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Two-Time Measurement of Entropy Transfer in Markovian Quantum Dynamics

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Abstract. We consider a protocol for the two-time measurement of entropic observables in quantum open systems driven out of thermal equilibrium by coupling to several heat baths. We concentrate on the Markovian approximation of the time-evolution and relate the expected value of the so defined entropy variations with the well-known expression of entropy production due to Lebowitz and Spohn. We do so under the detailed balance condition and, as a byproduct, we show that the probabilities of outcomes of two-time measurements are given by a continuous time Markov process determined by the Lindblad generator of the Markovian quantum dynamics.

1 Introduction

In this note, we consider open quantum systems in the Markovian approximation, and more specifically entropy transfer in such systems out of thermal equilibrium. We define entropy variation by the two-time measurements of certain entropic observables and relate the resulting quantities with the entropy production as defined by Lebowitz and Spohn [SL78, JPW14]. Entropy production is of prime interest in nonequilibrium statistical mechanics. Obviously, the two-time measurement approach to entropy production can also be considered beyond the Markovian approximation. In the Hamiltonian framework of open quantum systems, where the joint dynamics of the system and its extended environment is considered, modular theory, which only surfaces in the Markovian case, provides a rich mathematical structure. We refer the interested reader to [BBJ$^+$23, BBJ$^+$b, BBJ$^+$a] for a detailed exposition. However, given the wide usage of the Markovian approximation in physics, we feel that a discussion in the latter context is appropriate.

The two-time measurement protocol was first used in the context of quantum systems out of equilibrium in [Kur00], and involves the following procedure: Initially, say at time 0, a measurement of a given observable is performed on the system, resulting in a new state determined by the outcome of this first measurement. This new state then evolves according to quantum dynamics up to some later time t, after which a second measurement of the same observable is performed. This determines the quantum mechanical probabilities to get an outcome at time t, given the outcome at time 0, and hence the probabilities of the variations of the observable between times 0 and t.

The definition of entropy production for a Markovian quantum dynamics generated by a Lindblad operator was motivated by physical considerations on the entropy balance relation for

systems interacting with one or several thermal reservoirs in [SL78]. It was generalized to arbitrary Markovian quantum dynamics, or quantum dynamical semi-groups, in [Spo78]. See also [JPW14] for a detailed account on this topic.

Our goal is to relate this approach to the more operational one involving a two-time measurement of entropic observables. We consider in particular non equilibrium situations characterized by the fact that the Lindblad generator of the Markovian quantum dynamics is given by a sum of individual sub-Lindbladians admitting different invariant states. In all cases, we need these sub-Lindbladians to satisfy the detailed balance condition. As a byproduct, we remark that the outcomes of two-time measurements probabilities are given by a continuous time Markov process on a finite state space determined by the Lindblad generator of the Markovian quantum dynamics, under generic hypotheses. This allows us to express the properties of the quantum measurements in terms of classical data.

Related approaches of the entropy production for different quantum dynamics and various models have been proposed in the physics literature, see [FGM23] and the references therein for a recent account.

The paper is organized as follows. After setting up Markovian quantum dynamics in Section 2, Section 3 gathers properties of the two-time measurement protocols of a quantum observable. We discuss the detailed balance condition in Section 4 and recall the original Lebowitz-Spohn definition of entropy production and its main properties in Section 5. We then state and prove our main result regarding the relation of entropy production and two-time measurement of the entropy observable in Section 6. Then, in Section 7, we turn to the definition of the classical Markov process related to the two-time measurement protocol. In Section 8, we invoke this classical Markov process to investigate some specific properties of the two-time measurement protocol when the environment of the system is in thermal equilibrium. We elaborate on the use of the underlying classical Markov process to express the moments generating function of the latter in terms of classical properties under certain circumstances in Section 9. The paper closes with Section 10, where we provide an example, the so-called quantum reset model.

2 Markovian description of nonequilibrium open quantum systems

We consider a quantum system with finite dimensional Hilbert space $\mathscr{H}$. Observables of the system are elements of $\mathscr{O}$, the C^*-algebra of all linear operators on $\mathscr{H}$, and we denote by $\mathscr{B}(\mathscr{O})$ the set of *super-operators, i.e.,* linear operators on $\mathscr{O}$. Below, $\mathscr{T}$ denotes the vector space of linear operators on $\mathscr{H}$ equipped with the trace norm $\|A\|_1 = \mathrm{tr}(\sqrt{A^*A})$. We introduce a duality bracket on $\mathscr{T} \times \mathscr{O}$ by setting

$$\langle A|B\rangle = \mathrm{tr}(A^*B),$$

and denote the adjoint of a linear map $\mathscr{M} : \mathscr{T} \to \mathscr{T}$ with respect to this duality by $\mathscr{M}^\dagger \in \mathscr{B}(\mathscr{O})$. States of our system are described by density matrices, *i.e.,* elements of the convex set

$$\mathscr{S} = \{\rho \in \mathscr{T} \mid \rho \geq 0, \mathrm{tr}(\rho) = 1\}.$$

A state $\rho \in \mathscr{S}$ is said to be faithful, written $\rho > 0$, whenever $\mathrm{Ker}\,\rho = \{0\}$.

The effective evolution equation of states within the Markovian framework is

$$\dot\rho(t) = \mathscr{L}(\rho(t)), \quad t \in [0,\infty), \quad \rho(0) = \rho_0 \in \mathscr{S},$$

where the generator is the *Lindbladian*

$$\mathscr{L}(\,\cdot\,) = -\mathrm{i}[H,\cdot\,] + \sum_l \left(\Gamma_l \cdot \Gamma_l^* - \frac{1}{2}\{\Gamma_l^*\Gamma_l, \cdot\,\}\right). \tag{1}$$

In the above equation, the sum is over a finite set of indices, H and the Γ_l are all elements of $\mathscr{O}$, with H being self-adjoint. The Hamiltonian part of the Lindbladian (1), represented by the commutator with H, describes the state's evolution in the absence of an environment. The dissipator, which is the second term in (1), encodes the global effect of the environment on the evolution.

The family $(\mathrm{e}^{t\mathscr{L}})_{t\geq 0}$ is a norm continuous semi-group of completely positive trace-preserving (CPTP) contraction on $\mathscr{T}$, defining a *Markov quantum dynamics* (MQD) on $\mathscr{S}$, see [Lin76, GKS76]. Any element of $\mathscr{S}\cap\mathrm{Ker}\,\mathscr{L}$ represents a *steady state* of this MQD. The set of such states is never empty, and reduces to a singleton if it contains a faithful element [Fri78]. The MQD is called *relaxing* if

$$\lim_{t\to\infty}\mathrm{e}^{t\mathscr{L}}(\rho_0)=\rho^+$$

holds for some $\rho^+\in\mathrm{Ker}\,\mathscr{L}$ and all $\rho_0\in\mathscr{S}$. In this case, $\mathrm{Ker}\,\mathscr{L}$ is the one-dimensional subspace of $\mathscr{T}$ spanned by ρ^+. A general algebraic condition on the Kraus operators Γ_l which ensures the relaxing property has been obtained in [Spo77].

Even though the representation (1) of a Lindbladian can sometimes be deduced from physics, e.g., through the Davies weak coupling limit [Dav74, Dav76], this representation is in general not unique. For the Lindbladians occurring in the following, we tacitly assume that such a representation has been chosen.

In this note we will assume, without further mention, that the environment of the system of interest consists of several reservoirs, inducing the following structure of the Lindbladian $\mathscr{L}$.

(RS) (1) The Lindbladian $\mathscr{L}$ can be decomposed as

$$\mathscr{L}=\sum_{j\in\mathscr{J}}\mathscr{L}_j, \tag{2}$$

each *sub-Lindbladian* $\mathscr{L}_j$ generating the MQD of the system coupled to a reservoir $\mathscr{R}_j$.

(2) Each MQD $(\mathrm{e}^{t\mathscr{L}_j})_{t\geq 0}$ has a unique faithful steady state denoted by ρ_j^+.

3 Two-Time Measurement Protocol

Let us consider here the two-time measurement protocol (2TMP for short) of a quantum observable $S=S^*\in\mathscr{O}$, for states that vary in time according to the MQD $(\mathrm{e}^{t\mathscr{L}})_{t\geq 0}$ generated by a Lindbladian $\mathscr{L}$. This procedure yields statistical information on the variation of the observable S with time.

The initial state is $\rho_0\in\mathscr{S}$ and the observable S admits the spectral decomposition[1]

$$S=\sum_{s\in\mathrm{spec}(S)}sP_s.$$

A measurement of S in the state ρ_0 has the outcome $s\in\mathrm{spec}(S)$ with probability[2]

$$\mathbb{P}_{\rho_0}(S_0=s)=\langle\rho_0|P_s\rangle=\mathrm{tr}(P_s\rho_0P_s), \tag{3}$$

[1] $\mathrm{spec}(A)$ denotes the spectrum of $A\in\mathscr{O}$.

