The number of spanning trees for Sierpiński graphs and data center networks

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Abstract—The number of spanning trees is an important graph invariant related to different topological and dynamic properties of the graph, such as its reliability, synchronization capability and diffusion properties. In 2007, Chang et al. proposed two conjectures on the number of spanning trees of Sierpiński triangle graphs and its spanning tree entropy. In this paper, we completely confirm these conjectures. For data center networks $\mathcal{D}_{k,n}$, we get the exact formula for $k = 1$, and upper and lower bounds for $k \geq 2$. Our results allow also the calculation of the spanning tree entropy of Sierpiński graphs and data center networks.

Index Terms—Spanning tree, Data center network, Sierpiński graph, Tree entropy, Complex networks.

1 Introduction

The problem of determining the number of spanning tree originates from electrical network due to Kirchhoff in the analysis of electric circuits [14]. Enumerating spanning trees of a graph, applied in mathematics [3, 24, 27, 32, 38], chemistry [21, 25], physics [2, 31, 33], and computer science [36, 39–41], has been extensively studied for more than 150 years (see [19]). This parameter is related to the network reliability [4] and its synchronization [31], and the study of random walks [22]. Ozeki and Yamashita [26] gave a survey of spanning trees.

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All graphs considered in this paper are undirected, finite and simple. We refer to the book [1] for graph theoretical notation and terminology not described here. A *graph* G is denoted by an ordered pair $(V(G), E(G))$ where $V(G)$ is the set of *vertices* of G and $E(G)$ is the set of *edges* of G. A graph F is called a *subgraph* of a graph G if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. A *spanning subgraph* of a graph G is a subgraph obtained by edge deletions only. A *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. An *acyclic graph* is one that contains no cycles. A connected acyclic graph is called a *tree*. If this tree is a spanning subgraph, it is called a *spanning tree* of the graph. For a graph G, let $\tau(G)$ denotes the number of spanning trees of G.

The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is the number of neighbours of v in G. We denote by $\delta(G)$ and $\Delta(G)$ the *minimum and maximum degrees* of the vertices of G. The *neighborhood* $N_G(v)$ of the vertex v of G is defined by $N_G(v) = \{u \in V \mid uv \in E\}$. A path or cycle of length k is called a k-path or k-cycle, respectively. A 3-cycle is often called a *triangle*. A *complete graph* K_n is a simple graph on n vertices in which any two vertices are adjacent. The *union* $G \cup H$ of two graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *line graph* $L(G)$ of a graph G is the graph whose vertex set is $V(L(G)) = E(G)$ and two vertices of $L(G)$ are adjacent if and only if the corresponding two edges in G are adjacent to the same vertex in G. A *semi-regular graph*, denoted by $R_{a,b}$, is a bipartite graph with a bipartition $V(R_{a,b}) = U \cup W$ such that all vertices have degree a in U and have degree b in W. A *subdivision* of G is a graph obtained from G by replacing some edges with pairwise internally disjoint paths. Let $S(G)$ be the

subdivision of G, which is obtained from G by inserting a new vertex to each edge of G.

With the number of spanning trees of graph G, the *spanning tree entropy* is defined as

$$z(G) = \lim_{n\to\infty} \frac{\ln \tau(G)}{|V(G)|};$$

see [19, 36] for more details.

1.1 Sierpiński graphs

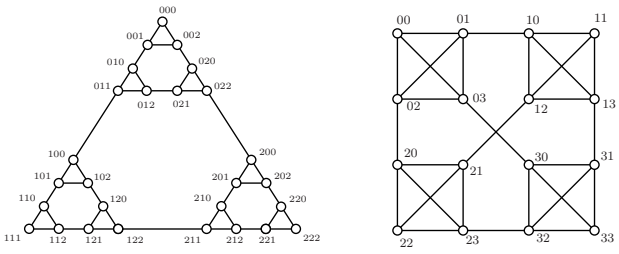
The interest for Sierpiński graphs comes from many different sources such as games like the Chinese rings or the tower of Hanoi, topology, physics, the study of interconnection networks, and elsewhere. For the details on the Sierpiński graphs, we refer to the survey paper [13].

The *k-dimensional Sierpiński graph*, denoted by $\mathcal{S}(n,k)$, is defined on the vertex set $\{0,1,\ldots,k\}^n$, two different vertices $u=(i_1,i_2,\ldots,i_n)$ and $v=(j_1,j_2,\ldots,j_n)$ being *adjacent* if and only if there exists an $h\in\{1,2,\ldots,n\}$ such that

(i) for all t, we have $t<h \Rightarrow i_t=j_t$,
(ii) $i_h\neq j_h$,
(iii) for all t, we have $t>h \Rightarrow i_t=j_h$ and $j_t=i_h$.

Note that $\mathcal{S}(0,k)$ is a complete graph K_1.

Example 1.1. *For $n=3$ and $k=2$, we have $V(\mathcal{S}(3,2)) = \{(i_1,i_2,i_3)\,|\,i_1,i_2,i_3\in\{0,1,2\}\}$ and $E(\mathcal{S}(3,2)) = \{\{(i_1,i_2,i_3),(j_1,j_2,j_3)\}\,|\,i_1\neq j_1, i_2=i_3=j_1,j_2=j_3=i_1$ or $i_1=j_1,i_2\neq j_2,i_3=j_2,j_3=i_2$ or $i_3\neq j_3,i_1=j_1,i_2=j_2\}$. The graph is shown in Figure 1 (a).*



(a) $\mathcal{S}(3,2)$ (b) $\mathcal{S}(2,3)$

Figure 1: Graphs $\mathcal{S}(3,2)$ and $\mathcal{S}(2,3)$.

Example 1.2. *For $n=2$ and $k=3$, we have $V(\mathcal{S}(2,3)) = \{(i_1,i_2)\,|\,i_1,i_2\in\{0,1\}\}$ and $E(\mathcal{S}(2,3)) = \{\{(i_1,i_2),(j_1,j_2)\}\,|\,i_1\neq j_1,i_2=j_1,j_2=i_1$ or $i_1=j_1,i_2\neq j_2\}$. The graph $\mathcal{S}(2,3)$ is shown in Figure 1 (b).*

The following observation is immediate.

Observation 1.1. *The order and size of $\mathcal{S}(n,k)$ are $(k+1)^n$ and $(k+1)((k+1)^n-1)/2$, respectively.*

The *k-dimensional Sierpiński triangle graph*, denoted by $\mathcal{ST}(n,k)$, is obtained from the k-dimensional Sierpiński graph by contracting all non-clique edges. Therefore, for any adjacent vertices $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$ in $\mathcal{S}(n,k)$, if the number of different vector components of $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$ is at least one, then we contract the vertices $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$ to be one vertex $(i_1,i_2,\ldots,i_n)$ or $(j_1,j_2,\ldots,j_n)$. Then the vertices in $\mathcal{ST}(n,k)$ are all $(i_1,i_2,\ldots,i_n)$ or $(j_1,j_2,\ldots,j_n)$, except the vertices $(i_1,i_2,\ldots,i_n)$, where $i_1=i_2=\cdots=i_n=i$ and $0\le i\le k$. Note that $\mathcal{ST}(0,k)$ is a complete graph K_1.

The *k-dimensional regularize Sierpiński graph*, denoted by ${}^+\mathcal{S}(n,k)$, is defined on the vertex set $\{0,1,\ldots,k\}^n\cup\{w\}$ and the edge set $E(\mathcal{S}(n,k))\cup\{\{w,i^n\}\}$, where $i\in\{0,1,\ldots,k\}$.

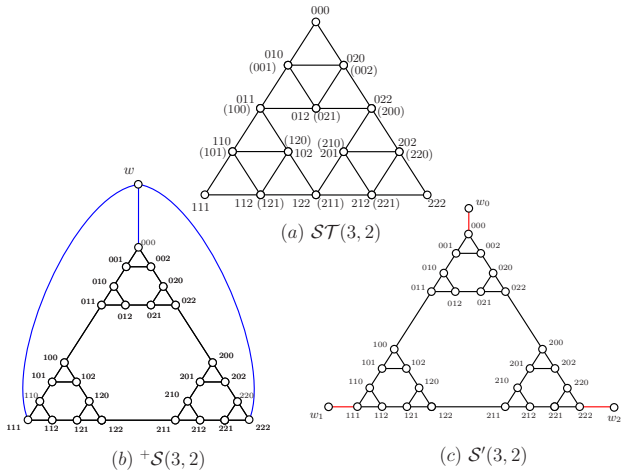
Let $\mathcal{S}'(n,k)$ be a graph obtained from $\mathcal{S}(n,k)$ by attaching one pendent edge to each vertex v of $\mathcal{S}(n,k)$, where the degree of each vertex is k. The graphs $\mathcal{ST}(3,2)$, ${}^+\mathcal{S}(3,2)$, $\mathcal{S}'(3,2)$ are shown in Figure 2.

The following observations are immediate.

Observation 1.2. (1) *The order and size of $\mathcal{S}'(n,k)$ are $(k+1)^n+k+1$ and $(k+1)((k+1)^n+1)/2$, respectively.*

(2) *There are exactly $k+1$ vertices of degree 1 in $\mathcal{S}'(n,k)$, and the degree of other $(k+1)^n$ vertices are exactly $k+1$.*

Observation 1.3. *For $n\ge 1$ and $k\ge 1$, we have $\tau(\mathcal{S}(n,k)) = \tau(\mathcal{S}'(n,k))$.*



(a) $\mathcal{ST}(3,2)$ (b) ${}^+\mathcal{S}(3,2)$ (c) $\mathcal{S}'(3,2)$

Figure 2: Graphs $\mathcal{ST}(3,2)$, ${}^+\mathcal{S}(3,2)$, and $\mathcal{S}'(3,2)$.

