{Z}_\varepsilon} \left(\fint_{I^\varepsilon_{k_1} \times I^\varepsilon_{k_2}} \varphi dx \right) 1_{T^k_\varepsilon}.$$

Then:

$$\lim_{\varepsilon \to 0} \fint_{T^\varepsilon} |\varphi - \varphi^\varepsilon|^2 dx = 0.$$

Proof. See Lemma 2.5 of [6]. □

Following estimate (26), we introduce the notation:

$$\frac{1}{\gamma^\varepsilon} := \varepsilon^2 \ln \frac{1}{r_\varepsilon} \tag{28}$$

and we asssume that:

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon = \gamma \in]0, \infty[. \tag{29}$$

Proposition 3.6. *For every $(\varphi^\sharp, \varphi) \in \mathcal{C}_c(\Omega) \times \mathcal{C}_c(\Omega)^3$, define:*

$$\vec{\varphi} = \begin{pmatrix} \varphi^\sharp \\ \varphi \end{pmatrix}, \quad m^\varepsilon(\vec{\varphi})(x) = \sum_{k \in \mathbb{Z}_\varepsilon} m^k_\varepsilon(\vec{\varphi})(x) 1_{Y^\varepsilon_k}(x)$$

where: $\forall k \in \mathbb{Z}_\varepsilon$, $\forall x \in \Omega$,

$$m_\varepsilon^k(\vec{\varphi})(x) = \begin{pmatrix} \fint_{D_{r_\varepsilon}^{k,\varepsilon}} \varphi^\sharp(y', x_3) dy' 1_{Y_k^\varepsilon \setminus \tilde{Y}_k^\varepsilon}(x) \\ \\ \fint_{D_{r_\varepsilon}^{k,\varepsilon}} \varphi(y', x_3) dy' 1_{\tilde{Y}_k^\varepsilon}(x) \end{pmatrix}, \quad M^\varepsilon(\vec{\varphi})(x) = \sum_{k \in \mathbb{Z}_\varepsilon} M_\varepsilon^k(x) 1_{T_k^\varepsilon}(x)$$

where: $\forall k \in \mathbb{Z}_\varepsilon$, $\forall x \in \Omega$,

$$M_\varepsilon^k(\vec{\varphi})(x) =$$

$$= \begin{pmatrix} \left(\fint_{I_{k_1}^\varepsilon \times I_{k_2}^\varepsilon} \varphi^\sharp(y', x_3) dy' 1_{I_k^\varepsilon \setminus \tilde{I}_k^\varepsilon}(x_3) + \varepsilon^{-2} \int_{I_{k_1}^\varepsilon \times I_{k_2}^\varepsilon \setminus D_{\varepsilon \varrho_\varepsilon^k(x_3)}^{k,\varepsilon}} \varphi^\sharp(y', x_3) dy' 1_{\tilde{I}_k^\varepsilon}(x_3) \right) 1_{Y_k^\varepsilon \setminus \tilde{Y}_k^\varepsilon}(x) \\ \\ \varepsilon^{-2} \int_{D_{\varepsilon \varrho_\varepsilon^k(x_3)}^{k,\varepsilon}} \varphi(y', x_3) dy' 1_{\tilde{Y}_k^\varepsilon}(x) \end{pmatrix}$$

Then: $\forall (\varphi^\sharp, \varphi) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)^3$,

$$\fint_\Omega |m^\varepsilon(\vec{\varphi})|^2 dx \leq \fint_{T^\varepsilon} |\vec{\varphi}|^2 dx, \tag{30}$$

$$\fint_{T^\varepsilon} |M^\varepsilon(\vec{\varphi})|^2 dx \leq \fint_\Omega |\vec{\varphi}|^2 dx. \tag{31}$$

Then, operators m_ε *and* M_ε *are linked by duality identities:*

$$\fint_\Omega m_\varepsilon(\vec{\varphi}) \vec{\psi} dx = \fint_{T^\varepsilon} \vec{\varphi} M_\varepsilon(\vec{\psi}) dx, \quad \forall (\varphi^\sharp, \varphi), (\psi^\sharp, \psi) \in \mathcal{C}_c(\Omega) \times \mathcal{C}_c(\Omega)^3. \tag{32}$$

Proof. See [12]. □

Remark 2. The Mean Value Theorem yields: $\forall (\varphi^\sharp, \varphi) \in \mathcal{C}_c(\Omega) \times \mathcal{C}_c(\Omega)^3$,

$$\lim_{\varepsilon \to 0} |m_\varepsilon(\vec{\varphi}) - \vec{\varphi}|_\infty = 0, \tag{33}$$

$$\lim_{\varepsilon \to 0} |M_\varepsilon(\vec{\varphi}) - \vec{\varphi}|_\infty = 0. \tag{34}$$

Proposition 3.7. *For every* $(\varphi^\sharp, \varphi) \in L^2(0, T; \mathcal{C}_c(\Omega)) \times L^2(0, T; \mathcal{C}_c(\Omega)^3)$,

$$\lim_{\varepsilon \to 0} \int_0^T \fint_{T^\varepsilon} |\vec{\varphi} - M_\varepsilon(\vec{\varphi})|^2 dx = 0.$$

Proof. See Lemma 3.6 of [13]. □

In Propositions 3.8 and 3.9 below, we show that two-scale limits of Theorem 3.1 are completed by macroscopic limits arising from the ducts Ω'^{ε}. To that aim, we introduce modified averages (36) (37) (38) of the control-zone method specific to the Stokes problem.

Proposition 3.8. *Assume that (29) holds true and that*

$$\lim_{\varepsilon\to 0}\frac{\ln R_\varepsilon}{\ln r_\varepsilon}=0. \tag{35}$$

Set:

$$\vec{u}^{\varepsilon}=\begin{pmatrix}1_{\Omega^\varepsilon}p^\varepsilon\\ 1_{\tilde{\Omega}^\varepsilon_\sharp}u^\varepsilon\end{pmatrix},\quad \vec{u}^{\varepsilon}_\sharp=\begin{pmatrix}1_{\Omega^\varepsilon}p^\varepsilon\\ 1_{\tilde{\Omega}^\varepsilon}u^\varepsilon\\ 1_{\Omega'^\varepsilon}u^\varepsilon\end{pmatrix},\quad \forall\varepsilon>0.$$

$$G^{\varepsilon,k}_{r_\varepsilon}(\vec{u}^{\varepsilon}_\sharp)(x_3)=\begin{pmatrix}G^{k,\varepsilon}_{r_\varepsilon}(p)1_{I^\varepsilon_{k_3}\setminus\tilde{I}^\varepsilon_{k_3}}(x_3)+G^{k,\varepsilon}_{\varepsilon\varrho^k_\varepsilon(x_3)}(p)1_{\tilde{I}^\varepsilon_{k_3}}(x_3)\\ G^{k,\varepsilon}_{r_\varepsilon}(u)1_{\tilde{I}^\varepsilon_{k_3}\setminus I'^\varepsilon_{k_3}}(x_3)\\ G^{k,\varepsilon}_{r_\varepsilon}(u)1_{I'^\varepsilon_{k_3}}(x_3)\end{pmatrix} \tag{36}$$

$$m^k_\varepsilon(\vec{u}^{\varepsilon}_\sharp)(x)=\begin{pmatrix}\fint_{D^{k,\varepsilon}_{r_\varepsilon}}p(y',x_3)dy'1_{Y^\varepsilon_k\setminus\tilde{Y}^\varepsilon_k(x)}\\ \fint_{D^{k,\varepsilon}_{r_\varepsilon}}u(y',x_3)dy'1_{\tilde{Y}^\varepsilon_k\setminus Y'^\varepsilon_k}(x)\\ \fint_{D^{k,\varepsilon}_{r_\varepsilon}}u(y',x_3)dy'1_{Y'^\varepsilon_k}(x)\end{pmatrix}, \tag{37}$$

$$G^{\varepsilon}_{r_\varepsilon}(\vec{u}^{\varepsilon}_\sharp)(x)=\sum_{k\in\mathbb{Z}_\varepsilon}G^{\varepsilon,k}_r(\vec{u}^{\varepsilon}_\sharp)(x_3)1_{Y^\varepsilon_k}(x),\quad m^\varepsilon(\vec{u}^{\varepsilon}_\sharp)(x)=\sum_{k\in\mathbb{Z}_\varepsilon}m^k_\varepsilon(\vec{u}^{\varepsilon}_\sharp)(x)1_{Y^k_\varepsilon}(x) \tag{38}$$

Then, there exist $(q,u',v)\in\in L^2(\Omega^T)\times L^2(\Omega^T)^3\times L^2(\Omega^T)^3$ *such that:*

$$\frac{1_{\Omega'^\varepsilon}}{|\Omega'^\varepsilon|}u^\varepsilon dx\overset{\star}{\rightharpoonup}v\frac{dx}{|\Omega|}\quad in\quad L^\infty(0,T;\mathcal{M}_b(\overline{\Omega})) \tag{39}$$

$$\frac{1_{\tilde{\Omega}^\varepsilon\cap T^\varepsilon}}{|\tilde{\Omega}^\varepsilon\cap T^\varepsilon|}u^\varepsilon dx\overset{\star}{\rightharpoonup}u'\frac{dx}{|\Omega|}\quad in\quad L^\infty(0,T;\mathcal{M}_b(\overline{\Omega})) \tag{40}$$

$$\frac{1_{\Omega^\varepsilon_\flat}}{|\Omega^\varepsilon_\flat|}p^\varepsilon dx\overset{\star}{\rightharpoonup}q\frac{dx}{|\Omega|}\quad in\quad L^\infty(0,T;\mathcal{M}_b(\overline{\Omega})) \tag{41}$$

$$m_\varepsilon(\vec{u}^\varepsilon_\sharp) \rightharpoonup \vec{v} := \begin{pmatrix} (1-2R)q \\ Ru' \\ Rv \end{pmatrix} \quad in \quad L^2(\Omega^T) \tag{42}$$

Moreover:

$$G^\varepsilon_{R_\varepsilon}(\vec{u}^\varepsilon) \to \begin{pmatrix} p^m \\ u^r \end{pmatrix} \quad in \quad L^2(\Omega^T) \tag{43}$$

Proof. See [12] □

Proposition 3.9. *Let $(\varphi^\sharp, \varphi) \in \mathcal{C}_c(\Omega) \times \mathcal{C}_c(\Omega)^3$. Set:*

$$\vec{\varphi} = \begin{pmatrix} \varphi^\sharp \\ \varphi \end{pmatrix}, \quad \vec{\varphi}^\varepsilon_\sharp = \begin{pmatrix} \varphi^\sharp 1_{\Omega^\varepsilon} \\ \varphi 1_{\tilde{\Omega}^\varepsilon_\sharp} \end{pmatrix}$$

and

$$\tilde{M}^\varepsilon(\vec{\varphi}) = \sum_{k\in\mathbb{Z}_\varepsilon} 1_{Y^\varepsilon_k}(x) \begin{pmatrix} \fint_{I^\varepsilon_{k_1}\times I^\varepsilon_{k_2}} \varphi^\sharp(y', x_3)dy' 1_{I^\varepsilon_{k_3}\setminus \tilde{I}^\varepsilon_{k_3}}(x_3) + \varepsilon^{-2}\displaystyle\int_{I^\varepsilon_{k_1}\times I^\varepsilon_{k_2}\setminus D^{k_3}_{\varepsilon\varrho^{k_3}_\varepsilon(x_3)}} \varphi^\sharp(y', x_3)dy' 1_{\tilde{I}^\varepsilon_{k_3}}(x_3) \\ \varepsilon^{-2}\displaystyle\int_{D^{k_3}_{\varepsilon\varrho^{k_3}_\varepsilon(x_3)}} \varphi(y', x_3)dy' 1_{\tilde{I}^\varepsilon_{k_3}}(x_3) \end{pmatrix}$$

$$= \sum_{k\in\mathbb{Z}_\varepsilon} 1_{Y^\varepsilon_k}(x) \underbrace{\left(1_{I^\varepsilon_{k_3}\setminus\tilde{I}^\varepsilon_{k_3}}(x_3) \begin{pmatrix} \fint_{I^\varepsilon_{k_1}\times I^\varepsilon_{k_2}} \varphi^\sharp(y', x_3)dy' \\ 0 \end{pmatrix} + 1_{\tilde{I}^\varepsilon_{k_3}}(x_3) \fint_{I^\varepsilon_{k_1}\times I^\varepsilon_{k_2}} \vec{\varphi}^\varepsilon_\sharp(y', x_3)dy' \right)}_{=:\tilde{M}^k_\varepsilon(\vec{\varphi})}$$