[2] In the following, the subscript to the observable S refers to the time at which the measurement is performed.

and, according to the reduction postulate, the state undergoes the transformation[3]

$$\rho_0 \mapsto \frac{P_s \rho_0 P_s}{\mathrm{tr}(P_s \rho_0 P_s)}$$

immediately after the measurement. After evolving this state for a time $t > 0$, a second measurement of the observable S has the outcome $s' \in \mathrm{spec}(S)$ with probability

$$\mathbb{P}_{\rho_0}(S_t = s' | S_0 = s) = \frac{\langle \mathrm{e}^{t\mathscr{L}}(P_s \rho_0 P_s) | P_{s'} \rangle}{\mathrm{tr}(P_s \rho_0 P_s)}. \tag{4}$$

According to Bayes rule, the joint law for the outcome of the two-time measurement is

$$\mathbb{P}_{\rho_0}(S_0 = s \,\&\, S_t = s') = \langle \mathrm{e}^{t\mathscr{L}}(P_s \rho_0 P_s) | P_{s'} \rangle. \tag{5}$$

Let us note here that this formula leads to the following expression for the law of the outcomes of the second measurement

$$\mathbb{P}_{\rho_0}(S_t = s') = \sum_{s \in \mathrm{spec}(S)} \langle \mathrm{e}^{t\mathscr{L}}(P_s \rho_0 P_s) | P_{s'} \rangle = \langle \mathrm{e}^{t\mathscr{L}}(\mathrm{Diag}_S(\rho_0)) | P_{s'} \rangle, \tag{6}$$

where Diag_S denote the CPTP map defined on $\mathscr{T}$ by

$$T \mapsto \mathrm{Diag}_S(T) = \sum_s P_s T P_s.$$

Remark 3.1. We will denote by the same symbol the linear map on $\mathscr{O}$ defined by the same formula. With this convention, Diag_S is the self-adjoint ($\mathrm{Diag}_S^\dagger = \mathrm{Diag}_S$) projection onto the commutant $\{S\}' = \{T \in \mathscr{T} \mid [T, S] = 0\}$ of S. For later reference, we observe that Diag_S actually only depends on the spectral projections of S so that for any injection $F : \mathrm{spec}(S) \mapsto \mathbb{R}$, we have $\mathrm{Diag}_S = \mathrm{Diag}_{F(S)}$.

Note also the *decoherence effect* of the first measurement: If $[\rho, S] \neq 0$ and $t > 0$, the right-hand side of (6) is in general different from $\langle \mathrm{e}^{t\mathscr{L}}(\rho_0) | P_{s'} \rangle$, which is the probability for a measurement of S performed at time t to have the outcome s' if the system was started at time 0 in the state ρ_0, *without* measuring S at time zero.

From Relation (5), we immediately derive the law $\mathbb{Q}^t_{\rho_0}$ of the variation of S during the time interval $[0, t]$ according to the two-time measurement of S in the state ρ_0

$$\mathbb{Q}^t_{\rho_0}(\Delta S = \sigma) = \sum_{\substack{s, s' \in \mathrm{spec}(S) \\ s' - s = \sigma}} \langle \mathrm{e}^{t\mathscr{L}}(P_s \rho_0 P_s) | P_{s'} \rangle. \tag{7}$$

The moment generating function of the random variable ΔS thus defined is given by

$$e^t_{\rho_0}(\alpha) = \mathbb{E}^t_{\rho_0}(\mathrm{e}^{\alpha \Delta S}) = \sum_{s, s' \in \mathrm{spec}(S)} \mathrm{e}^{\alpha(s'-s)} \langle \mathrm{e}^{t\mathscr{L}}(P_s \rho_0 P_s) | P_{s'} \rangle.$$

Performing the summation over s' yields

$$e^t_{\rho_0}(\alpha) = \sum_{s \in \mathrm{spec}(S)} \mathrm{e}^{-\alpha s} \langle \mathrm{e}^{t\mathscr{L}}(P_s \rho_0 P_s) | \mathrm{e}^{\alpha S} \rangle,$$

and hence

$$e^t_{\rho_0}(\alpha) = \sum_{s \in \mathrm{spec}(S)} \langle \mathrm{e}^{t\mathscr{L}}(P_s \rho_0 P_s \mathrm{e}^{-\alpha S}) | \mathrm{e}^{\alpha S} \rangle = \langle \mathrm{e}^{t\mathscr{L}}(\mathrm{Diag}_S(\rho_0) \mathrm{e}^{-\alpha S}) | \mathrm{e}^{\alpha S} \rangle.$$

[3] For our purpose, outcomes with zero probability are irrelevant.

The expected value of ΔS is given by[4]

$$\mathbb{E}^t_{\rho_0}(\Delta S) = (\partial_\alpha e^t_{\rho_0})(0) = \langle e^{t\mathscr{L}}(\mathrm{Diag}_S(\rho_0))|S\rangle - \langle e^{t\mathscr{L}}(\mathrm{Diag}_S(\rho_0)S)|\mathbb{I}\rangle$$
$$= \langle e^{t\mathscr{L}}(\mathrm{Diag}_S(\rho_0))|S\rangle - \langle\rho_0|S\rangle, \tag{8}$$

where we used the fact that the dual semi-group $e^{t\mathscr{L}^\dagger}$ is unit-preserving, and the cyclicity of the trace. Finally, the expected value of ΔS writes as a difference of quantum mechanical expectation values of the observable S in states at time t and time 0 where the initial state is indeed ρ_0, but the state at time t has been affected by the decoherence induced by the first measurement.

We note for later reference that

$$e^t_{\rho_0}(\alpha) = \langle e^{t\mathscr{L}_\alpha}(\mathrm{Diag}_S(\rho_0))|\mathbb{I}\rangle, \tag{9}$$

where $\mathscr{L}_\alpha(\,\cdot\,) = \mathscr{L}(\,\cdot\, e^{-\alpha S})e^{\alpha S}$. This should be compared with [JPW14, Definition (6)], which is not obviously related to two-time measurements.

Further, assuming the MQD to be relaxing to the steady state ρ^+, we get the following large t limits of the above quantities

$$\lim_{t\to\infty}\langle e^{t\mathscr{L}}(P_s\rho_0P_s)|P_{s'}\rangle = \langle\rho^+|P_{s'}\rangle\langle\rho_0|P_s\rangle,$$
$$\lim_{t\to\infty}\mathbb{P}_{\rho_0}(S_t = s'|S_0 = s) = \langle\rho^+|P_{s'}\rangle,$$
$$\lim_{t\to\infty}\mathbb{Q}^t_{\rho_0}(\Delta S = \delta) = \sum_{\substack{s,s'\in\mathrm{spec}(S)\\ s'-s=\delta}}\langle\rho^+|P_{s'}\rangle\langle\rho_0|P_s\rangle, \tag{10}$$
$$\lim_{t\to\infty} e^t_{\rho_0}(\alpha) = \langle\rho^+|e^{\alpha S}\rangle\langle\rho_0|e^{-\alpha S}\rangle,$$
$$\lim_{t\to\infty}\mathbb{E}^t_{\rho_0}(\Delta S) = \langle\rho^+ - \rho_0|S\rangle.$$

The RHS of the last relation is the difference of the QM expectation of S in the limiting state at time $t=\infty$, ρ^+, and in the initial state ρ_0.

4 Detailed Balance

Let us briefly recall the notion of detailed balance (DB for short) in the Lindblad context, we refer the reader to [Aga73, Ali76, FU07, FU10] for details.

The dual $\mathscr{L}^\dagger$ of the general Lindblad operator (1) takes the form

$$\mathscr{L}^\dagger(\,\cdot\,) = \mathrm{i}[H,\,\cdot\,] - \frac{1}{2}\{\Phi(\mathbb{I}),\,\cdot\,\} + \Phi(\,\cdot\,), \tag{11}$$

where $\Phi(\,\cdot\,) = \sum_l \Gamma_l^*\,\cdot\,\Gamma_l \in \mathscr{B}(\mathscr{O})$ is a Completely Positive (CP) map. With a slight abuse of language, we will say that Φ is the CP-map associated to $\mathscr{L}$.

For a faithful state $\rho > 0$, we introduce the following ρ-inner product on $\mathscr{O}$,

$$\langle A|B\rangle_\rho := \mathrm{tr}(\rho A^* B).$$

[4] $\mathbb{I}$ denotes the unit of $\mathscr{O}$.

We denote by $\mathscr{M}^\rho$ the adjoint of a super-operator $\mathscr{M} \in \mathscr{B}(\mathscr{O})$ w.r.t. this inner product, that is

$$\langle A|\mathscr{M}(B)\rangle_\rho = \langle \mathscr{M}^\rho(A)|B\rangle_\rho,$$

for all $A, B \in \mathscr{O}$.This is easily seen to be equivalent to

$$\mathscr{M}^\rho(A) = \mathscr{M}^\dagger(A\rho)\rho^{-1}. \tag{12}$$

We will say that $\mathscr{M}$ is ρ-self-adjoint whenever $\mathscr{M}^\rho = \mathscr{M}$.

