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Nastasia Henry, Alain Venditti

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Aix-Marseille Univ., CNRS, AMSE, Marseille, France

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ABSTRACT

To what extent protectionism affects growth and (de)stabilizes the economies? Although the impact of protectionism on growth has been widely explored without reaching a consensus, few has been said on its impact on macroeconomic stability. The present paper attempts to gauge more precisely its implications using a Barro-type (Barro, 1990) endogenous growth model with public debt and credit constraint where tariffs are a proxy of protectionism. Our main result is to show that when the debt level is high, and the share of foreign goods in total consumption is large enough, increasing tariffs may have a destabilizing effect generating some expectation coordination failures between multiple equilibria. We also exhibit some trade-off between tariffs and growth as tariffs are beneficial only to the low growth equilibrium which may only appear when the international interest rate is low enough. Finally, focusing on the local stability property, we show that the high BGP is always characterized by local indeterminacy, while the low BGP is always a saddle point. We then prove that tariffs may be responsible for the existence of large self-fulfilling fluctuations.

1. Introduction

This present paper aims to contribute to the understanding of the consequences of protectionism, proxied by tariffs (i.e. tax on imported goods), on economic growth and macroeconomic stability. Despite the promotion of international trade through various trade agreements over the last 30 years, some countries have implemented protectionist measures in response to economic shocks. A particular example is provided by the 2000s energy crisis between 2003–2008, during which the price of oil increased from 25\$ in 2002 to 160\$ in 2008. Such a productivity shock, possibly associated with a significant public debt level, has raised concerns about sustainability and led to calls for tax reforms to reduce structural deficits. More recently, the 2017 Trump election led to a strong increase in tariffs from 2018 for various goods imported from China, Europe, and other countries. In general, the central argument for protectionism (as the rise of tariffs) is to promote the production and consumption of domestic goods, and thus a relocalization of productive activities in the country. The expected effect is then to boost employment and growth but also to provide additional fiscal resources for the government. However, the impact of protectionism

has not been precisely identified yet. As discussed in the literature review provided in the next section, the implications of protectionism are still unclear. On the one hand, its impacts on economic growth have been widely explored, without reaching a consensus on the theoretical side, while being identified as detrimental to growth on the empirical side.¹ On the other hand, its implications on macroeconomic stability have been mainly disregarded.

The contributions of the paper are threefold. We first want to give clues on the possible (de)stabilizing role of protectionism, especially when economies are indebted. Then, we aim to clarify the impact of tariffs on economic growth. Finally, we want to gauge the link between tariffs and public debt and check whether there is an interplay between both.

We consider a small open economy where endogenous growth is driven by public spending as in Barro (1990).² We use the same basic formulation as in Modesto et al. (2021) where public debt is financed through taxes and external borrowing. A collateral constraint à la Hirano and Yanagawa (2017) (see also Farhi and Tirole, 2012) allows us to consider that households' borrowing on the international market

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^{*} Correspondence to: AMSE, 5 Boulevard Bourdet - CS 50498 13205, Marseille Cedex 01, France.

E-mail addresses: nastasia.henry@univ-amu.fr (N. Henry), alain.venditti@univ-amu.fr (A. Venditti).

¹ See e.g. Furceri et al. (2019).

² See also Morimoto et al. (2016).

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is limited, the loans provided by the rest of the world being proportional to the physical capital invested in the home country. Contrary to Modesto et al. (2021), we do not consider that the representative household derives utility from holding domestic bonds. However, we assume that she consumes a basket of goods, composed of domestic and foreign goods. The novelty of our approach is to add an international trade dimension through tax on imported goods.

We analyse two cases depending on the level of public debt: low and high public debt. Contrary to the standard framework of a closed economy,³ we show that when public debt is low, two BGPs exist if the international interest rate is low enough, even if the share of foreign goods into consumption and the size of tariffs remain limited.⁴ On the contrary, when the international interest rate is large, the only sustainable steady state is the low one as the debt burden is too strong to sustain a larger equilibrium associated to large government expenditures.

When public debt is high instead, we provide the main conclusions of our paper. We first show that there exists a unique stationary equilibrium in two cases: under a low international interest rate if the tax on imports and/or the share of the foreign good into total utility are low enough, and under a large international interest rate, even if the share of foreign goods and tariffs are large. In the first case, a large debt-output ratio is not compatible with the government budget constraint and cannot be sustained under a too low steady state which is then ruled out. In the second case, the debt burden is too large to allow households to coordinate on a larger equilibrium, which is necessarily characterized by larger government expenditures which cannot be sustained by the tax resources. The only sustainable steady state is then the low one characterized by moderate government spendings.

Second, we prove that multiplicity can be obtained when tariffs and the share of foreign goods in total consumption are high enough and the international interest rate is sufficiently low. Indeed, two steady states can exist, characterized by low and high growth. In such a configuration, households consume an important share of goods that are highly taxed. The government therefore earns some extra revenues that allow to sustain both a low equilibrium characterized by a low growth rate and a high equilibrium characterized by a high growth rate which remains compatible with a large debt since the debt burden is limited due to the low interest rate. In this case, there is a potential for expectations coordination problem. Indeed, the crowding out effect on private investment generated by the large debt can be more than compensated by government expenditures allowed by tariff income. It follows that even if agents expect a low growth, the related equilibrium can be self-fulfilled as they expect the government will be able to sustain the large debt burden from the tariff income and the low interest rate. We then conclude that under a low enough international interest rate, when the debt level is high, and the share of foreign goods in total consumption is large enough, increasing tariffs may have a dramatic destabilizing effect generating some expectation coordination failure between multiple equilibria and the possible existence of large self-fulfilling fluctuations

We also provide a comparative statics exercise focusing on the impact of tariffs on stationary equilibria. We emphasize that the high balanced growth rate is always negatively affected by tariffs while the low balanced growth rate is always positively affected. This difference takes its roots from the growth's origins. At the high steady state, growth is driven by productive spending from the government which is relatively large compared to capital. At the low steady state, growth is driven by private capital and thus private investment instead. Any increase in tariffs has two opposite effects. Firstly, households

can dedicate less revenue to productive investment. But at the same time, the government experiences additional resources that are used to increase public spending. Along the high growth equilibrium, the small increase of tariffs relative to the size of government spending has a limited impact on the government spending capacities that weakly increase while the tariffs strongly impact the households' income. The first effect is then dominant and as a result, growth declines. Along the low growth equilibrium on the contrary, the first effect is dominated by the second one since the increase of tariffs generates a relatively large increase in government spending that compensates the negative impact on household income. Our results therefore suggest the existence of a trade-off between tariffs and growth. Tariffs may enhance growth for the low equilibrium while they harm the high equilibrium.

Focusing on the local stability property, we show that the high BGP is always characterized by local indeterminacy, while the low BGP is always a saddle point. In the case of a high BGP, assume that along an equilibrium path the agents expect an increase in the growth rate. Due to the access to the international market and the low international interest rate, they may borrow, consume and invest more (preventing a crowding out effect) and they will expect a higher public spending. Then, for given tariffs, the government's revenue can increase significantly leading to an increase in public spending that generates a higher growth rate. The expectations are therefore self-fulfilling.

The low BGP on the contrary is characterized by different properties: low growth rate and thus a lower public debt spending to capital ratio. The credit constraint is then more tightened since the collateral needed to borrow is relatively low. The inflows of capital thus remain limited. Assume again that along an equilibrium agents are expecting an increase in the growth rate. Being more constrained, the agents consume less and decrease their investment in productive capital. Unable to rely on tariffs in this case, the government does not invest enough and growth cannot increase since growth, consumption and tariffs' income remain low. The expectations cannot be self-fulfilling and the equilibrium remains locally determinate.

The rest of this paper proceeds as follows: Section 2 proposes a literature review. Section 3 presents the model. In Section 4, we study the existence and possible multiplicity of BGPs, while Section 5 is dedicated to some comparative statics. Section 6 analyses the local stability and Section 7 concludes. All technical details are relegated to Appendix.

2. Related literature

Our paper relates and contributes to several strands of the literature on the link between international trade, macroeconomic stability and economic growth. In these literature strands the focus is given to the role of trade openness. Nishimura et al. (2010) consider a two-country, two-good, two-factor general equilibrium model with sector-specific externalities and show that some country's expectation-driven fluctuations can spread throughout the world once trade opens even if the other country has determinacy under autarky.⁵ Globalization and market integration then have destabilizing effects on a country's competitive equilibrium.⁶ On the contrary, Doi et al. (2007) formulate a two-country endogenous growth model, which explains the joint determination of long-run trade patterns and world growth rates and proves the existence and local stability of a continuum of balanced growth paths with different patterns and volumes of international trade. The destabilizing effect does not hold here: any equilibrium path starting

⁵ For some references on expectation-driven fluctuations, see Le Van et al. (2007).

⁶ See also Ghiglini (2007). Le Riche et al. (2022) derive similar results in a one-sector model of differentiated products with productive labour externalities, considering two OLG countries, one with wage rigidity and the other with full employment.

³ As studied by Chéron et al. (2019), Futagami et al. (2008), Maebayashi et al. (2017) and Minea and Villieu (2013).

⁴ The same result is found in Modesto et al. (2021).

from historically given initial international distribution of physical and human capitals converges to a balanced growth path in the continuum.

A destabilizing effect is also obtained using the Compensation hypothesis (Iversen, 2001; Down, 2007; Kim, 2007; Ehrlich and Hearn, 2014). According to this view, higher exposure to trade leads to less domestic macroeconomic stability as soon as trade partners share risks. The latter induces a higher demand for compensation through more transfers. Open countries expand security programs that inflate public expenditures, making countries even more vulnerable to shocks. In the same vein, Krugman (1993) shows that following trade openness, countries tend to specialize more in their production, increasing the geographical concentration of an industry, and being then more subject to regional shock.

On the empirical side, some evidence suggests that the link between trade openness and macroeconomic stability (i.e. growth volatility) is not straightforward, other factors playing a role. Bejan (2006) emphasizes the role of the government size in an econometric analysis based on 111 countries. Developed and developing countries exhibit different patterns. Trade openness allows for smooth volatility in developed countries whereas developing countries experience more volatility. Jansen (2004) and Cavallo and Frankel (2008) shed light on other factors, such as export concentration and product diversification for example.

Concerning the impact of trade on growth, some papers in the literature consider the role of trade openness. Ho (2017) examines the effect of externalities on the consequences of financial market globalization in a two-country growth model augmented with domestic credit market imperfections. He finds that depending on the externalities formation, financial market globalization can improve growth at the world level, or in the rich country only, and may in some cases imply that both the rich and the poor countries become locked in a stage with no meaningful growth. For the impact of tariffs on the growth rate, the results are often not conclusive. Osang and Pereira (1996) consider a small open economy where growth is endogenously driven by human capital accumulation. They examine the effects of an unanticipated increase in one of the tariff rates under different replacement regimes: a lump-sum transfer (LST) or an investment tax credit (ITC). An increase in the tariff of the consumption good is shown to not affect growth in the LST scenario while it positively affects growth under an ITC as the accumulation of capital is accelerated.⁷ Naito (2003) examines how a revenue-neutral tariff reform affects growth in an endogenous growth small open economy model with two final goods. In contrast to the previous paper, tariff reforms have ambiguous effects on growth, depending on the pattern of trade and the elasticities of substitution between the inputs and consumption of final goods. Closer to our framework, Osang and Turnovsky (2000) also analyse the effects of consumption and investment tariffs on growth. Conversely to the previous work, they develop an endogenous growth model in which the economy faces restricted access to the world capital market. A higher consumption tariff, by reducing the growth rate of consumption, harms the long-run growth rate. On the empirical side, Furceri et al. (2019) use a local projection method on a data set composed of 150 countries over the period 1963–2014 and emphasize the detrimental effect of protectionism on economic growth. Following the implementation of protectionist measures, the economy experiences a rise in unemployment and inequalities, together with a significant decrease in labour productivity. All in all, long-run growth declines.⁸

⁷ See also Chaudhry (2011) where the innovation degree of the export sector and the quality of institutions are key.

