

# Supplementary materials to: “Robin Hood model versus Sheriff of Nottingham model: transfers in population dynamics”

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## A Spaces of measures

Let  $X$  be a Polish space, that is complete metric space  $(X, d)$  which is separable (i.e., there exists a countable dense subset). As an example for  $X$  one may consider any closed subset of  $\mathbb{R}^n$  endowed with the standard metric  $d(x, y) = \|x - y\|$  induced by  $\|\cdot\|$  a norm on  $\mathbb{R}^n$ .

Recall that the Borel  $\sigma$ -algebra of  $X$  is the set  $\mathcal{B}(X) \subset \mathcal{P}(X)$  (the  $\sigma$ -algebra generated by the open subsets of  $X$ ) of all parts of  $X$  that can be obtained by countable union, countable intersection, and difference of open sets [4, Vol II Chap 6 section 6.3].

We define  $\mathcal{M}(X)$  the space of measures on  $X$  starting with the positive measures. A map  $\mu : \mathcal{B}(X) \rightarrow \mathbb{R}^+$  is a **positive measure**, if it is **additive** (or a **countably additive**). That is,

$$\mu \left( \bigcup_{n \in \mathbb{N}} B_n \right) = \sum_{n \in \mathbb{N}} \mu(B_n),$$

for any countable collection of disjoint Borel sets  $B_n \in \mathcal{B}(X)$  (where the empty set may occur infinitely many times). In the following, a countably additive measure will be called **Borel measure**.

A positive measure is **finite** if

$$\mu(X) < +\infty.$$

A **signed** measure  $\mu$  is the difference between two positive measures

$$\mu = \mu^+ - \mu^-$$

where  $\mu^+$  and  $\mu^-$  are both positive finite measures.

**Definition A.1.** *The set  $\mathcal{M}(X)$  is the space of all the signed finite measures  $\mu$ .*

Given a signed measure  $\mu$ , the Hahn decomposition theorem [4, Vol. I Theorem 3.1.1 p. 175] gives a decomposition of the space  $X$  into two subsets  $X^+$  and  $X^-$  on which  $\mu$  has constant sign.

**Theorem A.2** (Hahn decomposition). *Let  $\mu$  be a signed measure on a measurable space  $(X, \mathcal{B}(X))$ . Then, there exist disjoint sets  $X^+, X^- \in \mathcal{B}(X)$  such that  $X^+ \cup X^- = X$ , and for all  $A \in \mathcal{B}(X)$ , one has*

$$\mu(A \cap X^-) \leq 0 \text{ and } \mu(A \cap X^+) \geq 0.$$

Considering for example  $\mu = \delta_0 - \delta_2$  with  $X = \{0, 1, 2\}$ , we deduce that the Hahn decomposition is not unique in general. But the Hahn decomposition allows us to define the *positive part*  $\mu^+$  and the *negative part*  $\mu^-$  of a signed measure  $\mu$ :

$$\mu^-(A) := -\mu(A \cap X^-) \text{ and } \mu^+(A) := \mu(A \cap X^+), \text{ for all } A \in \mathcal{B}(X). \quad (\text{A.1})$$

Let us prove that  $\mu^+$  is uniquely defined, the proof for  $\mu^-$  being similar. Indeed, if we consider  $\tilde{X}^+ \cup \tilde{X}^- = X$  another Hahn decomposition for  $\mu$ . Then we have

$$\mu(X^+ \cap \tilde{X}^-) = 0, \text{ and } \mu(\tilde{X}^+ \cap X^-) = 0,$$

since both quantities are simultaneously positive and negative.

