



HAL
open science

The g-extra connectivity of graph products

Zhao Wang, Yaping Mao, Sun-Yuan Hsieh, Ralf Klasing, Yuzhi Xiao

► **To cite this version:**

Zhao Wang, Yaping Mao, Sun-Yuan Hsieh, Ralf Klasing, Yuzhi Xiao. The g-extra connectivity of graph products. *Journal of Computer and System Sciences*, 2024, 145, pp.103552. 10.1016/J.JCSS.2024.103552 . hal-04675801

HAL Id: hal-04675801

<https://hal.science/hal-04675801v1>

Submitted on 22 Aug 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The g -extra connectivity of graph products ^{*}

Zhao Wang[†], Yaping Mao[‡], Sun-Yuan Hsieh[§], Ralf Klasing[¶], Yuzhi Xiao^{||}

Abstract

Connectivity is one of important parameters for the fault tolerant of an inter-connection network. In 1996, Fàbrega and Fiol proposed the concept of g -extra connectivity. A subset of vertices S is said to be a *cutset* if $G - S$ is not connected. A cutset S is called an R_g -cutset, where g is a non-negative integer, if every component of $G - S$ has at least $g + 1$ vertices. If G has at least one R_g -cutset, the g -extra connectivity of G , denoted by $\kappa_g(G)$, is then defined as the minimum cardinality over all R_g -cutsets of G . In this paper, we first obtain the exact value of g -extra connectivity for the lexicographic product of two general graphs. Next, the upper and lower sharp bounds of g -extra connectivity for the Cartesian product of two general graphs are given. In the end, we apply our results on grid graphs and 2-dimensional generalized hypercubes.

Keywords: Connectivity; g -extra connectivity; Cartesian product; Lexicographic product.

AMS subject classification 2020: 05C40; 05C05; 05C76.

^{*}Supported by the National Science Foundation of China (Nos. 11601254, 11551001, 11161037, 61763041, 11661068, and 11461054) and the Science Found of Qinghai Province (Nos. 2016-ZJ-948Q, and 2014-ZJ-907) and the Qinghai Key Laboratory of Internet of Things Project (2017-ZJ-Y21).

[†]College of Science, China Jiliang University, Hangzhou 310018, China. wangzhao@mail.bnu.edu.cn

[‡]Corresponding author: School of Mathematics and Statistis, and Academy of Plateau Science and Sustainability, Qinghai Normal University, Xining, Qinghai 810008, China. maoyaping@ymail.com

[§]Department of Computer Science and Information Engineering, National Cheng Kung University, Tainan 701, Taiwan hsiehsy@mail.ncku.edu.tw

[¶]Université de Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR 5800, Talence, France. ralf.klasing@labri.fr

^{||}School of Coputer, Qinghai Normal University, Xining, Qinghai 810008, China. qh_xiaoyuzhi@139.com

1 Introduction

For a graph G , let $V(G)$, $E(G)$, $e(G)$, \overline{G} , and $diam(G)$ denote the set of vertices, the set of edges, the size, the complement, and the diameter of G , respectively. A subgraph H of G is a graph with $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and the endpoints of every edge in $E(H)$ belonging to $V(H)$. For any subset X of $V(G)$, let $G[X]$ denote the subgraph induced by X ; similarly, for any subset F of $E(G)$, let $G[F]$ denote the subgraph induced by F . We use $G - X$ to denote the subgraph of G obtained by removing all the vertices of X together with the edges incident with them from G ; similarly, we use $G - F$ to denote the subgraph of G obtained by removing all the edges of F from G . If $X = \{v\}$ and $F = \{e\}$, we simply write $G - v$ and $G - e$ for $G - \{v\}$ and $G - \{e\}$, respectively. The *degree* of a vertex v in a graph G , denoted by $deg_G(v)$, is the number of edges of G incident with v . Let $\delta(G)$ and $\Delta(G)$ be the minimum degree and maximum degree of the vertices of G , respectively. The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$. The *union* $G \cup H$ of two graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left.