Consider a pair $(\rho, \mathscr{L})$ where $\mathscr{L}$ is a Lindbladian and $\rho \in \mathscr{S} \cap \operatorname{Ker}\mathscr{L}$. The following condition essentially characterizes an open system in contact with a reservoir in thermal equilibrium (see [KFGV77]).

> **(DB)** $(\rho, \mathscr{L})$ satisfies the detailed balance condition if $\rho > 0$ and the CP-map associated to $\mathscr{L}$ is ρ-self-adjoint.

In the context of **(RS)**, the next assumption further specifies the inverse temperature $\beta_j > 0$ of reservoir $\mathscr{R}_j$.

> **(KMS)** For each pair $(\mathscr{L}_j, \rho_j^+)$ of Assumption **(RS)**, one has[5]
>
> $$\mathscr{L}_j^\dagger(\,\cdot\,) = \mathrm{i}[H, \cdot\,] - \frac{1}{2}\{\Phi_j(\mathbb{1}), \cdot\,\} + \Phi_j(\,\cdot\,), \qquad \rho_j^+ = \mathrm{e}^{-\beta_j(H-F_j)}.$$

For our purpose, the following simple consequence of the detailed balance condition will be important:

Lemma 4.1. *Assume* **(DB)** *holds for* $(\rho, \mathscr{L})$*, then,*

$$\mathrm{Diag}_\rho \circ \mathscr{L} = \mathscr{L} \circ \mathrm{Diag}_\rho. \tag{13}$$

Remark 4.2. (i) By Remark 3.1, the statement is equivalent to $\mathrm{Diag}_\rho \circ \mathscr{L}^\dagger = \mathscr{L}^\dagger \circ \mathrm{Diag}_\rho$.

(ii) As the proof shows, the dissipator part of $\mathscr{L}^\dagger(\cdot)$, $-\frac{1}{2}\{\Phi(\mathbb{1}), \cdot\,\} + \Phi(\,\cdot\,)$ is ρ-self-adjoint under Assumption **(DB)**.

Proof. Let Φ be the CP-map associated to $\mathscr{L}$ and define $\Delta_\rho \in \mathscr{B}(\mathscr{O})$ by[6]

$$\Delta_\rho(X) = \rho X \rho^{-1}.$$

The cyclicity of the trace implies that that $\Delta_\rho^\dagger = \Delta_\rho$. Moreover, under **(DB)**, see [JPW14, CM17],

$$[\Phi, \Delta_\rho] = 0. \tag{14}$$

This relation is the root of the following identities

$$[\Phi(\mathbb{1}), \rho] = 0, \qquad [H, \rho] = 0. \tag{15}$$

Indeed, applied to the unit $\mathbb{1}$, (14) immediately yields the first identity. It follows that

$$0 = \mathscr{L}(\rho) = -\mathrm{i}[H, \rho] - \Phi(\mathbb{1})\rho + \Phi^\dagger(\rho),$$

[5] $F_j = -\beta_j^{-1}\log\mathrm{tr}(\mathrm{e}^{-\beta_j H})$ is the free energy of the system at inverse temperature β_j.

[6] Δ_ρ is the modular operator of ρ.

and since, by Relation (12) and **(DB)**, $\Phi^\dagger(\rho) = \Phi^\rho(\mathbb{1})\rho = \Phi(\mathbb{1})\rho$, the second identity in (15) follows.

From (15), it easily follows that the first two terms in (11) commute with Diag_ρ. It remains to show that Φ commutes with Diag_ρ as well. To this end, we first deduce from (14) that Φ commutes with the spectral projections of Δ_ρ. Now clearly $1 \in \mathrm{spec}(\Delta_\rho)$ with $\mathrm{Ker}(\Delta_\rho - \mathrm{Id}) = \{\rho\}'$. It follows from Remark 3.1 that Diag_ρ is the spectral projection of Δ_ρ to its eigenvalue 1 which ends the proof of (13).

Finally, the properties $\Phi = \Phi^\rho$ together with (15) show that under **(DB)**, the dissipator of $\mathscr{L}^\dagger$ is ρ-self-adjoint, which justifies Remark 4.2-(ii). □

Remark 4.3. More generally, one can consider the inner products on $\mathscr{O}$ defined by $\langle A|B\rangle_{\rho_s} := \mathrm{tr}(\rho^s A^* \rho^{1-s} B)$, where $s \in [0,1]$. A pair $(\rho, \mathscr{L})$ with $\mathrm{Ker}\,\mathscr{L} \ni \rho > 0$ satisfies the ρ_s-Detailed Balance Condition whenever the associated CP map Φ is self-adjoint w.r.t. this inner product. It appears that the conclusion of Lemma 4.1 holds true if the pair $(\rho, \mathscr{L})$ satisfies the ρ_s-Detailed Balance Condition for some $s \in [0,1] \setminus \{1/2\}$. Indeed, [CM17, Lemma 2.5] ensures $[\Phi, \Delta_\rho] = 0$, from which the first identity in (15) follows. One uses the expression

$$\Phi(\,\cdot\,) = \Phi^{\rho_s}(\,\cdot\,) = \rho^{s-1}\Phi^\dagger(\rho^{1-s}\,\cdot\,\rho^s)\rho^{-s}$$

acting on the unit $\mathbb{1}$ to deduce

$$\Phi^\dagger(\rho) = \rho^{1-s}\Phi(\mathbb{1})\rho^s = \rho\Delta_\rho^{-s}(\Phi(\mathbb{1})) = \rho\Phi(\Delta_\rho^{-s}(\mathbb{1})) = \rho\Phi(\mathbb{1}),$$

which implies $[H, \rho] = 0$ as above. The rest of the proof is identical. Thus, for $s \in [0,1] \setminus \{1/2\}$ the various notions of detailed balance are all equivalent for our purposes.

This is not true for the special case $s = 1/2$, which corresponds to the so called *KMS detailed balance condition,* as the following counter example stemming from [FU07, Example 38] shows.

The Hilbert space is $\mathscr{H} = \mathbb{C}^2$ and the dual Lindbladian

$$\mathscr{L}^\dagger(X) = \mathrm{i}[H, X] - \frac{1}{2}\{\Phi(\mathbb{1}), X\} + \Phi(X)$$

acts on 2×2 matrices. With σ_j, $j \in \{1,2,3\}$ the Pauli matrices, and $\sigma_0 = \mathbb{1}$, the example is defined by

$$H = \kappa\omega\sigma_1, \quad \Gamma = \sqrt{1-\kappa^2}\sigma_0 + \mathrm{i}r\sigma_1 + s\sigma_2 + \sigma_3, \quad \Phi(X) = \Gamma^* X \Gamma, \quad \Phi^\dagger(\rho) = \Gamma\rho\Gamma^*,$$

where, $\kappa \in (0,1)$, $\omega, s, r \in \mathbb{R}$. The relation with the notation in [FU07] is $\kappa = 2\sqrt{\nu(1-\nu)}$ and $\kappa\omega = \Omega$, with the restriction $\nu \in (0, 1/2)$ which ensures that $1 - 2\nu = \sqrt{1-\kappa^2}$. Setting

$$s = \omega\frac{1+\kappa}{1-\kappa}, \qquad r = \omega\sqrt{\frac{1+\kappa}{1-\kappa}},$$

ensures that

$$\rho = \frac{1}{2}\left(\sigma_0 - \sqrt{1-\kappa^2}\sigma_3\right) = \begin{pmatrix} \nu & 0 \\ 0 & 1-\nu \end{pmatrix}$$

is invariant

$$\mathscr{L}(\rho) = 0,$$

and that

$$\Delta_\rho^{1/2}\Gamma^* = \rho^{1/2}\Gamma^*\rho^{-1/2} = \Gamma.$$

The latter implies that Φ is $\rho_{1/2}$-self-adjoint

$$\Phi^{\rho_{1/2}} = \Phi$$

so that the KMS-detailed balance condition is satisfied, but one finds that

$$\mathrm{i}[H, \rho] = -\kappa\sqrt{1-\kappa^2}\sigma_1 \neq 0, \quad \text{and} \quad (\mathscr{L}\circ\mathrm{Diag}_\rho - \mathrm{Diag}_\rho\circ\mathscr{L})(\sigma_1) = -4\omega\frac{\kappa}{1-\kappa}\sigma_3 \neq 0.$$

5 Entropy production of MQD

The relative entropy of a state $\mu \in \mathscr{S}$ w.r.t. another state ν is defined by the expression

$$\mathrm{Ent}(\mu|\nu) = \begin{cases} \langle \mu | \log(\mu) - \log(\nu) \rangle, & \text{if } \mathrm{Ker}(\nu) \subset \mathrm{Ker}(\mu); \\ +\infty, & \text{otherwise,} \end{cases}$$

which is the immediate extension of the relative entropy of two probability measures to the non-commutative setting of quantum mechanics. It satisfies $\mathrm{Ent}(\mu|\nu) \geq 0$, with equality iff $\mu = \nu$, as well as Uhlmann's monotonicity theorem

$$\mathrm{Ent}(\phi(\mu)|\phi(\nu)) \leq \mathrm{Ent}(\mu|\nu)$$

for any CPTP map ϕ.