Then:

$$|\tilde{M}^\varepsilon(\vec{\varphi})|_\Omega \le |\vec{\varphi}|_\Omega \tag{44}$$

and

$$\lim_{\varepsilon\to 0} |\tilde{M}^\varepsilon(\vec{\varphi}) - \vec{\varphi}^\varepsilon_\sharp|_\infty = 0. \tag{45}$$

Proof. Let $k \in \mathbb{Z}_\varepsilon$. There holds:

$$|\tilde{M}^\varepsilon(\varphi)|^2_{Y^k_\varepsilon} \le 2\left(1_{I^\varepsilon_{k_3}\setminus\tilde{I}^\varepsilon_{k_3}}(x_3) \fint_{I^\varepsilon_{k_1}\times I^\varepsilon_{k_2}} |\varphi^\sharp|^2 dy' + 1_{\tilde{I}^\varepsilon_{k_3}}(x_3) \fint_{I^\varepsilon_{k_1}\times I^\varepsilon_{k_2}} |\vec{\varphi}|^2 dy' \right)$$

which yields (44). Moreover: $\forall x \in \Omega$,

$$\tilde{M}_\varepsilon(\vec{\varphi})(x) - \vec{\varphi}^\varepsilon(x) = \sum_{k\in\mathbb{Z}_\varepsilon} 1_{Y^\varepsilon_k}(x) \left(1_{I^\varepsilon_{k_3}\setminus\tilde{I}^\varepsilon_{k_3}}(x_3) \begin{pmatrix} \fint_{I^\varepsilon_{k_1}\times I^\varepsilon_{k_2}} \varphi^\sharp(y', x_3)dy' - \varphi^\sharp(x', x_3) \\ 0 \end{pmatrix} + \right.$$

$$+1_{\tilde{I}^\varepsilon_{k_3}}(x_3)\left(\fint_{I^\varepsilon_{k_1}\times I^\varepsilon_{k_2}}\bar{\varphi}^\varepsilon_\sharp(y',x_3)dy'-\bar{\varphi}^\varepsilon_\sharp(x',x_3)\right)\Bigg)$$

which yields (45). □

4. The heat equation in a cylindrical ring

4.1. Preliminary computations

Passing to the limit as $\varepsilon \to 0$ in the variational formulation (9)–(10) require the construction of adequate with test functions taking into account vanishing ducts. To that aim, we introduce the following fundamental problem. Let $b > 0$. Set

$$W_0^\varepsilon(\rho)=\begin{cases}\dfrac{\ln(\rho)-\ln(R_\varepsilon)}{\ln(r_\varepsilon)-\ln(R_\varepsilon)} & \text{if} \quad r_\varepsilon<\rho<R_\varepsilon,\\ 1 & \text{if} \quad 0\le\rho\le r_\varepsilon,\\ 0 & \text{if} \quad \rho\ge R_\varepsilon,\end{cases} \tag{46}$$

and let $W^\varepsilon \in L^2(0,T;H^1_0(\mathcal{C}_{R_\varepsilon}))$ denote the solution of:

$$\frac{\partial W^\varepsilon}{\partial t}-\Delta W^\varepsilon=0 \quad \text{in} \quad \mathcal{C}_{R_\varepsilon}\setminus\mathcal{C}_{r_\varepsilon}\times]0,T[, \tag{47}$$

$$\rho^\varepsilon\frac{\partial W^\varepsilon}{\partial t}-b\Delta W^\varepsilon=0 \quad \text{in} \quad \mathcal{C}_{r_\varepsilon}\times]0,T[, \tag{48}$$

$$[W^\varepsilon]=\left[k^\varepsilon\frac{\partial W^\varepsilon}{\partial n}\right]=0 \quad \text{on} \quad \mathcal{C}_{r_\varepsilon}\times]0,T[. \tag{49}$$

$$W^\varepsilon(0)=W_0^\varepsilon \quad \text{in} \quad \mathcal{C}_{R_\varepsilon}\setminus\mathcal{C}_{r_\varepsilon}. \tag{50}$$

where

$$\rho^\varepsilon=\frac{a}{\pi|\Omega|}\left(\frac{\varepsilon}{r_\varepsilon}\right)^2 \tag{51}$$

$$k^\varepsilon=\begin{cases} b & \text{in} \quad \mathcal{C}_{r_\varepsilon},\\ 1 & \text{in} \quad \mathcal{C}_{R_\varepsilon},\end{cases} \tag{52}$$

.

Proposition 4.1. *Let (J_0,Y_0) denote a fundamental system of:*

$$\rho w''+w'+\rho w=0 \tag{53}$$

satisfying:

$$J_0(0) = 1, \quad J_0'(0) = 0, \quad \lim_{t\to+\infty} Y_0(t) = \lim_{t\to+\infty} Y_0'(t) = +\infty. \tag{54}$$

Then, (55)

$$W^\varepsilon(x,t) = W_0^\varepsilon(\rho) + \int_0^t (h_\varepsilon \star S_\varepsilon(\rho))(s)ds \tag{55}$$

where h_ε and S_ε are defined through their Laplace transform for $\mathrm{Re}(p) > 0$ by:

$$\hat{S}_\varepsilon(p) = \begin{cases} \dfrac{J_0\left(\varepsilon\alpha\sqrt{-p}\dfrac{\rho}{r_\varepsilon}\right)}{J_0(\varepsilon\alpha\sqrt{-p})} & \text{if} \quad 0 \le \rho < r_\varepsilon, \\ \dfrac{Y_0(\sqrt{-p}R_\varepsilon)J_0(\sqrt{-p}\,\rho) - J_0(\sqrt{-p}R_\varepsilon)Y_0(\sqrt{-p}\,\rho)}{Y_0(\sqrt{-p}R_\varepsilon)J_0(\sqrt{-p}r_\varepsilon) - J_0(\sqrt{-p}R_\varepsilon)Y_0(\sqrt{-p}r_\varepsilon)} & \text{if} \quad r_\varepsilon < \rho < R_\varepsilon, \end{cases} \tag{56}$$

$$\hat{h}_\varepsilon(p)\left(b\left.\frac{\partial\hat{S}^\varepsilon}{\partial\rho}\right|_{\rho=r_\varepsilon^-} - \left.\frac{\partial\hat{S}^\varepsilon}{\partial\rho}\right|_{\rho=r_\varepsilon^+}\right) = \frac{1}{r_\varepsilon(\ln(r_\varepsilon) - \ln(R_\varepsilon))}. \tag{57}$$

where we set:

$$\alpha = \sqrt{\frac{a}{\pi|\Omega|b}} \tag{58}$$

Moreover

$$\lim_{\varepsilon\to0} \hat{h}_\varepsilon(p) = \frac{-\gamma}{\gamma + \dfrac{\alpha^2 bp}{2}} =: \hat{h}_0. \tag{59}$$

Proof. By definition of the Laplace transform:

$$\hat{f}(x,p) := \mathcal{F}(f)(x,p) = \int_0^{+\infty} f(x,t)e^{-pt}dt, \quad p \in \mathbb{C}, \quad x \in \Omega, \quad f \in \mathcal{C}_c^\infty(\Omega \times \mathbb{R}^+)$$

there holds:

$$\mathcal{F}(f')(p) = -f(0) + p\hat{f}(p),$$

from which we deduce that we may look for W^ε of the form:

$$W^\varepsilon(x,t) = W_0^\varepsilon(x) + \int_0^t h_\varepsilon \star S_\varepsilon(x)ds$$

where:

$$\hat{W}^{\varepsilon}(x,p) = \underbrace{\frac{W_0^{\varepsilon}}{p}}_{=\mathcal{F}(W_0^{\varepsilon})} + \mathcal{F}\left(\int_0^t h_{\varepsilon} \star S_{\varepsilon}(x)ds\right)$$

Notice that

$$\hat{h}_{\varepsilon}\hat{S}^{\varepsilon} = \mathcal{F}(h_{\varepsilon} \star S_{\varepsilon}) = p\mathcal{F}\left(\int_0^t h_{\varepsilon} \star S_{\varepsilon}(x)ds\right)$$

Then:

$$p\hat{W}^{\varepsilon}(x,p) = W_0^{\varepsilon}(x) + \hat{h}_{\varepsilon}(p)\hat{S}^{\varepsilon}(x,p) \tag{60}$$

satisfies:

$$\Delta\hat{S}^{\varepsilon} = \begin{cases} \left(\dfrac{\varepsilon}{r_{\varepsilon}}\right)^2 \alpha^2 p\hat{S}^{\varepsilon} & \text{if} \quad 0 < \rho < r_{\varepsilon}, \\ \\ p\hat{S}^{\varepsilon} & \text{if} \quad r_{\varepsilon} < \rho < R_{\varepsilon}, \end{cases}$$

Let $r_{\varepsilon} < \rho < R_{\varepsilon}$ and set: $\forall(\theta, z) \in \mathbb{R}^2$,

$$\hat{S}^{\varepsilon} := R(\rho)\Theta(\theta)Z(z)$$

Direct computation yields:

$$\frac{(\rho R')'}{\rho R} + \frac{1}{\rho^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = p$$

from which we deduce that:

$$\frac{(\rho R')'}{\rho R} + \frac{1}{\rho^2}\frac{\Theta''}{\Theta} = p - \frac{Z''}{Z} =: l_1$$

that is:

$$Z'' + (p - \lambda_1)Z = 0.$$

Assuming that $Z \equiv 1$ yields:

$$\lambda_1 = p$$

and then:

$$\rho\frac{(\rho R')'}{R} - p\rho^2 = -\frac{\Theta''}{\Theta} =: \lambda_2$$

that is:

$$\Theta'' + \lambda_2 \Theta = 0 \tag{61}$$

and

$$\rho^2 R'' + \rho R' - (p\rho^2 + \lambda_2) R = 0. \tag{62}$$

Set $p = -\omega_0^2 < 0$ and $\lambda_2 = n^2$, $n \in \mathbb{N}$ in (62). In particular, $\omega_0 := \sqrt{-p}$ is the square root of positive real part. Then:

$$\hat{S}^\varepsilon(\rho, \theta) = \sum_{n \geq 0} (a_n J_n(\sqrt{-p}\rho) + b_n Y_n(\sqrt{-p}\rho)) \Theta_n(\theta)$$

where (J_n, Y_n) is a fundamental system of the Bessel equation:

$$\rho^2 w'' + \rho w' + (\rho^2 - n^2) w = 0$$

made of the Bessel function J_n and its associated Neumann function Y_n, $n \geq 0$, and where Θ_n solves (61) with $\lambda_2 = n^2$. The requirement:

$$\hat{S}^\varepsilon(r_\varepsilon^+) \equiv 1$$

implies that

$$a_n = b_n = 0, \quad \forall n \geq 1.$$

There remains:

$$\hat{S}^\varepsilon = a_0 J_0(\sqrt{-p}\rho) + b_0 Y_0(\sqrt{-p}\rho)$$

where constants a_0, $b_0 \in \mathbb{R}$ are defined by the other requirement: $\hat{S}^\varepsilon(R_\varepsilon) = 0$, namely:

$$\hat{S}^\varepsilon(\rho) = \frac{Y_0(\sqrt{-p}R_\varepsilon) J_0(\sqrt{-p}\rho) - J_0(\sqrt{-p}R_\varepsilon) Y_0(\sqrt{-p}\rho)}{Y_0(\sqrt{-p}R_\varepsilon) J_0(\sqrt{-p}r_\varepsilon) - J_0(\sqrt{-p}R_\varepsilon) Y_0(\sqrt{-p}r_\varepsilon)}, \quad r_\varepsilon < \rho < R_\varepsilon. \tag{63}$$

The same arguments yield $\hat{S}^\varepsilon$ for $0 < \rho < r_\varepsilon$, that is (56).