⁸ See also Bairoch (1972) and Eichengreen (1981) where protectionism affects negatively economic growth but only in the short run.

3. Theoretical framework

Our framework builds on the Barro (1990) model where production benefits from externalities due to public spending and on the Modesto et al. (2021) model of a small open economy with public debt and borrowing constraints. We consider a decentralized, continuous time intertemporal model of a small open economy composed of three types of agents: a large number of identical competitive firms, a constant population of identical infinitely lived households and a government. Firms and households operate in competitive markets, they are price takers. The government levies tariffs on imported consumption good, taxes on the global output of the country and issues public debt. Tariffs, tax revenues and debt are used to produce public services affecting the aggregate production function. We assume that domestic households can borrow on the international market subject to a borrowing constraint based on domestic private physical capital as collateral. Since we consider a small open economy, the price of imported goods and the international interest rate are taken as given. All the prices are expressed in units of the domestically produced good, the numéraire good.

3.1. Production

We consider a perfectly competitive economy where the final output Y is produced using capital K and labour L . As in Barro (1990), the production also benefits from an externality due to public services G , and is given by $Y = K^s(LG)^{1-s}$. Public spending is thus the driver of endogenous growth.⁹ Denoting output per capita as $y = Y/L$ and capital per capita as $k = K/L$, the production function becomes¹⁰:

$$y = k^s G^{1-s} \tag{1}$$

Profit maximization of firms yields the rental rate of capital $r(t)$ and the wage rate $w(t)$ as given by:

$$r(t) = sx(t)^{1-s} \tag{2}$$

$$w(t) = (1 - s)x(t)^{1-s}k(t), \tag{3}$$

where $x \equiv G/k$ is the government expenditure to capital ratio.

3.2. Households

The infinitely-lived households derive utility from a consumption bundle, $c(t)$, composed by domestic and imported foreign goods. Each consumer is initially endowed with an initial stock of private physical capital, supplies inelastically one unit of labour and the size of the population is normalized to one. Households can save through capital $k(t)$ and domestic public debt $B_h(t)$, and they can borrow from foreign savers through the international asset $d(t)$. For convenience, we will then consider negative values for the amount of international borrowing $d(t)$. Domestic public bonds and international assets are freely traded on international markets, whereas capital used in production is not mobile. As we will see later on, contrary to the formulation of Modesto et al. (2021), we do not assume that some utility is derived from holding domestic public bonds. As a result, the representative

⁹ See also Boucekkine and Ruiz-Tamarit (2008) and Brito and Venditti (2010) for Lucas-type endogenous growth models based on human capital.

¹⁰ We could consider instead that the government expenditure finances public infrastructures which would constitute the public capital that complements private capital. In such a case however, denoting by k_p the stock of public capital per capita, the production function should be $y = k^s k_p^{1-s}$ and we should introduce a relationship between public spending and public capital such that $\dot{k}_p(t) = G(t) - \delta_p k_p(t)$ with δ_p the public capital depreciation rate (see for instance Maebayashi et al., 2017). Such a formulation would introduce additional complexities without leading to significantly different qualitative results.

household will get higher returns from holding physical capital $k(t)$ and thus will decide $B_h(t) = 0$.

Besides this portfolio choice, households consume a consumption bundle that is a combination of foreign and domestic goods, denoted $c_h(t)$ and $c_f(t)$ respectively, where the two goods are imperfect substitutes. As in [Osang and Pereira \(1996\)](#), the consumption good $c(t)$ is expressed as follow:

$$c(t) = c_f(t)^\alpha c_h(t)^{1-\alpha} \tag{4}$$

with $\alpha \in [0, 1]$ the share of imported foreign goods in total consumption. In our model, we introduce tariffs τ_c that are imposed on the foreign good. Normalizing the price of the domestic good to unity, we denote P^* the price of the imported good,¹¹ and we express the total consumption spending of the household as:

$$P_c(t)c(t) = c_h(t) + (1 + \tau_c)P^*c_f(t), \tag{5}$$

Maximizing (4) subject to (5), leads to

$$c_h(t) = (1 - \alpha)P_c(t)c(t) \text{ and } c_f(t) = \frac{\alpha P_c(t)c(t)}{(1 + \tau_c)P^*}, \tag{6}$$

which implies a constant price P_c such that

$$P_c(t) = \left(\frac{1}{\alpha}\right)^\alpha \left(\frac{1}{1-\alpha}\right)^{1-\alpha} (1 + \tau_c)^\alpha P^{*\alpha} \equiv P_c. \tag{7}$$

Since P^* is exogenous, we will in the rest of the paper choose its value to normalize the price P_c to 1.

As we have assumed above, the infinitely-lived households save through capital $k(t)$ and domestic public debt $B_h(t)$ and borrow from foreign savers through some international asset. Considering \bar{d} the per capita net foreign asset (NFA) expressed in terms of foreign goods, we derive that $d = P^*\bar{d}$ represents the NFA expressed in domestic good. τ is the tax rate on income which is assumed to be constant and such that $\tau \in [0, 1)$. The international interest rate is denoted by r^* , expressed in domestic goods and we assume that it is constant and strictly positive. Physical capital being non mobile, its return is provided by the domestic real interest rate $r(t)$. On the contrary, domestic bonds and foreign assets (debt) hold by households are both traded on the international financial markets. As a result, their return is provided by the international interest rate r^* . The budget constraint of the representative household can then be expressed as:

$$c(t) + \dot{k}(t) + \dot{d}(t) + \dot{B}_h(t) = (1 - \tau)r(t)k(t) + w(t) + r^*d(t) + r^*B_h(t). \tag{8}$$

In this economy, following [Hirano and Yanagawa \(2017\)](#),¹² we assume that because of frictions in the international financial market, the representative households can pledge at most a fraction θ of the future return from investment to creditors, where $\theta \in [0, 1]$ is a parameter that captures credit market imperfection.¹³ Thus, debt repayment cannot exceed the pledgeable value $\theta(1 - \tau)r(t)k(t)$ to establish credibility in debt contracts. Considering that the amount of international borrowing is formulated as negative values, the borrowing constraint is stated as:

$$r^*d(t) \geq -\theta(1 - \tau)r(t)k(t). \tag{9}$$

We will consider in the rest of the analysis that this constraint is binding.

The intertemporal maximization program of a representative agent is then given by:

$$\max_{c(t), B_h(t), k(t), d(t)} \int_{t=0}^{+\infty} e^{-\rho t} \ln c(t) dt \tag{10}$$

¹¹ The price of the imported good is exogenously determined and supposed to be constant.

¹² See also [Farhi and Tirole \(2012\)](#) and [Kiyotaki and Moore \(1997\)](#).

¹³ See also [Boucekkine et al. \(2015, 2017\)](#) for additional references with similar formulations.

$$\text{s.t. (8), (9), } B_h(t) \geq 0 \text{ and } k(0) \text{ given.} \tag{11}$$

where $\rho > 0$ corresponds to the discount rate. In our framework, the portfolio decisions of the households are based on three assets: domestic public debt, physical capital for saving and foreign assets for borrowing. These assets are imperfect substitutes, letting the residents not be indifferent between holding physical capital, international assets and domestic public debt since borrowing on the international markets requires capital as collateral.

Our model is built on a unique dynamical equation corresponding to the budget constraint of the representative household. Therefore, it is more convenient to solve the model using the standard method of calculus of variations based on the consideration of the Euler equation. Let us then introduce the following Lagrangian:

$$\begin{aligned} \mathcal{L} = & e^{-\rho t} \ln \left((1 - \tau)r(t)k(t) + w(t) + r^*d(t) + r^*B_h(t) - \dot{k}(t) \right. \\ & \left. - \dot{d}(t) - \dot{B}_h(t) \right) \\ & + \lambda(t)[r^*d(t) + \theta(1 - \tau)r(t)k(t)] + \mu(t)B_h(t). \end{aligned}$$

$\lambda(t)$ corresponds to the Lagrange multiplier associated to the borrowing constraint while $\mu(t)$ is the Lagrange multiplier of the domestic holding of public debt.

The first order conditions are derived from the Euler equation $\frac{\partial \mathcal{L}}{\partial \omega} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\omega}}$, with $\omega = \{k, d, B_h\}$:

$$\frac{e^{-\rho t}(1 - \tau)r(t)}{c(t)} + \lambda(t)(1 - \tau)\theta r(t) = \frac{e^{-\rho t}}{c(t)} \left(\rho + \frac{\dot{c}(t)}{c(t)} \right) \tag{12}$$

$$\left(\frac{e^{-\rho t}}{c(t)} + \lambda(t) \right) r^* = \frac{e^{-\rho t}}{c(t)} \left(\rho + \frac{\dot{c}(t)}{c(t)} \right) \tag{13}$$

$$e^{-\rho t} \frac{r^*}{c(t)} + \mu(t) = \frac{e^{-\rho t}}{c(t)} \left(\rho + \frac{\dot{c}(t)}{c(t)} \right) \tag{14}$$

Any solution needs also to satisfy the transversality conditions:

$$\lim_{t \rightarrow +\infty} e^{-\rho t} \omega(t) \frac{\partial \mathcal{L}}{\partial \dot{\omega}(t)} = 0 \Leftrightarrow \lim_{t \rightarrow +\infty} e^{-\rho t} \frac{\omega(t)}{c(t)} = 0 \tag{15}$$

with $\omega(t) = \{k(t), d(t), B_h(t)\}$.

