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# Robin Hood model versus Sheriff of Nottingham model: transfers in population dynamics

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## Abstract

We study the problem of transfers in a population structured by a continuous variable corresponding to the quantity being transferred. The model takes the form of an integro-differential equations with kernels corresponding to the specific rules of the transfer process. We focus our interest on the well-posedness of the Cauchy problem in the space of measures. We characterize transfer kernels that give a continuous semiflow in the space of measures and derive a necessary and sufficient condition for the stability of the space  $L^1$  of integrable functions. We construct some examples of kernels that may be particularly interesting in economic applications. Our model considers blind transfers of economic value (e.g. money) between individuals. The two models are the “Robin Hood model”, where the richest individual unconditionally gives a fraction of their wealth to the poorest when a transfer occurs, and the other extreme, the “Sheriff of Nottingham model”, where the richest unconditionally takes a fraction of the poorest’s wealth. Between these two extreme cases is a continuum of intermediate models obtained by interpolating the kernels. We illustrate those models with numerical simulations and show that any small fraction of the “Sheriff of Nottingham” in the transfer rules leads to a segregated population with extremely poor and extremely rich individuals after some time. Although our study is motivated by economic applications, we believe that this study is a first step towards a better understanding of many transfer phenomena occurring in the life sciences.

## 1 Introduction

Transfer phenomena are fundamental processes in population dynamics, influencing a wide range of biological and social systems. These phenomena manifest in various forms, such as predation, parasitism, cooperation, and sexual reproduction, where the transfer typically involves energy, proteins, or genetic material. Social interactions, including opinion formation and economic exchanges, can also be viewed as transfers, with economic transactions being the primary focus of this study.

Their mathematical formulations have a long history. In the context of mathematical physics, a fundamental approach has been proposed by Boltzmann, in which interacting particles are viewed

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as members of a continuum of population density. In these models transfer of physical quantities from one particle to another is modulated by a kernel function that specifies the transfer process. Many examples have been explored, and a review is found in Perthame [17]. While these models share a structural similarity with the ones we propose here, Boltzmann-type models include a kinetic term linking the position and speed of particles, which makes their analysis quite intricate; here the models we propose do not include such term and we will analyze them by other methods.

In populations dynamics, the transfer of genetic material has been increasingly studied in the past decades. In Magal and Webb [15] and Magal [14], the authors introduced a model devoted to transfers of genetic material in which parent cells exchange genetic material to form their offspring. While they relied on the example of *Helicobacter pylori*, these models are applicable to a much wider range of species subject to mutation, selection and recombination. More recently, several authors have studied population dynamics model including a mutation and selection (among which [9, 1, 6, 5]), and models derived from the *Fisher infinitesimal model* that provides a microscopic basis for transfer models of sexual reproduction [2, 7] have also attracted some attention.

Some social phenomena can also be studied by the use of transfer models. Toscani [18] used a transfer model closely related to the one introduced here in a model of opinion formation. Such a model is sometimes called “kinetic model” even in the absence of an actual kinetic term; the literature around kinetic models for opinion formation is developed [16, 11, 19]. These models have also been proposed to model economic phenomena, where the quantity being transferred is an abstract value (money, stocks, etc.) [8, 10, 3]. The model we call “Robin-Hood” is very close to the Cordier, Pareschi and Toscani model [8], but we did not find a model similar to the “Sheriff of Nottingham” model in the literature.

This article studies models inspired by the life sciences with potential economic applications. Here we use a similar idea to model the transfer of richness between individuals. Before we introduce the mathematical concepts rigorously, let us explain our model’s idea in a few words. We study a population of individuals who possess a certain transferable quantity (for example, money) and exchange it according to a rule expressed for two individuals chosen randomly in the population. This rule is encoded in an abstract integration kernel but we also propose explicit examples that we call “Robin Hood” (RH) and “Sheriff of Nottingham” (SN) model, in reference to the well-known folk tale. In the Robin Hood model, the richer gives a fraction of its wealth to the poorest. On the contrary, in the Sheriff of Nottingham model, the richest takes a fraction of the poorest’s wealth, possibly through debt mechanism (we consider that people’s wealth may become negative). We also consider distributed versions of those models, in which the result of the interaction is given by a probability distribution instead of just the exchange of a fixed fraction. All those models are conservative, meaning that the total wealth in the population remains constant in time.

We introduce several novelties compared to the existing literature. Our first result is the existence of solutions as a continuous homogeneous semiflow on the space of measures, as well as the well-posedness of the problem in a strong sense, for a rather general class of interaction kernels (Theorem 4.3); when in the existing literature we could consult, the existence of solutions is always understood in a weak sense. We also provide a simple characterization of kernels that leave positively invariant the  $L^1$  space of integrable functions, that is to say the measures that are absolutely continuous for the Lebesgue measure. This invariance may occur even when the kernel possesses some singularities (that is the case for the RH and SN models), as long as the singularities occur in a negligible for the integration variables. Finally, we compute the transfer operators corresponding to the RH and SN models and show their well-posedness in  $L^1$  as well as the set of measures, by applying our previous results.

The plan of the paper is the following. In Section 2 we introduce some notations, the functional setting of the paper, and our main assumptions. In Section 3, we present the RH model, which corresponds to cooperative transfers, and the SN model, which corresponds to competitive transfers. These two examples illustrate the problem and will show how the rules at the individual level can be expressed using a proper kernel  $K$ . In Section 4, we prove that under Assumption 2.1, the operator  $B$  maps  $\mathcal{M}(I) \times \mathcal{M}(I)$  into  $\mathcal{M}(I)$  and is a bounded bi-linear operator. Due to the boundedness of  $B$ , we will deduce that (2.1)-(2.2) generates a unique continuous semiflow on

the space of positive measures  $\mathcal{M}_+(I)$ . In Section 5, we consider the restriction of the system (2.1)-(2.2) to  $L_+^1(I)$ . In Section 6, we run individual-based stochastic simulations of the mixed RH and SN model. The paper is complemented with a Supplementary Materials file (denoted by the letter A) in which we recall some classical results about measure theory.

## 2 Notations, functional setting, and assumptions

Let  $I \subset \mathbb{R}$  be a closed interval of  $\mathbb{R}$ . Depending on the situation, we may consider in the paper that  $I$  is bounded or unbounded. Let  $\mathcal{M}(I)$  be the space of measures on  $I$ . It is well known that  $\mathcal{M}(I)$  endowed with the norm

$$\|u\|_{\mathcal{M}(I)} = \int_I |u|(dx), \forall u \in \mathcal{M}(I),$$

is a Banach space. Here  $|u| = u^+ + u^-$  is the total variation of the measure  $u$ , and  $u^+$  and  $u^-$  are the positive and negative parts of  $u$ . We refer the reader to the Supplementary Materials file for a precise definition.

An alternative to define the norm of a measure is the following (see Proposition A.8 in the Supplementary Materials)

$$\|u\|_{\mathcal{M}(I)} = \sup_{\phi \in \text{BC}(I): \|\phi\|_\infty \leq 1} \int_I \phi(x)u(dx), \forall u \in \mathcal{M}(I).$$

A finite measure on an interval  $I \subset \mathbb{R}$  (bounded or not) is, therefore, a bounded linear form on  $\text{BC}(I)$ , the space of bounded and continuous functions from  $I$  to  $\mathbb{R}$ . Since  $\mathcal{M}(I)$  endowed with its norm is a Banach space, we deduce that  $\mathcal{M}(I)$  is a closed subset of  $\text{BC}(I)^*$ . When the interval  $I$  is not bounded, Example A.9 in the Supplementary Materials shows that

$$\mathcal{M}(I) \neq \text{BC}(I)^*.$$

Let us recall that  $\mathcal{B}(I)$  is the  $\sigma$ -algebra generated by all the open subsets of  $I$  is called the **Borel  $\sigma$ -algebra**. A subset  $A$  of  $I$  that belongs to  $\mathcal{B}(I)$  is called a **Borel set**. For any Borel subset  $A \subset I$ , the quantity

$$\int_A u(t, dx) = u(t, A),$$

is the number of individuals having their transferable quantity  $x$  in the domain  $A \subset I$ . Therefore,

$$\int_I u(t, dx) = u(t, I),$$

is the total number of individuals at time  $t$ .