Therefore we have

$$\begin{aligned} \mu(X^+ \cap A) &= \mu\left(X^+ \cap \left((A \cap \tilde{X}^+) \cup (A \cap \tilde{X}^-)\right)\right) \\ &= \mu(A \cap \tilde{X}^+ \cap X) + \mu(A \cap \tilde{X}^- \cap X^+) \\ &= \mu(A \cap \tilde{X}^+ \cap X) \\ &= \mu(A \cap \tilde{X}^+ \cap X^+) + \mu(A \cap \tilde{X}^- \cap X^+) \\ &= \mu\left(\tilde{X}^+ \cap \left((A \cap X^+) \cup (A \cap X^-)\right)\right) = \mu(\tilde{X}^+ \cap A). \end{aligned}$$

This shows that  $\mu^+$  defined by (A.1) is unique (i.e.  $\mu^+$  is independent of the Hahn decomposition).

The *total variation* of  $\mu$  (see [4, Vol. I Definition 3.1.4 p.176]) is

$$|\mu| = \mu^+ + \mu^-.$$

The space  $\mathcal{M}(X)$  of signed finite measures over  $X$ , is a Banach space endowed with the *total variation norm*

$$\|\mu\|_{\mathcal{M}(X)} := \int_X |\mu|(dx).$$

We refer again to Bogachev [4, Vol. I Theorem 4.6.1] for this result.

First, we check that the positive part, negative part and total variation are continuous on  $\mathcal{M}(X)$ .

**Lemma A.3.** *Let  $(X, \mathcal{B}(X))$  be a measurable space. The maps  $\mu \mapsto \mu^+$ ,  $\mu \mapsto \mu^-$  and  $\mu \mapsto |\mu|$  are 1-Lipschitz continuous on  $\mathcal{M}(X)$  equipped with  $\|\cdot\|_{\mathcal{M}(X)}$ . That is,*

$$\begin{aligned} \|\mu_1^+ - \mu_2^+\|_{\mathcal{M}(X)} &\leq \|\mu_1 - \mu_2\|_{\mathcal{M}(X)}, \\ \|\mu_1^- - \mu_2^-\|_{\mathcal{M}(X)} &\leq \|\mu_1 - \mu_2\|_{\mathcal{M}(X)}, \\ \||\mu_1| - |\mu_2|\|_{\mathcal{M}(X)} &\leq \|\mu_1 - \mu_2\|_{\mathcal{M}(X)}. \end{aligned}$$

*Proof.* Let  $\mu_1, \mu_2 \in \mathcal{M}(X)$  be given. We introduce the Hahn decompositions of  $X$  with respect to  $\mu_1$  and  $\mu_2$ , respectively:  $X =: X_1^+ \cup X_1^-$  and  $X =: X_2^+ \cup X_2^-$ , so that  $X_1^+$  is the support of  $\mu_1^+$ ,  $X_1^-$  is the support of  $\mu_1^-$ ,  $X_2^+$  is the support of  $\mu_2^+$ , and  $X_2^-$  is the support of  $\mu_2^-$ .

We also introduce the Hahn decomposition of  $X$  for  $|\mu_1| - |\mu_2|$ ,  $X =: Y^+ \cup Y^-$ . Then,

$$\begin{aligned} \||\mu_1| - |\mu_2|\|_{\mathcal{M}(X)} &= (|\mu_1| - |\mu_2|)^+(X) + (|\mu_1| - |\mu_2|)^-(X) \\ &= |\mu_1|(Y^+) - |\mu_2|(Y^+) + |\mu_2|(Y^-) - |\mu_1|(Y^-) \\ &= \mu_1^+(Y^+) + \mu_1^-(Y^+) - \mu_2^+(Y^+) - \mu_2^-(Y^+) \quad (\text{A.2}) \\ &\quad + \mu_2^+(Y^-) + \mu_2^-(Y^-) - \mu_1^+(Y^-) - \mu_1^-(Y^-). \quad (\text{A.3}) \end{aligned}$$