Using (12) and (13), the Lagrange multiplier associated to the borrowing constraint is given by:

$$\lambda(t) = \frac{e^{-\rho t}}{c(t)} \left(\frac{(1-\tau)r(t)-r^*}{r^*-\theta(1-\tau)r(t)} \right), \tag{16}$$

and we easily get a condition that ensures a binding borrowing constraint with $\lambda(t) > 0$:

$$(1 - \tau)r(t) > r^* > \theta(1 - \tau)r(t). \tag{17}$$

We consider in the rest of the paper that condition (17) is satisfied and thus that

$$d(t) = -\frac{\theta}{r^*}(1 - \tau)r(t)k(t) \tag{18}$$

On top of the transversality condition, the foreign debt holding by households $d(t)$ needs also to satisfy the No Ponzi Game (NPG) condition

$$\lim_{t \rightarrow +\infty} d(t)e^{-r^*t} = 0 \tag{19}$$

The Cobb–Douglas formulation (1) implies $r(t)k(t) = sy(t)$ and, as we will show later on, we easily get along a Balanced Growth Path $\dot{d}(t)/d(t) = \dot{y}(t)/y(t) = \dot{k}(t)/k(t) = \dot{c}(t)/c(t)$. Substituting (16) in (12), we obtain the consumption growth rate:

$$\frac{\dot{c}(t)}{c(t)} = \frac{(1-\tau)r(t)(1-\theta)}{1-\theta(1-\tau)r(t)/r^*} - \rho \tag{20}$$

and the NPG condition for the households is equivalent to

$$\frac{(1-\tau)r(t)(1-\theta)}{1-\theta(1-\tau)r(t)/r^*} - \rho < r^* \tag{21}$$

In our model three assets coexist as they are, *a priori*, not perfect substitutes. This property explains why the growth rate of consumption

is not constant. If the assets would have been perfect substitutes, assuming then that private capital $k(t)$ is freely tradable on the international market, the consumption growth rate would have been constant and equal to $(1 - \tau)r^* - \rho$ with $r(t) = r^*$.¹⁴

On the contrary, as borrowing on the international market is subject to a collateral constraint which is binding, the decision to hold physical capital and domestic public debt depends on a trade-off. Using (20), we derive indeed the expected return of capital:

$$\frac{(1-\tau)r(t)(1-\theta)}{1-\theta(1-\tau)r(t)/r^*} \tag{22}$$

We clearly observe that the marginal benefit of investing one unit of capital is equal to $(1 - \tau)r(t)(1 - \theta)$ while the expected cost of investing one unit of capital is given by $1 - \theta(1 - \tau)r(t)/r^*$. The allocation between physical capital, domestic debt and the foreign asset then depends on this ratio. Glancing out the second Lagrange multiplier $\mu(t)$, we may then derive the optimal holding of domestic debt $B_h(t)$. Using indeed (13), (14) and (16) under condition (17), we find that $\lambda(t)r^* = \mu(t) > 0$. Hence, we prove from Kuhn–Tucker conditions that domestic agents do not hold any domestic bonds (i.e. $B_h(t) = 0$), and thus all public debt is held only by foreigners, i.e. $B(t) = B_f(t)$. Domestic households can save with physical capital which is non-tradable and characterized by a higher net return compared to domestic bonds, i.e. $(1 - \tau)r(t) > r^*$, since the borrowing constraint is binding. The domestic public asset is then strictly dominated by capital and the optimal holding of domestic bonds is equal to zero.

Notice on the contrary that foreign households are indifferent between holding the international asset and domestic public debt because both assets have the same international return r^* .

3.3. Government

The government levies tax on production and on imported goods and issues debt to finance public spending G . The government budget constraint is then given by:

$$\dot{B}(t) = G(t) + r^*B(t) - \tau(u(t) + r(t)k(t)) - \tau_c P^* c_f(t), \tag{23}$$

where $\dot{B}(t)$ corresponds to the newly-issued government bonds, $G(t)$ the amount of public expenditures in time t and $r^*B(t)$ the debt repayment. A priori, the total amount of bonds $B(t)$ is split between domestic holding $B_h(t)$ and foreign holding $B_f(t)$, namely $B(t) = B_h(t) + B_f(t)$. But as shown previously $B_h(t) = 0$ and we have $B(t) = B_f(t)$. Considering \tilde{B}_f the amount of domestic bonds held by foreigners expressed in terms of foreign goods, we derive that $B_f = P^* \tilde{B}_f$ represents the amount of domestic bonds held by foreigners expressed in domestic good.

In most small open economies, debt stability is a major concern for governments as bankruptcy may lead to an economic crisis. We will follow Futagami et al. (2008), introducing a debt stabilization rule based on a long-run target determined by the government. However, contrary to Futagami et al. (2008),¹⁵ where the government is assumed to adjust its debt-to-capital ratio, we proceed as in Minea and Villieu (2013), Chéron et al. (2019) and Morimoto et al. (2016) and consider that the government adjusts its debt-to-GDP ratio according to the following rule¹⁶:

$$\frac{\dot{b}(t)}{b(t)} = -\phi \left(1 - \frac{b^*}{b(t)} \right), \text{ with } \phi > 0 \tag{24}$$

where $b \equiv B/y$ is the ratio of debt over GDP. Two policy parameters, b^* and ϕ , allow the government to design its policy. Firstly, b^* represents the long-run target to be reached such that any difference with this

¹⁴ Morimoto et al. (2016) consider a small open economy where all assets are perfectly substitutable and reach this conclusion.

¹⁵ See also Maebayashi et al. (2017).

¹⁶ See also Nishimura et al. (2015) for a different formulation where the debt-to-GDP ratio is assumed to be constant and the tax rate on GDP is non-linear.

threshold requires a debt adjustment. Secondly, the adjustment of public debt is calibrated with the policy parameter ϕ . If $b(t) > b^*$, the government reduces its debt by 100ϕ percent of the difference $b(t) - b^*$. If ϕ is large (small), the government adjusts $b(t)$ to the target at a fast (slow) pace.

Once these policy parameters have been set, the evolution of the public debt-to-GDP ratio is perfectly characterized: it converges to the stationary state b^* . But this should also have some repercussions on the dynamics of public debt $B(t)$ which has to satisfy the government budget constraint (23). Since taxes and tariffs are given, the only adjustment variable is then $G(t)$. In this way we obtain a feedback rule that links the amount of public spending $G(t)$ at any time t with the other dynamic variables involved in Eq. (23). As a result Eqs. (23) and (24) are perfectly consistent.

Remark 1. There is a vast literature proposing different fiscal rules ensuring debt stabilization in models with a monetary authority that sets the nominal interest rate by following a Taylor rule as in Schmitt-Grohé (2004) and Schmitt-Grohé and Uribe (2004, 2007). For instance, Benhabib and Eusepi (2005) consider two different fiscal policies. The first one is a balanced budget rule that keeps the total amount of real debt constant. The second is a fiscal rule requiring taxes to respond to deviations of real bonds from a target. In the case of a “passive” fiscal rule, the growth rate of government debt is lower than the real interest rate and thus, the government sets fiscal policy to satisfy its intertemporal budget constraint. In the case of an “active” fiscal rule, the government conducts fiscal policy disregarding the effects on its intertemporal budget constraint so that the price level needs to adjust to guarantee the solvency of the fiscal authority. Our simpler formulation, which does not take into account a monetary authority, is actually similar to a “passive” fiscal rule based on direct debt adjustments by the government implying a growth rate of public debt which will be shown to be equal to the growth rate of GDP.

3.4. Intertemporal equilibrium

Before describing the intertemporal equilibrium, we need to examine the market-clearing conditions required in each period t : the markets of the two productive factors, i.e. capital and labour. The good market requires the amount of consumption plus that of investment and public expenditure to be equal to the total production plus net import. In the domestic bonds markets, supply coincides with public liabilities, and finally the foreign liabilities market.

The market clearing conditions in the inputs markets are derived from the expressions of the equilibrium prices $r(t)$ and $w(t)$ as given by (2) and (3) and imply

$$y(t) = r(t)k(t) + w(t).$$

The good market clearing condition allows to derive the amount of exports $ex(t)$ as

$$ex(t) = y(t) - \dot{k}(t) - c_h(t) - G(t).$$

Since imports are obviously given by

$$im(t) = P^* c_f(t),$$

we can derive the balance of payment $BP(t)$ equilibrium (recalling that the debt based on the foreign asset $d(t)$ is formulated as negative values):

$$BP(t) = ex(t) - im(t) + r^*d(t) - \dot{d}(t) - r^*B_f(t) + \dot{B}_f(t) = 0$$

or equivalently:

$$\dot{d}(t) = y(t) - \dot{k}(t) - c_h(t) - G(t) - P^*c_f(t) + r^*d(t) - r^*B_f(t) + \dot{B}_f(t)$$

Substituting this expression in the representative household’s budget constraint (8) using (5) with $P_c = 1$ and the fact that $B_h(t) = 0$ yields

$$\dot{B}_f(t) + \tau_c P^* c_f(t) + \tau y(t) = G(t) + r B_f(t)$$

Solving this expression with respect to $G(t)$ using $\dot{B}(t) = \dot{B}_f(t)$ obviously shows that, as a direct consequence of the Walras law, the government budget constraint (23) is satisfied. Since $b(t) = B(t)/y(t)$, we then derive using (24) that

$$\frac{\dot{b}(t)}{b(t)} = \frac{\dot{B}_f(t)}{B_f(t)} - \frac{\dot{y}(t)}{y(t)} = -\phi \left(1 - \frac{b^*}{b(t)}\right)$$

Once the growth rate of $y(t)$ will be determined, this equation is cleared with the growth rate of public debt hold by foreign households $\dot{B}_f(t)/B_f(t)$ equal to $\dot{y}(t)/y(t)$ along a Balanced Growth Path with $b(t) = b^*$. As for domestic households, on top of the transversality condition, the government debt and thus the domestic debt holding of foreign households have also to satisfy the NPG condition

$$\lim_{t \rightarrow +\infty} B(t)e^{-r^*t} = \lim_{t \rightarrow +\infty} B_f(t)e^{-r^*t} = 0 \tag{25}$$

Along a Balanced Growth Path, this property is ensured again if condition (21) holds.

Let us then denote $v = c/y$ the consumption as a proportion of GDP.

Lemma 1. *The intertemporal equilibrium is determined by the following three dynamical equations:*

$$\begin{aligned} \frac{\dot{x}(t)}{x(t)} &= \frac{\left(\frac{r^* - \theta(1-\tau)sx(t)^{1-s}}{r^*}\right) \left[\frac{x(t)^s - \tau}{b(t)} + r - \frac{\alpha\tau_c}{1+\tau_c} \frac{v(t)}{b(t)} + \phi \left(1 - \frac{b^*}{b(t)}\right)\right] - x(t)^{1-s}[(1-\tau)(1-\theta)s - v(t)]}{(1-s)} \\ &\equiv X(b(t), x(t), v(t)) \\ \frac{\dot{v}(t)}{v(t)} &= \frac{r^*(1-\tau)sx(t)^{1-s}(1-\theta)}{r^* - \theta(1-\tau)sx(t)^{1-s}} - \rho - \frac{x(t)^s - \tau}{b(t)} - r^* + \frac{\alpha\tau_c}{1+\tau_c} \frac{v(t)}{b(t)} - \phi \left(1 - \frac{b^*}{b(t)}\right) \\ &\equiv V(b(t), x(t), v(t)) \\ \frac{\dot{b}(t)}{b(t)} &= -\phi \left(1 - \frac{b^*}{b(t)}\right), \text{ with } \phi > 0. \end{aligned}$$

Proof. See Appendix A.1

We have a three-dimensional dynamical system which involves three variables which are a priori all jumpable: $b(t)$, $x(t)$ and $v(t)$. The initial stocks of public debt $B(0)$ and capital per capita $k(0)$ are both predetermined. However, considering that $y(t) = x(t)^{1-s}k(t)$, we easily derive that the possible jumps of $x(0)$ and $b(0)$ at time $t = 0$ are not independent. Indeed the ratio $b(0) = B(0)/k(0) = B(0)/(x(0)^{1-s}k(0))$ cannot jump freely once $x(0)$ has been chosen. It follows that among our three variables, only two can jump freely in the dynamical system provided in Lemma 1, $x(t)$ and $v(t)$, and thus $b(t)$ has to be considered as a pre-determined variable. Any intertemporal equilibrium path also needs to satisfy the transversality conditions (described in Section 3.2).

4. The balanced growth path (BGP): uniqueness versus multiplicity

In this section, we analyse the conditions for the existence of unique or multiple BGPs. We show that the results are driven by four elements. The share of foreign goods in total consumption and tariffs are central to the existence of a Balanced Growth Path. Depending on their values, they promote multiplicity. The level of the debt-to-GDP ratio is also crucial, as initially shown by Modesto et al. (2021). We finally shed light on the importance of the international interest rate which has a strong impact on the existence and multiplicity BGPs.