1.1 g -extra connectivity

The g -extra connectivity has been an object of interest for many years, and it was firstly introduced by Fàbrega and Fiol [9]. A subset of vertices S is said to be a *cutset* if $G - S$ is not connected. A cutset S is called an R_g -*cutset*, where g is a non-negative integer, if every component of $G - S$ has at least $g + 1$ vertices. If G has at least one R_g -cutset, the g -*extra connectivity* of G , denoted by $\kappa_g(G)$, is then defined as the minimum cardinality over all R_g -cutsets of G . Clearly, $\kappa_0(G) = \kappa(G)$ for any connected non-complete graph G . So the g -extra connectivity can be viewed as a generalization of the traditional connectivity, and it can more accurately evaluate the reliability and fault tolerance for large-scale parallel processing systems accordingly. In [21], Wang et al. obtained the exact values of g -extra connectivity of some special graphs. Graphs with $\kappa_g(G) = 1, 2, 3$ and trees with $\kappa_g(T_n) = n - 2g - 2$ was characterized, respectively, where G is a general graph and T_n is a tree of order n . Three extremal results for the g -extra connectivity was also obtained. For more research on g -extra connectivity and related topic, we refer to [1, 4, 5, 6, 7, 9, 11, 12, 17, 20, 22, 23, 27, 28].

1.2 Graph products

Product networks were proposed based upon the idea of using the cross product as a tool for “combining” two known graphs with established properties to obtain a new one that inherits properties from both [3]. In graph theory, Cartesian product, lexicographical product are two of main products, each with its own set of applications and theoretical interpretations.

- The *Cartesian product* of two graphs G and H , written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$.

- The *lexicographic product* $G \circ H$ of graphs G and H has the vertex set $V(G \circ H) = V(G) \times V(H)$. Two vertices $(u, v), (u', v')$ are adjacent if $uu' \in E(G)$, or if $u = u'$ and $vv' \in E(H)$.

For more details, we refer to [13].

1.3 Application backgrounds

With the rapid development of VLSI technology, a multiprocessor system may contain thousands of nodes, and some of them may be faulty when the system is implemented. As the number of processors in a system increases, the possibility that its processors may be come faulty also increases. Because designing such systems without defects is nearly impossible, reliability and fault tolerance are two of the most critical concerns of multiprocessor systems [25].

By the definition proposed by Esfahanian [8], a multiprocessor system is fault tolerant if it can remain functional in the presence of failures. Two basic functionality criteria have received considerable attention. The first criterion for a system to be regarded as functional is whether the network logically contains a certain topological structure. This is the problem that occurs when embedding one architecture into another [15, 24]. This approach involves using system-wide redundancy and reconfiguration. The second functionality criterion considers a multiprocessor system functional if a fault-free communication path exists between any two fault-free nodes; that is, the topological structure of the multiprocessor system remains connected in the presence of certain failures. Thus, connectivity and edge connectivity are two major measurements of this criterion [24]. However, these two parameters tacitly assume that all vertices that are adjacent to, or all edges that are incident to, the same vertex can potentially fail simultaneously. This is practically impossible in some network applications. To address this deficiency, two specific terms forbidden faulty set and forbidden faulty edge

set are introduced. The vertices in a forbidden faulty set or the edges in a forbidden faulty edge set cannot fail simultaneously.

The monotone property of $\kappa_g(G)$ for non-negative integer g is true.

Lemma 1.1. [21] *Let g be a non-negative integer, and let G be a connected graph. Then*

$$\kappa_g(G) \leq \kappa_{g+1}(G)$$

The range of the integer g can be determined immediately.

Lemma 1.2. [21] *Let g be a non-negative integer. If G has its g -extra connectivity, then*

$$0 \leq g \leq \left\lfloor \frac{n-3}{2} \right\rfloor \text{ and } e(G) \leq \binom{n}{2} - (g+1)^2.$$

The following upper and lower bounds are immediate.

Proposition 1.1. [21] *Let g be a non-negative integer and let G be a connected graph of order n such that $0 \leq g \leq \left\lfloor \frac{n-\kappa(G)-2}{2} \right\rfloor$. Then*

$$\kappa(G) \leq \kappa_g(G) \leq n - 2g - 2.$$

Moreover, the upper and lower bounds are sharp.

2 Main results

In this section, let G and H be two connected graphs with $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H) = \{v_1, v_2, \dots, v_m\}$, respectively. Then $V(G * H) = \{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, where $*$ denotes lexicographic product operation, strong product operation, Cartesian product operation or direct product operation. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u_i, v) \mid 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u, v_j) \mid 1 \leq j \leq m\}$.

2.1 Lexicographic product

Yang and Xu [26] investigated the classical connectivity of the lexicographic product of two graphs.

Theorem 2.1. [26] *Let G and H be two graphs. If G is non-trivial, non-complete and connected, then*

$$\kappa(G \circ H) = \kappa(G)|V(H)|.$$

We can get the exact value of $\kappa_g(G \circ H)$.