In [2], Chang, Chen and Yang obtained the number of spanning trees of k-dimensional Sierpiński triangle

graph $\mathcal{ST}(n,k)$, where $k=2,3,4$, and then they proposed the following conjectures for general k.

Conjecture 1.1. [2] *The number of spanning trees on the k-dimensional Sierpiński triangle graph $\mathcal{ST}(n,k)$ is given by*

$$\tau(\mathcal{ST}(n,k)) = 2^{\alpha}(k+1)^{\beta}(k+3)^{\gamma},$$

where the exponents are $\alpha = \frac{k-1}{2}((k+1)^{n-1}-1)$, $\beta = \frac{k-1}{2k}((k+1)^n + k(n-1)-1)$, $\gamma = \frac{k-1}{2k}((k+1)^{n-1} - k(n-1)-1)$.

Conjecture 1.2. [2] *The asymptotic growth constant for the k-dimensional Sierpiński triangle graph $\mathcal{ST}(n,k)$ is*

$$\frac{k-1}{k(k+1)}\left(k\ln 2 + (k+1)\ln(k+1) + \ln(k+3)\right).$$

In [43], Zhang, Wu, Li and Comellas got the number of spanning trees of $\mathcal{S}(n,2)$.

In Sections 2 and 4, we show that these conjectures are both true; see Theorem 2.4 and Corollary 4.2. We also obtain the numbers of spanning trees of k-dimensional Sierpiński graph and k-dimensional regularize Sierpiński graph in Section 2; see Theorems 2.3 and 2.6.

1.2 Data center networks: construction and properties

Given a positive integer n, we use $[n]$ and $\langle n\rangle$ to denote the sets $\{1,2,\ldots,n\}$ and $\{0,1,2,\ldots,n\}$, respectively. For every integers $k\geq 0$ and $n\geq 2$, we use $\mathcal{D}_{k,n}$ to denote the k-dimensional DCell with n-port switches. $\mathcal{D}_{0,n}$ is isomorphic to the complete graph on n vertices.

We use $t_{k,n}$ to denote the number of vertices in $\mathcal{D}_{k,n}$ with $t_{k,n} = t_{k-1,n}\times(t_{k-1,n}+1)$. Note that $t_{0,n}=n$. Let $V_{k,n} = \{u_k u_{k-1}\ldots u_0 \,|\, u_i \in \langle t_{i-1,n}\rangle \text{ and } i\in\langle k\rangle\}$, and $V^l_{k,n} = \{u_k u_{k-1}\ldots u_l \,|\, u_i\in\langle t_{i-1,n}\rangle \text{ and } i\in\{l,l+1,\ldots,k\}\}$ for every $l\in[k]$. Clearly, $|V_{k,n}| = t_{k,n}$ and $|V^l_{k,n}| = t_{k,n}/t_{l-1,n}$. We now give the definition of $\mathcal{D}_{k,n}$.

Definition 1. *The k-dimensional data center network $\mathcal{D}_{k,n}$ is the graph with vertex set $V_{k,n}$, where a vertex $u = u_k u_{k-1}\ldots u_i\ldots u_0$ is adjacent to a vertex $v = v_k v_{k-1}\ldots v_i\ldots v_0$ if and only if there is an integer l such that*

(1) $u_k u_{k-1}\ldots u_l = v_k v_{k-1}\ldots v_l$,
(2) $u_{l-1}\neq v_{l-1}$,
(3) $u_{l-1} = v_0 + \sum_{j=1}^{l-2}(v_j\times t_{j-1,n})$ *and* $v_{l-1} = u_0 + \sum_{j=1}^{l-2}(u_j\times t_{j-1,n}) + 1$ *with* $l>1$.

It is clear from the definition that $\mathcal{D}_{k,n}$ is an $(n+k-1)$-regular graph with $t_{k,n}$ vertices. In fact, $\mathcal{D}_{k,n}$ can be constructed recursively by $t_{k-1,n}+1$ disjoint copies of $\mathcal{D}_{k-1,n}$, called $(k-1)$-*dimensional subgraphs* of $\mathcal{D}_{k,n}$, where $\mathcal{D}^i_{k-1,n}$ denotes the i-th copy. For each pair of copies of $\mathcal{D}_{k-1,n}$, say $\mathcal{D}^i_{k-1,n}$ and $\mathcal{D}^j_{k-1,n}$, there is a unique edge between them. The edges between vertices in the same copy of $\mathcal{D}_{k-1,n}$ are called *internal edges* and the edges between vertices in disjoint copies of $\mathcal{D}_{k-1,n}$ are called *cross edges*. It follows that each vertex u of $\mathcal{D}^i_{k-1,n}$ has only one neighbor that is not in $D^i_{k-1,n}$, called the *cross neighbor* of u. Clearly, the set of (all) cross edges of $\mathcal{D}_{k,n}$ forms a perfect matching of $\mathcal{D}_{k,n}$ for $k\geq 1$ and $n\geq 2$. Figure 3 depicts several data center networks with small parameters k and n.

The following theorem describes the upper and lower bounds of the number of vertices in $\mathcal{D}_{k,n}$.

Theorem 1.1. [9] *For any integers $k\geq 0$ and $n\geq 2$, the number $t_{k,n}$ of vertices in $D_{k,n}$ satisfies*

$$\left(n+\frac{1}{2}\right)^{2^k} - \frac{1}{2} \leq t_{k,n} \leq (n+1)^{2^k} - 1.$$

Theorem 1.2. [35] *Let $\mathcal{D}_{h,n}$ be the data center network with $h\geq 0$ and $n\geq 2$.*

(1) *$\mathcal{D}_{0,n}$ is a complete graph with n vertices labeled as $0,1,2,\ldots,n-1$ respectively.*

(2) *For $h\geq 1$, $\mathcal{D}_{h,n}$ consists of $t_{h-1,n}+1$ copies of $\mathcal{D}_{h-1,n}$, denoted by $\mathcal{D}^i_{h-1,n}$, for each $i\in\langle t_{h-1,n}\rangle$. There is only one edge between $\mathcal{D}^i_{h-1,n}$ and $\mathcal{D}^j_{h-1,n}$ for any $i,j\in I_{h,n}$ and $i\neq j$, where $I_{h,n} = \langle t_{h-1,n}\rangle$. It implies that the outside neighbours of vertices in $\mathcal{D}^i_{h-1,n}$ belong to different copies of $\mathcal{D}^j_{h-1,n}$s for $i\neq j$ and $i,j\in I_{h,n}$.*

(3) *For any two distinct vertices u,v in $\mathcal{D}^i_{h-1,n}$,*

$$N_{\mathcal{D}^{I_{h,n}\setminus\{i\}}_{h-1,n}}(u)\cap N_{\mathcal{D}^{I_{h,n}\setminus\{i\}}_{h-1,n}}(v) = \varnothing \text{ and } \left|N_{\mathcal{D}^{I_{h,n}\setminus\{i\}}_{h-1,n}}(u)\right| = 1.$$

A server-centric data center network called DCell was proposed by Guo et al. in [9], which can support millions of servers with high network capacity by only using commodity switches. As such, data center networks are good models for designing interconnection networks. However, it is important to make sure that these networks have good properties that are essential for interconnection networks. Indeed, there are many papers on the data center networks with respect to a number of vulnerability and reliability measures [11, 12, 17, 18, 23, 28].

In Section 3, for data center networks $\mathcal{D}_{k,n}$, we get the exact formula for $k=1$, and upper and lower bounds for $k\geq 2$. In Section 4, we calculate the entropy of spanning trees for k-dimensional Sierpiński graph, k-dimensional Sierpiński triangle graphs and data center network $\mathcal{D}_{1,n}$.

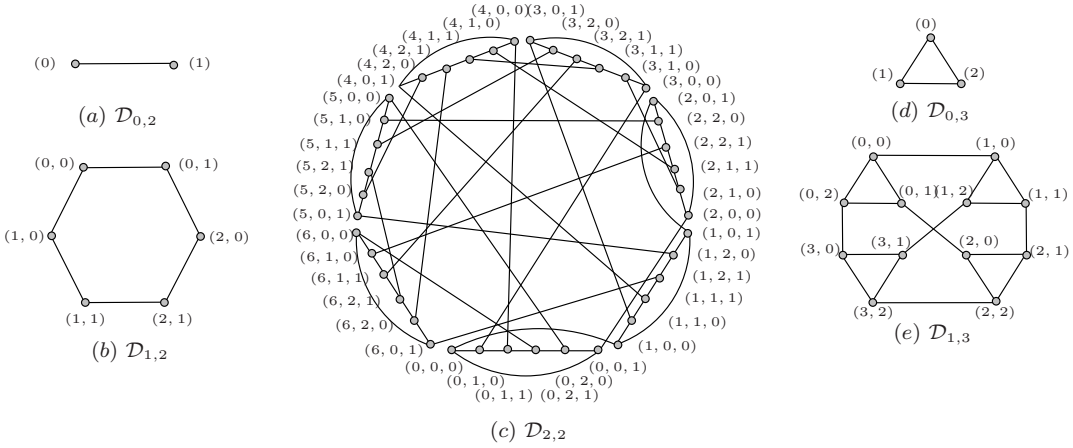


Figure 3: Several data center networks with small parameters k and n.

