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# On the connectedness of arithmetic hyperplanes

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**Abstract.** Discrete geometry is a geometry specific to computers that studies  $\mathbb{Z}^d$  structures. It appears naturally in image analysis or 3D printing. Our goal is to find efficient algorithms to characterise these geometric structures and their properties.

We are interested in a fundamental structure of discrete geometry, the arithmetic hyperplanes, and more particularly in their connectedness. Many works have studied a connectedness defined from the neighbourhood by faces and have allowed to observe a percolation phenomenon. These studies have also allowed to decide the connectedness of a plane in an efficient way. We propose an extension of these results in the case of connectivity defined from general neighbourhoods.

Beyond the new concepts that this extension requires, the main contribution of the paper lies in the use of analysis to solve this arithmetic problem and in the design of an algorithm that decides the general connectedness problem. The study of the thickness of connectedness as a function reveals discontinuities at each rational point. However, a much more regular underlying structure appears in the irrational case. Thus, the consideration of irrational vectors allows a simpler approach to the connectedness of arithmetic hyperplanes.

**Keywords:** discrete geometry, discrete hyperplane, algorithm, connectedness, connecting thickness, irrational, continuity.

## 1 Introduction

Discrete geometry studies the behaviour of various  $\mathbb{Z}^d$  structures via a geometric approach. For example, continuous  $\mathbb{R}^d$  objects can be discretised and then studied in the discrete  $\mathbb{Z}^d$  space, which fundamentally changes their properties. Lines and planes, and more generally hyperplanes and affine subspaces, are fundamental structures in both discrete and Euclidean geometries. Indeed, they allow to locally approach discrete or continuous varieties. The notion of discrete line was defined arithmetically in 1991 by J.-P. Réveillès in [8] and then extended to any finite dimension by E. Andres in [1].

We are interested in the structure of arithmetic hyperplanes, and more particularly in their connectedness. If it is immediate that affine subspaces are connected in Euclidean geometry, it is quite different in discrete geometry. The connectedness of hyperplanes has essentially been studied for the neighbourhood by facets [2,3,4,6]. These works have allowed to reveal a percolation phenomenon on the connectedness and have also established a link with the numeration bases.

Our paper generalises these results to the case of a connectedness induced by any neighbourhood. This problem was treated in 2002 by Y. Gérard in [5], but only in the case of rational hyperplanes and by a very different approach. Some of the results obtained for facet connectedness extend naturally to general connectedness, but others require the introduction of new concepts. In particular, we show the appearance of a percolation phenomenon in the general case, excepted for a finite number of so-called pathological vectors. There is however a critical connecting thickness  $\Omega(v)$ , beyond which the plane is always connected.

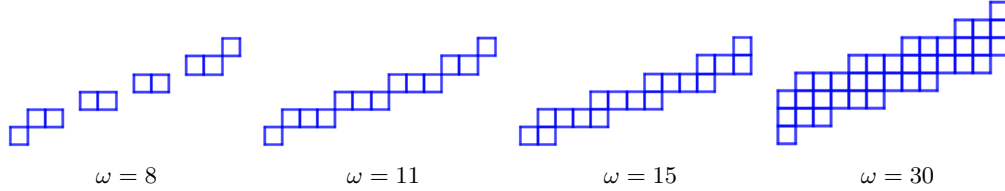
Beyond the generalisation of known results, the main contribution of the article lies in a change of point of view: we study  $\Omega$  not as a combinatorial object but as a function with analytic properties. The numerical plot of the  $\Omega$  function shows discontinuity peaks at every rational point. However, there is an underlying regular structure. If the integers facilitate the representations in machine and the questions of computability, they induce however a strong combinatorial complexity. This difficulty can be seen in the  $\Omega$  function. The main theorem of the article (theorem 7.8) is the proof that the restriction to irrational  $\overline{\Omega}$  of  $\Omega$  is continuous. We then deduce from the analytic properties of the function  $\overline{\Omega}$  some arithmetic properties on  $\Omega$  and an algorithm for computing this critical thickness.

## 2 Discrete hyperplanes

We work in the Euclidean space  $\mathbb{R}^d$  ( $d \geq 2$ ) provided with its canonical scalar product. In 1991, Jean-Pierre Réveillès defined in [8] the notion of discrete line (see Figure 1) as being the set of points with integer coordinates between 2 parallel lines:

$$\mathcal{D}(a, b, \mu, \omega) \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq ax + by + \mu < \omega\}.$$

Each point  $(x, y)$  of  $\mathbb{Z}^2$  can be represented by a square of side 1 centred at this point, namely by the square  $[x - 1/2, x + 1/2] \times [y - 1/2, y + 1/2]$ .



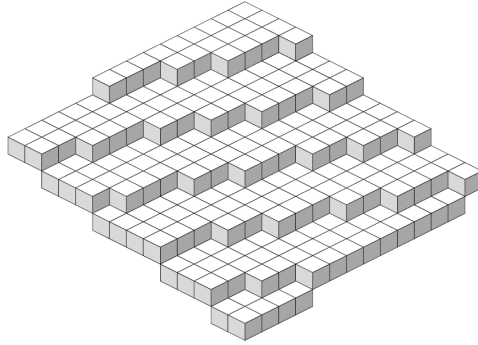
**Fig. 1.** Discrete lines of parameters  $(a, b, \mu) = (-4, 11, 7)$  of variable thickness  $\omega$ .

This definition extends naturally to any finite dimension [1].

**Definition 2.1 (Arithmetic hyperplane)** Let  $v \in \mathbb{R}^d$  with  $v \neq 0$ , and  $\omega, \mu \in \mathbb{R}$ . The arithmetic hyperplane of normal vector  $v$ , shift  $\mu$  and thickness  $\omega$  is the set

$$\mathbb{P}(v, \mu, \omega) \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^d \mid 0 \leq \langle x, v \rangle + \mu < \omega\}.$$

An arithmetic hyperplane (see Figure 2) is thus the set of integer points between two parallel real hyperplanes. The parameter  $v$  corresponds to the inclination of the real hyperplanes,  $\omega$  to the distance between them and  $\mu$  is a shift from the origin. Each point  $x \in \mathbb{Z}^d$  can be represented by a hypercube centred at  $x$  and of side 1, namely by the hypercube  $\mathcal{B}_\infty(x, 1/2)$ .



**Fig. 2.** A portion of an arithmetic hyperplane.

We are interested in the connectedness properties of arithmetic hyperplanes. Our study has led us to distinguish two types of vectors: rational ones and irrational ones. In particular, we define the notion with gcd of a vector which naturally appears in the questions we ask ourselves.

**Definition 2.2 (Rational/Irrational vector, gcd)** Given a non-zero vector  $v \in \mathbb{R}^d$ , we call  $\text{gcd}(v)$  the largest real  $\alpha$ , if it exists, such that  $\frac{1}{\alpha}v \in \mathbb{Z}^d$ . Otherwise, we take  $\text{gcd}(v) = 0$ . We say that a vector  $v$  is rational if  $\text{gcd}(v) > 0$ . Otherwise, it is said to be irrational.

**Remark 2.3** *The rational character of a vector  $v$  can also be defined from the dimension on  $\mathbb{Q}$  of the space generated by the coordinates of  $v$ . Thus, a vector  $v$  is rational iff  $\dim_{\mathbb{Q}}(v) \stackrel{\text{def}}{=} \dim_{\mathbb{Q}}(\text{Vect}(v_1, \dots, v_d)) = 1$ .*

**Example 2.4** 1.  $\gcd((4, 6, 10)) = 2$ ;  
 2.  $\gcd((1, \sqrt{2}, 1 + \sqrt{2})) = 0$ ;  
 3.  $\gcd((\pi, 2\pi, 4\pi)) = \pi$ .

The rational and irrational characteristics of hyperplanes can be seen in the behaviour of their fibres.

**Definition 2.5 (Fibre)** *The level  $\alpha$  fibre for  $\alpha \in [0, \omega[$  of a plane  $\mathbb{P}(v, \mu, \omega)$  is  $\{x \in \mathbb{Z}^d \mid \langle x, v \rangle + \mu = \alpha\}$ .*

**Proposition 2.6** *Let  $\omega > 0$ ,  $v \in \mathbb{R}^d$  with  $v \neq 0$  and  $\mu \in \mathbb{R}$ . Then  $v$  is rational iff  $\mathbb{P}(v, \mu, \omega)$  contains a finite number of non-empty fibres.*

*Proof.*  $\Rightarrow$ : let us assume that  $v$  is rational and let us say  $\alpha \stackrel{\text{def}}{=} \gcd(v) > 0$ . If  $x \in \mathbb{Z}^d$  then  $\langle x, v \rangle + \mu \in \mu + \alpha\mathbb{Z}$ . As  $[0, \omega[ \cap (\mu + \alpha\mathbb{Z})$  is finite,  $\mathbb{P}(v, \mu, \omega)$  admits a finite number of non-empty fibres.

$\Leftarrow$ : by contradiction, let us suppose  $v$  irrational. From then on,  $\{\langle x, v \rangle\}_{x \in \mathbb{Z}^d}$  is dense in  $\mathbb{R}$  so the set of non-empty fibre levels of  $\mathbb{P}(v, \mu, \omega)$  is dense in  $[0, \omega[$ . So  $\mathbb{P}(v, \mu, \omega)$  has an infinite number of non-empty fibres.  $\square$

### 3 Connectedness by facets

One of the most natural connectivities is surely the facet connectedness, induced by the facet neighbourhood relation: two hypercubes are neighbours iff they touch at the facets.

**Definition 3.1** *Two points  $x, y \in \mathbb{Z}^d$  are said to be neighbours by facets iff  $\|y - x\|_1 = 1$ .*

**Remark 3.2** *In the rest of this article, we will say that a plane is connected when it is connected (between 2 points of the plane, there is a path respecting the neighbourhood relation) and **non empty**. This modification makes it easier to state the connectivity properties of arithmetic hyperplanes.*

**Problem 3.3** *Given  $v, \mu, \omega$ , how do we decide whether the plane  $\mathbb{P}(v, \mu, \omega)$  is connected by facets?*

**Example 3.4** *Given an arithmetic hyperplane, the facet-neighbour relationship induces a graph. The vertices are the points of the plane and an edge is present between 2 points if they are neighbours by facets. Thus, the piece of hyperplane in Figure 3 has a tree as its induced graph (and is therefore connected). The colours correspond to 3 children of the root.*

A first fundamental result, obtained in [2], is the existence of a percolation phenomenon which appears when  $\omega$  varies.

**Theorem 3.5 ([2])** *Let  $v \in \mathbb{R}^d$  with  $v \neq 0$  and  $\mu \in \mathbb{R}$ . There exists a critical thickness  $\Omega(v, \mu)$  such that*

1.  $\forall \omega < \Omega(v, \mu)$ ,  $\mathbb{P}(v, \mu, \omega)$  is not connected by facets;
2.  $\forall \omega > \Omega(v, \mu)$ ,  $\mathbb{P}(v, \mu, \omega)$  is connected by facets.

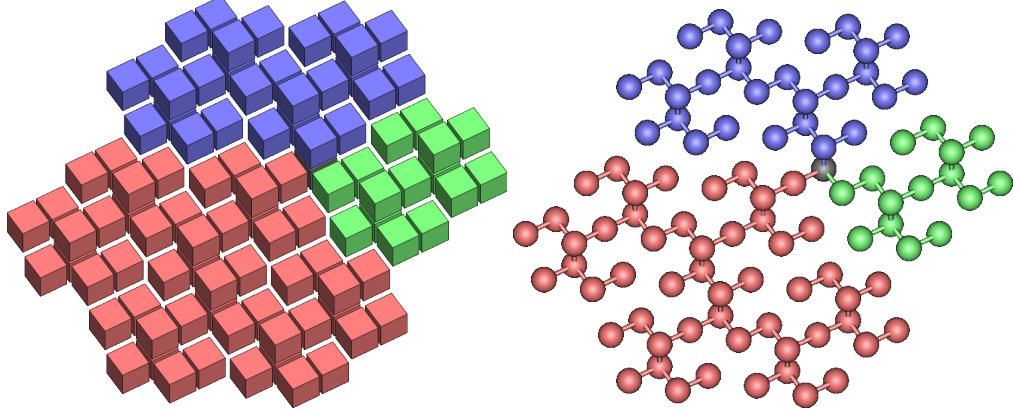
Once this result is established, two questions remain:

1. how to calculate  $\Omega(v, \mu)$ ?
2. what is the behaviour at the critical thickness?

The paper also gives relatively fine bounds on the connecting thickness and shows that its  $\mu$  dependence is very small.

**Proposition 3.6 ([2])** *Let  $v \in \mathbb{R}^d$  with  $v \neq 0$  and  $\mu \in \mathbb{R}$ . We note  $\lambda(v) \stackrel{\text{def}}{=} \min(\{|v_i| \mid i \in \llbracket 1, d \rrbracket, v_i \neq 0\})$ . So:*

1.  $\|v\|_{\infty} \leq \Omega(v, \mu) \leq \|v\|_{\infty} + \lambda(v)$ .



**Fig. 3.** A piece of arithmetic hyperplane and its induced graph. Note that the induced graph is a tree whose branches are represented by 3 different colours.

$$2. \Omega(v, \mu) = \begin{cases} \Omega(v, 0) + (\mu \bmod \gcd(v)) & \text{if } v \text{ is rational;} \\ \Omega(v, 0) & \text{if } v \text{ is irrational.} \end{cases}$$

In particular, in the irrational case,  $\Omega$  does not depend on  $\mu$  and, in the rational case, its dependence in  $\mu$  is piecewise affine. We therefore allow ourselves the abuse of notation  $\Omega(v) \stackrel{\text{d\'ef}}{=} \Omega(v, 0)$ .

Finally, the paper proposes a simple algorithm to compute  $\Omega(v)$ , from a gcd algorithm and some properties of the connecting thickness. We define for that, for  $i \in \llbracket 1, d \rrbracket$ ,  $\pi_i$  the orthogonal projection parallel to  $e_i$  on  $\text{Vect}((e_j)_{j \neq i})$

**Proposition 3.7** *Let  $v$  be a non-zero vector of  $\mathbb{R}^d$ . Then  $\Omega(v) = \Omega(|v|)$  where  $|v| \stackrel{\text{d\'ef}}{=} (|v_1|, \dots, |v_d|)$ .*

**Theorem 3.8 ([2])** *Let  $v$  be a non negative non-zero vector of  $\mathbb{R}^d$ .*

1. *if  $v$  has a zero coordinate  $v_i$  then  $\Omega(v) = \Omega(\pi_i(v))$ .*
2. *if  $v$  has the maximum coordinate  $v_k$  and if  $\|\pi_k(v)\|_1 \leq v_k$  then  $\Omega(v) = v_k + \gcd(\pi_k(v)) - \gcd(v)$ .*
3. *if  $v$  has the minimum coordinate  $v_i$  then  $\Omega(v) = \Omega(v - v_i(\mathbb{1} - e_i)) + v_i$  where  $\mathbb{1} \stackrel{\text{d\'ef}}{=} (1, 1, \dots, 1)$ .*

**Algorithm:** Fully subtractive algorithm [2].

**Input:**  $v$  a non-zero vector.

**Output:**  $\Omega(v)$ .

$d \leftarrow$  dimension of  $v$ ;

$v \leftarrow |v|$ ;

$\Omega \leftarrow 0$ ;

**while**  $d > 1$  **do**

**if** there exists  $k$  such that  $\|\pi_k(v)\|_1 \leq v_k$  **then**

Return( $v_k + \gcd(\pi_k(v)) - \gcd(v) + \Omega$ )

**if**  $v$  has a zero coordinate  $v_i$  **then**

$v \leftarrow \pi_i(v)$ ;

$d \leftarrow d - 1$

**else**

$i \leftarrow$  index of a minimum coordinate of  $v$ ;

$v \leftarrow v - v_i(\mathbb{1} - e_i)$  where  $\mathbb{1} = (1, 1, \dots, 1)$ ;

$\Omega \leftarrow \Omega + v_i$

**end.**

The previous algorithm terminates for almost any vector and returns its connecting thickness. The set  $\mathcal{K}_d$  of vectors for which the algorithm does not terminate and never performs a projection is the set of Kraaikamp and Meester, introduced in [7]. The authors prove in particular that  $\mathcal{K}_d$  is Lebesgue-negligible.

