



HAL
open science

Schur function minimization under Nevanlinna-Pick constraints: a convex approach

Gibin Bose, David Martínez Martínez, Martine Olivi, Fabien Seyfert

► **To cite this version:**

Gibin Bose, David Martínez Martínez, Martine Olivi, Fabien Seyfert. Schur function minimization under Nevanlinna-Pick constraints: a convex approach. 2024. hal-04675421

HAL Id: hal-04675421

<https://hal.science/hal-04675421v1>

Preprint submitted on 22 Aug 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

SCHUR FUNCTION MINIMIZATION UNDER NEVANLINNA-PICK CONSTRAINTS: A CONVEX APPROACH.

GIBIN BOSE *, DAVID MARTINEZ MARTINEZ†, MARTINE OLIVI AND FABIEN SEYFERT‡

Abstract In this paper we consider a Schur minimization problem in a frequency band under various constraints, which are relevant in engineering applications. These constraints include NP-interpolation conditions, a degree bound and the location of spectral zeros. We propose a relaxation of this initial problem, transforming it into a convex optimization problem. The relaxation concerns the degree constraint, but the degree remains bounded and the problem becomes tractable by nonlinear semi-definite programming techniques. The solution provides a bound on the minimal achievable norm for a Schur function satisfying the initial problem at a given degree. We illustrate our results with an application to the matching problem in filter design. We show that our method significantly improve on the Fano's bound, well-known in this context.

1. Introduction. Our interest in the optimization problem studied in this paper stems from two important system design issues: sensitivity minimization in robust control and broadband matching in circuit theory. In both problems, to improve the functioning of the system, a particular transfer function has to be optimized over a prescribed frequency band. The problem of broadband matching is to design a matching network in order to maximize the power transmitted to a given load in a frequency band. In this band, the reflection of the global system must therefore be minimized (see Section 7). In robust control, the sensitivity function has to be as small as possible at low frequencies - where the power of the reference signal is usually concentrated - to ensure a good tracking accuracy as well as disturbance rejection (see [2, chap. 23]).

The relevance for these problems of Nevanlinna-Pick (NP) interpolation, which consists in finding a Schur function to meet a finite set of interpolation conditions on the disk or the half-plane, has been early pointed: in the 50's for broadband matching [24] and in the 80's for robust control [25]. In broadband matching, a set of NP-interpolation conditions translate the limitations imposed on the global reflection due to the presence of the load. In robust control, interpolation constraints ensure internal stability, while a norm bound guarantees performance and robustness [2, chap. 23]. Both applications lead to a same optimization problem.

Schur optimization problem: *find a Schur function f with minimum infinite norm in a frequency band, which satisfies a set of NP-interpolation constraints.*

In engineering, additional constraints must be taken into consideration. First of all, it is important to maintain the complexity of design objects as small as possible. The complexity of a linear system is captured by the McMillan degree of its transfer function. A degree constraint must thus be added to the set of NP-interpolation constraints.

The last constraint amounts to prescribe the spectral zeros of f , that is the n stable zeros of $1 - ff^*$. It presents some interest for the design problems that we consider. For example, placing a spectral zero close to the imaginary axis at some frequency, is a classical technique in sensitivity shaping. Indeed, it raises the modulus of the sensitivity function close to its upper bound at that frequency. Then, by the so-called waterbed effect, the sensitivity will be lowered in other parts of the spectrum [21]. In filter design too, more economic and effective filters can be designed by controlling the location of spectral zeros, called transmission zeros in this context. They are used to improve the near-band rejection slopes and thus the selectivity of the filter [9]. In the matching problem, in addition, part of the transmission zeros of the global reflection to be optimized are known, they comes from the cascade connection with

*DESIGN ENGINEER AT FARRAN, CORK, IRLANDE

†SENIOR RADIO FREQUENCY TECHNOLOGIES ENGINEER AT HUAWEI, STOCKHOLM, SUÈDE

‡RESEARCHERS AT INRIA, SOPHIA ANTIPOLIS, FRANCE

the load.

It turns out that spectral zeros also play a fundamental role in the NP-interpolation problem with degree constraint. In a collection of papers (see e.g. [8]), it was established that given $n + 1$ interpolation conditions such that the Pick matrix is positive definite, the couple of polynomials (p, q) such that p/q is an interpolant of degree at most n are in bijective correspondence with polynomials of degree $2n$, positive on the boundary of the analyticity domain (unit circle, real or complex line). Given a positive polynomial R , the corresponding interpolant p/q is the unique minimizer of a weighted entropy functional and it satisfies, $qq^* - pp^* = R$.

In the same spirit, we propose to fix the spectral zeros by imposing the polynomial numerator R of $1 - ff^*$. However in our case, an interpolant whose prescribed degree exceeds the number of interpolation conditions is searched, leaving room for optimization. This situation has been addressed in [16], with an application to the sensitivity shaping problem: given $n + 1$ interpolation conditions and the Pick matrix being positive definite, the rational Schur interpolants f of degree at most $n + m$ are parametrized by the $n + m$ stable zeros of $1 - ff^*$, but extra parameters have to be fixed to uniquely determine the interpolant. It should be noticed that fixing R allows the convex relaxation of our optimization problem.

In Section 2, we put together all these constraints, fix the notations and state our optimization problem. The convex relaxation of this problem is presented in section 3 and studied in Section 4. It takes the form of a SDP problem, for which numerical implementations are available. In Section 5 we generalize our results to the case where the interpolation problem also includes boundary interpolation data. Finally, we illustrate our method with an application to the broadband matching problem in Section 7. This convex relaxation provides new bounds for the best "matching" level achievable at a given degree.

GeneralProblem

2. A Schur optimization problem with NP-constraint. The Hardy space H^∞ of bounded holomorphic functions in Π^+ , the open right half-plane, is the general framework of our study. A function which belongs to the closed unit ball \mathcal{S} of H^∞ , is called a *Schur function*. We denote by \mathbb{P}_{2N}^+ the set of nonnegative polynomials on the imaginary axis of degree at most $2N$, which is regarded as embedded in \mathbb{R}^{2N+1} . For a rational matrix-valued function F , we define its *paraconjugate* F^* by

{paraconjugate}

$$(1) \quad F^*(s) = F(-\bar{s})^*, \quad s \in \mathbb{C}$$

where the superscript "*" stands for transpose-conjugate. If p is a polynomial, its paraconjugate p^* has the same degree as p and roots symmetric to those of p , with respect to the imaginary axis. A polynomial is called *stable* (resp. *stable in the broad sense*) if it has no root in Π^+ (resp. $\overline{\Pi^+}$, the closed right half-plane). A function of the form q^*/q where q is a stable polynomial of degree d , is called a *Blaschke product* of degree d .

The classical **Nevanlinna-Pick problem** [2, sect. 18.1] is, given a collection of distinct interpolation points z_1, z_2, \dots, z_M in Π^+ , with a collection of complex numbers v_1, v_2, \dots, v_M , to find a Schur function satisfying

{IC}

$$(2) \quad f(z_j) = v_j, \quad j = 1, \dots, M.$$

The set $\sigma = \{(z_j, v_j); 1 \leq j \leq M\}$ is called the *data set*; the NP-problem $\mathcal{P}(\sigma)$ and its set of solutions $\mathcal{S}(\sigma)$.

The following result is well-known.

nevanlinnaPickTheorem

THEOREM 1 (Nevanlinna-Pick). *The problem $\mathcal{P}(\sigma)$ has a solution if and only if the Pick matrix $\Lambda(\sigma)$ given by*

{Pick}

$$(3) \quad \Lambda(\sigma)_{ij} = \frac{1 - v_i \bar{v}_j}{z_i + \bar{z}_j}, \quad 1 \leq i, j \leq M$$

is positive semi-definite ($\Lambda(\sigma) \geq 0$).

1. If $\Lambda(\sigma) > 0$ (positive definite), the problem is indeterminate: it has infinitely many solutions.
2. If $\Lambda(\sigma)$ is singular, and has rank $k < M$, the problem has a unique solution which is a Blaschke product of exact degree k .

An interesting proof of this result can be found in [14, sect. 1.2], in the case of the analogous interpolation problem for the unit disk.

We denote by

$$(4) \quad \|f\|_{\infty, \mathbb{I}} = \sup_{jw \in \mathbb{I}} |f(jw)|$$

the uniform norm over \mathbb{I} , a finite union of disjoint compact intervals of the imaginary axis.

We consider the following **Schur optimization problem with NP-constraints**: given

- $\sigma = \{(z_j, v_j); 1 \leq j \leq M\}$, a data set for a NP-problem $\mathcal{P}(\sigma)$ such that

$$\{\text{Pickpositive}\} \quad (5) \quad \Lambda(\sigma) > 0$$

- $R \in \mathbb{P}_{2N}^+$, a nonnegative polynomial with no root in \mathbb{I} , $N > M$,
- solve the minimization problem

$$\{\text{GOP}\} \quad (6) \quad \min_{f \in \mathcal{S}_N^R(\sigma)} \|f\|_{\infty, \mathbb{I}},$$

where $\mathcal{S}_N^R(\sigma)$ is the set of Schur rational functions $f \in \mathcal{S}(\sigma)$ of degree at most N and having a polynomial fraction representation of the form $f = \frac{p}{q}$, where p and q are polynomials of degree at most N , such that $qq^* - pp^* = R$, and q is stable broad sense.

If ξ is a root of R , then for $f \in \mathcal{S}_N^R(\sigma)$, $|f(\xi)| = 1$. Since the goal is to minimize the uniform norm over \mathbb{I} , it is better to discard the polynomials R with a root in \mathbb{I} . Also note that the polynomial fraction representation of $f \in \mathcal{S}_N^R(\sigma)$ needs not be irreducible. In particular, if r is a stable spectral factor of R ($R = rr^*$), a root of q on the imaginary line, is also root of p and r , with the same multiplicity (because $|p|^2 + |r|^2 = |q|^2$ on $j\mathbb{R}$). More generally, recall that a rational Schur function has no poles on the imaginary axis.

If the interpolation values are all zero, the function $f = 0$ belongs to $\mathcal{S}_N^R(\sigma)$ and the problem has a trivial solution. We will assume in the sequel that at least one of the values v_j 's is not zero.

PROPOSITION 2. $\mathcal{S}_N^R(\sigma)$ is nonempty.

Proof. Since $\Lambda(\sigma) > 0$, the problem $\mathcal{P}(\sigma)$ has infinitely many solutions. We shall use the following result (see e.g. [15, 8, 3]): for any nonzero polynomial $\eta(z)$, of degree at most $M - 1$, stable in the broad sense, there exists a couple of polynomials (π, χ) of degree at most $M - 1$, χ stable in the broad sense, such that

- $f = \frac{\pi}{\chi}$ is Schur,
- f is a solution to $\mathcal{P}(\sigma)$
- $\chi\chi^* - \pi\pi^* = \eta\eta^*$.

Let r be a spectral factor of R ($R = rr^*$), stable in the broad sense. Choose for η a polynomial factor of r , so that the polynomials $p = \pi r / \eta$ and $q = \chi r / \eta$ have degree at most N . Then $f = \frac{p}{q}$ belongs to $\mathcal{S}_N^R(\sigma)$. \square

3. Convex relaxation of the problem.

3.1. The set of admissible gain functions. Following the terminology used in system theory, we call *gain function* a function which is the modulus of a H^∞ function.

Let $f = \frac{p}{q} \in \mathcal{S}_N^R(\sigma)$ be a non-zero function, p and q being two polynomials, such that q is stable in the broad sense and $qq^* - pp^* = R$. Then,

$$|f(jw)|^2 = \frac{P}{P+R}(jw), \quad w \in \mathbb{R},$$

where $P = pp^*$ belongs to $\mathbb{P}_{2N}^+ \setminus \{0\}$.

For any P in $\mathbb{P}_{2N}^+ \setminus \{0\}$, consider the function

$$(7) \quad G_P(jw) = \sqrt{\frac{P}{P+R}}(jw), \quad w \in \mathbb{R}.$$

Remark 3. The quotient in (7) is well-defined and non negative, even at those jw such that $P(jw) = R(jw) = 0$. A polynomial positive on the imaginary axis having even multiplicity roots on the imaginary axis, let $m_R(jw)$ (resp. $m_P(jw)$) be half the multiplicity of jw as a root of R (resp. P). Then,

- if $m_P(jw) < m_R(jw)$, then $G_P(jw) = 1$,
- if $m_P(jw) = m_R(jw)$, then $0 < G_P(jw) < 1$,
- if $m_P(jw) > m_R(jw)$, then $G_P(jw) = 0$.