Theorem 2.2. *Let G, H be two connected graphs of order n, m , respectively, such that G is not complete.*

(1) *If $0 \leq g \leq m - 1$, then*

$$\kappa_g(G \circ H) = \kappa(G)m.$$

(2) *If $km < g + 1 \leq (k + 1)m$, then*

$$\kappa_g(G \circ H) = |A|m,$$

where A is a minimum vertex subset of G such that the resulting graph of $G - A$ is not connected and each connected component has at least $(k + 1)$ vertices.

Proof. (1) From Theorem 2.1, $\kappa_g(G \circ H) \geq \kappa(G \circ H) = \kappa(G)m$. It suffices to show that $\kappa_g(G \circ H) \leq \kappa(G)m$. Let $\kappa(G) = x$. From the definition of $\kappa(G)$, there exists $X \subseteq V(G)$ and $|X| = x$ such that $G - X$ is not connected. Without loss of generality, let $X = \{u_1, u_2, \dots, u_x\}$. Let

$$Y = H(u_1) \cup H(u_2) \cup \dots \cup H(u_x).$$

It is clear that $|Y| = xm$ and $(G \circ H) - Y$ is not connected and each component has at least $m \geq g + 1$ vertices. So $\kappa_g(G \circ H) \leq \kappa(G)m$.

(2) From the definition of A , $G - A$ is not connected and each connected component of $G - A$ has at least $k + 1$ vertices. Let $|A| = x$. Without loss of generality, let $A = \{u_{i_1}, u_{i_2}, \dots, u_{i_x}\}$. Let

$$S = V(H(u_{i_1})) \cup V(H(u_{i_2})) \cup \dots \cup V(H(u_{i_x})).$$

Then $|S| = mx$. Clearly, $(G \circ H) - S$ is not connected and each connected component of $(G \circ H) - S$ has at least $(k + 1)m$ vertices, and hence $\kappa_g(G \circ H) \leq |S| = mx = |A|m$.

It suffices to show $\kappa_g(G \circ H) \geq |A|m$. Suppose $\kappa_g(G \circ H) \leq |A|m - 1$. For any $X \subseteq V(G \circ H)$ with $|X| \leq |A|m - 1$, we suppose that $(G \circ H) - X$ is not connected.

Let $H(u_{i_1}), H(u_{i_2}), \dots, H(u_{i_r})$ such that $\bigcup_{j=1}^r H(u_{i_j}) \subseteq X$ and r is maximized. Since $|X| \leq |A|m - 1$, it follows that $r \leq |A| - 1$. Then $G - \{u_{i_1}, u_{i_2}, \dots, u_{i_r}\}$ is not connected. Let C_1, C_2, \dots, C_y be the components of $G - \{u_{i_1}, u_{i_2}, \dots, u_{i_r}\}$. From the minimality of A , there exists some component of $G - \{u_{i_1}, u_{i_2}, \dots, u_{i_r}\}$, say C_p , having at most k vertices. Let $V(C_p) = \{u_{j_1}, u_{j_2}, \dots, u_{j_t}\}$. Then there exists a component of $(G \circ H) - X$ such that it is a subgraph of $C_p \circ H$, and hence $|V(C_p \circ H)| \leq km < g + 1$. \square

2.2 Cartesian product

Sabidussi [18] derived the following theorem on the connectivity of Cartesian product graphs.

Theorem 2.3. [18] *Let G and H be two connected graphs. Then*

$$\kappa(G \square H) \geq \kappa(G) + \kappa(H).$$

But we mention that it is incorrectly claimed that $\kappa(G \square H) = \kappa(G) + \kappa(H)$ holds for any connected G and H , see [13] (p-308). Let G be a graph obtained from two triangles by identifying one vertex in each of them. Then $\kappa(G) = 1$, $\kappa(G \square G) = 4 > 2 = \kappa(G) + \kappa(G)$, see [13] (p-309). In [19], Špacapan obtained the following result.

Theorem 2.4. [19] *Let G and H be two nontrivial graphs. Then*

$$\kappa(G \square H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\}.$$

We can get the following upper and lower bounds of $\kappa_g(G \square H)$.