Notice that:

$$\left.\frac{\partial \hat{S}^\varepsilon}{\partial \rho}\right|_{\rho = r_\varepsilon^-} = \frac{\varepsilon\alpha\sqrt{-p}}{r_\varepsilon} \frac{J_0'(\varepsilon\alpha\sqrt{-p})}{J_0(\varepsilon\alpha\sqrt{-p})} \tag{64}$$

with:

$$J_0(t) = 1 - \frac{t^2}{4} + o(t^2) \underset{\to 0}{\to} 1, \tag{65}$$

$$J_0'(t) \underset{t \to 0}{\sim} -\frac{t}{2}. \tag{66}$$

Substitution of (65)–(66) into (64) yields:

$$\left.\frac{\partial \hat{S}^\varepsilon}{\partial \rho}\right|_{\rho=r_\varepsilon^-} \underset{\varepsilon\to 0}{\sim} \frac{\varepsilon^2}{2r_\varepsilon}\alpha^2 p \tag{67}$$

Relying on the asymptotics:

$$Y_0(t) = 2J_0(t)\ln\frac{t}{2} + O(1) \underset{t\to 0}{\sim} 2\ln t \tag{68}$$

$$Y_0'(t) = 2J_0'(t)\ln\frac{t}{2} + \frac{2}{t}J_0(t) + O(1) \underset{t\to 0}{\sim} \frac{2}{t} \tag{69}$$

we first infer that:

$$\left(Y_0(\sqrt{-p}R_\varepsilon)J_0(\sqrt{-p}r_\varepsilon) - J_0(\sqrt{-p}R_\varepsilon)Y_0(\sqrt{-p}r_\varepsilon)\right)^{-1} \underset{\varepsilon\to 0}{\sim} \frac{-1}{2\ln\frac{r_\varepsilon}{R_\varepsilon}} \tag{70}$$

and

$$Y_0(\sqrt{-p}R_\varepsilon)J_0'(\sqrt{-p}r_\varepsilon) - J_0(\sqrt{-p}R_\varepsilon)Y_0'(\sqrt{-p}r_\varepsilon) \underset{\varepsilon\to 0}{\sim} -\sqrt{-p}r_\varepsilon\ln R_\varepsilon - \frac{2}{\sqrt{-p}r_\varepsilon}$$

$$= -\frac{1}{r_\varepsilon}\left(\sqrt{-p}r_\varepsilon^2\ln R_\varepsilon + \frac{2}{\sqrt{-p}}\right) \tag{71}$$

with

$$r_\varepsilon^2\ln R_\varepsilon = \left(\frac{r_\varepsilon}{R_\varepsilon}\right)^2 R_\varepsilon^2\ln R_\varepsilon \underset{\varepsilon\to 0}{\to} 0 \tag{72}$$

Substitution of (70), (71) and (72) into (63) yields:

$$\left.\frac{\partial \hat{S}^\varepsilon}{\partial \rho}\right|_{\rho=r_\varepsilon^+} \underset{\varepsilon\to 0}{\sim} \frac{1}{r_\varepsilon\ln\frac{r_\varepsilon}{R_\varepsilon}}. \tag{73}$$

Notice that

$$\left.\frac{\partial W_0^\varepsilon}{\partial \rho}\right|_{\rho=r_\varepsilon^+} = \frac{1}{r_\varepsilon\ln\frac{r_\varepsilon}{R_\varepsilon}} \tag{74}$$

Then substitution of (67), (73) and (74) into the definition (57) of $\hat{h}_\varepsilon$ yields:

$$b\hat{h}_\varepsilon\frac{\varepsilon^2\alpha^2}{2r_\varepsilon}p \underset{\varepsilon\to 0}{\sim} \frac{1+\hat{h}_\varepsilon}{r_\varepsilon\ln\frac{r_\varepsilon}{R_\varepsilon}}$$

that is:

$$\hat{h}_\varepsilon \left(\frac{\alpha^2 bp}{2} - \underbrace{\frac{1}{\varepsilon^2 \ln \frac{r_\varepsilon}{R_\varepsilon}}}_{\underset{\varepsilon\to 0}{\to} -\gamma} \right) \underset{\varepsilon\to 0}{\sim} \frac{1}{\varepsilon^2 \ln \frac{r_\varepsilon}{R_\varepsilon}} \underset{\varepsilon\to 0}{\sim} \frac{1}{\varepsilon^2 \ln r_\varepsilon} \underset{\varepsilon\to 0}{\sim} -\gamma$$

which ends the proof of (59). □

Proposition 4.2. *There holds:*

$$\left.\frac{\partial W^\varepsilon}{\partial \rho}\right|_{\rho=R_\varepsilon} = \frac{F_\varepsilon(t)}{R_\varepsilon \ln\left(\frac{r_\varepsilon}{R_\varepsilon}\right)} \tag{75}$$

where:

$$p\hat{F}_\varepsilon(p) \underset{\varepsilon\to 0}{\sim} 1 + \hat{h}_\varepsilon \underset{\varepsilon\to 0}{\to} 1 + \hat{h}_0 =: p\hat{F}_0 \tag{76}$$

Proof. Notice that

$$\left.\frac{\partial W_0^\varepsilon}{\partial \rho}\right|_{\rho=R_\varepsilon} = \frac{1}{R_\varepsilon \ln \frac{r_\varepsilon}{R_\varepsilon}}.$$

Moreover, relying on (70) and (63), we obtain:

$$\left.\frac{\partial \hat{S}^\varepsilon}{\partial \rho}\right|_{\rho=R_\varepsilon} \underset{\varepsilon\to 0}{\sim} -\frac{1}{2\ln \frac{r_\varepsilon}{R_\varepsilon}} \left(Y_0(\sqrt{-p}R_\varepsilon) J_0'(\sqrt{-p}R_\varepsilon) - J_0(\sqrt{-p}R_\varepsilon) Y_0'(\sqrt{-p}R_\varepsilon) \right).$$

where substitution of the asymptotics (65), (66), (68), (69) yields:

$$Y_0(\sqrt{-p}R_\varepsilon) J_0'(\sqrt{-p}R_\varepsilon) - J_0(\sqrt{-p}R_\varepsilon) Y_0'(\sqrt{-p}R_\varepsilon) \underset{\varepsilon\to 0}{\sim} -\sqrt{-p}R_\varepsilon \ln R_\varepsilon - \frac{2}{\sqrt{-p}R_\varepsilon}$$

$$= -\frac{1}{R_\varepsilon}\left(\sqrt{-p}R_\varepsilon^2 \ln R_\varepsilon + \frac{2}{\sqrt{-p}} \right) \underset{\varepsilon\to 0}{\sim} -\frac{2}{\sqrt{-p}R_\varepsilon}$$

from which we deduce that

$$\left.\frac{\partial \hat{S}^\varepsilon}{\partial \rho}\right|_{\rho=R_\varepsilon} \underset{\varepsilon\to 0}{\sim} \frac{1}{R_\varepsilon \ln \frac{r_\varepsilon}{R_\varepsilon}}$$

From (60) we infer that

$$p \left.\frac{\partial \hat{W}^\varepsilon}{\partial \rho}\right|_{\rho=R_\varepsilon} = \left.\frac{\partial \hat{W}_0^\varepsilon}{\partial \rho}\right|_{\rho=R_\varepsilon} + \hat{h}_\varepsilon \left.\frac{\partial \hat{S}^\varepsilon}{\partial \rho}\right|_{\rho=R_\varepsilon} \underset{\varepsilon\to 0}{\sim} \frac{1+\hat{h}_\varepsilon}{R_\varepsilon \ln \frac{r_\varepsilon}{R_\varepsilon}}.$$

which yields the result. □

Set:

$$w^\varepsilon(x,t) = \sum_{k\in\mathbb{Z}_\varepsilon} W^\varepsilon(|x' - \varepsilon k'|, t) 1_{\mathcal{C}^{k,\varepsilon}_{R_\varepsilon}}(x). \tag{77}$$

resp.:

$$w_0^\varepsilon(x,t) = \sum_{k\in\mathbb{Z}_\varepsilon} W_0^\varepsilon(|x' - \varepsilon k'|) 1_{\mathcal{C}^{k,\varepsilon}_{R_\varepsilon}}(x). \tag{78}$$

Proposition 4.3. *The sequence $(w^\varepsilon)_\varepsilon$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ and*

$$w^\varepsilon \underset{\varepsilon\to 0}{\to} 0 \quad in \quad L^\infty(0,T;L^2(\Omega)).$$

4.2. Computation of the tangential component

As our problem lives in the three-dimensional space, it requires the introduction of a tangential component which eventually reduces to its angular component along $\vec{e}_\theta$ to ensure that it remains incompressible. More precisely, let:

$$\overrightarrow{W_0^\varepsilon} = W^\varepsilon_{0,\theta}(\rho)\vec{e}_\theta + W^\varepsilon_{0,3}(\rho)\vec{e}_3$$

where $W^\varepsilon_{0,3}$ is defined by (46) and where

$$W^\varepsilon_{0,\theta}(\rho) = \begin{cases} \dfrac{r_\varepsilon R_\varepsilon}{R_\varepsilon^2 - r_\varepsilon^2}\left(\dfrac{R_\varepsilon}{\rho} - \dfrac{\rho}{R_\varepsilon}\right) & \text{if} \quad r_\varepsilon < \rho < R_\varepsilon, \\ 1 & \text{if} \quad 0 \le \rho \le r_\varepsilon, \\ 0 & \text{if} \quad \rho \ge R_\varepsilon, \end{cases} \tag{79}$$

and let $(\overrightarrow{W}^\varepsilon, P^\varepsilon = 0) \in L^2(0,T;H_0^1(\mathcal{C}_{R_\varepsilon})) \times L^2(0,T;L_0^2(\mathcal{C}_{R_\varepsilon}))$ denote the solution of:

$$\mathrm{div}\overrightarrow{W}^\varepsilon = 0$$

$$\frac{\partial \overrightarrow{W}^\varepsilon}{\partial t} - \Delta \overrightarrow{W}^\varepsilon = 0 \quad \text{in} \quad \mathcal{C}_{R_\varepsilon} \setminus \mathcal{C}_{r_\varepsilon} \times]0,T[, \tag{80}$$

$$\rho^\varepsilon \frac{\partial \overrightarrow{W}^\varepsilon}{\partial t} - b\Delta \overrightarrow{W}^\varepsilon + \vec{\nabla} P^\varepsilon = 0 \quad \text{in} \quad \mathcal{C}_{r_\varepsilon} \times]0,T[, \tag{81}$$

$$[W^\varepsilon] = \left[k_\varepsilon \frac{\partial W^\varepsilon}{\partial n}\right] = 0 \quad \text{on} \quad D_{r_\varepsilon} \times]0,T[. \tag{82}$$

$$\overrightarrow{W}^{\varepsilon}(0) = \overrightarrow{W}_0^{\varepsilon} \quad \text{in} \quad \mathcal{C}_{R_\varepsilon} \setminus \mathcal{C}_{r_\varepsilon}. \tag{83}$$

where

$$k_\varepsilon = \begin{cases} 1 & \text{in} \quad \mathcal{C}_{R_\varepsilon} \setminus \mathcal{C}_{r_\varepsilon}, \\ b & \text{in} \quad \mathcal{C}_{r_\varepsilon}, \end{cases}$$