We decompose further  $Y^+ = (Y^+ \cap X_1^+) \cup (Y^+ \cap X_1^-)$  to obtain

$$\begin{aligned} \mu_1^+(Y^+) + \mu_1^-(Y^+) - \mu_2^+(Y^+) - \mu_2^-(Y^+) &= \mu_1(Y^+ \cap X_1^+) - \mu_1(Y^+ \cap X_1^-) \\ &\quad - |\mu_2|(Y^+ \cap X_1^+) - |\mu_2|(Y^+ \cap X_1^-), \end{aligned} \quad (\text{A.4})$$

and

$$\begin{aligned} \mu_1(Y^+ \cap X_1^+) - |\mu_2|(Y^+ \cap X_1^+) &= \mu_1(Y^+ \cap X_1^+) - \mu_2^+(Y^+ \cap X_1^+) - \mu_2^-(Y^+ \cap X_1^+) \\ &\leq \mu_1(Y^+ \cap X_1^+) - \mu_2^+(Y^+ \cap X_1^+) + \mu_2^-(Y^+ \cap X_1^+) \\ &= \mu_1(Y^+ \cap X_1^+) - \mu_2(Y^+ \cap X_1^+) \\ &\leq |\mu_1 - \mu_2|(Y^+ \cap X_1^+), \end{aligned}$$

similarly

$$\begin{aligned}
-\mu_1(Y^+ \cap X_1^-) - |\mu_2|(Y^+ \cap X_1^-) &= -\mu_1(Y^+ \cap X_1^-) - \mu_2^+(Y^+ \cap X_1^-) - \mu_2^-(Y^+ \cap X_1^-) \\
&\leq -\mu_1(Y^+ \cap X_1^-) + \mu_2^+(Y^+ \cap X_1^-) - \mu_2^-(Y^+ \cap X_1^-) \\
&= \mu_2(Y^+ \cap X_1^-) - \mu_1(Y^+ \cap X_1^-) \\
&\leq |\mu_1 - \mu_2|(Y^+ \cap X_1^-),
\end{aligned}$$

so finally (A.4) becomes

$$\begin{aligned}
(|\mu_1| - |\mu_2|)(Y^+) &= \mu_1^+(Y^+) + \mu_1^-(Y^+) - \mu_2^+(Y^+) - \mu_2^-(Y^+) \\
&\leq |\mu_1 - \mu_2|(Y^+ \cap X_1^+) + |\mu_1 - \mu_2|(Y^+ \cap X_1^-) \quad (\text{A.5}) \\
&= |\mu_1 - \mu_2|(Y^+).
\end{aligned}$$

By a similar argument using this time the decomposition  $Y^- = (Y^- \cap X_2^+) \cup (Y^- \cap X_2^-)$ , we obtain

$$\begin{aligned}
(|\mu_1| - |\mu_2|)(Y^-) &= \mu_2^+(Y^-) + \mu_2^-(Y^-) - \mu_1^+(Y^-) - \mu_1^-(Y^-) \\
&\leq |\mu_1 - \mu_2|(Y^- \cap X_2^+) + |\mu_1 - \mu_2|(Y^- \cap X_2^-) \quad (\text{A.6}) \\
&= |\mu_1 - \mu_2|(Y^-).
\end{aligned}$$

Finally, combining (A.5) and (A.6) into (A.2)-(A.3), we have

$$\begin{aligned}
\| |\mu_1| - |\mu_2| \|_{\mathcal{M}(X)} &\leq |\mu_1 - \mu_2|(Y^+) + |\mu_1 - \mu_2|(Y^-) \\
&= |\mu_1 - \mu_2|(Y^+) + |\mu_1 - \mu_2|(Y^-) \\
&= |\mu_1 - \mu_2|(X) \\
&= \| \mu_1 - \mu_2 \|_{\mathcal{M}(X)}.
\end{aligned}$$

We have proved that  $\mu \mapsto |\mu|$  is 1-Lipschitz. Since  $\mu^+ = \frac{1}{2}(|\mu| + \mu)$  and  $\mu^- = \frac{1}{2}(|\mu| - \mu)$ , both  $\mu \mapsto \mu^+$  and  $\mu \mapsto \mu^-$  are also 1-Lipschitz. The proof is completed.  $\blacksquare$

We have the following lemma.