The operator of transfer  $T : \mathcal{M}_+(I) \rightarrow \mathcal{M}_+(I)$  is defined by

$$T(u)(dx) := \begin{cases} \frac{B(u, u)(dx)}{\int_I u(d\sigma)}, & \text{if } u \in \mathcal{M}_+(I) \setminus \{0\}, \\ 0, & \text{if } u = 0, \end{cases}$$

where  $B : \mathcal{M}(I) \times \mathcal{M}(I) \rightarrow \mathcal{M}(I)$  is a bounded bi-linear map defined by

$$B(u, v)(dx) := \iint_{I^2} K(dx, x_1, x_2)u(dx_1)v(dx_2),$$

or equivalently, defined by

$$B(u, v)(A) := \iint_{I^2} K(A, x_1, x_2)u(dx_1)v(dx_2),$$

for each Borel set  $A \subset I$ , and each  $u, v \in \mathcal{M}(I)$ . The assumptions on the kernel  $K$  will be made precise later in Assumption 2.1.

Let  $T : \mathcal{M}_+(I) \rightarrow \mathcal{M}_+(I)$  be a transfer operator. Let  $\tau > 0$  be the rate of transfers. We assume the time between two transfers follows an exponential law with a mean value of  $1/\tau$ . Two individuals will be involved once the transfer time has elapsed, so the transfer rate will have to be doubled (i.e, equal to  $2\tau$ ). The model of transfers is an ordinary differential equation on the space of positive measures

$$\partial_t u(t, dx) = 2\tau T(u(t, \cdot))(dx) - 2\tau u(t, dx), \forall t \geq 0. \quad (2.1)$$

The equation (2.1) should be complemented with the measure-valued initial distribution

$$u(0, dx) = \phi(dx) \in \mathcal{M}_+(I). \quad (2.2)$$

A solution of (2.1), will be a continuous function  $u : [0, +\infty) \rightarrow \mathcal{M}_+(I)$ , satisfying the fixed point problem

$$u(t, dx) = \phi(dx) + \int_0^t 2\tau T(u(\sigma, \cdot))(dx) - 2\tau u(\sigma, dx) d\sigma, \quad (2.3)$$

where the above integral is a Riemann integral in  $(\mathcal{M}(I), \|\cdot\|_{\mathcal{M}(I)})$ .

To prove the positivity of the solution (2.1), one may prefer to use the fixed point problem

$$u(t, dx) = e^{-2\tau t} \phi(dx) + \int_0^t e^{-2\tau(t-s)} 2\tau T(u(\sigma, \cdot))(dx) d\sigma, \quad (2.4)$$

which is equivalent to (2.3).

For each  $t \geq 0$ ,

$$u(t, dx) \in \mathcal{M}_+(I),$$

is understood as a measure-valued population density at time  $t$ .

Based on the examples of kernels presented in Section 3, we can make the following assumptions.

**Assumption 2.1.** *We assume the kernel  $K$  satisfies the following properties*

- (i) *For each Borel set  $A \in \mathcal{B}(I)$ , the map  $(x_1, x_2) \mapsto K(A, x_1, x_2)$  is Borel measurable.*
- (ii) *For each  $(x_1, x_2) \in I \times I$ , the map  $A \in \mathcal{B}(I) \mapsto K(A, x_1, x_2)$  is a probability measure.*

Assumption 2.1 is relatively weak and can be satisfied by many kernels. In particular, it is the case for kernels  $K(dz, x_1, x_2) = K(z, x_1, x_2)dy$  that are given by a nonnegative integrable function of  $\mathbb{R}^3$  (with respect to the Lebesgue measure).

We will often associate Assumption 2.1 with the following.

**Assumption 2.2.** *For each  $(x_1, x_2) \in I \times I$ ,*

$$\int_I x K(dx, x_1, x_2) = \frac{x_1 + x_2}{2}.$$

### 3 Examples of transfer kernels

Constructing a transfer kernel is a difficult problem. In this section, we propose two examples of transfer models. Some correspond to existing examples in the literature, while others seem new.

### 3.1 Robin Hood model: (the richest give to the poorest)

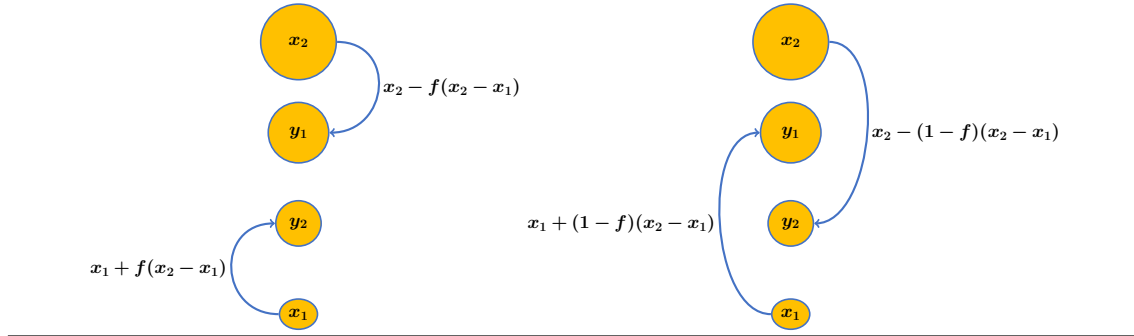
The poorest gains a fraction  $f$  of the difference  $|x_2 - x_1|$ , and the richest loses a fraction  $f$  of the difference  $|x_2 - x_1|$

$$K_1(dx, x_1, x_2) := \frac{1}{2} (\delta_{x_2 - f(x_2 - x_1)}(dx) + \delta_{x_1 - f(x_1 - x_2)}(dx)),$$

where  $f \in (0, 1)$  is fixed.

In Figure 1, we explain why we need to consider a mean value of two Dirac masses in the kernel  $K_1$ . Consider a transfer between two individuals, and consider  $x_1$  and  $x_2$  (respectively  $y_1$  and  $y_2$ ) the values of the transferable quantities before transfer (respectively after transfer). When we define  $K_1$  we need to take a mean value, because we can not distinguish if  $x_1$  and  $x_2$  are ancestors of  $y_1$  or  $y_2$ . In other words, we can not distinguish between the two cases: 1)  $x_1 \rightarrow y_1$  and  $x_2 \rightarrow y_2$ ; and 2)  $x_1 \rightarrow y_2$  and  $x_2 \rightarrow y_1$ ; (where  $x \rightarrow y$  means  $x$  becomes  $y$ ).

**Figure 1** In the figure, a transfer between two individuals, and consider  $x_1$  and  $x_2$  (respectively  $y_1$  and  $y_2$ ) the values of the transferable quantities before transfer (respectively after transfer). We plot a transfer whenever  $x_2 > x_1$ , and  $f \in [0, 1/2]$  (on the left hand side), and  $(1 - f) \in [1/2, 1]$  (on the right hand side). We observe that the values  $y_1$  and  $y_2$  are the same on both sides.