The function G_P is a gain function. Thus, there exists an outer function $U_P \in H^\infty$ such that

$$(8) \quad |U_P(jw)| = G_P(jw), \quad U_P(1) > 0,$$

which is Schur and uniquely defined by the normalization condition (see [14, Chap. II, Thm. 4.4]). It can be computed using the spectral factorization of a positive polynomial ([3, Prop. 2]):

PROPOSITION 4. To any non zero $P \in \mathbb{P}_{2N}^+ \setminus \{0\}$ one can associate a polynomial \check{p} of degree N , stable in the broad sense, such that

$$(9) \quad P = \check{p}\check{p}^*.$$

The polynomial \check{p} is unique up to a multiplicative unimodular constant, and if P has exact degree $2N$, then \check{p} has exact degree N .

We define the map

$$(10) \quad \varphi : P \mapsto \varphi(P),$$

where $\varphi(P)$ is the unique solution to (9) such that $\varphi(P)(1) > 0$. The map φ is continuous.

The outer function U_P is thus given by

$$(11) \quad U_P = \frac{\varphi(P)}{\varphi(P+R)}.$$

The two following inequalities will be useful in the sequel:

PROPOSITION 5. Let P_1 and P_2 two polynomials in $\mathbb{P}_{2N}^+ \setminus \{0\}$ and let $P = \alpha P_1 + (1 - \alpha)P_2$ for $0 < \alpha < 1$. We then have

$$(12) \quad \alpha G_{P_1}(jw) + (1 - \alpha) G_{P_2}(jw) \leq G_P(jw), \quad w \in \mathbb{R},$$

as well as

$$(13) \quad \alpha G_{P_1}(jw)^2 + (1 - \alpha) G_{P_2}(jw)^2 \leq G_P(jw)^2, \quad w \in \mathbb{R}.$$

At w such that $R(jw) \neq 0$, equalities holds if and only if $P_1(jw) = P_2(jw)$.

Proof. First consider $w \in \mathbb{R}$ such that $R(jw) \neq 0$. The first inequality follows from the strict concavity of the map $x \mapsto \sqrt{\frac{x}{x+c}}$, $c \in \mathbb{R}^*$. The second one is proved similarly, using the strict concavity of the map $x \mapsto \frac{x}{x+c}$, $c \in \mathbb{R}^*$.

Now consider w such that $R(jw) = 0$. We may assume without loss of generality that $m_{P_1}(jw) \leq m_{P_2}(jw)$, so that $m_P(jw) = m_{P_1}(jw)$ and consider the following cases:

- $m_R(jw) < m_{P_1}(jw)$. Then $G_{P_1}(jw) = G_{P_2}(jw) = G_P(jw) = 0$, and we get two trivial equalities.
- $m_R(jw) = m_{P_1}(jw)$. Simplifying the polynomials P_1, P_2, P and R by the factor $(z - jw)^{2m_R(jw)}$, we get four positive polynomials $\tilde{P}_1, \tilde{P}_2, \tilde{P}$ and \tilde{R} such that $\tilde{P} = \alpha\tilde{P}_1 + (1 - \alpha)\tilde{P}_2$ and

$$G_{P_i}(jw) = \sqrt{\frac{\tilde{P}_i}{\tilde{P}_i + \tilde{R}}}(jw), \quad i = 1, 2; \quad G_P(jw) = \sqrt{\frac{\tilde{P}}{\tilde{P} + \tilde{R}}}(jw),$$

while $\tilde{R}(jw) \neq 0$. Then, the first part of the proof applies.

- $m_{P_1}(jw) < m_R(jw) \leq m_{P_2}(jw)$. Then $G_{P_1}(jw) = G_P(jw) = 1$, while $0 \leq G_{P_2}(jw) < 1$, and the two inequalities are satisfied.
- $m_{P_1}(jw) \leq m_{P_2}(jw) < m_R(jw)$. Then $G_{P_1}(jw) = G_{P_2}(jw) = G_P(jw) = 1$, and the two equalities are satisfied. \square

DEFINITION 6. We say that a gain function of the form G_P is admissible if there exists a rational Schur function f solving $\mathcal{P}(\sigma)$ and such that

$$(14) \quad |f(jw)| \leq G_P(jw), \quad w \in \mathbb{R}.$$

We define $\mathcal{H}_N^R(\sigma)$ to be the set of polynomials P in $\mathbb{P}_{2N}^+ \setminus \{0\}$ such that the gain function G_P defined by (7) is admissible.

3.2. A convex optimization problem. The main idea of our approach is to optimize directly over the set of admissible gain functions $G_P, P \in \mathcal{H}_N^R(\sigma)$, rather than over the set $\mathcal{S}_N^R(\sigma)$. This approach leads to a convex relaxation of the initial problem (6), which can be stated as a **Polynomial optimization problem**:

$$(15) \quad \min_{P \in \mathcal{H}_N^R(\sigma)} \Psi(P)$$

where, for $P \in \mathcal{H}_N^R(\sigma)$,

$$(16) \quad \Psi(P) = \left\| \frac{P}{R} \right\|_{\infty, \mathbb{I}} = \sup_{jw \in \mathbb{I}} \frac{P}{R}(jw).$$

Note that minimizing $\Psi(P)$ is equivalent to minimizing the infinite norm of G_P .

For any $f = \frac{p}{q} \in \mathcal{S}_N^R(\sigma)$, it is clear that $P = pp^*$ belongs to $\mathcal{H}_N^R(\sigma)$, so that this polynomial optimization problem can be viewed as a relaxation of (6). We shall see that $\mathcal{H}_N^R(\sigma)$ possesses some interesting properties which make this new problem much more tractable. Moreover, Ψ is a quasiconvex function [7, section 3.4].

First remark that, $\mathcal{S}_N^R(\sigma)$ being nonempty by Proposition 2, $\mathcal{H}_N^R(\sigma)$ is nonempty too.

PROPOSITION 7. The set $\mathcal{H}_N^R(\sigma)$ is convex.

Proof. Let P_1 and P_2 be two polynomials in $\mathcal{H}_N^R(\sigma)$, and let $P = \alpha P_1 + (1 - \alpha)P_2$ for some $\alpha \in [0, 1]$. We want to prove that $G_P(s)$ is admissible.

By definition of $\mathcal{H}_N^R(\sigma)$, there exists two Schur functions f_1 and f_2 such that

$$|f_1(jw)| \leq G_{P_1}(jw), \quad |f_2(jw)| \leq G_{P_2}(jw), \quad w \in \mathbb{R}.$$

Let $f = \alpha f_1 + (1 - \alpha)f_2$. Obviously, f is a Schur function solving $\mathcal{P}(\sigma)$. Moreover, for any $w \in \mathbb{R}$, using (12), we have

$$|f(jw)| \leq \alpha |f_1(jw)| + (1 - \alpha) |f_2(jw)| \leq \alpha G_{P_1}(jw) + (1 - \alpha) G_{P_2}(jw) \leq G_P(jw),$$

so that $P \in \mathcal{H}_N^R(\sigma)$. \square

To clarify the link between the two optimization problems, (6) and (15), we investigate the following question: what can be said about a rational function f in $\mathcal{S}(\sigma)$ satisfying

$$\{\text{Padmissible}\} \quad (17) \quad |f(jw)| \leq G_P(jw), \quad w \in \mathbb{R},$$

for some $P \in \mathcal{H}_N^R(\sigma)$?

If U_P denotes the outer function defined by (8), then $g = f/U_P$ is analytic in Π^+ and satisfies

$$|f(jw)| \leq |U_P(jw)|, \quad w \in \mathbb{R}.$$

For a.e w in \mathbb{R} , those such that $R(jw) \neq 0$ and $P(jw) \neq 0$, $|U_P(jw)| \neq 0$, and $|g(jw)| \leq 1$.

The function $g = f/U_P$ is thus a Schur function and the interpolation conditions that f should satisfy for P to be admissible, carry over g :

$$\{\text{NP4B}\} \quad (18) \quad g(z_k) = v_k(P) := \frac{v_k}{U_P(z_k)}, \quad k = 1, \dots, M.$$

Since $U_P(z_k)$ is well-defined, finite and is not zero (since U_P is outer and z_k belongs to Π^+), the interpolation value $v_k(P)$ is a finite number possibly greater than 1. We thus get a new NP-interpolation problem in which the interpolation values depend on P .

The degree of U_P being at most N , we finally have:

$\{\text{fP}\}$ LEMMA 8. *A positive polynomial P is in $\mathcal{H}_N^R(\sigma)$, if and only if the NP-interpolation problem (18) is solvable.*

If so, let B be the Blaschke product solution of it. Then, the Schur function $f_P = BU_P$, of degree at most $M+N$, solves $\mathcal{P}(\sigma)$ and satisfies

$$|f_P(jw)| = G_P(jw), \quad w \in \mathbb{R}.$$

Remark 9. Let $B = \frac{b^*}{b}$ be the Blaschke product solution to (18), where b is a stable polynomial of degree at most M , then $f_P = \frac{b^*}{b} \frac{\varphi(P)}{\varphi(P+R)}$ belongs to $\mathcal{S}_{N+M}^{bb^*R}(\sigma)$. The constraint on the degree we had in $\mathcal{S}_N^R(\sigma)$ has been relaxed, but we do gain an important property, the convexity $\mathcal{H}_N^R(\sigma)$.

$\{\text{HNRclosed}\}$

3.3. $\mathcal{H}_N^R(\sigma)$ is closed.

$\{\text{HNRproperties}\}$

PROPOSITION 10. *The set $\mathcal{H}_N^R(\sigma)$ is closed.*

Proof. Let P_n be a sequence of polynomials in $\mathcal{H}_N^R(\sigma)$. \mathbb{P}_{2N}^+ being closed, it converges to some $P \in \mathbb{P}_{2N}^+$. Since P_n belongs to $\mathcal{H}_N^R(\sigma)$, there exists by Lemma 8 a Schur function f_n of degree at most $N+M$ such that

$$\{\text{Pnadmissible}\} \quad (19) \quad |f_n(jw)| = G_{P_n}(jw), \quad w \in \mathbb{R}.$$

As a basis of the set \mathbb{P}_{N+M} of polynomials of degree at most $M+N$, we pick the Lagrange interpolation polynomials L_n , $n = 0, 1, \dots, N+M$, associated with $M+N+1$ points on the imaginary axis, say $jw_0, jw_1, \dots, jw_{M+N}$. To each $n = 0, 1, \dots, N+M$, we have that $L_n(jw_k) = \delta_{n,k}$, the Kronecker delta function. The coordinates of $p \in \mathbb{P}_{N+M}$ in this basis are $(p(jw_0), p(jw_1), \dots, p(jw_{M+N}))$. Even if all norms are equivalent, to fix the ideas, we consider the 2-norm

$$\|p\|^2 = \sum_{n=0}^{N+M} |p(jw_n)|^2.$$

Let $f_n = \frac{p_n}{q_n}$, where p_n and q_n are polynomials of degree at most $N+M$, represented by their coordinates in our basis, and q_n is stable in the broad sense.

Since $q_n/\|q_n\|$ is bounded, we may assume, upon taking a subsequence if necessary, that $q_n/\|q_n\|$ converges to $q \in \mathbb{P}_{N+M}$, such that $\|q\| = 1$. We now prove that q has no root in Π^+ . Assume for contradiction, that q has some root, say $\xi \in \Pi^+$, with multiplicity m . This root being necessarily isolated, there exists $\rho > 0$ such that the closed disk $D(\xi, \rho) = \{z/|z - \xi| \leq \rho\}$ is included in Π^+ and its boundary, the circle γ , contain no other root of q . Since $q_n/\|q_n\|$ converges uniformly to q on every compact subset of \mathbb{C} , and $\{|q(z)|, z \in \gamma\}$ is bounded below by a positive number, for n large enough

$$|q_n(z)/\|q_n\| - q(z)| < |q(z)|, \quad z \in \gamma.$$

By Rouché's theorem, $q_n/\|q_n\|$ and q should have the same number of roots in $D(\xi, \rho)$, counting multiplicities, which contradicts the stability of q_n .