**Lemma A.4.** *Let  $(X, \mathcal{B}(X))$  be a measurable space. The subset  $\mathcal{M}_+(X)$  is a positive cone of  $\mathcal{M}(X)$ . That is,*

- (i)  $\mathcal{M}_+(X)$  is a closed and convex subset of  $\mathcal{M}(X)$ .
- (ii)  $\lambda m \in \mathcal{M}_+(X)$ ,  $\forall \lambda \geq 0, \forall m \in \mathcal{M}_+(X)$ .
- (iii)  $\mathcal{M}_+(X) \cap -\mathcal{M}_+(X) = \{0_{\mathcal{M}(X)}\}$ .

*Proof.* Proof of (i). By Lemma A.3, the map  $\mu \mapsto \mu^-$  is continuous, and

$$\mathcal{M}_+(X) = \{\mu \in \mathcal{M}(X) : \mu^- = 0\}.$$

The property (ii) is trivial, since  $(\lambda m)(A) = \lambda m(A), \forall A \in \mathcal{B}(X)$ .

Proof of (iii). Let  $\mu \in \mathcal{M}_+(X) \cap -\mathcal{M}_+(X)$ . We observe that  $\mu \in \mathcal{M}_+(X)$  implies  $\mu^- = 0$ . Next  $\mu \in -\mathcal{M}_+(X)$  is equivalent to  $-\mu \in \mathcal{M}_+(X)$ , and it follows that  $(-\mu)^- = \mu^+ = 0$ . We conclude that  $\mu = \mu^+ - \mu^- = 0$ , and (iii) is proved.  $\blacksquare$

When  $\mu \in \mathcal{M}(X)$  is a given measure (not necessarily finite), one can define the space of integrable functions quotiented by the equivalence  $\mu$ -almost everywhere,  $L^1(X, \mu)$ . It is a Banach space [4, Vol. I Theorem 4.1.1 p.250] equipped with the norm

$$\|f\|_{L^1(X, \mu)} = \int_X |f(x)| |\mu|(dx).$$

For each  $f \in L^1(X, \mu)$ , the product measure  $m(dx) = f(x)\mu(dx)$  is defined by

$$m(A) = \int_A f(x)\mu(dx), \forall A \in \mathcal{B}(X),$$

and this measure satisfies

$$\|m\|_{\mathcal{M}(X)} = \int_X |f(x)| |\mu|(dx) = \|f\|_{L^1(X, \mu)}.$$

It follows from its Banach space property, that  $L^1(X, \mu)$  is a closed subspace of  $\mathcal{M}(X)$ . Remark that it is still true when  $X = I$  is an interval and  $\mu(dx) = dx$  is the Lebesgue measure, in which case  $L^1(X, \mu) = L^1(I)$  is the usual space of  $L^1$  functions.

Let us recall the Radon-Nikodym Theorem for signed measures [4, Vol. I Theorem 3.2.2 p.178]. We first recall the notion of absolute continuity [4, Vol. I Definition 3.2.1 (i) p.178].

**Definition A.5** (Absolute continuity). *Let  $(X, \mathcal{B}(X))$  be a measurable space, and  $\mu, \nu \in \mathcal{M}(X)$  be two signed measures. The measure  $\nu$  is **absolutely continuous** with respect to  $\mu$  (notation:  $\nu \ll \mu$ ) if for any Borel subset  $A \in \mathcal{B}(X)$ ,  $|\mu|(A) = 0$  implies  $|\nu|(A) = 0$ .*

**Theorem A.6** (Radon-Nikodym). *Let  $(X, \mathcal{B}(X))$  be a measurable space and  $\mu, \nu \in \mathcal{M}(X)$ . The measure  $\nu$  is absolutely continuous with respect to  $\mu$  if there exists a  $\mu$ -integrable function  $f \in L^1(X, \mu)$ , such that*

$$\nu(A) = \int_A f(x)\mu(dx), \forall A \in \mathcal{B}(X).$$

Next, we consider the following formula

$$\|u\|_{\mathcal{M}(X)} = \sup_{\phi \in C(\mathbb{R}): \|\phi\|_\infty \leq 1} \int_X \phi(x)u(dx), \forall u \in \mathcal{M}(I).$$

where  $X$  is a Polish space.