A BGP is a steady state of the dynamical system provided in Lemma 1, i.e. is a stationary solution (b, v, x) solving $\dot{b}(t) = \dot{v}(t) = \dot{x}(t) = 0$. Along the BGP, the following equality is satisfied:

$$\Gamma(x) = H(x), \tag{26}$$

where:

$$\Gamma(x) \equiv \frac{r^*(1-\tau)sx^{1-s}(1-\theta)}{r^* - \theta(1-\tau)sx^{1-s}} - \rho, \tag{27}$$

$$H(x) \equiv \frac{x^s - \tau}{b^*} + r^* - \frac{\alpha\tau_c \left[(1-\tau)(1-s)x^{1-s} + \rho \frac{r^* - \theta(1-\tau)sx^{1-s}}{r^*} \right]}{(1+\tau_c)b^*x^{1-s}} \tag{28}$$

Recall from Eq. (20) that $\Gamma(x)$ corresponds to the growth rate of consumption. Moreover $H(x)$ can be expressed as follows:

$$H(x) = \frac{1}{b^*} \left[x^s - \tau - \frac{\alpha\tau_c}{(1+\tau_c)} v + r^* b^* \right] \tag{29}$$

with v the consumption-to-GDP ratio. Recall now that a primary surplus (deficit) is obtained if and only if $\tau y + \tau_c P^* c_f - G > (<) 0$. It is therefore immediate to derive that a stationary solution x features a primary surplus if $x^s - \tau - \alpha\tau_c v / (1 + \tau_c) < 0$, and a primary deficit if $x^s - \tau - \alpha\tau_c v / (1 + \tau_c) > 0$. As a result the first part of the expression between brackets in (29) corresponds to the primary balance. Considering that $r^* b^*$ corresponds to the debt burden, $H(x)$ is a measure of the total deficit of the government. Therefore, Eq. (26) shows that at the equilibrium, the growth rate should be proportional to the total deficit.

Using (2) and the binding constraint (9) we necessarily have $\underline{x} < x < \bar{x}$ with:

$$\bar{x} \equiv \left(\frac{r^*}{(1-\tau)s\theta}\right)^{\frac{1}{1-s}} \text{ and } \underline{x} \equiv \left(\frac{r^*}{(1-\tau)s}\right)^{\frac{1}{1-s}}. \tag{30}$$

We introduce the following assumption:

Assumption 1. $r^* \in (\rho, \bar{r})$, with $\bar{r} \equiv \tau^{\frac{1-s}{s}}(1-\tau)s$, $\tau < 1/2$ and $\theta > s/(2-s)$.

The condition $r^* > \rho$ ensures that the growth rate (27) is positive whatever the value of θ , the restriction on the tax rate τ is empirically justified, and the condition on θ simplifies the analysis, ensuring that $\Gamma(x)$ is a convex function.¹⁷

Consider then the transversality conditions:

$$\lim_{t \rightarrow +\infty} e^{-\rho t} k(t)/c(t) = 0 \text{ and } \lim_{t \rightarrow +\infty} e^{-\rho t} d(t)/c(t) = 0. \tag{31}$$

Along a BGP characterized by a constant growth rate $\Gamma(x)$ as given by (27) with $\underline{x} < x < \bar{x}$ and such that $k(t)/k(t) = d(t)/d(t) = \dot{y}(t)/y(t) = \dot{c}(t)/c(t)$, we get $k(t) = k(0)e^{\Gamma(x)t}$, $d(t) = d(0)e^{\Gamma(x)t}$, $y(t) = y(0)e^{\Gamma(x)t}$ and $c(t) = c(0)e^{\Gamma(x)t}$. It follows therefore that

$$\lim_{t \rightarrow +\infty} e^{-\rho t} k(0)/c(0) = 0 \text{ and } \lim_{t \rightarrow +\infty} e^{-\rho t} d(0)/c(0) = 0 \tag{32}$$

and the transversality conditions hold along a BGP. Moreover, the growth rate of public debt hold by foreign households satisfies $\dot{B}(t)/B(t) = \dot{B}_f(t)/B_f(t) = \Gamma(x)$ and thus also satisfies its transversality condition. Finally, the NPG condition for domestic households and the government as given by (21) is satisfied along a BGP if $\Gamma(x) < r^*$, or equivalently if

$$x < \left(\frac{r^*(r^* + \rho)}{(1-\tau)s(r^* + \rho\theta)}\right)^{\frac{1}{1-s}} \equiv x^{max} \tag{33}$$

Straightforward computations then show that $x^{max} \in (\underline{x}, \bar{x})$. It follows that any admissible steady state x^* solution of Eq. (26) must satisfy $x^* \in (\underline{x}, x^{max})$ for the NPG condition to hold.

4.1. The case of large debt: a fundamental role of tariffs

In the following Proposition, we first consider the case of a high enough debt-output ratio. We show that there is a unique BGP, as illustrated in Fig. 1, if the share α of imported foreign goods in total consumption is low enough, or if the tariff τ_c on imported goods is low enough, or if the international interest rate r^* is large. On the contrary, for large values of both α and τ_c , two BGPs may occur when the international interest rate r^* is low enough.

Proposition 1. *Under Assumption 1, there exist $\bar{\rho} > 0$, $\bar{\theta} \in (0, 1)$, $\bar{b} > \hat{b} > 0$, $1 > \bar{\alpha}(\theta) > \hat{\alpha}(\theta) > 0$, $\bar{\tau}_c(\theta) > \hat{\tau}_c(\theta) > 0$ and $\hat{r} \in (\rho, \bar{r})$ such that when $\rho < \bar{\rho}$, $b^* \in (\hat{b}, \bar{b})$ and $\theta \in (0, \bar{\theta})$, the following cases hold:*

¹⁷ All our results on the existence, uniqueness and multiplicity of the steady states could be obtained even under $\theta < s(2-s)$, but at the cost of cumbersome technical details.

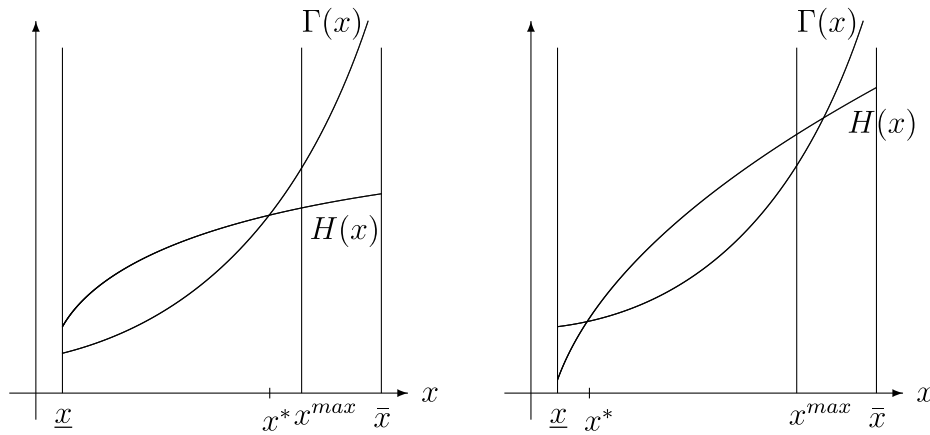


Fig. 1. Uniqueness of BGP with $r^* \in (\rho, \hat{r})$ or $r^* \in (\hat{r}, \bar{r})$.

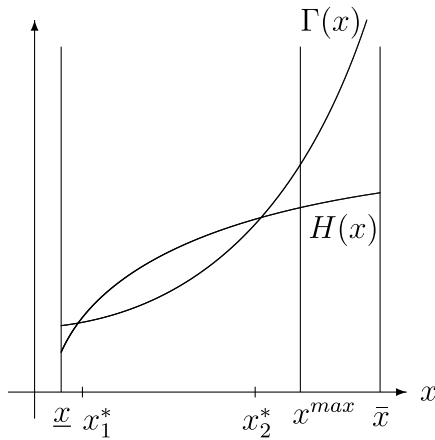


Fig. 2. Multiplicity of BGP with $r^* \in (\rho, \hat{r})$.

1. There is a unique steady state $x^* \in (\underline{x}, x^{max})$ solution of (26) in the following cases:

- (a) for any $\tau_c \geq 0$ if $r^* \in (\rho, \hat{r})$ and $\alpha \leq \bar{\alpha}(\theta)$,
- (b) if $r^* \in (\rho, \hat{r})$, $\alpha > \bar{\alpha}(\theta)$ and $\tau_c < \bar{\tau}_c(\theta)$,
- (c) if $r^* \in (\hat{r}, \bar{r})$, $\alpha \in (\bar{\alpha}(\theta), \bar{\alpha}(\theta))$ and $\tau_c > \bar{\tau}_c(\theta)$,
- (d) if $r^* \in (\hat{r}, \bar{r})$, $\alpha > \bar{\alpha}(\theta)$ and $\tau_c \in (\bar{\tau}_c(\theta), \bar{\tau}_c(\theta))$.

2. There exists $\epsilon > 0$ such that there are two steady states $x_1^*, x_2^* \in (\underline{x}, x^{max})$ solutions of (26), with $x_1^* < x_2^*$, if $r^* \in (\rho, \hat{r})$, $\alpha > \bar{\alpha}(\theta)$ and $\tau_c \in (\bar{\tau}_c(\theta), \bar{\tau}_c(\theta) + \epsilon)$.

Moreover, any steady state is always characterized by a primary surplus.

Proof. See Appendix A.2.

In a closed economy with perfectly substitutable assets (public debt and capital), it has been shown by Minea and Villieu (2013) that the multiplicity of BGPs is ruled out under a log-linear utility function in consumption. Indeed, if agents expect an increase in public expenditures, this will induce a higher future income. To finance this increase in public spending, a larger debt emission is required, which crowds out private investment harming future income. Then, expectations may not be self-fulfilling and uniqueness is obtained.

In contrast, in our framework, multiplicity is driven first by the coexistence of two key mechanisms, (i) the inflow of international assets and (ii) the existence of an investment multiplier due to the credit constraint with collateral, and second, by the existence of tariffs. As a small open economy can import international funds, the crowding-out effect is no longer relevant. Therefore, higher public spending may

now be compatible with an increase in productive investment. The effect on growth is magnified by the collateral role of capital which generates an investment multiplier. In this case, an expected increase in public spending can be self-fulfilling because of higher future income and growth, which sustain a long-run equilibrium with larger public spending. However, in Modesto et al. (2021), where households derive utility from holding domestic bonds, it is shown that this mechanism strongly depends on the size of public debt. Indeed, in the case of high debt, they show that no matter the preference parameter for domestic debt, the uniqueness of the BGP holds. A too large debt requires a large enough growth rate to be sustainable.

As Proposition 1 makes clear, the international trade dimension is central in our framework. When public debt is high, the share of foreign goods and tariffs drive the existence of two BGPs. We note that when domestic households do not consume an important share of goods, or when tariffs are low enough, a unique BGP exists. This configuration is similar to the framework of Modesto et al. (2021). We also show that the value of the international interest rate r^* is crucial as uniqueness also holds when r^* is too large. As illustrated by Fig. 1, a high steady state is associated with a low interest rate r^* while a low steady state arises under a large r^* . This property is crucial to explain why uniqueness or multiplicity may arise.

