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► **To cite this version:**

David Arturo Man Castillo, Stéphane Junca. Computing the Elo Stationary Distribution for Two Players. 2024. hal-04675789

HAL Id: hal-04675789

<https://hal.science/hal-04675789v1>

Preprint submitted on 22 Aug 2024

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Computing the Elo Stationary Distribution for Two Players

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Abstract—The Elo rating system, originally developed by Arpad Elo, is a widely used method for assessing the skill levels of players in two-player games. This research focuses on analyzing the stationary distribution of player ratings within the Elo model. We developed a numerical algorithm to approximate the cumulative distribution function (CDF) of the stationary distribution and conducted extensive simulations to validate the accuracy and applicability of the algorithm.

Our study included a grid convergence analysis, symmetry tests, and an examination of the "Elo conjecture", which posits a relationship between rating differences and match outcomes. The results confirmed the accuracy of the numerical scheme and revealed that the stationary distribution exhibits key properties, including symmetry around its mean and a dependence of variance on the parameter K , which controls the adjustment speed of ratings. These findings provide valuable insights into the Elo rating system's underlying dynamics, offering a foundation for both theoretical advancements and practical applications.

Keywords: Elo rating system, Markov chain, stationary distribution, cumulative distribution function, numerical algorithm.

I. INTRODUCTION

The Elo rating system, developed by Arpad Elo, is a widely used method for calculating the relative skill levels of players in two-player games, such as chess [1]. Despite its widespread use, the Elo rating system has certain limitations and unresolved problems. The extension of the Elo rating system is currently an active area of research, for instance, for players of varying rating strength [2]–[5], to improve the system [6],

for ranks' dynamic [7], for educational applications [8]–[11], for a large number of players [12], [13]. Moreover, recent studies have also explored the system's convergence properties and the application of contraction methods in tournament settings [14], [15]. However, the theoretical framework assuming constant strength between two players remains incomplete. The objective of this research is to shed new light on this latest case.

This study aims to explore the stationary distribution of player ratings in the Elo system [16], [17] and to investigate Elo's assumption that the expected score depends on the rating difference. More details on the results presented in this work can be found in the Research Report [18].

II. MATHEMATICAL FRAMEWORK

This section outlines the theoretical basis of the Elo rating system, its dynamics, and the mathematical constructs utilized in this study. The Elo rating system for two players calculates the relative skill levels of competitors in head-to-head games. The key equations that describe the update mechanism for players' ratings after a match are as follows:

A. Rating Update Equations

For two players, i and j , the ratings after a match are updated according to the following equations:

$$R_i^{n+1} = R_i^n + K (S_{ij}^n - b(R_i^n - R_j^n)) \quad (1)$$

$$R_j^{n+1} = R_j^n + K (S_{ji}^n - b(R_j^n - R_i^n)) \quad (2)$$

where the sum of the scores $S_{ij}^n + S_{ji}^n = 1$ represents the outcome of the game. The function b

is known as the bonus-malus function, which is defined as

$$b(x) = \frac{1}{1 + 10^{-x/400}} \in [0, 1]. \quad (3)$$

Which satisfies the same symmetry condition as the scores, $b(x) + b(-x) = 1$.

This symmetries leads to a fundamental property of the Elo rating system: the conservation of the total mass of Elo points:

$$R_i^{n+1} + R_j^{n+1} = R_i^n + R_j^n = R_i^0 + R_j^0 = 2\rho^0, \quad (4)$$

where ρ^0 is the average rating. Therefore, the dynamics for two players are governed by the difference in ratings:

$$\Delta_{ij}^n = R_i^n - R_j^n, \quad (5)$$

since

$$R_i^n = \rho^0 + \Delta_{ij}^n, \quad R_j^n = \rho^0 - \Delta_{ij}^n. \quad (6)$$

The evolution of the rating difference is given by:

$$\Delta_{ij}^{n+1} = \Delta_{ij}^n + 2K(S_{ij}^n - b(\Delta_{ji}^n)). \quad (7)$$

B. Players with Constant Strength

In this study, we assume that players maintain constant strength over time. Only two outcomes are possible after each encounter: a win or a loss. Thus, the score S_{ij}^n for match n can be modeled by a sequence of independent Bernoulli trials with a fixed probability distribution:

$$S_{ij}^n = 0 \quad \text{or} \quad 1. \quad (8)$$

Let $p = p_{ij}$ and $q = 1 - p$ be the probabilities of player i winning or losing against player j , respectively. In this framework, the Elo rating system behaves as a homogeneous Markov chain [20]. The focus is on the distribution of Δ_{ij}^n in its limiting form.

C. Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of the Elo distribution of Δ_{ij}^n is [18], [19]:

$$F_n(x) = P(\Delta_{ij}^n \leq x). \quad (9)$$

To update the CDF for the next game, the recurrence relation (7) is rewritten as follows [18]:

$$F_{n+1}(x) = p \cdot F_n(h(x - 2K)) + q \cdot F_n(h(x)), \quad (10)$$

where $h(x)$ is the inverse function of $g(x) = x - 2Kb(x)$. The function g is increasing for sufficiently small K [18] and governs the dynamics by rewriting (7) as:

$$\Delta_{ij}^{n+1} = g(\Delta_{ij}^n) + 2K S_{ij}^n. \quad (11)$$

The dynamics (10) can be interpreted through the transition operator \mathcal{L} [20]:

$$F_{n+1} = \mathcal{L}(F_n). \quad (12)$$

The stationary distribution F is then the CDF of Δ_{ij} , the limit of Δ_{ij}^n , fixed by \mathcal{L} :

$$F = \mathcal{L}(F), \quad (13)$$

which requires solving the functional equation for all $x \in \mathbb{R}$:

$$F(x) = p \cdot F(h(x - 2K)) + q \cdot F(h(x)), \quad (14)$$

with the conditions that F is nondecreasing and satisfies the boundary conditions:

$$\lim_{x \rightarrow -\infty} F(x) = 0, \quad \lim_{x \rightarrow +\infty} F(x) = 1. \quad (15)$$

III. METHODOLOGY

The goal of this section is to detail the computational approach used to approximate F , the CDF of the stationary distribution in the Elo rating system. This involves discretizing \mathbb{R} and linearize $\mathcal{L}(\cdot)$. The following steps outline the methodology employed [18]:

A. Discretization and Mesh Generation

To approximate the stationary distribution, we discretize the domain of possible rating differences, x , into a finite set of points. Specifically, we create a mesh $(x_i)_{i=0}^N$, where $x_i = i\delta x$ and δx is the mesh spacing over a finite interval $[-M, M]$. The inverse function $h(x)$ required for the update equations is computed using the Newton-Raphson method. This approach allows us to accurately determine $h(x_i)$ and $h(x_i - 2K)$.

B. Interpolation and Matrix Construction

Given that F is only known on the discrete grid, we estimate $F(h(x_i))$ at any non-mesh point using linear interpolation within the interval $x_j < h(x_i) < x_{j+1}$. If x_j or x_{j+1} fall outside the grid, $F(x_j)$ is approximated using the boundary conditions derived from (15).

Next, we construct a matrix \mathbf{A} that approximates the linear relationship described by (14) on the grid. The vector \mathbf{F} represents the values of the CDF on the grid, $(F(x_i))_{i=0}^N$, and satisfies the following equation:

$$\mathbf{F} = \mathbf{A}\mathbf{F} + \mathbf{b} + \mathcal{O}(\delta x)^2. \quad (16)$$

The vector \mathbf{b} represents the boundary conditions at the grid edges. The primary source of error, $\mathcal{O}(\delta x)^2$, arises from the linear interpolation process, with a smaller contribution from the approximation of values outside the grid, which are estimated using the limits as x approaches $\pm\infty$. This approximation is valid provided that the chosen interval $[-M, M]$ is sufficiently large.

C. Solving for the Stationary Distribution

By neglecting the interpolation error, the stationary distribution can be approximated by solving the linear system:

$$(\mathbf{I} - \mathbf{A})\mathbf{F} = \mathbf{b}. \quad (17)$$

IV. RESULTS

A. Stationary Distribution Computational Algorithm

The computed CDF for different p and q values in Figure 1 are nondecreasing, which illustrates

a desirable property of our numerical scheme. Moreover, the observed expected values and variances align well with real-world scenarios, further validating the approach.