2 Results for Sierpiński graphs

In 2013, Yan [37] proved the following result for line graphs.

Theorem 2.1. [37] *Let G be a connected graph with $n+s$ vertices and $m+s$ edges in which n vertices have degree k and s vertices have degree 1. Then*

$$\tau(L(G)) = 2^{m-n+1}k^{m+s-n-1}\tau(G).$$

Recall that $S(G)$ is the subdivision of G, which is obtained from G by inserting a new vertex to each edge of G.

Dong and Yan [7] established a relation between $\tau(G)$ and $\tau(L(S(G)))$.

Theorem 2.2. [7] *Let G be a connected graph of order $n+s$ and size $m+s$ in which s vertices are of degree 1 and all others are of degree k. Then*

$$\tau(L(S(G))) = k^{m+s-n-1}(k+2)^{m-n+1}\tau(G).$$

Let ψ be a bijection between sets A and B. For $a \in A$ and $b \in B$, we say that *a corresponds to b* or *b corresponds to a* if $\psi(a) = b$ and $\psi^{-1}(b) = a$.

Lemma 2.1. *For $n \geq 2$ and $k \geq 1$, we have*

$$L(S(\mathcal{S}'(n-1,k))) = \mathcal{S}'(n,k).$$

Proof: From the definition, we have $V(\mathcal{S}'(n,k)) = \{0,1,\ldots,k\}^{(n)} \cup \{w_0, w_1, \ldots, w_k\}$ and $E(\mathcal{S}'(n,k)) = E(\mathcal{S}(n,k)) \cup \{w_i(i_1,i_2,\ldots,i_n) \,|\, i_1 = i_2 = \cdots = i_n = i,\ 0 \leq i \leq k\}$, where $(i_1,i_2,\ldots,i_n) \in \{0,1,\ldots,k\}^{(n)}$. Clearly, $((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n))$ is an edge of $\mathcal{S}(n,k)$ such that $(i_1,i_2,\ldots,i_n)$, $(j_1,j_2,\ldots,j_n) \in \{0,1,\ldots,k\}^{(n)} \subseteq V(\mathcal{S}'(n,k))$ if and only if there exists an $h \in \{1,2,\ldots,n\}$ such that

(i) for all t, we have $t < h \Rightarrow i_t = j_t$,
(ii) $i_h \neq j_h$,
(iii) for all t, we have $t > h \Rightarrow i_t = j_h$ and $j_t = i_h$.

Recall that $S(\mathcal{S}'(n,k))$ is the subdivision graph of $\mathcal{S}'(n,k)$ satisfying the following conditions.

- $((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n))$ is the vertex inserted on the edge $((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n))^*$ between $(i_1,i_2,\ldots,i_n) \in \{0,1,\ldots,k\}^{(n)}$ and $(j_1,j_2,\ldots,j_n) \in \{0,1,\ldots,k\}^{(n)}$.
- $(w_i,(i_1,i_2,\ldots,i_n))$ is the vertex inserted on the edge $(w_i,(i_1,i_2,\ldots,i_n))^*$ between w_i and $(i_1,i_2,\ldots,i_n) \in \{0,1,\ldots,k\}^{(n)}$, where $0 \leq i \leq k$, where $i_1 = i_2 = \cdots = i_n = i$.

Table 1. The corresponding relation of elements in $E(S(\mathcal{S}'(n,k)))$ and $V(\mathcal{S}'(n+1,k))$.

$E(S(\mathcal{S}'(n,k)))$	$V(\mathcal{S}'(n+1,k))$
$((i_1,\ldots,i_n),((i_1,\ldots,i_n),(j_1,\ldots,j_n)))^*$	$(i_1,\ldots,i_n,j_h)$
$(w_i,(w_i,(i_1,\ldots,i_n)))^*$	w_i
$((i_1,\ldots,i_n),(w_i,(i_1,\ldots,i_n)))^*$	$(i_1,\ldots,i_n,i_{n+1})$

We now construct a bijection between the edge set $E(S(\mathcal{S}'(n,k)))$ and the vertex set $V(\mathcal{S}'(n+1,k))$ as follows.

- For any edge $((i_1,i_2,\ldots,i_n),((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n)))^* \in E(S(\mathcal{S}'(n,k)))$, we know that the hth component is the first different component between $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$. Then let the edge $((i_1,i_2,\ldots,i_n),((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n)))^*$ correspond to the vertex $(i_1,i_2,\ldots,i_n,\ j_h) \in V(\mathcal{S}'(n+1,k))$, where $(i_1,i_2,\ldots,i_n) \in V(S(\mathcal{S}'(n,k)))$ and $((i_1,i_2,\ldots,i_n),\ (j_1,j_2,\ldots,\ j_n))) \in V(S(\mathcal{S}'(n,k)))$.

- The path induced by the three vertices w_i, $(w_i,(i_1,i_2,\ldots,i_n))$, and $(i_1,i_2,\ldots,i_n)$ $(i_1=i_2=\cdots=i_n=i$ and $0\leq i\leq k)$ in $S(\mathcal{S}'(n,k))$ is a pendant path P_3. Let $(i_1,i_2,\ldots,i_n,i_{n+1})\in V(\mathcal{S}'(n+1,k))$ be the corresponding vertex of the edge $((w_i,(i_1,i_2,\ldots,i_n)),(i_1,i_2,\ldots,i_n))^*\in E(S(\mathcal{S}'(n,k)))$, where $i_1=i_2=\cdots=i_n=i_{n+1}=i$ and $0\leq i\leq k$. Let $w_i\in V(\mathcal{S}'(n+1,k))$ be the corresponding vertex of the edge $(w_i,(w_i,(i_1,i_2,\ldots,i_n)))^*\in E(S(\mathcal{S}'(n,k)))$, where $i_1=i_2=\cdots=i_n=i$ and $0\leq i\leq k$.
- For any vertex $(i_1,i_2,\ldots,i_n,i_{n+1})\in V(\mathcal{S}'(n+1,k))-\{w_i\,|\,0\leq i\leq k\}-\{(i_1,i_2,\ldots,i_{n+1})\,|\,i_1=i_2=\cdots=i_{n+1}=i,0\leq i\leq k\}$, there exits some h such that $i_h\neq i_{h+1}$ and $i_{h+1}=\cdots=i_n=i_{n+1}$. Then the vertex $(i_1,i_2,\ldots,i_n,i_{n+1})$ corresponds to the edge $((i_1,i_2,\ldots,i_n),((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_{h-1},j_h,\ldots,j_n)))^*$, where $(i_1,i_2,\ldots,i_{h-1})=(j_1,j_2,\ldots,j_{h-1})$, $j_h=i_{n+1}$ and $j_{h+1}=j_{h+2}=\cdots=j_n=i_h$.
- For any vertex $(i_1,i_2,\ldots,i_n,i_{n+1})\in\{(i_1,i_2,\ldots,i_{n+1})\,|\,i_1=i_2=\cdots=i_{n+1}=i,0\leq i\leq k\}$, let $(i_1,i_2,\ldots,i_n,i_{n+1})$ correspond to the edge $((i_1,i_2,\ldots,i_n),(w_i,(i_1,i_2,\ldots,i_n)))^*$, where $i_1=i_2=\cdots=i_n=i$ and $0\leq i\leq k$. Let the vertex w_i in $\mathcal{S}'(n+1,k)$ correspond to the edge $(w_i,(w_i,(i_1,i_2,\ldots,i_n)))^*$, where $i_1=i_2=\cdots=i_n=i$ and $0\leq i\leq k$.

It remains to consider the correspondence between the adjacent relation of two edges in $S(\mathcal{S}'(n,k))$ and the adjacent relation of two vertices in $\mathcal{S}'(n+1,k)$.

Except for the path induced by the three vertices w_i, $(w_i,(i_1,i_2,\ldots,i_n))$, and $(i_1,i_2,\ldots,i_n)$ $(i_1=i_2=\cdots=i_n=i$ and $0\leq i\leq k)$ in $S(\mathcal{S}'(n,k))$, for any two edges

$$e = ((i_1,i_2,\ldots,i_n),((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n)))^*,$$

$$e' = ((i'_1,i'_2,\ldots,i'_n),((i'_1,i'_2,\ldots,i'_n),(j'_1,j'_2,\ldots,j'_n)))^*,$$

if they are adjacent, then the common vertex between them is $((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n))$ or $(i_1,i_2,\ldots,i_n)$.

Furthermore, we have the following facts.

Fact 1. *If the common vertex is $(i_1,i_2,\ldots,i_n)$, then*

$$e'=((i_1,i_2,\ldots,i_n),((i_1,i_2,\ldots,i_n),(j'_1,j'_2,\ldots,j'_n)))^*.$$

From the definition of $\mathcal{S}'(n+1,k)$, the corresponding vertices $(i_1,i_2,\ldots,i_n,j_h)$ and $(i_1,i_2,\ldots,i_n,\ j'_{h'})$ of e,e' are adjacent in $\mathcal{S}'(n+1,k)$, respectively, where the hth component is the first different component between $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$ and the h'th component is the first different component between $(i_1,i_2,\ldots,i_n)$ and $(j'_1,j'_2,\ldots,j'_n)$.