Proposition 4.4. *Set*

$$\alpha = \sqrt{\frac{a}{\pi|\Omega|b}}.$$

Then:

$$\overrightarrow{W}^{\varepsilon}(x,t) = W_\theta^\varepsilon(\rho)\vec{e}_\theta + W_3^\varepsilon(\rho)\vec{e}_3$$

where W_3^ε and W_θ^ε are defined by (55)–(76) and (85)–(87) resp. The same arguments yield:

$$W_\theta^\varepsilon = W_{0,\theta}^\varepsilon(\rho) + \int_0^t (h_\theta^\varepsilon \star S_\theta^\varepsilon(\rho))(s)ds \tag{84}$$

where h_θ^ε and S_θ^ε are defined through their Laplace transform for $\mathrm{Re}(p) > 0$ by:

$$\hat{S}_\theta^\varepsilon(p) = \begin{cases} \dfrac{J_1\left(\varepsilon\frac{\alpha\sqrt{-p}}{r_\varepsilon}\rho\right)}{J_1(\varepsilon\alpha\sqrt{-p})} & \text{if} \quad 0 \le \rho < r_\varepsilon, \\ \dfrac{Y_1(\sqrt{-p}R_\varepsilon)J_1(\sqrt{-p}\rho) - J_1(\sqrt{-p}R_\varepsilon)Y_1(\sqrt{-p}\rho)}{Y_1(\sqrt{-p}R_\varepsilon)J_1(\sqrt{-p}r_\varepsilon) - J_1(\sqrt{-p}R_\varepsilon)Y_1(\sqrt{-p}r_\varepsilon)} & \text{if} \quad r_\varepsilon < \rho < R_\varepsilon, \end{cases} \tag{85}$$

where (J_1, Y_1) is a fundamental system of solutions of the Bessel equation:

$$\rho^2 w'' + \rho w' + (\rho^2 - 1)w = 0 \tag{86}$$

made of the Bessel function J_1 and its associated Neumann function Y_1 satisfying:

$$J_1(0) = 0, \quad J_1'(0) = 1. \tag{87}$$

Moreover:

$$\hat{h}_\theta^\varepsilon \underset{\varepsilon\to 0}{\sim} -\frac{1}{1+b} =: \hat{h}_0. \tag{88}$$

Proof. We ate interested in componentwise tangential solutions, namelyn we look for

$$\vec{W}^{\varepsilon} = W_\rho^\varepsilon \vec{e}_\rho,$$

resp.:

$$\vec{W}^\varepsilon = W_\theta^\varepsilon \vec{e}_\theta.$$

Direct computation yields that the general solution $\vec{W}^\varepsilon = W_\rho^\varepsilon + W_\theta^\varepsilon$ solves the system:

$$\rho^\varepsilon \frac{\partial W_\rho^\varepsilon}{\partial t} - k_\varepsilon \left(\Delta W_\rho^\varepsilon - \frac{2}{\rho^2} \frac{\partial W_\theta^\varepsilon}{\partial \theta} - \frac{W_\rho^\varepsilon}{\rho^2} \right) = 0$$

$$\rho^\varepsilon \frac{\partial W_\theta^\varepsilon}{\partial t} - k_\varepsilon \left(\Delta W_\theta^\varepsilon + \frac{2}{\rho^2} \frac{\partial W_\rho^\varepsilon}{\partial \theta} - \frac{W_\theta^\varepsilon}{\rho^2} \right) = 0$$

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho W_\rho^\varepsilon) + \frac{1}{\rho} \frac{\partial W_\theta^\varepsilon}{\partial \theta} = 0$$

In particular:

$$W_\theta^\varepsilon = 0 \Rightarrow \frac{\partial}{\partial \rho} (\rho W_\rho^\varepsilon) = 0$$

from which there results:

$$W_\rho^\varepsilon = \frac{C}{\rho}$$

and the requirement $W_\rho^\varepsilon|_{\rho = R_\varepsilon} = 0$ leads to $W_\rho^\varepsilon = 0$ for $r_\varepsilon < \rho < R_\varepsilon$, which contradicts $W_\rho^\varepsilon|_{\rho = r_\varepsilon} = 1$. So $\vec{W}^\varepsilon = W_\rho^\varepsilon \vec{e}_\rho$ is not a solution and we are refered to the componentwise solution:

$$\vec{W}^\varepsilon = W_\theta^\varepsilon \vec{e}_\theta.$$

Then, the problem reduces to: find W_θ^ε solution of

$$\Delta W^\varepsilon - \frac{W_\theta^\varepsilon}{\rho^2} = \begin{cases} pW_\theta^\varepsilon & \text{if} \quad r_\varepsilon < \rho < R_\varepsilon, \\ \left(\dfrac{\varepsilon \alpha}{r_\varepsilon} \right)^2 pW_\theta^\varepsilon & \text{if} \quad 0 < \rho < r_\varepsilon, \end{cases}$$

$$\frac{\partial W_\theta^\varepsilon}{\partial \theta} = 0$$

The same arguments as in Proposition 4.1 yield that

$$p\hat{W}_\theta^\varepsilon = W_{0,\theta}^\varepsilon + \hat{h}_\theta^\varepsilon \hat{S}_\theta^\varepsilon$$

with $\hat{h}_\theta^\varepsilon$ defined by (85). Relying on the asymptotics:

$$J_1(t) \underset{\varepsilon\to 0}{\sim} \frac{t}{2}, \quad J_1'(t) \underset{\varepsilon\to 0}{\sim} \frac{1}{2}$$

$$Y_1(t) \underset{\varepsilon\to 0}{\sim} \frac{-2}{\pi t}, \quad Y_1'(t) \underset{\varepsilon\to 0}{\sim} \frac{2}{\pi t^2}$$

we find that

$$\left.\frac{\partial \hat{S}_\theta^\varepsilon}{\partial \rho}\right|_{\rho=r_\varepsilon^-} = \frac{\varepsilon\alpha\sqrt{-p}}{r_\varepsilon}\frac{J_1'(\varepsilon\alpha\sqrt{-p})}{J_1(\varepsilon\alpha\sqrt{-p})} \underset{\varepsilon\to 0}{\sim} \frac{1}{r_\varepsilon}$$

$$\big(Y_1(\sqrt{-p}R_\varepsilon)J_1(\sqrt{-p}r_\varepsilon) - J_1(\sqrt{-p}R_\varepsilon)Y_1(\sqrt{-p}r_\varepsilon)\big)^{-1} \underset{\varepsilon\to 0}{\sim} \frac{\pi r_\varepsilon}{R_\varepsilon}$$

$$Y_1(\sqrt{-p}R_\varepsilon)J_1'(\sqrt{-p}r_\varepsilon) - J_1(\sqrt{-p}R_\varepsilon)Y_1'(\sqrt{-p}r_\varepsilon) \underset{\varepsilon\to 0}{\sim} -\frac{R_\varepsilon}{\pi\sqrt{-p}\, r_\varepsilon^2}$$

$$\Rightarrow \left.\frac{\partial \hat{S}_\theta^\varepsilon}{\partial \rho}\right|_{\rho=r_\varepsilon^+} \underset{\varepsilon\to 0}{\sim} -\frac{1}{r_\varepsilon}$$

Noticing that

$$\left.\frac{\partial W_{0,\theta}^\varepsilon}{\partial \rho}\right|_{\rho=r_\varepsilon^+} = -\frac{1}{r_\varepsilon}\left(\frac{R_\varepsilon^2 + r_\varepsilon^2}{R_\varepsilon^2 - r_\varepsilon^2}\right) \underset{\varepsilon\to 0}{\sim} -\frac{1}{r_\varepsilon}$$

we infer that

$$b\frac{\hat{h}_\theta^\varepsilon}{r_\varepsilon} \underset{\varepsilon\to 0}{\sim} -\frac{(1+\hat{h}_\theta^\varepsilon)}{r_\varepsilon} \Rightarrow \hat{h}_\theta^\varepsilon \underset{\varepsilon\to 0}{\sim} -\frac{1}{1+b} =: \hat{h}_\theta^0$$

□

Proposition 4.5. *There holds:*

$$p\left.\frac{\partial \hat{W}_\theta^\varepsilon}{\partial \rho}\right|_{\rho=R_\varepsilon} = -\frac{2b}{(1+b)}\frac{r_\varepsilon}{R_\varepsilon^2}. \tag{89}$$

Proof. Direct computation yields:

$$p\frac{\partial \hat{W}_\theta^\varepsilon}{\partial \rho} = \frac{\partial W_{0,\theta}^\varepsilon}{\partial \rho} + \hat{h}_\theta^\varepsilon \frac{\partial \hat{S}_\theta^\varepsilon}{\partial \rho}$$

with:

$$\left.\frac{\partial W^{\varepsilon}_{0,\theta}}{\partial \rho}\right|_{\rho=R_\varepsilon} = -\frac{2r_\varepsilon}{(R^2_\varepsilon - r^2\varepsilon)} \underset{\varepsilon\to 0}{\sim} -\frac{2r_\varepsilon}{R^2_\varepsilon}$$

$$Y_1(\sqrt{-p}R_\varepsilon)J_1'(\sqrt{-p}R_\varepsilon) - J_1(\sqrt{-p}R_\varepsilon)Y_1'(\sqrt{-p}R_\varepsilon) \underset{\varepsilon\to 0}{\sim} -\frac{2}{\pi\sqrt{-p}R_\varepsilon}$$

$$\Rightarrow \left.\frac{\partial \hat{S}^{\varepsilon}_{\theta}}{\partial \rho}\right|_{\rho=R_\varepsilon} \underset{\varepsilon\to 0}{\sim} -\frac{2r_\varepsilon}{R^2_\varepsilon}$$

from which we deduce that

$$p\left.\frac{\partial \hat{W}^{\varepsilon}_{\theta}}{\partial \rho}\right|_{\rho=R_\varepsilon} \underset{\varepsilon\to 0}{\sim} -\frac{2b}{(1+b)}\frac{r_\varepsilon}{R^2_\varepsilon}$$

□

Remark 3. Notice that:

$$\frac{r_\varepsilon}{R^2_\varepsilon}R_\varepsilon \ln\left(\frac{r_\varepsilon}{R_\varepsilon}\right) = \frac{r_\varepsilon}{R_\varepsilon}\ln\left(\frac{r_\varepsilon}{R_\varepsilon}\right) \underset{\varepsilon\to 0}{\to} 0.$$

that is:

$$\frac{r_\varepsilon}{R^2_\varepsilon} = o\left(\frac{1}{R_\varepsilon \ln\left(\frac{r_\varepsilon}{R_\varepsilon}\right)}\right) \quad \text{as} \quad \varepsilon \to 0. \tag{90}$$

Set:

$$\vec{w}^{\varepsilon}(x,t) = \sum_{k\in\mathbb{Z}_\varepsilon} \overrightarrow{W}^{\varepsilon}(|x' - \varepsilon k'|, t)1_{\mathcal{C}^{k,\varepsilon}_{R_\varepsilon}}(x). \tag{91}$$

resp.:

$$\vec{w}^{\varepsilon}_0(x,t) = \sum_{k\in\mathbb{Z}_\varepsilon} \overrightarrow{W}^{\varepsilon}_0(|x' - \varepsilon k'|)1_{\mathcal{C}^{k,\varepsilon}_{R_\varepsilon}}(x). \tag{92}$$

Remark 4. [20]

$$J_\lambda(z) = \frac{1}{\pi}\int_0^\pi \cos(\lambda t - z\sin t)dt,$$

$$Y_1(z) = \lim_{\lambda\to 1}\frac{\cos(\pi\lambda)J_\lambda(z) - J_{-\lambda}(z)}{\sin(\pi\lambda)}$$

Proposition 4.6. *Set:*

$$\vec{w}^{\varepsilon}(x,t)=\sum_{k\in\mathbb{Z}_{\varepsilon}}\vec{W}^{\varepsilon}(|x'-\varepsilon k'|,t)1_{\mathcal{C}^{k,\varepsilon}_{R_{\varepsilon}}}(x). \tag{93}$$

resp.:

$$\vec{w}^{\varepsilon}_{0}(x,t)=\sum_{k\in\mathbb{Z}_{\varepsilon}}\vec{W}^{\varepsilon}_{0}(|x'-\varepsilon k'|)1_{\mathcal{C}^{k,\varepsilon}_{R_{\varepsilon}}}(x). \tag{94}$$

The sequence $(\vec{w}^{\varepsilon})_{\varepsilon}$ *is uniformly bounded in* $L^{\infty}(0,T;L^{2}(\Omega))\cap L^{2}(0,T;H^{1}_{0}(\Omega))$ *and*

$$\vec{w}^{\varepsilon}\underset{\varepsilon\to 0}{\to}0\quad in\quad L^{\infty}(0,T;L^{2}(\Omega)).$$

5. The limit problem

5.1. The new functional setting

In this Section we pass to the limit as ε tends to zero in the variational problem (9)–(10) with two-scale convergence results of Theorem 3.1 in mind.