The entropy production (EP for short[7]) of the MQD $(\mathrm{e}^{t\mathscr{L}})_{t\geq 0}$ in the state ρ was defined in [SL78] as

$$\mathrm{EP}(\rho) = \sum_{j\in\mathscr{J}} \mathrm{EP}_j(\rho), \qquad \mathrm{EP}_j(\rho) = -\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ent}(\mathrm{e}^{t\mathscr{L}_j}(\rho)|\rho_j^+)\Big|_{t=0}, \tag{16}$$

where each $\mathrm{EP}_j(\rho)$ represents the entropy production due to the interaction of the system with the j^{th} reservoir. Note that since $\rho_j^+ = \mathrm{e}^{t\mathscr{L}_j}(\rho_j^+)$, it follows from Uhlmann's theorem that the function

$$t \mapsto \mathrm{Ent}(\mathrm{e}^{t\mathscr{L}_j}(\rho)|\rho_j^+)$$

is monotone decreasing, so that $\mathrm{EP}_j(\rho) \geq 0$. As proven in [Spo78, SL78], the map

$$\mathscr{S} \ni \rho \mapsto \mathrm{EP}_j(\rho) \in [0,\infty]$$

is convex. It is given by

$$\mathrm{EP}_j(\rho) = \langle \mathscr{L}_j(\rho) | \log(\rho_j^+) - \log(\rho) \rangle, \tag{17}$$

where the first term $\langle \mathscr{L}_j(\rho)|\log(\rho_j^+)\rangle$ is finite since $\rho_j^+ > 0$. Whenever $0 \in \mathrm{spec}(\rho)$, the second term should be computed in the eigenbasis $\{\phi_r\}$ of ρ,

$$\langle \mathscr{L}_j(\rho)|\log(\rho)\rangle = \sum_r (\phi_r, \mathscr{L}(\rho)\phi_r)\log(r)$$

with the convention[8]

$$a\log 0 = \begin{cases} 0 & \text{if } a = 0; \\ -\infty & \text{otherwise.} \end{cases}$$

Considering the von Neumann entropy

$$S(\rho) = -\langle \rho | \log(\rho) \rangle \geq 0,$$

of a state $\rho \in \mathscr{S}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} S(\mathrm{e}^{t\mathscr{L}}(\rho)) = -\langle \mathscr{L}(\mathrm{e}^{t\mathscr{L}}(\rho))|\log(\mathrm{e}^{t\mathscr{L}}(\rho))\rangle = -\sum_{j\in\mathscr{J}} \langle \mathscr{L}_j(\mathrm{e}^{t\mathscr{L}}(\rho))|\log(\mathrm{e}^{t\mathscr{L}}(\rho))\rangle$$

[7] It should more accurately be called entropy production *rate*.

[8] Note that whenever $\phi \in \mathrm{Ker}\,\rho$, one must have $(\phi, \mathscr{L}(\rho)\phi) \geq 0$.

with the same convention. It immediately follows from (17) that

$$\frac{\mathrm{d}}{\mathrm{d}t}S(\mathrm{e}^{t\mathscr{L}}(\rho)) = \mathrm{EP}(\mathrm{e}^{t\mathscr{L}}(\rho)) - \sum_{j\in\mathscr{J}}\langle\mathscr{L}_j(\mathrm{e}^{t\mathscr{L}}(\rho))|\log(\rho_j^+)\rangle, \tag{18}$$

To interpret this relation, we express the j^{th}-term of the sum in its right-hand side using the dual maps $\mathscr{L}_j^\dagger \in \mathscr{B}(\mathscr{O})$ and the observable

$$S_j^+ = -\log(\rho_j^+) \tag{19}$$

representing the entropy of the j^{th}-reservoir. By duality,

$$-\langle\mathscr{L}_j(\mathrm{e}^{t\mathscr{L}}(\rho))|\log(\rho_j^+)\rangle = \langle\mathscr{L}_j(\mathrm{e}^{t\mathscr{L}}(\rho))|S_j^+\rangle = \langle\mathrm{e}^{t\mathscr{L}}(\rho)|\mathscr{L}_j^\dagger(S_j^+))\rangle = \langle\mathrm{e}^{t\mathscr{L}}(\rho)|\mathscr{I}_j^+\rangle,$$

where $\mathscr{I}_j^+ = \mathscr{L}_j^\dagger(S_j^+) = \partial_t \mathrm{e}^{t\mathscr{L}_j^\dagger}(S_j^+)|_{t=0}$ is the entropy flux observable out of this reservoir. Thus, the identity (18) becomes the entropy balance relation

$$\frac{\mathrm{d}}{\mathrm{d}t}S(\mathrm{e}^{t\mathscr{L}}(\rho)) = \mathrm{EP}(\mathrm{e}^{t\mathscr{L}}(\rho)) + \sum_{j\in\mathscr{J}}\langle\mathrm{e}^{t\mathscr{L}}(\rho)|\mathscr{I}_j^+\rangle, \tag{20}$$

which expresses the rate of change in the entropy of the system as the sum of the entropy production rate and the total entropy flux out of the reservoirs.

In the special case of a steady state $\rho^+ \in \mathscr{S}\cap\mathrm{Ker}\,\mathscr{L}$, the entropy is constant and the entropy balance reduces to

$$\mathrm{EP}(\rho^+) = -\sum_{j\in\mathscr{J}}\langle\rho^+|\mathscr{I}_j^+\rangle. \tag{21}$$

Finally, in case the decomposition (2) is trivial, Formula (17) becomes

$$0 \le \mathrm{EP}(\rho) = \langle\mathscr{L}(\rho)|\log(\rho^+) - \log(\rho)\rangle. \tag{22}$$

The entropy observable is $S^+ = -\log(\rho^+)$, the entropy flux observable is $\mathscr{I}^+ = \mathscr{L}^\dagger(S^+)$. Note that $\mathrm{EP}(\rho^+) = 0 = \langle\rho^+|\mathscr{I}^+\rangle$ in this case.

As noted above, it may happen that the time derivative of $S(\mathrm{e}^{t\mathscr{L}}(\rho))$ and hence $\mathrm{EP}(\mathrm{e}^{t\mathscr{L}}(\rho))$ become infinite, even for faithful ρ. The following proposition gives sufficient conditions that exclude this behavior.

Proposition 5.1. *Assume the Lindbladian $\mathscr{L}$ to be relaxing to a faithful state ρ^+.*

(i) For all faithful $\rho\in\mathscr{S}$ and all $t\ge 0$, $\partial_t S(\mathrm{e}^{t\mathscr{L}}(\rho)) < \infty$.

(ii) For all $\rho\in\mathscr{S}$ there exists $T(\rho) > 0$ such that $\partial_t S(\mathrm{e}^{t\mathscr{L}}(\rho)) < \infty$ for all $t > T(\rho)$.

6 Link between EP and 2TMP

We can now establish the link between the Lebowitz-Spohn EP defined by (16) and the 2TMP of the entropic observable (19), under DB conditions on the sub-Lindbladians $\mathscr{L}_j$. We start with the simplest case where the decomposition (2) is trivial, *i.e.,* there is only one reservoir.

Lemma 6.1. *Suppose that* **(DB)** *holds for the pair* $(\rho^+, \mathscr{L})$ *and set* $S^+ = -\log\rho^+$. *For any* $\rho_0 \in \mathscr{S}$, *one has*

$$\mathbb{E}^t_{\rho_0}(\Delta S^+) = \langle e^{t\mathscr{L}}(\rho_0)|S^+\rangle - \langle\rho_0|S^+\rangle.$$

Thus, there is no decoherence effect of the first measurement and the expected 2TMP is the difference of the QM expectation value of S^+ *in the state* $e^{t\mathscr{L}}(\rho_0)$ *at time* t *and in the initial state* ρ_0. *In particular*

$$\mathbb{E}^t_{\rho_0}(\Delta S^+) = \int_0^t \langle e^{s\mathscr{L}}(\rho_0)|\mathscr{I}^+\rangle ds = S(e^{t\mathscr{L}}(\rho_0)) - S(\rho_0) - \int_0^t \mathrm{EP}(e^{s\mathscr{L}}(\rho_0))ds, \tag{23}$$

where $\mathscr{I}^+ = \mathscr{L}^\dagger(S^+)$.