In Figure 1, we use the following equalities

$$y_1 = x_2 - f(x_2 - x_1) = x_1 + (1 - f)(x_2 - x_1),$$

and

$$y_2 = x_2 - (1 - f)(x_2 - x_1) = x_1 + (1 - f)(x_2 - x_1).$$

The values  $y_1$  and  $y_2$  are the same after a given transfer if we choose  $f$  or  $1 - f$ . In other words, we can not distinguish if  $x_1$  and  $x_2$  are ancestors of  $y_1$  or  $y_2$ . This explains the mean value of Dirac masses in the kernel  $K_1$ . It follows, that this kind of kernel will preserve the support of the initial distribution, so we can restrict to any closed bounded interval  $I \subset \mathbb{R}$ . This transfer kernel corresponds to the one proposed to build the recombination operator in Magal and Webb [15]. An extended version with friction was proposed by Hinow, Le Foll, Magal, and Webb [12].

Here we can compute  $B_1(u, v)$  explicitly when  $I = \mathbb{R}$ ,  $u \in L^1(\mathbb{R})$  and  $v \in L^1(\mathbb{R})$ . Indeed let  $\varphi \in C_c(I)$  be a compactly supported test function. Then

$$\begin{aligned} \int_{\mathbb{R}} \varphi(x) B_1(u, v)(dx) &= \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}} \int_{x \in \mathbb{R}} \varphi(x) K_1(dx, x_1, x_2) u(x_1) dx_1 u(x_2) dx_2 \\ &= \iint_{\mathbb{R} \times \mathbb{R}} \frac{1}{2} (\varphi(x_2 - f(x_2 - x_1)) + \varphi(x_1 - f(x_1 - x_2))) u(x_1) v(x_2) dx_1 dx_2 \\ &= \frac{1}{2} \left( \iint_{\mathbb{R} \times \mathbb{R}} \varphi((1 - f)x_2 + fx_1) u(x_1) v(x_2) dx_1 dx_2 \right. \\ &\quad \left. + \iint_{\mathbb{R} \times \mathbb{R}} \varphi((1 - f)x_1 + fx_2) u(x_1) v(x_2) dx_1 dx_2 \right) \end{aligned}$$

$$= \frac{1}{2} \left( \iint_{\mathbb{R} \times \mathbb{R}} \varphi((1-f)x_2 + fx_1) u(x_1) v(x_2) dx_1 dx_2 + \iint_{\mathbb{R} \times \mathbb{R}} \varphi((1-f)x_2 + fx_1) u(x_2) v(x_1) dx_1 dx_2 \right).$$

Next, by using the change of variable

$$\begin{cases} x_1 = x - (1-f)\sigma \\ x_2 = x + f\sigma \end{cases} \Leftrightarrow \begin{cases} \sigma = x_2 - x_1 \\ x = (1-f)x_2 + fx_1 \end{cases}$$

we obtain

$$\int_{\mathbb{R}} \varphi(x) B_1(u, v)(dx) = \frac{1}{2} \left( \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} u(x - (1-f)\sigma) v(x + f\sigma) d\sigma dx + \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} u(x + f\sigma) v(x - (1-f)\sigma) d\sigma dx \right),$$

and we obtain

$$B_1(u, v)(x) = \frac{1}{2} \left( \int_{\mathbb{R}} u(x - (1-f)\sigma) v(x + f\sigma) d\sigma + \int_{\mathbb{R}} v(x - (1-f)\sigma) u(x + f\sigma) d\sigma \right).$$

We conclude that the transfer operator restricted to  $L^1_+(\mathbb{R})$  is defined by

$$T_1(u)(x) = \begin{cases} \frac{\int_{\mathbb{R}} u(x - (1-f)\sigma) u(x + f\sigma) d\sigma}{\int_{\mathbb{R}} u(x) dx}, & \text{if } u \in L^1_+(\mathbb{R}) \setminus \{0\}, \\ 0, & \text{if } u = 0. \end{cases}$$

**Remark 3.1.** *One may also consider the case where the fraction transferred varies in function of the distance between the poorest and richest before transferred. This problem was considered by Hinow, Le Foll, Magal and Webb [12], and in the case*

$$K_1(dx, x_1, x_2) := \frac{1}{2} \left( \delta_{x_2 - f(|x_2 - x_1|)(x_2 - x_1)}(dx) + \delta_{x_1 - f(|x_2 - x_1|)(x_1 - x_2)}(dx) \right),$$

where  $f : [0, +\infty) \rightarrow [0, 1]$  is a continuous function.

### 3.2 Sheriff of Nottingham model: (the poorest give to the richest)

The poorest loses a fraction  $f$  of the difference  $|x_2 - x_1|$ , and the richest gains a fraction  $f$  of the difference  $|x_2 - x_1|$

$$K_2(dx, x_1, x_2) := \frac{1}{2} \left( \delta_{x_2 + f(x_2 - x_1)}(dx) + \delta_{x_1 + f(x_1 - x_2)}(dx) \right),$$

where  $f \in (0, 1)$  is fixed.

This kind of kernel will expand the support of the initial distribution to the whole real line, therefore we can not restrict to bounded intervals  $I \subset \mathbb{R}$ . **In this case, the only possible choice for the interval is  $I = \mathbb{R}$ .**

In Figure 2, we explain why we need to consider a mean value of two Dirac masses in the kernel  $K_1$ . Indeed, we have

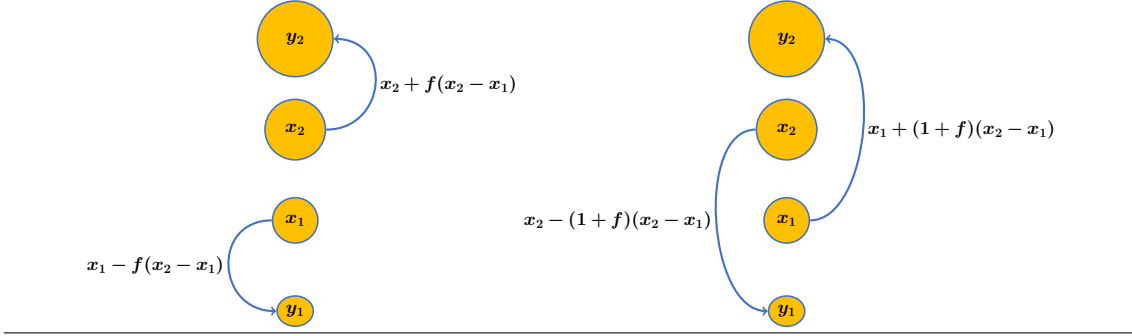
$$y_2 = x_2 + f(x_2 - x_1) = x_1 + (1+f)(x_2 - x_1),$$

and

$$y_1 = x_2 - (1+f)(x_2 - x_1) = x_1 - f(x_2 - x_1).$$

Therefore, the transferable quantities  $y_1$  and  $y_2$  after a given transfer, are the same with  $f$  or  $1+f$ .

**Figure 2** In the figure, the two values before transfers are  $x_1$  and  $x_2$ , and the two values after transfer are  $y_1$  and  $y_2$ . We plot a transfer whenever  $x_2 > x_1$ , and  $f \in [0, 1]$  (on the left hand side), and  $f$  is replaced by  $1 + f$  (on the right hand side). We observe that the values  $y_1$  and  $y_2$  are the same on both sides.