Set:

$$X=H\times L^{2}(\Omega^{T};\tilde{H}^{1}_{\rm per}(\tilde{Y})),\quad M=\{q\in L^{2}_{0}(\Omega,L^{2}_{\rm per}(Y)),\ q^{m}=\tilde{q}^{m}\in H^{1}(\Omega)\},$$

where:

$$H=\{u\in L^{2}(\Omega^{T}\times Y)\cap L^{2}(\Omega^{T},H_{0}({\rm div},\Omega)),\quad \tilde{u}\in H_{0}({\rm div},\Omega),$$

$$u^{r}=\tilde{u}^{r}\in L^{2}(0,T,H^{1}(\Omega))\cap L^{2}(0,T,H_{0}({\rm div},\Omega))\}$$

Then, X is a Hilbert space with respect to the scalar product:

$$((v,z),(\varphi,\psi))_{X}:=\int_{\Omega^{T}\times Y_{m}}v\varphi dxdydt+\int_{\Omega^{T}}{\rm div}(\tilde{v}){\rm div}(\tilde{\varphi})dxdt+$$

$$+\int_{\Omega^{T}\times\tilde{Y}}(\nabla v+\nabla_{y}z)(\nabla\varphi+\nabla_{y}\psi)dxdydt+\int_{\Omega\times\tilde{\Gamma}}(v-(v\cdot\tilde{n}_{y})\tilde{n}_{y})\varphi d\sigma dx$$

In the sequel, we set: : $\forall((v,z),(\varphi,\psi))\in X\times X$, $\forall\pi\in M$

$$a((v,z),(\varphi,\psi))=\int_{\Omega\times Y_{m}}Av\cdot\varphi dxdy+\beta^{2}\int_{\Omega\times\tilde{Y}}(\nabla v+\nabla_{y}z)(\nabla\varphi+\nabla_{y}\psi)dxdy+$$

$$+\int_{\Omega\times\tilde{\Gamma}}\alpha(v-(v\cdot\tilde{n})\tilde{n})\cdot\varphi dxd\sigma$$

$$b(\pi,(v,z)) = -\int_{\Omega\times Y}\pi\mathrm{div}(v)dxdy + \int_{\Omega\times\tilde{\Gamma}}\pi^m(z\cdot\tilde{n}_y)dxd\sigma - \int_{\Omega\times\tilde{Y}}\pi\mathrm{div}_y(z)dxdy.$$

We begin with an auxiliary convergence result involving external forces.

Proposition 5.1. *Set:*

$$\mathcal{F}_\varepsilon(\varphi) = \int_{\Omega\setminus\Omega'^\varepsilon} f\varphi dx + \fint_{\Omega'^\varepsilon} g\varphi dx, \quad \forall\varphi\in H_0^1(\Omega).$$

Then:

$$\mathcal{F}_\varepsilon \rightharpoonup \mathcal{F} \tag{95}$$

where

$$\mathcal{F}(\varphi) = \int_{\Omega} f\varphi dx + \fint_{\Omega} g\varphi dx, \quad \forall\varphi\in H_0^1(\Omega). \tag{96}$$

Proof. See [12]. □

5.2. The Control zone method revisited

Proposition 5.2. *Let $\varphi\in\mathcal{C}_c^1(\Omega)$. Then:*

$$\int_{\tilde{\Omega}_\sharp^\varepsilon}(-k_\varepsilon\Delta u^\varepsilon+\nabla p^\varepsilon)\star\vec{w}^\varepsilon m_\varepsilon(\varphi)dx - \int_{\tilde{\Omega}_\sharp^\varepsilon} u^\varepsilon\star(-k_\varepsilon\Delta\vec{w}^\varepsilon)m_\varepsilon(\varphi)dx+$$

$$-\int_{\tilde{\Gamma}^\varepsilon\cup\Sigma^\varepsilon}\alpha_\varepsilon(u^\varepsilon-(u^\varepsilon\cdot\tilde{n}^\varepsilon)\tilde{n}^\varepsilon)\star\vec{w}^\varepsilon m_\varepsilon(\varphi)d\sigma \underset{\varepsilon\to 0}{\rightharpoonup} -2\pi\gamma F_0\star u_3^r\cdot\varphi \quad in \quad L^2(0,T),$$

where

$$F_0 := 1+\int_0^t h_0(s)ds. \tag{97}$$

is defined after:

$$\hat{F_0} := \frac{1+\hat{h}_0}{p} = \frac{\alpha^2 b}{2\gamma+\alpha^2 bp} = \frac{1}{\frac{2\gamma}{\alpha^2 b}+p} \underset{(58)}{=} \frac{1}{\frac{2\pi\gamma|\Omega|}{a}+p} \tag{98}$$

Proof. Let $\varphi\in\mathcal{C}_c^1(\Omega)^3$. There holds:

$$\int_{\tilde{\Omega}_\sharp^\varepsilon}(-k_\varepsilon\Delta u^\varepsilon+\nabla p^\varepsilon)\star\vec{w}^\varepsilon m_\varepsilon(\varphi)dx = \int_{\tilde{\Omega}_\sharp^\varepsilon} u^\varepsilon\star(-k_\varepsilon\Delta\vec{w}^\varepsilon)m_\varepsilon(\varphi)+\int_{\tilde{\Omega}_\sharp^\varepsilon} k_\varepsilon\frac{\partial u^\varepsilon}{\partial x_3}\star\vec{w}^\varepsilon m_\varepsilon\left(\frac{\partial\varphi}{\partial x_3}\right)dx+$$

$$-\int_{\tilde{\Omega}_\sharp^\varepsilon} p^\varepsilon\star\underbrace{\mathrm{div}(\vec{w}^\varepsilon)}_{=0}m_\varepsilon(\varphi)dx-\int_{\tilde{\Omega}_\sharp^\varepsilon} p^\varepsilon\star w_3^\varepsilon m_\varepsilon\left(\frac{\partial\varphi}{\partial x_3}\right)dx-\int_{\tilde{\Gamma}^\varepsilon\cup\Sigma^\varepsilon}\alpha_\varepsilon(u^\varepsilon-(u^\varepsilon\cdot\tilde{n}^\varepsilon)\tilde{n}^\varepsilon)\star\vec{w}^\varepsilon m_\varepsilon(\varphi)d\sigma+$$

$$\underbrace{-\int_{\partial\mathcal{C}_\varepsilon} k_\varepsilon \frac{\partial u^\varepsilon}{\partial n^\varepsilon} \star \vec{w}^\varepsilon m_\varepsilon(\varphi) + \int_{\partial\mathcal{C}_\varepsilon} p^\varepsilon n^\varepsilon \star \vec{w}^\varepsilon m_\varepsilon(\varphi)}_{=0} + \int_{\partial\mathcal{C}_\varepsilon} u^\varepsilon \star k_\varepsilon \frac{\partial \vec{w}^\varepsilon}{\partial n^\varepsilon} m_\varepsilon(\varphi)$$

with:

$$\left| \int_{\tilde{\Omega}^\varepsilon_\sharp} k_\varepsilon \frac{\partial u^\varepsilon}{\partial x_3} \star \vec{w}^\varepsilon m_\varepsilon \left(\frac{\partial \varphi}{\partial x_3} \right) dx \right| \leq C |\nabla u^\varepsilon|_{\tilde{\Omega}^\varepsilon_\sharp} |\vec{w}^\varepsilon|_\Omega \left\| \frac{\partial \varphi}{\partial x_3} \right\|_\infty \leq C |\vec{w}^\varepsilon|_{\tilde{\Omega}^\varepsilon_\sharp} \underset{\varepsilon \to 0}{\to} 0,$$

$$\left| \int_{\tilde{\Omega}^\varepsilon_\sharp} p^\varepsilon \star w_3^\varepsilon m_\varepsilon \left(\frac{\partial \varphi}{\partial x_3} \right) dx \right| \leq C |p^\varepsilon|_{\tilde{\Omega}^\varepsilon_\sharp} |w_3^\varepsilon|_\Omega \left\| \frac{\partial \varphi}{\partial x_3} \right\|_\infty \leq C |w_3^\varepsilon|_\Omega \underset{\varepsilon \to 0}{\to} 0,$$

$$\int_{\partial\mathcal{C}_\varepsilon} u^\varepsilon \star k_\varepsilon \frac{\partial \vec{w}^\varepsilon}{\partial n^\varepsilon} m_\varepsilon(\varphi) = \int_{\mathrm{C}_{R_\varepsilon}} u^\varepsilon \star \frac{\partial \vec{w}^\varepsilon}{\partial n^\varepsilon} m_\varepsilon(\varphi) d\sigma =$$

$$= \sum_{k \in \mathbb{Z}^\varepsilon} R_\varepsilon \int_0^{2\pi} \int_{\varepsilon k_3 - \frac{\varepsilon}{2}}^{\varepsilon k_3 + \frac{\varepsilon}{2}} u^\varepsilon|_{\rho = R_\varepsilon} \star \left. \frac{\partial \vec{w}^\varepsilon}{\partial \rho} \right|_{\rho = R_\varepsilon} m_\varepsilon^k(\varphi) dx_3 d\theta$$

$$\underset{(75),(76),(89)}{=} \sum_{k \in \mathbb{Z}^\varepsilon} 2\pi R_\varepsilon \int_{\varepsilon k_3 - \frac{\varepsilon}{2}}^{\varepsilon k_3 + \frac{\varepsilon}{2}} G^{\varepsilon,k}_{R_\varepsilon}(u^\varepsilon) \star \vec{F}^\varepsilon m_\varepsilon^k(\varphi) dx_3$$

where:

$$\vec{F}^\varepsilon = \begin{pmatrix} -\dfrac{2r_\varepsilon}{(R_\varepsilon^2 - r_\varepsilon^2)} \\ -\dfrac{2r_\varepsilon}{(R_\varepsilon^2 - r_\varepsilon^2)} \\ \dfrac{F_3^\varepsilon}{R_\varepsilon \ln \frac{r_\varepsilon}{R_\varepsilon}} \end{pmatrix}.$$

There follows:

$$\int_{C^\varepsilon_{R_\varepsilon}} u^\varepsilon \star \frac{\partial \vec{w}^\varepsilon}{\partial \rho} m_\varepsilon(\vec{\varphi}) d\sigma = \frac{2\pi R_\varepsilon}{\varepsilon^2} \vec{F}^\varepsilon \star \int_\Omega G^\varepsilon_{R_\varepsilon}(u^\varepsilon) m_\varepsilon(\varphi) dx.$$

where:

$$\frac{2\pi}{\varepsilon^2 \ln \frac{r_\varepsilon}{R_\varepsilon}} F_3^\varepsilon \star \int_\Omega G^\varepsilon_{R_\varepsilon}(u_3^\varepsilon) m_\varepsilon(\varphi) dx \underset{\varepsilon \to 0}{\sim} 2\pi \gamma_\varepsilon F_3^\varepsilon \star \int_\Omega G^\varepsilon_{R_\varepsilon}(u_3^\varepsilon) m_\varepsilon(\varphi) dx$$

$$\underset{\varepsilon \to 0}{\rightharpoonup} 2\pi \gamma F_0 \star \int_\Omega u_3^r \varphi dx.$$

Moreover, taking into account (90) we infer that:

$$\lim_{\varepsilon\to 0}\frac{4\pi r_\varepsilon R_\varepsilon}{\varepsilon^2(R_\varepsilon^2-r_\varepsilon^2)}1\star\int_\Omega G^\varepsilon_{R_\varepsilon}(u_i^\varepsilon)\otimes m_\varepsilon(\varphi)dx=0,\quad i\in\{\rho,\theta\}.$$

We conclude as in Proposition 4.7 of [13].