An equivalent statement is proved in [4, Vol.II Theorem 7.9.1 p.108] with far more general assumptions.

Here, we give a more elementary proof when  $X$  is Polish. We rely on the Borel-regularity of Borel measures that we recall first. The following statement is exactly [4, Vol. I Theorem 1.4.8 p.30] when  $X \subset \mathbb{R}^n$ , and in general it is an easy consequence of the fact that all Borel measures are Radon in a Polish space [4, Vol. II Theorem 7.1.7 p.70].

**Theorem A.7** (Approximations of Borel measures). *Let  $(X, d)$  be a Polish space, and let  $\mu$  be a Borel measure on  $X$ . Then, for any Borel set  $B \subset X$ , and any  $\varepsilon > 0$ , there exists an open subset  $U_\varepsilon \subset X$ , and a compact subset  $K_\varepsilon \subset X$ , such that*

$$K_\varepsilon \subset B \subset U_\varepsilon, \text{ and } \mu(U_\varepsilon \setminus K_\varepsilon) \leq \varepsilon.$$

Now we have the following result.

**Proposition A.8.** *Let  $(X, d)$  be a Polish space. For any measure  $\mu \in \mathcal{M}(X)$ , we have*

$$\|\mu\|_{\mathcal{M}(X)} = \sup_{\phi \in C(X): |\phi| \leq 1} \int_X \phi(x) \mu(dx).$$

*Proof.* Let  $\mu^+$  and  $\mu^-$  be the positive and negative part of  $\mu$  and  $X^+, X^-$  the support of  $\mu^+$  and  $\mu^-$ , respectively. By Theorem A.7 applied to  $|\mu|$ , there exists  $K_\varepsilon^+ \subset X^+ \subset U_\varepsilon^+$  with  $K_\varepsilon^+$  compact and  $U_\varepsilon^+$  open such that

$$|\mu|(U_\varepsilon^+ \setminus K_\varepsilon^+) \leq \frac{\varepsilon}{4},$$

so

$$\begin{aligned} \mu^+(X^+) &= |\mu|(X^+) = |\mu|(K_\varepsilon^+ \cup (X^+ \setminus K_\varepsilon^+)) \\ &\geq \mu(K_\varepsilon^+) - |\mu|(U_\varepsilon^+ \setminus K_\varepsilon^+) \\ &= \mu^+(K_\varepsilon^+) - \frac{\varepsilon}{4}. \end{aligned}$$

Similarly we can find  $K_\varepsilon^-$  compact and  $U_\varepsilon^-$  open such that

$$|\mu|(U_\varepsilon^- \setminus K_\varepsilon^-) \leq \frac{\varepsilon}{4}, \text{ so } \mu^-(X^-) \geq \mu^-(K_\varepsilon^-) - \frac{\varepsilon}{4}.$$

Recall that the distance between a point  $x$  and a subset  $B \subset X$  is defined as

$$d(x, B) = \inf_{y \in B} |x - y|.$$

Consider

$$d_+ = \min_{y \notin U_\varepsilon^+} d(y, K_\varepsilon^+) > 0, \text{ and } d_- = \min_{y \notin U_\varepsilon^-} d(y, K_\varepsilon^-) > 0.$$

Define  $d = \min(d_-, d_+)$ . Then

$$\phi^+(x) = \rho\left(\frac{\text{dist}(x, K_\varepsilon^+)}{d}\right), \text{ and } \phi^-(x) = \rho\left(\frac{\text{dist}(x, K_\varepsilon^-)}{d}\right),$$

where  $\rho$  is truncation map

$$\rho(u) = \begin{cases} e^{u^2/(u^2-1)}, & \text{if } |u| < 1, \\ 0, & \text{if } |u| \geq 1. \end{cases}$$