For the behaviour at the connecting thickness (the critical case of the percolation phenomenon), this amounts to studying the set  $\mathcal{K}_d$  and to studying an associated numbering system, as presented in [4] and [3].

## 4 General connectedness

As the case of facet connectedness is dealt with, it may be useful to look at other neighbourhoods. In 2D, we are generally interested in facet connectedness and vertex connectedness. In 3D, facet connectedness, edge connectedness and vertex connectedness are most often used. In  $d$  dimension, Eric Andres [1] defines  $d$  neighbourhood relations by faces, according to the dimension of the intersection of cubes representing 2 neighbouring points. One can even extend these notions to more exotic connectivities, such as the knight step, with any finite set of jumps allowed. Hereafter, for  $a, b \in \mathbb{Z}$ , the notation  $\llbracket a, b \rrbracket$  refers to the set of integers  $\{a, a+1, \dots, b\}$ .

**Definition 4.1** Let  $k \in \llbracket 0, d-1 \rrbracket$  and  $x, y \in \mathbb{Z}^d$ . We say that  $x$  and  $y$  are  $k$ -neighbours iff  $\|y - x\|_\infty = 1$  and  $\|y - x\|_1 \leq d - k$ .

**Definition 4.2** Let  $\mathcal{V}$  be a finite subset of  $\mathbb{Z}^d$ . We say that  $\mathcal{V}$  is a neighbourhood iff  $\mathcal{V}$  is symmetric (i.e.  $\forall x \in \mathcal{V}, -x \in \mathcal{V}$ ) and generates  $\mathbb{Z}^d$  as a group. From then on, we say that  $x, y \in \mathbb{Z}^d$  are  $\mathcal{V}$ -neighbours iff  $y - x \in \mathcal{V}$  which we denote by  $x \leftrightarrow_{\mathcal{V}} y$ . In this case, if we call  $u \stackrel{\text{def}}{=} y - x \in \mathcal{V}$ , we also say that  $u$  is a movement from  $x$  to  $y$ .

**Definition 4.3** A non-empty subset  $A$  of  $\mathbb{Z}^d$  is  $\mathcal{V}$ -connected iff  $\forall x, y \in A$ , there exist  $x = x_0, x_1, \dots, x_k = y \in A$  such that  $\forall i < k, x_i$  and  $x_{i+1}$  are  $\mathcal{V}$ -neighbours (we note  $x \leftrightarrow_{\mathcal{V}}^* y$  iff it exists a  $\mathcal{V}$ -path from  $x$  to  $y$ ). Note that the empty set is considered non connected for simplicity in the theorems and definitions.

The symmetrical character of  $\mathcal{V}$  ensures the symmetry of the  $\mathcal{V}$ -neighbour relationship. The generative character of  $\mathcal{V}$  ensures that the whole space  $\mathbb{Z}^d$  is  $\mathcal{V}$ -connected. By abuse, we will sometimes speak of connectedness or neighbour without specifying the neighbourhood when it is clearly identified.

**Example 4.4** 1.  $\mathcal{V}_f \stackrel{\text{def}}{=} \overline{\mathcal{B}}_\infty(0, 1) \cap \mathbb{Z}^d = \{\pm e_i\}_{i \in \llbracket 1, d \rrbracket}$  (where  $\overline{\mathcal{B}}_\infty(0, 1)$  is the unit ball closed for the infinite norm) is the neighbourhood of facet connectedness;  
 2.  $\mathcal{V}_s \stackrel{\text{def}}{=} \overline{\mathcal{B}}_1(0, 1) \cap \mathbb{Z}^d$  is the neighbourhood of the vertex connectedness (see Figure 4);  
 3.  $\mathcal{V}_c \stackrel{\text{def}}{=} \{\pm(1, 2), \pm(2, 1), \pm(-1, 2), \pm(-2, 1)\}$  is the knight-neighbourhood in dimension 2, corresponding to the movements of a knight on a chess board.

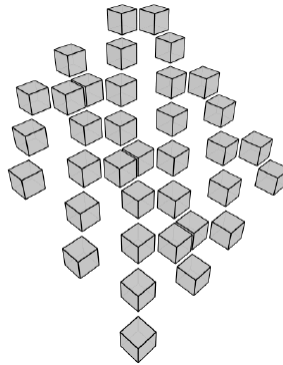
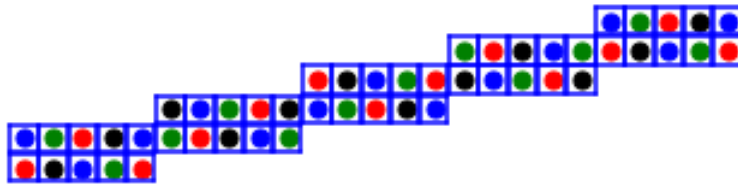


Fig. 4. A vertex-connected plane.



**Fig. 5.** A knight-connected line.

In the case of the knight-neighbourhood in Figure 5, we look at an example with the discrete line of slope  $1/5$  and zero shift. The colours correspond to the 4 connected components obtained for the set of jumps  $\{\pm(2, 1), \pm(2, -1)\}$ .

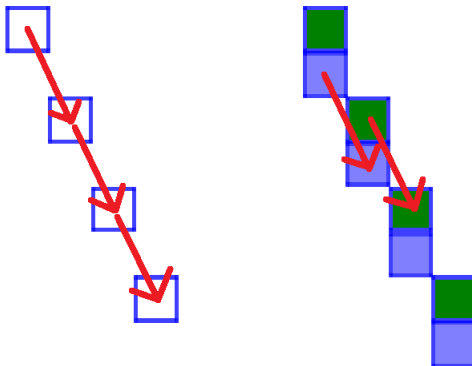
By adding the jump  $\pm(1, 2)$ , we can link the red and black, green and red, blue and green, black and blue components. So the line is  $\mathcal{V}_c$ -connected. Note by the way that we did not use the jump  $\pm(-1, 2)$  to show connectedness, which intuitively means that it is useless in the neighbourhood (a notion which will be formally defined in part 5).

**Problem 4.5** *Given a plane  $\mathbb{P}(v, \mu, \omega)$  and a neighbourhood  $\mathcal{V}$ , we wish to decide whether  $\mathbb{P}(v, \mu, \omega)$  is  $\mathcal{V}$ -connected.*

We have started by extending some known results for facet connectedness: existence of the connecting thickness, bounds on it, dependence on shift, reduction. However, this generalisation requires some additional assumptions or modifications, often due to the complexity of the  $\mathcal{V}$ -induced lattice.

#### 4.1 Connecting thickness

A first counter-example to the existence of a connecting thickness appears immediately (see Figure 6) when we look at the knight-neighbourhood  $\mathcal{V}_c$  for the vector  $v = (2, 1)$  (and  $\mu = 0$ ). At thickness 1 (figure on the left), the line is connected by knight. On the other hand, at thickness 2 (figure on the right), we observe 2 distinct connected components. This contradicts the existence of a percolation phenomenon for connectedness. However, we can easily characterise the pathological vectors and show that they are finite in number. The pathological case appears indeed only when, for  $\omega > 0$  small enough, the plane contains only one connected fibre. This is only possible in the case where  $v$  is rational and the level 0 fibre  $v^\perp \cap \mathbb{Z}^d$  is connected. Increasing the thickness makes a new fibre appear which is not necessarily linked to the previous one.



**Fig. 6.** A straight line with no connecting thickness.

**Definition 4.6** Let  $\mathcal{V}$  be a neighbourhood and  $v$  be a non-zero vector. We say that  $v$  is  $\mathcal{V}$ -pathological iff  $v$  is rational and  $v^\perp \cap \mathbb{Z}^d$  is  $\mathcal{V}$ -connected.

**Example 4.7**

1. In the example presented above for  $v = (2, 1)$ ,  $v^\perp \cap \mathbb{Z}^d = \text{Vect}_{\mathbb{Z}}((1, -2))$  which is  $\mathcal{V}_c$ -connected. Therefore  $(2, 1)$  is  $\mathcal{V}_c$ -pathological.
2. In facet connectedness, the  $e_i$ 's are pathological.

In order to prove the existence of a  $\mathcal{V}$ -connecting thickness in the non-pathological case, we need some small intermediate results.

**Definition 4.8** Let  $\mathcal{V}$  be a neighbourhood and  $v \in \mathbb{R}^d \setminus \{0\}$ . We pose  $\mathcal{V}^+ \stackrel{\text{def}}{=} \{u \in \mathcal{V} \mid \langle u, v \rangle > 0\}$ . We then define  $\lambda_{\mathcal{V}}(v) \stackrel{\text{def}}{=} \min_{u \in \mathcal{V}^+} \langle u, v \rangle$  and  $\|v\|_{\mathcal{V}} \stackrel{\text{def}}{=} \max_{u \in \mathcal{V}} |\langle u, v \rangle|$ .

Note that, as  $\mathcal{V}$  generates  $\mathbb{Z}^d$ ,  $\|\cdot\|_{\mathcal{V}}$  is a norm on  $\mathbb{R}^d$ .

**Lemma 4.9** Let  $\mathcal{V}$  be a neighbourhood and  $v$  be a non-pathological vector. Let  $\omega \in \mathbb{R}_+$  such that  $\mathbb{P}(v, \mu, \omega)$  is connected. Then  $\omega > \lambda_{\mathcal{V}}(v)$ .

*Proof.* We pose  $\lambda \stackrel{\text{def}}{=} \lambda_{\mathcal{V}}(v)$ . Let us suppose by contradiction  $\omega \leq \lambda$ . We will show that  $\mathbb{P}(v, \mu, \omega)$  contains exactly one non empty fibre. Intuitively, this comes from the fact that it is impossible to move in  $\mathbb{P}(v, \mu, \omega)$  using  $u \in \mathcal{V}$  such that  $\langle u, v \rangle \neq 0$ . Let us take  $x, y \in \mathbb{P}(v, \mu, \omega)$  neighbours. It exists  $u \in \mathcal{V}$  such that  $y = x + u$ . As  $x, y \in \mathbb{P}(v, \mu, \omega)$ ,  $\lambda \geq \omega > |\langle y - x, v \rangle| = |\langle u, v \rangle|$ . This imposes, by definition of  $\lambda$ ,  $\langle u, v \rangle = 0$  hence  $\langle x, v \rangle = \langle y, v \rangle$ .

Let  $G$  be the graph induced by the  $\mathcal{V}$ -neighbourhood relation on  $\mathbb{P}(v, \mu, \omega)$ . As  $\mathbb{P}(v, \mu, \omega)$  is  $\mathcal{V}$ -connected,  $G$  is also connected. Moreover,  $\mathbb{P}(v, \mu, \omega)$  is non-empty and therefore contains an element  $x_0$ . By connectedness, any point  $x$  of  $G$  is at a finite distance from  $x_0$  in the graph, which we denote  $d_G(x_0, x)$ . Recall also that  $\forall x, y \in G$ , if  $x$  and  $y$  are neighbours then  $\langle x, v \rangle = \langle y, v \rangle$ . It is therefore easy to show by recurrence on  $d_G(x_0, x)$  that for all  $x \in G$ , we have  $\langle x, v \rangle = \langle x_0, v \rangle$ , and so  $x - x_0 \in v^\perp$ .

We thus showed that  $\mathbb{P}(v, \mu, \omega) \subset (x_0 + v^\perp) \cap \mathbb{Z}^d$ . The reciprocal inclusion is trivial, so  $\mathbb{P}(v, \mu, \omega) = (x_0 + v^\perp) \cap \mathbb{Z}^d$ . As  $\mathbb{P}(v, \mu, \omega)$  is connected,  $(x_0 + v^\perp) \cap \mathbb{Z}^d$  also is and therefore  $v^\perp \cap \mathbb{Z}^d$  as well. Finally,  $\mathbb{P}(v, \mu, \omega) = x_0 + (v^\perp \cap \mathbb{Z}^d)$  has a single non-empty fibre so  $v$  is rational. This contradicts the non-pathological character of  $v$ .  $\square$

**Lemma 4.10 (inclusion lemma)** Let  $\mu_1, \omega_1, \mu_2, \omega_2 \in \mathbb{R}$  such that  $\mathbb{P}(v, \mu_1, \omega_1) \subset \mathbb{P}(v, \mu_2, \omega_2)$ . It is further assumed that  $\mathbb{P}(v, \mu_1, \omega_1)$  is connected and that  $\omega_1 > \lambda$ . Then  $\mathbb{P}(v, \mu_2, \omega_2)$  is connected.

*Proof.* We pose  $\lambda \stackrel{\text{def}}{=} \lambda_{\mathcal{V}}(v)$  and  $u_0 \in \mathcal{V}$  such that  $\langle u_0, v \rangle = \lambda$ . Let  $x, y \in \mathbb{P}(v, \mu_2, \omega_2)$ . There exist  $k, \ell \in \mathbb{Z}$  such that  $\langle x, v \rangle + \mu_1 - k\lambda \in [0, \lambda[$  and  $\langle y, v \rangle + \mu_1 - \ell\lambda \in [0, \lambda[$ . Thus, by posing  $x_0 \stackrel{\text{def}}{=} x - ku_0$  and  $y_0 \stackrel{\text{def}}{=} y - \ell u_0$ , as  $\lambda < \omega_1$ , we obtain that  $x_0, y_0 \in \mathbb{P}(v, \mu_1, \omega_1)$  (and by the way that  $x_0, y_0 \in \mathbb{P}(v, \mu_2, \omega_2)$ ).

As  $\mathbb{P}(v, \mu_1, \omega_1)$  is connected, there exists a path from  $x_0$  to  $y_0$  with values in  $\mathbb{P}(v, \mu_1, \omega_1)$  and thus in  $\mathbb{P}(v, \mu_2, \omega_2)$  (by inclusion). Moreover,  $x \leftrightarrow^* x_0$  and  $y_0 \leftrightarrow^* y$  in  $\mathbb{P}(v, \mu_2, \omega_2)$ .

We have thus constructed a path from  $x$  to  $y$  in  $\mathbb{P}(v, \mu_2, \omega_2)$ . So  $\mathbb{P}(v, \mu_2, \omega_2)$  is connected.  $\square$

**Proposition 4.11** Let  $v \in \mathbb{R}^d$  be non-pathological and  $\omega \in \mathbb{R}$  such that  $\mathbb{P}(v, \mu, \omega)$  is  $\mathcal{V}$ -connected. If  $\omega' \geq \omega$  then  $\mathbb{P}(v, \mu, \omega')$  is  $\mathcal{V}$ -connected.

*Proof.* As  $v$  is non-pathological and  $\mathbb{P}(v, \mu, \omega)$  is connected, we have  $\omega > \lambda$ .

We then apply the inclusion lemma. Indeed, we have  $\mathbb{P}(v, \mu, \omega) \subset \mathbb{P}(v, \mu, \omega')$ , so the connectedness of  $\mathbb{P}(v, \mu, \omega)$  implies that of  $\mathbb{P}(v, \mu, \omega')$ .

We thus obtain that the set  $I \stackrel{\text{def}}{=} \{\omega \in \mathbb{R}_+ \mid \mathbb{P}(v, \mu, \omega) \text{ is connected}\}$  is an unbounded interval of  $\mathbb{R}_+$ . By posing  $\Omega(v, \mu) \stackrel{\text{def}}{=} \inf(I)$ , we obtain the existence of the connecting thickness.  $\square$

**Theorem 4.12** Let  $\mathcal{V}$  be a neighbourhood and  $v$  a non-pathological vector. Let  $\mu \in \mathbb{R}$ . Then  $(v, \mu)$  admits a connecting thickness  $\Omega_{\mathcal{V}}(v, \mu)$ , i.e.  $\forall \omega < \Omega_{\mathcal{V}}(v, \mu)$ ,  $\mathbb{P}(v, \mu, \omega)$  is not  $\mathcal{V}$ -connected and  $\forall \omega > \Omega_{\mathcal{V}}(v, \mu)$ ,  $\mathbb{P}(v, \mu, \omega)$  is  $\mathcal{V}$ -connected.