Here again we can compute  $B_2(u, v)$  explicitly when  $I = \mathbb{R}$ ,  $u \in L^1(\mathbb{R})$  and  $v \in L^1(\mathbb{R})$ . Indeed let  $\varphi \in C_c(I)$  be a compactly supported test function. Then

$$\begin{aligned}
 \int_{\mathbb{R}} \varphi(x) B_2(u, v)(dx) &= \int_{x_1 \in \mathbb{R}} \int_{x_2 \in \mathbb{R}} \int_{x \in \mathbb{R}} \varphi(x) K_2(dx, x_1, x_2) u(x_1) dx_1 u(x_2) dx_2 \\
 &= \iint_{\mathbb{R} \times \mathbb{R}} \frac{1}{2} (\varphi(x_2 + f(x_2 - x_1)) + \varphi(x_1 + f(x_1 - x_2))) u(x_1) v(x_2) dx_1 dx_2 \\
 &= \frac{1}{2} \left( \iint_{\mathbb{R} \times \mathbb{R}} \varphi((1+f)x_2 - fx_1) u(x_1) v(x_2) dx_1 dx_2 \right. \\
 &\quad \left. + \iint_{\mathbb{R} \times \mathbb{R}} \varphi((1+f)x_1 - fx_2) u(x_1) v(x_2) dx_1 dx_2 \right) \\
 &= \frac{1}{2} \left( \iint_{\mathbb{R} \times \mathbb{R}} \varphi((1+f)x_2 - fx_1) u(x_1) v(x_2) dx_1 dx_2 \right. \\
 &\quad \left. + \iint_{\mathbb{R} \times \mathbb{R}} \varphi((1+f)x_2 - fx_1) u(x_2) v(x_1) dx_1 dx_2 \right).
 \end{aligned}$$

Next, by using the change of variable

$$\begin{cases} x_1 = x - (1+f)\sigma \\ x_2 = x - f\sigma \end{cases} \Leftrightarrow \begin{cases} \sigma = x_2 - x_1 \\ x = (1+f)x_2 - fx_1 \end{cases}$$

we obtain

$$\begin{aligned}
 \int_{\mathbb{R}} \varphi(x) B_2(u, v)(dx) &= \frac{1}{2} \left( \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} u(x - (1+f)\sigma) v(x - f\sigma) d\sigma dx \right. \\
 &\quad \left. + \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} u(x - f\sigma) v(x - (1+f)\sigma) d\sigma dx \right),
 \end{aligned}$$

and we obtain

$$B_2(u, v)(x) = \frac{1}{2} \left( \int_{\mathbb{R}} u(x - (1+f)\sigma) v(x - f\sigma) d\sigma + \int_{\mathbb{R}} v(x - (1+f)\sigma) u(x - f\sigma) d\sigma \right).$$

We conclude that the transfer operator restricted to  $L^1_+(\mathbb{R})$  is defined by

$$T_2(u)(x) = \begin{cases} \frac{\int_{\mathbb{R}} u(x - (1+f)\sigma) u(x - f\sigma) d\sigma}{\int_{\mathbb{R}} u(x) dx}, & \text{if } u \in L^1_+(\mathbb{R}) \setminus \{0\}, \\ 0, & \text{if } u = 0. \end{cases}$$



## 4 Understanding (2.1) in the space of measures

Before anything, we need to define  $B(u, v)(dx)$  whenever  $u$  and  $v$  are finite measures on  $I$ .

**Theorem 4.1.** *Let Assumption 2.1 be satisfied. Define*

$$B(u, v)(A) := \iint_{I^2} K(A, x_1, x_2) u(dx_1) v(dx_2), \quad (4.1)$$

for each Borel set  $A \subset I$ , and each  $u, v \in \mathcal{M}(I)$ .

Then  $B$  maps  $\mathcal{M}(I) \times \mathcal{M}(I)$  into  $\mathcal{M}(I)$ , and satisfies the following properties

(i)

$$\|B(u, v)\|_{\mathcal{M}(I)} \leq \|u\|_{\mathcal{M}(I)} \|v\|_{\mathcal{M}(I)}, \forall u, v \in \mathcal{M}(I).$$

(ii)

$$B(u, v) \in M_+(I), \forall u, v \in M_+(I),$$

(iii)

$$\int_I B(u, v)(dx) = \int_I u(dx_1) \int_I v(dx_2), \forall u, v \in M_+(I).$$

If moreover Assumption 2.2 is satisfied, then the following property also holds.

(iv)

$$\int_I x B(u, v)(dx) = \frac{\int_I x_1 u(dx_1) \int_I v(dx_2) + \int_I u(dx) \int_I x_2 v(dx_2)}{2}, \forall u, v \in M_+(I).$$

*Proof.* Let  $u \in \mathcal{M}(I)$ ,  $v \in \mathcal{M}(I)$  and define

$$w(dx) := B(u, v)(dx)$$

by (4.1). Let  $(A_n)_{n \in \mathbb{N}}$  a collection of pairwise disjoint Borel-measurable sets in  $I$ . We want to prove that

$$w\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} w(A_n). \quad (4.2)$$

We have

$$w\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \iint_{I^2} K\left(\bigcup_{n \in \mathbb{N}} A_n, x_1, x_2\right) u(dx_1) v(dx_2) = \iint_{I^2} \sum_{n \in \mathbb{N}} K(A_n, x_1, x_2) u(dx_1) v(dx_2).$$

In order to change the order of summation between the integral and the sum, we will use Fubini's Theorem [4, Vol. I Theorem 3.4.4 p.185] and Tonelli's Theorem [4, Vol. I Theorem 3.4.5 p.185].

We consider  $P \subset \mathcal{B}(I^2)$  the support of the positive part of  $u \otimes v$  (as given by [4, Theorem 3.1.1 p. 175]). That is

$$\mathbb{1}_P u \otimes v \in M_+(I^2), \text{ and } -\mathbb{1}_{P^c} u \otimes v \in M_+(I^2),$$

where  $\mathbb{1}_P$  (respectively  $\mathbb{1}_{P^c}$ ) is the indicator functions of  $P$ , that is  $\mathbb{1}_P(x) = 1$  if  $x \in P$  else  $\mathbb{1}_P(x) = 0$  (respectively the indicator function of  $P^c = I \setminus P$  the complement set of  $P$ ).

We consider the maps defined for all  $n \in \mathbb{N}$ , and all  $x_1, x_2 \in I$ ,

$$f_n(x_1, x_2) := \mathbb{1}_P K(A_n, x_1, x_2), \quad (4.3)$$

and

$$f(n, x_1, x_2) := f_n(x_1, x_2).$$

Then by Assumption 2.1-(i), for each integer  $n \in \mathbb{N}$  the map  $f_n$  is Borel measurable (i.e., measurable with respect to the Borel  $\sigma$ -algebra), and for each Borel set  $B \subset \mathbb{R}$ , we have

$$f^{-1}(B) = \bigcup_{n \in \mathbb{N}} \{n\} \times f_n^{-1}(B).$$

Consequently,  $f$  is measurable for the  $\sigma$ -algebra  $\mathcal{P}(\mathbb{N}) \otimes \mathcal{B}(I^2)$ , the smallest  $\sigma$ -algebra in  $\mathcal{P}(\mathbb{N} \times I^2)$  that contains all rectangles  $\mathcal{N} \times B$  where  $\mathcal{N} \in \mathcal{P}(\mathbb{N})$  and  $B \in \mathcal{B}(I^2)$ .