Proof. The expression (8) and Lemma 4.1 imply

$$\begin{aligned}\mathbb{E}^t_{\rho_0}(\Delta S^+) &= \langle e^{t\mathscr{L}}(\mathrm{Diag}_{S^+}(\rho_0))|S^+\rangle - \langle\rho_0|S^+\rangle\\ &= \langle \mathrm{Diag}_{S^+}(e^{t\mathscr{L}}(\rho_0))|S^+\rangle - \langle\rho_0|S^+\rangle\\ &= \langle e^{t\mathscr{L}}(\rho_0)|\mathrm{Diag}_{S^+}(S^+)\rangle - \langle\rho_0|S^+\rangle\\ &= \langle e^{t\mathscr{L}}(\rho_0)|S^+\rangle - \langle\rho_0|S^+\rangle.\end{aligned} \tag{24}$$

Applying the fundamental theorem of calculus to the right-hand side of the last identity, we get

$$\mathbb{E}^t_{\rho_0}(\Delta S^+) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\langle e^{s\mathscr{L}}(\rho_0)|S^+\rangle ds = \int_0^t \langle \mathscr{L}(e^{s\mathscr{L}}(\rho_0))|S^+\rangle ds = \int_0^t \langle e^{s\mathscr{L}}(\rho_0)|\mathscr{L}^\dagger(S^+)\rangle ds.$$

The last equality in (23) now follows from the entropy balance relation (20). □

Remark 6.2. (i) By differentiation, (23) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}^t_{\rho_0}(\Delta S^+) = -\mathrm{EP}(e^{t\mathscr{L}}(\rho_0)) + \frac{\mathrm{d}}{\mathrm{d}t}S(e^{t\mathscr{L}}(\rho_0)).$$

(ii) Further, assuming $\lim_{t\to\infty} e^{t\mathscr{L}}(\rho_0) = \rho^+$, we have $\int_0^\infty \mathrm{EP}(e^{s\mathscr{L}}(\rho_0))ds = \mathrm{Ent}(\rho_0|\rho^+)$ and

$$\lim_{t\to\infty}\mathbb{E}^t_{\rho_0}(\Delta S^+) = S(\rho^+) - S(\rho_0) - \int_0^\infty \mathrm{EP}(e^{s\mathscr{L}}(\rho_0))ds = \langle\rho^+ - \rho_0|S^+\rangle.$$

Thanks to $\mathrm{EP}(\rho) \geq 0$, we deduce immediately the following upper bounds on the 2TMP expectation of the entropic variable ΔS^+ and its time derivative in terms of the variation of the von Neumann entropy and its derivative.

Corollary 6.3. *Under the assumptions of Lemma 6.1, the 2TMP expectation of the entropic observable* $S^+ = -\log\rho^+$ *in the state* ρ_0 *satisfies*

$$S(e^{t\mathscr{L}}(\rho)) - S(\rho) \geq \mathbb{E}^t_{\rho_0}(\Delta S^+).$$

Turning to the general multi-reservoir case, we apply the two-time measurement protocol to the MQD $(e^{t\mathscr{L}_j})_{t\geq 0}$ introduced by **(RS)** and to the entropy observables S^+_j defined in (19). In this context, for $j \in \mathscr{J}$, we set

$$\mathbb{Q}^t_{j,\rho_0}(\Delta S^+_j = \sigma) = \sum_{\substack{s,s'\in\mathrm{spec}(S_j)\\ s'-s=\sigma}} \langle e^{t\mathscr{L}_j}(P_{j,s}\rho_0 P_{j,s})|P_{j,s'}\rangle, \qquad S^+_j = \sum_{s\in\mathrm{spec}(S^+_j)} sP_{j,s} \tag{25}$$

and call the random variable ΔS^+_j the *two-time measurement entropy production* of the j^{th}-reservoir.

Theorem 6.4. *Suppose that the Lindbladian $\mathscr{L}$ satisfies Assumption* **(RS)** *and is such that, for all $j \in \mathscr{J}$, the pair $(\rho_j^+, \mathscr{L}_j)$ satisfies* **(DB)**. *Considering the entropic observables $S_j^+ = -\log(\rho_j^+)$ and $\mathscr{I}_j^+ = \mathscr{L}_j^\dagger(S_j^+)$, for any $j \in \mathscr{J}$, $\rho_0 \in \mathscr{S}$ and $t \geq 0$, one has*

$$\mathbb{E}^t_{j,\rho_0}(\Delta S_j^+) = \int_0^t \langle \mathrm{e}^{s\mathscr{L}_j}(\rho_0)|\mathscr{I}_j^+\rangle \mathrm{d}s = S(\mathrm{e}^{t\mathscr{L}_j}(\rho_0)) - S(\rho_0) - \int_0^t \mathrm{EP}_j(\mathrm{e}^{s\mathscr{L}_j}(\rho_0))\mathrm{d}s, \tag{26}$$

where $\mathbb{E}^t_{j,\rho_0}$ denotes the expectation w.r.t. the law (25). *In particular, if $\rho^+ \in \mathrm{Ker}\,\mathscr{L}$, then*

$$\mathrm{EP}(\rho^+) = -\lim_{t\downarrow 0} \sum_{j\in\mathscr{J}} \mathbb{E}^t_{j,\rho^+}\left(\frac{\Delta S_j^+}{t}\right). \tag{27}$$

Proof. Formula (26) follows from Lemma 6.1, and more precisely Relation (23), applied to the pair $(\rho_j^+, \mathscr{L}_j)$. Differentiating the first equality in (26) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{j\in\mathscr{J}} \mathbb{E}^t_{j,\rho^+}(\Delta S_j^+)\Big|_{t=0} = \sum_{j\in\mathscr{J}} \langle \rho^+|\mathscr{I}_j^+\rangle,$$

and (27) follows from (21). □

Remark 6.5. (i) The differential version of (26) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}^t_{j,\rho_0}(\Delta S_j^+) = \frac{\mathrm{d}}{\mathrm{d}t} S(\mathrm{e}^{t\mathscr{L}_j}(\rho_0)) - \mathrm{EP}_j(\mathrm{e}^{t\mathscr{L}_j}(\rho_0)).$$

(ii) For $j \in \mathscr{J}$, one has

$$\mathrm{EP}_j(\mathrm{e}^{t\mathscr{L}_j}(\rho_0)) = -\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ent}(\mathrm{e}^{t\mathscr{L}_j}(\rho_0)|\rho_j^+),$$

so that

$$\sum_{j\in\mathscr{J}} \int_0^\infty \mathrm{EP}_j(\mathrm{e}^{s\mathscr{L}_j}(\rho_0))\mathrm{d}s = \sum_{j\in\mathscr{J}} \mathrm{Ent}(\rho_0|\rho_j^+)$$

is finite, as opposed to the total entropy production

$$\sum_{j\in\mathscr{J}} \int_0^t \mathrm{EP}_j(\mathrm{e}^{s\mathscr{L}}(\rho_0))\mathrm{d}s = t\,\mathrm{EP}(\rho^+) + o(t), \quad (t\to\infty)$$

which diverges if $\mathrm{EP}(\rho^+) > 0$.

(iii) We note that also

$$\lim_{t\to\infty} \sum_{j\in\mathscr{J}} \mathbb{E}^t_{j,\rho_0}(\Delta S_j^+) = \sum_{j\in\mathscr{J}} \langle \rho_j^+ - \rho_0|S_j^+\rangle$$

is finite

(iv) Under the additional Assumption **(KMS)**, one has $S_j^+ = \beta_j(H - F_j)$ and hence $\mathscr{I}_j^+ = \beta_j \mathscr{Q}_j^+$, where $\mathscr{Q}_j^+ = \mathscr{L}_j^\dagger(H)$ is the observable describing the heat current out of the j^{th} reservoir. We then have

$$\mathbb{E}^t_{j,\rho_0}(\Delta S_j^+) = \beta_j \int_0^t \langle \mathrm{e}^{s\mathscr{L}_j}(\rho_0)|\mathscr{Q}_j^+\rangle \mathrm{d}s.$$

7 A Classical Markov Chain

In this section, we further elaborate on the 2TMP under the DB condition, showing that a classical continuous time Markov chain can be naturally associated to the two-time measurement process.

We first recall well known facts about homogeneous continuous time Markov chains, mainly to set the notation. See, e.g., [Nor97] for more details. Such a process $(X_t)_{t\geq 0}$ defined on a finite state space Σ is completely characterized by the initial probabilities of each state $s \in \Sigma$

$$\mathbb{P}(X_0 = s) = \pi_s(0),$$

and by the probability to find the process in the state $s' \in \Sigma$ at time $t \geq 0$, given its state $s \in \Sigma$ at time 0,

$$\mathbb{P}(X_t = s' | X_0 = s) = P_{ss'}(t).$$

Denoting by $\mathbf{1} \in \mathbb{R}^\Sigma$ the column vector whose entries are all set to 1, the initial probabilities can be seen as a dual/row vector $\pi(0) = (\pi_s(0)) \in \mathbb{R}^{\Sigma *}$ normalized by $\pi(0)\mathbf{1} = 1$, and the transition probabilities form a time dependent matrix acting on $\mathbb{R}^\Sigma$, $P(t) = (P_{ss'}(t))$. This matrix is stochastic, *i.e.,* such that $P(t)\mathbf{1} = \mathbf{1}$ for all $t \geq 0$, and the family $(P(t))_{t\geq 0}$ is a semi-group

$$P(t) = \mathrm{e}^{tQ},$$

generated by the so-called transition rate matrix $Q = (Q_{ss'})$ which satisfies $Q_{ss'} \geq 0$ for $s \neq s'$ and $Q\mathbf{1} = 0$. The time-t probability vector $\pi(t) = (\pi_s(t))$, with $\pi_s(t) = \mathbb{P}(X_t = s)$, is given by

$$\pi(t) = \pi(0)P(t).$$

Hence, an invariant probability vector π^{inv} is characterized by $\pi^{\mathrm{inv}}Q = 0$. Since $Q\mathbf{1} = 0$, the Perron-Frobenius theorem ensures the existence of an invariant probability.