Fact 2. *If the common vertex is $((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n))$, then*

$$e'=((j_1,j_2,\ldots,j_n),((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n)))^*.$$

Since the common vertex $((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n))$ is a new vertex which is inserted in the edge $((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n))^*$ of $\mathcal{S}'(n,k)$, then the vertices $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$ are adjacent in $\mathcal{S}'(n,k)$. Therefore, there exists an h such that

(i) for all t, we have $t<h\Rightarrow i_t=j_t$,
(ii) $i_h\neq j_h$,
(iii) for all t, we have $t>h\Rightarrow i_t=j_h$ and $j_t=i_h$.

Then the number of different vector components of $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$ is $n+1-h$. Thus, the corresponding vertices $(i_1,i_2,\ldots,i_n,j_h)$ and $(j_1,j_2,\ldots,j_n,i_h)$ of e,e' are adjacent in $\mathcal{S}'(n+1,k)$.

For the edges on the pendent path P_3, we have the following.

- The edge $(w_i,(w_i,(i_1,i_2,\ldots,i_n)))^*$, where $i_1=i_2=\cdots=i_n=i$, $0\leq i\leq k$, is adjacent to the edge $((i_1,i_2,\ldots,i_n),(w_i,(i_1,i_2,\ldots,i_n)))^*$ in $S(\mathcal{S}'(n,k))$ if and only if the vertex w_i is adjacent to the vertex $(i_1,i_2,\ldots,i_n,i_{n+1})$ in $\mathcal{S}'(n+1,k)$, where $i_1=i_2=\cdots=i_n=i_{n+1}=i$, $0\leq i\leq k$.
- The edge $((i_1,i_2,\ldots,i_n),(w_i,(i_1,i_2,\ldots,i_n)))$ is adjacent to the edge

$$((i_1,i_2,\ldots,i_n),((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n)))^*$$

in $S(\mathcal{S}'(n,k))$ if and only if the vertex $(i_1,i_2,\ldots,i_n,i_{n+1})$ is adjacent to the vertex $(i_1,i_2,\ldots,i_n,j_h)$ in $\mathcal{S}'(n+1,k)$, where $i_1=i_2=\cdots=i_n=i_{n+1}=i$, $0\leq i\leq k$.

From the above arguments, we have $L(S(\mathcal{S}'(n-1,k)))=\mathcal{S}'(n,k)$. The result follows. □

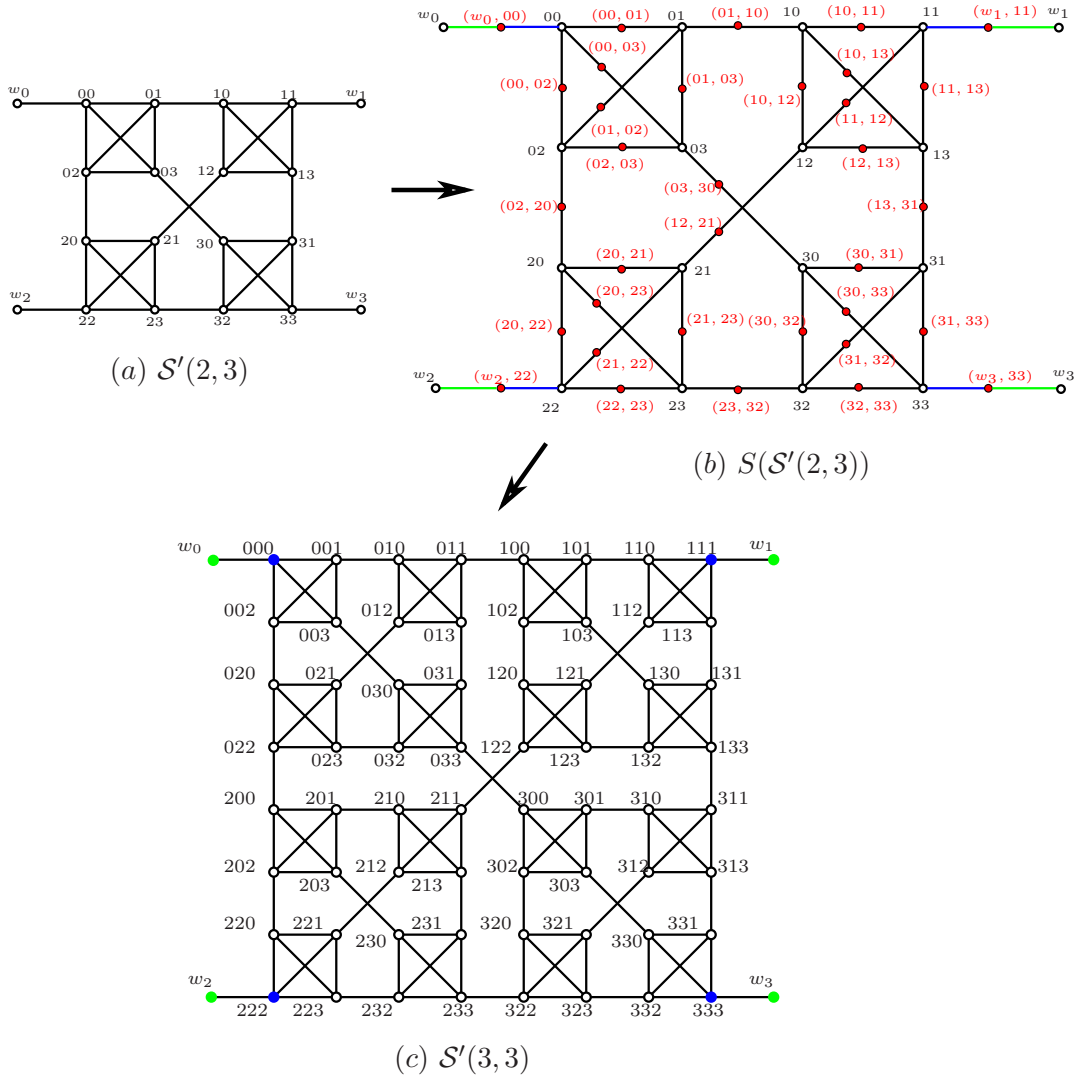
Figure 4 shows that $S(\mathcal{S}'(2,3))$ is the subdivision graph of $\mathcal{S}'(2,3)$ and the graph $\mathcal{S}'(3,3)$ is the line graph of $S(\mathcal{S}'(2,3))$.

Lemma 2.2. *For $n\geq 1$ and $k\geq 1$, we have*

$$L(\mathcal{S}'(n,k))=\mathcal{ST}(n+1,k).$$

Proof: We now construct a bijection between the edge set $E(\mathcal{S}'(n,k))$ and the vertex set $V(\mathcal{ST}(n+1,k))$ as follows.

- For any edge $((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n))^*\in E(\mathcal{S}'(n,k))$, we know that the hth component is the first different component between $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$. Let



(a) $\mathcal{S}'(2,3)$

(b) $S(\mathcal{S}'(2,3))$

(c) $\mathcal{S}'(3,3)$

Figure 4: The relation between $\mathcal{S}'(3,3)$ and $S(\mathcal{S}'(2,3))$.

the edge $((i_1, i_2, \ldots, i_n), (j_1, j_2, \ldots, j_n))^*$ correspond to the vertex $(i_1, i_2, \ldots, i_n, j_h)$ or $(j_1, j_2, \ldots, j_n, i_h)$ in $V(\mathcal{ST}(n+1, k))$, where $(i_1, i_2, \ldots, i_n), (j_1, j_2, \ldots, j_n) \in V(\mathcal{S}'(n, k))$.

- The edge $(w_i, (i_1, i_2, \ldots, i_n))^*$ in $\mathcal{S}'(n, k)$, where $i_1 = i_2 = \cdots = i_n = i$ and $0 \leq i \leq k$, is the corresponding edge of the vertex $(i_1, i_2, \ldots, i_n, i_{n+1})$ in $\mathcal{ST}(n+1, k)$, where $i_1 = i_2 = \cdots = i_n = i_{n+1} = i$ and $0 \leq i \leq k$.
- For any vertex $(i_1, i_2, \ldots, i_n, i_{n+1}) \in V(\mathcal{ST}(n+1, k)) - \{(i_1, i_2, \ldots, i_{n+1}) \,|\, i_1 = i_2 = \cdots = i_{n+1} = i, 0 \leq i \leq k\}$, there exists some h such that $i_h \neq i_{h+1}$ and $i_{h+1} = \cdots = i_n = i_{n+1}$, then the vertex $(i_1, i_2, \ldots, i_n, i_{n+1})$ corresponds to the edge $((i_1, i_2, \ldots, i_n), (j_1, j_2, \ldots, j_{h-1}, j_h, \ldots, j_n))^*$, where $(i_1, i_2, \ldots, i_{h-1}) = (j_1, j_2, \ldots, j_{h-1})$, $j_h = i_{n+1}$ and $j_{h+1} = j_{h+2} = \cdots = j_n = i_h$.
- For any vertex $(i_1, i_2, \ldots, i_n, i_{n+1}) \in \{(i_1, i_2, \ldots, i_{n+1}) \,|\, i_1 = i_2 = \cdots = i_{n+1} = i, 0 \leq i \leq k\}$, let $(i_1, i_2, \ldots, i_n, i_{n+1})$ correspond to the edge $(w_i, (i_1, i_2, \ldots, i_n))^*$, where $i_1 = i_2 = \cdots = i_n = i$ and $0 \leq i \leq k$.