Apart from the pathological cases, we thus find the phenomenon of percolation observed for facet connectedness. The case of pathological vectors is discussed in the next section. The aim of the rest of the article is to obtain properties on  $v \mapsto \Omega_{\mathcal{V}}(v, 0)$  and to manage to calculate this function. Indeed, we will see in part 4.3 that the dependence of  $\Omega_{\mathcal{V}}$  on  $\mu$  is very weak and that we can limit our study to the case  $\mu = 0$ .

## 4.2 Pathological vectors

We consider here the case of pathological vectors for a given  $v$  neighbourhood. Recall that a vector  $v$  is said to be pathological iff  $v$  is rational and  $v^{\perp} \cap \mathbb{Z}^d$  is  $\mathcal{V}$ -connected. As  $\forall \alpha > 0, \mathbb{P}(v, \mu, \omega) = \mathbb{P}(\alpha v, \alpha \mu, \alpha \omega)$ , we can return to the case where the pathological vectors studied have a gcd equal to 1 (thus are integers).

**Proposition 4.13** *Let  $v \in \mathbb{R}^d$  with  $v \neq 0$ . Then  $v$  is pathological iff there exists  $\mathcal{V}_0 \subset \mathcal{V}$  of rank  $d - 1$  such that  $v^{\perp} \cap \mathbb{Z}^d = \text{Vect}_{\mathbb{Z}}(\mathcal{V}_0)$ .*

*Proof.*  $\Rightarrow$ : let us suppose that  $v$  is pathological. If we multiply  $v$  by a scalar, we can assume  $\text{gcd}(v) = 1$ . As  $v$  is rational,  $v^{\perp} \cap \mathbb{Z}^d$  is a lattice of dimension  $d - 1$ . Consider  $\mathcal{V}_0 \stackrel{\text{def}}{=} \mathcal{V} \cap v^{\perp}$ . From then on,  $\text{Vect}_{\mathbb{Z}}(\mathcal{V}_0) \subset v^{\perp} \cap \mathbb{Z}^d$ . Now,  $v^{\perp} \cap \mathbb{Z}^d$  is  $\mathcal{V}$ -connected. Moreover, no movement by a vector in  $v^{\perp} \cap \mathbb{Z}^d$  is possible in  $v^{\perp} \cap \mathbb{Z}^d$ . Therefore  $v^{\perp} \cap \mathbb{Z}^d$  is  $\mathcal{V}_0$ -connected. Thus,  $v^{\perp} \cap \mathbb{Z}^d \subset \text{Vect}_{\mathbb{Z}}(\mathcal{V}_0)$ , which concludes.

$\Leftarrow$ : let us suppose that there exists  $\mathcal{V}_0 \subset \mathcal{V}$  of rank  $d - 1$  such that  $v^{\perp} \cap \mathbb{Z}^d = \text{Vect}_{\mathbb{Z}}(\mathcal{V}_0)$ . As  $\mathcal{V}_0$  is of rank  $d - 1$ , it is the same for  $v^{\perp} \cap \mathbb{Z}^d$  so  $v$  is rational. Moreover,  $v^{\perp} \cap \mathbb{Z}^d = \text{Vect}_{\mathbb{Z}}(\mathcal{V}_0)$  is  $\mathcal{V}_0$ -connected hence  $\mathcal{V}$ -connected. Therefore  $v$  is pathological.  $\square$

The preceding proposition thus makes it possible to characterise the pathological vectors algebraically: they are the vector products of  $\mathcal{V}_0 \subset \mathcal{V}$  of rank  $d - 1$ . We thus obtain that the pathological vectors are, to the nearest multiplication by a scalar, in finite number.

**Proposition 4.14** *Let  $v$  be a pathological vector with  $\text{gcd } 1$ . Then  $v$  admits a connecting thickness iff there exists  $u \in \mathcal{V}$  such that  $\langle u, v \rangle = 1$ . In this case (named semi-pathological case),  $\Omega_{\mathcal{V}}(v, 0) = 0$ . (the case  $\mu \neq 0$  is treated in Part 4.3).*

*Proof.*  $\Rightarrow$ : suppose that  $v$  has a connecting thickness. Then, as  $\mathbb{P}(v, 0, 1) = v^{\perp} \cap \mathbb{Z}^d$  is  $\mathcal{V}$ -connected,  $\mathbb{P}(v, 0, 2)$  is also  $\mathcal{V}$ -connected. So the level 0 and level 1 fibres of  $\mathbb{P}(v, 0, 2)$  are connected. Since  $\mathbb{P}(v, 0, 2)$  contains only 2 non-empty fibres, this imposes the existence of a  $u \in \mathcal{V}$  such that  $\langle u, v \rangle = 1$ .

$\Leftarrow$ : let us suppose that there exists  $u \in \mathcal{V}$  such that  $\langle u, v \rangle = 1$ . Let  $\omega > 0$ . The plane  $\mathbb{P}(v, 0, \omega)$  contains a finite number of non empty fibres. The level 0 fibre,  $v^{\perp} \cap \mathbb{Z}^d$  is  $\mathcal{V}$ -connected because  $v$  is pathological. Moreover, for all  $n$ ,  $u$  connects the level  $n$  fibre to the level  $n + 1$  fibre. Thus  $\mathbb{P}(v, 0, \omega)$  is connected. Therefore  $v$  has a connecting thickness. Moreover, we have shown that  $\forall \omega > 0, \mathbb{P}(v, 0, \omega)$  is  $\mathcal{V}$ -connected. Therefore  $\Omega(v, 0) = 0$ .  $\square$

**Theorem 4.15** *Let  $v$  be a pathological vector (with  $\text{gcd } 1$ ). Then  $I_v \stackrel{\text{def}}{=} \{\omega > 0 \mid \mathbb{P}(v, 0, \omega) \text{ is not connected}\}$  is an interval.*

*Proof.* In the case where there exists  $u \in \mathcal{V}$  such that  $\langle u, v \rangle = 1$ , by the previous proposition,  $I_v = \emptyset$  hence the result.

Otherwise (a totally pathological case), we pose  $\lambda = \lambda_{\mathcal{V}}(v)$ . As  $\forall u \in \mathcal{V}, \langle u, v \rangle \neq 1$ , we have  $\lambda > 1$ . Let  $\omega \in ]1, \lambda]$ .  $\mathbb{P}(v, 0, \omega)$  contains at least 2 non-empty fibres (the one of level 0 and the one of level 1) and 2 fibres of the plane are never linked. So  $\mathbb{P}(v, 0, \omega)$  is not connected so  $]1, \lambda] \subset I_v$ . For  $\omega \in ]\lambda, +\infty[$ , the inclusion lemma 4.10 allows us to show that there exists  $\Omega(v, 0) \geq \lambda$  such that:  $\forall \omega \in ]\lambda, \Omega(v, 0)[$ ,  $\mathbb{P}(v, 0, \omega)$  is not connected ( $\omega \in I_v$ ) and  $\forall \omega > \Omega(v, 0)$ ,  $\mathbb{P}(v, 0, \omega)$  is connected. Therefore  $I_v = ]1, \Omega(v, 0)[$  or  $I_v = ]1, \Omega(v, 0)]$  which are intervals.  $\square$

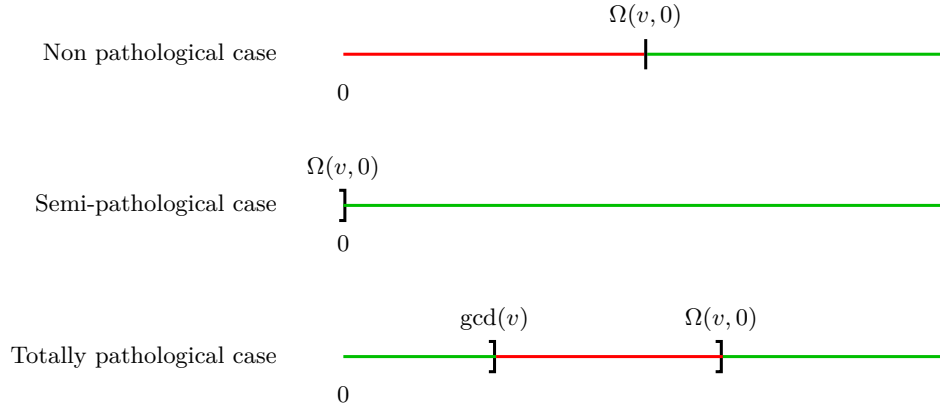
**Definition 4.16** *In the case of pathological vectors  $v$ , we pose*

$$\Omega(v, 0) \stackrel{\text{def}}{=} \sup(I_v) = \sup(\{\omega > 0 \mid \mathbb{P}(v, 0, \omega) \text{ is not connected}\})$$

*which extends the concept of connecting thickness to the pathological case.*

A summary of the 3 different possible cases can be seen in Figure 7:

1. The vector is non-pathological. Then we observe a percolation phenomenon: the plane is non-connected, then connected for  $\omega > \Omega(v, 0)$ .
2. The vector is pathological and  $\exists u \in \mathcal{V}, \langle u, v \rangle = 1$  (semi-pathological case). The plane is then always connected.
3. The vector is pathological and  $\forall u \in \mathcal{V}, \langle u, v \rangle \neq 1$  (totally pathological case). The plane is not connected for  $\omega \in ]\gcd(v), \Omega(v, 0)]$  and connected for all other  $\omega > 0$ .



**Fig. 7.** The different cases of existence of the connecting thickness  $\Omega(v, 0)$ . The green interval corresponds to the connected part, the red to the non-connected part.

**Remark 4.17** *The totally pathological case never appears in the case of facet connectedness. Indeed, the condition  $v^\perp \cap \mathbb{Z}^d = \text{Vect}_{\mathbb{Z}}(\mathcal{V}_0)$  imposes that  $v$  has a single non-zero coordinate, hence  $v = \alpha e_i$ . Then  $\langle e_i, v \rangle = \alpha = \gcd(v)$ : we are thus in the semi-pathological case.*

### 4.3 Shift dependency

We wish here to study the dependency in  $\mu$  of  $\Omega$ . We distinguish between the rational case and the irrational one. The proof for rational vectors is based on Bézout's theorem, whereas the proof for irrational vectors uses a density result.

**Proposition 4.18** *Let  $v$  be an irrational vector. Then  $\Omega(v, \mu) = \Omega(v, 0)$ .*

*Proof.* We show that  $\mu \mapsto \Omega(v, \mu)$  is a constant function. Let  $\mu_1, \mu_2 \in \mathbb{R}$ ,  $\omega \in \mathbb{R}_+$  and  $\varepsilon > 0$ . We suppose that  $\mathbb{P}(v, \mu_1, \omega)$  is connected and let us show that  $\mathbb{P}(v, \mu_2, \omega + \varepsilon)$  also is.

As  $v$  is irrational,  $\{\langle x, v \rangle\}_{x \in \mathbb{Z}^d}$  is dense in  $\mathbb{R}$ . There is thus  $x_0 \in \mathbb{Z}^d$  such that  $\mu_2 = \langle x_0, v \rangle + \mu_1 + \gamma$  where  $\gamma \in [0, \varepsilon[$ .

$$\begin{aligned}
 \mathbb{P}(v, \mu_2, \omega + \varepsilon) &= \{x \in \mathbb{Z}^d \mid 0 \leq \langle x, v \rangle + \mu_2 < \omega + \varepsilon\} \\
 &= \{x \in \mathbb{Z}^d \mid 0 \leq \langle x, v \rangle + \langle x_0, v \rangle + \mu_1 + \gamma < \omega + \varepsilon\} \\
 &= \{x \in \mathbb{Z}^d \mid 0 \leq \langle x + x_0, v \rangle + \mu_1 + \gamma < \omega + \varepsilon\} \\
 &= \{x \in \mathbb{Z}^d \mid 0 \leq \langle x, v \rangle + \mu_1 + \gamma < \omega + \varepsilon\} - x_0 \\
 &= \mathbb{P}(v, \mu_1 + \gamma, \omega + \varepsilon) - x_0.
 \end{aligned}$$

Now  $\mathbb{P}(v, \mu_1, \omega) \subset \mathbb{P}(v, \mu_1 + \gamma, \omega + \varepsilon)$ . Indeed, if  $0 \leq \langle x, v \rangle + \mu_1 < \omega$  then  $0 \leq \gamma \leq \langle x, v \rangle + \mu_1 + \gamma < \omega + \gamma < \omega + \varepsilon$ . As  $\mathbb{P}(v, \mu_1, \omega)$  is connected, we deduce, by the inclusion lemma 4.10, that  $\mathbb{P}(v, \mu_1 + \gamma, \omega + \varepsilon)$  also is. Therefore, by translation,  $\mathbb{P}(v, \mu_2, \omega + \varepsilon)$  is connected. Thus, taking the inf, we obtain  $\Omega(v, \mu_1) = \Omega(v, \mu_2)$ , which concludes.  $\square$

**Proposition 4.19** *Let  $v$  be a rational vector with  $\gcd 1$  and  $\mu_0 = \mu \bmod 1$ . Then  $\mathbb{P}(v, \mu, \omega)$  is connected if  $\mathbb{P}(v, 0, \omega - \mu_0)$  is connected.*

*Proof.* By Bézout's theorem, there exists  $x_0 \in \mathbb{Z}^d$  such that  $\langle x_0, v \rangle = 1$ . There exists  $k \in \mathbb{Z}$  and  $\mu_0 \in [0, 1[$  such that  $\mu = k + \mu_0$ .

$$\begin{aligned}
\mathbb{P}(v, \mu, \omega) &= \{x \in \mathbb{Z}^d \mid 0 \leq \langle x, v \rangle + \mu < \omega\} \\
&= \{x \in \mathbb{Z}^d \mid 0 \leq \langle x, v \rangle + k + \mu_0 < \omega\} \\
&= \{x \in \mathbb{Z}^d \mid 0 \leq \langle x + kx_0, v \rangle + \mu_0 < \omega\} \\
&= \{x \in \mathbb{Z}^d \mid -\mu_0 \leq \langle x + kx_0, v \rangle < \omega - \mu_0\} \\
&= \{x \in \mathbb{Z}^d \mid -\mu_0 \leq \langle x, v \rangle < \omega - \mu_0\} - kx_0 \\
&= \{x \in \mathbb{Z}^d \mid 0 \leq \langle x, v \rangle < \omega - \mu_0\} - kx_0 \text{ because } \mu_0 \in [0, 1[ \\
&= \mathbb{P}(v, 0, \omega - \mu_0) - kx_0.
\end{aligned}$$

Therefore, by translation,  $\mathbb{P}(v, \mu, \omega)$  is connected if  $\mathbb{P}(v, 0, \omega - \mu_0)$  is connected.  $\square$

From the two previous propositions, the following theorem can be deduced:

**Theorem 4.20** *Let  $v \in \mathbb{R}^d$  with  $v \neq 0$  and  $\mu \in \mathbb{R}$ .*

*Then  $\Omega(v, \mu) = \begin{cases} \Omega(v, 0) + (\mu \bmod \gcd(v)) & \text{if } v \text{ is rational;} \\ \Omega(v, 0) & \text{if } v \text{ is irrational.} \end{cases}$*

Given the piecewise affine dependence on the  $\mu$  shift, we can restrict ourselves to the study of  $\Omega(v) \stackrel{\text{d\'ef}}{=} \Omega(v, 0)$ .

## 5 Useful neighbourhoods and bounds on the connecting thickness

### 5.1 Usefulness

In the case of the knight step, we saw that some vectors were not used for connectedness. Generally, even for large  $\mathcal{V}$ -neighbourhoods, when the vector  $v$  is fixed, we seldom use all the  $u \in \mathcal{V}$ -jumps allowed to show connectedness. We fix a vector  $v$  for all this part.