Proposition 7.1. *Let $(\rho, \mathscr{L})$ satisfy* **(DB)** *and ρ_0 be a faithful initial state. Assume that the entropic observable $S = -\log\rho$ has simple spectrum, with spectral decomposition $S = \sum_s sP_s$. Then, there exists a continuous time Markov chain $(X_t)_{t\geq 0}$ on the state space $\Sigma = \mathrm{spec}(S)$ such that the pair of random variables (X_0, X_t) has the same statistics as the outcome of the two measurements of S at time 0 and t. The chain $(X_t)_{t\geq 0}$ has the initial probability vector*

$$\pi_s(0) = \langle \rho_0 | P_s \rangle, \tag{28}$$

and transition matrix

$$P_{ss'}(t) = \mathbb{P}_{\rho_0}(S_t = s' | S_0 = s) = \langle \mathrm{e}^{t\mathscr{L}}(P_s) | P_{s'} \rangle, \tag{29}$$

where $\mathbb{P}^{\rho_0}$ is defined in (4). The corresponding transition rate matrix Q is given by

$$Q_{ss'} = \langle P_s | \Phi(P_{s'}) \rangle - \delta_{ss'} \langle P_s | \Phi(\mathbb{I}) \rangle \tag{30}$$

where Φ is the CP map associated to $\mathscr{L}$.

Remark 7.2. (i) The conditional probability (4) takes the ρ_0-independent form (29) since, for rank one spectral projections, $P_s\rho_0 P_s = P_s\,\mathrm{tr}(P_s\rho_0 P_s)$ for all s. In this case, the choice of initial state ρ_0 only manifests itself in the initial probabilities (28).

(ii) Further, assuming the MQD generated by $\mathscr{L}$ to be relaxing, we deduce from Relation (10) that

$$\lim_{t\to\infty} \mathbb{P}_{\rho_0}(S_t = s' | S_0 = s) = \langle \rho | P_{s'} \rangle = \langle \mathrm{e}^{-S} | P_{s'} \rangle = \mathrm{e}^{-s'}.$$

Hence, the invariant probability vector π^{inv} of the Markov chain satisfies $\pi^{\mathrm{inv}}_s = \mathrm{e}^{-s}$. It corresponds to initial states ρ_0 such that $\mathrm{Diag}_\rho(\rho_0) = \rho$.

(iii) From Definition (30) we derive, starting from $\mathrm{e}^{-s}P_s = \rho P_s$ and invoking the DB condition $\Phi^\rho = \Phi$,

$$\begin{aligned}\mathrm{e}^{-s}Q_{ss'} &= \langle \rho P_s | \Phi(P_{s'})\rangle - \delta_{ss'}\langle \rho P_s|\Phi(\mathbb{I})\rangle \\ &= \langle P_s|\Phi(P_{s'})\rangle_\rho - \delta_{ss'}\langle P_s|\Phi(\mathbb{I})\rangle_\rho \\ &= \langle \Phi(P_s)|P_{s'}\rangle_\rho - \delta_{ss'}\langle P_{s'}|\Phi(\mathbb{I})\rangle_\rho \\ &= \langle \rho P_{s'}|\Phi(P_s)\rangle - \delta_{ss'}\langle \rho P_{s'}|\Phi(\mathbb{I})\rangle = \mathrm{e}^{-s'}Q_{s's},\end{aligned}$$

which is the classical detailed balance condition for a Markov chain. In matrix form, it reads

$$RQ = Q^T R, \tag{31}$$

where $R_{ss'} = \delta_{ss'}\mathrm{e}^{-s}$. Hence, $P(t) = \mathrm{e}^{tQ}$ has the same property:

$$RP(t)R^{-1} = \mathrm{e}^{tRQR^{-1}} = \mathrm{e}^{tQ^T} = P(t)^T. \tag{32}$$

Note that (31) implies that Q is self-adjoint with respect to the inner product on $\mathbb{R}^\Sigma$ defined by R, so that, in particular, $\mathrm{spec}(Q) \subset (-\infty, 0]$.

(iv) Since under the DB condition, $[\rho, H] = 0$ and by assumption the spectrum of ρ is simple, the transition matrix (29) is identical to that of the two-time measurement protocol of the energy observable, in case the spectrum of H is simple as well.

Proof. The first statement (28) reformulates Relation (3). Considering (4) and Remark 7.2-(i), it remains to show that the RHS of Relation (29), *i.e.,* Formula $P_{ss'}(t) = \langle \mathrm{e}^{t\mathscr{L}}(P_s)|P_{s'}\rangle$ defines a semi-group $(P(t))_{t\geq 0}$ of stochastic matrices generated by a matrix Q satisfying Relation (30). We first remark that the stochasticity follows directly from the first equality in (29). Differentiating Relation (29), we get

$$\dot{P}_{ss'}(t) = \langle \mathscr{L}(\mathrm{e}^{t\mathscr{L}}(P_s))|P_{s'}\rangle = \langle \mathrm{e}^{t\mathscr{L}}(P_s)|\mathscr{L}^\dagger(P_{s'})\rangle \tag{33}$$

The simplicity of $\mathrm{spec}(S)$ gives that $\mathrm{Ran}(\mathrm{Diag}_S) = \mathrm{span}\{P_s \mid s \in \Sigma\}$. By Remark 3.1-(ii), it follows from **(DB)** that

$$\mathscr{L}^\dagger(P_{s'}) = \mathscr{L}^\dagger \circ \mathrm{Diag}_S(P_{s'}) = \mathrm{Diag}_S \circ \mathscr{L}^\dagger(P_{s'}) = \sum_{s''} L_{s''s'}P_{s''}$$

for some matrix L. Hence, Relation (33) becomes $\dot{P}(t) = P(t)L$, and since it immediately follows from (29) that $P(0) = I$, we conclude that $P(t) = \mathrm{e}^{tL}$. It remains to identify L with the RHS of (30). Since $\langle P_{s'}|P_s\rangle = \delta_{ss'}$, we deduce, using (11),

$$L_{s''s'} = \langle P_{s''}|\mathscr{L}^\dagger(P_{s'})\rangle = \langle P_{s''}|\mathrm{i}[H, P_{s'}] - \tfrac{1}{2}\{\Phi(\mathbb{I}), P_{s'}\} + \Phi(P_{s'})\rangle.$$

From the proof of Lemma 4.1, we know that $[H, P_{s'}] = [\Phi(\mathbb{I}), P_s] = 0$, so that, invoking the cyclicity of the trace,

$$L_{s''s'} = \langle P_{s''}| - \Phi(\mathbb{I})P_{s'} + \Phi(P_{s'})\rangle = -\delta_{s's''}\langle P_{s'}|\Phi(\mathbb{I})\rangle + \langle P_{s''}|\Phi(P_{s'})\rangle = Q_{s''s'}.$$

□

8 2TMP Properties Inherited from Markov Processes

We take advantage here of the representation of outcomes of quantum measurement processes of the observable $S^+ = -\log\rho$ as a classical Markov process to get further insight on the 2TMP distribution $\mathbb{Q}^t_{\rho_0}$ of that observable, in case ρ is a steady state of the Lindbladian $\mathscr{L}$, under the DB condition.

In this section we work under assumptions ensuring the validity of Proposition 7.1, namely: the Lindblad operator $\mathscr{L}$ admits a faithful steady state ρ such that the pair $(\rho,\mathscr{L})$ satisfies the **(DB)** condition. The entropic observable $S = -\log\rho$ is assumed to have simple spectrum Σ and the spectral decomposition $S = \sum_{s\in\Sigma} sP_s$.

We will further assume the following genericity hypothesis on S:

(GenS) The numbers $s'-s$ where $s,s'\in\Sigma$ and $s\neq s'$ are all distinct.

Under this set of hypotheses, for any faithful initial state ρ_0, the 2TMP law $\mathbb{Q}^{\rho_0}_t$ satisfies

$$\mathbb{Q}^t_{\rho_0}(\Delta S = s'-s) = \begin{cases} \langle\rho_0|P_s\rangle\langle \mathrm{e}^{t\mathscr{L}}(P_s)|P_{s'}\rangle & \text{if } s'\neq s;\\ \sum_{s\in\Sigma}\langle\rho_0|P_s\rangle\langle \mathrm{e}^{t\mathscr{L}}(P_s)|P_s\rangle & \text{otherwise,}\end{cases}$$

since any non-vanishing variation ΔS, corresponds to a unique pair $(s',s)\in\Sigma^2$, thanks to Condition **(GenS)**, while a zero variation implies $s'=s$.