On the one hand, to understand why uniqueness occurs under these conditions, we rewrite the government budget constraint (20) as $\Gamma(x)b^* = G/y - \tau + r^*b^* - \frac{\alpha\tau_c}{(1+\tau_c)}v$. When τ_c and/or α is low, for a too low x , growth is not sufficient to allow the repayment of a high level of debt, even if the international interest rate is low as in cases 1.(a) and (b). Hence, a large debt-output ratio is not compatible with the government budget constraint and cannot be sustained, so a too low steady state cannot exist. We then recover the result of Modesto et al. (2021) and the uniqueness of a large BGP. Similarly, and even if the share of foreign goods into consumption and tariffs are large, when the international interest rate is too strong as in cases 1.(c) and (d), the debt burden is too large to allow households to coordinate on a larger equilibrium necessarily characterized by larger government expenditures which cannot be sustained by the tax resources. The only sustainable steady state is then the low one characterized by moderate government spendings.¹⁸

On the other hand, multiplicity can be obtained when tariffs and the share of foreign goods in total consumption are high enough and the international interest rate is sufficiently low. In fact, two steady states

¹⁸ It is worth noticing however that if the international interest rate is high and the share of foreign goods into consumption is large, too large tariffs may prevent the existence of any equilibria. Indeed, tariffs are equivalent to externalities, and too large externalities are detrimental to the existence of equilibrium.

can exist, characterized by low and high growth as shown in Fig. 2. In such a configuration, households consume an important share of goods that are highly taxed. The government therefore earns some extra revenues that allow it to sustain both a low equilibrium characterized by a low growth rate and a high equilibrium characterized by a high growth rate which remain compatible with a large debt since the debt burden is limited due to the low interest rate. Henceforth, when two BGPs coexist, the economy may be located at either the low (x_1^*) or high growth (x_2^*) steady state. In this case, there is a potential for expectations coordination problem. Indeed, the crowding out effect on private investment generated by the large debt can be more than compensated by government expenditures allowed by tariff income. So, even if agents expect a low growth, the related equilibrium can be self-fulfilled as they expect the government will be able to sustain the large debt burden from the tariff income and the low interest rate. We then conclude that under a low enough international interest rate, when the debt level is high, and the share of foreign goods in total consumption is large enough, increasing tariffs may have a dramatic destabilizing effect generating some expectation coordination failure between multiple equilibria and the possible existence of large self-fulfilling fluctuations.

It is also important to notice that no matter whether uniqueness or multiplicity hold, any steady state is necessarily characterized by a primary surplus. This property is derived from the NPG condition which ensures that the government will reimburse its debt in the long run. Such a capacity must rely on a primary surplus allowing at least to cover the debt burden.

4.2. The case of low debt: tariffs do not really matter

Let us focus now on the case where public debt is low, i.e. $b^* < \hat{b}$. Unlike the previous case, we show that the uniqueness or multiplicity of BGPs do not depend on tariffs.

Proposition 2. Under Assumption 1, let $b^* < \hat{b}$. Then there exist $\tilde{\alpha}(\theta) \in (0, 1)$, $\tilde{\tau}_c(\theta) > 0$ and $\hat{r} \in (\rho, \bar{r})$ such that when either $\alpha \leq \tilde{\alpha}(\theta)$ or $\alpha > \tilde{\alpha}(\theta)$ and $\tau_c < \tilde{\tau}_c(\theta)$,

1. There is a unique steady state $x^* \in (\underline{x}, x^{max})$ solution of (26) for any $r^* \in (\hat{r}, \bar{r})$.
2. There exists $\epsilon > 0$ such that there are two steady states $x_1^*, x_2^* \in (\underline{x}, x^{max})$ solutions of (26), with $x_1^* < x_2^*$, if $r^* \in (\hat{r} - \epsilon, \hat{r})$.

Moreover, any steady state is always characterized by a primary surplus.

Proof. See Appendix A.3

Interestingly, when public debt is low enough, two BGPs exist if the international interest rate is low enough, even though the share of foreign goods into consumption and the size of tariffs remain limited.¹⁹ To understand this result, let us consider again the government budget constraint (20) rewritten as $\Gamma(x)b^* = G/y - \tau + r^*b^* - \frac{\alpha\tau_c}{(1+\tau_c)}v$. If the debt-output ratio is low enough, the government budget constraint is sustainable even with a low growth rate, at least as long as the international interest rate is low enough. This then explains the existence of the low steady state (x_1^*). In contrast, at a high steady state (x_2^*), the growth rate is high enough to sustain the government budget whatever the level of debt. Given a sufficiently low level of public debt, the multiplicity of BGPs is essentially explained by the coexistence of the same two key mechanisms: (i) the inflow of international assets and (ii) the existence of an investment multiplier due to the credit constraint with collateral. Compared to the case with high debt, the role of tariffs income is here not crucial as it just appears as a complement of resources allowing to support the two steady states. We observe indeed

¹⁹ As already mentioned previously, if tariffs are too large there may not exist any steady state.

that even if $\alpha = 0$, the two steady states still exist as initially shown by Modesto et al. (2021) (under a low preference parameter for domestic debt). It is however important to notice that the multiplicity disappears under a too large international interest rate and the only sustainable steady state is the low one. The intuition is the same as in the case of high debt: the debt burden is then too large to sustain a larger equilibrium associated to large government expenditures. However, as we will see in the next section, tariffs have a crucial role on the value of the long-run growth rate.

5. Comparative statics: a trade-off between tariffs and growth

We now provide comparative statics. We focus on the behaviour of the equilibrium when tariffs increase, considering the two equations characterizing the intertemporal equilibrium, namely $\Gamma(x)$ and $H(x)$. As already mentioned, $\Gamma(x)$ corresponds to the growth rate of the economy, while $H(x)$ corresponds to the total deficit of the government (as shown by (29)). Notice that only $H(x)$ depends on tariffs and its impact appears to be negative.

Proposition 3. Under Assumption 1, consider Propositions 1 and 2.

1. In all cases with a unique steady state $x^* \in (\underline{x}, x^{max})$ and $r^* \in (\rho, \hat{r})$,

$$\frac{\partial x^*}{\partial \tau_c} < 0 \tag{34}$$

2. In all cases with a unique steady state $x^* \in (\underline{x}, x^{max})$ and $r^* \in (\hat{r}, \bar{r})$,

$$\frac{\partial x^*}{\partial \tau_c} > 0 \tag{35}$$

3. In all cases with two steady states $x_1^*, x_2^* \in (\underline{x}, x^{max})$ with $x_1^* < x_2^*$ and $r^* \in (\rho, \hat{r})$,

$$\frac{\partial x_1^*}{\partial \tau_c} > 0 \text{ and } \frac{\partial x_2^*}{\partial \tau_c} < 0 \tag{36}$$

Proof. See Appendix A.4

In Figs. 3 and 4, we depict an increase of tariffs:

The increase of tariffs allows for a decrease in the deficit of the government. However, it has different impacts on growth depending on the amount of public spending relative to capital. At the low steady state x_1^* with a relatively low public spending over capital, any increase of tariffs is pro-growth. At the high steady state x_2^* , with relatively large public spending over capital, this increase is detrimental to growth. We argue that this difference comes from the origin of growth. At x_2^* , growth is driven by productive spending from the government which is relatively large compared to capital. On the contrary, at x_1^* , growth is driven by private capital and thus private investment. Following an increase in tariffs two effects counteract. Firstly, households can dedicate less revenue to productive investment and this harms growth. At the same time, the government experiences additional resources that are used to increase public spending, boosting growth. The source of growth now matters. At x_2^* , even though growth comes mainly from productive spending, the small increase of tariffs relative to the size of government spending has a limited impact on the government spending capacities that weakly increase while the tariffs strongly impact the households' income. Therefore, the first effect dominates the second one and growth declines. At x_1^* on the contrary, the first effect is dominated by the second one. Indeed, even though growth is driven by private investment, the increase in tariffs generates a relatively large increase in government spending that compensates for the negative impact on household income. Our results therefore suggest the existence of a trade-off between tariffs and growth. Tariffs may enhance growth for the low equilibrium while they harm the high equilibrium.

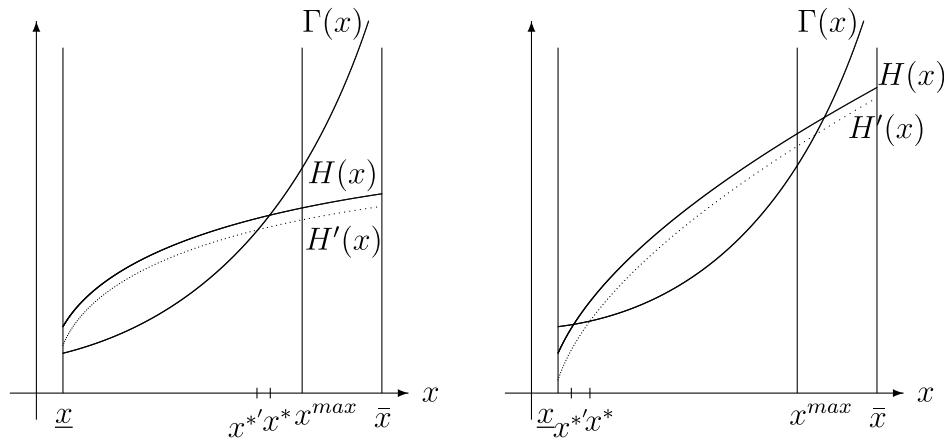


Fig. 3. The impact of an increase of τ_c under uniqueness with $r^* \in (\rho, \hat{r})$ or $r^* \in (\hat{r}, \bar{r})$. Note: In this figure we display the effects of increasing tariffs. Dotted curve represents $H(x)$ following an increase of tariffs, while the solid $H(x)$ curve represents $H(x)$ before the increase of tariffs.

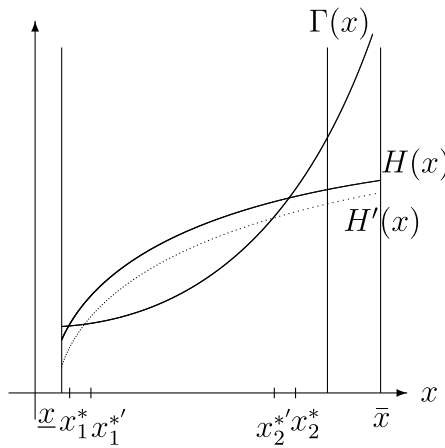


Fig. 4. The impact of an increase of τ_c under multiplicity with $r^* \in (\rho, \hat{r})$.

6. Local stability analysis

We now investigate the local stability properties of the BGPs. Let us consider the three-dimensional dynamical system as given in Lemma 1 and that can be written as follows:

$$\begin{aligned} \dot{b}(t) &= -\phi(b(t) - b^*) \\ \dot{v}(t) &= V(b(t), x(t))v(t) \\ \dot{x}(t) &= X(b(t), v(t), x(t))x(t). \end{aligned} \tag{37}$$

For the local stability analysis, we linearize this three-dimensional dynamical system around each steady state. From the eigenvalues of the Jacobian matrix, we determine the properties of the steady states. Since the equation which is driving public debt is linear, we derive that $-\phi$ is a negative eigenvalue. As explained in Section 3.4, $b(t)$ has to be considered as a pre-determined variable as only $x(t)$ and $v(t)$ can jump freely at date $t = 0$. It follows that the Blanchard–Khan conditions ensure that if the two remaining eigenvalues have a positive real part, then the steady state (b^*, v^*, x^*) is *saddle-point stable*, while, if λ_2 and/or λ_3 has negative real part, then the steady state is *locally*

Table 1
Local stability analysis.