Now, f_n being a rational Schur function, for any $w \in \mathbb{R}$, $|p_n(jw)| \leq |q_n(jw)|$, and in particular $|p_n(jw_k)| \leq |q_n(jw_k)|$ so that $\|p_n\| \leq \|q_n\|$. The sequence $p_n/\|q_n\|$ is thus bounded, and upon taking a subsequence if necessary, it converges to some polynomial $p \in \mathbb{P}_{N+M}$ such that, for any $w \in \mathbb{R}$, $|p(jw)| \leq |q(jw)|$.

Clearly, the function $f = \frac{p}{q}$ is Schur and since each f_n satisfies the interpolation conditions (2), f also. Moreover, going to the limit into (19), we get

$$|f(jw)| = G_P(jw), \quad w \in \mathbb{R},$$

which concludes the proof. \square

3.4. Concavity of the Pick constraint. Let P be any positive polynomial in $\mathbb{P}_{2N}^+ \setminus \{0\}$. We denote by $\sigma(P)$ the data set

$$(20) \quad \sigma(P) = \{(z_j, v_j(P)); 1 \leq j \leq M\}, \quad \text{where } v_j(P) = \frac{v_j}{U_P(z_j)}, \quad j = 1, \dots, M.$$

Lemma 8 asserts that $P \in \mathcal{H}_N^R(\sigma)$ if and only if the NP-problem with data set $\sigma(P)$ is solvable, or equivalently, if and only if the Pick matrix

$$(21) \quad \Delta(P) = \left(\frac{1 - v_i(P)\overline{v_j(P)}}{z_i + \bar{z}_j} \right)_{1 \leq i, j \leq M}.$$

is positive semi-definite [Garnett, 14, Theorem 2.2].

The set $\mathcal{H}_N^R(\sigma)$ can thus be alternatively defined by

$$(22) \quad \mathcal{H}_N^R(\sigma) = \{P \in \mathbb{P}_{2N}^+ \setminus \{0\}, \Delta(P) \geq 0\}.$$

Let \mathbb{H}_M be the set of $M \times M$ Hermitian matrices and \mathbb{H}_M^+ , the subset of nonnegative ones. Then,

PROPOSITION 11. *The matrix valued function*

$$\Delta: \begin{array}{l} \mathbb{P}_{2N}^+ \setminus \{0\} \\ P \end{array} \mapsto \begin{array}{l} \mathbb{H}_M \\ \rightarrow \Delta(P) \end{array}$$

maps $\mathcal{H}_N^R(\sigma)$ to \mathbb{H}_M^+ and is strictly concave: for all $\alpha, 0 < \alpha < 1$, and all P_1 and P_2 , distinct polynomials in \mathbb{P}_{2N}^+ :

$$\Delta(\alpha P_1 + (1 - \alpha) P_2) > \alpha \Delta(P_1) + (1 - \alpha) \Delta(P_2).$$

Proof. Let $P = \alpha P_1 + (1 - \alpha) P_2$ and let us prove the positivity of

$$\Lambda = \Delta(P) - \alpha \Delta(P_1) - (1 - \alpha) \Delta(P_2).$$

The entries of Λ are computed as follows:

$$\begin{aligned}\Lambda_{ij} &= \frac{1 - v_i(\mathbf{P}) \overline{v_j(\mathbf{P})}}{z_i + \bar{z}_j} - \alpha \frac{1 - v_i(\mathbf{P}_1) \overline{v_j(\mathbf{P}_1)}}{z_i + \bar{z}_j} - (1 - \alpha) \frac{1 - v_i(\mathbf{P}_2) \overline{v_j(\mathbf{P}_2)}}{z_i + \bar{z}_j} \\ &= \frac{-v_i(\mathbf{P}) \overline{v_j(\mathbf{P})} + \alpha v_i(\mathbf{P}_1) \overline{v_j(\mathbf{P}_1)} + (1 - \alpha) v_i(\mathbf{P}_2) \overline{v_j(\mathbf{P}_2)}}{z_i + \bar{z}_j} \\ &= v_i(\mathbf{P}) \frac{\alpha \frac{U_P(z_i)}{U_{P_1}(z_i)} \frac{\overline{U_P(z_j)}}{\overline{U_{P_1}(z_j)}} + (1 - \alpha) \frac{U_P(z_i)}{U_{P_2}(z_i)} \frac{\overline{U_P(z_j)}}{\overline{U_{P_2}(z_j)}} - 1}{z_i + \bar{z}_j} \bar{v}_j(\mathbf{P}).\end{aligned}$$

Let

$$(23) \quad u_i = \left[\sqrt{\alpha} \frac{U_P}{U_{P_1}}(z_i) \quad \sqrt{1 - \alpha} \frac{U_P}{U_{P_2}}(z_i) \right], \quad i = 1, \dots, M.$$

and let D be the diagonal matrix $D = \text{diag}(v_1(\mathbf{P}), v_2(\mathbf{P}), \dots, v_M(\mathbf{P}))$.

We have $\Lambda = D\Pi D^*$, where

$$\Pi = \left(\frac{u_i u_j^* - 1}{z_i + \bar{z}_j} \right)_{1 \leq i, j \leq M}.$$

This matrix is a Pick matrix, but associated with a *tangential interpolation problem* (see [2, Example 18.5.1], [13, Chap. 5]). This NP-problem consists in finding a (2×1) -matrix function $F(s)$ satisfying the interpolation conditions:

$$(24) \quad u_i F(z_i) = 1, \quad i = 1, \dots, M.$$

The rational matrix-valued function

$$(25) \quad F(s) = \begin{bmatrix} \sqrt{\alpha} \frac{U_{P_1}}{U_P}(s) \\ \sqrt{1 - \alpha} \frac{U_{P_2}}{U_P}(s) \end{bmatrix}$$

clearly satisfies the tangential interpolation conditions (24). From (13), $F(s)$ is a strictly contractive vector-valued function a.e. on the imaginary axis. Being in addition analytic in the open right half-plane, it is strictly Schur. Thus, the matrix Π , has to be positive definite (see e.g. [2, Theorem 18.5.3]). Finally, Λ has to be positive too, which concludes the proof. \square

study

4. Study of the quasiconvex optimization problem. Consider the image of $\mathcal{H}_N^R(\sigma)$ under the action of the objective function Ψ defined in (16)

$$\Psi(\mathcal{H}_N^R(\sigma)) = \{\Psi(P), P \in \mathcal{H}_N^R(\sigma)\}.$$

Since $\mathcal{H}_N^R(\sigma)$ is nonempty, $\Psi(\mathcal{H}_N^R(\sigma))$ is a nonempty subset of \mathbb{R}^+ . Thus it has an infimum

$$(26) \quad \hat{L} = \inf_{P \in \mathcal{H}_N^R(\sigma)} \Psi(P).$$

solvability

PROPOSITION 12. *Problem (15) is solvable. Moreover, we have*

$$\forall f \in \mathcal{S}_N^R(\sigma), \quad \|f\|_{\infty, \mathbb{I}} \geq \sqrt{\frac{\hat{L}}{\hat{L} + 1}},$$

so that $\sqrt{\frac{\hat{L}}{\hat{L} + 1}}$ is a lower bound for the optimal value of the initial optimization problem (6).

Proof. Since $\Psi(\mathcal{H}_N^R(\sigma))$ has an infimum \hat{L} , there exists a sequence of polynomials (P_n) , $P_n \in \mathcal{H}_N^R(\sigma)$ such that $\Psi(P_n) \leq \hat{L} + 1/n$, $n \in \mathbb{N}^*$, so that the sequence (P_n) is bounded with respect to the uniform norm over \mathbb{I} ,

$$\|P_n\|_{\infty, \mathbb{I}} \leq (\hat{L} + 1)\|R\|_{\infty, \mathbb{I}}.$$

Thus it is bounded with respect to any norm, since they are all equivalent on a finite dimensional vector space. Since $\mathcal{H}_N^R(\sigma)$ is closed, we may thus extract from (P_n) a subsequence converging to some $\hat{P} \in \mathcal{H}_N^R(\sigma)$. The infimum of Ψ over $\mathcal{H}_N^R(\sigma)$ is achieved and thus the problem is solvable.

We have seen that if $f = \frac{p}{q} \in \mathcal{S}_N^R(\sigma)$, then $P_f = pp^* \in \mathcal{H}_N^R(\sigma)$. Thus, we have

$$\|f\|_{\infty, \mathbb{I}} = \sqrt{\frac{P_f}{P_f + R}} \geq \min_{P \in \mathcal{H}_N^R(\sigma)} \sqrt{\frac{P}{P + R}} = \sqrt{\frac{\hat{L}}{\hat{L} + 1}}. \quad \square$$

PROPOSITION 13. *At an optimal point \hat{P} , the Pick matrix $\Delta(\hat{P})$ is singular.*

Proof. Let \hat{P} in $\mathcal{H}_N^R(\sigma)$ be an optimum of Ψ and assume that the Pick matrix is positive definite: $\Delta(\hat{P}) > 0$.

First observe that, for $k = 1, \dots, L$, the map $\alpha \mapsto U_{\alpha \hat{P}}(z_k)$, $\alpha \in \mathbb{R}^*$ is continuous. This follows from the continuity of the map φ defined in (9), and from the fact that the denominator of $U_{\alpha \hat{P}}$ is stable broad sense and thus do not vanish at z_k .

The map $\alpha \in \mathbb{R}^* \mapsto P \mapsto \Delta(\alpha \hat{P})$ is also continuous. For α , $0 < \alpha < 1$, close enough to 1 the matrix $\Delta(\alpha \hat{P})$ is then positive, while $\Psi(\alpha \hat{P}) < \Psi(\hat{P})$, which contradicts the minimality of \hat{P} . Thus $\Delta(\hat{P})$ has to be singular. \square

COROLLARY 14. *Let \hat{P} be an optimal point and let \hat{B} be the unique Blaschke product solving the NP-problem with data set $\sigma(\hat{P})$. Then, $f_{\hat{P}} = \hat{B}U_{\hat{P}}$ has degree*

$$\deg f_{\hat{P}} = N + \text{rank } \Delta(\hat{P}) \leq N + M - 1,$$

and solves (6)

PROPOSITION 15. *Problem (15) has a unique optimal point.*

Proof. Assume that (15) has two distinct optimal points P_1 and P_2 and let $P = \alpha P_1 + (1 - \alpha)P_2$, $0 < \alpha < 1$. Since P_1 and P_2 belong to $\mathcal{H}_N^R(\sigma)$, there exists B_1 and B_2 , two Blaschke products such that for $i = 1, 2$, the Schur function $f_i = B_i U_{P_i}$ satisfies the interpolation conditions (2). The function $f = \alpha f_1 + (1 - \alpha)f_2$ also satisfies the interpolation conditions (2). From (12), we have at any $w \in \mathbb{R}$ such that $R(jw) \neq 0$, and thus a.e.

$$|f(jw)| \leq \alpha |U_{P_1}(jw)| + (1 - \alpha) |U_{P_2}(jw)| < |U_P(jw)|.$$

The quotient map f/U_P is thus strictly Schur and satisfies the interpolation conditions (18), so that $\Delta(P) > 0$. But this is a contradiction with Proposition 13 because P is an optimal point for (15). Indeed,

$$P(jw) = \alpha P_1(jw) + (1 - \alpha)P_2(jw), \quad w \in \mathbb{R},$$

so that $\Psi(P) \leq \hat{L}$, and thus $\Psi(P) = \hat{L}$. \square

5. Generalization to a mixed NP-interpolation problem. We now consider a more general NP-interpolation problem, in which interpolation nodes may be located on the boundary of the analyticity domain, namely on the imaginary axis or at ∞ . This case is particularly relevant when filter design applications are under consideration, as we'll see in section 7.

MIP

5.1. Mixed interpolation problem. A rational function f , analytic on $\overline{\Pi}^+$, admits an *angular derivative* at z_0 on the imaginary axis, noted $f'(z_0)$, which is the nontangential limit at z_0 of f' , or equivalently, the nontangential limit of $\frac{f(s)-f(z_0)}{s-z_0}$ at z_0 . If f is a Schur function, and $f(z_0)$ has modulus one, the quantities

$$(27) \quad \rho_{z_0}(f) = -\frac{f'(z_0)}{f(z_0)}, \quad \text{if } z_0 = jw_0,$$

$$(28) \quad \rho_{z_0}(f) = \lim_{s \rightarrow \infty} \frac{s^2 f'(s)}{f(s)}, \quad \text{if } z_0 = \infty,$$

are positive numbers (see [2, Theorem 21.1.4]). The limit $s \rightarrow \infty$ has to be understood as a nontangential limit.