Theorem 2.5. *Let G, H be two non-complete connected graphs of order n, m , respectively, where $n \geq m \geq 3$. Then*

$$\kappa_g(G \square H) \geq \min\{\kappa(G)m, \kappa(H)n, \delta(G) + \delta(H)\}.$$

Furthermore, (1) if $0 \leq g \leq m - 1$, then

$$\kappa_g(G \square H) \leq \min\{\kappa(G)m, \kappa(H)n, (g+1)\delta(G) + (m-g-1), (g+1)\delta(H) + (n-g-1)\};$$

(2) if $m \leq g \leq n - 1$, then

$$\kappa_g(G \square H) \leq \min\{\kappa(H)n, (g+1)\delta(H) + (n-g-1)\};$$

(3) if $km < g + 1 \leq (k+1)m$, then

$$\kappa_g(G \square H) \leq |A|m,$$

where A is a minimum vertex subset of G such that the resulting graph of $G - A$ is not connected and each connected component has at least $(k+1)$ vertices.

(4) if $kn < g + 1 \leq (k+1)n$, then

$$\kappa_g(G \square H) \leq |B|n,$$

where B is a minimum vertex subset of H such that the resulting graph of $H - B$ is not connected and each connected component has at least $(k+1)$ vertices.

Moreover, the upper and lower bounds are sharp.

Proof. From Theorem 2.4 and Proposition 1.1, we have

$$\kappa_g(G \square H) \geq \kappa(G \square H) = \min\{\kappa(G)m, \kappa(H)n, \delta(G) + \delta(H)\}.$$

(1) If $1 \leq g \leq m - 1$, then it suffices to show that

$$\kappa_g(G \square H) \leq \min\{\kappa(G)m, \kappa(H)n, (g+1)\delta(G) + (m-g-1), (g+1)\delta(H) + (n-g-1)\}.$$

From the definition of $\kappa(G)$, there exists $X \subseteq V(G)$ and $|X| = \kappa(G)$ such that $G - X$ is not connected. Let C_1, C_2, \dots, C_r be the components of $G - X$. Then $|V(C_i)| \geq 1$ for each i ($1 \leq i \leq r$). Let $Y = X \square H$. Then $(G \square H) - Y$ is not connected and $C_i \square H$ ($1 \leq i \leq r$) are all components of $(G \square H) - Y$. Since $|V(C_i \square H)| = |V(C_i)||V(H)| \geq |V(H)| = m$ and $1 \leq g \leq m - 1$, it follows that $|V(C_i \square H)| \geq g + 1$, and hence $\kappa_g(G \square H) \leq |Y| = |X|m = \kappa(G)m$. Similarly, we have $\kappa_g(G \square H) \leq \kappa(H)n$.

It is clear that there exists a vertex u in G such that $\deg_G(u) = \delta(G)$. Let $N_G(u) = \{u_1, u_2, \dots, u_\delta\}$, and let $(u, v_1), (u, v_2), \dots, (u, v_{g+1})$ be the vertices such that their induced subgraph is connected in $H(u)$. Let

$$Z = \{(u_i, v_j) \mid 1 \leq i \leq \delta, 1 \leq j \leq g+1\} \cup \{(u, v_j) \mid g+2 \leq j \leq m\}.$$

Then $(G \square H) - Z$ is not connected. Note that $\deg_G(u_i) \geq 2$ for each u_i ($1 \leq i \leq \delta$). If $Z \cap H(u_i) = \emptyset$, then the component containing $H(u_j)$ in $(G \square H) - Z$ has at least $g + 1$ vertices. If $Z \cap H(u_i) \neq \emptyset$, then without loss of generality, we suppose $Z \cap H(u_1) \neq \emptyset$. If there exists some vertex $u_x \in N_G(u_1)$ such that $u_x \notin \{u, u_2, \dots, u_\delta\}$, then the component containing $H(u_1) - Z$ in $(G \square H) - Z$ must contain $H(u_x)$. If $N_G(u_1) \subseteq \{u, u_2, \dots, u_\delta\}$, then there exist some u_j ($2 \leq j \leq \delta$) such that $N_G(u_j) \not\subseteq \{u, u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_\delta\}$ since G is not complete, and hence there exists some vertex u_y such that $u_y u_j \in E(G \square H)$ and $u_y \notin \{u, u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_\delta\}$. Furthermore, the component containing $H(u_1) - Z$ in $(G \square H) - Z$ must contain $H(u_y)$, and it has at least $g + 1$ vertices. If $u_i = u$, then the component containing $H(u) - Z$ in $(G \square H) - Z$ has at least $g + 1$ vertices. We conclude that

$$\kappa_g(G) \leq |Z| = (g+1)\delta(G) + (m-g-1).$$

Similarly, we have

$$\kappa_g(G) \leq (g+1)\delta(H) + (n-g-1),$$

as desired.

(2) It follows from (1) similarly.