BGP existence	Local stability properties
Unique BGP	Locally indeterminate if $r^* \in (\rho, \hat{r})$ or saddle-point if $r^* \in (\hat{r}, \bar{r})$
Two BGPs	Global indeterminacy

indeterminate. Following the conditions presented in Propositions 1 and 2, we formulate the following results:

Proposition 4. Under Assumption 1, consider Propositions 1 and 2. Then the following results hold:

1. In all cases where uniqueness holds, the steady state $x^* \in (\underline{x}, x^{max})$ is either locally indeterminate if $r^* \in (\rho, \hat{r})$ or saddle-point stable if $r^* \in (\hat{r}, \bar{r})$.
2. In all cases with two steady states $x_1^*, x_2^* \in (\underline{x}, x^{max})$ with $x_1^* < x_2^*$, the high steady state x_2^* is locally indeterminate and the low one x_1^* is saddle-point stable.

Proof. See Appendix A.5

When the balance growth path is unique, the steady state is locally indeterminate and characterized by sunspot fluctuations around it when the international interest rate r^* is low, while it is saddle-point stable when the international interest rate r^* is large. If two BGPs coexist, the lowest steady state x_1^* is a saddle-point stable while x_2^* is locally indeterminate. Global indeterminacy with large expectation-driven fluctuations then occurs under multiplicity.

We can resume the local stability analysis in Table 1:

The main result of this paper is then to show that when debt is large, the fact that domestic households do consume an important share of foreign goods under large enough tariffs is the key ingredient to explain global indeterminacy, i.e. the existence of expectations coordination failures that may lead to the occurrence of large sunspot fluctuations.

Let us present the mechanisms at stake in the two cases: uniqueness and multiplicity. We first discuss the existence of self-fulfilling expectations when the BGP is *unique* under a low enough international interest rate. In this case the BGP is characterized by a large growth rate. Consider an equilibrium along which agents suddenly formulate expectations about a possible higher growth rate. Since assets

are imperfect substitutes and due to the access to the international market, agents may borrow, consume and invest more (preventing a crowding out effect). The investment multiplier allows to achieve a higher growth rate which leads to increased expenditure of the government. Moreover, the consumption of foreign goods increases, tariff income is quite large, and the debt burden is limited due to the low interest rate. All these mechanisms highlight the possibility of self-fulfilling expectations leading to multiple transitional paths and expectation-driven fluctuations when the BGP is unique.

When the BGP is *unique* under a large international interest rate, the economy is now characterized by a low growth rate and thus by a low government spending-to-capital ratio. The credit constraint is then more tightened since the collateral needed to borrow is relatively lower. Assume again that along an equilibrium, agents are expecting an increase in the growth rate. Being constrained to get fewer inflows of capital, they cannot increase significantly their consumption and their investment in productive capital. At the same time, the government becomes more constrained, being unable to rely on sufficient revenues, since growth, consumption and tariffs' income remain low while the debt burden is large. As a result, the government does not invest more and growth cannot increase. Therefore, the expectations cannot be self-fulfilling.

When *two* BGPs coexist, a low steady state x_1^* coexists with a high one, x_2^* . Nevertheless, x_1^* and x_2^* keep the same local stability properties, x_1^* being saddle-point stable while x_2^* is locally indeterminate, as all the mechanisms previously mentioned remain the same. However, due to expectations coordination failures already mentioned earlier, the existence of global indeterminacy with two steady states leads to possible large self-fulfilling fluctuations.

7. Conclusion

In this paper, we have considered a small open economy with endogenous growth driven by public spending where agents consume some foreign goods subject to tariffs and are affected by some borrowing constraint. The government finances its expenditures through taxes on income, imported goods and debt issuance. Tariffs are here considered as a proxy of protectionism.

Our main result has been to prove that when debt is sufficiently high, the trade dimension is central. Indeed, under a low enough international interest rate, when agents consume a sufficiently large share of the foreign good and tariffs are high enough, two BGPs coexist while uniqueness holds without foreign good consumption. The high BGP is always negatively affected by some tariff increases and is always locally indeterminate. On the contrary, the low BGP is always positively affected by some increase in tariffs and is always saddle-point stable. We then show that tariffs may have a dramatic destabilizing effect generating some expectation coordination failure between multiple equilibria and the possible existence of large self-fulfilling fluctuations. We also exhibit some trade-off between tariffs and growth as tariffs are beneficial only to the low growth equilibrium.

CRedit authorship contribution statement

Nastasia Henry: Formal analysis. **Alain Venditti:** Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix

A.1. Proof of Lemma 1

Using the equilibrium prices (2) and (3), the constraint on public debt (24) and $G/y = x^s$, we derive from Eq. (23)

$$x(t) = \frac{\dot{B}(t)}{k(t)} + \tau x(t)^{1-s} + \frac{\tau_c P^* c_f(t)}{k(t)} - \frac{rB(t)}{k(t)} \tag{38}$$

Let us introduce the variable $v \equiv c/y$. From (6) we get

$$\frac{P^* c_f(t)}{k(t)} = \frac{\alpha \tau_c}{(1+\tau_c)} \frac{c(t)}{k(t)} = \frac{\alpha \tau_c}{1+\tau_c} v(t)x(t)^{1-s}$$

Substituting this expression into (38) and using the fact that $b/b = B/B - \dot{y}/y$ give the growth rate of production:

$$\frac{\dot{y}(t)}{y(t)} = \frac{x(t)^s - \tau}{b(t)} + r^* - \frac{\alpha \tau_c}{1+\tau_c} \frac{v(t)}{b(t)} + \phi \left(1 - \frac{b^*}{b(t)} \right) \tag{39}$$

and the growth rate of consumption (20) can be rewritten as:

$$\Gamma(x(t)) \equiv \frac{\dot{c}(t)}{c(t)} = \frac{\dot{c}_h(t)}{c_h(t)} = \frac{r^*(1-\tau)sx(t)^{1-s}(1-\theta)}{r^s - \theta(1-\tau)sx(t)^{1-s}} - \rho. \tag{40}$$

Moreover, a binding credit constraint (9) means that $d = -\theta(1-\tau)sy/r$ and also implies that $\dot{d}/d = \dot{y}/y$. Using these results and (8), we then derive the growth rate of capital:

$$\begin{aligned} \frac{\dot{k}(t)}{k(t)} &= x(t)^{1-s} [(1-\tau)(1-\theta)s - v(t)] + \frac{\theta(1-\tau)s}{r^*} x(t)^{1-s} \frac{\dot{y}(t)}{y(t)} \\ &= x(t)^{1-s} [(1-\tau)(1-\theta)s - v(t)] \\ &\quad + \frac{\theta(1-\tau)s}{r^*} x(t)^{1-s} \left[\frac{x(t)^s - \tau}{b(t)} + r^* - \frac{\alpha \tau_c}{1+\tau_c} \frac{v(t)}{b(t)} + \phi \left(1 - \frac{b^*}{b(t)} \right) \right] \end{aligned} \tag{41}$$

Note that $x = G/k = (y/k)^{\frac{1}{1-s}}$. Using (39) and (40), we easily get:

$$\begin{aligned} \frac{\dot{x}(t)}{x(t)} &= \frac{\left(\frac{r^s - \theta(1-\tau)sx(t)^{1-s}}{r^*} \right) \left[\frac{x(t)^s - \tau}{b(t)} + r^* - \frac{\alpha \tau_c}{1+\tau_c} \frac{v(t)}{b(t)} + \phi \left(1 - \frac{b^*}{b(t)} \right) \right]}{(1-s)} \\ &\quad - \frac{x(t)^{1-s} [(1-\tau)(1-\theta)s - v(t)]}{(1-s)} \\ &\equiv X(b(t), v(t), x(t)). \end{aligned} \tag{42}$$

Recalling that $v \equiv c/y$, we finally get:

$$\begin{aligned} \frac{\dot{v}(t)}{v(t)} &= \frac{r^*(1-\tau)sx(t)^{1-s}(1-\theta)}{r^s - \theta(1-\tau)sx(t)^{1-s}} - \rho - \frac{x(t)^s - \tau}{b(t)} - r^* + \frac{\alpha \tau_c}{1+\tau_c} \frac{v(t)}{b(t)} - \phi \left(1 - \frac{b^*}{b(t)} \right) \\ &\equiv V(b(t), v(t), x(t)). \end{aligned} \tag{43}$$

A.2. Proof of Proposition 1

From (27)–(28) we derive:

$$\Gamma'(x) = \frac{r^{s-2}(1-\tau)s(1-s)(1-\theta)x^{-s}}{[r^s - \theta(1-\tau)sx^{1-s}]^2} > 0 \text{ and } H'(x) = s \frac{x^{s-1}}{b^*} + \frac{\rho \alpha \tau_c (1-s)}{(1+\tau_c)b^* x^{2-s}} > 0 \tag{44}$$

We also easily see that $H''(x) < 0$, while $\Gamma''(x)$ has the same sign than $\theta(1-\tau)(2-s)x^{1-s} - r^*$. Since $x > \underline{x}$, $\Gamma''(x) > 0$ is ensured by Assumption 1. In addition, we have:

$$\begin{aligned} \Gamma(\bar{x}) &= +\infty, \quad \Gamma(x^{max}) = r^* \text{ and } \Gamma(\underline{x}) = r^* - \rho > 0 \\ H(\bar{x}) &= \frac{\left(\frac{r^*}{(1-\tau)s\theta} \right)^{\frac{s}{1-s} - \tau} b^*}{+} - \frac{\alpha \tau_c (1-\tau)(1-s)}{(1+\tau_c)b^*} \\ H(x^{max}) &= \frac{\left(\frac{r^*(r^* + \rho)}{(1-\tau)s(r^* + \rho\theta)} \right)^{\frac{s}{1-s} - \tau} b^*}{+} - \frac{\alpha \tau_c (1-\tau)}{(1+\tau_c)b^*(r^* + \rho)} \\ &\quad \times [(1-s)(r^* + \rho) + \rho s(1-\theta)] \\ H(\underline{x}) &= \frac{\left(\frac{r^*}{(1-\tau)s} \right)^{\frac{s}{1-s} - \tau} b^*}{+} - \frac{\alpha \tau_c (1-\tau)}{(1+\tau_c)b^* r^*} [(1-s)r^* + \rho s(1-\theta)] \end{aligned}$$

We start to analyse the case of uniqueness and we then focus on the case with multiplicity.

1 - Since $\Gamma(x)$ is convex and $H(x)$ is concave, there is a unique solution $x \in (\underline{x}, \bar{x})$ in two configurations: (i) if $\Gamma(x^{max}) > H(x^{max})$ and $\Gamma(\underline{x}) < H(\underline{x})$, or (ii) if $\Gamma(x^{max}) < H(x^{max})$ and $\Gamma(\underline{x}) > H(\underline{x})$. We consider in Proposition 1 the case of a large enough debt target b^* as this is the only configuration where tariffs have an impact of the number of equilibria.