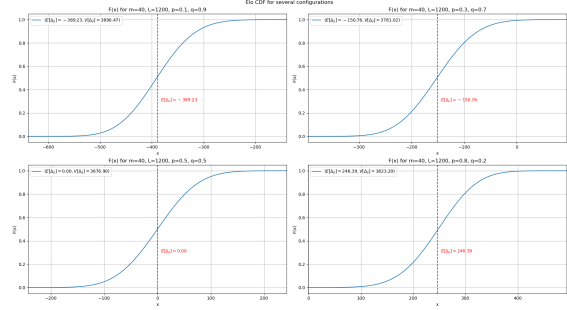


Fig. 1. Computed CDF for different p and q values. Each plot represents the CDF function derived from the Elo rating system for different winning probabilities.

B. Grid Convergence Study

The grid convergence study [18], [21] in Figure 2 shows an approximate second-order convergence, indicated by a regression slope of 1.98. This demonstrates the reliability and accuracy of the numerical scheme in approximating the stationary distribution.

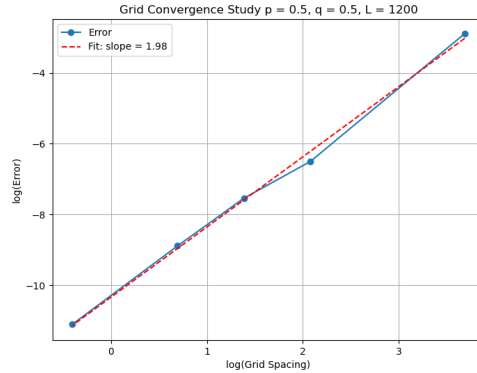


Fig. 2. Grid convergence study. The graph shows second-order convergence with a regression slope of 1.98. It plots the log of the error against the log of the grid spacing.

C. Symmetry Study

Let μ denote the expected value of the stationary distribution, $\mu = \int_{\mathbb{R}} x dF(x)$. The symmetry test involves checking whether the CDF satisfies the relation $F(\mu + x) + F(\mu - x) = 1$. Figures 3, 4 confirms that the stationary distribution appears symmetric. This symmetry is particularly evident for $p = 0.5$, a case where symmetry is theoretically proven [16], [18]. For other values of p , the symmetry is less certain but still suggested by the results. The error decreases as the grid spacing δx decreases as shown in Figure 4, correlating with the precision of the expectation μ . Notably, for $p = 0.5$, where the expectation $\mu = 0$ is theoretically known, the method demonstrates particularly high accuracy.

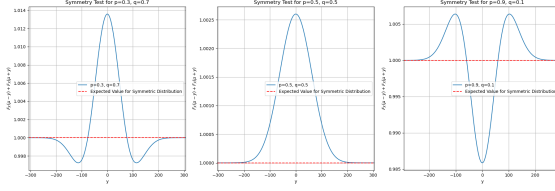


Fig. 3. Symmetry test for different p and q values. The plot shows $F(\mu + x) + F(\mu - x)$ in blue compared to the constant value 1 in red.

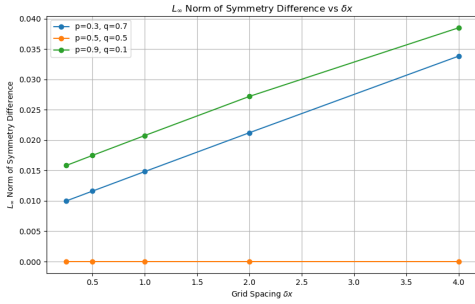


Fig. 4. Symmetry test error analysis: the maximum deviation $\|F(\mu + x) + F(\mu - x) - 1\|_{\infty}$ versus grid spacing δx , for various p and q values. The results suggest that the stationary distribution is symmetric, particularly for $p = 0.5$.

D. $\mathbb{E}[\Delta_{ij}]$ and $\mathbb{V}[\Delta_{ij}]$ vs. K

The expected value $\mathbb{E}[\Delta_{ij}]$ demonstrates a minimal dependence on K , to the extent that it can

be considered effectively independent, while the variance $\mathbb{V}[\Delta_{ij}]$ exhibits a clear linear dependence on K , as illustrated in Figures 5 and 6.

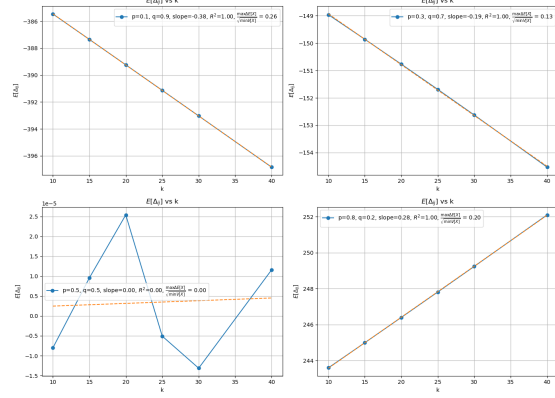


Fig. 5. Expected value $\mathbb{E}[\Delta_{ij}]$ vs. parameter K . The analysis shows that this statistical measure is independent of the parameter K for different values of p .