It remains to consider the correspondence between the adjacent relation of two edges in $\mathcal{S}'(n, k)$ and the adjacent relation of two vertices in $\mathcal{ST}(n+1, k)$. Except for the edge induced by the vertices w_i and $(i_1, i_2, \ldots, i_n)$ $(i_1 = i_2 = \cdots = i_n = i$ and $0 \leq i \leq k)$ in $\mathcal{S}'(n, k)$, for any two edges

$$e = ((i_1, i_2, \ldots, i_n), (j_1, j_2, \ldots, j_n))^*,$$

$$e' = ((i'_1, i'_2, \ldots, i'_n), (j'_1, j'_2, \ldots, j'_n))^*,$$

if they are adjacent, then the common vertex between them is $(i_1, i_2, \ldots, i_n)$ or $(j_1, j_2, \ldots, j_n)$.

Table 2. The corresponding relation of elements in $E(\mathcal{S}'(n,k))$ and $V(\mathcal{ST}(n+1,k))$.

$E(\mathcal{S}'(n,k))$	$V(\mathcal{ST}(n+1,k))$
$((i_1,\ldots,i_n),(j_1,\ldots,j_n))^*$	$(i_1,\ldots,i_n,j_h)((j_1,\ldots,j_n,i_h))$
$(w_i,(i_1,\ldots,i_n))^*$	$(i_1,\ldots,i_n,i_{n+1})$

Furthermore, we have the following facts.

Fact 3. *If the common vertex is $(i_1,i_2,\ldots,i_n)$, then $e'=((i_1,i_2,\ldots,i_n),(j'_1,j'_2,\ldots,j'_n))^*$ and the corresponding vertices $(i_1,i_2,\ldots,i_n,j_h)$ or $(j_1,j_2,\ldots,j_n,i_h)$ of e, and $(i_1,i_2,\ldots,i_n,j'_{h'})$ or $(j'_1,j'_2,\ldots,j'_n,i_{h'})$ of e' are adjacent in $\mathcal{ST}(n+1,k)$, where the hth component is the first different component between $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$ and the h′th component is the first different component between $(i_1,i_2,\ldots,i_n)$ and $(j'_1,j'_2,\ldots,j'_n)$.*

Fact 4. *If the common vertex is $(j_1,j_2,\ldots,j_n)$, then $e'=((i'_1,i'_2,\ldots,i'_n),(j_1,j_2,\ldots,j_n))^*$ and the corresponding vertices $(i_1,i_2,\ldots,i_n,j_h)$ or $(j_1,j_2,\ldots,j_n,i_h)$ of e, and $(i'_1,i'_2,\ldots,i'_n,j_{h'})$ or $(j_1,j_2,\ldots,j_n,i'_{h'})$ of e' are adjacent in $\mathcal{ST}(n+1,k)$, respectively, where the hth component is the first different component between $(i_1,i_2,\ldots,i_n)$ and $(j_1,j_2,\ldots,j_n)$ and the h′th component is the first different component between $(i_1,i_2,\ldots,i_n)$ and $(j'_1,j'_2,\ldots,j'_n)$.*

For the edge $(w_i,(i_1,i_2,\ldots,i_n))^*$ in $\mathcal{S}'(n,k)$, where $i_1=i_2=\cdots=i_n=i$, $0\le i\le k$, we have the edge $(w_i,(i_1,i_2,\ldots,i_n))^*$ is adjacent to the edge $((i_1,i_2,\ldots,i_n),(j_1,j_2,\ldots,j_n))^*$ in $\mathcal{S}'(n,k)$ if and only if the vertex $(i_1,i_2,\ldots,i_n,i_{n+1})$ is adjacent to the vertex $(i_1,i_2,\ldots,i_n,j_h)$ or $(j_1,j_2,\ldots,j_n,i_h)$ in $\mathcal{ST}(n+1,k)$, where $i_1=i_2=\cdots=i_n=i_{n+1}=i$, $0\le i\le k$.

From the above arguments, we have $L(\mathcal{S}'(n,k))=\mathcal{ST}(n+1,k)$. □

Figure 5 shows that the graph $\mathcal{ST}(3,2)$ is the line graph of $\mathcal{S}'(2,2)$.

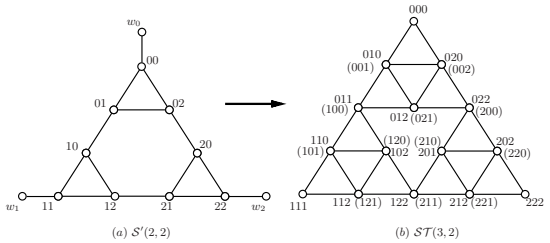


Figure 5: The relation between $\mathcal{S}'(2,2)$ and $\mathcal{ST}(3,2)$.

We are now in a position to show the number of spanning trees of k-dimensional Sierpiński graph and k-dimensional Sierpiński triangle graph.

Theorem 2.3. *The number of spanning trees on the k-dimensional Sierpiński graph $\mathcal{S}(n,k)$ is given by*

$$\tau(\mathcal{S}(n,k))=(k+1)^{\frac{k-1}{2k}((k+1)^n+kn-1)}(k+3)^{\frac{k-1}{2k}((k+1)^n-kn-1)}.$$

Proof: From Observation 1.2, we know that $|V(\mathcal{S}'(n,k))|=(k+1)^n+k+1$, $|E(\mathcal{S}'(n,k))|=(k+1)((k+1)^n+1)/2$ and there are exactly $k+1$ vertices of degree 1 in $\mathcal{S}'(n,k)$, and the degree of other $(k+1)^{n-1}$ vertices are exactly $k+1$. From Observation 1.3, we have $\tau(\mathcal{S}'(n,k))=\tau(\mathcal{S}(n,k))$. From Lemma 2.1, $\mathcal{S}'(n,k)$ is the line graph of $S(\mathcal{S}'(n-1,k))$. From Theorem 2.2, we have

$$\begin{aligned}
\tau(\mathcal{S}(n,k)) &= \tau(\mathcal{S}'(n,k))\\
&= (k+1)^{\frac{(k+1)((k+1)^{(n-1)}-1)}{2}+(k+1)-(k+1)^{(n-1)}-1}\\
&\quad\cdot(k+3)^{\frac{(k+1)((k+1)^{(n-1)}-1)}{2}-(k+1)^{(n-1)}+1}\\
&\quad\cdot\tau(\mathcal{S}'(n-1,k))\\
&= (k+1)^{\frac{(k+1)((k+1)^{(n-1)}-1)}{2}+(k+1)-(k+1)^{(n-1)}-1}\\
&\quad\cdot(k+3)^{\frac{(k+1)((k+1)^{(n-1)}-1)}{2}-(k+1)^{(n-1)}+1}\\
&\quad\cdot\tau(\mathcal{S}(n-1,k)).
\end{aligned}$$

Since $\tau(\mathcal{S}(0,k))=1$, it follows that

$$\begin{aligned}
\tau(\mathcal{S}(1,k)) &= (k+1)^{k-1}(k+3)^0\tau(\mathcal{S}(0,k))=(k+1)^{k-1}\\
\tau(\mathcal{S}(2,k)) &= (k+1)^{\frac{k(k+1)}{2}-1}(k+3)^{\frac{k(k+1)}{2}-k}\tau(\mathcal{S}(1,k))\\
&= (k+1)^{\frac{k(k+1)}{2}+k-2}(k+3)^{\frac{k(k+1)}{2}-k}\\
&\vdots\\
\tau(\mathcal{S}(n,k)) &= (k+1)^{\frac{(k+1)((k+1)^{(n-1)}-1)}{2}+(k+1)-(k+1)^{(n-1)}-1}\\
&\quad\cdot(k+3)^{\frac{(k+1)((k+1)^{(n-1)}-1)}{2}-(k+1)^{(n-1)}+1}\\
&\quad\cdot\tau(\mathcal{S}(n-1,k))\\
&= (k+1)^{\sum_{i=1}^{n}(\frac{(k+1)((k+1)^{i-1}-1)}{2}+(k+1)-(k+1)^{i-1}-1)}\\
&\quad\cdot(k+3)^{\sum_{i=1}^{n}(\frac{(k+1)((k+1)^{i-1}-1)}{2}-(k+1)^{i-1}+1)}\\
&\quad\tau(\mathcal{S}(0,k))\\
&= (k+1)^{\frac{(k-1)((k+1)^n-1)}{2k}+\frac{k-1}{2}n}\\
&\quad\cdot(k+3)^{\frac{(k-1)((k+1)^n-1)}{2k}-\frac{k-1}{2}n}.
\end{aligned}$$

□

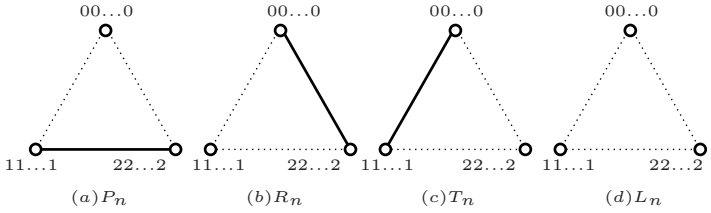


Figure 6: Graphs P_n, R_n, T_n and L_n.

Theorem 2.4. *The number of spanning trees on the k-dimensional Sierpiński triangle graph $\mathcal{ST}(n,k)$ is given by*

$$\tau(\mathcal{ST}(n,k)) = 2^{\alpha}(k+1)^{\beta}(k+3)^{\gamma},$$

where the exponents are $\alpha = \frac{k-1}{2}((k+1)^{n-1}-1)$, $\beta = \frac{k-1}{2k}((k+1)^n + k(n-1)+k-1)$, $\gamma = \frac{k-1}{2k}((k+1)^{n-1} - k(n-1)-1)$.