(i) Consider the condition $\Gamma(x) < H(x)$. When $\alpha = 0$ or $\tau_c = 0$, $\Gamma(x) < H(x)$ if and only if $b^* > \hat{b}$ with

$$\hat{b} \equiv \frac{\tau - \left[\frac{r^*}{(1-\tau)s} \right]^{\frac{s}{1-s}}}{\rho} \quad (45)$$

Consider now $\alpha > 0$ and $\tau_c > 0$. Two cases then arise:

(a) $\Gamma(x) < H(x)$ for any $\tau_c \in (0, 1)$ if $\alpha < \tilde{\alpha}(\theta)$ with

$$\tilde{\alpha}(\theta) \equiv \frac{r^* \rho (b^* - \hat{b})}{(1-\tau)[(1-s)r^* + \rho s(1-\theta)]}$$

Moreover, straightforward computations show that for any $\theta \in (0, 1)$, $\tilde{\alpha}(\theta) < 1$ if $b^* \in (\hat{b}, \bar{b}_1)$ with

$$\bar{b}_1 = \hat{b} + \frac{(1-\tau)(1-s)}{\rho} \quad (46)$$

(b) On the contrary if $\alpha > \tilde{\alpha}(\theta)$, $\Gamma(x) < H(x)$ for any $\tau_c < \bar{\tau}_c(\theta)$ with

$$\bar{\tau}_c(\theta) \equiv \frac{r^* \rho (b^* - \hat{b})}{(1-\tau)[(1-s)r^* + \rho s(1-\theta)](\alpha - \tilde{\alpha}(\theta))}$$

Consider now the condition $\Gamma(x^{max}) = r^* > H(x^{max})$ which is equivalent to

$$h(r^*) = \left(\frac{r^*(r^* + \rho)}{(1-\tau)s(r^* + \rho\theta)} \right)^{\frac{s}{1-s}} - \tau - \frac{\alpha \tau_c (1-\tau)}{(1+\tau_c)(r^* + \rho)} [(1-s)(r^* + \rho) + \rho s(1-\theta)] < 0 \quad (47)$$

Notice that $h(r^*)$ is strictly increasing. As $r^* > \rho$, we easily derive that inequality (47) holds when $r^* = \rho$ if and only if

$$\rho < \frac{(1-\tau)s(1+\theta)}{2} \left(\tau + \frac{\alpha \tau_c (1-\tau)}{2(1+\tau_c)} [2(1-s) + s(1-\theta)] \right)^{\frac{1-s}{s}} \equiv \bar{\rho}$$

Moreover, since $r^* < \bar{r}$ we have

$$h(\bar{r}) = \tau \left[\left(\frac{\frac{1-s}{\tau} (1-\tau)s + \rho}{\tau \frac{1-s}{\tau} (1-\tau)s + \rho\theta} \right)^{\frac{s}{1-s}} - 1 \right] - \frac{\alpha \tau_c (1-\tau) \left[(1-s)\tau \frac{1-s}{\tau} (1-\tau)s + \rho(1-s\theta) \right]}{(1+\tau_c) \left(\tau \frac{1-s}{\tau} (1-\tau)s + \rho \right)} > 0$$

if

$$\alpha < \frac{1+\tau_c}{\tau_c} \frac{\tau \left[\left(\frac{\frac{1-s}{\tau} (1-\tau)s + \rho}{\tau \frac{1-s}{\tau} (1-\tau)s + \rho\theta} \right)^{\frac{s}{1-s}} - 1 \right] \left(\tau \frac{1-s}{\tau} (1-\tau)s + \rho \right)}{(1-\tau) \left[(1-s)\tau \frac{1-s}{\tau} (1-\tau)s + \rho(1-s\theta) \right]}$$

Since $(1+\tau_c)/\tau_c \in (1, +\infty)$ we then get $h(\bar{r}) > 0$ for any $\tau_c \in (0, 1)$ if

$$\alpha < \frac{\tau \left[\left(\frac{\frac{1-s}{\tau} (1-\tau)s + \rho}{\tau \frac{1-s}{\tau} (1-\tau)s + \rho\theta} \right)^{\frac{s}{1-s}} - 1 \right] \left(\tau \frac{1-s}{\tau} (1-\tau)s + \rho \right)}{(1-\tau) \left[(1-s)\tau \frac{1-s}{\tau} (1-\tau)s + \rho(1-s\theta) \right]} \equiv \tilde{\alpha}(\theta)$$

But we may also have $h(\bar{r}) > 0$ when $\alpha > \tilde{\alpha}(\theta)$ if

$$\tau_c < \frac{\tau \left[\left(\frac{\frac{1-s}{\tau} (1-\tau)s + \rho}{\tau \frac{1-s}{\tau} (1-\tau)s + \rho\theta} \right)^{\frac{s}{1-s}} - 1 \right] \left(\tau \frac{1-s}{\tau} (1-\tau)s + \rho \right)}{(1-\tau) \left[(1-s)\tau \frac{1-s}{\tau} (1-\tau)s + \rho(1-s\theta) \right] (\alpha - \tilde{\alpha}(\theta))} \equiv \bar{\tau}_c(\theta)$$

We need to show that $1 > \tilde{\alpha}(\theta) > \bar{\alpha}(\theta)$. We notice that $\tilde{\alpha}(1) = 0 < \bar{\alpha}(1)$ and $\tilde{\alpha}(0) > \bar{\alpha}(0)$ if $b^* < \bar{b}_2$ with

$$\bar{b}_2 = \hat{b} + \frac{\tau \left[\left(\frac{\frac{1-s}{\tau} (1-\tau)s + \rho}{\tau \frac{1-s}{\tau} (1-\tau)s} \right)^{\frac{s}{1-s}} - 1 \right] \left(\tau \frac{1-s}{\tau} (1-\tau)s + \rho \right)}{(1-s)\tau \frac{1-s}{\tau} (1-\tau)s + \rho} \quad (48)$$

Let us then denote $\bar{b} = \min\{\bar{b}_1, \bar{b}_2\}$. It follows that if $b^* \in (\hat{b}, \bar{b})$, there exists $\bar{\theta} \in (0, 1)$ such that $\tilde{\alpha}(\theta) > \bar{\alpha}(\theta)$ for any $\theta \in (0, \bar{\theta})$. Assumption 1 ensures $\tilde{\alpha}(\theta) < 1$. We also need to compare $\bar{\tau}_c(\theta)$ and $\tilde{\tau}_c(\theta)$. Obvious computations easily show that if $b^* \in (\hat{b}, \bar{b})$ and $\theta \in (0, \bar{\theta})$, then $\bar{\tau}_c(\theta) > \tilde{\tau}_c(\theta)$. Therefore, if $\rho < \bar{\rho}$, $\theta \in (0, \bar{\theta})$, $b^* \in (\hat{b}, \bar{b})$ and either $\alpha \leq \tilde{\alpha}(\theta)$, or $\alpha \in (\tilde{\alpha}(\theta), 1)$ and $\tau_c < \tilde{\tau}_c(\theta)$, there exists $\hat{r} \in (\rho, \bar{r})$ such that $\Gamma(x^{max}) = r^* > H(x^{max})$ if and only if $r^* \in (\rho, \hat{r})$. It is worth noticing that when $\alpha \in (\tilde{\alpha}(\theta), 1)$ and $\tau_c = \tilde{\tau}_c(\theta)$, then $\hat{r} = \bar{r}$. It follows

therefore that when $\alpha \in (\tilde{\alpha}(\theta), 1)$ and $\tau_c \geq \tilde{\tau}_c(\theta)$, $h(r^*) < 0$ and thus $\Gamma(x^{max}) = r^* > H(x^{max})$ for any $r^* \in (\rho, \bar{r})$.

(ii) We study now the conditions $\Gamma(x^{max}) < H(x^{max})$ and $\Gamma(x) > H(x)$. Under $b^* > \hat{b}$, $\Gamma(x) > H(x)$ holds if $\alpha > \tilde{\alpha}(\theta)$ and $\tau_c > \tilde{\tau}_c(\theta)$. Moreover, assuming also that $\alpha < \tilde{\alpha}(\theta)$ or $\alpha > \tilde{\alpha}(\theta)$ and $\tau_c < \tilde{\tau}_c(\theta)$, $\Gamma(x^{max}) = r^* < H(x^{max})$ holds if and only if $r^* \in (\hat{r}, \bar{r})$.

2. The existence of two steady states can only be obtained if $\Gamma(x^{max}) > H(x^{max})$ and $\Gamma(x) > H(x)$. As shown previously, under $b^* > \hat{b}$, this requires $r^* \in (\rho, \hat{r})$, $\alpha \in (\tilde{\alpha}(\theta), \bar{\alpha}(\theta))$ and $\tau_c > \tilde{\tau}_c(\theta)$, or $\alpha \in (\tilde{\alpha}(\theta), 1)$ and $\tau_c \in (\tilde{\tau}_c(\theta), \bar{\tau}_c(\theta))$. Notice that if $\tau_c = \tilde{\tau}_c(\theta)$ then $\Gamma(x) = H(x)$ and \bar{x} is a steady state. By continuity, there exists $\epsilon > 0$ such that if $\tau_c \in (\tilde{\tau}_c(\theta), \tilde{\tau}_c(\theta) + \epsilon)$, there exist two stationary solutions $x_1, x_2 \in (\underline{x}, \bar{x})$ of (26) with $x_1 < x_2$.

Steady state characterization. We characterize the steady state of the economy. Considering (39) and (40) and solving for v yields:

$$v = \frac{1}{x^{1-s}} \left[(1-\tau)(1-s)x^{1-s} + \rho \frac{r^* - \theta(1-\tau)sx^{1-s}}{r^*} \right] \quad (49)$$

We can express the primary deficit as:

$$DP(x) = x^s - \tau - \frac{\alpha \tau_c}{(1+\tau_c)} \left[(1-\tau)(1-s)x^{1-s} + \rho \frac{r^* - \theta(1-\tau)sx^{1-s}}{r^*} \right] \quad (50)$$

It is easy to notice that the equality $\Gamma(x) = H(x)$ can be equivalently written $[\Gamma(x) - r^*]b^* = DP(x)$. Since any steady state has to satisfy the NPO condition, namely $\Gamma(x) < r^*$, we conclude that any steady state is always characterized by a primary surplus.

A.3. Proof of Proposition 2

Consider the computations provided in the proof of Proposition 1. If $b^* < \hat{b}$ we get $\Gamma(x) > H(x)$ for any $\alpha \in (0, 1)$ and any $\tau_c > 0$. Moreover, we have $\Gamma(x^{max}) = r^* < H(x^{max})$ for any $r^* \in (\hat{r}, \bar{r})$ if either $\alpha \leq \tilde{\alpha}(\theta)$, or $\alpha > \tilde{\alpha}(\theta)$ and $\tau_c < \tilde{\tau}_c(\theta)$. In this case there exists a unique steady state. Under either $\alpha \leq \tilde{\alpha}(\theta)$, or $\alpha > \tilde{\alpha}(\theta)$ and $\tau_c < \tilde{\tau}_c(\theta)$, since when $r^* = \hat{r}$ we have $\Gamma(x^{max}) = r^* = H(x^{max})$, we conclude that there exists $\epsilon > 0$ such that when $r^* \in (\hat{r} - \epsilon, \hat{r})$, $\Gamma(x^{max}) = r^* > H(x^{max})$ and there necessarily exist two steady states. As shown in Proposition 1, any steady state is also necessarily characterized by a primary surplus.