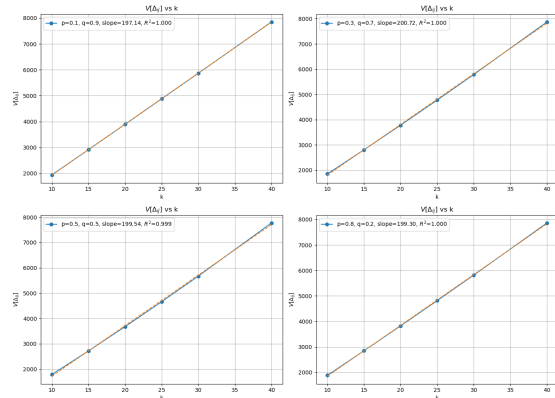


Fig. 6. Variance $\mathbb{V}[\Delta_{ij}]$ vs. parameter K . The analysis shows that the variance depends linearly on the parameter K for different values of p .

E. Elo Conjecture Study

Arpad Elo's model assumes that the difference in ratings predicts the probability of winning. Given that the true strength of player i is represented by $\lim_{n \rightarrow \infty} \mathbb{E}(R_i^n)$ [13], [22], we reformulate this conjecture in terms of the stationary

distribution of the rating differences. The conjecture posits that:

$$\mathbb{E}[b(\Delta_{ij})] = b(\mathbb{E}[\Delta_{ij}]). \quad (18)$$

This equality is mathematically valid for $p = 0.5$ [18]. For other values of p , we hypothesize that the conjecture is not exactly true due to the nonlinearity of the function b . The error in the conjecture can be estimated as:

$$|\mathbb{E}(b(\Delta_{ij})) - b(\mathbb{E}(\Delta_{ij}))| = |\mathbb{E}(b(\mu + (\Delta_{ij} - \mu))) - b(\mu)| \leq L \mathbb{E}(|\Delta_{ij} - \mu|),$$

where L is a Lipschitz constant for b , approximately 0.001. This residual error is independent of δx , as shown in Figure 7. Despite this small error, the Elo conjecture is practically true, with an error around 10^{-3} .

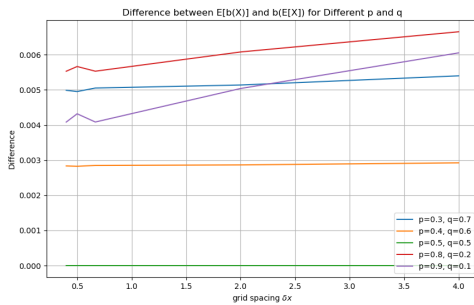


Fig. 7. Error of the Elo conjecture vs. grid spacing. The decrease in error as the grid spacing becomes finer suggests that the conjecture may hold approximately.

V. CONCLUSION

Our study validates the accuracy and reliability of the numerical scheme used to approximate the stationary distribution in the Elo rating system. The analysis of the computed stationary distribution reveals several significant properties that emerge from the numerical results. The outcomes of this research highlight key aspects of the distribution’s behavior and suggest areas where further mathematical validation and refinement of the numerical approach are needed:

- A. The numerical scheme preserves the monotonicity of the cumulative distribution function.
- B. The scheme exhibits an approximate second-order convergence rate.
- C. The stationary distribution appears symmetric, particularly evident when $p = 0.5$.
- D. The expected value of the rating difference is independent of the K -factor, whereas the variance shows a linear dependence on K .
- E. The Elo conjecture, while not perfectly exact for all p , is practically valid with a small margin of error, sufficient for predicting match outcomes.

Future research could focus on mathematical proofs of the observed properties and the development of more precise numerical techniques. Additionally, exploring the tail behavior of the distribution and comparing it with the normal distribution could provide further insights into the stationary distribution properties. Finally, incorporating stochastic algorithms alongside deterministic approaches could offer a broader validation of the results.

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Code Available on [GitHub](#)

Thursday 22nd August, 2024