By definition we have  $\phi^+(x)$  and  $\phi^-(x)$  are continuous maps, and

$$\phi^+(x) \begin{cases} = 0, & \text{if } x \notin U_\varepsilon^+, \\ = 1, & \text{if } x \in K_\varepsilon^+ \\ \in [0, 1], & \text{otherwise,} \end{cases} \quad \text{and } \phi^-(x) \begin{cases} = 0, & \text{if } x \notin U_\varepsilon^-, \\ = 1, & \text{if } x \in K_\varepsilon^- \\ \in [0, 1], & \text{otherwise.} \end{cases}$$

Consider  $\phi(x) := \phi^+(x) - \phi^-(x)$ , then we have

$$\begin{aligned} \int_X \phi(x) \mu(dx) &= \int_X \phi^+(x) \mu(dx) - \int_X \phi^-(x) \mu(dx) \\ &= \int_{K_\varepsilon^+} \phi^+(x) \mu(dx) + \int_{U_\varepsilon^+ \setminus K_\varepsilon^+} \phi^+(x) \mu(dx) \\ &\quad - \int_{K_\varepsilon^-} \phi^-(x) \mu(dx) - \int_{U_\varepsilon^- \setminus K_\varepsilon^-} \phi^-(x) \mu(dx) \\ &\geq \mu(K_\varepsilon^+) - \int_{U_\varepsilon^+ \setminus K_\varepsilon^+} \phi^+(x) |\mu|(dx) - \mu(K_\varepsilon^-) - \int_{U_\varepsilon^- \setminus K_\varepsilon^-} \phi^-(x) |\mu|(dx) \\ &\geq \mu^+(K_\varepsilon^+) + \mu^-(K_\varepsilon^-) - \frac{\varepsilon}{2} \\ &\geq \mu^+(X^+) - \frac{\varepsilon}{4} + \mu^-(X^-) - \frac{\varepsilon}{4} - \frac{\varepsilon}{2} \\ &= |\mu|(X) - \varepsilon = \|\mu\|_{\mathcal{M}(X)} - \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary, we have proved that

$$\sup_{\phi \in C(X) : \sup_{x \in X} |\phi(x)| \leq 1} \int_X \phi(x) \mu(dx) \geq \|\mu\|_{\mathcal{M}(X)}.$$

The converse inequality follows from the comparison of integrals  $\int_X \phi(x) \mu(dx) \leq \|\phi\|_\infty \int_X 1 \mu(dx)$ . Proposition A.8 is proved.  $\blacksquare$

**Example A.9** (A bounded linear form that is not a measure). *The space of measures on non-compact metric space  $(X, d)$  is not a dual space of the continuous functions or bounded sequences in the present case. Indeed, consider the example (taken from the book of [4]),  $X = \mathbb{N}$  endowed with the standard metric  $d(n, m) = |n - m|$ . Due to the additive property of measures, any measure on  $\mathbb{N}$  must be a linear form. That is,*

$$\mu(f) = \int_{\mathbb{N}} f(n) \mu(dn) = \sum_{n=1}^{\infty} \mu_n f_n,$$

whenever  $f \in l^\infty(\mathbb{N}, \mathbb{R})$  the space of bounded sequence, which is a Banach space endowed with the standard supremum norm  $\|f\|_\infty = \sup_{n \geq 1} |f_n|$ .

Next, if we consider the linear form

$$x^*(f) = \lim_{n \rightarrow \infty} f_n,$$

defined for the converging sequences. By the Hahn Banach theorem,  $x^*$  has a continuous extension to the space of bounded sequence (endowed with the standard supremum norm), and this extension is not a measure. Therefore the dual space  $l^\infty(\mathbb{N}, \mathbb{R})^*$  is a larger than  $\mathcal{M}(X)$  the space of measure on  $X = \mathbb{N}$ .