Moreover, there exists functions $\tilde{N}\in\mathcal{C}^1_c(\Omega,\mathcal{C}^1_{\rm per}(\tilde{Y}))$ such that

$$\tilde{N}^\varepsilon(x):=\tilde{N}\left(\frac{x}{\varepsilon}\right)=\tilde{n}^\varepsilon\quad\text{on}\quad\tilde{\Gamma}^\varepsilon\cup\Sigma^\varepsilon.$$

Set:

$$G^\varepsilon_{ij}=(u_i^\varepsilon-(u^\varepsilon\cdot\tilde{N}^\varepsilon)\tilde{N}_i^\varepsilon)\tilde{N}_j^\varepsilon,\quad\forall i,j\in\{1,2,3\}.$$

Then:

$$\int_{\tilde{\Gamma}^\varepsilon\cup\Sigma^\varepsilon}\alpha_\varepsilon(u^\varepsilon-(u^\varepsilon\cdot\tilde{n}^\varepsilon)\tilde{n}^\varepsilon)\star\vec{w}^\varepsilon m_\varepsilon(\varphi)d\sigma=\int_{\tilde{\Gamma}^\varepsilon\cup\Sigma^\varepsilon}\alpha_\varepsilon G^\varepsilon_{ij}\tilde{n}_j^\varepsilon\star w_i^\varepsilon m_\varepsilon(\varphi)d\sigma=$$

$$=\int_{\tilde{\Omega}^\varepsilon_\sharp}\frac{\partial G^\varepsilon_{ij}}{\partial x_j}\star w_i^\varepsilon m_\varepsilon(\varphi)dx$$

where:

$$\left|\int_{\tilde{\Omega}^\varepsilon_\sharp}\frac{\partial G^\varepsilon_{ij}}{\partial x_j}\star w_i^\varepsilon m_\varepsilon(\varphi)dx\right|\le C|\nabla u^\varepsilon|_{\tilde{\Omega}^\varepsilon}|\vec{w}^\varepsilon|_\Omega\|\varphi\|_\infty\le C|\vec{w}^\varepsilon|_\Omega\underset{\varepsilon\to 0}{\to}0,$$

that is:

$$\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}^\varepsilon\cup\Sigma^\varepsilon}\alpha_\varepsilon(u^\varepsilon-(u^\varepsilon\cdot\tilde{n}^\varepsilon)\tilde{n}^\varepsilon)\star\vec{w}^\varepsilon m_\varepsilon(\varphi)d\sigma=0.$$

We conclude as in Proposition 4.7 of [13]. □

Remark 5. From Definition (58) we deduce that:

$$\hat{F}_0(p)=\frac{1}{\dfrac{2\pi\gamma}{a}|\Omega|+p}.$$

There results:

$$F_0(t)=e^{-\frac{2\pi\gamma}{a}|\Omega|t},\quad\forall t>0.\tag{99}$$

5.3. The limit problem

Now, we are in a position to pass to the limit as $\varepsilon\to 0$ in the variational formulation (9)–(10). We begin with tests functions from the two-scale homogenization theory,

postponing to Propositions 5.4 and 5.5 the identification of non local limits v and q resp.

Proposition 5.3. *Set:*

$$\tilde{m} = \frac{|\tilde{Y}|}{|Y|}.$$

Then, the limit $((u,w),(v,w'),(p,q))$ *is solution of:* $(u,w) \in \mathcal{C}^1(0,T;X)$, $(p,q) \in \mathcal{C}^1(0,T;M) \times \mathcal{C}^1(0,T;L^2(\Omega))$ *and* $\forall (\varphi,\psi) \in X$, $\forall \pi \in M$,

$$\tilde{m}\frac{d}{dt}(u^r,\varphi^r)_\Omega + R\frac{a}{|\Omega|}\frac{d}{dt}(v,\varphi^r)_\Omega + (1-\tilde{m})\int_\Omega \fint_{Y^m} Au^m\varphi^m dxdy +$$

$$+\tilde{m}\beta^2\int_\Omega \fint_{\tilde{Y}} (\nabla u^r + \nabla_y w)(\nabla\varphi^r + \nabla_y\psi)dxdy + \int_\Omega\int_{\tilde{\Gamma}} \alpha(u^r - (u^r\cdot\tilde{n}_y)\tilde{n}_y)\varphi^r dxd\sigma +$$

$$-(1-\tilde{m})\int_\Omega p^m \mathrm{div}(\tilde{\varphi}^m)dx - \tilde{m}\int_\Omega \tilde{p}^r\mathrm{div}(\varphi^r)dx - \tilde{m}\int_\Omega\fint_{\tilde{Y}}(p^r-p^m)\mathrm{div}_y\psi dxdy =$$

$$= (1-\tilde{m})\int_\Omega f\tilde{\varphi}^m dx + \tilde{m}\int_\Omega f\varphi^r dx + R\fint_\Omega g\varphi^r dx. \tag{100}$$

$$\left(\frac{1}{2}-R\right)a\frac{d}{dt}\fint_\Omega q\pi^m dx + \frac{(1-\tilde{m})}{E_m}\frac{d}{dt}\int_\Omega p^m\pi^m dx + \frac{\tilde{m}}{E_r}\frac{d}{dt}\int_\Omega\fint_{\tilde{Y}} p^r\pi^r dxdy +$$

$$+(1-\tilde{m})\int_\Omega \pi^m\mathrm{div}(\tilde{u}^m)dx + \tilde{m}\int_\Omega\tilde{\pi}^r\mathrm{div}(u^r)dx + \tilde{m}\int_\Omega\fint_{\tilde{Y}}(\pi^r-\pi^m)\mathrm{div}_y(w)dxdy = 0. \tag{101}$$

Proof. Notice that (9)–(10) equivalently reads: find $(u^\varepsilon,p^\varepsilon) \in L^2(0,T;U^\varepsilon) \times L^2(0,T;L^2_0(\Omega)$ solution of: $\forall\phi\in\mathcal{C}^1_c(0,T)$, $\forall v\in U_\varepsilon$, $\forall q\in L^2_0(\Omega)$,

$$-\int_0^T \phi'(t)(u^\varepsilon(t),v)_\varepsilon dt + \int_0^T a_\varepsilon(u^\varepsilon(t),v)\phi(t)dt + \int_0^T b_\varepsilon(p^\varepsilon(t),v)\phi(t)dt = \int_0^T (g^\varepsilon,v)_\Omega + \phi(0)(u_0^\varepsilon,v)_\varepsilon,$$

$$\int_0^T \phi'(t)(p^\varepsilon(t),q)^\sharp_\varepsilon dt + \int_0^T b_\varepsilon(p^\varepsilon(t),v)\phi(t)dt + \phi(0)(p_0^\varepsilon,q)^\sharp_\varepsilon, = 0.$$

Let $\phi\in\mathcal{C}^1_c([0,T[)$, $\varphi\in\mathcal{D}(\Omega,\mathcal{C}^\infty_{\mathrm{per}}(Y))$, $\psi\in\mathcal{D}(\Omega,\mathcal{C}^\infty_{\mathrm{per}}(\tilde{Y}))$ such that $(\varphi,\psi)\in X$. Let $\hat{\psi}\in\mathcal{D}(\Omega,\tilde{H}_{\mathrm{per}}(\mathrm{div}_y,Y))$ denote any extension of ψ and set: $\forall x\in\Omega$,

$$\varphi^\varepsilon(x) := \varphi\left(x,\frac{x}{\varepsilon}\right), \quad \psi^\varepsilon(x) := \hat{\psi}\left(x,\frac{x}{\varepsilon}\right), \tag{102}$$

$$\Phi^\varepsilon := \varphi^\varepsilon + \varepsilon\psi^\varepsilon + w_0^\varepsilon(\varphi^\flat - \varphi^r - \varepsilon\psi^\varepsilon). \tag{103}$$

Direct computation yields:

$$\int_\Omega p^\varepsilon \mathrm{div}(\varphi^\varepsilon + \varepsilon\psi^\varepsilon)dx = \int_\Omega p^\varepsilon((\mathrm{div}\varphi)^\varepsilon + (\mathrm{div}_y\psi)^\varepsilon + \varepsilon(\mathrm{div}\psi)^\varepsilon)dx$$

$$\underset{\varepsilon\to 0}{\to} \int_{\Omega\times Y} p(\mathrm{div}\varphi + \mathrm{div}_y\psi)dxdy = -b(p,(\varphi,\psi)) \quad \text{a.e. in} \quad]0,T[.$$

There holds:

$$\int_{\Omega^\varepsilon} u^\varepsilon(\varphi^\varepsilon + \varepsilon\psi^\varepsilon)dx \underset{\varepsilon\to 0}{\to} \int_{\Omega^T\times Y_m} u^m\varphi^m dxdy,$$

$$\lim_{\varepsilon\to 0}\int_{\tilde\Omega^\varepsilon} \nabla u^\varepsilon\nabla(\varphi^r + \varepsilon\psi^\varepsilon)dx = \int_{\Omega\times\tilde Y}(\nabla u + \nabla_y w)(\nabla\varphi^r + \nabla_y\psi)dxdydt,$$

and:

$$\left| b\fint_{\Omega'^\varepsilon} \nabla u^\varepsilon\nabla(\varphi^r + \varepsilon\psi^\varepsilon)dx\right| \le b|\nabla u^\varepsilon|_{\tilde\Omega^\varepsilon_\sharp}|\nabla\varphi^r|_{\Omega'^\varepsilon} \le C|\nabla\varphi^r|_{\Omega'^\varepsilon} \underset{\varepsilon\to 0}{\to} 0$$

$$\fint_{\Omega'^\varepsilon} u^\varepsilon(\varphi^r + \varepsilon\psi^\varepsilon)dx = \fint_{\Omega'^\varepsilon} u^\varepsilon\varphi^r dx + \varepsilon\fint_{\Omega'^\varepsilon} u^\varepsilon\psi^\varepsilon dx$$

with:

$$\left|\fint_{\Omega'^\varepsilon} u^\varepsilon\psi^\varepsilon dx\right| \le \left(\fint_{\Omega'^\varepsilon}|u^\varepsilon|^2dx\right)^{\frac12}\left(\fint_{\Omega'^\varepsilon}|\psi^\varepsilon|^2dx\right)^{\frac12} \underset{(27)}{\le} C\left(\fint_{\Omega'^\varepsilon}|\psi^\varepsilon|^2dx\right)^{\frac12} \le C\|\psi\|_\infty$$

and

$$\fint_{\Omega'^\varepsilon} u^\varepsilon\varphi^r dx \underset{\substack{\varepsilon\to 0\\(39)}}{\to} \fint_\Omega v\varphi^r dx$$

This yields (100).