In this section, we consider a mixed type of interpolation problem, combining interior and boundary interpolation nodes.

Mixed NP-interpolation problem : Given

- distinct interpolation points $z_k \in j\mathbb{R} \cup \infty$, together with interpolation values v_k , $|v_k| = 1$ and positive numbers ρ_k , for $k = 1, \dots, L$,
- distinct interpolation points $z_k \in \Pi^+$, together with interpolation values v_k , $|v_k| < 1$, for $k = L+1, \dots, M$,

find a Schur function f such that

$$(29) \quad f(z_k) = v_k, \quad \text{and} \quad \rho_{z_k}(f) \leq \rho_k, \quad k = 1, \dots, L \quad (z_k \in j\mathbb{R} \cup \infty)$$

and

$$(30) \quad f(z_k) = v_k, \quad k = L+1, \dots, M \quad (z_k \in \Pi^+).$$

We denote by $\tilde{\sigma}$ the data set for this problem, called $\mathcal{P}(\tilde{\sigma})$. It is the union of a data set σ for the classical NP-problem and a data set σ_{∂} for the boundary NP-problem:

$$(31) \quad \sigma_{\partial} = \{(z_k, v_k, \rho_k); 1 \leq k \leq L\}, \quad \sigma = \{(z_k, v_k); L+1 \leq k \leq M\}.$$

A Pick matrix $\Lambda(\tilde{\sigma})$ is associated with this mixed problem, as follows:

$$(32) \quad \Lambda(\tilde{\sigma})_{i,i} = \begin{cases} \rho_i, & i = 1, \dots, L \\ \frac{1 - |v_i|^2}{z_i + \bar{z}_i}, & i = L+1, \dots, M \end{cases}$$

and for $i \neq j$,

$$(33) \quad \Lambda(\tilde{\sigma})_{i,j} = \begin{cases} \frac{1 - v_i \bar{v}_j}{z_i + \bar{z}_j}, & z_i, z_j \neq \infty \\ 1 - v_i \bar{v}_j, & z_i = \infty \text{ or } z_j = \infty \end{cases}.$$

Note that, except for the first L diagonal entries, the entries corresponding to boundary interpolation nodes are the limits of those in (3) when the interpolation points tend nontangentially to the boundary; for $z_i = \infty$, we consider an interpolation point of the form $1/\delta_i$, $\delta_i \in \Pi^+$, and multiply the i th row and the i th column of (3) by $1/\delta_i$ and $1/\bar{\delta}_i$ respectively, before we let δ_i tend nontangentially to zero. $\Lambda(\tilde{\sigma})$ can also be obtained from its analog for the unit disk (see Appendix A).

The boundary NP-interpolation problem, in which all the interpolation nodes are on the boundary of the analyticity domain (case $L = M$) has been widely studied in the literature. Sarason interpolation

problem [22, Theorem 1] provides a criterion for this problem (he called $\partial NPS'$), in the context of Schur function on the unit disk. Sarason's paper shows that $(\partial NPS')$ is very close to the classical Nevanlinna-Pick problem in that its solutions have the same basic structure.

The statement that we need, which is a refinement of Sarason boundary interpolation theorem, was stated in [1, Th.3.3.], but still in the framework of the unit disk. However, it is well-known that the two problems (for the unit disk and for the right half-plane), can be shown equivalent using a Moebius transformation (since the proof is hard to find in the literature, we give it in Appendix A).

THEOREM 16. *The problem $\mathcal{P}(\tilde{\sigma})$ has a solution if and only if the associated Pick matrix $\Lambda(\tilde{\sigma})$ is positive semi-definite.*

1. If $\Lambda(\tilde{\sigma}) > 0$, the problem is indeterminate: it has infinitely many solutions,
2. If $\Lambda(\tilde{\sigma})$ is singular, and has rank $k < M$, the problem has a unique solution which is a Blaschke product of degree **at most** k .

For the proof, [1] refers to [4, Theorem 2.5] and to [10, Theorem 5.2] for a solution to the analogous interpolation problem for the upper half-plane. In the indeterminate case, the set of Schur solutions can be parametrized using linear fractional representations. Such a parametrization of the set of solutions is given in [2, Th 21.1.2] for the unit disk and [2, Th 21.1.5] for the half-plane, and is extended in [2, Th 21.4.1] to a mixed problem for matrix-valued functions. The degenerate case was also studied in [5, Theorem 2.2 (2)], but only for the boundary case. Bolotnikov's paper can be used to provide a formula for the unique Blaschke product solution in the degenerate boundary case (see Appendix B). It should be noted that despite the similarity of Theorem 1 and Theorem 16, there is a notable difference when nodes on the boundary are allowed: the degree of the Blaschke product solution in the determinate case can be *strictly less* than the rank of the Pick matrix. Appendix B provides some details on this point.

5.2. Schur optimization problem with boundary Nevanlinna-Pick constraints. We consider in this section the following **Schur optimization problem with mixed NP-constraints**: given

- $\tilde{\sigma}$ a data set of a mixed NP-problem as in (31) such that

$$\{Pickpositive\} \quad (34) \quad \Lambda(\tilde{\sigma}) > 0$$

- $R \in \mathbb{P}_{2N}^+$ a nonnegative polynomial such that $R(z_k) = 0$, for $k = 1, \dots, L$ and with no roots in \mathbb{I} , solve the minimization problem

$$\{GOPmixed\} \quad (35) \quad \min_{f \in \mathcal{S}_N^R(\tilde{\sigma})} \|f\|_{\infty, \mathbb{I}},$$

where $\mathcal{S}_N^R(\tilde{\sigma})$ denotes the set of rational Schur functions f of degree at most N , solving $\mathcal{P}(\tilde{\sigma})$, and having a polynomial fraction representation of the form $f = \frac{p}{q}$ with $qq^* - pp^* = R$.

Note that the boundary interpolation points cannot belong to \mathbb{I} .

PROPOSITION 17. $\mathcal{S}_N^R(\tilde{\sigma})$ is nonempty.

Proof. The matrix $\Lambda(\tilde{\sigma})$ is block-partitioned as

$$\Lambda(\tilde{\sigma}) = \begin{bmatrix} \Lambda(\sigma_{\partial}) & \Lambda_{12} \\ \Lambda_{21} & \Lambda(\sigma) \end{bmatrix},$$

where σ_{∂} is the data set associated with the boundary NP-problem and σ the data set associated with the classical NP-problem as in (31). We first assume that no interpolation point lies at ∞ .

Since $\Lambda(\tilde{\sigma}) > 0$, we also have $\Lambda(\sigma_{\partial}) > 0$ and the problem $\mathcal{P}(\sigma_{\partial})$ has infinitely many solutions. In particular, all the solutions for which equality holds in (29) are parametrized by a linear fractional transformation (see [2, Th. 21.1.5])

$$\{FTsolutions\} \quad (36) \quad f = (\Theta_{11}g + \Theta_{12})(\Theta_{21}g + \Theta_{22})^{-1}$$

where g is a Schur function for which $\Theta_{21}g + \Theta_{22}$ has *simple* poles at the points z_1, z_2, \dots, z_L . The J -inner matrix function

$$\Theta(s) = (\Theta_{ij}(s))_{1 \leq i, j \leq 2}$$

is built from the interpolation data. It is of the form

$$\Theta(s) = \frac{1}{r_0^*} \begin{bmatrix} \varepsilon q_0^* & p_0 \\ \varepsilon p_0^* & q_0 \end{bmatrix}$$

where $r_0(s) = \prod_{j=1}^L (s - z_j)$, ε is a unit complex number, p_0 and q_0 are polynomials of degree L , possibly not coprime, such that

$$q_0 q_0^* - p_0 p_0^* = r_0 r_0^*.$$

The polynomial $R_0 = r_0 r_0^*$ divides R , since by assumption $R(z_k) = 0$ for $k = 1, \dots, L$, and we call R_1 the quotient. For f to belong to $\mathcal{S}_N^R(\tilde{\sigma})$, g needs to satisfy of the interpolation conditions

$$(37) \quad g(z_k) = (v_k \Theta_{21}(z_k) - \Theta_{11}(z_k))^{-1} (\Theta_{12}(z_k) - v_k \Theta_{22}(z_k)) = w_k, \quad k = L+1, \dots, M.$$

Let $\sigma_1 = \{(z_k, w_k), k = L+1, \dots, M\}$ be the data set associated with these interpolation conditions.

We first prove that the Pick matrix $\Lambda(\sigma_1)$ given by

$$\Lambda(\sigma_1) = \left[\frac{1 - w_i \bar{w}_j}{z_i + \bar{z}_j} \right]_{L+1 \leq i, j \leq M}.$$

is positive definite.

For $i = L+1, \dots, M$, putting $x_i = v_i \Theta_{21}(z_i) - \Theta_{11}(z_i)$, we get from (37)

$$x_i (1 - w_i \bar{w}_j) \bar{x}_j = \begin{bmatrix} 1 & -v_i \end{bmatrix} \Theta(z_i) J \Theta(z_j)^* \begin{bmatrix} 1 \\ -\bar{v}_j \end{bmatrix}.$$

The matrix-valued function $\Theta(z)$ in (36) can be expressed in terms of the interpolation data (see [2, Th. 21.1.5]) as

$$\Theta(s) = I - C(sI + A^*)^{-1} \Lambda(\sigma_\partial)^{-1} C^* J, \quad C = \begin{bmatrix} 1 & 1 \dots & 1 \\ \bar{v}_1 & \bar{v}_2 \dots & \bar{v}_L \end{bmatrix}, \quad A = \text{diag}(z_1, z_2, \dots, z_L).$$

and being J -unitary, satisfies ([2, Formula (6.1.8)])

$$\Theta(z_i) J \Theta(z_j)^* = J - (z_i + \bar{z}_j) C(z_i I + A^*)^{-1} \Lambda(\sigma_\partial)^{-1} (\bar{z}_j I + A)^{-1} C^*,$$

so that

$$(38) \quad x_i \frac{1 - w_i \bar{w}_j}{z_i + \bar{z}_j} \bar{x}_j = \Lambda(\sigma)_{ij} - \begin{bmatrix} 1 & -v_i \end{bmatrix} C(z_i I + A^*)^{-1} \Lambda(\sigma_\partial)^{-1} (\bar{z}_j I + A)^{-1} C^* \begin{bmatrix} 1 \\ -\bar{v}_j \end{bmatrix}.$$

It is easily checked that if c_j , for $j = L+1, \dots, M$, denotes the j th column of Λ_{12} , then

$$c_j = \begin{bmatrix} 1 & -v_{L+j} \end{bmatrix} C(z_{L+j} I + A^*)^{-1},$$

so that, if $X = \text{diag}(x_{L+1}, x_{L+2}, \dots, x_M)$, then from (38)

$$X \Lambda(\sigma_1) X^* = \Lambda(\sigma_\partial) - \Lambda_{21} \Lambda(\sigma)^{-1} \Lambda_{12}.$$

Thus, $X \Lambda(\sigma_1) X^*$ is the Schur complement of $\Lambda(\sigma)$ and has to be definite positive, as well as $\Lambda(\sigma_1)$. This result is well-known, at least for the classical NP-problem.

Then, from Proposition 2, there exists $f_1 = \frac{p_1}{q_1}$ solution to $\mathcal{P}(\sigma_1)$ and such that

$$q_1 q_1^* - p_1 p_1^* = R_1.$$

Let

$$f = \frac{\varepsilon q_0^* p_1 + p_0 q_1}{\varepsilon p_0^* p_1 + q_0 q_1} := \frac{p}{q}$$

be the solution of $\mathcal{P}(\tilde{\sigma})$ obtained from (36) with $g = f_1$. It clearly satisfies $qq^* - pp^* = R$ and has degree less than N , and thus belongs to $\mathcal{S}_N^R(\tilde{\sigma})$.