**Definition 5.1** *Let  $\mathcal{V}$  be a neighbourhood. A pair of vectors  $\{-u, u\}$  of  $\mathcal{V}$  is said to be useless iff, for  $u' = u$  or  $u' = -u$ , there exist  $u_1, \dots, u_k \in \mathcal{V} \setminus \{-u, u\}$  (the  $u_i$ 's are not necessarily distinct) such that:*

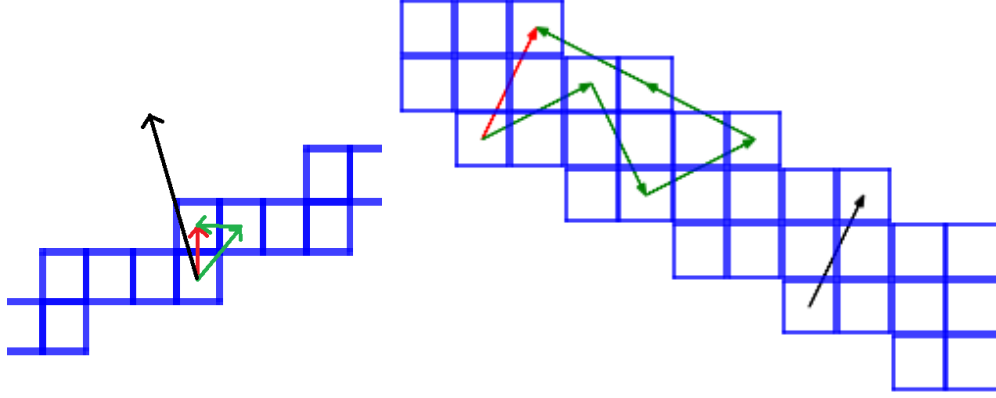
1.  $u' = \sum_{i=1}^k u_i$ .
2.  $\forall j \in \llbracket 0, k \rrbracket, 0 \leq \sum_{i=1}^j \langle u_i, v \rangle \leq \langle u', v \rangle$ .

*Since the neighbourhood vectors  $u$  work in pairs  $\{-u, u\}$ , we say by abuse that  $u$  is useless iff the pair  $\{-u, u\}$  is useless. Finally, a vector is said to be useful iff it is not useless.*

**Example 5.2** *In Figure 8 (left), we are interested in a straight line for vertex connectedness and the jump  $e_2$ . If this jump is possible in the line, it can however be decomposed into 2 jumps  $e_2 = (e_2 + e_1) - e_1$  with smaller scalar products. So the jump  $e_2$  is useless.*

**Example 5.3** *In Figure 8 (right), we are interested in the line  $\mathbb{P}((1, 2), 0, 6)$  and the knight jump  $u = (1, 2)$ . We can show the uselessness of  $u$  thanks to the following relation (the first line represents the uselessness relation of  $u$  and the second calculates the scalar products along the way):*

$$\begin{array}{cccccc}
(1, 2) & = & (2, 1) & + & (1, -2) & + & (2, 1) & + & (-2, 1) & + & (-2, 1) \\
5 & | & 4 & & 1 & & 5 & & 5 & & 5
\end{array}$$



**Fig. 8.** Two examples of useless vectors (in red). On the left with vertex connectivity, on the right with knight connectivity. The normal vector is shown in black.

Note that the notion of usefulness depends on the vector  $v$  considered.

**Definition 5.4** A  $\mathcal{V}$ -neighbourhood is said to be useful iff all its vectors are useful.

The computation of a useful sub-neighbourhood of  $\mathcal{V}$  is not immediate. An algorithm deciding the usefulness of a vector is proposed in the following section on Yan Gérard's algorithm. Note also that there is no uniqueness of a useful neighbourhood for  $v$ . There is also no uniqueness of the size of a useful neighbourhood for  $v$ .

**Proposition 5.5** Let  $\mathcal{V}$  be a neighbourhood and  $u \in \mathcal{V}$  a useless vector. Then  $\forall \mu, \omega, \mathbb{P}(v, \mu, \omega)$  is  $\mathcal{V}$ -connected iff it is  $\mathcal{V}'$ -connected, where  $\mathcal{V}' \stackrel{\text{def}}{=} \mathcal{V} \setminus \{-u, u\}$ .

*Proof.*  $\Leftarrow$  is trivial because  $\mathcal{V}' \subset \mathcal{V}$ .

$\Rightarrow$ : Let  $x, y \in \mathbb{P}(v, \mu, \omega)$ . There is a  $\mathcal{V}$ -path from  $x$  to  $y$  in  $\mathbb{P}(v, \mu, \omega)$ . It remains to find a  $\mathcal{V}'$ -path. By immediate induction on the length of the path, it is enough to show the result in the case where  $x \leftrightarrow_{\mathcal{V}} y$ . The other cases being trivial, it is enough to show the result when  $y - x \in \{-u, u\}$ . Finally, should we exchange  $x$  and  $y$ , we can suppose  $y = x + u$ .

Since  $u$  is useless, there exist  $u_1, \dots, u_k \in \mathcal{V}'$  such that:

1.  $u = \sum_{i=1}^k u_i$ .
2.  $\forall j \in \llbracket 0, k \rrbracket, 0 \leq \sum_{i=1}^j \langle u_i, v \rangle \leq \langle u, v \rangle$ .

For  $j \in \llbracket 0, k \rrbracket$ , we pose  $x_j = x + \sum_{i=1}^j u_i$ . Thus:

1.  $x = x_0$ .
2.  $y = x_k$  by 1).
3.  $\forall j \in \llbracket 0, k-1 \rrbracket, x_j \leftrightarrow_{\mathcal{V}'} x_{j+1}$  because  $u_{j+1} \in \mathcal{V}'$ .
4.  $\forall j \in \llbracket 1, k \rrbracket, 0 \leq \langle x, v \rangle + \mu \leq \langle x_j, v \rangle + \mu \leq \langle y, v \rangle + \mu < \omega$  by 2).

We have thus found a path from  $x$  to  $y$  in  $\mathbb{P}(v, \mu, \omega)$ . So  $\mathbb{P}(v, \mu, \omega)$  is  $\mathcal{V}'$ -connected.  $\square$

As every neighbourhood contains a useful neighbourhood, the previous property allows us to restrict the study of connectedness to the case of useful neighbourhoods. This allows to limit enormously the size of the neighbourhood and the associated combinatorics. For example, for vertex connectedness in dimension 4 (where  $|\mathcal{V}| = 80$ ), all vectors admit a useful neighbourhood of size 8.

## 5.2 A case of reduction to connectedness by facets

An interesting case is when a vector admits a useful neighbourhood of size  $2d$ , we can then return to the case of facet connectedness.

**Theorem 5.6** *Let  $\mathcal{V} = \{\pm u_i\}_{i \in \llbracket 1, d \rrbracket}$  a neighbourhood of size  $2d$ . We pose  $\varphi \stackrel{\text{def}}{=} \begin{cases} \mathbb{R}^d \rightarrow \mathbb{R}^d \\ w \mapsto (\langle u_i, w \rangle)_{i \in \llbracket 1, d \rrbracket} \end{cases}$ .*

*Then  $\forall \omega, \mathbb{P}(v, 0, \omega)$  is  $\mathcal{V}$ -connected iff  $\mathbb{P}(\varphi(v), 0, \omega)$  is connected by facets.*

*Proof.* Let  $\omega \in \mathbb{R}_+$ . Since  $\{u_i\}$  is of size  $d$  and generates  $\mathbb{Z}^d$ ,  $\{u_i\}_{i \in \llbracket 1, d \rrbracket}$  is a basis of  $\mathbb{Z}^d$  so  $\varphi \in GL_d(\mathbb{Z})$ . We pose  $f \stackrel{\text{def}}{=} {}^t\varphi^{-1}$ . Thus, for all  $x \in \mathbb{Z}^d$ ,  $\langle f(x), \varphi(v) \rangle = \langle {}^t\varphi^{-1}(x), \varphi(v) \rangle = \langle x, v \rangle$ . Therefore  $x \in \mathbb{P}(v, 0, \omega)$  iff  $f(x) \in \mathbb{P}(\varphi(v), 0, \omega)$ . So  $f$  is a bijection of  $\mathbb{P}(v, 0, \omega)$  in  $\mathbb{P}(\varphi(v), 0, \omega)$ .

To conclude, since  ${}^t\varphi(e_i) = u_i$  and therefore  $f(u_i) = e_i$ , it is thus enough to show that for  $x, y \in \mathbb{Z}^d$ ,  $x$  and  $y$  are  $\mathcal{V}$ -neighbours iff  $f(x)$  and  $f(y)$  are  $(d-1)$ -neighbours. So  $f$  is a graph isomorphism between  $\mathbb{P}(v, 0, \omega)$  and  $\mathbb{P}(\varphi(v), 0, \omega)$ . Hence  $\mathbb{P}(v, 0, \omega)$  is connected iff  $\mathbb{P}(\varphi(v), 0, \omega)$  is.  $\square$

Note that Theorem 5.6 holds independently of the choice  $u_i$ , or  $-u_i$ , in the definition of  $\varphi$ , the facet connectedness being invariant by change of sign of the coordinates.

The procedure is as follows. Given a vector  $v$  and a neighbourhood  $\mathcal{V}$ , we start by computing a useful neighbourhood  $\mathcal{U}$  for  $v$ . If  $\mathcal{U}$  is of size  $2d$  then we are back to a facet connectedness problem which is already solved. This method allows to treat some neighbourhoods:

1. the  $(d-2)$ -connectedness in dimension  $d$ ;
2. the  $k$ -connectedness in dimensions 3 and 4.

**Counter-example 5.7** *For 2-connectedness in dimension 5, the vector  $v = (4, 6, 12, 14, 23)$  admits a useful neighbourhood of size 12 but none of size 10. We cannot therefore compute  $\Omega(v)$  with this method.*

**Counter-example 5.8** *For the knight-neighbourhood, no vector has a useful neighbourhood of size 4 (all are of size 6). Therefore, no thickness can be calculated with this method.*

## 5.3 Bounds on $\Omega$

The aim here is to efficiently bound  $\Omega_{\mathcal{V}}(v)$ . Recall that  $\|v\|_{\mathcal{V}} = \max_{u \in \mathcal{V}} |\langle u, v \rangle|$  and that  $\lambda_{\mathcal{V}}(v) = \min_{u \in \mathcal{V}, \langle u, v \rangle > 0} \langle u, v \rangle$ .

Then, there exists  $u_0 \in \mathcal{V}$  such that  $\lambda \stackrel{\text{def}}{=} \lambda(v) = \langle u_0, v \rangle$ .

**Proposition 5.9** *Let  $\omega \geq \|v\|_{\mathcal{V}} + \lambda$ . Then  $\mathbb{P}(v, 0, \omega)$  is  $\mathcal{V}$ -connected.*

*Idea of proof:* Let  $x, y \in \mathbb{P}(v, 0, \omega)$ . As  $\mathcal{V}$  generates  $\mathbb{Z}^d$ , there exists a path  $u_1, \dots, u_n$  from  $x$  to  $y$  in  $\mathbb{Z}^d$ . To add  $u_i$  (see Figure 9), if this leads out of  $\mathbb{P}(v, 0, \omega)$ , we get back via jumps from  $u_0$  to a fibre of  $\mathbb{P}(v, 0, \lambda)$ . Then the jump  $u_i$  becomes authorised. Once all the jumps  $u_i$  are done, we come back to  $y$  via  $u_0$ .

*Proof.* Let  $x, y \in \mathbb{P}(v, 0, \omega)$ . As  $\mathcal{V}$  generates the space  $\mathbb{Z}^d$ , there exists  $u_1, \dots, u_n \in \mathcal{V}$  such that  $y - x = \sum_{i=1}^n u_i$ .

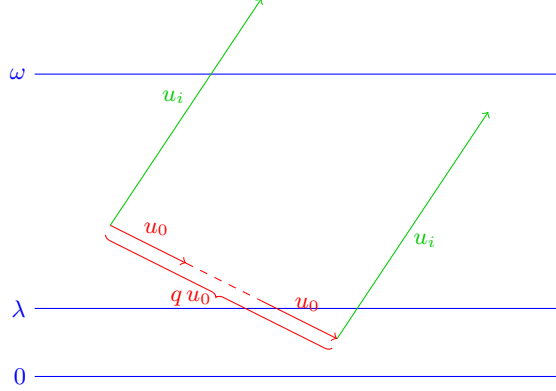
By reordering the  $u_i$ , we can assume that there exists  $M(y - x) \in \mathbb{N}$  such that

$$\forall i \in \llbracket 1, n \rrbracket, u_i \in \{-u_0, u_0\} \Leftrightarrow i > M(y - x).$$

In other words, we place the vectors  $u_0$  and  $-u_0$  at the end of the sum. Let us show that  $x$  and  $y$  are  $\mathcal{V}$ -connected in  $\mathbb{P}(v, 0, \omega)$  by induction on  $M(y - x)$ .

If  $M(y - x) = 0$  then there exists  $q \in \mathbb{Z}$  such that  $y - x = qu_0$ . Should we exchange  $x$  and  $y$ , we can suppose  $q \in \mathbb{N}$ . We then pose, for  $j \in \llbracket 0, q \rrbracket$ ,  $x_j \stackrel{\text{def}}{=} x + ju_0$ . From then on, the  $x_j$  form a  $\mathcal{V}$ -path from  $x$  to  $y$ . Moreover,  $\forall j \in \llbracket 0, q \rrbracket$ ,  $0 \leq \langle x, v \rangle \leq \langle x_j, v \rangle \leq \langle y, v \rangle < \omega$  so this path has values in  $\mathbb{P}(v, 0, \omega)$ .

If  $M(y - x) > 0$  then  $u_1 \notin \{-u_0, u_0\}$ . Should we exchange  $x$  and  $y$ , we can suppose  $\langle v, u_1 \rangle \geq 0$ . There exists  $q \in \mathbb{N}$  such that  $\langle x, v \rangle - q\lambda \in [0, \lambda[$ . Consequently, by posing  $x' \stackrel{\text{def}}{=} x - qu_0$ ,  $x' \in \mathbb{P}(v, 0, \omega)$  (because  $\lambda \leq \omega$ ) and is  $\mathcal{V}$ -connected to  $x$  in  $\mathbb{P}(v, 0, \omega)$ .



**Fig. 9.** Illustration of the upper bound proof. To move  $u_i$ , we return to the layer  $[0, \lambda[$  using the vector  $u_0$ .

Moreover, by posing  $x'' \stackrel{\text{def}}{=} x + u_1$ ,  $0 \leq \langle x', v \rangle \leq \langle x', v \rangle + \langle u_1, v \rangle < \lambda + \langle u_1, v \rangle \lambda + \|v\|_{\mathcal{V}}$ . Therefore  $x'' \in \mathbb{P}(v, 0, \omega)$  and is  $\mathcal{V}$ -connected to  $x'$  in  $\mathbb{P}(v, 0, \omega)$ , thus to  $x$  by transitivity.

Now  $y - x'' = \sum_{i=2}^n u_i + q u_0$  so  $M(y - x'') = M(y - x) - 1$ . By induction hypothesis, one deduces that  $x''$  and  $y$  are connected in  $\mathbb{P}(v, 0, \omega)$ . It is thus also the case for  $x$  and  $y$ . Thus,  $\mathbb{P}(v, 0, \omega)$  is  $\mathcal{V}$ -connected.  $\square$

We deduce from this result that  $\Omega(v) \leq \|v\|_{\mathcal{V}} + \lambda(v)$ . The assumption of usefulness of  $\mathcal{V}$  is not necessary for the proof of the upper bound. The presence of useless vectors only increases (unnecessarily) the  $\|v\|_{\mathcal{V}}$  norm. Restricting ourselves to a useful neighbourhood thus allows to improve the upper bound on  $\Omega(v)$ .

**Proposition 5.10** *Let  $u \in \mathcal{V}$  such that  $\Omega(v) < \langle u, v \rangle$ . It is further assumed that  $v$  is not semi-pathological. Then  $u$  is useless.*

*Proof.* We pose  $\omega \stackrel{\text{def}}{=} \langle u, v \rangle$ . Let  $\omega > \Omega(v)$ . As  $v$  is not semi-pathological,  $\lambda \leq \Omega(v) < \omega$ . As  $\omega = \langle u, v \rangle$ ,  $0 \in \mathbb{P}(v, 0, \omega)$  and  $u \notin \mathbb{P}(v, 0, \omega)$ .