(3) Let $A = \{u_1, u_2, \dots, u_r\}$, and let $X = H(u_1) \cup H(u_2) \cup \dots \cup H(u_r)$. Since $G - X$ is not connected and each component has at least $k + 1$ vertices, it follows that $(G \square H) - X$ is not connected and each component has at least $(k + 1)m \geq g + 1$ vertices, and hence $\kappa_g(G \square H) \leq rm = |A|m$.

(4) It follows from (3) similarly. \square

3 Applications

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of Cartesian and lexicographical product networks.

3.1 Two-dimensional grid graph

A *two-dimensional grid graph* is the Cartesian product $P_n \square P_m$ of path graphs on m and n vertices; see Figure 1 (a) for the case $m = 3$. For more details on grid graph, we refer to [2, 14]. The network $P_n \circ P_m$ is the graph lexicographical product $P_n \circ P_m$ of path graphs on m and n vertices; see Figure 1 (b) for the case $m = 3$. For more details on $P_n \circ P_m$, we refer to [16].

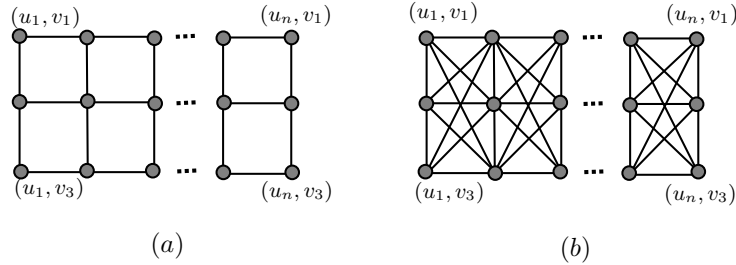


Figure 1: (a) Two-dimensional grid graph $P_n \square P_3$; (b) The network $P_n \circ P_3$.

The *boundary* of $P_n \circ P_m$, say B , is defined as the subgraph of $P_n \square P_m$ induced by the vertices in $B_1 \cup B_2 \cup B_3 \cup B_4$, where $B_1 = \{(u_1, v_j) \mid 1 \leq j \leq m\}$, $B_2 = \{(u_n, v_j) \mid 1 \leq j \leq m\}$, $B_3 = \{(u_i, v_1) \mid 1 \leq i \leq n\}$, and $B_4 = \{(u_i, v_m) \mid 1 \leq i \leq n\}$. We call B_1, B_2, B_3, B_4 are sections of B .

Proposition 3.1. For network $P_n \square P_m$ ($n \geq m \geq 3$),

$$\kappa_g(P_n \square P_m) = \begin{cases} \min\{t \mid g \leq \frac{t(t-1)}{2} - 1\}, & \text{if } 0 \leq g \leq \frac{m(m-1)}{2} - 1; \\ m, & \text{if } \frac{m(m-1)}{2} \leq g \leq \lfloor \frac{mn-m-2}{2} \rfloor. \end{cases}$$

Proof. Suppose $0 \leq g \leq \frac{m(m-1)}{2} - 1$. Let $t_0 = \min\{t \mid g \leq \frac{t(t-1)}{2} - 1\}$. Let $P_n = u_1 u_2 \dots u_n$ and $P_m = v_1 v_2 \dots v_m$. Then $V(P_n \square P_m) = \{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. Choose

$$X = \{(u_i, v_j) \mid i + j = t_0 + 1, 1 \leq i \leq n, 1 \leq j \leq m\}.$$

It is clear that $(P_n \square P_m) - X$ is not connected, and each component has at least $g + 1$ vertices, and $|X| = t_0$. So we have $\kappa_g(P_n \square P_m) \leq t_0 = \min\{t \mid g \leq \frac{t(t-1)}{2} - 1\}$.

It suffices to show that $\kappa_g(P_n \square P_m) \geq \min\{t \mid g \leq \frac{t(t-1)}{2} - 1\}$. We have the following claim.

Claim 1. *For any $X \subseteq V(P_n \square P_m)$ with $|X| = x \leq m - 1$, if $P_n \square P_m - X$ is not connected, then there exists a connected component C such that $|V(C)| \leq \frac{x(x-1)}{2}$.*

Proof. Let C_1, C_2, \dots, C_r be all the components of $(P_n \square P_m) - X$. We first assume that there exists a component, say C_1 , in $(P_n \square P_m) - X$ such that $V(C_1) \cap V(B) = \emptyset$. Then there is no edges between $\bigcup_{i=2}^r C_i$ and C_1 , and $N_{P_n \square P_m}(C_1) \subseteq X$. Since $|X| = x$, it follows that $|V(C_1)| \leq \lceil \frac{x^2}{16} \rceil \leq \frac{x(x-1)}{2}$, as desired.