As for (101), we set:

$$\pi^\varepsilon(x) = \pi\left(x,\frac{x}{\varepsilon}\right), \quad \Theta^\varepsilon(x) = (1 - w_0^\varepsilon(x))\pi^\varepsilon(x) + w_0^\varepsilon(x)\pi^\flat(x), \quad \forall x\in\Omega.$$

for some $\pi\in M$. Then:

$$(p^\varepsilon,\pi^\varepsilon)^\sharp_\varepsilon = \int_{\Omega\setminus\Omega^\varepsilon_\flat} p^\varepsilon\pi^\varepsilon dx + \fint_{\Omega^\varepsilon_\flat} p^\varepsilon\pi^\flat dx$$

with:

$$\int_{\Omega\setminus\Omega_\flat^\varepsilon} p^\varepsilon \pi^\varepsilon \to \int_\Omega \fint_Y p\pi dxdy$$

$$\fint_{\Omega_\flat^\varepsilon} p^\varepsilon \pi^\varepsilon dx = \fint_{\Omega_\flat^\varepsilon} p^\varepsilon \pi^m dx \underset{\varepsilon\to 0}{\to} \fint_\Omega q\pi^m dx$$

Moreover:

$$\int_\Omega \pi^\varepsilon \mathrm{div}(u^\varepsilon)dx = \int_{\Omega\setminus(\mathcal{C}_\varepsilon\cup T^\varepsilon)} \pi^\varepsilon \mathrm{div}(u^\varepsilon)dx \underset{\substack{\varepsilon\to 0\\(17)}}{\to} \int_\Omega \fint_Y \pi \mathrm{div} u dxdy = -b(\pi,(u,w)).$$

We conclude as in [16]. □

Proposition 5.4. *Assume that there exists $v_0 \in L^2(\Omega^T)$ such that: $\forall \varphi \in \mathcal{C}_c^1(\Omega^T)$,*

$$\int_0^T \fint_{\Omega'^\varepsilon} u_0^\varepsilon \varphi dx \to R\int_0^T \fint_\Omega v_0\varphi dxdt.$$

Then, the limit v is solution of:

$$\frac{a}{|\Omega|}\frac{\partial v}{\partial t} + 2\pi\gamma\left(v - \frac{u_3^r}{Ra}\vec{e}_3\right) = 2\pi\gamma v_0 + \frac{g}{|\Omega|} - \frac{2\pi\gamma}{a}\int_0^t g(s)ds$$

Proof. Direct computation yields:

$$-\int_{\tilde\Omega_\sharp^\varepsilon} \rho^\varepsilon u_0^\varepsilon w^\varepsilon m_\varepsilon(\varphi)dx + \int_{\tilde\Omega_\sharp^\varepsilon} \rho^\varepsilon u^\varepsilon w_0^\varepsilon m_\varepsilon(\varphi)dx + \int_{\tilde\Gamma^\varepsilon\cup\Sigma^\varepsilon} \alpha_\varepsilon(u^\varepsilon - (u^\varepsilon\cdot\tilde n^\varepsilon)\tilde n^\varepsilon)\star w^\varepsilon m_\varepsilon(\varphi)d\sigma +$$

$$+\int_{\tilde\Omega_\sharp^\varepsilon} (-k_\varepsilon\Delta u^\varepsilon + \nabla p^\varepsilon)\star w^\varepsilon m_\varepsilon(\varphi)dx - \int_{\tilde\Omega_\sharp^\varepsilon} u^\varepsilon \star (-k_\varepsilon\Delta w^\varepsilon)m_\varepsilon(\varphi)dx =$$

$$= \int_{\Omega\setminus\Omega'^\varepsilon} f\star w^\varepsilon m_\varepsilon(\varphi)dx + \fint_{\Omega'^\varepsilon} g\star 1 m_\varepsilon(\varphi)$$

with

$$\left|\int_{\tilde\Omega_\sharp^\varepsilon\setminus\Omega'^\varepsilon} u_0^\varepsilon w^\varepsilon m_\varepsilon(\varphi)dx\right| \le |u_0^\varepsilon|_\Omega |w^\varepsilon|_\Omega \|\varphi\|_\infty \le C|w^\varepsilon|_\Omega \underset{\varepsilon\to 0}{\to} 0$$

$$\fint_{\Omega'^\varepsilon} u_0^\varepsilon w^\varepsilon m_\varepsilon(\varphi)dx = \fint_{\Omega'^\varepsilon} u_0^\varepsilon m_\varepsilon(\varphi)dx \underset{\varepsilon\to 0}{\to} R\fint_\Omega v_0\varphi dx$$

$$\left|\int_{\tilde{\Omega}^\varepsilon_\sharp\setminus\Omega'^\varepsilon} u^\varepsilon w_0^\varepsilon m_\varepsilon(\varphi)dx\right| \leq |u^\varepsilon|_\Omega |w_0^\varepsilon|_\Omega \|\varphi\|_\infty \leq C|w_0^\varepsilon|_\Omega \underset{\varepsilon\to 0}{\to} 0$$

$$⨍_{\Omega'^\varepsilon} u^\varepsilon w_0^\varepsilon m_\varepsilon(\varphi)dx = ⨍_{\Omega'^\varepsilon} u^\varepsilon m_\varepsilon(\varphi)dx \underset{\varepsilon\to 0}{\to} R⨍_\Omega v\varphi dx$$

$$\left|\int_{\tilde{\Omega}^\varepsilon_\sharp\setminus\Omega'^\varepsilon} f\star w^\varepsilon m_\varepsilon(\varphi)dx\right| \leq \ C|w^\varepsilon|_\Omega \underset{\varepsilon\to 0}{\to} 0$$

$$⨍_{\Omega'^\varepsilon} (g\star 1)m_\varepsilon(\varphi)dx \underset{\varepsilon\to 0}{\to} R⨍_\Omega (g\star 1)\varphi dx.$$

Then, Proposition 5.2 implies:

$$\frac{1}{|\Omega|}v - 2\pi\gamma F_0 \star u_3^r \vec{e}_3 = \int_0^t g(s)ds + \frac{1}{|\Omega|}v_0$$

and we conclude by derivation of this identity. □

Proposition 5.5. *The limit (p^m, q) satisfies:*

$$\frac{a}{|\Omega|}\frac{\partial q}{\partial t} + 2\pi\gamma\left(q - \left(\frac{1}{2} - R\right)^{-1}\frac{p^m}{a}\right) = 2\pi\gamma q_0 - \frac{b}{|\Omega|}\mathrm{div}(f) + 2\pi\gamma\frac{b}{a}\int_0^t \mathrm{div}(f)ds$$

Proof. Let $\phi \in \mathcal{D}(\Omega)$. Direct computation yields:

$$\int_{\tilde{\Omega}^\varepsilon_\sharp} \mathrm{div}(u^\varepsilon)\star w_3^\varepsilon m_\varepsilon(\phi)dx = \int_{\Omega^\varepsilon} k_\varepsilon\frac{\partial p^\varepsilon}{\partial t}\star w_3^\varepsilon m_\varepsilon(\phi)dx + \int_\Omega \mathrm{div}(u^\varepsilon)\star w_3^\varepsilon m_\varepsilon(\phi)dx =$$

$$= \int_{\Omega^\varepsilon} k_\varepsilon\frac{\partial p^\varepsilon}{\partial t}\star w_3^\varepsilon m_\varepsilon(\phi)dx + \int_{\Omega^\varepsilon} k_\varepsilon(-\Delta p^\varepsilon + \mathrm{div}(f))\star w_3^\varepsilon m_\varepsilon(\phi)dx + \int_{\tilde{\Omega}^\varepsilon_\sharp} \mathrm{div}(u^\varepsilon)\star w_3^\varepsilon m_\varepsilon(\phi)dx =$$

$$= \underbrace{\int_{\Omega^\varepsilon} p^\varepsilon \star k_\varepsilon\frac{\partial w_3^\varepsilon}{\partial t}m_\varepsilon(\phi)dx + \int_{\Omega^\varepsilon} p^\varepsilon\star(-k_\varepsilon\Delta w_3^\varepsilon)m_\varepsilon(\phi)dx}_{=0} + \int_{\Omega^\varepsilon} k_\varepsilon\mathrm{div}(f)\star w_3^\varepsilon m_\varepsilon(\phi)dx +$$

$$+ \int_{\tilde{\Omega}^\varepsilon_\sharp} \mathrm{div}(u^\varepsilon)\star w_3^\varepsilon m_\varepsilon(\phi)dx + \int_{\Omega^\varepsilon} \rho_\varepsilon p^\varepsilon \star w_{0,3}^\varepsilon m_\varepsilon(\phi)dx - \int_{\Omega^\varepsilon} \rho_\varepsilon p_0^\varepsilon \star w_3^\varepsilon m_\varepsilon(\phi)dx +$$

$$+ \int_{\Omega^\varepsilon} k_\varepsilon\frac{\partial p^\varepsilon}{\partial x_3}\star w_3^\varepsilon m_\varepsilon\left(\frac{\partial\phi}{\partial x_3}\right)dx + \int_{\partial\mathcal{C}_\varepsilon} p^\varepsilon|_{\Omega_\flat^\varepsilon}\star k_\varepsilon\frac{\partial w_3^\varepsilon}{\partial n_{x'}}m_\varepsilon(\phi) +$$

$$-\int_{\tilde{\Omega}_\sharp^\varepsilon} u^\varepsilon \star \nabla w_3^\varepsilon m_\varepsilon(\phi)dx - \int_{\tilde{\Omega}_\sharp^\varepsilon} u_3^\varepsilon \star w_3^\varepsilon m_\varepsilon\left(\frac{\partial \phi}{\partial x_3}\right)dx$$

where

$$\left|\int_{\tilde{\Omega}_\sharp^\varepsilon} \mathrm{div}(u^\varepsilon) \star w_3^\varepsilon m_\varepsilon(\phi)dx\right| \leq |\nabla u^\varepsilon|_{\tilde{\Omega}_\sharp^\varepsilon} |w_3^\varepsilon|_{\mathcal{C}_\varepsilon} \|\phi\|_\infty \leq C|w_3^\varepsilon|_{\mathcal{C}_\varepsilon} \underset{\varepsilon\to 0}{\to} 0$$

$$\left|\int_{\Omega^\varepsilon} k_\varepsilon \frac{\partial p^\varepsilon}{\partial x_3} \star w_3^\varepsilon m_\varepsilon\left(\frac{\partial \phi}{\partial x_3}\right)dx\right| \leq C|\nabla p^\varepsilon|_{\Omega^\varepsilon} |w_3^\varepsilon|_{\mathcal{C}_\varepsilon} \left\|\frac{\partial \phi}{\partial x_3}\right\|_\infty \leq C|w_3^\varepsilon|_{\mathcal{C}_\varepsilon} \underset{\varepsilon\to 0}{\to} 0$$

$$\left|\int_{\tilde{\Omega}_\sharp^\varepsilon} u^\varepsilon \star \nabla w_3^\varepsilon m_\varepsilon(\phi)dx\right| \leq |u^\varepsilon|_{\tilde{\Omega}_\sharp^\varepsilon \cap \mathcal{C}_\varepsilon} |\nabla w^\varepsilon|_{\mathcal{C}_\varepsilon} \|\phi\|_\infty \leq C|u^\varepsilon|_{\tilde{\Omega}_\sharp^\varepsilon \cap \mathcal{C}_\varepsilon} \underset{\varepsilon\to 0}{\to} 0$$

$$\left|\int_{\tilde{\Omega}_\sharp^\varepsilon} u_3^\varepsilon \star w_3^\varepsilon m_\varepsilon\left(\frac{\partial \phi}{\partial x_3}\right)dx\right| \leq C|u^\varepsilon|_{\tilde{\Omega}_\sharp^\varepsilon} |w_3^\varepsilon|_{\mathcal{C}_\varepsilon} \left\|\frac{\partial \phi}{\partial x_3}\right\|_\infty \leq C|w_3^\varepsilon|_{\mathcal{C}_\varepsilon} \underset{\varepsilon\to 0}{\to} 0$$

$$\int_{\Omega^\varepsilon} \rho_\varepsilon p^\varepsilon \star w_{0,3}^\varepsilon m_\varepsilon(\phi)dx \underset{\varepsilon\to 0}{\to} \left(\frac{1}{2} - R\right) a \fint_\Omega q \star \phi dx$$

$$\int_{\Omega^\varepsilon} \rho_\varepsilon p_0^\varepsilon \star w_3^\varepsilon m_\varepsilon(\phi)dx \underset{\varepsilon\to 0}{\to} \left(\frac{1}{2} - R\right) a \fint_\Omega q_0 \star \phi dx.$$

$$\int_{\Omega^\varepsilon} k_\varepsilon \mathrm{div}(f) \star w_3^\varepsilon m_\varepsilon(\phi)dx \underset{\varepsilon\to 0}{\to} b \fint_\Omega \mathrm{div}(f) \star 1 \phi dx$$