There exists  $q \in \mathbb{N}$  such that  $\langle u, v \rangle - q\lambda \in [0, \lambda[$ . We pose  $u' \stackrel{\text{def}}{=} u - q u_0$ . As  $\lambda < \omega$ ,  $u' \in \mathbb{P}(v, 0, \omega)$ . By connectedness of  $\mathbb{P}(v, 0, \omega)$ , there exists a path from 0 to  $u'$  in  $\mathbb{P}(v, \mu, \omega)$  formed of vectors  $u_1, u_2, \dots, u_k \in \mathcal{V}$ .

Let  $i \in \llbracket 1, k \rrbracket$ . By posing  $x \stackrel{\text{def}}{=} \sum_{j=1}^{i-1} u_j$  and  $y \stackrel{\text{def}}{=} x + u_i$ , we notice that  $x, y \in \mathbb{P}(v, 0, \omega)$ . Therefore  $0 \leq \langle x, v \rangle < \omega$  and  $0 \leq \langle x, v \rangle + \langle u_i, v \rangle < \omega$ . Consequently,  $|\langle u_i, v \rangle| < \omega$ . Thus, as  $\langle u_i, v \rangle < \omega$ ,  $u_i$  cannot be in  $\{-u, u\}$ .

Let us show that  $u$  is useless, by showing items 1) and 2) of the definition. For all  $i \in \llbracket 1, k \rrbracket$ ,  $u_i \in \mathcal{V}'$ .

1.  $u = u' + q u_0 = \sum_{i=1}^k u_i + q u_0$ .
2. (a) For  $j \in \llbracket 0, k \rrbracket$ , as  $\sum_{i=1}^j u_i \in \mathbb{P}(v, 0, \omega)$ , we have  $0 \leq \sum_{i=1}^j \langle u_i, v \rangle < \omega = \langle u, v \rangle$ .  
 (b) For  $j \in \llbracket 0, q \rrbracket$ ,  $0 \leq \sum_{i=1}^k \langle u_i, v \rangle + j \langle u_0, v \rangle = \langle u', v \rangle + j \langle u_0, v \rangle \leq \langle u', v \rangle + q \langle u_0, v \rangle = \langle u, v \rangle$ .

So  $u$  is useless.  $\square$

From this proposition, we deduce that, if  $\mathcal{V}$  is useful then  $\Omega(v) \geq \langle u, v \rangle$ . Taking the max,  $\Omega(v) \geq \|v\|_{\mathcal{V}}$ . We could thus prove the following theorem.

**Theorem 5.11** *Let  $v$  be a useful neighbourhood and  $v$  a non semi-pathological vector. Then*

$$\|v\|_{\mathcal{V}} \leq \Omega(v) \leq \|v\|_{\mathcal{V}} + \lambda_{\mathcal{V}}(v).$$

## 5.4 Other bounds

Even if the bounds obtained previously are quite satisfactory, one observes experimentally, in the zones where the useful neighbourhoods are of size greater than  $2d$ , that the thickness is often close to  $2\lambda(v)$ . This observation led us to find a bound that involves the quantity  $2\lambda(v)$ . The idea is that the thickness  $2\lambda$  corresponds to the case where no point of the plane is isolated.

**Proposition 5.12** *Let  $v$  be an irrational vector and  $\mathcal{V}$  a neighbourhood. Let  $\omega < 2\lambda(v)$ . Then  $\mathbb{P}(v, 0, \omega)$  contains an isolated fibre for  $\mathcal{V}$  (in the sense that this fibre is not connected to any other fibre in the plane).*

*Proof.* we consider  $\lambda(v)$  and we pose  $I \stackrel{\text{d\'ef}}{=} ]\omega - \lambda, \lambda[$  which is a non-empty open set because  $\omega < 2\lambda$ . As  $v$  is irrational, by density of  $\{\langle x, v \rangle\}_{x \in \mathbb{Z}^d}$ , there exists  $x \in \mathbb{Z}^d$  such that  $\langle x, v \rangle \in I$ . Thus,  $x \in \mathbb{P}(v, 0, \omega)$ . Let  $u \in \mathcal{V}^+$ .  $\langle x + u, v \rangle = \langle x, v \rangle + \langle u, v \rangle > (\omega - \lambda) + \lambda = \omega$ .  $\langle x - u, v \rangle = \langle x, v \rangle - \langle u, v \rangle < \lambda - \lambda = 0$ . So the fibre of  $x$  is isolated in  $\mathbb{P}(v, 0, \omega)$ .  $\square$

**Proposition 5.13** *Let  $v$  be a rational vector with  $\gcd 1$  and  $\mathcal{V}$  a neighbourhood. Let  $\omega < 2\lambda(v) - 1$ . Then  $\mathbb{P}(v, 0, \omega)$  contains an isolated fibre for  $\mathcal{V}$ .*

*Proof.* Consider  $\lambda \stackrel{\text{d\'ef}}{=} \lambda(v)$  and look at the level  $\lambda - 1$  fibre. It is non-empty by Bézout's theorem. Let  $x$  be a point of this fibre. We have  $x \in \mathbb{P}(v, 0, \omega)$ . Let  $u \in \mathcal{V}^+$ .  $\langle x + u, v \rangle = \langle x, v \rangle + \langle u, v \rangle \geq (\lambda - 1) + \lambda = 2\lambda - 1 > \omega$ .  $\langle x - u, v \rangle = \langle x, v \rangle - \langle u, v \rangle \leq (\lambda - 1) - \lambda = -1 < 0$ . So the level  $\lambda - 1$  fibre is isolated in  $\mathbb{P}(v, 0, \omega)$ .  $\square$

Apart from the semi-pathological case (where the plane  $\mathbb{P}(v, 0, \omega)$  can contain a single non-empty fibre for  $\omega > \Omega(v)$ ), we thus obtain another lower bound on  $\Omega$ .

**Theorem 5.14** *Let  $v$  be a non semi-pathological vector and  $\mathcal{V}$  a neighbourhood. Then  $\Omega(v) \geq 2\lambda_{\mathcal{V}}(v) - \gcd(v)$ . In particular, in the irrational case,  $\Omega(v) \geq 2\lambda_{\mathcal{V}}(v)$ .*

## 6 Algorithm of Yan Gérard

In his 2002 article [5], Y. Gérard shows the decidability of the connectedness problem of rational planes for any neighbourhood. Although the algorithm does not terminate in the irrational case, it allows however to restrict to the case of totally irrational  $v$  vectors, i.e. verifying  $\dim_{\mathbb{Q}}(v) = d$ .

We give here a brief description of the algorithm in the case we are interested in. The idea is to make a breadth-first search of  $\mathbb{P}(v, 0, \omega)$  whose fibres are represented by a graph  $G$ . When we find  $x, y$  which are on the same fibre, we add  $y - x \in v^\perp$  to a set  $H$  which is a generator of the reachable vectors of  $v^\perp$ . Connectedness is concluded if all non-empty fibres are reached by an element of  $G$  and  $\text{Vect}_{\mathbb{Z}}(H) = v^\perp \cap \mathbb{Z}^d$ .

Note that this algorithm terminates only in the rational case. For irrational vectors, there exist infinitely many fibres. Since this is a breadth-first search, the sets  $G$  and  $H$  obtained at the limit allow us to conclude that the system is connected, but they are not computable in finite time.

This algorithm can be easily adapted to decide whether or not a vector  $u$  is useless for  $v$  with the neighbourhood  $\mathcal{V}$ . We define for that  $\bar{\mathbb{P}}(v, 0, \omega) \stackrel{\text{d\'ef}}{=} \{x \in \mathbb{Z}^d \mid 0 \leq \langle x, v \rangle \leq \omega\}$  (we authorise the level  $\omega$  fibre). Therefore,  $u$  is useless if  $u$  is accessible from 0 in  $\bar{\mathbb{P}}(v, 0, \langle u, v \rangle)$  with the neighbourhood  $\mathcal{V} \setminus \{-u, u\}$ .

Given the graph  $G$  and the set  $H$  returned by Yan Gérard's algorithm,  $u$  is useless iff the fibre of  $u$  appears in  $G$  via a point  $x$  and  $u - x$  is generated by  $H$ . We can therefore compute a useful neighbourhood of a vector in the rational case (the irrational case requires to go to the limit in the algorithm).

**Theorem 6.1** *The uselessness problem of a vector is decidable in the rational case. We can therefore calculate a useful neighbourhood of a vector.*

**Algorithm:** Algorithm of Yan Gérard [5].

**Input:**  $v$  a non-zero vector,  $\mathcal{V}$  neighbourhood,  $\omega > 0$ .

**Output:** The graph of the connected component of 0, a boolean indicating whether  $\mathbb{P}(v, 0, \omega)$  is connected.

$G \leftarrow$  empty graph;

$li \leftarrow 0$  (list of vertices to be visited, with their predecessor in the breadth-first search);

$H \leftarrow \emptyset$  (generator of reachable vectors of  $v^\perp$ );

**while**  $li \neq []$  **do**

    Extract the first element  $x$  of  $li$ ;

**if** there exists  $y \in G$  such that  $y - x \in v^\perp$  (the fibre of  $x$  has already been visited) **then**

**if**  $y - x$  is generated by  $H$  **then**

            Do nothing ( $x$  is already counted in the connected component of 0)

**else**

            Add  $y - x$  to  $H$ ;

**else**

        (the fibre of  $x$  is new);

        Add  $x$  to  $G$  and connect it to the other vertices according to the relation  $\mathcal{V}$ ;

**for**  $u \in \mathcal{V}$  **do**

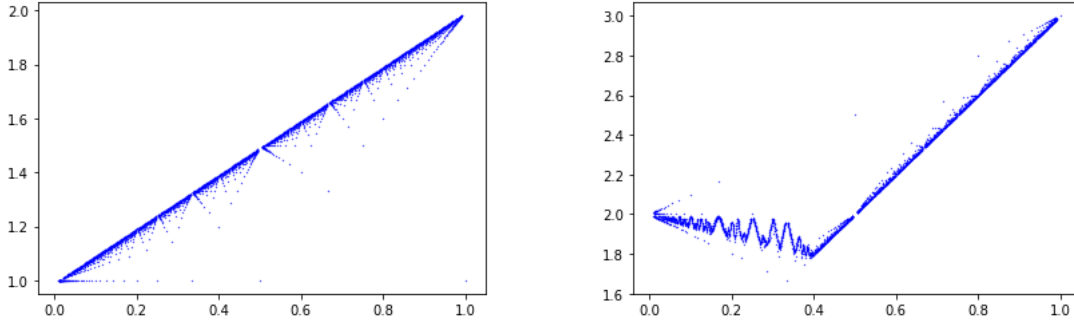
**if**  $\langle x + u, v \rangle \in [0, \omega[$  **then**

                Add  $x + u$  to the end of  $li$ .

$b_1 \leftarrow |G|$  = number of fibres in  $\mathbb{P}(v, 0, \omega)$ ;

$b_2 \leftarrow "H \text{ generates } v^\perp \cap \mathbb{Z}^d"$ ;

Return  $G, H, b_1$  and  $b_2$ .



**Fig. 10.** The connecting thickness by facets in 2D (on the left) and for the knight-neighbourhood (on the right).

## 7 Analytical approach

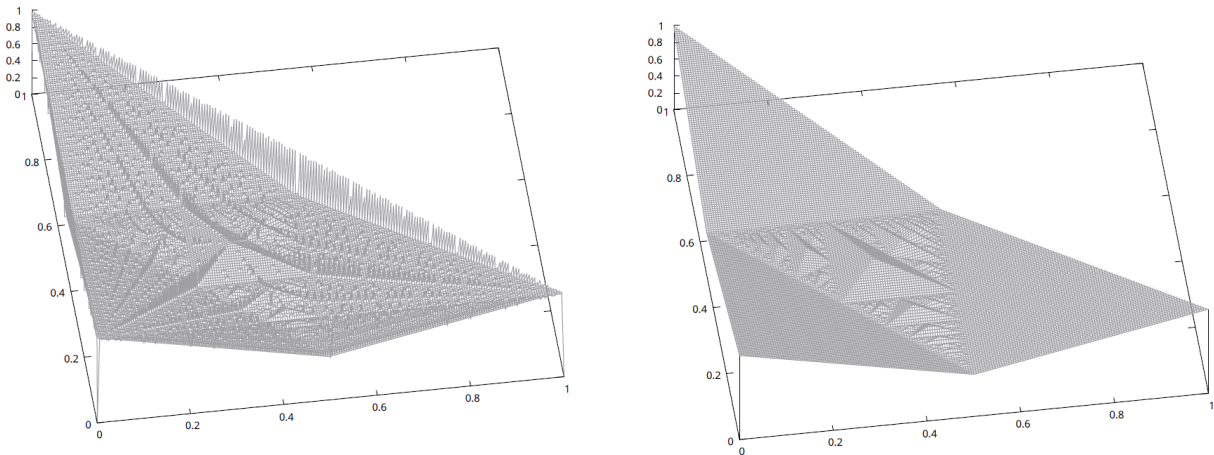
The main idea of the article is to see  $\Omega$  not as a combinatorial object but as a function that can be studied analytically. We were thus able to draw some representations of  $\Omega : \mathbb{R}^d \rightarrow \mathbb{R}$ . To plot  $\Omega$ , we use its homogeneity property:  $\Omega(\lambda v) = |\lambda| \Omega(v)$ . We can therefore restrict ourselves to a cut of  $\mathbb{R}^d$  by an affine hyperplane to know the values of the function at any point. In Figure 10 (left), we plot the connecting thickness by facets in 2D (Abscissa:  $\alpha$ , Ordinate:  $\Omega((\alpha, 1))$ ). Figure 10 (right) focuses on the knight-neighbourhood.

It can be noted that the function is discontinuous in general. It is therefore, at first sight, difficult to propose an analytical study of this function. However, in the case of Figure 10 (left), we have an explicit expression for  $\Omega$ :  $\Omega((x, y)) = |x| + |y| - \gcd(x, y)$ . The discontinuities thus come from the gcd function, and can be avoided by restricting  $\Omega$  to irrational vectors. Indeed, if  $x$  and  $y$  are independent on  $\mathbb{Q}$  then  $\Omega((x, y)) = |x| + |y|$ .

For  $\Omega$  by facets in dimension 3 (Abscissa:  $(x, y)$ , Ordinate:  $\Omega((x, y, 1 - x - y))$ ), shown in Figure 11 (left), finding a closed formula is less obvious. One still observes discontinuity peaks, which line up in the form of combs. Below the figure, a continuous underlying shape can be seen.

Again, the peaks come from the arithmetic relations between the coordinates of the vectors. By considering totally irrational vectors (having a dimension on  $\mathbb{Q}$  equal to  $d$ ), we remove these relations which make appear





**Fig. 11.** Graphical representation of  $(x, y) \mapsto \Omega(x, y, 1 - x - y)$  (left) and underlying structure (right) for the facet connectedness.

gcd. We then obtain the continuous underlying structure of the  $\Omega$  function and we can observe its periodic character (see Figure 11 (right)).

The structure that appears can be explained from the fully subtractive algorithm. The outer plane is a zone where we conclude directly thanks to the condition  $|v_i| + |v_j| \leq |v_k|$ . The thickness returned is then  $|v_k|$  (we remove the gcd which comes from the dependency relations between vector coordinates).

For the periodic zone, the central pyramid corresponds to a zone where we make a reduction step before concluding (by falling back on a plane). For the intermediate pyramids, one reduces to the central pyramid, then concludes after another reduction. The size of the pyramids thus represents the number of reductions necessary to conclude. The limit points of the structure form the Kraaikamp and Meester set (points for which the algorithm cannot conclude in a finite number of reductions).

We show in the rest of this section the continuity of the underlying structure. We deduce from this result an algorithm for computing  $\Omega$  for any vector.

## 7.1 Path Bounding Theorem

In order to show the regularity of  $\Omega$  restricted to irrationals, it is essential to first obtain a control on the length of the paths in a plane. Given 2 points  $x, y \in \mathbb{P}(v, \mu, \omega)$ , we are interested in the link between their Euclidean distance  $\|y - x\|_2$  and their distance  $d(x, y)$  in the graph induced by the plane and the neighbourhood  $\mathcal{V}$ .