Hence, we have the following consequence of the classical detailed balance condition (32): for all $s,s'\in\Sigma$

$$\mathrm{e}^{-s}\mathbb{Q}^t_{\rho_0}(\Delta S = s'-s)\langle\rho_0|P_{s'}\rangle = \mathrm{e}^{-s'}\mathbb{Q}^t_{\rho_0}(\Delta S = s-s')\langle\rho_0|P_s\rangle.$$

Equivalently, the ratio of the 2TMP probability to measure $s'-s$ and that to measure $-(s'-s)$, for $s'\neq s$, only depends on $\mathrm{e}^{-(s'-s)}$ and on the ratio of the probabilities of outcomes of measures of S in the initial state $\rho_0>0$, for all $t>0$:

$$\frac{\mathbb{Q}^t_{\rho_0}(\Delta S = s'-s)}{\mathbb{Q}^t_{\rho_0}(\Delta S = -(s'-s))} = \mathrm{e}^{-(s'-s)}\,\frac{\langle\rho_0|P_s\rangle}{\langle\rho_0|P_{s'}\rangle}.$$

Choosing the initial state $\rho_0>0$ so that $\mathrm{Diag}_S\rho_0=\rho$ we get, for any $\sigma\neq 0$,

$$\mathbb{Q}^t_{\rho_0}(\Delta S=-\sigma) = \mathbb{Q}^t_{\rho_0}(\Delta S=\sigma),$$

and in particular $\mathbb{E}^t_{\rho_0}(\Delta S)=0$.

9 Moment generating function of $\mathbb{Q}^t_{\rho_0}$

As an application of the previous Section, we consider the moment generating function of the 2TMP distribution $\mathbb{Q}^t_{\rho_0}$ for the entropic observable $S=-\log(\rho)$, where $\rho>0$ is a steady state of the MQD generated by $\mathscr{L}$, under the assumption **(DB)** on the pair $(\rho,\mathscr{L})$. Expressions for this moment generating function have been derived in the literature, see *e.g.* [JPW14]. The objective here is to express this moment generating function in terms of the quantities defining the classical Markov process attached to the 2TMP.

Proposition 9.1. *Let $\rho_0 > 0$ and assume* **(DB)** *holds for $(\rho, \mathscr{L})$, $S = -\log\rho$ having simple spectrum Σ and spectral decomposition $S = \sum_{s\in\Sigma} sP_s$. Denoting by $d_0 \in \mathbb{R}^{\Sigma *}$ the row vector $(\langle\rho_0|P_s\rangle)_{s\in\Sigma}$ and by R the matrix introduced in Remark 7.2-(iii), we have*

$$e^t_{\rho_0}(\alpha) = \mathbb{E}^t_{\rho_0}(\mathrm{e}^{\alpha\Delta S}) = d_0 R^\alpha \mathrm{e}^{tQ} R^{-\alpha}\mathbf{1}. \tag{34}$$

Proof. Combining Relation (9) with Lemma 4.1, the moment generating function can be expressed as

$$e^t_{\rho_0}(\alpha) = \langle \mathrm{e}^{t\mathscr{L}_\alpha}(\rho_0)|\mathbb{I}\rangle.$$

in terms of the deformed Lindbladian $\mathscr{L}_\alpha$ defined after (9). When $\dim P_s = 1$, for all $s \in \Sigma$, Formula (34) is shown by making use of $R^\alpha = \mathrm{Diag}(\mathrm{e}^{-\alpha s})$ and Proposition 7.1 to get

$$\begin{aligned} e^t_{\rho_0}(\alpha) &= \sum_{s,s'\in\Sigma} \mathrm{e}^{\alpha(s'-s)}\langle P_s|\rho_0\rangle P_{ss'}(t) \\ &= \sum_{s,s'\in\Sigma} \langle P_s|\rho_0\rangle \left(R^\alpha \mathrm{e}^{tQ} R^{-\alpha}\right)_{ss'} = d_0 R^\alpha \mathrm{e}^{tQ} R^{-\alpha}\mathbf{1}. \end{aligned}$$

□

Note that since $(\alpha, t) \mapsto e^t_{\rho_0}(\alpha)$ is analytic on $\mathbb{C}^2$, Lemma 6.1 can be rephrased as

$$\frac{\partial^2}{\partial t \partial\alpha} e^t_{\rho_0}(\alpha)\Big|_{(t,\alpha)=(0,0)} = \langle\rho_0|\mathscr{I}\rangle,$$

where, making use of (34), we have

$$\langle\rho_0|\mathscr{I}\rangle = -d_0 Q \log R\mathbf{1}.$$

The concrete expressions for $e^t_{\rho_0}(\alpha)$ provided in Proposition 9.1 allow for explicit computations of all moments of $\mathbb{Q}^{\rho_0}_t$, which we do not develop further. For more results about the moments generating function $e^t_{\rho_0}(\alpha)$ in case $\mathscr{L} = \sum_{j\in\mathscr{J}} \mathscr{L}_j$ we refer the reader to [JPW14].

10 Example

As an example, we consider the so called Quantum Reset Model (QRM), which is simple enough so that the quantities introduced above can be computed explicitly. The QRM is an effective Lindbladian evolution equation arising in different guises, which is of interest in the study of so-called entanglement machines and yet simple enough to allow for a mathematical analysis. Key properties of certain QRMs are studied in [HJ21, HJ24], including entropy production. We refer to these papers for more details and consider the simplest of their versions that corresponds to the present setup.

On a d-dimensional Hilbert space $\mathscr{H}$, the generator of the QRM is the Lindbladian $\mathscr{L}$ defined by

$$\mathscr{L}(\rho) = -\mathrm{i}[H, \rho] + \Gamma(T\,\mathrm{tr}(\rho) - \rho), \tag{35}$$

where $\Gamma > 0$ and $T \in \mathscr{S}$. The Hamiltonian part of the generator, $H = H^* \in \mathscr{O}$, is arbitrary so far, and we denote its repeated eigenvalues by $e_1, e_2, \ldots, e_d$. For simplicity, we further make a genericity hypothesis on the spectrum of $\mathrm{ad}_H(\cdot) = [H, \cdot]$ similar to **(GenS)**:

(Bohr) The Bohr spectrum $\mathrm{spec}(\mathrm{ad}_H) \setminus \{0\}$ is simple.

We recall some properties of the generator (35) of use to us, referring the reader to [HJ21, HJ24] for details.

Under **(Bohr)**, the spectrum of the QRM Lindbladian is

$$\mathrm{spec}(\mathscr{L}) = \{0\} \cup \{-\Gamma - \mathrm{i}\alpha \mid \alpha \in \mathrm{spec}(\mathrm{ad}_H)\},$$

where 0 is simple with eigenspace spanned by

$$\rho^+ = \Gamma\big(\mathrm{i}\,\mathrm{ad}_H + \Gamma\big)^{-1}(T) \in \mathscr{S}, \tag{36}$$

$-\Gamma$ has the $d-1$ dimensional eigenspace spanned by the traceless elements of the commutant $\{H\}' = \mathrm{Ker}(\mathrm{ad}_H)$, and the $d(d-1)$ distinct eigenvalues with non-zero imaginary parts $-\Gamma + \mathrm{i}\alpha$, $\alpha \in \mathrm{spec}(\mathrm{ad}_H) \setminus \{0\}$, appear as complex conjugate pairs. Moreover, one has the explicit expression,

$$\mathrm{e}^{t\mathscr{L}}(\rho_0) = \langle \rho_0 | \mathbb{I} \rangle \rho^+ + \mathrm{e}^{-t\Gamma}\mathrm{e}^{-\mathrm{i}tH}\big(\rho_0 - \langle \rho_0 | \mathbb{I} \rangle \rho^+\big)\mathrm{e}^{\mathrm{i}tH}, \tag{37}$$

for any $\rho_0 \in \mathscr{T}$. In particular, the MQD generated by $\mathscr{L}$ is relaxing with asymptotic state ρ^+. It is a simple matter to check that the CP map associated to $\mathscr{L}$ is given by

$$\Phi(X) = \Gamma \langle T | X \rangle \mathbb{I}.$$

The following properties are proven in [HJ24]:

Lemma 10.1. *For any $H = H^*$, and $T \in \mathscr{T}$, the linear map $\rho_H : \mathscr{T} \to \mathscr{T}$*

$$T \mapsto \rho_H(T) = \big(\mathrm{i}\,\mathrm{ad}_H + \Gamma\big)^{-1}(T),$$

see (36), is CPTP and such that $T > 0 \Longrightarrow \rho_H(T) > 0$.

For $T > 0$, the Lindbladian $\mathscr{L}$ defined by (35) and its asymptotic state ρ^+ given by (36) it holds:

*(i) The pair $(\rho^+, \mathscr{L})$ satisfies the detailed balance condition **(DB)** if and only if $[H, T] = 0$.*

*(ii) If condition **(DB)** holds, $\rho^+ = T$ and the EP (22) of the MQD generated by $\mathscr{L}$ in the state ρ reads*

$$\mathrm{EP}(\rho) = \Gamma(\mathrm{Ent}(T|\rho) + \mathrm{Ent}(\rho|T))$$

for all faithful $\rho \in \mathscr{S}$, so that $\mathrm{EP}(\rho) = 0$ if and only if $\rho = T$.