If one of the interpolation points lie at ∞ , we can use a Moebius transform to bring the problem back to the previous case. If none of the interpolation points is 0, we can take $s \mapsto 1/s$. Otherwise, all the Moebius transform preserving the right half-plane being of the form

$$\mu(s) = \frac{as - jb}{jcs + d}, \quad a, b, c, d \in \mathbb{R}, \quad \text{and } ad - bc > 0,$$

we can chose $c = 1$, to avoid sending ∞ to ∞ , and d such that jd is different from any of the finite interpolation points, (otherwise, ∞ would once again be an interpolation point). \square

5.3. Convex relaxation of the problem.

We shall proceed as in Section 3.

HNRtdef

DEFINITION 18. The set $\mathcal{H}_N^R(\tilde{\sigma})$ is defined as the set of polynomials P in \mathbb{P}_{2N}^+ such that the gain function G_P defined by (7) is admissible, i.e. there exists a rational Schur function $f(s)$ solving $\mathcal{P}(\tilde{\sigma})$ and satisfying (17):

$$|f(jw)| \leq G_P(jw), \quad w \in \mathbb{R}.$$

At the boundary interpolation points z_k , $k = 1, \dots, L$, the function f having modulus one, since it solves $\mathcal{P}(\tilde{\sigma})$, $G_P(z_k)$ has modulus one too. Thus, the zero polynomial does not belong to $\mathcal{H}_N^R(\tilde{\sigma})$, and in view of Remark 3 any $P \in \mathcal{H}_N^R(\tilde{\sigma})$ should satisfy $m_P(z_k) < m_R(z_k)$, for $k = 1, \dots, L$.

Any $P \in \mathcal{H}_N^R(\tilde{\sigma})$ thus belongs to the set

{setPsigmat}

$$(39) \quad \mathbb{P}_{2N}^+(\tilde{\sigma}) = \{P \in \mathbb{P}_{2N}^+ \setminus \{0\}, m_P(z_k) < m_R(z_k), k = 1, \dots, L\}.$$

The **polynomial optimization problem** that we now consider is

nvexProblem}

$$(40) \quad \min_{P \in \mathcal{H}_N^R(\tilde{\sigma})} \Psi(P),$$

where Ψ is still defined by (16). As previously, the set $\mathcal{H}_N^R(\tilde{\sigma})$ is convex. The proof is similar to that of Proposition 7.

Now, let $P \in \mathcal{H}_N^R(\tilde{\sigma})$ and let $f \in \mathcal{S}(\tilde{\sigma})$ satisfy (17). The function $g = f/U_P$ is again Schur, and since f solves $\mathcal{P}(\tilde{\sigma})$, g satisfies the interpolation conditions:

{IntCond4B}

$$(41) \quad g(z_k) = v_k(P) := \frac{v_k}{U_P(z_k)}, \quad k = 1, \dots, M$$

and

Condition4B}

$$(42) \quad 0 \leq \rho_{z_k}(g) \leq \rho_k(P) := \rho_k - \rho_{z_k}(U_P), \quad k = 1, \dots, L,$$

Remark 19. Note that for $k = 1, \dots, L$, $|v_k(P)| = 1$ and condition (42) follows from the relation $\rho_{z_k}(f) = \rho_{z_k}(g) + \rho_{z_k}(U_P)$.

To any $P \in \mathbb{P}_{2N}^+(\tilde{\sigma})$, we associate the data set

$$\{\text{dataSetPt}\} \quad (43) \quad \tilde{\sigma}(P) = \{(z_k, v_k(P), \rho_k(P)); 1 \leq k \leq L\} \cup \{(z_k, v_k(P)); L+1 \leq k \leq M\}.$$

The fact that $\mathcal{H}_N^R(\tilde{\sigma})$ is included in $\mathbb{P}_{2N}^+(\tilde{\sigma})$, implies that the data set $\tilde{\sigma}(P)$ has exactly the same structure that $\tilde{\sigma}$: same number of interpolation conditions on the boundary, as well as in Π^+ .

The Pick matrix associated with $\tilde{\sigma}(P)$, $\tilde{\Delta}(P)$, is given by (32) and (33), namely

$$\{\text{PickPbDiag}\} \quad (44) \quad \tilde{\Delta}(P)_{i,i} = \begin{cases} \rho_i(P), & i = 1, \dots, L \\ \frac{1 - |v_i(P)|^2}{z_i + \bar{z}_i}, & i = L+1, \dots, M \end{cases}$$

and for $i \neq j$,

$$\{\text{PickPbExtraDiag}\} \quad (45) \quad \tilde{\Delta}(P)_{i,j} = \begin{cases} \frac{1 - v_i(P)\overline{v_j(P)}}{z_i + \bar{z}_j}, & z_i, z_j \neq \infty \\ 1 - v_i(P)v_j(P), & z_i = \infty \text{ or } z_j = \infty \end{cases}.$$

$\{\text{fPt}\}$ LEMMA 20. A positive polynomial P is in $\mathcal{H}_N^R(\tilde{\sigma})$, if and only if $P \in \mathbb{P}_{2N}^+(\tilde{\sigma})$ and the interpolation problem with data set $\tilde{\sigma}(P)$ is solvable ($\tilde{\Delta}(P) \geq 0$).

Let B be the Blaschke product solution of it. Then the Schur function $f_P = BU_P$, of degree at most $M+N$, solves $\mathcal{P}(\tilde{\sigma})$ and satisfies

$$|f_P(jw)| = G_P(jw), \quad w \in \mathbb{R}.$$

We can prove as in Section 3.3 that the set $\mathcal{H}_N^R(\tilde{\sigma})$ is closed. It can be defined alternatively by

$$\mathcal{H}_N^R(\tilde{\sigma}) = \{P \in \mathbb{P}_{2N}^+(\tilde{\sigma}), \tilde{\Delta}(P) \geq 0\}.$$

$\{\text{PicktConvex}\}$

PROPOSITION 21. The matrix valued function

$$\begin{aligned} \tilde{\Delta} : \mathbb{P}_{2N}^+(\tilde{\sigma}) &\mapsto \mathbb{H}_M \\ P &\rightarrow \tilde{\Delta}(P) \end{aligned}$$

maps $\mathcal{H}_N^R(\tilde{\sigma})$ to \mathbb{H}_M^+ and is strictly concave: for any α , $0 < \alpha < 1$, and P_1 and P_2 two distinct polynomials in \mathbb{P}_{2N}^+ :

$$\tilde{\Delta}(\alpha P_1 + (1-\alpha)P_2) > \alpha \tilde{\Delta}(P_1) + (1-\alpha)\tilde{\Delta}(P_2).$$

Proof. The proof of Proposition 11 can be easily adapted. For $i = 1, \dots, L$, the diagonal entries of $\Lambda = \tilde{\Delta}(P) - \alpha \tilde{\Delta}(P_1) - (1-\alpha)\tilde{\Delta}(P_2)$ are now given by

$$\Lambda_{ii} = \rho_i(P) - \alpha \rho_i(P_1) - (1-\alpha)\rho_i(P_2),$$

that is, using (42) and the fact that $|v_i(P)| = 1$:

$$\Lambda_{ii} = v_i(P) (\alpha \rho_{z_i}(U_{P_1}) + (1-\alpha)\rho_{z_i}(U_{P_2}) - \rho_{z_i}(U_P)) \overline{v_i(P)}.$$

For $i, j = 1, \dots, M$, $i \neq j$,

$$\Lambda_{ij} = v_i(P) \frac{u_i u_j^* - 1}{z_i + \bar{z}_j} \overline{v_j(P)},$$

except if $z_i = \infty$, in which case

$$\Lambda_{ij} = v_i(P) (u_i u_j^* - 1) \overline{v_j(P)}, \quad \text{and} \quad \Lambda_{ji} = v_j(P) (u_j u_i^* - 1) \overline{v_i(P)},$$

where u_i is still given by (23).

Again, the matrix Λ can be written $\Lambda = D^* \Pi D$, where D is the diagonal matrix

$$D = \text{diag}(v_1(P), \dots, v_M(P))$$

and Π has the form of a Pick matrix associated with a tangential boundary interpolation problem [2, Chap. 21]. Namely, the problem is to find a matrix-valued Schur function $F(s)$ satisfying the interpolation conditions: for $i = 1, \dots, L$ (at the boundary points)

$$\text{TICon} \quad (46) \quad u_i F(z_i) = 1$$

$$\text{TICAD} \quad (47) \quad -u_i F'(z_i) = \alpha \rho_{z_i}(U_{P_1}) + (1 - \alpha) \rho_{z_i}(U_{P_2}) - \rho_{z_i}(U_P)$$

and for $i = L + 1, \dots, M$

$$\{\text{TICin}\} \quad (48) \quad u_i F(z_i) = 1.$$

For $i = 1, \dots, L$, observe that since P_1, P_2 and P belong to $\mathbb{P}_{2N}^+(\tilde{\sigma})$, we have $|U_{P_1}(z_i)| = |U_{P_2}(z_i)| = |U_P(z_i)| = 1$, and thus the vector u_i has norm 1.

The vector valued function $F(s)$ defined by (25) is still a strictly Schur function and we shall see that it solves this interpolation problem. Indeed, it clearly satisfies (46) and (48). Moreover, for $i = 1, \dots, L$, we have

$$\text{TICin} \quad (49) \quad -u_i F'(s) = -\alpha \frac{U_P(z_i)}{U_{P_1}(z_i)} \frac{U_{P_1}'(s) U_P(s) - U_{P_1}(s) U_P'(s)}{U_P(s)^2}$$

$$(50) \quad -(1 - \alpha) \frac{U_P(z_i)}{U_{P_2}(z_i)} \frac{U_{P_2}'(s) U_P(s) - U_{P_2}(s) U_P'(s)}{U_P(s)^2},$$

and as s goes to z_i , the non-tangential limit of this expression is equal to the second member of (47).

From Theorem 16, the nontangential interpolation problem having a strictly Schur solution, the matrix Λ has to be positive definite. \square

5.4. Study of the quasiconvex optimization problem. Thanks to Theorem 16, the results of Section 3 are still valid in the context of a NP-problem including boundary angular derivative conditions.

PROPOSITION 22. *Problem (40) is solvable and its infimum \hat{L} provides a lower bound for the optimization problem (35) over $\mathcal{S}_N^R(\tilde{\sigma})$, namely $\sqrt{\frac{\hat{L}}{\hat{L}+1}}$. Moreover,*

- at an optimal point, \hat{P} , the associated Pick matrix $\tilde{\Delta}(\hat{P})$ is singular, and the function $f_{\hat{P}}$ defined in Lemma 20 is a solution of (35) of degree at most $N + M - 1$;
- problem (40) has a unique optimal point.

Proof. The proof of the first point is similar to that of Proposition 12.

For the second point, let \hat{P} in $\mathcal{H}_N^R(\sigma)$ be an optimum of Ψ and assume that the Pick matrix is positive definite: $\tilde{\Delta}(\hat{P}) > 0$.

We first prove the continuity of the map $\alpha \mapsto \tilde{\Delta}(\alpha \hat{P})$, for $\alpha \in \mathbb{R}^*$. For $k = L + 1, \dots, M$, the evaluation map $\alpha \in \mathbb{R}^* \mapsto U_{\alpha \hat{P}}(z_k)$ is continuous as previously. For $k = 1, \dots, L$, the point z_k lies on the real line and since $\hat{P} \in \mathbb{P}_{2N}^+(\tilde{\sigma})$, $m_{\hat{P}}(z_k) < m_R(z_k)$. Then, the map $\alpha \mapsto U_{\alpha \hat{P}}(z_k)$ is continuous since

$$U_{\alpha \hat{P}} = \frac{\varphi(\alpha \hat{P})}{\varphi(\alpha \hat{P} + R)},$$

and after simplification by the factor $(z - z_k)^{m_{\hat{P}}(z_k)}$ eventually, the denominator of $U_{\alpha \hat{P}}$ does not vanish at z_k . Likewise, the map $\alpha \in \mathbb{R}^* \mapsto \rho_{z_k}(\alpha \hat{P})$ is continuous.

For α , $0 < \alpha < 1$, close enough to 1 the matrix $\tilde{\Delta}(\alpha\hat{P})$ is then positive, while $\Psi(\alpha\hat{P}) < \Psi(\hat{P})$, which contradicts the minimality of \hat{P} . Thus $\Delta(\hat{P})$ has to be singular.