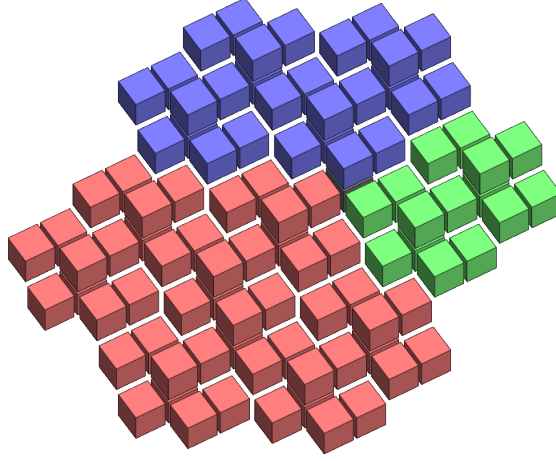
We can first notice that  $\|y - x\|_2 \leq C d(x, y)$  for a constant  $C$  independent of  $x$  and  $y$ . We simply use a triangle inequality for this: “If two points are close in the graph then they are close in Euclidean distance.” However, the converse is not true, as the example in Figure 12.

The plane considered here is a tree, so there is one and only one path between 2 points  $x$  and  $y$ . By removing the point 0 from the plane, we obtain 3 connected components of the plane. The path between 2 points  $x$  and  $y$  which are in 2 different components therefore necessarily passes through 0. Now, we can find such points  $x, y$  at Euclidean distance  $\sqrt{2}$  and  $d(x, y) = d(x, 0) + d(y, 0)$  is unbounded. We cannot therefore control a priori the length of the paths in the plane with the Euclidean norm.

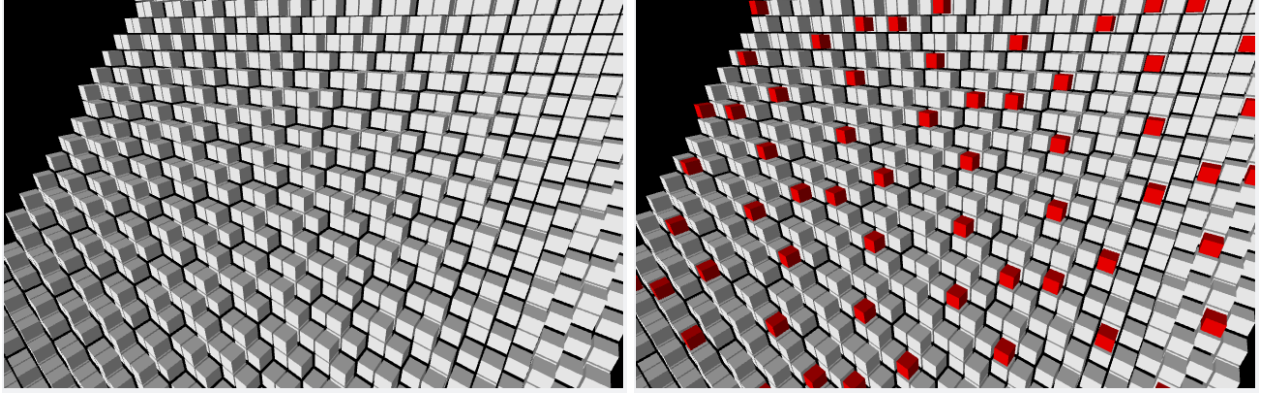
However, this problem can be solved by increasing the thickness very slightly (see Figure 13). If the plane  $\mathbb{P}(v, \mu, \omega)$  is connected, one can limit the length of the paths between 2 close points in Euclidean distance in the plane  $\mathbb{P}(v, \mu, \omega + \varepsilon)$ : one thus avoids critical cases like that of Figure 13 (left).

The paths are then said to be bounded. The study of path bounding is the objective of this section.

**Definition 7.1** *Let  $v$  be a non-zero vector of  $\mathbb{R}^d$ . Let  $\omega, r, \varepsilon > 0$  and  $\mu \in \mathbb{R}$ . We say that an integer  $M$  bounds the paths for  $v, \mu, \omega, r, \varepsilon$  iff*



**Fig. 12.** A facet-connected hyperplane. The different colours are the connected components of  $\mathbb{P} \setminus \{0\}$ .



**Fig. 13.** A plane at connecting thickness (left) and at a greater thickness (right).

$\forall x, y \in \mathbb{P}(v, \mu, \omega)$ , if  $\|y - x\|_2 \leq r$  then there exists a path from  $x$  to  $y$  in  $\mathbb{P}(v, \mu, \omega + \varepsilon)$  of length less than  $M$ .

We note in particular  $M(v, \mu, \omega, r, \varepsilon)$  the smallest integer verifying this property (if it exists, otherwise we will take  $M(v, \mu, \omega, r, \varepsilon) = \infty$ ).

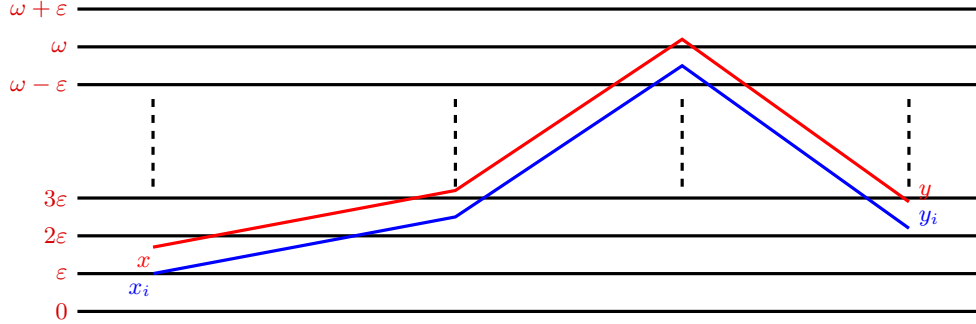
**Remark 7.2** The  $\omega$  dependence of  $M$  is not necessary because an interesting case is the one where  $\omega = \Omega(v, \mu)$  (the plane at the thickness is not always connected, but this problem disappears when we add  $\varepsilon$ ). As  $r$  will often be fixed later on, we will also allow ourselves to omit it in the expression of  $M$ . We will thus allow ourselves the abuse of writing  $M(v, \mu, \varepsilon)$ .

A disjunction of cases arises between the irrational and rational planes. In the first case, we use density results to conclude, in the second the Bézout's theorem. The 2 proofs are however analogous, we simply replace the  $\varepsilon$  with  $\gcd(v)$ . In both cases, we first show that the paths are bounded in the case of zero shift, before showing that this is true for any shift, as well as the independence in  $\mu$  of  $M$ .

**Proposition 7.3 (Irrational path bounding)** *Let  $v$  be an irrational vector and  $\omega > 0$  such that  $\mathbb{P}(v, 0, \omega)$  is connected. Let  $\varepsilon > 0$  and  $r \in \mathbb{R}_+$ . Then there exists  $M \in \mathbb{R}$  which bounds the paths for  $v, (\mu = 0), \omega, r, \varepsilon$ .*

*Idea of proof:* The formal and complete proof is in appendix B.1. To show the theorem, we cut the interval  $[0, \omega[$  into layers of size  $\varepsilon$  (see Figure 14). We then take, by density,  $x_1 \in \mathbb{Z}^d$  such that  $\langle x_1, v \rangle = \varepsilon$ . The layer  $[i\varepsilon, (i+1)\varepsilon[$  is represented by the point  $x_i \stackrel{\text{d\'ef}}{=} ix_1$ . We then find a bound for the  $x_i$ 's because this set of points

is finite. We then want to return to this set of points for any  $x \in \mathbb{P}(v, 0, \omega)$ . We take  $i$  such that  $\langle x_i, v \rangle = i\varepsilon$ . The path from  $x$  to  $y$  then corresponds to a path from  $x_i$  to  $y_i$  which is bounded. The division into layers ensures moreover that the path from  $x$  to  $y$  does not go out of the  $\mathbb{P}(v, 0, \omega + \varepsilon)$  plane.



**Fig. 14.** Illustration of the proof by layers. The horizontal axis represents the progress in the path, the vertical axis the evolution of the scalar product with  $v$ . The red and the blue paths are plotted parallel because the 2 paths follow the same neighbourhood vectors.

This result is generalised for any shift.

**Proposition 7.4** *Let  $v$  be an irrational vector and  $\mu, \omega > 0$  such that  $\mathbb{P}(v, \mu, \omega)$  is connected. Let  $\varepsilon > 0$  and  $r \in \mathbb{R}_+$ . Then there exists  $M$  which bounds the paths for  $v, \mu, \omega, r, \varepsilon$ . In addition,  $M(v, 0, \frac{\varepsilon}{2}) \geq M(v, \mu, \varepsilon)$ .*

*Proof.* As  $v$  is irrational, for any interval  $I$  of  $\mathbb{R}$ , there exists, by density,  $\mu' \in I$  such that  $\mathbb{P}(v, \mu', \omega)$  is a translate of  $\mathbb{P}(v, \mu, \omega)$ . By taking  $I = ]0, \frac{\varepsilon}{2}]$ , we can suppose without loss of generality  $\mu \in ]0, \frac{\varepsilon}{2}]$ .

As  $\mathbb{P}(v, \mu, \omega)$  is connected and  $v$  is irrational,  $\mathbb{P}(v, 0, \omega + \frac{\varepsilon}{4})$  is also connected. Thus, by irrational path bounding (zero shift), there exists  $M$  such that  $\forall x, y \in \mathbb{P}(v, 0, \omega)$ , if  $\|y - x\|_2 \leq r$  then there exists a path from  $x$  to  $y$  in  $(v, 0, \omega + \frac{\varepsilon}{2})$  of length less than  $M$ . We notice by the way that  $M$  does not depend on  $\mu$ . It is enough to notice that  $\mathbb{P}(v, \mu, \omega) \subset \mathbb{P}(v, 0, \omega + \frac{\varepsilon}{2}) \subset \mathbb{P}(v, \mu, \omega + \varepsilon)$  to conclude.  $\square$

In the rational case, the proofs are the same as in the irrational case, by replacing the density arguments by Bézout and by properly handling the shifts. We can therefore refer to Figure 14 by replacing  $\varepsilon$  with  $\gcd(v)$  to reproduce the same arguments.

**Proposition 7.5 (Rational paths bounding)** *Let  $v$  be a rational vector and  $\omega > 0$  such that  $\mathbb{P}(v, 0, \omega)$  is connected. Let  $r \in \mathbb{R}_+$ . Then there exists  $M$  which bounds the paths for  $v, (\mu = 0), \omega, r, (\varepsilon = 0)$ .*

*Proof.* in appendix B.2.

This result also generalises to any shift.

**Proposition 7.6** *Let  $v$  be a rational vector and  $\mu, \omega > 0$  such that  $\mathbb{P}(v, 0, \omega)$  is connected. Let  $r \in \mathbb{R}_+$ . Then there exists  $M$  bounding the paths for  $v, \mu, \omega, r, (\varepsilon = \bar{\mu})$  with  $\bar{\mu} \stackrel{\text{def}}{=} \mu \bmod \gcd(v)$ . In addition,  $M(v, 0, 0) \geq M(v, \mu, \bar{\mu})$ .*

*Proof.* We pose  $\lambda = \lambda_v(v)$  and we take  $u_0 \in \mathcal{V}$  such that  $\lambda = \langle u_0, v \rangle$ . As  $v$  is rational,  $\mathbb{P}(v, \bar{\mu}, \omega)$  is a translate of  $\mathbb{P}(v, \mu, \omega)$ . One can thus suppose  $0 \leq \mu < \gcd(v)$ .

As  $\mathbb{P}(v, 0, \omega)$  is connected, by the path bounding proposition (zero shift) 7.5, there exists  $M$  such that:  $\forall x, y \in \mathbb{P}(v, 0, \omega)$ , if  $\|y - x\|_2 \leq r$  then there exists a path from  $x$  to  $y$  in  $\mathbb{P}(v, 0, \omega)$  of length less than  $M$ . We notice by the way that  $M$  does not depend on  $\mu$ . We conclude by noticing that  $\mathbb{P}(v, \mu, \omega + \mu) = \mathbb{P}(v, 0, \omega)$  because  $0 \leq \mu < \gcd(v)$ .  $\square$

From all these propositions, we get a general theorem of path bounding.

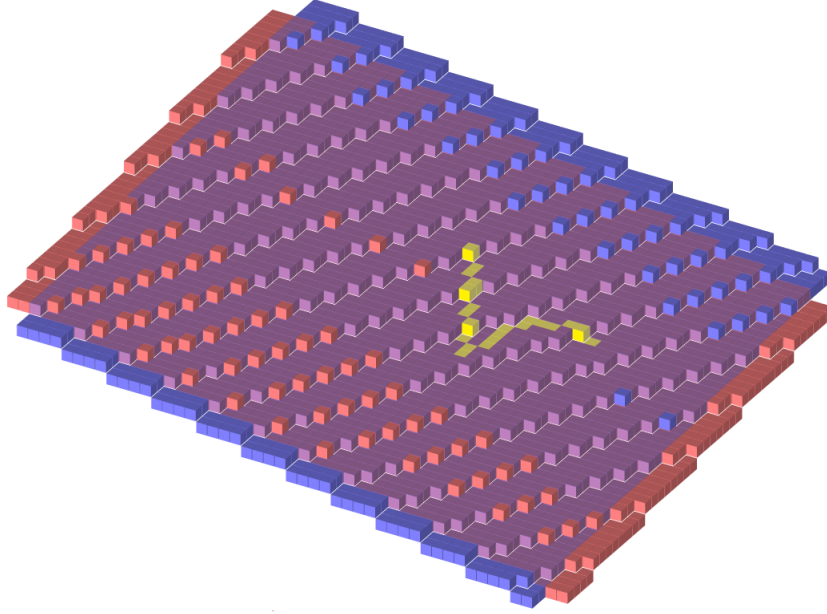
**Theorem 7.7** *Let  $v$  be a non-zero vector and  $\mu, \omega > 0$  such that  $\mathbb{P}(v, 0, \omega)$  is connected. Let  $r, \varepsilon > 0$ . Then there exists  $M$  independent of  $\mu$  and bounding the paths for  $v, \mu, \omega, r, \gcd(v) + \varepsilon$ .*

## 7.2 A semi-continuity property

In this part, we seek to prove a first semi-continuity inequality (a half continuity). For  $v$  a non-zero vector and  $\varepsilon > 0$ , we wish to show that for any  $v'$  sufficiently close to  $v$ ,  $\Omega(v') \leq \Omega(v) + \varepsilon + \gcd(v)$ , i.e. that the thickness of  $v'$  does not exceed the thickness of  $v$  by much.

**Theorem 7.8 (Semi-continuity)** *Let  $v \in \mathbb{R}^d$  with  $v \neq 0$  and  $\varepsilon > 0$ . Then there exists  $\eta > 0$  such that  $\forall v' \in \mathbb{R}^d, \|v - v'\|_2 \leq \eta \Rightarrow \Omega(v') \leq \Omega(v) + \gcd(v) + 16\varepsilon$ .*

*Idea of proof:* The formal and complete proof can be found in the appendix B.3. We suppose by contradiction, for  $v'$  close to  $v$ , that  $\Omega(v') > \Omega(v) + 16\varepsilon + \gcd(v)$ . We place ourselves at a thickness  $\omega_0 \in ]\Omega(v) + \gcd(v), \Omega(v')[$ . As  $\mathbb{P}(v', 0, \omega_0)$  is not connected, we can (via a lemma) exhibit  $x$  and  $y$  in 2 different connected components of  $\mathbb{P}(v', 0, \omega_0)$  but close in distance in  $\mathbb{Z}^d$ . We will then consider  $\mathbb{P}(v, \mu, \omega_0)$ , which is connected, to superpose it (like in Figure 15) on top of this zone in an intelligent way (i.e. by making  $\langle x, v \rangle + \mu \approx \langle x, v' \rangle$ ). This will give us a path between  $x$  and  $y$  of bounded length thanks to the path bounding theorem 7.7. The scalar products of the points traversed with  $v$  and  $v'$  are close, because  $\langle x, v \rangle + \mu \approx \langle x, v' \rangle$  and because the path has bounded length. This therefore ensures that there is a path from  $x$  to  $y$  in the plane  $\mathbb{P}(v', 0, \omega_0)$ .