Let  $c$  be the counting measure on  $\mathbb{N}$ ,  $c := \sum_{n \in \mathbb{N}} \delta_n$ . By Tonelli's Theorem [4, Vol. I Theorem 3.4.5 p.185], since  $f$  is nonnegative, and  $c$  and  $(u \otimes v)_+$  are nonnegative  $\sigma$ -finite measures, and

$$\begin{aligned} \int_{I^2} \int_{\mathbb{N}} f(n, x_1, x_2) c(dn) (u \otimes v)_+(dx_1 dx_2) &= \int_{I^2} \sum_{n \in \mathbb{N}} \mathbb{1}_P(x_1, x_2) K(A_n, x_1, x_2) (u \otimes v)_+(dx_1 dx_2) \\ &= \iint_{I^2} \mathbb{1}_P(x_1, x_2) K \left( \bigcup_{n \in \mathbb{N}} A_n, x_1, x_2 \right) u(dx_1) v(dx_2) \\ &\leq \iint_{I^2} \mathbb{1}_P(x_1, x_2) K(I^2, x_1, x_2) u(dx_1) v(dx_2) \\ &= (u \otimes v)(P) < +\infty. \end{aligned}$$

We conclude that  $f \in L^1(c \otimes (u \otimes v)_+)$  and therefore by Fubini's Theorem [4, Vol. I Theorem 3.4.4 p.185] we have

$$\int_{I^2} \int_{\mathbb{N}} f(n, x_1, x_2) c(dn) (u \otimes v)_+(dx_1 dx_2) = \int_{\mathbb{N}} \int_{I^2} f(n, x_1, x_2) c(dn) (u \otimes v)_+(dx_1 dx_2).$$

This means that

$$\iint_{I^2} \sum_{n \in \mathbb{N}} \mathbb{1}_P K(A_n, x_1, x_2) u(dx_1) v(dx_2) = \sum_{n \in \mathbb{N}} \iint_{I^2} \mathbb{1}_P K(A_n, x_1, x_2) u(dx_1) v(dx_2).$$

By similar arguments we can show that

$$\iint_{I^2} \sum_{n \in \mathbb{N}} (-\mathbb{1}_{P^c}) K(A_n, x_1, x_2) u(dx_1) v(dx_2) = \sum_{n \in \mathbb{N}} \iint_{I^2} (-\mathbb{1}_{P^c}) K(A_n, x_1, x_2) u(dx_1) v(dx_2).$$

Thus we obtain

$$\begin{aligned} \iint_{I^2} \sum_{n \in \mathbb{N}} K(A_n, x_1, x_2) u(dx_1) v(dx_2) &= \iint_{I^2} \sum_{n \in \mathbb{N}} (\mathbb{1}_P + \mathbb{1}_{P^c}) K(A_n, x_1, x_2) u(dx_1) v(dx_2) \\ &= \iint_{I^2} \sum_{n \in \mathbb{N}} \mathbb{1}_P K(A_n, x_1, x_2) u(dx_1) v(dx_2) + \iint_{I^2} \sum_{n \in \mathbb{N}} \mathbb{1}_{P^c} K(A_n, x_1, x_2) u(dx_1) v(dx_2) \\ &= \sum_{n \in \mathbb{N}} \iint_{I^2} \mathbb{1}_P K(A_n, x_1, x_2) u(dx_1) v(dx_2) + \sum_{n \in \mathbb{N}} \iint_{I^2} \mathbb{1}_{P^c} K(A_n, x_1, x_2) u(dx_1) v(dx_2) \\ &= \sum_{n \in \mathbb{N}} \iint_{I^2} K(A_n, x_1, x_2) u(dx_1) v(dx_2) = \sum_{n \in \mathbb{N}} w(A_n). \end{aligned}$$

We have proved (4.2) for any family of pairwise disjoint Borel sets  $(A_n)$ , hence  $w$  is a measure. Moreover  $w$  is finite because

$$\int_I |w|(dx) = |w|(I) \leq \iint_{I^2} K(I, x_1, x_2) |u \otimes v|(dx_1 dx_2) = \iint_{I^2} |u|(dx_1) |v|(dx_2) = \|u\|_{\mathcal{M}(I)} \|v\|_{\mathcal{M}(I)}.$$

We have proved that

$$\|B(u, v)\|_{\mathcal{M}(I)} \leq \|u\|_{\mathcal{M}(I)} \|v\|_{\mathcal{M}(I)}.$$

Hence  $B(u, v)$  is a continuous bilinear map on  $\mathcal{M}(X)$ .

To prove (iii), we use Fubini's theorem in the formula (4.1) of  $B(u, v)$  as follows:

$$\begin{aligned}\int_I B(u, v)(dx) &= B(u, v)(I) = \iint_{I^2} K(I, x_1, x_2) u(dx_1) v(dx_2) \\ &= \iint_{I^2} u(dx_1) v(dx_2) \\ &= \int_I u(dx) \int_I v(dx),\end{aligned}$$

because  $K(I, x_1, x_2) = 1$  by assumption.

To prove (iv), we use Fubini's theorem applied to the formula (4.1) of  $B(u, v)$ :

$$\begin{aligned}\int_I xB(u, v)(dx) &= \iint_{I^2} \int_I xK(dx, x_1, x_2) u(dx_1) v(dx_2) \\ &= \iint_{I^2} \frac{x_1 + x_2}{2} u(dx_1) v(dx_2) \\ &= \frac{\int_I x_1 u(dx_1) \int_I v(dx_2) + \int_I u(dx_1) \int_I x_2 v(dx_2)}{2},\end{aligned}$$

because  $\int_I xK(dx, x_1, x_2) = \frac{x_1 + x_2}{2}$  by Assumption 2.2. ■

As a consequence of Theorem 4.1, the map  $B$  is a bounded and bi-linear operator from  $\mathcal{M}(I) \times \mathcal{M}(I)$  to  $\mathcal{M}(I)$ . Moreover  $B$  maps  $M_+(I) \times M_+(I)$  into  $M_+(I)$ . To investigate the Lipschitz property of  $T : M_+(I) \rightarrow M_+(I)$ , it is sufficient to observe that (here for short we replace  $\|\cdot\|_{\mathcal{M}(I)}$  by  $\|\cdot\|$ )

$$\begin{aligned}\|T(u) - T(v)\| &= \left\| \|u\|^{-1} B(u, u - v) + (\|u\|^{-1} - \|v\|^{-1}) B(u, v) + \|v\|^{-1} B(u - v, v) \right\| \\ &\leq \left\| \|u\|^{-1} B(u, |u - v|) + (\|u\|^{-1} - \|v\|^{-1}) B(u, v) + \|v\|^{-1} B(|u - v|, v) \right\| \\ &\leq 2 \|u - v\| + \left| \|u\|^{-1} - \|v\|^{-1} \right| \|u\| \|v\| \\ &\leq 2 \|u - v\| + \left| \|u\| - \|v\| \right| \\ &\leq 3 \|u - v\|,\end{aligned}$$

therefore we obtain the following proposition.

**Proposition 4.2.** *Let Assumption 2.1 be satisfied. The operator  $T$  map  $M_+(I)$  into itself, and  $T$  satisfies the following properties*

- (i)  $T : M_+(I) \rightarrow M_+(I)$  is Lipschitz continuous.
- (ii)  $T$  is positively homogeneous. That is,

$$T(\lambda u) = \lambda T(u), \forall \lambda \geq 0, \forall u \in M_+(I).$$

- (iii)  $T$  preserves the total mass of individuals. That is,

$$\int_I T(u)(dx) = \int_I u(dx), \forall u \in M_+(I).$$

If moreover Assumption 2.2 holds, then

- (iv)  $T$  preserves the total mass of transferable quantity. That is,

$$\int_I xT(u)(dx) = \int_I xu(dx), \forall u \in M_+(I).$$

Therefore we obtain the following theorem.