Next, we assume that for each component C_i ($1 \leq i \leq r$), $V(C_i) \cap V(B) \neq \emptyset$. Since $|X| = x \leq m - 1$, it follows that there exists a component C_j satisfying one of the following.

- There exists one section of B , say B_1 , such that $V(C_j) \cap V(B_1) \neq \emptyset$ and $V(C_j) \cap V(B_k) = \emptyset$ for $k = 2, 3, 4$.
- There exist two adjacent sections of B , say B_1, B_2 , such that $V(C_j) \cap V(B_i) \neq \emptyset$ for $i = 1, 2$ and $V(C_j) \cap V(B_k) = \emptyset$ for $k = 3, 4$.

It is clear that $|V(C_j)| \leq \frac{x(x-1)}{2}$, as desired. □

Claim 2. $\kappa_g(P_n \square P_m) = t_0$ for $0 \leq g \leq \frac{m(m-1)}{2} - 1$.

Proof. Assume, to the contrary, that $\kappa_g(P_n \square P_m) \leq t_0 - 1$. From Claim 1, for any $X \subseteq V(P_n \square P_m)$ with $|X| = x \leq m - 1$, if $(P_n \square P_m) - X$ is not connected, then there exists a connected component C such that $|V(C)| \leq \frac{(t_0-1)(t_0-2)}{2} < g$, a contradiction. □

From Claim 2, we have $\kappa_g(P_n \square P_m) = t_0 = \min\{t \mid g \leq \frac{t(t-1)}{2} - 1\}$.

Suppose $\frac{m(m-1)}{2} \leq g \leq \lfloor \frac{mn-m-2}{2} \rfloor$. If n is even, then we choose $X = \{(u_{\frac{n}{2}}, v_i) \mid 1 \leq i \leq m\}$. If n is odd, then we choose $X = \{(u_{\lfloor \frac{n-1}{2} \rfloor}, v_i) \mid 1 \leq i \leq \lceil \frac{m}{2} \rceil\} \cup \{(u_{\lceil \frac{n-1}{2} \rceil}, v_i) \mid \lceil \frac{m}{2} \rceil + 1 \leq i \leq m\}$. It is clear that $(P_n \square P_m) - X$ is not connected, and each component has at least $g + 1$ vertices. So we have $\kappa_g(P_n \square P_m) \leq m$.

Claim 3. $\kappa_g(P_n \square P_m) = m$ for $\frac{m(m-1)}{2} \leq g \leq \lfloor \frac{mn-m-2}{2} \rfloor$.

Proof. Assume, to the contrary, that $\kappa_g(P_n \square P_m) \leq m-1$. From Claim 1, for any $X \subseteq V(P_n \square P_m)$ with $|X| = x \leq m-1$, if $(P_n \square P_m) - X$ is not connected, then there exists a connected component C such that $|V(C)| \leq \frac{(m-1)(m-2)}{2} < g$, a contradiction. \square

From Claim 3, the result follows for $\frac{m(m-1)}{2} \leq g \leq \lfloor \frac{mn-m-2}{2} \rfloor$. \square

The following corollary is immediate from Theorem 2.2.

Corollary 3.1. For network $P_n \circ P_m$ ($n \geq m \geq 3$), $\kappa_g(P_n \circ P_m) = m$ for $0 \leq g \leq \lfloor \frac{mn-m-2}{2} \rfloor$.

3.2 2-dimensional generalized hypercube

Let K_m be a clique of m vertices, $m \geq 2$. An n -dimensional generalized hypercube [3, 10] is the product of n cliques.

Proposition 3.2. For network $K_n \square K_m$ ($n \geq m \geq 2$) and $0 \leq g \leq \frac{mn}{4} - 1$, we have

$$\kappa_g(K_n \square K_m) = \min\{(m - y_1)x_1 + (n - x_1)y_1, (m - y_2)x_2 + (n - x_2)y_2\},$$

where

$$y_1 = \left\lceil \sqrt{\frac{m(g+1)}{n}} \right\rceil, \quad x_1 = \left\lceil \frac{g+1}{y_1} \right\rceil$$

and

$$y_2 = \left\lceil \sqrt{\frac{m(g+1)}{n}} \right\rceil (y_2 \neq 0), \quad x_2 = \left\lceil \frac{g+1}{y_2} \right\rceil.$$