**Fig. 15.** Illustration of the superposition of hyperplanes  $\mathbb{P}(v, \mu, \omega_0)$  (in blue) and  $\mathbb{P}(v', 0, \omega_0)$  (in red) around  $x$  and  $y$  (in yellow) which are connected by a path (in yellow).

We notice, in the formal proof, that the value of  $\eta$  depends on  $v$  only via the constant  $M$  of the path bounding. In the next section, we obtain a uniform value for  $M$  (i.e. not depending on  $v$  on a compact), which then allows to find a uniform value of  $\eta$ . From then on, we can invert the quantifiers and prove the double continuity inequality of  $\Omega$  on a compact.

## 7.3 Relative continuity

The aim of this part is to obtain a uniform bounding of  $M$  on a compact. This bounding is possible in the case of irrational vectors, by restricting ourselves to certain vectors  $v'$  possessing the same dependence relations as  $v$ . We can look at the proof in the case of totally irrational vectors (i.e. verifying  $\dim_{\mathbb{Q}}(v) = d$ ), which simplifies the matter.

**Definition 7.9** Let  $v$  be an irrational vector. We call the quotient set of  $v$ , the set  $Q(v) \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^d \mid \langle v, x \rangle = 0\} = v^\perp \cap \mathbb{Z}^d$ . This is the set of dependence relations with integer coefficients verified by  $v$ .

**Example 7.10** 1.  $Q((1, \sqrt{2}, 1 + \sqrt{2})) = \text{Vect}_{\mathbb{Z}}((-1, -1, 1))$ .

2.  $Q((1, \sqrt{2}, \pi)) = \{0\}$ .

3.  $Q((1, 2, 3)) = \text{Vect}_{\mathbb{Z}}((-2, 1, 0), (-3, 0, 1))$ .

**Remark 7.11** In the case of totally irrational vectors,  $Q(v) = \{0\}$  and the assumptions on  $Q(v)$  can therefore be omitted in the theorems.

**Proposition 7.12** Let  $v$  be an irrational vector. Let  $\omega > 0$  such that  $\mathbb{P}(v, 0, \omega)$  is connected. Let  $r, \varepsilon > 0$ . Then there exist  $\eta > 0$  and  $M \in \mathbb{R}$  such that:

$\forall v' \in \mathbb{R}^d$ , if  $\|v - v'\|_2 \leq \eta$  and  $Q(v) \subset Q(v')$  then  $M$  bounds the paths for  $v', (\mu = 0), \omega, r, \varepsilon$ .

**Remark 7.13** If  $v$  is totally irrational, the hypothesis  $Q(v) \subset Q(v')$  is always verified: we thus have a bounding of  $M$  on a compact.

*Idea of proof:* The complete and formal proof is in appendix B.4. We reuse the same scheme as in the path bounding theorem 7.7, this time separating into layers of size  $\frac{\varepsilon}{2}$ . We can then find  $M$  in the same way as before. To show that  $M$  fits on an open set around  $v$ , we simply have to notice that the paths used from  $x_i$  (which are in finite number) pass through a point  $x$  of zero scalar product only for elements  $x$  of  $Q(v)$ . Therefore, by taking  $v'$  close to  $v$  such that  $Q(v) \subset Q(v')$ , these points will always remain in  $\mathbb{P}(v', 0, \omega)$ .

Moreover, as  $M(v, \mu, \varepsilon) \leq M(v, 0, \frac{\varepsilon}{2})$ , we deduce the following theorem:

**Theorem 7.14 (Uniform path bounding)** Let  $v$  be an irrational vector,  $\omega > 0$  such that  $\mathbb{P}(v, 0, \omega)$  is connected,  $r, \varepsilon > 0$  and  $\mu \in \mathbb{R}$ . Then there exist  $\eta > 0$  and  $M \in \mathbb{R}$  (independent of  $\mu$ ) such that:

$\forall v' \in \mathbb{R}^d$ , if  $\|v - v'\|_2 \leq \eta$  and  $Q(v) \subset Q(v')$  then  $M$  bounds the paths for  $v', \mu, \omega, r, \varepsilon$ .

Since  $M$  is uniform on a compact around  $v$ , the proof of the semi-continuity theorem 7.8 provides a double inequality. We thus obtain the following theorem.

**Theorem 7.15 (Relative continuity)** Let  $v$  be irrational and  $\varepsilon > 0$ . Then there exists  $\eta > 0$  such that  $\forall v' \in \mathbb{R}^d$ , if  $\|v - v'\|_2 \leq \eta$  and  $Q(v) \subset Q(v')$  then  $|\Omega(v) - \Omega(v')| \leq \text{gcd}(v') + \varepsilon$ .

This shows in particular that the restriction of  $\Omega$  to totally irrational vectors (verifying  $\dim_{\mathbb{Q}}(v) = d$ ) is continuous.

## 7.4 Calculation of the connecting thickness

We can thus deduce from this study the main theorem of the article, which offers a sequential characterisation of relative continuity.

**Theorem 7.16 (Relative continuity)** Let  $v$  be an irrational vector. Let  $(v_n)_{n \in \mathbb{N}}$  be a sequence of vectors converging to  $v$  and such that  $\forall n \in \mathbb{N}, Q(v) \subset Q(v_n)$ . Then  $\lim_{n \rightarrow +\infty} \Omega(v_n) = \Omega(v)$ .

The proof of the theorem is based on the following lemma.

**Lemma 7.17** Let  $(v_n)_{n \in \mathbb{N}}$  be a sequence of vectors converging to an irrational vector  $v$ . Then  $\lim_{n \rightarrow +\infty} \text{gcd}(v_n) = 0$ .

*Proof.* We pose  $\forall n \in \mathbb{N}, u_n \stackrel{\text{def}}{=} \text{gcd}(v_n)$ . As  $(v_n)_{n \in \mathbb{N}}$  converges, it is bounded. So the sequence  $u$  is also bounded. Let us then show that the sequence  $u$  admits 0 as the only adherence value to conclude. Let  $\alpha$  be an adherence value of  $u$  and suppose by contradiction  $\alpha \neq 0$ . Should we extract, we can suppose without loss of generalities that  $u_n \rightarrow \alpha$ . The sequence  $u$  being ultimately strictly positive, should we have to extract, we can suppose it strictly positive.

Let us then consider the sequence  $w = \left(\frac{1}{u_n} v_n\right)_{n \in \mathbb{N}}$  (which is well defined thanks to the preceding remark).

This sequence is, by definition of the gcd, with values in  $\mathbb{Z}^d$  which is closed. Moreover,  $w$  converges to  $\frac{1}{\alpha} v$ , so  $\frac{1}{\alpha} v \in \mathbb{Z}^d$ . Therefore  $\text{gcd}(v) \geq \alpha > 0$ : contradiction.  $\square$

*Proof. of the theorem.* Let  $(v_n)_{n \in \mathbb{N}}$  be a sequence converging to  $v$  and such that  $\forall n \in \mathbb{N}, Q(v) \subset Q(v_n)$  and  $\varepsilon > 0$ . By the preceding lemma, the sequence  $(\gcd(v_n))_{n \in \mathbb{N}}$  converges to 0. So there exists a rank  $n_0$  such that  $\forall n \geq n_0, \gcd(v_n) \leq \frac{\varepsilon}{2}$ .

By the relative continuity theorem 7.15, there exists  $\eta > 0$  such that  $\forall v' \in \mathbb{R}^d$ , is  $\|v - v'\|_2 \leq \eta$  and  $Q(v) \subset Q(v')$  then  $|\Omega(v) - \Omega(v')| \leq \gcd(v') + \frac{\varepsilon}{2}$ .

By convergence of  $(v_n)_{n \in \mathbb{N}}$  towards  $v$ , there exists a rank  $n_1$  such that  $\forall n \geq n_1, \|v - v_n\|_2 \leq \eta$ .

Moreover, as  $\forall n, Q(v) \subset Q(v_n)$ , we deduce that  $\forall n \geq n_1, |\Omega(v_n) - \Omega(v)| \leq \gcd(v_n) + \frac{\varepsilon}{2}$ .

By posing  $N = \max(n_0, n_1)$ , we obtain that  $\forall n \geq N, |\Omega(v_n) - \Omega(v)| \leq \gcd(v_n) + \frac{\varepsilon}{2} \leq \varepsilon$ .

Therefore  $\lim_{n \rightarrow +\infty} \Omega(v_n) = \Omega(v)$ .  $\square$

To calculate the connecting thickness of an irrational vector, one thus starts by taking a sequence  $(v_n)_{n \in \mathbb{N}}$  of rational vectors converging towards  $v$  and respecting  $\forall n, Q(v) \subset Q(v_n)$ . We then calculate, by Yan Gérard's algorithm,  $(\Omega(v_n))_{n \in \mathbb{N}}$ , which is a sequence of rationals. Taking the limit, we obtain the desired value  $\Omega(v)$ .

## 7.5 Reductions and periodic shapes

Looking at the graph of  $\bar{\Omega}$ , one can observe numerous periodic zones, as shown in Figure 11. In the case of facet connectedness, the presence of these areas is explained by the fully subtractive algorithm and the reductions that occur in it. We give here sufficient (but far from necessary) conditions for such reductions to be observed in the case of any neighbourhood  $\mathcal{V}$ . The areas where these reductions apply appear graphically as periodicities.

In the case of facet connectedness, the correction of the totally subtractive algorithm proved in [2] came from the following reduction: if  $v$  is a positive vector of minimal coordinate  $\lambda = v_i > 0$  then, by posing  $v' \stackrel{\text{def}}{=} v - \lambda(1 - e_i)$  (where  $1 = (1, \dots, 1)$ ), we obtain

$$\forall \omega \in \mathbb{R}, \mathbb{P}(v, 0, \omega) \text{ is connected} \Leftrightarrow \mathbb{P}(v', 0, \omega - \lambda) \text{ is connected}.$$

We wish to generalise this process for any neighbourhood. Let  $v \in \mathbb{R}^d$  with  $v \neq 0$  and  $\mathcal{V}$  a neighbourhood. We suppose moreover that  $\forall u \in \mathcal{V}, \langle u, v \rangle \neq 0$ . We recall that  $\mathcal{V}^+ \stackrel{\text{def}}{=} \{u \in \mathcal{V} \mid \langle u, v \rangle > 0\}$  and  $\lambda \stackrel{\text{def}}{=} \min_{u \in \mathcal{V}^+} \langle u, v \rangle$ . Finally we take  $u_0 \in \mathcal{V}$  such that  $\lambda = \langle u_0, v \rangle$ .

**Definition 7.18** Let  $\Delta \in \mathbb{Z}^d$ . We say that  $\Delta$  induces a valid reduction for  $v$  iff

$$\forall \omega \geq 2\lambda, \mathbb{P}(v, 0, \omega) \text{ is connected} \Leftrightarrow \mathbb{P}(v - \lambda\Delta, 0, \omega - \lambda) \text{ is connected}.$$

**Theorem 7.19 (reduction)** We suppose that there exists  $\Delta \in \mathbb{Z}^d$  such that  $\langle u_0, \Delta \rangle = 0$  and  $\forall u \in \mathcal{V}^+ \setminus \{u_0\}, \langle u, \Delta \rangle = 1$ . Then  $\Delta$  induces a valid reduction for  $v$ .

*Proof.* The proof is available in Appendix A.1. This is essentially a transcript in the general case of the reduction theorem of [2] established for the connectedness by facets.  $\square$

This theorem gives access to vectors  $\Delta$  inducing a valid reduction for  $v$ . One can for example conclude for the 2-connectedness in dimension 5 around the vector  $(4, 6, 12, 14, 23)$ . However, the assumptions of the theorem remain extremely strong. For example, the theorem never applies for the knight-neighbourhood.

For the connectedness by facets, we deduced from the reduction theorem that  $\Omega(v) = \lambda + \Omega(v - \lambda\Delta)$ . Here, the conclusion is slightly more subtle, because of the valid reduction definition.

**Theorem 7.20** Let  $\Delta$  induce a valid reduction for  $v$ . We pose  $v' \stackrel{\text{def}}{=} v - \lambda\Delta$ . Then  $\Omega(v) = \max(2\lambda - \gcd(v), \Omega(v') + \lambda)$ .

*Proof.* The demonstration is available in appendix A.2. It consists essentially in a translation of “ $\forall \omega \geq 2\lambda, \mathbb{P}(v, 0, \omega) \text{ is connected} \Leftrightarrow \mathbb{P}(v - \lambda\Delta, 0, \omega - \lambda) \text{ is connected}$ ” in terms of connecting thickness, via the study of different cases.

## 8 Conclusion

In this article, we have extended known notions and results on the connectedness of arithmetic hyperplanes. Already known for facet adjacency, the phenomenon of percolation observed on connectedness extends to all neighbourhoods. This phenomenon reveals a critical thickness  $\Omega$  that has the same bounds and the same dependence on translation as in the case of facet connectedness.

We have also defined and studied the notion of useful neighbourhood, which sometimes allows us to reduce the problem to the already solved case of facet connectedness. This approach greatly reduces the size of the neighbourhoods considered and concludes for face connectivities in dimensions 3 and 4. However, it remains limited, being ineffective for certain vectors in dimensions higher than 5 and for more exotic neighbourhoods such as knight's movement.

The major contribution of the article comes from an analytical approach that focuses mainly on irrational vectors. The graph of the function  $\Omega$  exhibits spikes and combs at each rational point due to the presence of greatest common divisors (gcds). Removing rational vectors and integer dependency relations between the coordinates of a vector yields a more regular version of the function, resulting in a periodic structure. The main theorem of this paper shows the continuity of this underlying function. We can deduce an algorithm for computing  $\Omega$  from this result.

More generally, an analytical approach and a focus on the irrational case can eliminate the combinatorial complexity with gcds. Many problems in discrete geometry can then be simplified using this viewpoint.

Despite these numerous results, many questions still remain unanswered.

We would like, in particular, to obtain a general reduction algorithm like the fully subtractive algorithm for facet connectedness. The reduction theorem that we have proved has very strong conditions and does not apply in all cases. However, we observe reductions for the knight's movement that we need to explain and prove.

The proof of relative continuity does not provide explicit bounds, due to the upper bound on paths. Therefore, our algorithm for computing  $\Omega$  converges, but the convergence rate is still unknown.

Finally, the behaviour with respect to thickness, for facet connectedness, had shown a deep connection with numeration systems and automata. We thus hope that the study of behaviour with respect to thickness in the general case may lead to other numeration systems.

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## A Reductions

### A.1 Proof of the reduction theorem

**Lemma:** Let  $\omega \geq 2\lambda$  and  $x \in \mathbb{P}(v, 0, \omega)$ . Then either  $x \in \mathbb{P}(v, 0, \omega - \lambda)$  or  $x - u_0 \in \mathbb{P}(v, 0, \omega - \lambda)$ .



*Proof.* Let  $\omega \geq 2\lambda$  and  $x \in \mathbb{P}(v, 0, \omega)$ . We set  $\omega' \stackrel{\text{def}}{=} \omega - \lambda$ . Assume  $x \notin \mathbb{P}(v, 0, \omega')$  and consider  $x - u_0$ .  $\langle x - u_0, v \rangle = \langle x, v \rangle - \lambda < \omega - \lambda = \omega'$ . Moreover,  $\langle x, v \rangle \geq \omega'$  so  $\langle x - u_0, v \rangle = \langle x, v \rangle - \lambda \geq \omega' - \lambda = \omega - 2\lambda \geq 0$  by assumption. Therefore,  $x - u_0 \in \mathbb{P}(v, 0, \omega')$ . Thus, if  $x \in \mathbb{P}(v, 0, \omega)$ , then either  $x \in \mathbb{P}(v, 0, \omega')$  or  $x - u_0 \in \mathbb{P}(v, 0, \omega')$ .  $\square$

**Theorem (reduction) 7.19:** We assume that there exists  $\Delta \in \mathbb{Z}^d$  such that:  $\langle u_0, \Delta \rangle = 0$  and  $\forall u \in \mathcal{V}^+ \setminus u_0, \langle u, \Delta \rangle = 1$ . Then,  $\Delta$  induces a valid reduction for  $v$ .