**Theorem 4.3.** *Let Assumption 2.1 be satisfied. Then the Cauchy problem is*

$$\partial_t u(t, dx) = 2\tau T(u(t))(dx) - 2\tau u(t, dx), \quad (4.4)$$

with

$$u(0, dx) = \phi(dx) \in M_+(I). \quad (4.5)$$

The Cauchy problem generates a unique continuous homogeneous semiflow  $t \rightarrow S(t)\phi$  on  $M_+(I)$ . That is

(i) **(Semiflow property)**

$$S(0)\phi = \phi \text{ and } S(t)S(s)\phi = S(t+s)\phi, \forall t, s \geq 0, \forall \phi \in M_+(I).$$

(ii) **(Continuity)** The map  $(t, \phi) \rightarrow S(t)\phi$  is a continuous map from  $[0, +\infty) \times M_+(I)$  to  $M_+(I)$ .

(iii) **(Homogeneity)**

$$S(t)\lambda\phi = \lambda S(t)\phi, \forall t \geq 0, \forall \lambda \geq 0, \forall \phi \in M_+(I).$$

(iv) **(Preservation of the total mass of individuals)** The total mass of individuals is preserved

$$\int_I S(t)(\phi)(dx) = \int_I \phi(dx), \forall t \geq 0, \forall \phi \in M_+(I).$$

(v) **(From transfer rate 1/2 to any transfer rate  $\tau > 0$ )** If we define  $S^*(t)$  the semi-flow generated by (4.4)-(4.5) whenever  $\tau = 1/2$ , then

$$S(t) = S^*(2\tau t), \forall t \geq 0.$$

If moreover Assumption 2.2 holds, then

(vi) **(Preservation of the total mass of transferable quantity)** The total mass of transferable quantity is preserved

$$\int_I xS(t)(\phi)(dx) = \int_I x\phi(dx), \forall t \geq 0, \forall \phi \in M_+(I).$$

**Remark 4.4.** Let  $\mathcal{F} : \mathcal{M}(I) \rightarrow \mathbb{R}$  be a positive bounded linear form on  $\mathcal{M}(I)$ . We can consider for example

$$\mathcal{F}(u) = \int_I f(x)u(dx),$$

where  $f : I \rightarrow \mathbb{R}$  a bounded and positive continuous map on  $I$ .

Then  $(t, \phi) \mapsto U(t)\phi$  define on  $[0, +\infty) \times M_+(I)$  by

$$U(t)u = \frac{S(t)\phi}{1 + \int_0^t \mathcal{F}(S(\sigma)\phi)d\sigma},$$

is the unique solution of the Cauchy problem

$$u'(t) = 2\tau T(|u(t)|)(dx) - 2\tau u(t, dx) - \mathcal{F}(u(t))u(t, dx),$$

with

$$u(0, dx) = \phi(dx) \in M_+(I).$$

More detailed arguments can be found in Magal and Webb [15], and Magal [13].

**Remark 4.5.** The rate of transfers  $\tau(x)$  may vary in function of  $x$  the transferable quantity. In that case, we obtain the following model

$$\partial_t u(t, x) = T(2\tau(\cdot)u(t, \cdot))(x) - 2\tau(x)u(t, x), \text{ for } x \in \mathbb{R},$$

with

$$u(0, dx) = \phi(dx) \in \mathcal{M}(I).$$

## 5 Understanding (2.1) in $L^1(\mathbb{R})$

Recall that a Borel subset  $A \in \mathcal{B}(I)$  is said to be **negligible** if and only if  $A$  has a null Lebesgue measure. We consider the following assumption on the kernel  $K$ .

**Assumption 5.1.** *For each negligible subset  $A \in \mathcal{B}(I)$ , the set*

$$\mathcal{N}(A) := \{(x_1, x_2) \in I \times I : K(A, x_1, x_2) \neq 0\},$$

*is negligible.*

We note that an equivalent way to state Assumption 5.1 is the following: for each negligible subset  $A \in \mathcal{B}(I)$ , we have

$$\iint_{I \times I} K(A, x_1, x_2) dx_1 dx_2 = 0.$$

Thanks to the Radon-Nikodym Theorem (which is recalled in the Supplementary Materials as Theorem A.6), we have the following characterization of kernels that define a bilinear mapping from  $L^1(I) \times L^1(I)$  to  $L^1(I)$ .

**Proposition 5.2.** *Let Assumption 2.1 be satisfied. Then we have*

$$B(u, v) \in L^1(I) \text{ for all } (u, v) \in L^1(I) \times L^1(I)$$

*if, and only if, Assumption 5.1 is satisfied.*

*Proof.* For simplicity, here we call  $\mathcal{L}$  the one-dimensional Lebesgue measure, that is to say  $\mathcal{L}(A) = \int_A dx$ ; and  $\mathcal{L}^2 := \mathcal{L} \otimes \mathcal{L}$  the two-dimensional Lebesgue measure in  $\mathbb{R}^2$ .

Let  $u, v \in L^1(I)$  be given. Suppose that

$$\mathcal{L}^2(\mathcal{N}(A)) = 0,$$

for each  $A \in \mathcal{B}(I)$  with  $\mathcal{L}(A) = 0$ . Then by definition (see (4.1)),

$$B(u, v)(A) = \iint_{I \times I} K(A, x_1, x_2) u(x_1) dx_1 v(x_2) dx_2 = 0,$$

since  $(x_1, x_2) \mapsto K(A, x_1, x_2)$  is equal to zero  $\mathcal{L}^2$ -almost everywhere in  $\mathbb{R}^2$  by assumption. Therefore  $B(u, v)$  is absolutely continuous with respect to the Lebesgue measure  $\mathcal{L}$ , and by the Radon-Nikodym Theorem A.6, we can find function  $f \in L^1(I)$  such that

$$B(u, v)(dx) = f(x) dx,$$

which is equivalent to

$$B(u, v) \in L^1(I).$$

Conversely, assume that  $B(u, v) \in L^1(I)$  for any  $(u, v) \in L^1(I)^2$ . If  $I$  is bounded then  $1 \in L^1(I)$ , so taking  $u = v = 1$ , and  $B(1, 1)(dx) = f(x) dx$  with  $f \in L^1(I)$  gives

$$B(u, v)(A) = \iint_{I \times I} K(A, x_1, x_2) dx_1 dx_2 = \int_A f(x) dx = 0,$$

whenever  $\mathcal{L}(A) = 0$ , and we are done.

Let us consider the case when  $I$  is not bounded. Assume that  $A \in \mathcal{B}(I)$  is negligible. Define

$$u_n(x) = v_n(x) = \mathbb{1}_{[-n, n] \cap I}(x)$$

where  $x \mapsto \mathbb{1}_E(x)$  is the indicator function of the set  $E$ .

Then, we have by assumption,

$$B(u_n, v_n) = f_n(x) dx$$

for some  $f_n \in L^1(I)$ .

Moreover

$$B(u_n, v_n)(A) = \iint_{(I \cap [-n, n]) \times (I \cap [-n, n])} K(A, x_1, x_2) dx_1 dx_2 = \int_A f_n(x) dx = 0,$$

thus

$$\mathcal{L}(\mathcal{N}(A) \cap [-n, n]^2) = 0 \text{ for all } n \in \mathbb{N}.$$

Finally since we have an increase sequence of subsets, we obtain

$$\mathcal{L}(\mathcal{N}(A)) = \mathcal{L}\left(\mathcal{N}(A) \cap \bigcup_{n \in \mathbb{N}} [-n, n]^2\right) = \lim_{n \rightarrow +\infty} \mathcal{L}(\mathcal{N}(A) \cap [-n, n]^2) = 0,$$

and the proof is completed.  $\blacksquare$

Since the norm in the space of measure coincides with the  $L^1$  norm for an  $L^1$  function, we deduce that  $T$  maps  $L_+^1(I)$  into  $L_+^1(I)$  into itself, and the following statements are consequences of Theorem 4.3, and Proposition 5.2.