Proof: From Observation 1.2, we know that $|V(\mathcal{S}'(n-1,k))| = (k+1)^{n-1}+k+1$, $|E(\mathcal{S}'(n-1,k))| = \frac{(k+1)((k+1)^{n-1}+1)}{2}$ and there are exactly $k+1$ vertices of degree 1 in $\mathcal{S}'(n-1,k)$, and the degree of other $(k+1)^{n-1}$ vertices are exactly $k+1$. From Observation 1.3, we have $\tau(\mathcal{S}'(n-1,k)) = \tau(\mathcal{S}(n-1,k))$. From Lemma 2.2, $\mathcal{ST}(n,k)$ is the line graph of $\mathcal{S}'(n-1,k)$. From Theorems 2.3 and 2.1, the number of spanning trees of $\mathcal{ST}(n,k)$ is

$$\begin{aligned}
\tau(\mathcal{ST}(n,k)) &= (k+1)^{(\frac{(k+1)((k+1)^{n-1}-1)}{2}+k+1-(k+1)^{n-1}-1)} \\
&\quad \cdot 2^{(\frac{(k+1)((k+1)^{n-1}-1)}{2}-(k+1)^{n-1}+1)} \\
&\quad \cdot \tau(\mathcal{S}'(n-1,k)) \\
&= (k+1)^{(\frac{(k+1)((k+1)^{n-1}-1)}{2}+k+1-(k+1)^{n-1}-1)} \\
&\quad \cdot 2^{(\frac{(k+1)((k+1)^{n-1}-1)}{2}-(k+1)^{n-1}+1)} \\
&\quad \cdot \tau(\mathcal{S}(n-1,k)) \\
&= 2^{\frac{k-1}{2}((k+1)^{n-1}-1)} \\
&\quad \cdot (k+1)^{\frac{(k+1)((k+1)^{n-1}-1)}{2}+k+1-(k+1)^{n-1}-1} \\
&\quad \cdot (k+1)^{\frac{(k-1)((k+1)^{n-1}-1)}{2k}+\frac{k-1}{2}(n-1)} \\
&\quad \cdot (k+3)^{\frac{(k-1)((k+1)^{n-1}-1)}{2k}-\frac{k-1}{2}(n-1)} \\
&= 2^{\frac{k-1}{2}((k+1)^{n-1}-1)} \\
&\quad \cdot (k+1)^{\frac{k-1}{2k}((k+1)^n+k(n-1)+k-1)} \\
&\quad \cdot (k+3)^{\frac{k-1}{2k}((k+1)^{n-1}-k(n-1)-1)}.
\end{aligned}$$

□

Let P_n, R_n, T_n be the sets of spanning subgraphs of $\mathcal{S}(n,2)$, each of which consists of two trees with the outmost vertex $00\ldots0, 11\ldots1, 22\ldots2$ belonging to one tree while the other two outmost vertices being in the second tree, respectively. Let L_n denote the set of s-panning subgraphs of $\mathcal{S}(n,2)$, each of which contains three trees with every outmost vertex in a different tree. These four types of spanning subgraphs are illustrated in Figure 6.

Let p_n, r_n, t_n and l_n denote the cardinality of sets P_n, R_n, T_n and L_n.

Theorem 2.5. [43] *The number of spanning trees $\tau(\mathcal{S}(n,2))$ and the number of spanning subgraphs p_n, r_n, t_n and l_n are*

$$\begin{aligned}
\tau(\mathcal{S}(n,2)) &= 3^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}5^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}}, \\
p_n &= \frac{1}{6}\frac{5^n-3^n}{5^n}3^{\frac{1}{4}3^n-\frac{1}{2}n+\frac{3}{4}}5^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}, \\
l_n &= \frac{1}{4}(3^n-5^n)^2 3^{\frac{1}{4}3^n-\frac{3}{2}n+\frac{3}{4}}5^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}}.
\end{aligned}$$

By symmetry, $r_n = t_n = p_n$.

Theorem 2.6. *The number of the spanning trees on ${}^{+}\mathcal{S}(n,2)$ is*

$$\tau({}^{+}\mathcal{S}(n,2)) = 3\tau(\mathcal{S}(n,2)) + 6p + l,$$

where

$$\tau(\mathcal{S}(n,2)) = 3^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}5^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}},$$

$$p = \frac{1}{6}\frac{5^n-3^n}{5^n}3^{\frac{1}{4}3^n-\frac{1}{2}n+\frac{3}{4}}5^{\frac{1}{4}3^n+\frac{1}{2}n-\frac{1}{4}}$$

and $l = \frac{1}{4}(3^n-5^n)^2 3^{\frac{1}{4}3^n-\frac{3}{2}n+\frac{3}{4}}5^{\frac{1}{4}3^n-\frac{1}{2}n-\frac{1}{4}}$.

Proof: Notice that the number of spanning tree on ${}^{+}\mathcal{S}(n,2)$ can be enumerated by the case in Figure 7. Then we have

$$\begin{aligned}
\tau({}^{+}\mathcal{S}(n,2)) &= 3\tau(\mathcal{S}(n,2)) + 2p_n + 2r_n + 2t_n + l_n \\
&= 3\tau(\mathcal{S}(n,2)) + 6p_n + l_n.
\end{aligned}$$

□

3 Results for data center networks

Methods for computing the number of spanning trees, such as the well-known Matrix-Tree Theorem and technique of electrical networks are not easy for the larger

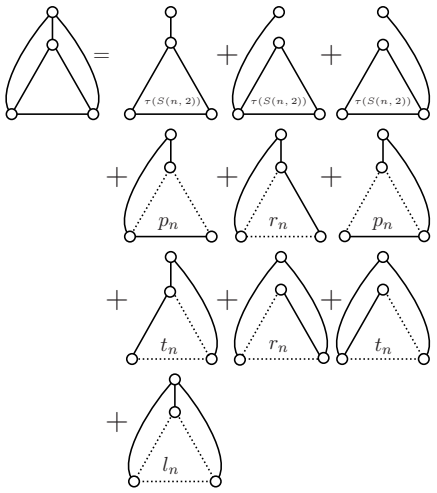


Figure 7: Illustration of the configurations needed to find $\tau(^{+}\mathcal{S}(n,2))$.

number of vertices of the graph. Naturally it is necessary to find other methods to produce exact expressions for the number of spanning trees of graphs and networks. Some bounds and algorithms can be found in [6, 10, 29].

In this section, we give the results for the number of spanning trees of data center networks.

3.1 Number of spanning trees of $\mathcal{D}_{1,n}$

A well known result on the number of spanning trees, called Cayley Theorem, is that $\tau(K_n) = n^{n-2}$. Note that $\mathcal{D}_{0,n}$ is isomorphic to K_n, so $\tau(\mathcal{D}_{0,n}) = n^{n-2}$. The line graph of a graph G, denoted by $L(G)$, is defined as the graph with $V(L(G)) = E(G)$ and $E(L(G)) = \{v_{e_i}v_{e_j} : e_ie_j \in E(G)\}$.

In [44], Zhang et al. gave a succinct relation of the number of spanning trees between a regular graph and the line graph of its subdivision.

For $\tau(\mathcal{D}_{1,n})$, we have the following result by Theorem 2.2.

Corollary 3.1. *For $n \geq 2$, we have*

$$\tau(\mathcal{D}_{1,n}) = n^{\frac{1}{2}(n^2-n-4)}(n+1)^{n-1}(n+2)^{\frac{1}{2}(n^2-n)}.$$

Proof: Since $\mathcal{D}_{1,n}$ is the line graph of $S(K_{n+1})$, it follows from Theorem 2.2 that

$$\tau(\mathcal{D}_{1,n}) = n^{\frac{1}{2}(n^2-n-4)}(n+1)^{n-1}(n+2)^{\frac{1}{2}(n^2-n)},$$

as desired. □

Spectrum graph theory plays an important role in counting the number of spanning trees. Let us introduce the method to the proof of Corollary 3.1 by the idea in [44].

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *Laplacian matrix* of G, denoted by $Q(G)$, is a $n \times n$ matrix whose entries q_{ij} are given by

$$q_{ij} = \begin{cases} d_G(v_i), & \text{if } i = j; \\ -1, & \text{if } i \neq j \text{ and } v_iv_j \in E(G); \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.2. [14] **(Matrix-Tree-Theorem)** *Let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the Laplacian eigenvalues of a connected graph G of order n. Then*

$$\tau(G) = \frac{1}{n}\lambda_2\lambda_3\cdots\lambda_n.$$

For a matrix M of order n, the *characteristic polynomial* of M on λ, denoted by $P(M,\lambda)$, is defined as $det(\lambda I - M)$, where I is an identity matrix of order n. For simplicity, we write $P(G,\lambda)$ for short of $P(Q(G),\lambda)$, where G is a graph.

Theorem 3.3. [15] *For any d-regular graph G, we have*

$$P(S(G),\lambda) = (-1)^{e(G)}(2-\lambda)^{e(G)-\nu(G)}P(G,\lambda(d+2-\lambda)).$$

where $e(G)$ and $\nu(G)$ are the size and order of G, respectively.

A relation of spectrum between a semi-regular graph and its line graph was obtained by Mohar in [20].