Proof. Let $V(K_n) = \{u_1, u_2, \dots, u_n\}$ and $V(K_m) = \{v_1, v_2, \dots, v_m\}$. Then $V(G \square H) = V(K_n \square K_m) = \{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. Choose

$$X_1 = \left\{ (u_i, v_j) \mid 1 \leq i \leq x_1, y_1 + 1 \leq j \leq m \right\} \cup \left\{ (u_i, v_j) \mid x_1 + 1 \leq i \leq n, 1 \leq j \leq y_1 \right\}$$

and

$$X_2 = \left\{ (u_i, v_j) \mid 1 \leq i \leq x_2, y_2 + 1 \leq j \leq m \right\} \cup \left\{ (u_i, v_j) \mid x_2 + 1 \leq i \leq n, 1 \leq j \leq y_2 \right\}.$$

It is clear that $(K_n \square K_m) - X_1$ and $(K_n \square K_m) - X_2$ are not connected, and each component has at least $g+1$ vertices. So we have

$$\kappa_g(K_n \square K_m) \leq \min\{(m - y_1)x_1 + (n - x_1)y_1, (m - y_2)x_2 + (n - x_2)y_2\}.$$

For each $X \subseteq V(G \square H)$, if $G \square H - X$ is not connected, then let C_1, C_2, \dots, C_r be the components of $G \square H - X$, where $|V(C_i)| \geq g + 1$ for each i ($1 \leq i \leq r$). Let

$$V(C_1) = \{(u_{i_a}, v_{j_b}) \mid 1 \leq a \leq x, 1 \leq b \leq y\}.$$

It is clear that

$$\left\{ (u_{i_c}, v_{j_d}) \mid 1 \leq c \leq x, y + 1 \leq d \leq m \right\} \cup \left\{ (u_{i_c}, v_{j_d}) \mid x + 1 \leq c \leq n, 1 \leq d \leq y \right\} \subseteq X.$$

Then $|X| \geq (m - y)x + (n - x)y$. Under the conditions $xy \geq g + 1$, $(m - y)(n - x) \geq g + 1$, the function $f(x, y) = (m - y)x + (n - x)y$ get the minimum value $\min\{(m - y_1)x_1 + (n - x_1)y_1, (m - y_2)x_2 + (n - x_2)y_2\}$. \square

To show the sharpness of Theorem 2.5, we consider the following examples.

Example 1. For $g = 0$, the lower bound is sharp, just from Theorem 2.4. For $0 \leq g \leq n - 2$, let $G = P_2$ and $H = P_n$. Clearly, $\kappa_g(G \square H) = 2 = \min\{\kappa(G)m, \kappa(H)n, \delta(G) + \delta(H)\}$. This implies that the lower bound of Theorem 2.5 is sharp.

Example 2. For $m = 3$ and $0 \leq g \leq m - 1$, let $G = K_n$ and $H = P_m$. Clearly, $\kappa_g(K_n \square P_m) = n = \min\{\kappa(G)m, \kappa(H)n, (g + 1)\delta(G) + (m - g - 1), (g + 1)\delta(H) + (n - g - 1)\}$. This means that (1) of Theorem 2.5 is sharp.

Example 3. For $m \leq g \leq n - 1$, let $G = K_n$ and $H = P_m$. It is clear that $\kappa_g(K_n \square P_m) = n = \min\{\kappa(H)n, (g + 1)\delta(H) + (n - g - 1)\}$. This means that (2) of Theorem 2.5 is sharp.

Example 4. For $km < g + 1 \leq (k + 1)m$, let $G = P_n$ and $H = P_m$. From Proposition 3.1, $\kappa_g(P_n \square P_m) = m = |A|m$. This means that (3) of Theorem 2.5 is sharp.

Remark 1. If $n = m$, then the results in (3) and (4) of Theorem 2.5 are same. So (4) of Theorem 2.5 is sharp.

4 Concluding remark

In this paper, we study the exact value of g -extra connectivity for the lexicographic product of two general graphs, which will give some ideas to study special networks and graphs. It is also interesting to study the other products, like strong and direct products.