Finally, concerning the last point, assume that (40) has two distinct optimal points P_1 and P_2 and let $P = \alpha P_1 + (1 - \alpha)P_2$, $0 < \alpha < 1$. Note that P belongs to $\mathbb{P}_{2N}^+(\tilde{\sigma})$.

Since P_1 and P_2 belong to $\mathcal{H}_N^R(\tilde{\sigma})$, there exists B_1 and B_2 , two Blaschke products such that for $i = 1, 2$, the Schur function $f_i = B_i U_{P_i}$ satisfies the interpolation conditions (30) and (29). The function $f = \alpha f_1 + (1 - \alpha)f_2$ clearly satisfies (30). Moreover, for $k = 1, \dots, L$, and $z_k \in j\mathbb{R}$, or $z_k = \infty$,

$$\rho_{z_k}(\alpha f_1 + (1 - \alpha)f_2) = \frac{\alpha f_1'(z_k) + (1 - \alpha)f_2'(z_k)}{\alpha f_1(z_k) + (1 - \alpha)f_2(z_k)},$$

and since $f_1(z_k) = f_2(z_k) = v_k$,

$$\rho_{z_k}(\alpha f_1 + (1 - \alpha)f_2) = \alpha \rho_{z_k}(f_1) + (1 - \alpha)\rho_{z_k}(f_2) \leq \rho_k,$$

so that f also satisfies the interpolation conditions (29).

From (12), at w such that $R(jw) \neq 0$,

$$|f(jw)| \leq \alpha |U_{P_1}(jw)| + (1 - \alpha) |U_{P_2}(jw)| < |U_P(jw)|.$$

Inequality holds since P_1 and P_2 are distincts. The quotient map f/U_P is thus strictly Schur and satisfies the interpolation conditions (41) and (42), so that $\tilde{\Delta}(P) > 0$. But this is a contradiction with the second point, because P is an optimal point for (40). Indeed,

$$P(jw) = \alpha P_1(jw) + (1 - \alpha)P_2(jw), \quad w \in \mathbb{R},$$

so that $\Psi(P) \leq \hat{L}$, and thus $\Psi(P) = \hat{L}$. □

6. Numerical implementation. The polynomial optimization problems stated at (15) and (40) are semi-definite programming problems (SDP for short) and existing nonlinear semidefinite programming techniques can be used to solve them. Polynomials in \mathbb{P}_{2N}^+ can be characterized by means of $(N + 1) \times (N + 1)$ positive semi-definite matrices ([12, Theorem 2.5]). Positivity and admissibility constraints are implemented using the standard logarithmic barrier and a special penalty function [23, Section 8.2].

An implementation was carried out by D. Martinez-Martinez, as part of his thesis [20, Chapter 7 and 8]. This work gave rise to a toolbox PUMA [19] dedicated to the broadband matching problem. In what follows, we will illustrate our method in this context.

7. Application to impedance matching.

7.1. The impedance matching problem. A crucial task in RF applications is to design a matching network as in Figure 1, in order to maximize the power transmitted to a load, whose impedance Z_L is known, over a wide frequency band Ω .

The matching problem is classically stated using a scattering description [17, 24]. In this setting, the *reflection coefficient* of the load is modeled by a rational Schur function L_{11} .

The matching network to be designed is described by a 2×2 lossless matrix, the scattering matrix,

$$(51) \quad F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}.$$

Its entries are Schur functions and $F(j\omega)$ is unitary at each frequency $\omega \in \mathbb{R}$.

The reflection coefficient of the overall system is given by the chaining operation

$$S_{11} = F_{11} + \frac{F_{12}F_{21}L_{11}}{1 - F_{22}L_{11}},$$

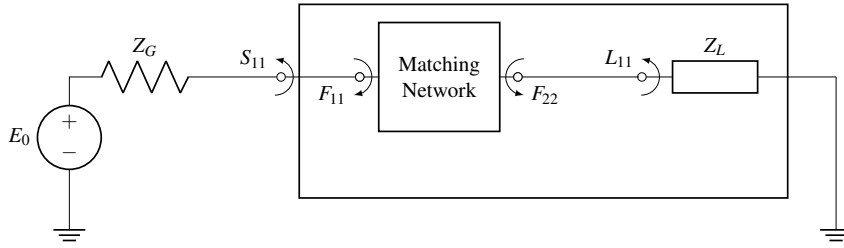


FIG. 1. Matching network connected to a one-port load

fig:System

and the matching problem amounts to minimize its modulus $|S_{11}(j\omega)|$ over the frequency band Ω . The problem is then to understand the constraints, that is the limitations imposed on S_{11} due to the presence of the load.

These theoretical limitations take the form of interpolation conditions, and the best way to derive them, is to use the powerful Darlington's theory [11]. This theory asserts that any reflection coefficient (here L_{11}) can be viewed as the reflection coefficient at port 1 of a lossless network (Darlington's equivalent model) closed at port 2 on an arbitrary resistance. Mathematically speaking, Darlington's model is represented by a 2×2 lossless minimal extension of L_{11} . It has the following polynomial structure, called Belevitch form:

$$(52) \quad L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} \frac{p}{q} & -\varepsilon \frac{r^*}{q} \\ \frac{r}{q} & \varepsilon \frac{p^*}{q} \end{bmatrix}, \quad qq^* - pp^* = rr^*.$$

Note that the existence of such an extension is guaranteed by the Schur character of L_{11} and the fact it is *rational*. A minimal degree extension is not unique. The freedom is in the choice of

- the distribution between L_{12} and L_{21} of the *spectral zeros* of L_{11} (the zeros of $1 - L_{11}L_{11}^*$ in the closed right half-plane or at ∞),
- a unitary constant factor ε , common to L_{12} and L_{22} .

Note that L_{22} is uniquely determined from L_{11} up to this factor.

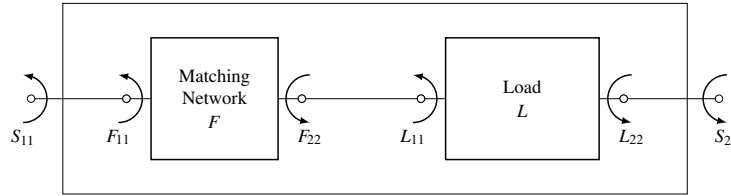


FIG. 2. Darlington equivalent model and scattering parameters

fig:Global_system

The *global scattering matrix* S (see Figure 2) is obtained from the cascade connection of F with L , or chaining operation $F \circ L$,

$$(53) \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = F \circ L := \begin{bmatrix} F_{11} + \frac{F_{12}F_{21}L_{11}}{1-F_{22}L_{11}} & \frac{F_{12}L_{12}}{1-F_{22}L_{11}} \\ \frac{F_{21}L_{21}}{1-F_{22}L_{11}} & L_{22} + \frac{L_{12}L_{21}F_{22}}{1-F_{22}L_{11}} \end{bmatrix}.$$

It is a minimal lossless extension of the global reflection S_{11} , so that

$$|S_{22}(j\omega)| = |S_{11}(j\omega)|, \quad \omega \in \mathbb{R}.$$

The matching problem thus amounts to minimize $|S_{22}(j\omega)|$ over the frequency band Ω . Moreover, whatever the filter, the cascade connection with the given load imposes to S some interpolation conditions, which in fact concern the scalar Schur function S_{22} .

DEFINITION 23. *Given a 2×2 lossless function L , we will say that a Schur function S_{22} is feasible if there exists a lossless extension S of S_{22} , and a lossless scattering matrix F such that $S = F \circ L$, i.e. such that L is de-chainable from S . We denote by $\mathcal{F}(L)$ the set of feasible Schur functions.*

Let z_1, z_2, \dots, z_M be the distinct spectral zeros of L_{11} (including ∞). These are the so-called *transmission zeros* of the load and we assume here that they are all simple. In view of (53), if S is feasible, then S_{22} satisfies

1. The transmission zeros of the load are *spectral zeros* of S_{22} :

$$\{\text{condTZ}\} \quad (54) \quad 1 - S_{22}(z_k)S_{22}^*(z_k) = 0, \quad k = 1, \dots, M.$$

2. For $k = 1, \dots, M$

$$\{\text{condin}\} \quad (55) \quad S_{22}(z_k) = L_{22}(z_k).$$

3. For k such that $z_k \in j\mathbb{R} \cup \infty$

$$\{\text{condon}\} \quad (56) \quad \rho_{z_k}(S_{22}) \leq \rho_{z_k}(L_{22}).$$

These necessary conditions happen to be also sufficient. If S_{22} satisfies (54), (55) and (56), then it is feasible: $S_{22} \in \mathcal{F}(L)$. This result was first proved by Youla in [24], but the first derivations of limiting conditions using Darlington's global approach date back to the pioneer works of Bode and Fano. It is now classical analytic interpolation theory.

The interpolation conditions (55) and (56) provides a data set of the form $\tilde{\sigma}$ (see Section 5), where

- the interpolation points are the transmission zeros, the z_k 's for $k = 1, \dots, M$
- the interpolation values are interpolation values of L_{22} : $v_k = L_{22}(z_k)$ for $k = 1, \dots, M$
- if $|z_k| = 1$, the angular derivatives are $\rho_k = \rho_{z_k}(L_{22})$.

In addition, as mentioned in the introduction, it is important for manufacturing reasons to maintain the degree of the filter as small as possible, and thus, to limit the degree of S_{22} below a prescribed degree, N . Moreover, the transmission zeros are used to shape the response of the filter. It is thus classical, if $S_{22} = \frac{p}{q}$ to prescribe a polynomial $R = qq^* - pp^*$ with no roots in the interval $\mathbb{I} = j\Omega$ and such that $R(z_k) = 0$, for $k = 1, \dots, L$.

The matching problem can now be stated, in terms of S_{22} , as the **Schur optimization problem** stated in (35), namely:

$$\min_{f \in \mathcal{S}_N^R(\tilde{\sigma})} \|f\|_{\infty, \mathbb{I}}.$$

Remark 24. Since $\mathcal{P}(\tilde{\sigma})$ has a strictly Schur solution, namely L_{22} , $\Lambda(\tilde{\sigma})$ is positive definite as required in (34).

8. Illustrative matching problems. For simplicity, we assume that the bandwidth is normalized to be the interval $\Omega = [-1, 1]$.

We first consider a matching problem, with a load of degree 1, its reflection coefficient being

$$L_{11}(s) = \frac{s}{s+1}.$$

It has a single transmission zero at ∞ . We choose the Darlington extension so that $L_{22}(s) = L_{11}(s)$. The corresponding data set is of the form $\tilde{\sigma} = \{(z_1, v_1, \rho_1)\}$, with

$$z_1 = \infty; \quad v_1 = L_{22}(\infty) = 1; \quad \rho_1 = \rho_{\infty}(L_{22}) = 1.$$

For $P \in \mathbb{P}_{2N}^+$ of exact degree $2N$, the Pick matrix is the 1×1 matrix $\tilde{\Delta}(P) = [\rho_\infty(L_{22}) - \rho_\infty(U_P)]$, so that the only constraint is

$$(57) \quad \rho_\infty(U_P) \leq \rho_\infty(L_{22}).$$

We solve the quasi-convex optimization problem (40) with $N = 5$ and the polynomial $R(s) = s^4 + 1.5s^3 + 1.9s^2 + 1.22s + 0.42$. It has been chosen according to the classical filter design rules, in order to conveniently distribute the transmission zeros of the global network.

The optimal value is reached at $\hat{P} = s^5 + 1.1s^4 + 1.7s^3 + 1.1s^2 + 0.58s + 0.13$. The corresponding Blaschke factor being $B = 1$, we get a feasible global reflection S_{22} of degree $N = 5$. It is plotted on figure 3.

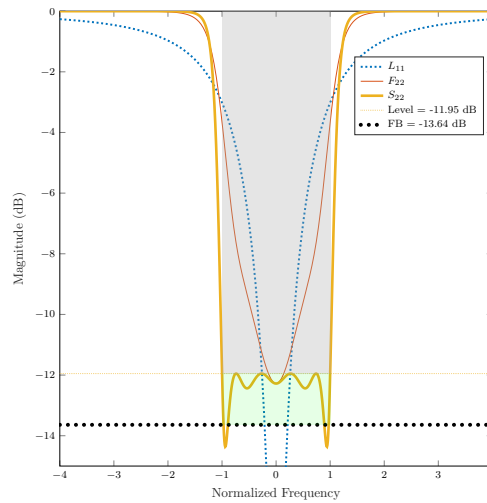


FIG. 3. Optimal system reflection obtained by convex optimization; Load degree = 1, Global system degree = 5; Reflection coefficients: L_{11} antenna, S_{22} global system, F_{22} matching circuit. FB is the Fano bound.