Theorem 3.4. [20] *For any semi-regular graph $R_{a,b}$, we have*

$$\begin{aligned} P(L(R_{a,b}),\lambda) &= (-1)^{\nu(R_{a,b})}(\lambda-(a+b))^{e(R_{a,b})-\nu(R_{a,b})} \\ &\quad \cdot P(R_{a,b}, a+b-\lambda). \end{aligned}$$

We now introduce the proof idea for the number of spanning trees of $\mathcal{D}_{1,n}$. Note that $\mathcal{D}_{1,n}$ is the line graph of the semi-regular graph $R_{n,2}$ obtained from K_{n+1} by subdividing each edge, i.e., $\mathcal{D}_{1,n} = L(R_{n,2}) = L(S(K_{n+1}))$. It is easy to see that $\nu(R_{n,2}) = \frac{1}{2}(n+1)(n+2)$, $e(R_{n,2}) = \nu(\mathcal{D}_{1,n}) = n(n+1)$ and $P(K_{n+1},\lambda) = \lambda(\lambda-(n+1))^n$. By Theorem 3.4, we have

$$\begin{aligned} P(\mathcal{D}_{1,n},\lambda) &= (-1)^{\nu(R_{n,2})}(\lambda-(n+2))^{e(R_{n,2})-\nu(R_{n,2})} \\ &\quad \cdot P(R_{n,2}, n+2-\lambda) \\ &= (-1)^{\frac{(n+1)(n+2)}{2}}(\lambda-(n+2))^{\frac{1}{2}(n+1)(n-2)} \\ &\quad \cdot P(R_{n,2}, n+2-\lambda). \end{aligned}$$

By Theorem 3.3, we have

$$\begin{aligned}
&P(R_{n,2}, n+2-\lambda)\\
&= (-1)^{e(K_{n+1})}(\lambda-n)^{e(K_{n+1})-\nu(K_{n+1})}P(K_{n+1}, \lambda(n+2-\lambda)\\
&= (-1)^{\frac{n(n+1)}{2}}(\lambda-n)^{\frac{1}{2}(n+1)(n-2)}\lambda(n+2-\lambda)\\
&\cdot(\lambda(n+2-\lambda)-(n+1))^n\\
&= (-1)^{\frac{(n+1)(n+2)}{2}}(\lambda-n)^{\frac{1}{2}(n+1)(n-2)}\lambda(\lambda-(n+2))(\lambda-1)^n\\
&\cdot(\lambda-(n+1))^n,
\end{aligned}$$

and hence

$$\begin{aligned}
P(\mathcal{D}_{1,n}, \lambda) &= \lambda(\lambda-1)^n(\lambda-n)^{\frac{1}{2}(n^2-n)-1}\\
&\quad\cdot(\lambda-(n+1))^n(\lambda-(n+2))^{\frac{1}{2}(n^2-n)}.
\end{aligned}$$

Therefore, all spectra $\lambda_1, \lambda_2, \cdots, \lambda_{n(n+1)}$ of $Q(\mathcal{D}_{1,n})$ are

$$\begin{aligned}
&Spec(Q(\mathcal{D}_{1,n}))\\
&= \begin{pmatrix} 0 & 1 & n & n+1 & n+2 \\ 1 & n & \frac{1}{2}(n^2-n)-1 & n & \frac{1}{2}(n^2-n) \end{pmatrix}.
\end{aligned}$$

By Matrix-Tree-Theorem, we have

$$\tau(\mathcal{D}_{1,n}) = n^{\frac{1}{2}(n^2-n-4)}(n+1)^{n-1}(n+2)^{\frac{1}{2}(n^2-n)},$$

as required.

The following table shows the number of spanning trees of $\mathcal{D}_{1,n}$.

Table 3. Number of spanning trees of $\mathcal{D}_{1,n}$.

	$n=2$	$n=3$	$n=4$	$n=5$
$\mathcal{D}_{1,n}$	6	6000	1492992000	143003094806250000

3.2 Number of spanning trees of $\mathcal{D}_{k,n}$

We first present a lower bound of $\tau(\mathcal{D}_{k,n})$ for $k \geq 2$.

Lemma 3.1. *For $k \geq 1$, we have*

$$\tau(\mathcal{D}_{k,n}) \geq 2(t_{k-1,n})^{t_{k-1,n}-1}\tau(\mathcal{D}_{k-1,n})^{t_{k-1,n}}.$$

Proof: Note that

$$V(\mathcal{D}_{k,n}) = \bigcup_{i=0}^{t_{k-1,n}} V(\mathcal{D}^i_{k-1,n})$$

and $V(\mathcal{D}^i_{k-1,n}) \cap V(\mathcal{D}^j_{k-1,n}) = \emptyset$ for any $0 \leq i < j \leq t_{k-1,n}$. For each i $(0 \leq i \leq t_{k-1,n})$, there are $\tau(\mathcal{D}_{k-1,n})$ spanning trees in $\mathcal{D}^i_{k-1,n}$. Let T'_i be a spanning tree of $\mathcal{D}^i_{k-1,n}$. Then $\bigcup_{i=0}^{t_{k-1,n}} T'_i$ is a spanning forest of $\mathcal{D}_{k,n}$. The *reduced graph* H of $\mathcal{D}_{k,n}$ is constructed by contracting each $\mathcal{D}^i_{k-1,n}$ by a single vertex u_i $(0 \leq i \leq t_{k-1,n})$. Observe that H is a complete graph of order $t_{k-1,n}+1$ and $V(H) = \{u_i \,|\, 0 \leq i \leq t_{k-1,n}\}$. Then the number of spanning trees of H is $(t_{k-1,n}+1)^{t_{k-1,n}-1}$.

Now let us consider the case where the spanning trees of $\mathcal{D}_{k,n}$ contains no edges of $\mathcal{D}^l_{k-1,n}$ for the unique l with $0 \leq l \leq t_{k-1,n}$. Let $H' = H - u_l$. Then H' is a complete graph of order $t_{k-1,n}$ and $V(H') = \{u_i \,|\, i \neq l,\ 0 \leq i \leq t_{k-1,n}\}$. Then the number of spanning trees of H' is $(t_{k-1,n})^{t_{k-1,n}-2}$, and hence

$$\begin{aligned}
\tau(\mathcal{D}_{k,n}) &\geq (t_{k-1,n}+1)^{t_{k-1,n}-1}\tau(\mathcal{D}_{k-1,n})^{t_{k-1,n}+1}\\
&\quad+(t_{k-1,n}+1)(t_{k-1,n})^{t_{k-1,n}-2}\tau(\mathcal{D}_{k-1,n})^{t_{k-1,n}}\\
&\geq 2(t_{k-1,n})^{t_{k-1,n}-1}\tau(\mathcal{D}_{k-1,n})^{t_{k-1,n}},
\end{aligned}$$

as desired. □

Theorem 3.5. *The number of spanning trees of $\mathcal{D}_{k,n}$, $k \geq 1$, $n \geq 1$, is*

$$\tau(\mathcal{D}_{k,n}) \geq 2^p \prod_{i=1}^{k-1} t_{i,n}^q n^{\frac{1}{2}r(n^2-n-4)}(n+1)^{r(n-1)}(n+2)^{\frac{1}{2}r(n^2-n)},$$

where $p = 1 + \sum_{i=2}^{k-1}\prod_{j=i}^{k-1} t_{j,n}$, $q = (t_{i,n}-1)\prod_{j=i+1}^{k-1} t_{j,n}$ and $r = \prod_{i=1}^{k-1} t_{i,n}$.

Proof: We prove this theorem by induction on k. If $k = 2$, then it follows from Lemma 3.1 that $\tau(\mathcal{D}_{2,n}) \geq 2(t_{1,n})^{t_{1,n}-1}\tau(\mathcal{D}_{1,n})^{t_{1,n}}$. Suppose that the result holds for $k = m-1$, that is,

$$\begin{aligned}
&\tau(\mathcal{D}_{m-1,n})\\
&\geq 2^{1+\sum_{i=2}^{m-2}\prod_{j=i}^{m-2} t_{j,n}} \prod_{i=1}^{m-2} t_{i,n}^{(t_{i,n}-1)\prod_{j=i+1}^{m-2} t_{j,n}}\\
&\cdot\left(n^{\frac{1}{2}(n^2-n-4)}(n+1)^{n-1}(n+2)^{\frac{1}{2}(n^2-n)}\right)^{\prod_{i=1}^{m-2} t_{i,n}}.
\end{aligned}$$

For $k = m$, we have

$$\begin{aligned}
&\tau(\mathcal{D}_{m-1,n})^{t_{m-1,n}}\\
&\geq 2^{t_{m-1,n}+\sum_{i=2}^{m-2}\prod_{j=i}^{m-1} t_{j,n}} \prod_{i=1}^{m-2} t_{i,n}^{(t_{i,n}-1)\prod_{j=i+1}^{m-1} t_{j,n}}\\
&\cdot n^{\frac{1}{2}(n^2-n-4)\prod_{i=1}^{m-1} t_{i,n}}\left((n+1)^{n-1}(n+2)^{\frac{1}{2}(n^2-n)}\right)^{\prod_{i=1}^{m-1} t_{i,n}},
\end{aligned}$$

and hence

$$\begin{aligned}
&\tau(\mathcal{D}_{m,n})\\
&\geq 2(t_{m-1,n})^{t_{m-1,n}-1}\tau(\mathcal{D}_{m-1,n})^{t_{m-1,n}}\\
&\geq 2^{1+\sum_{i=2}^{m-1}\prod_{j=i}^{m-1} t_{j,n}} \prod_{i=1}^{m-1} t_{i,n}^{(t_{i,n}-1)\prod_{j=i+1}^{m-1} t_{j,n}}\\
&\cdot\left(n^{\frac{1}{2}(n^2-n-4)}(n+1)^{n-1}(n+2)^{\frac{1}{2}(n^2-n)}\right)^{\prod_{i=1}^{m-1} t_{i,n}},
\end{aligned}$$

as desired. □

Next, we use a recent result, due to Klee et al., to get an upper bound of $\tau(\mathcal{D}_{k,n})$ for $k \geq 2$.