We conclude as in Proposition 5.4. □

Corollary 5.6. *The limit $((p^m, q), \tilde{u}^m)$ is solution of:*

$$\frac{a}{|\Omega|}\frac{\partial q}{\partial t} + 2\pi\gamma\left(q - \left(\frac{1}{2} - R\right)^{-1}\frac{p^m}{a}\right) = 2\pi\gamma q_0 - \frac{b}{|\Omega|}\mathrm{div}(f) + 2\pi\gamma\frac{b}{a}\int_0^t \mathrm{div}(f)ds \quad in \quad \Omega^T. \tag{104}$$

$$\left(\frac{1}{2} - R\right)\frac{a}{|\Omega|}\frac{\partial q}{\partial t} + (1-\tilde{m})\left(\frac{1}{E_m}\frac{\partial p^m}{\partial t} + \mathrm{div}(\tilde{u}^m)\right) = \tilde{m}\fint_{\tilde{Y}} \mathrm{div}_y(w)dy. \tag{105}$$

Proof. See (101). □

Corollary 5.7. *The limit $(p^r, (u^r, w))$, resp. $((p^m, \tilde{p}^r, q), (\tilde{u}^m, u^r))$, is solution of:*

$$\frac{1}{E_r}\frac{\partial p^r}{\partial t} + \mathrm{div}_y w + \mathrm{div}(u^r) = 0 \quad in \quad \Omega^T \times \tilde{Y}. \tag{106}$$

resp.:

$$\left(\frac{1}{2}-R\right)\frac{a}{|\Omega|}\frac{\partial q}{\partial t}+(1-\tilde{m})\left(\frac{1}{E_m}\frac{\partial p^m}{\partial t}+\mathrm{div}(\tilde{u}^m)\right)+\tilde{m}\left(\frac{1}{E_r}\frac{\partial \tilde{p}^r}{\partial t}+\mathrm{div}(u^r)\right)=0 \quad in \quad \Omega^T \tag{107}$$

Proof. See (101), resp. (104) and (105). □

Proposition 5.8. *The limit problem reads:*

$$(1-\tilde{m})\left(\frac{1}{E_m}\frac{\partial p^m}{\partial t}-\Delta p^m\right)+\tilde{m}\frac{F}{G}(p^m-\tilde{p}^r)+2\pi\gamma\left(\frac{p^m}{a}-\left(\frac{1}{2}-R\right)q\right)=$$

$$=-(1-\tilde{m})\mathrm{div}(f)-\left(\frac{1}{2}-R\right)\left(2\pi\gamma q_0-\frac{b}{|\Omega|}\mathrm{div}(f)+2\pi\gamma\frac{b}{a}\int_0^t\mathrm{div}(f)ds\right) \quad in \quad \Omega^T \tag{108}$$

$$\frac{a}{|\Omega|}\frac{\partial q}{\partial t}+2\pi\gamma\left(q-\left(\frac{1}{2}-R\right)^{-1}\frac{p^m}{a}\right)=2\pi\gamma q_0-\frac{b}{|\Omega|}\mathrm{div}(f)+2\pi\gamma\frac{b}{a}\int_0^t\mathrm{div}(f)ds$$

$$\frac{1}{E_r}\frac{\partial \tilde{p}^r}{\partial t}+\frac{(F-1)}{G}(\tilde{p}^r-p^m)=0 \tag{109}$$

$$p^m=\tilde{p}^r+G\mathrm{div}(u^r) \tag{110}$$

$$\frac{a}{|\Omega|}\frac{\partial v}{\partial t}+2\pi\gamma\left(v-\frac{u_3^r}{Ra}\vec{e}_3\right)=2\pi\gamma v_0+\frac{g}{|\Omega|}-\frac{2\pi\gamma}{a}\int_0^t g(s)ds \tag{111}$$

$$\frac{\partial u^r}{\partial t}-2\pi\gamma\frac{R}{\tilde{m}}\left(v-\frac{u_3^r}{Ra}\vec{e}_3\right)-\mathrm{div}(a^{\mathrm{hom}}\nabla u^r)+\nabla p^m+Cu^r=$$

$$=f-2\pi\gamma\frac{R}{\tilde{m}}v_0+\frac{R}{\tilde{m}}\left(\frac{2\pi\gamma}{a}\int_0^t g(s)ds-\frac{g}{|\Omega|}\right) \tag{112}$$

where constants F and G are defined after cell problems:

$$\int_{\tilde{Y}}\nabla_y w^{ij}\nabla_y\psi dy-\int_{\tilde{Y}}\pi^{ij}\mathrm{div}_y\psi dy=-\int_{\tilde{Y}}\frac{\partial\psi_i}{\partial y_j}dy, \quad \forall\psi\in H^1_{\mathrm{per}}(\tilde{Y}). \tag{113}$$

$$\mathrm{div}_y w^{ij}+F\delta_{ij}=0, \tag{114}$$

$$\fint_{\tilde{Y}} \pi^{ij} dy + G\delta_{ij} = 0 \tag{115}$$

$$a^{\hom}_{k\ell,ms} = \fint_{\tilde{Y}} \left(\delta_{ki}\delta_{\ell j} + \frac{\partial w_i^{k\ell}}{\partial y_j}\right)\left(\delta_{mi}\delta_{sj} + \frac{\partial w_i^{ms}}{\partial y_j}\right) dy - \fint_{\tilde{Y}} \pi^{k\ell}(\delta_{ms} + \mathrm{div}_y w^{ms}) dy \tag{116}$$

and where the matrix C is definded by

$$C_{ij} = \int_{\tilde{\Gamma}} \alpha(\delta_{ij} - \tilde{n}_{yj}\tilde{n}_{yi}) d\sigma_y.$$

Proof. Setting $\varphi = \varphi^\flat = \psi^\flat = 0$ in (100) yields:

$$\tilde{m}\beta^2 \int_\Omega \fint_{\tilde{Y}} (\nabla u^r + \nabla_y w)\nabla_y \psi dx dy - \tilde{m} \int_\Omega \fint_{\tilde{Y}} (p^r - p^m)\mathrm{div}_y \psi dx dy = 0$$

from which there results: $w = \frac{\partial u_k^r}{\partial x_\ell} w^{k\ell}(y), \quad p^r = p^m + \frac{\partial u_k^r}{\partial x_\ell}\pi^{k\ell}(y)$ with $(w^{k\ell}, \pi^{k\ell})$ solution of (113). Morover, assumption (114), resp. (115) implies:

$$\fint_{\tilde{Y}} \mathrm{div}_y w = -F\mathrm{div}(u^r) \tag{117}$$

resp.(110). Substituting (117) into (106), we infer that: $\frac{1}{E_r}\frac{\partial p^r}{\partial t} + (1-F)\mathrm{div}(u^r) = 0$ which, taking into account (110), leads to (109).

Notice that setting $A = I$ and $\varphi^r = \varphi^\flat = \psi = \psi^\flat = 0$ in (100) also yields:

$$\tilde{u}^m = -\nabla p^m + f \tag{118}$$

Then, substitution of (118), (117) and (110) into (105) yields (108). Setting

$$\psi = \frac{\partial \varphi_i^r}{\partial x_j} w^{ij},$$

in (100), we obtain (112). □

References

[1] Allaire, G.: Homogenization of the Stokes flow in a connected porous medium, Asymptotic Analysis **2**, 203-222 (1989).

[2] G.I. Barenblatt, Y.P. Zheltov and I.N. Kochina, On basic conceptions of the theory of homogeneous fluids seepage in fractured rocks (in Russian), Prikl.Mat. i Mekh. 24 (1960) 852-864.

[3] Barenblatt, G.I., Entov, V.M., Ryzhik, V.M.: Theory of Fluid Flows Through Natural Rocks, Kluwer Acad. Pub., Dordrecht, 1990.

[4] Beavers, G.S., Joseph, D.D.: Boundary conditions at a naturally permeable wall, J. Fluid Mech. **30**, 197–207 (1967).

[5] M. Bellieud, G. Bouchitté, Homogenization of elliptic problems in a fiber reinforced structure. Non local effects, *Ann. Scuola Norm. Sup. Pis Cl. Sci.(4)* **26 (3)**, (1998) 407–436.
[6] Bentalha, F. and Gruais, I. and Poliševski, D., Homogenization of a conductive suspension in a Stokes-Boussinesq flow, *Appl. Anal.*, **85 (6-7)**, 2006,811–830.
[7] F. Bentalha, I. Gruais, D. Polisevski, Diffusion process in a rarefied binary structure, *Revue Roumaine de Mathématiques Pures et Appliquées*, **52(1)**, 2007, 9–34.
[8] F. Bentalha, I. Gruais, D. Poliševski, Diffusion in a highly rarefied binary structure of general periodic shape, *Applicable Analysis* **87 (6)** (2008), 635–655.
[9] C. Boutin, Acoustics of porous media with inner resonators, *J. Acoust. Soc. Am.* **143 (6)** (2013), 4717–4729.
[10] Cioranescu, D., Saint Jean-Paulin, J.: Homogenization in open sets with holes, J. Math. Anal. Appl. **71**(2), 590–607 (1979).
[11] Ene, H.I., Poliševski, D.: Model of diffusion in partially fissured media, Z.A.M.P. **53**, 1052–1059 (2002).
[12] I. Gruais, Modelling of poro-acoustic media. Preprint hal-04667947 (2024).
[13] I. Gruais, D. Poliševski, Homogenizing a critical binary structure of finite diffusivities, *Asymptot. Anal.* **55** (2007), 85–101.
[14] I. Gruais, D. Poliševski, Homogenizing media containing a highly conductive honeycomb substructure, *Asymptot. Anal.*, **67 (1-2)**, (2010), 33–43.
[15] I. Gruais, D. Poliševski, Asymptotic heat equation for crossing superconductive thin walls, *Appl. Anal.*, **91 (11)**, (2012), 2029–2043.
[16] I. Gruais, D. Poliševski, Fluid flows through fractured porous media along Beavers-Joseph interfaces, *J. Math. Pures Appl. (9)* **102 (3)** (2014), 482–497.
[17] I. Gruais, D. Poliševski, and F.A. Stanescu, The effective permeability of fractured porous media subject to the Beavers-Joseph contact law, *Asymptot. Anal.*, **90 (3-4)** (2014), 267–280.
[18] I. Gruais, D. Poliševski, Homogenization of fluid-porous interface coupling in a biconnected fractured media, *Appl. Anal.*, **94 (8)**, (2015),1736–1747.
[19] I. Gruais, D. Poliševski, Model of two-temperature convective transfer in porous media, *Z. Angew. Math. Phys.*, **68 (6)**, 2017, Paper No. 143, 11,
[20] L. Guillopé, Analyse Fonctionnelle. Cours de l'Ecole Polytechnique de l'Université de Nantes, 2006–2007
[21] Jones, I.P.: Low Reynolds number flow past a porous spherical shell, Proc Camb. Phil. Soc. **73**, 231–238 (1973).
[22] M. Neuss-Radu, Some extensions of two-scale convergence. C.R. Acad. Sci. Paris, t. 322, Serie I (1996) 899-904.
[23] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, S.I.A.M. J.Math.Anal. 20 (1989) 608-623.
[24] Saffman, P.G.: On the boundary condition at the interface of a porous medium. Stud. Appl. Math. **50**, 93–101 (1971).
[25] E. Sanchez-Palencia, Non-Homogeneous Media and Vibration Theory. Lecture Notes in Physics, Vol. 127, Springer-Verlag, Berlin, 1980.
[26] L. Tartar, Incompressible fluid flow in a porous medium. Convergence of the homogenization process. Appendix of [25].