Hence we assume from now on that $T > 0$ and that **(DB)** holds, so that for all $\rho_0 \in \mathscr{S}$,

$$\mathrm{e}^{t\mathscr{L}}(\rho_0) = \mathrm{e}^{-\Gamma t}\mathrm{e}^{-\mathrm{i}tH}\rho_0\mathrm{e}^{\mathrm{i}tH} + T(1 - \mathrm{e}^{-\Gamma t}),$$

which shows that $\mathrm{e}^{t\mathscr{L}}$ is positivity improving. In particular, under these assumptions, the entropic observable of the QRM model is $S = -\log T$, and for any $\rho_0 = \mathrm{Diag}_S(\rho_0) \in \mathscr{S}$ we have $[H, \rho_0] = 0$, so that

$$\mathrm{e}^{t\mathscr{L}}(\mathrm{Diag}_S(\rho_0)) = T + \mathrm{e}^{-t\Gamma}(\mathrm{Diag}_S(\rho_0) - T). \tag{38}$$

In particular, the matrix elements of $P(t)$ read with $S = \sum_s sP_s$

$$P_{ss'}(t) = \langle \mathrm{e}^{t\mathscr{L}}(P_s) | P_{s'} \rangle = \mathrm{e}^{-s'}(1 - \mathrm{e}^{-t\Gamma}) + \mathrm{e}^{-t\Gamma}\delta_{ss'}.$$

This yieds the spectral decomposition of $P(t)$ in term of the vectors $\mathbf{1}$ and $\pi^+ = (\mathrm{e}^{-s})$ s.t. $\pi^+\mathbf{1} = 1$, $(\mathbf{1}\pi^+)_{ss'} = \mathrm{e}^{-s'}$,

$$P(t) = \mathrm{e}^{-t\Gamma}(\mathbb{I} - \mathbf{1}\pi^+) + \mathbf{1}\pi^+,$$

and therefore

$$Q = -\Gamma(\mathbb{I} - \mathbf{1}\pi^+), \quad \text{so that} \quad Q_{ss'} = \Gamma(\mathrm{e}^{-s'} - \delta_{ss'}).$$

In turn, we obtain the sought for explicit 2TMP distribution $\mathbb{Q}^t_{\rho_0}$

Proposition 10.2. *Under the hypotheses of Section 8, in particular* **(GenS)**, *we have for the QRM model* (35), *for any state* $\rho_0 > 0$ *and* $S = -\log T$,

$$\mathbb{Q}^t_{\rho_0}(\Delta S = \sigma) = \begin{cases} (1-\mathrm{e}^{-t\Gamma})\langle \rho_0 | P_s\rangle \mathrm{e}^{-s'} & \text{if } \sigma = s'-s \neq 0; \\ (1-\mathrm{e}^{-t\Gamma})\langle \rho_0 | T\rangle + \mathrm{e}^{-t\Gamma} & \text{otherwise.} \end{cases}$$

Remark 10.3. (i) Without assuming **(GenS)**, one has

$$\mathbb{E}^t_{\rho_0}(\Delta S) = (1-\mathrm{e}^{-t\Gamma})\langle T - \mathrm{Diag}_S(\rho_0)|S\rangle,$$
$$e^t_{\rho_0}(\alpha) = \langle \rho_0 | T^\alpha\rangle\langle T^{1-\alpha}|\mathbb{I}\rangle(1-\mathrm{e}^{-t\Gamma}) + \mathrm{e}^{-t\Gamma}.$$

The first formula is a straightforward consequence of (24) and (38), whereas the second one stems from (37).

(ii) In case $\mathrm{Diag}_S(\rho_0) = T$, we recover from the previous formula that $\mathbb{E}_{\mathbb{Q}^{\rho_0}_t}(\Delta S) = 0$, in accordance with (24).

Finally, to tackle the many-reservoir setup characterized by different dissipators, within the context of QRM , we consider the following construction, see [HJ24].

Let $\mathscr{J}$ be a finite set of indices, for $j \in \mathscr{J}$ let $\lambda_j \in \mathbb{R}$ be such that $\sum_{j\in\mathscr{J}} \lambda_j = 1$, and set

$$\mathscr{L}_j(\rho) = -\mathrm{i}[\lambda_j H, \rho] + \Gamma_j(T_j\,\mathrm{tr}(\rho) - \rho),$$

where the dissipator is characterized by $T_j \in \mathscr{S}$ and the coupling rates $\Gamma_j > 0$. Then, the full Lindbladian $\mathscr{L} = \sum_{j\in\mathscr{J}} \mathscr{L}_j$ is given by

$$\mathscr{L}(\rho) = -\mathrm{i}[H,\rho] + \Gamma(T\,\mathrm{tr}(\rho) - \rho)$$

where

$$\Gamma = \sum_{j\in\mathscr{J}} \Gamma_j > 0, \qquad T = \frac{1}{\Gamma}\sum_{j\in\mathscr{J}} \Gamma_j T_j \in \mathscr{S},$$

and reduces to (35). Thus, assuming $[H, T_j] = 0$ for all $j \in \mathscr{J}$, **(DB)** holds for $(T_j, \mathscr{L}_j)$ and $(T, \mathscr{L})$. Thus, Theorem 6.4 applies to yield with $S^+_j = -\log(T_j)$ and Proposition 10.2

$$\mathrm{EP}(T) = -\sum_{j\in\mathscr{J}} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}^t_{j,T}(\Delta S^+_j)\Big|_{t=0} = \sum_{j\in\mathscr{J}} \Gamma_j(\mathrm{Ent}(T|T_j) + \mathrm{Ent}(T_j|T)).$$

Similarly, by Relation (26)

$$\sum_{j\in\mathscr{J}}\int_0^\infty \mathrm{EP}_j(\mathrm{e}^{s\mathscr{L}_j}(T))\mathrm{d}s = -\sum_{j\in\mathscr{J}} \mathbb{E}_{\mathbb{Q}^T_{\infty,j}}(\Delta S^+_j) + \sum_{j\in\mathscr{J}}(S(T_j) - S(T)) = \sum_{j\in\mathscr{J}} \mathrm{Ent}(T|T_j).$$

11 Proof of Proposition 5.1

Assume that the MQD generated by $\mathscr{L}$ is relaxing, the initial state ρ and the asymptotic state ρ^+ being both faithful. In our finite dimensional context, the map $\mathbb{C} \ni t \mapsto \rho(t) = \mathrm{e}^{t\mathscr{L}}(\rho)$ is entire, taking its values in the self-adjoint elements of $\mathscr{S}$ for $t \geq 0$. In particular, the spectral decomposition

$$\rho(t) = \sum_{h=1}^m p_h(t) P_h(t),$$

is such that the eigenvalues $p_h(t)$ and eigenprojections $P_h(t)$ are real-analytic functions of t, even at exceptional points where some eigenvalues coincide [Kat66, Chapter 2, Theorem 1.10]. Expressing the von Neumann entropy in terms of the function $[0,1] \ni p \mapsto \chi(p) = -p\log(p)$,[9]

$$S(\rho(t)) = \sum_{h=1}^{m} g_h \chi(p_h(t)), \qquad g_h = \operatorname{tr} P_h(t), \tag{39}$$

with the usual convention that $\chi(0) = 0$ and $\chi'(0^+) = +\infty$, we have that $S(\rho(t))$ is finite for all $t \geq 0$, but its time-derivative might diverge whenever some eigenvalues $p_h(t)$ vanish. We show that this does not happen.

First note that for small enough $t \geq 0$, $\rho(t) > 0$ by continuity, whereas the relaxing assumption ensures that $\rho(t) > 0$ for large enough t. Thus, we can restrict our attention to $t \in (-\delta, 1/\delta)$, for some $\delta > 0$ small enough. Let $t_0 \in (-\delta, 1/\delta)$ be such that some eigenvalue $p(t)$ vanishes at $t = t_0$, (dropping the index from the notation). We can focus on the corresponding contribution of that eigenvalue to (39). The non-negativity of $\rho(t)$ implies that in a neighborhood of t_0 the analytic function $p(t)$ factorizes as

$$p(t_0 + \tau) = \tau^{2p} r(\tau), \tag{40}$$

where $p \in \mathbb{N}^*$ and the function r is analytic near 0 and such that $a = r(0) > 0$. It follows that

$$\chi(p(t_0 + \tau)) = -ap\tau^{2p}(\log \tau^2 + O(1)), \quad (\tau \to 0),$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi(p(t))\bigg|_{t=t_0} = -\lim_{\tau \to 0} ap\tau^{2p-1}(\log \tau^2 + O(1)) = 0. \tag{41}$$

Altogether, we showed that the function $t \mapsto S(\rho(t))$ is of class C^1 on $(0,\infty)$. Since $\rho(0) > 0$ by assumption, it has a finite right derivative at $t = 0$. This proves part (i). Part (ii) follows immediately since $\rho(t) > 0$ for large enough t.

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