References

- [1] N.-W. Chang , C.-Y. Tsai , S.-Y. Hsieh, On 3-extra connectivity and 3-extra edge connectivity of folded hypercubes, *IEEE Trans. Comput.* 63(6) (2014), 1593–1599.
- [2] N.J. Calkin and H.S. Wilf, The number of independent sets in a grid graph, *SIAM J. Discrete Math.* 11(1) (1998), 54–60.
- [3] K. Day and A.-E. Al-Ayyoub, The cross product of interconnection networks, *IEEE Trans. Parall. Distr. Syst.* 8(2) (1997), 109–118.
- [4] E. Cheng, K. Qiu, Z. Shen, A general approach to deriving the g -good-neighbor conditional diagnosability of interconnection networks, *Theor. Comput. Sci.* 757 (2019), 56–67.
- [5] E. Cheng, K. Qiu, Z. Shen, Diagnosability of interconnection networks: past, present and future, *Int. J. Parallel Emergent Distributed Syst.* 35(1)(2020), 2–8.
- [6] E. Cheng, K. Qiu, Z. Shen: On diagnosability of interconnection networks, *Int. J. Unconv. Comput.* 13(3) (2017), 245–251.
- [7] E. Cheng, K. Qiu, Z. Shen, On the conditional diagnosability of hyper-butterfly graphs and related networks. *Parallel Process. Lett.* 26(1)(2016), 1650005.
- [8] A.H. Esfahanian, Generalized measures of fault tolerance with application to n -cube networks, *IEEE Trans. Comput.* 38(11) (1989), 1586–1591.
- [9] J. Fàbrega, M.A. Fiol, On the extra connectivity of graphs, *Discrete Math.* 155 (1996), 49–57.
- [10] P. Fragopoulou, S.G. Akl, and H. Meijer, Optimal communication primitives on the generalized hypercube network, *IEEE Trans. Parall. Distr. Comput.* 32(2)(1996), 173–187.
- [11] M.-M. Gu, R.-X. Hao, 3-extra connectivity of 3-ary n -cube networks, *Inf. Process. Lett.* 114(9) (2014), 486–491.
- [12] M. Gu, R. Hao, J. Liu, On the extra connectivity of k -ary n -cube networks, *Inter. J. Comput. Math.* 94(1) (2015), 95–106.
- [13] R. Hammack, W. Imrich, S. Klavžr, *Handbook of Product Graphs*, Second Edition, CRC Press, 2011.

- [14] A. Itai and M. Rodeh, The multi-tree approach to reliability in distributed networks, *Inform. Comput.* 79 (1988), 43–59.
- [15] F.T. Leighton, *Introduction to Parallel Algorithms and Architecture: Arrays, Trees, Hypercubes*, Morgan Kaufmann 1992.
- [16] Y. Mao, Path-connectivity of lexicographical product graphs, *Int. J. Comput. Math.* 93(1) (2016), 27–39.
- [17] Y. Ren, S. Wang, The tightly super 2-extra connectivity and 2-extra diagnosability of locally twisted cubes, *JOIN* 17(2) (2017), 1750006.
- [18] G. Sabidussi, Graphs with given group and given graph theoretical properties, *Canadian J. Math.* 9 (1957), 515–525.
- [19] S. Špacapan, Connectivity of Cartesian products of graphs, *Appl. Math. Lett.* 21 (2008), 682–685.
- [20] S. Wang, X. Ma, The g -extra connectivity and diagnosability of crossed cubes, *Appl. Math. Comput.* 336 (2018), 60–66.
- [21] Z. Wang, Y. Mao, S.-Y. Hsieh, On the g -extra connectivity of graphs, arXiv:1904.06527, 2019.
- [22] Z. Wang, Y. Mao, S.-Y. Hsieh, J. Wu, On the g -good-neighbor connectivity of graphs, *Theor. Comput. Sci.* 804 (2020) 139–148.
- [23] L. Xu, L. Lin, S. Zhou, and S. Hsieh, The extra connectivity, extra conditional diagnosability, and t/m -diagnosability of arrangement graphs, *IEEE Trans. Reliab.* 65(3) (2016), 1248–1262.
- [24] J. Xu, *Topological structure and analysis of interconnection networks*, Kluwer Academic Publishers, 2001.
- [25] J. Xu, J. Wang, and W. Wang, On super and restricted connectivity of some interconnection networks, *Ars Combin.* 94 (2010), 1–8.
- [26] C. Yang, J. Xu, Connectivity of lexicographic product and direct product of graphs, *Ars Combin.* 111 (2013), 3–12.
- [27] M. Zhang, J. Zhou, On g -extra connectivity of folded hypercubes, *Theor. Comput. Sci.* 593 (2015), 146–153.

- [28] J. Zhou, On g -extra connectivity of hypercube-like networks, *J. Comput. Syst. Sci.* 88 (2017), 208–219.