A.4. Proof of Proposition 3

Since $H(x) = \Gamma(x)$ at a steady state and tariffs appear only in $H(x)$, we have:

$$\frac{dx}{d\tau_c} = \frac{\partial H / \partial \tau_c}{\Gamma'(x) - H'(x)} = - \frac{\alpha \left[(1-\tau)(1-s)x^{1-s} + \rho \frac{r^* - \theta(1-\tau)sx^{1-s}}{r^*} \right]}{(1+\tau_c)^2 b^* x^{1-s} [\Gamma'(x) - H'(x)]} \quad (51)$$

In the case of a unique BGP x^* , we know that $H'(x) < \Gamma'(x)$ and thus $dx^*/d\tau_c < 0$. In the case of two steady states, the impact of the tariffs depends on the difference $H'(x) - \Gamma'(x)$. At x_1 , since $H'(x_1) > \Gamma'(x_1)$, we get $dx_1/d\tau_c > 0$, while at x_2 , since $H'(x_2) < \Gamma'(x_2)$, we get $dx_2/d\tau_c < 0$.

A.5. Proof of Proposition 4

Linearizing the dynamical system as given in Lemma 1 around a steady state (b^*, v^*, x^*) gives the following Jacobian matrix

$$J = \begin{pmatrix} -\phi & 0 & 0 \\ V_1(b^*, v^*, x^*)v^* & V_2(b^*, v^*, x^*)v^* & V_3(b^*, v^*, x^*)v^* \\ X_1(b^*, v^*, x^*)x^* & X_2(b^*, v^*, x^*)x^* & X_3(b^*, v^*, x^*)x^* \end{pmatrix}$$

We easily derive

$$\begin{aligned} D &= -\phi [V_2(b^*, v^*, x^*)X_3(b^*, v^*, x^*) - V_3(b^*, v^*, x^*) \\ &\quad \times X_2(b^*, v^*, x^*)] x^* v^* \\ \mathcal{T} &= -\phi + V_2(b^*, v^*, x^*)v^* + X_3(b^*, v^*, x^*)x^* \\ \mathcal{S} &= -\phi [V_2(b^*, v^*, x^*)v^* + X_3(b^*, v^*, x^*)x^*] \\ &\quad + [V_2(b^*, v^*, x^*)X_3(b^*, v^*, x^*) - V_3(b^*, v^*, x^*)X_2(b^*, v^*, x^*)] x^* v^* \end{aligned}$$

It follows that the eigenvalues of J are solution of the following polynomial

$$P(\lambda) = \lambda^3 - \mathcal{T}\lambda^2 + S\lambda - D$$

$$= (\lambda + \phi) \left[\lambda^2 - \lambda(\mathcal{T} + \phi) - \frac{D}{\phi} \right]$$

We then get three eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ such that

$$\lambda_1 = -\phi$$

$$\lambda_2 + \lambda_3 = \mathcal{T} + \phi = V_2(b^*, v^*, x^*)v^* + X_3(b^*, v^*, x^*)x^*$$

$$\lambda_2 \lambda_3 = -\frac{D}{\phi} = \left[V_2(b^*, v^*, x^*)X_3(b^*, v^*, x^*) - V_3(b^*, v^*, x^*) \times X_2(b^*, v^*, x^*) \right] x^* v^*$$

Note first that

$$X(b, v, x) = \frac{\left(\frac{r^* - \theta(1-\tau)sx^{1-s}}{r^*} \right) [\Gamma(x) - V(b, v, x)] - x^{1-s} [(1-\tau)(1-\theta s) - v]}{1-s}$$

Straightforward computations give

$$V_2(b^*, v^*, x^*) = \frac{\alpha\tau_c}{(1-\alpha)(1+\tau_c)b^*}$$

$$V_3(b^*, v^*, x^*) = \Gamma'(x^*) - \frac{sx^{*s-1}}{b^*}$$

$$X_2(b^*, v^*, x^*) = \frac{-\left(\frac{r^* - \theta(1-\tau)sx^{*s-1-s}}{r^*} \right) V_2(b^*, v^*, x^*) + \frac{x^{*1-s}}{1-\alpha}}{1-s}$$

$$X_3(b^*, v^*, x^*) = -\frac{\theta(1-\tau)sx^{*s-1-s}}{r^*} \Gamma(x^*) + \frac{\left(\frac{r^* - \theta(1-\tau)sx^{*s-1-s}}{r^*} \right) [\Gamma'(x^*) - V_3(b^*, v^*, x^*)]}{1-s} - x^{*s-1} [(1-\tau)(1-\theta s) - v]$$

with

$$\Gamma(x^*) = \frac{r^*(1-\theta)(1-\tau)sx^{*s-1-s}}{r^* - \theta(1-\tau)sx^{*s-1-s}} - \rho$$

Note that at the steady state we get

$$x^{*s-1} [(1-\tau)(1-\theta s) - v^*] = \left(\frac{r^* - \theta(1-\tau)sx^{*s-1-s}}{r^*} \right) \Gamma(x^*) \tag{52}$$

We then get after simplifications

$$X_3(b^*, v^*, x^*)x^* = \frac{s}{1-s} \left(\frac{r^* - \theta(1-\tau)sx^{*s-1-s}}{r^*} \right) \frac{x^{*s}}{b^*} - \Gamma(x^*)$$

Straightforward computations then yield

$$\lambda_2 \lambda_3 = \frac{x^{*2-s} v^*}{(1-s)(1+\alpha)} [H'(x^*) - \Gamma'(x^*)]$$

Consider **Propositions 1** and **2**. In the cases where there exists a unique steady state, we have the following two distinct configurations:

- when $r^* \in (\rho, \hat{r})$ we have $H'(x^*) - \Gamma'(x^*) < 0$. It follows that $\lambda_2 > 0$ and $\lambda_3 < 0$. Since $\lambda_1 = -\phi < 0$, we conclude that two eigenvalues are negative and one is positive implying that the steady state is locally indeterminate.

- when $r^* \in (\hat{r}, \bar{r})$ we have $H'(x^*) - \Gamma'(x^*) > 0$. It follows that λ_2 and λ_3 have the same sign. We then need to study the sign of $\lambda_2 + \lambda_3 = \mathcal{T} + \phi$ and thus to compute $V_2(b^*, v^*, x^*)v^* + X_3(b^*, v^*, x^*)x^*$. Let us write $X(b^*, v^*, x^*)$ as follows

$$X(b^*, v^*, x^*) = \frac{\Phi(x^*)}{1-s}$$

with $\Phi(x) = B(x) \left[A(x) - \frac{\alpha\tau_c v^*}{(1-\alpha)(1+\tau_c)b^*} \right] - x^{1-s} [(1-\tau)(1-\theta s) - v^*]$, and

$$A(x) = \frac{x^s - \tau}{b^*} + r^* + \phi \left(1 - \frac{b^*}{b} \right)$$

$$B(x) = \frac{r^* - \theta s(1-\tau)x^{1-s}}{r^*}$$

Recall that at the steady state $b^* = b$. We obviously get

$$X_3(b^*, v^*, x^*) = \frac{\Phi'(x^*)}{1-s}$$

From this we can compute

$$\Phi'(x^*) = B'(x^*) \left[A(x^*) - \frac{\alpha\tau_c v^*}{(1+\tau_c)b^*} \right] + A'(x^*)B(x^*) - (1-s)x^{*s-1} [(1-\tau)(1-\theta s) - v^*]$$

We easily derive

$$\mathcal{A}'(x^*) = s \frac{x^{*s-1}}{b^*} > 0$$

$$B'(x^*) = -\frac{\theta(1-s)s(1-\tau)x^{*s-1-s}}{r^*} < 0 \tag{53}$$

At the steady state we get $\Phi(x^*) = 0$ which implies

$$\frac{v^* x^{*s-1}}{1-\alpha} = x^{*s-1} (1-\tau)(1-\theta s) - \frac{B(x^*)}{x^*} \left[A(x^*) - \frac{\alpha\tau_c v^*}{(1+\tau_c)b^*} \right]$$

Substituting all this into the expression of $\Phi'(x^*)$ yields

$$\Phi'(x^*) = \mathcal{A}'(x^*)B(x^*) + \left[A(x^*) - \frac{\alpha\tau_c v^*}{(1+\tau_c)b^*} \right] \left[B'(x^*) - \frac{(1-s)}{x^*} B(x^*) \right]$$

Considering that

$$B'(x^*) - \frac{(1-s)}{x^*} B(x^*) = -\frac{\theta(1-s)s(1-\tau)x^{*s-1-s}}{r^*} - \frac{(1-s)}{x^*} \frac{r^* - \theta s(1-\tau)x^{*s-1-s}}{r^*} = -\frac{(1-s)}{x^*}$$

we get

$$\Phi'(x^*) = \mathcal{A}'(x^*)B(x^*) - \frac{(1-s)}{x^*} \left[A(x^*) - \frac{\alpha\tau_c v^*}{(1+\tau_c)b^*} \right]$$

Recalling that for any steady state we have $\Gamma(x^*) = A(x^*)$ and

$$\mathcal{A}'(x^*) = H'(x^*) - \frac{\rho\alpha\tau_c(1-s)}{(1+\tau_c)b^* x^{*2-s}}$$

$$\Gamma'(x^*) = \frac{(1-s)r^*}{x^* [r^* - \theta(1-\tau)sx^{*s-1-s}]} [\Gamma(x^*) + \rho] \tag{54}$$

we derive

$$\Phi'(x^*) = \frac{r^* - \theta s(1-\tau)x^{*s-1-s}}{r^*} \left[H'(x^*) - \Gamma'(x^*) - \frac{\rho\alpha\tau_c(1-s)}{(1+\tau_c)b^* x^{*2-s}} \right] + \frac{\rho(1-s)}{x^*}$$

and thus

$$X_3(b^*, v^*, x^*)x^* = \rho + \frac{r^* - \theta s(1-\tau)x^{*s-1-s}}{r^*(1-s)} \times \left[H'(x^*)x^* - \Gamma'(x^*)x^* - \frac{\rho\alpha\tau_c(1-s)}{(1+\tau_c)b^* x^{*1-s}} \right]$$

We then derive

$$\mathcal{T} + \phi = V_2(b^*, v^*, x^*)v^* + X_3(b^*, v^*, x^*)x^*$$

$$= \rho + \frac{\alpha\tau_c v^*}{(1+\tau_c)b^*} + \frac{r^* - \theta s(1-\tau)x^{*s-1-s}}{r^*(1-s)} \times \left[H'(x^*)x^* - \Gamma'(x^*)x^* - \frac{\rho\alpha\tau_c(1-s)}{(1+\tau_c)b^* x^{*1-s}} \right]$$

Using the expression of v^* as given by (49), we finally get

$$\mathcal{T} + \phi = \rho + \frac{r^* - \theta s(1-\tau)x^{*s-1-s}}{r^*(1-s)} [H'(x^*) - \Gamma'(x^*)]x^* + \frac{\alpha\tau_c(1-\tau)(1-s)}{(1+\tau_c)b^*}$$

Since in this case $H'(x^*) - \Gamma'(x^*) > 0$, we conclude that λ_2 and λ_3 are positive. Therefore $\lambda_1 = -\phi < 0$ is the unique negative root and the steady state is saddle-point stable.

Let us finally consider the cases where two steady states $x_1^* < x_2^*$ exist and are necessarily such that $\Gamma'(x_1^*) - H'(x_1^*) < 0$ and $\Gamma'(x_2^*) - H'(x_2^*) > 0$. It follows that the highest steady state x_2^* satisfies $-D/\phi < 0$ and is locally indeterminate, while the lowest steady state x_1^* satisfies $\Gamma'(x_1^*) - H'(x_1^*) < 0$ and is saddle-point stable.

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