Theorem 3.6. [16] *For a simple graph G, we have*

$$\tau(G) \leq \frac{1}{|V(G)|^2} \prod_{v \in V(G)} (d(v)+1).$$

Theorem 3.7. *For $k \geq 2$ and any n, we have*

$$\tau(\mathcal{D}_{k,n}) \leq \frac{(n+k)^{(n+1)^{2^k}-1}}{((n+1)^{2^k}-1)^2}.$$

Proof: Since $D_{k,n}$ is a $n+k-1$-regular graph, it follows from Theorem 3.6 that

$$\tau(\mathcal{D}_{k,n}) \leq \frac{(n+k)^{t_{k,n}}}{t_{k,n}^2}.$$

By Theorem 1.1, we have

$$\tau(\mathcal{D}_{k,n}) \leq \frac{(n+k)^{(n+1)^{2^k}-1}}{((n+1)^{2^k}-1)^2},$$

as desired. □

4 Spanning tree entropy

Zhang, Wu and Lin [41] compared the entropy of s-panning trees for Farey network, pseudofractal fractal web, the square lattice, and the 2-dimensional Sierpiński triangle graph. Comellas, Miralles, Liu and Zhang [5] compared the entropy of spanning trees for $M(t)$, the honeycomb lattice, the 4-8-8 lattice (bathroom tile), and 3-12-12 lattice. Zhang, Wu and Comellas [42] compared the entropy of spanning trees for the Apollonian network, 3-dimensional Sierpiński triangle graph, and the 3-dimensional hypercubic lattice.

We now calculate the entropy of spanning trees for k-dimensional Sierpiński graph, k-dimensional Sierpiński triangle graphs and data center network $\mathcal{D}_{1,n}$.

Corollary 4.1. *The spanning tree entropy of k-dimensional Sierpiński graph $\mathcal{S}(n,k)$ is $\frac{k-1}{2k}\ln(k+1)(k+3)$.*

Proof: From Observation 1.1 and Theorem 2.3, we get that

$$\begin{aligned}
&\frac{\ln \tau(\mathcal{S}(n,k))}{|V(\mathcal{S}(n,k))|} \\
=\ & \frac{\ln((k+1)^{\frac{k-1}{2k}((k+1)^n+kn-1)}(k+3)^{\frac{k-1}{2k}((k+1)^n-kn-1)})}{(k+1)^n} \\
=\ & \frac{\frac{k-1}{2k}((k+1)^n+kn-1)\ln(k+1)}{(k+1)^n} \\
& +\frac{\frac{k-1}{2k}((k+1)^n-kn-1)\ln(k+3)}{(k+1)^n},
\end{aligned}$$

and hence

$$z(\mathcal{S}(n,k)) = \lim_{n\to\infty} \frac{\ln \tau(\mathcal{S}(n,k))}{|V(\mathcal{S}(n,k))|} = \frac{k-1}{2k}\ln(k+1)(k+3),$$

as desired. □

From the definition of k-dimensional Sierpiński triangle graphs, the order of $\mathcal{ST}(n,k)$ is $\frac{(k+1)^n+k+1}{2}$.

Corollary 4.2. *The spanning tree entropy of k-dimensional Sierpiński triangle graph $\mathcal{ST}(n,k)$ is*

$$\frac{k-1}{k(k+1)}\left(k\ln 2+(k+1)\ln(k+1)+\ln(k+3)\right).$$

Proof: From Theorem 2.4, we have

$$\begin{aligned}
&\frac{\ln \tau(\mathcal{ST}(n,k))}{|V(\mathcal{ST}(n,k))|} \\
=\ & \frac{\ln(2^{\frac{k-1}{2}((k+1)^{n-1}-1)}(k+1)^{\frac{k-1}{2k}((k+1)^n+k(n-1)-1)}}{\frac{(k+1)^n+k+1}{2}} \\
& \cdot\frac{(k+3)^{\frac{k-1}{2k}((k+1)^{n-1}-k(n-1)-1)})}{\frac{(k+1)^n+k+1}{2}} \\
=\ & \frac{(k-1)((k+1)^{n-1}-1)\ln 2}{(k+1)^n+k+1} \\
& +\frac{(k-1)((k+1)^n+k(n-1)-1)\ln(k+1)}{k((k+1)^n+k+1)} \\
& +\frac{(k-1)((k+1)^{n-1}-k(n-1)-1)\ln(k+3)}{k((k+1)^n+k+1)},
\end{aligned}$$

and hence

$$\begin{aligned}
z &= \lim_{n\to\infty} \frac{\ln \tau(\mathcal{ST}(n,k))}{|V(\mathcal{ST}(n,k))|} \\
&= \frac{k-1}{k+1}\ln 2+\frac{k-1}{k}\ln(k+1)+\frac{k-1}{k(k+1)}\ln(k+3) \\
&= \frac{k-1}{k(k+1)}\left(k\ln 2+(k+1)\ln(k+1)+\ln(k+3)\right).
\end{aligned}$$

□

Note that the order of $\mathcal{D}_{1,n}$ is $n(n+1)$. Then we have the following corollary.

Corollary 4.3. *The spanning tree entropy of $\mathcal{D}_{1,n}$ is ∞.*

Proof: From Corollary 3.1, we have

$$\begin{aligned}
&\frac{\ln \tau(\mathcal{D}_{1,n})}{|V(\mathcal{D}_{1,n})|} \\
=\ & \frac{\ln(n^{\frac{1}{2}(n^2-n-4)}(n+1)^{n-1}(n+2)^{\frac{1}{2}(n^2-n)})}{n(n+1)} \\
=\ & \frac{\frac{1}{2}(n^2-n)\ln(n^2+2n)+(n-1)\ln(n+1)-4\ln n}{n(n+1)},
\end{aligned}$$

and hence,

$$z = \lim_{n\to\infty} \frac{\ln \tau(\mathcal{D}_{1,n})}{|V(\mathcal{D}_{1,n})|} = \infty.$$

□

Felker and Lyons [8] gave an interesting large-k asymptotic expansion of k-dimensional hypercubic lattice $\mathcal{L}_k$ to order $1/k^{14}$,

$$\begin{aligned} & z(\mathcal{L}_k) \\ = \; & \ln(2k) - \frac{1}{4k} - \frac{3}{16k^2} - \frac{3}{16k^2} - \frac{7}{32k^3} - \frac{45}{128k^4} \\ & - \frac{269}{384k^5} - \frac{805}{512k^6} - \frac{3615}{1024k^7} - \frac{23205}{4096k^8} - \frac{23205}{4096k^8} \\ & + \frac{144963}{10240k^9} + \frac{2187031}{8192k^{10}} + \frac{40225409}{16384k^{11}} + \frac{1277353077}{65536k^{12}} \\ & + \frac{668172164455}{458752k^{13}} + \frac{271891453119}{262144k^{14}} - O\left(\frac{1}{k^{15}}\right). \end{aligned}$$

The definition of k-dimensional hypercubic lattice $\mathcal{L}_k$ and the entropy of the spanning trees for $\mathcal{L}_k$ can be found in [30]. We can compare the asymptotic values of the entropy of spanning trees for k-dimensional Sierpiński triangle graphs $\mathcal{ST}(n,k)$, $(2k-1)$-dimensional Sierpiński graphs $\mathcal{S}(n,2k-1)$, and $(k+1)$-dimensional hypercubic lattice $\mathcal{L}_{k+1}$ with the same asymptotic average degree $2k$, where $2 \leq k \leq 6$; see Table 4.

Table 4. The spanning tree entropy of $\mathcal{ST}(n,k)$, $\mathcal{S}(n,2k-1)$ and $\mathcal{L}_{k+1}$.

k	$z(\mathcal{ST}(n,k))$	$z(\mathcal{S}(n,2k-1))$	$z(\mathcal{L}_{k+1})$
2	1.5851	1.0594	1.6685
3	2.4653	1.5485	1.9991
4	3.0824	1.8780	2.2424
5	3.5591	2.1278	2.4366
6	3.9477	2.3291	2.5987

From Table 4, the asymptotic value for the entropy of spanning trees of k-dimensional Sierpiński graphs $\mathcal{S}(n,k)$ is the lowest among these graphs with average degree $2k$, where $2 \leq k \leq 6$. This suggests that k-dimensional Sierpiński graphs $\mathcal{S}(n,k)$ would be less reliable to a random removal of edges than the graphs mentioned above. However, the degree distribution means that the reliability of graphs, and as we know, graphs with a scale-free degree distribution are more resilient than homogeneous graphs (like regular and exponential graphs, where all vertices are statistically identical), see [34]. By considering this, it would be of interest to study the connections among the spanning tree entropy of a graph and other relevant graph parameters, such as degree distribution, degree correlation, etc.

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