The case of a load of degree 1 is very particular. First, the relaxation does not increase the degree of the solution. Second, we only have one optimization constraint which is precisely Fano's bound, as we shall see.

Bode and Fano actually knew the theoretical limitations (54), (55), (56), but they formulated them in terms of integrals. For a transmission zero at infinity, such a formulation is particularly interesting. It relies on a classical integral equation, sometime called Bode's integral: let $\mu_1, \mu_2, \dots, \mu_k$ be the zeros in the right half-plane of a Schur function f , then

$$(58) \quad -\frac{1}{\pi} \int_{-\infty}^{\infty} \ln |f(j\omega)| d\omega = \rho_\infty(f) - 2 \sum_{j=1}^k \Re \mu_k.$$

In particular, if U is the outer factor of f ,

$$(59) \quad -\frac{1}{\pi} \int_{-\infty}^{\infty} \ln |f(j\omega)| d\omega = \rho_\infty(U).$$

Consequently, for $S_{22} \in \mathcal{F}(L)$, (57) is equivalent to

{FanoIntegral}

$$(60) \quad -\frac{1}{\pi} \int_{-\infty}^{\infty} \ln |S_{22}(j\omega)| d\omega \leq \rho_{\infty}(L_{22}).$$

This equation has a nice geometric interpretation. If $l = \sup_{\omega \in \Omega} |S_{22}(j\omega)|$, (60) yields

$$-\frac{1}{\pi} \int_{-1}^1 \ln(l) d\omega \leq -\frac{1}{\pi} \int_{-\infty}^{\infty} \ln |S_{22}(j\omega)| d\omega \leq \rho_{\infty}(L_{22}),$$

which provides a bound on the best achievable level l . The magnitude of a signal being expressed in dB, we get

{FanoBound}

$$(61) \quad 20 \log l \geq -\frac{10\pi}{\ln(10)} \rho_{\infty}(L_{22}).$$

where \log denotes the common logarithm (with base 10). The second term of the inequality is Fano's bound.

The green shaded rectangles on Figure 3 and Figure 4 measure the gap between the level achieved with our optimum and Fano's bound.

FigLoadDeg3

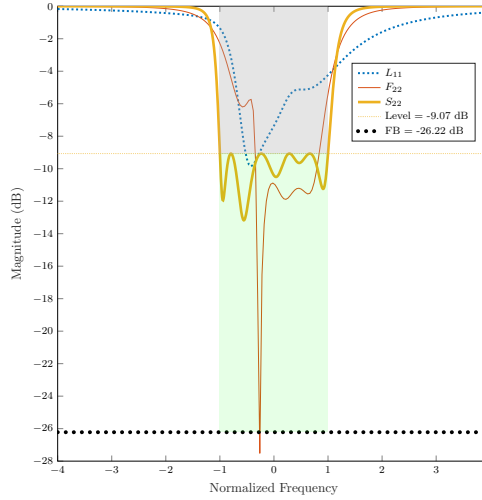


FIG. 4. Optimal system reflection obtained by convex optimization; Load degree = 3, Global system degree = 5; Reflection coefficients: L_{11} antenna, S_{22} global system, F_{22} matching circuit. FB is the Fano bound.

The situation is different with a load of degree 3. Consider

$$L_{11}(s) = \frac{-s^3 + (0.1 + 0.2j)s^2 + (0.09 + 0.35j)s + (0.046 + 0.2j)}{s^3 + (2.022 - 0.2j)s^2 + (1.449 + 0.25j)s + (0.483 - 0.023j)},$$

a load of degree 3 with transmission zeros at $(0.5 - 0.9j)$, $(0.3 + 0.3j)$, ∞ . We again choose L_{22} so that $L_{22}(\infty) = 1$. The results obtained by solving the quasi-convex optimization problem (40) are presented on

Figure 4. The optimal level that we get is far from Fano's bound, and the situation does not improve very much when increasing target degree. Fano's bound, because it only takes into account the interpolation condition at infinity, is not very tight when finite transmission zeros are present. At contrary, the bound that we provide in this work depends on the target degree and reflects all the interpolation conditions. Details on these two academic examples can be found in [6]. More realistic examples can be found there and also in [18]. They were all treated with the software [19].

Appendices.

A. NP-interpolation problem: from the half-plane setting to the disk setting. The mixed NP-interpolation problem $\mathcal{P}(\tilde{\sigma})$ considered in section 5.1, with interpolation data set $\tilde{\sigma} = \sigma_{\partial} \cup \sigma$, where

$$\sigma_{\partial} = \{(z_k, v_k, \rho_k); 1 \leq k \leq L\}, \quad \sigma = \{(z_k, v_k); L+1 \leq k \leq M\},$$

can be transformed into a mixed NP-Problem in the disk setting.

This is classically done using the Moebius transformation¹

$$m : s \mapsto z = (1-s)/(1+s),$$

which maps the right half-plane, Π^+ , to the unit disk, \mathbb{D} . Note that the inverse of m , which maps the unit disk to the right half-plane coincides with $m^{-1} : z \mapsto s = (1-z)/(1+z)$.

Let $t_k = m(z_k)$. For $k = 1, \dots, L$, t_k belongs to the unit circle, \mathbb{T} , while for $k = L+1, \dots, M$, t_k belongs to the open unit disk. Assume that $f \in \mathcal{S}$ solves $\mathcal{P}(\tilde{\sigma})$ and let $g = f \circ m^{-1}$. It is clear that g is a Schur function on the unit disk, and for $k = 1, \dots, M$, it satisfies the interpolation conditions

$$g(t_k) = f(z_k) = v_k.$$

For $k = 1, \dots, L$, these are boundary interpolation conditions on the unit circle, and for $t_k \neq -1$, a straightforward computation gives

$$g'(t_k) = -\frac{2}{(1+t_k)^2} f'(z_k).$$

In the disk setting, extra interpolation conditions bear on the so-called phasor derivative, the quantity

$$(62) \quad Ag(t_k) = \overline{t_k g(t_k)} g'(t_k),$$

which is positive ^{BGR} [2, Th.21.1.1]. Using the relation

$$\frac{2t_k}{(1+t_k)^2} = \frac{1-z_k^2}{2},$$

we get

$$Ag(t_k) = \frac{1-z_k^2}{2} \rho_{z_k}(f).$$

Note that $1-z_k^2$ is a positive number, since z_k lies on the imaginary axis.

If $t_k = -1$, corresponding to $z_k = \infty$, we have (since $\lim_{s \rightarrow \infty} f'(s) = 0$),

$$\lim_{z \rightarrow -1} z g'(z) = \lim_{s \rightarrow \infty} \frac{(1-s^2)}{2} f'(s) = -\lim_{s \rightarrow \infty} \frac{s^2}{2} f'(s),$$

so that

$$Ag(-1) = \frac{\rho_{\infty}(f)}{2}.$$

Finally, for $k = 1, \dots, L$, let

$$\gamma_k = \begin{cases} \frac{2t_k}{(1+t_k)^2} \rho_k, & t_k \neq -1, \\ \frac{\rho_k}{2}, & t_k = -1. \end{cases}$$

¹Moebius transformations (linear fractional transformations) are one-to-one, onto and conformal maps of the extended complex plane:

$$\text{for } s \in \mathbb{C} \cup \{\infty\}, \quad \mu(s) = \frac{as+b}{cs+d}, \quad a, b, c, d \in \mathbb{C} \text{ and } ad-bc \neq 0.$$

The function g satisfies a **mixed NP-interpolation problem on the disk** $\mathcal{P}(\tilde{\tau})$: given a data set $\tilde{\tau} = \tau_\partial \cup \tau$, where

$$(63) \quad \tau_\partial = \{(t_k, v_k, \gamma_k); t_k, v_k \in \mathbb{T}, \gamma_k > 0, 1 \leq k \leq L\}, \quad \tau = \{(t_k, v_k); t_k, v_k \in \mathbb{D}, L+1 \leq k \leq M\},$$

find a function g , schur on the disk and such that

$$(64) \quad g(t_k) = v_k, \quad k = 1, \dots, M.$$

and

$$(65) \quad Ag(t_k) \leq \gamma_k, \quad k = 1, \dots, L.$$

Conversely, if g solves $\mathcal{P}(\tilde{\tau})$, then f solves $\mathcal{P}(\tilde{\sigma})$, so that the two problems are equivalents.

Let us now stress the relation between the Pick matrices $\Lambda(\tilde{\sigma})$ and the Pick matrix \mathbb{P} , associated with problem $\mathcal{P}(\tilde{\tau})$:

$$(66) \quad \mathbb{P}_{ij} = \begin{cases} \gamma_i, & 1 \leq i = j \leq L \\ \frac{1 - v_i \bar{v}_j}{1 - t_i \bar{t}_j}, & \text{otherwise.} \end{cases}$$

First observe that for z_i, z_j in \mathbb{C} , we have

$$z_i + \bar{z}_j = 2 \frac{1 - t_i \bar{t}_j}{(1 + t_i)(1 + \bar{t}_j)}.$$

For i, j , such that $j \neq i$ and, neither z_i , nor z_j is equal to infinity,

$$\Lambda(\tilde{\sigma})_{ij} = \frac{1 - v_i \bar{v}_j}{z_i + \bar{z}_j} = \frac{(1 + t_i)}{\sqrt{2}} \mathbb{P}_{ij} \frac{(1 + \bar{t}_j)}{\sqrt{2}}.$$

For i, j , such that $j \neq i$ and $z_i = \infty$ (then $t_i = -1$, and $z_j \neq \infty$ by assumption),

$$\Lambda(\tilde{\sigma})_{ij} = 1 - v_i \bar{v}_j = \sqrt{2} \mathbb{P}_{ij} \frac{(1 + \bar{t}_j)}{\sqrt{2}}.$$

For $i = 1, \dots, L$, such that $z_i \neq \infty$,

$$\Lambda(\tilde{\sigma})_{ii} = \rho_i = \frac{(1 + t_i)^2}{2t_i} \gamma_i = \frac{(1 + t_i)}{\sqrt{2}} \mathbb{P}_{ii} \frac{(1 + \bar{t}_i)}{\sqrt{2}},$$

while if $z_i = \infty$,

$$\Lambda(\tilde{\sigma})_{ii} = \rho_i = 2\gamma_i = \sqrt{2} \mathbb{P}_{ii} \sqrt{2}.$$

The two matrices are $*$ -congruent, so that $\Lambda(\tilde{\sigma}) \geq 0$ if and only if $\mathbb{P} \geq 0$, and the two matrices have the same rank.

B. Boundary NP-interpolation problem on the disk. We consider here the boundary interpolation problem $\mathcal{P}(\tau_\partial)$ (case $L = M$). This problem, as well as its variant, $\hat{\mathcal{P}}(\tau_\partial)$, where equality is required in (65), were first studied in [22]. Sarason proved that problem $\mathcal{P}(\tau_\partial)$ has a solution if and only if the Pick matrix is positive semi definite, while problem $\hat{\mathcal{P}}(\tau_\partial)$ has a solution if and only if the Pick matrix is positive definite or minimally positive. Sarason's paper makes transparent why problem $\mathcal{P}(\tau_\partial)$ bears the closest resemblance to the classical NP-problem: both problems are indeterminate if and only if the Pick matrix is positive definite, and determinate whenever it is positive semi-definite and singular. We'll focus on the determinate case and prove part 2 of Theorem 16 in that particular case:

TheoremBNPD

PROPOSITION 25. Assume that the Pick matrix \mathbb{P} associated to problem $\mathcal{P}(\tau_\partial)$ is positive semi-definite, and has rank $n < L$. Then $\mathcal{P}(\tau_\partial)$ has a unique solution, which is a Blaschke product of degree at most n .

This statement can be found in [5, Th.2.6, (2)]. Bolotnikov's paper provides a parametrization of low Blaschke degree solutions to the interpolation conditions (64) only. In this representation of the solutions, phasar derivatives are used as parameters. We'll use it to prove Theorem 25 and to specify the exact degree of the solution from some characteristics of the Pick matrix.