**Theorem 5.3.** *Let Assumption 2.1 and 5.1 be satisfied and consider the Cauchy problem*

$$\partial_t u(t, x) = 2\tau T(u(t)) - 2\tau u(t, x), \quad (5.1)$$

with

$$u(0, x) = \phi(x) \in L_+^1(I). \quad (5.2)$$

The Cauchy problem (5.1)-(5.2) generates a unique semiflow which is the restriction of  $S(t)$  to  $L_+^1(I)$ . We deduce that

$$S(t)L_+^1(I) \subset L_+^1(I), \forall t \geq 0,$$

and the semiflow  $t \rightarrow S(t)\phi$  restricted to  $L_+^1(I)$  satisfies the following properties:

- (i) **(Continuity)** The map  $(t, \phi) \rightarrow S(t)\phi$  is a continuous map from  $[0, +\infty) \times L_+^1(I)$  to  $L_+^1(I)$ .
- (ii) **(Preservation of the total mass of individuals)** The total mass of individuals is preserved

$$\int_I S(t)(\phi)(x) dx = \int_I \phi(x) dx, \forall t \geq 0, \forall \lambda \geq 0, \forall \phi \in M_+(I).$$

If moreover Assumption 2.2 holds, then we have in addition

- (iii) **(Preservation of the total mass of transferable quantity)** The total mass of transferable quantity is preserved

$$\int_I x S(t)(\phi)(x) dx = \int_I x \phi(x) dx, \forall t \geq 0, \forall \lambda \geq 0, \forall \phi \in M_+(I).$$

**Example 5.4** (Robin Hood model). Let  $K(dx, x_1, x_2) = K_1(dx, x_1, x_2) = \frac{1}{2}(\delta_{x_2 - f(x_2 - x_1)}(dx) + \delta_{x_1 - f(x_1 - x_2)}(dx))$ . If  $A \in \mathcal{B}(I)$  has zero Lebesgue measure, then we have:

$$\begin{aligned} \mathcal{N}(A) &= \{(x_1, x_2) \in I \times I : K(A, x_1, x_2) > 0\} \\ &= \{(x_1, x_2) : x_2 - f(x_2 - x_1) = y \in A \text{ or } x_1 - f(x_1 - x_2) = z \in A\} \\ &= \left\{ (x_1, x_2) : x_1 = \frac{1-f}{1-2f}z - \frac{f}{1-2f}y \text{ and } \right. \\ &\quad \left. x_2 = \frac{1-f}{1-2f}y - \frac{f}{1-2f}z \text{ and } (y \in A \text{ or } z \in A) \right\} \\ &= \left\{ \left( \frac{1-f}{1-2f}z - \frac{f}{1-2f}y, \frac{1-f}{1-2f}y - \frac{f}{1-2f}z \right) : y \in A, z \in I \right\} \\ &\quad \cup \left\{ \left( \frac{1-f}{1-2f}z - \frac{f}{1-2f}y, \frac{1-f}{1-2f}y - \frac{f}{1-2f}z \right) : y \in I, z \in A \right\}. \end{aligned}$$

The two sets above have zero Lebesgue measure because they are the image of  $A \times I$  and  $I \times A$  by a linear invertible transformation. Therefore  $\mathcal{N}(A)$  has zero Lebesgue measure and we can apply Proposition 5.2.

**Example 5.5** (Sheriff of Nottingham model). Let  $K(dx, x_1, x_2) = K_2(dx, x_1, x_2) = \frac{1}{2}(\delta_{x_2+f(x_2-x_1)}(dx) + \delta_{x_1+f(x_1-x_2)}(dx))$ . If  $A \in \mathcal{B}(\mathbb{R})$  has zero Lebesgue measure, then we have:

$$\begin{aligned} \mathcal{N}(A) &= \{(x_1, x_2) \in I \times I : K(A, x_1, x_2) > 0\} \\ &= \{(x_1, x_2) : x_2 + f(x_2 - x_1) = y \in A \text{ or } x_1 + f(x_1 - x_2) = z \in A\} \\ &= \left\{ \left( \frac{1+f}{1+2f}z + \frac{f}{1+2f}y, \frac{1+f}{1+2f}y + \frac{f}{1+2f}z \right) : y \in A, z \in I \right\} \\ &\cup \left\{ \left( \frac{1+f}{1+2f}z + \frac{f}{1+2f}y, \frac{1+f}{1+2f}y + \frac{f}{1+2f}z \right) : y \in I, z \in A \right\}. \end{aligned}$$

The two sets above have zero Lebesgue measure because they are the image of  $A \times \mathbb{R}$  and  $\mathbb{R} \times A$  by a linear invertible transformation. Therefore  $\mathcal{N}(A)$  has zero Lebesgue measure and we can apply Proposition 5.2.

Similarly, the mixed Robin Hood and Sheriff of Nottingham model also define a bilinear mapping from  $L^1 \times L^1$  to  $L^1$ .

**Example 5.6** (Distributed Robin Hood or Sheriff of Nottingham models). The kernel of the distributed Robin Hood model consists in replacing the Dirac mass centered at 0, by  $x \rightarrow g(x) \in L^1(I)$  a density of probability centered at 0. That is

$$K_3(dx, x_1, x_2) = \frac{1}{2} \left\{ g(x - [x_2 - f(x_2 - x_1)]) + g(x - [x_1 - f(x_1 - x_2)]) \right\} dx.$$

Similarly, the kernel of the distributed Sheriff of Nottingham model is the following

$$K_4(dx, x_1, x_2) := \frac{1}{2} \left\{ g(x - [x_2 + f(x_2 - x_1)]) + g(x - [x_1 + f(x_1 - x_2)]) \right\} dx.$$

Let  $K(dx, x_1, x_2) = K(x, x_1, x_2)dx$  with  $K(x, x_1, x_2) \in L^1(I)$  for any  $(x_1, x_2) \in I \times I$ . Examples are the distributed Robin Hood model, distributed Sheriff of Nottingham model, and distributed mixed Robin Hood and Sheriff of Nottingham model. If  $A \in \mathcal{B}(I)$  has zero Lebesgue measure, then we have automatically

$$K(A, x_1, x_2) = \int_A K(x, x_1, x_2)dx = 0 \text{ for any } (x_1, x_2) \in I \times I, \text{ so } \mathcal{N}(A) = \emptyset.$$

Therefore we can apply Proposition 5.2.

## 6 Numerical simulation

We introduce  $p \in [0, 1]$ , the population's redistribution fraction. The parameter  $p$  is also the probability of applying the Robin Hood (RH) model during a transfer between two individuals. Otherwise, we use the Sheriff of Nottingham (SN) model with the probability  $1 - p$ . In that case, the model is the following

$$\partial_t u(t, dx) = 2\tau [pT_1(u(t))(dx) + (1-p)T_2(u(t))(dx)] - 2\tau u(t, dx), \quad (6.1)$$

with

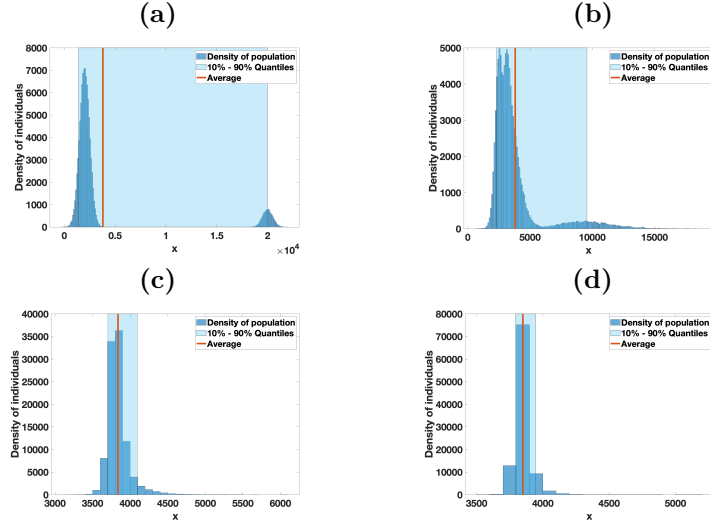
$$u(0, dx) = \phi(dx) \in M_+(I). \quad (6.2)$$

In Figures 3-7, we run an individual based simulation of the model (6.1)-(6.2). Such simulations are stochastic. We first choose a pair randomly following an exponential law with average  $1/\tau$ . Then we choose the RH model with a probability  $p$  and the SN model with a probability  $1-p$ . Then we apply the transfers rule described in section 3. To connect this problem with our description in the space of measures, we can consider an initial distribution that is a sum of Dirac masses.

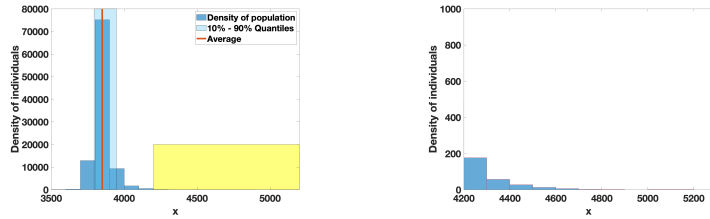
$$\phi(dx) = \sum_{i=1}^N \delta_{x_i}(x),$$

in which  $x_i$  is the value of the transferable quantity for individual  $i$  at  $t = 0$ .

**Figure 3** In this figure, we use  $p = 1$  (i.e. 100% RH model),  $f_1 = f_2 = 0.1$ ,  $1/\tau = 1$  years. We start the simulations with 100 000 individuals. The figures (a) (b) (c) (d) are respectively the initial distribution at time  $t = 0$ , and the distribution 10 years, 50 years and 100 years.

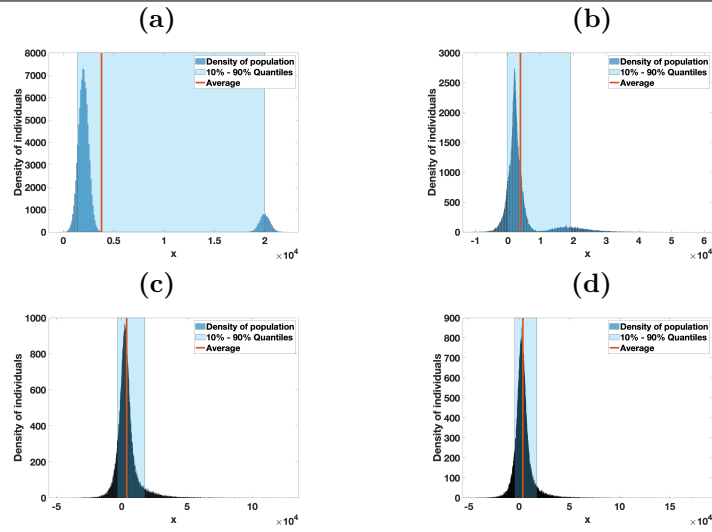


**Figure 4** In this figure, we zoom on the distribution for  $t = 100$  in Figure 3 (d). The figure on the right-hand side corresponds to the yellow region in the left figure.

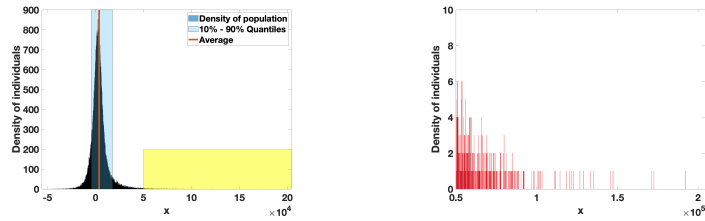




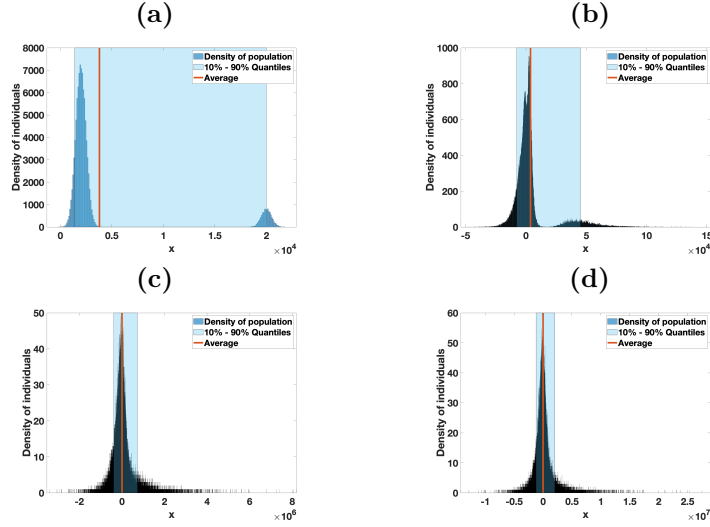
**Figure 5** In this figure, we use  $p = 0.5$  (i.e. 50% RH model and 50% SN model),  $f_1 = f_2 = 0.1$ ,  $1/\tau = 1$  years. We start the simulations with 100 000 individuals. The figures (a) (b) (c) (d) are respectively the initial distribution at time  $t = 0$ , and the distribution 10 years, 50 years and 100 years.



**Figure 6** In this figure, we zoom on the distribution for  $t = 100$  in Figure 5 (d). The figure on the right-hand side corresponds to the yellow region in the left figure.



**Figure 7** In this figure, we use  $p = 0$  (i.e. 0% RH model and 100% SN model),  $f_1 = f_2 = 0.1$ ,  $1/\tau = 1$  years. We start the simulations with 100 000 individuals. The figures (a) (b) (c) (d) are respectively the initial distribution at time  $t = 0$ , and the distribution 10 years, 50 years and 100 years.



**Figure 8** In this figure, we zoom on the distribution for  $t = 100$  in Figure 7 (d). The figure on the right-hand side corresponds to the yellow region in the left figure.

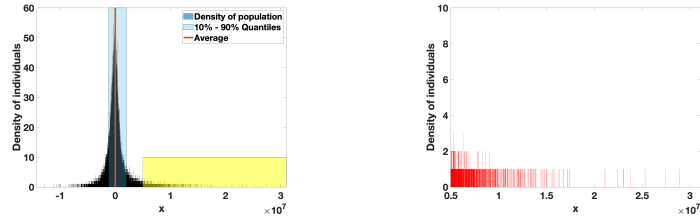


Figure 3 corresponds to the full RH model which corresponds to  $p = 1$ . In that case, the population density converges to a Dirac mass centered at the mean value. That is, everyone will ultimately have the same amount of transferable quantity.

Whatever the value of  $p$  strictly less than 1, the simulations can be summed up by saying that "there is always a sheriff in town". In Figures 5-7, the unit for  $x$ -axis changes from (a) to (d). We can see from that some rich guys will always become richer and richer. The SN model induces competition between the poorest individuals also, and the population ends up after 100 years with a lot of debts. In other words, the richest individuals are becoming richer, while the poorest are becoming poorer. The effect in changing the value of the parameter  $p \in [0, 1)$  is strictly positive, it seems that it is only a matter of time before we end up with a very segregated population. We observe a difference for the richest of two orders of magnitude between the case  $p = 0.5$  and  $p = 0$ . We conclude by observing that the smaller  $p$  is, the more the wealthiest individuals are rich.

**Data availability:** No data were produced for this study.

**Conflict of interest:** The authors declare no conflict of interest.

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