We may assume (permuting the interpolation data if necessary) that the leading principal submatrix, \mathbb{P}_n of \mathbb{P} , is positive definite. For $i, j \in \{n+1, \dots, L\}$, let

$$\{M_{i,j}\} \quad (67) \quad M_{\{i,j\}} = \begin{bmatrix} \mathbb{P}_n & u_j \\ u_i^* & p_{ij} \end{bmatrix},$$

be the principal submatrix of \mathbb{P} of size $n+1$, that lies in the rows numbered $1, 2, \dots, n, i$ and the columns numbered $1, 2, \dots, n, j$. Since $M_{\{i,j\}}$ is singular, we have

$$\{p_{ij}\} \quad (68) \quad p_{ij} = u_i^* \mathbb{P}_n^{-1} u_j, \quad i, j = n+1, \dots, L,$$

and in particular,

$$\{\gamma_{jj}\} \quad (69) \quad \gamma_j = u_j^* \mathbb{P}_n^{-1} u_j, \quad j = n+1, \dots, L.$$

Fix $j \in \{n+1, \dots, L\}$ and let

$$\{\text{thetaj}\} \quad (70) \quad \Theta_j(z) = \begin{bmatrix} \Theta_{11}^j & \Theta_{12}^j \\ \Theta_{21}^j & \Theta_{22}^j \end{bmatrix}$$

$$(71) \quad = I + (z - t_j) \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - zT^*)^{-1} \mathbb{P}_n^{-1} (t_j I - T)^{-1} \begin{bmatrix} E & -M \end{bmatrix},$$

where

$$T = \text{diag}(t_1, t_2, \dots, t_n); \quad E^* = \begin{bmatrix} 1 & & & 1 \end{bmatrix}; \quad M^* = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}.$$

Note that $\Theta_j(t_j) = I$.

According to [5, Th.3.5] the function

$$\{f_j\} \quad (72) \quad f_j(z) = T_{\Theta_j}(v_j) := \frac{\Theta_{11}^j(z)v_j + \Theta_{12}^j(z)}{\Theta_{21}^j(z)v_j + \Theta_{22}^j(z)},$$

is a finite Blaschke product of degree at most n , which satisfies conditions (64) for $k = 1, \dots, n$ and $k = j$ and (65) for $k = 1, \dots, n$.

We check now that f_j also satisfies interpolation condition (65) at t_j . First observe that, in view of (66),

$$\{u_j\} \quad (73) \quad u_j = (I - \bar{t}_j T)^{-1} (E - \bar{v}_j M).$$

Differentiating f_j as a quotient, we get

$$f_j'(z) = f_j(z) \left[\frac{(\Theta_{11}^j)'(z)v_j + (\Theta_{12}^j)'(z)}{\Theta_{11}^j(z)v_j + \Theta_{12}^j(z)} - \frac{(\Theta_{21}^j)'(z)v_j + (\Theta_{22}^j)'(z)}{\Theta_{21}^j(z)v_j + \Theta_{22}^j(z)} \right]$$

so that

$$f_j'(t_j) = v_j \left[\frac{(\Theta_{11}^j)'(t_j)v_j + (\Theta_{12}^j)'(t_j)}{v_j} - (\Theta_{21}^j)'(t_j)v_j - (\Theta_{22}^j)'(t_j) \right] = v_j \begin{bmatrix} 1 & -v_j \end{bmatrix} \Theta'(t_j) \begin{bmatrix} 1 \\ \bar{v}_j \end{bmatrix}.$$

Now, from (70),

$$\Theta'(t_j) = \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - t_j T^*)^{-1} \mathbb{P}_n^{-1} (t_j I - T)^{-1} \begin{bmatrix} E & -M \end{bmatrix},$$

so that,

$$\begin{aligned} f_j'(t_j) &= v_j \begin{bmatrix} 1 & -v_j \end{bmatrix} \Theta'(t_j) \begin{bmatrix} 1 \\ \bar{v}_j \end{bmatrix} \\ &= v_j (E^* - v_j M^*) (I - t_j T^*)^{-1} \mathbb{P}_n^{-1} (t_j I - T)^{-1} (E - \bar{v}_j M) \\ &= v_j \bar{t}_j u_j^* \mathbb{P}_n^{-1} u_j \\ &= v_j \bar{t}_j \gamma_j, \end{aligned}$$

thanks to (73) and (69). Finally,

$$A f_j(t_j) = \gamma_j,$$

and f_j solve the boundary interpolation problem \mathcal{P}_j with data set

$$\tau_j' = \{(t_k, v_k, \gamma_k); k \in \{1, \dots, n, j\}\}.$$

Recall that the solution of $\mathcal{P}(\tau_\partial)$ is unique and then, either f_j also satisfies the extra interpolation conditions (for $k \neq j, n < k \leq L$), or the full problem has no solution.

Let i , such that $n < i \leq L$ and $i \neq j$, and construct the corresponding solution f_i of \mathcal{P}_i as previously. First note that

$$\Theta_i(z) = \Theta_j(z)H,$$

where H is the J -unitary matrix given by

$$H = \Theta_i(t_j).$$

Then,

$$f_i(z) = T_{\Theta_i}(v_i) = T_{\Theta_j}(T_H(v_i)).$$

Now, let us show that $T_H(v_i) = v_j$, or equivalently that

$$\begin{bmatrix} 1 & -v_j \end{bmatrix} H \begin{bmatrix} v_i \\ 1 \end{bmatrix} = 0.$$

Since

$$H = I + (t_j - t_i) \begin{bmatrix} E^* \\ M^* \end{bmatrix} (I - t_j T^*)^{-1} \mathbb{P}_n^{-1} (t_i I - T)^{-1} \begin{bmatrix} E & -M \end{bmatrix},$$

we have

$$\begin{aligned} \begin{bmatrix} 1 & -v_j \end{bmatrix} H \begin{bmatrix} v_i \\ 1 \end{bmatrix} &= (v_i - v_j) + (t_j - t_i) (E^* - v_j M^*) (I - t_j T^*)^{-1} \mathbb{P}_n^{-1} (t_i I - T)^{-1} (E v_i - M) \\ &= (v_i - v_j) + (t_j - t_i) u_j^* \mathbb{P}_n^{-1} \bar{t}_i v_i u_i \\ &= v_i (1 - \bar{t}_i t_j) (p_{ji} - u_j^* \mathbb{P}_n^{-1} u_i) \\ &= 0, \end{aligned}$$

using (73) again and (68). Finally f_i coincides with f_j . For $i = n + 1, \dots, L$, the functions f_i are different representations of a same function f , which solves the boundary problem $\mathcal{P}(\tau_\partial)$.

Moreover, from [5, Th. 3.5 and Th. 3.6], we have:

COROLLARY 26. Let

$$\mathbb{P}_n^{-1} u_j = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix}.$$

The function f_j given by (72) can be rewritten

$$f_j(z) = v_j \frac{1 - (1 - z\bar{t}_j) \sum_{i=1}^n \frac{\Delta_i}{1 - z\bar{t}_i}}{1 - (1 - z\bar{t}_j) \sum_{i=1}^n v_j \bar{v}_i \frac{\Delta_i}{1 - z\bar{t}_i}}. \quad (74)$$

It has degree $n - |Z|$, where $|Z|$ denotes the cardinality of

$$Z = \{i, 1 \leq i \leq n; \Delta_i = 0\}.$$

Moreover, for $i = 1, \dots, n$, we have

$$|f'(t_i)| = \gamma_i \Leftrightarrow \Delta_i \neq 0. \quad (75)$$

REFERENCES

- [1] J. AGLER, Z. A. LYKOVA, AND N. J. YOUNG, *Finite Blaschke products and construction of rational Γ -inner functions*, J. Math. Anal. Appl., 447 (2017), pp. 1163–1196.
- [2] J. BALL, I. GOHBERG, AND L. RODMAN, *Interpolation of Rational Matrix Functions*, vol. 45 of Operator Theory: Advances and Applications, Birkhäuser, 1990.
- [3] L. BARATCHART, M. OLIVI, AND F. SEYFERT, *Generalized Nevanlinna-Pick interpolation on the boundary. Application to impedance matching*. Working paper or preprint: <https://hal.inria.fr/hal-01249330>, Dec. 2015, <https://hal.inria.fr/hal-01249330>.
- [4] V. BOLOTNIKOV, *On zeros of certain analytic functions*, Integral Equations Operator Theory, 69 (2011), pp. 203–215.
- [5] V. BOLOTNIKOV, *Boundary interpolation by finite blaschke products*, in Complex Analysis and Dynamical Systems, 2018. [arXiv:1609.09843](https://arxiv.org/abs/1609.09843).
- [6] G. BOSE, *H^∞ Approximation, Analytic Interpolation and Convex Optimisation : Application to Broadband Impedance Matching*, PhD thesis, Université Côte d’Azur, France, 2021.
- [7] S. BOYD AND L. VANDENBERGHE, *Convex Optimization*, Cambridge University Press, 2004.
- [8] C. BYRNES, T. T. GEORGIU, A. LINDQUIST, AND A. MEGRETSKI, *Generalized interpolation in H^∞ with a complexity constraint*, Trans. Amer. Math. Soc., 358 (2006), pp. 965–988.
- [9] R. CAMERON, *General coupling matrix synthesis methods for chebyshev filtering functions*, IEEE Transaction on Microwave Theory and Techniques, 47 (1999), pp. 433–442.
- [10] G. CHEN AND Y. HU, *Multiple nevanlinna-pick interpolation with both interior and boundary data and its connection with the power moment problem*, Linear Algebra Appl., 323 (2001), pp. 167–194.
- [11] S. DARLINGTON, *Synthesis of reactance 4-poles which produce a prescribed insertion loss characteristics*, J. Math. Physics, 18 (1939), pp. 257–355.
- [12] B. DUMITRESCU, *Positive Trigonometric Polynomials and Signal Processing Applications*, Springer Publishing Company, Incorporated, 2007.
- [13] H. DYM, *J-contractive matrix functions, reproducing kernel spaces and interpolation*, vol. 71 of CBMS lecture notes, American Mathematical Society, Rhode Island, 1989.
- [14] J. GARNETT, *Bounded Analytic Functions*, Academic Press, 1981.
- [15] T. GEORGIU, *The interpolation problem with a degree constraint*, IEEE Transactions on Automatic Control, (1999).
- [16] J. KARLSSON, T. T. GEORGIU, AND A. G. LINDQUIST, *The inverse problem of analytic interpolation with degree constraint and weight selection for control synthesis*, IEEE Transactions on Automatic Control, 55 (2010), pp. 405–418.
- [17] K. KUROKAWA, *Power waves and the scattering matrix*, IEEE transactions on Microwave Theory and Techniques, (1965).
- [18] D. M. MARTINEZ, *Methodologies and synthesis tools for functions filters loaded by complex impedances*, PhD thesis, Université de Limoges, France, 2019.
- [19] D. M. MARTINEZ, F. SEYFERT, A. COOMAN, AND M. OLIVI, *Software PUMA-HF*: <https://project.inria.fr/puma>, 2018.

tel-02273906

NL01

Sarason

Stingl

Youla_64

ZF83

- [20] D. MARTINEZ MARTINEZ, *Methodologies and synthesis tools for functions filters loaded by complex impedances*, theses, Université de Limoges, June 2019, <https://tel.archives-ouvertes.fr/tel-02273906>.
- [21] R. NAGAMURE AND A. LINDQUIST, *Optimal Control and Partial Differential Equations*, IOS Press, Amsterdam, 2001, ch. Sensitivity shaping in feedback control and analytic interpolation theory.
- [22] D. SARASON, *Nevanlinna–Pick interpolation with boundary data*, *Integral Equations Operator Theory*, 30 (1998), pp. 231–250.
- [23] M. STINGL, *On the solution of nonlinear semidefinite programs by augmented Lagrangian methods*, Shaker Verlag GmbH, 2006.
- [24] D. C. YOULA, *A new theory of broadband matching*, *IEEE Trans. Circuits Sys.*, CT11 (1964), pp. 30–50.
- [25] O. ZAMES AND B. A. FRANCIS, *Feedback, minimax sensitivity, and optimal robustness*, *IEEE Transactions on Automatic Control*, 28 (1983), pp. 585–600.