*Proof.* Let us start with some notations. We consider vectors as columns. We pose  $v' \stackrel{\text{def}}{=} v - \lambda\Delta$ . Then,

$$v' = v - \lambda\Delta = v - \langle u_0, v \rangle \Delta = v - \Delta^t u_0 v = (I - \Delta^t u_0) v.$$

We then pose  $\chi \stackrel{\text{def}}{=} I - \Delta^t u_0$ . Thus,  $v' = \chi(v)$ . Moreover  $\chi$  is bijective and,  $\chi^{-1} = I + \Delta^t u_0$  because  $\langle u_0, \Delta \rangle = 0$ . We pose  $f \stackrel{\text{def}}{=} {}^t \chi^{-1} = I + u_0 {}^t \Delta$ . As a result,  $f^{-1} = I - u_0 {}^t \Delta$ .

Here are some useful calculations for the future:

1.  $\langle u_0, v' \rangle = \langle u_0, v \rangle - \lambda \langle u_0, \Delta \rangle = \langle u_0, v \rangle = \lambda$ ;
2. if  $u \in \mathcal{V}^+ \setminus \{u_0\}$ ,  $\langle u, v' \rangle = \langle u, v \rangle - \lambda \langle u, \Delta \rangle = \langle u, v \rangle - \lambda$ ;
3.  $f(u_0) = u_0 + \langle \Delta, u_0 \rangle u_0 = u_0$ ;
4. if  $u \in \mathcal{V}^+ \setminus \{u_0\}$ ,  $f(u) = u + \langle \Delta, u \rangle u_0 = u + u_0$ .  $f^{-1}(u_0) = u_0 - \langle \Delta, u_0 \rangle u_0 = u_0$ ;
5. if  $u \in \mathcal{V}^+ \setminus \{u_0\}$ ,  $f^{-1}(u) = u - \langle \Delta, u \rangle u_0 = u - u_0$ ;
6. if  $x \in \mathbb{Z}^d$ ,  $\langle f(x), v' \rangle = \langle x, {}^t f(v') \rangle = \langle x, \chi^{-1}(v') \rangle = \langle x, v \rangle$ .

As a consequence of 6, for all  $\omega$ ,  $f : \mathbb{P}(v, 0, \omega) \rightarrow \mathbb{P}(v', 0, \omega)$  is a bijection.

Let us now move on to the proof. Let  $\omega \geq 2\lambda$ . We pose  $\omega' \stackrel{\text{def}}{=} \omega - \lambda$ . Let us show that  $\mathbb{P}(v, 0, \omega)$  is connected iff  $\mathbb{P}(v', 0, \omega')$  is connected.

$\Leftarrow$ : Let us assume that  $\mathbb{P}(v', 0, \omega')$  is connected. Let  $x, y \in \mathbb{P}(v, 0, \omega)$ . We look for a path from  $x$  to  $y$  in  $\mathbb{P}(v, 0, \omega)$ . By the lemma (because  $\omega \geq 2\lambda$ ),  $x$  or  $x - u_0$  is in  $\mathbb{P}(v, 0, \omega')$ . In the same way,  $y$  or  $y - u_0$  is in  $\mathbb{P}(v, 0, \omega')$ . As  $x - u_0 \leftrightarrow x$  and  $y - u_0 \leftrightarrow y$ , we can assume without loss of generality  $x, y \in \mathbb{P}(v, 0, \omega')$ .

We look at  $x' \stackrel{\text{def}}{=} f(x)$  and  $y' \stackrel{\text{def}}{=} f(y)$ . From then on,  $x', y' \in \mathbb{P}(v', 0, \omega')$ . By connectedness of  $x'$  to  $y'$ , there exists a path from  $x'$  to  $y'$  in  $\mathbb{P}(v', 0, \omega')$ . We wish to deduce that there exists a path from  $x$  to  $y$  in  $\mathbb{P}(v, 0, \omega)$ . By immediate induction on the size  $n$  of the path from  $x'$  to  $y'$ , it is enough to show the result in the case  $n = 1$  to conclude. Let us thus suppose that  $y' = x' + u$  for  $u \in \mathcal{V}$ . Should we exchange  $x$  and  $y$ , we can assume  $u \in \mathcal{V}^+$ . Then  $y - x = f^{-1}(y') - f^{-1}(x') = f^{-1}(u)$ . If  $u = u_0$  then  $y - x = f^{-1}(u_0) = u_0$  so  $x \leftrightarrow y$ . If  $u \neq u_0$  then  $y - x = f^{-1}(u) = u - u_0$ . We thus propose the following path  $x \leftrightarrow x + u \leftrightarrow x + u - u_0 = y$ . It remains to show that  $x + u \in \mathbb{P}(v, 0, \omega)$ .  $\langle x + u, v \rangle = \langle x, v \rangle + \langle u, v \rangle \geq \langle u, v \rangle \geq 0$  because  $u \in \mathcal{V}^+$ .  $\langle x + u, v \rangle = \langle f(x + u), v' \rangle = \langle x' + u + u_0, v' \rangle = \langle y' + u_0, v' \rangle = \langle y', v' \rangle + \lambda < \omega' + \lambda = \omega$ . Thus  $x + u \in \mathbb{P}(v, 0, \omega)$  which concludes.

$\Rightarrow$ : Let us assume that  $\mathbb{P}(v, 0, \omega)$  is connected. As  $\omega \geq \lambda$ , by the lemma, we can define:

$$\tilde{f} \stackrel{\text{def}}{=} \begin{cases} \mathbb{P}(v, 0, \omega) \rightarrow \mathbb{P}(v', 0, \omega') \\ x \mapsto \begin{cases} f(x) & \text{if } x \in \mathbb{P}(v, 0, \omega'); \\ f(x - u_0) & \text{if } x \notin \mathbb{P}(v, 0, \omega'). \end{cases} \end{cases}$$

By surjectivity of  $f$ ,  $\tilde{f}$  is also surjective. Let  $x', y' \in \mathbb{P}(v', 0, \omega')$ . By surjectivity of  $\tilde{f}$ , there exist  $x, y \in \mathbb{P}(v, 0, \omega)$  such that  $\tilde{f}(x) = x'$  and  $\tilde{f}(y) = y'$ .

By connectedness of  $\mathbb{P}(v, 0, \omega)$ , there exists a path from  $x$  to  $y$  in  $\mathbb{P}(v, 0, \omega)$ . We wish to deduce that there exists a path from  $x'$  to  $y'$  in  $\mathbb{P}(v', 0, \omega')$ . By immediate induction on the size  $n$  of the path from  $x$  to  $y$ , it is enough to show the result in the case  $n = 1$  to conclude. Let us suppose that  $y = x + u$  for  $u \in \mathcal{V}$ . Should we exchange  $x$  and  $y$ , we can suppose  $u \in \mathcal{V}^+$ . From then on  $y' - x' = \tilde{f}(y) - \tilde{f}(x)$ .

Note that  $\langle y, v \rangle - \langle x, v \rangle = \langle u, v \rangle \geq \lambda$ . Therefore  $\omega > \langle y, v \rangle = \langle x, v \rangle + \lambda$  and  $\langle x, v \rangle < \omega - \lambda = \omega'$ . Hence  $x \in \mathbb{P}(v, 0, \omega')$ .

**Case 1:**  $y \in \mathbb{P}(v, 0, \omega')$ . In this case,  $y' - x' = f(y) - f(x) = f(u)$ . If  $u = u_0$ ,  $y' - x' = f(u_0) = u_0$  so  $x' \leftrightarrow y'$ . Else,  $y' - x' = f(u) = u + u_0$ . We therefore propose the path  $x' \leftrightarrow x' + u \leftrightarrow x' + u + u_0 = y'$ . It



remains to be shown that  $x' + u \in \mathbb{P}(v', 0, \omega')$ .  $\langle x' + u, v' \rangle = \langle x', v' \rangle + \langle u, v' \rangle \geq \langle u, v' \rangle = \langle u, v \rangle - \lambda \geq 0$  by definition of  $\lambda$ .  $\langle x' + u, v' \rangle \leq \langle x' + u, v' \rangle + \lambda = \langle y', v' \rangle < \omega'$ . So  $x' + u \in \mathbb{P}(v', 0, \omega')$  which concludes.

**Case 2:**  $y \notin \mathbb{P}(v, 0, \omega')$ . Then,  $y' - x' = f(y - u_0) - f(x) = f(u) - f(u_0)$ . If  $u = u_0$  then  $y' - x' = 0$  so  $x' \leftrightarrow^* y'$ . Else  $y' - x' = u + u_0 - u_0 = u$  so  $x' \leftrightarrow y'$ .  $\square$

## A.2 Consequences of the reduction theorem 7.19

We start by proving a lemma before proving the theorem.

**Lemma:** Let  $\Delta$  be a vector inducing a valid reduction for  $v$ . Let  $v' \stackrel{\text{def}}{=} v - \lambda\Delta$ . Then  $\max(\Omega(v), 2\lambda) = \max(\Omega(v') + \lambda, 2\lambda)$ .

*Proof.* Case 1: let us suppose  $\Omega(v) \geq 2\lambda$ . Let us then show that  $\Omega(v) = \max(\Omega(v') + \lambda, 2\lambda)$  by double inequality, starting with the  $\geq$  direction.

Let  $\omega$  be such that  $\mathbb{P}(v, 0, \omega)$  is connected. Then  $\omega \geq \Omega(v) \geq 2\lambda$ . Therefore, by reduction,  $\mathbb{P}(v', 0, \omega - \lambda)$  is connected. Thus  $\omega - \lambda \geq \Omega(v')$ , i.e.  $\omega \geq \Omega(v') + \lambda$ . Taking the infimum, one deduces  $\Omega(v) \geq \Omega(v') + \lambda$ . Thus  $\Omega(v) \geq \max(\Omega(v') + \lambda, 2\lambda)$ . Let us now prove that  $\Omega(v) \leq \max(\Omega(v') + \lambda, 2\lambda)$ .

Case 1a: suppose  $\Omega(v') \geq \lambda$ . Let  $\omega'$  be such that  $\mathbb{P}(v', 0, \omega')$  is connected. Then  $\omega' \geq \Omega(v') \geq \lambda$  so  $\omega' + \lambda \geq 2\lambda$ . By reduction,  $\mathbb{P}(v, 0, \omega' + \lambda)$  is connected. So  $\omega' + \lambda \geq \Omega(v)$ , ie  $\omega' \geq \Omega(v) - \lambda$ . Taking the infimum, we obtain  $\Omega(v') \geq \Omega(v) - \lambda$  so  $\Omega(v) \leq \Omega(v') + \lambda = \max(\Omega(v') + \lambda, 2\lambda)$ .

Case 1b: suppose that  $\Omega(v') < \lambda$ . Then  $\mathbb{P}(v', 0, \lambda)$  is connected. As  $2\lambda \geq 2\lambda$ , by reduction,  $\mathbb{P}(v, 0, 2\lambda)$  is connected. So  $\Omega(v) \leq 2\lambda = \max(\Omega(v') + \lambda, 2\lambda)$ .

It has been shown that  $\max(\Omega(v') + \lambda, 2\lambda) = \Omega(v) = \max(\Omega(v), 2\lambda)$ .

Case 2: suppose  $\Omega(v) < 2\lambda$ . Then  $\mathbb{P}(v, 0, 2\lambda)$  is connected. As  $2\lambda \geq 2\lambda$ , by reduction,  $\mathbb{P}(v', 0, \lambda)$  is connected. So  $\Omega(v') \leq \lambda$ , ie  $\Omega(v') + \lambda \leq 2\lambda$ . So  $\max(2\lambda, \Omega(v') + \lambda) = 2\lambda = \max(\Omega(v), 2\lambda)$ .  $\square$

**Theorem 7.20:** Let  $\Delta$  induce a valid reduction for  $v$ . We pose  $v' \stackrel{\text{def}}{=} v - \lambda\Delta$ . Then  $\Omega(v) = \max(2\lambda - \gcd(v), \Omega(v') + \lambda)$ .

*Proof.* By the preceding proposition,  $\max(\Omega(v), 2\lambda) = \max(2\lambda, \Omega(v') + \lambda)$ .

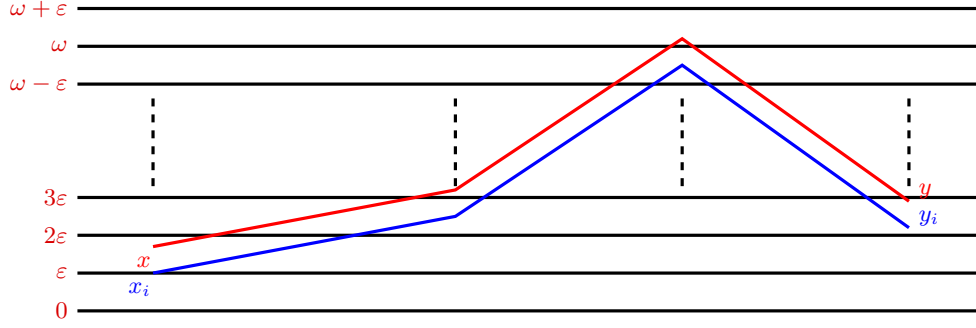
1. If  $v$  is irrational, then  $\Omega(v) \geq 2\lambda$  by Theorem 5.14. Therefore  $\Omega(v) = \max(2\lambda, \Omega(v') + \lambda)$ .
2. If  $v$  is rational, we can suppose without loss of generality that  $\gcd(v) = 1$ . Two cases arise:
  - (a) If  $\Omega(v') > \lambda$ :  $2\lambda < \Omega(v') + \lambda = \max(2\lambda, \Omega(v))$  so  $\Omega(v) = \Omega(v') + \lambda$ .
  - (b) otherwise,  $2\lambda = \max(2\lambda, \Omega(v))$  so  $\Omega(v) \leq 2\lambda$ . Now,  $2\lambda - 1 \leq \Omega(v)$  (by Theorem 5.14) therefore  $\Omega(v) \in \{2\lambda - 1, 2\lambda\}$ . In addition,

$$\begin{aligned} \Omega(v) = 2\lambda - 1 &\Leftrightarrow \mathbb{P}(v, 0, 2\lambda) \text{ is connected} \\ &\Leftrightarrow \mathbb{P}(v', 0, \lambda) \text{ is connected} \\ &\Leftrightarrow \Omega(v') < \lambda. \end{aligned}$$

BALANCE SHEET:

- in the irrational case,  $\Omega(v) = \max(2\lambda, \Omega(v') + \lambda)$ ;
- in the rational case ( $\gcd = 1$ ):  $\Omega(v) = \begin{cases} \Omega(v') + \lambda & \text{if } \Omega(v') > \lambda; \\ 2\lambda & \text{if } \Omega(v') = \lambda; \\ 2\lambda - 1 & \text{if } \Omega(v') < \lambda. \end{cases}$  So  $\Omega(v) = \max(2\lambda - 1, \Omega(v') + \lambda)$ .

In both cases, we have,  $\Omega(v) = \max(2\lambda - \gcd(v), \Omega(v') + \lambda)$ .  $\square$



**Fig. 16.** Illustration of the proof by layers. The horizontal axis represents the progress in the path, the vertical axis the evolution of the scalar product with  $v$ . The red and the blue paths are plotted parallel because the 2 paths follow the same neighbourhood vectors.

## B Analytical approach

### B.1 Path bounding: the irrational case

**Proposition 7.3:** Let  $v$  be an irrational vector and  $\omega > 0$  such that  $\mathbb{P}(v, 0, \omega)$  is connected. Let  $\varepsilon > 0$  and  $r \in \mathbb{R}_+$ . Then there exists  $M \in \mathbb{R}$  which bounds the paths for  $v, (\mu = 0), \omega, r, \varepsilon$ .

*Proof.* We invite the reader to refer to Figure 16 to follow the proof. Let  $v, \omega$  such that  $\mathbb{P}(v, 0, \omega)$  is connected. Let  $\varepsilon > 0$  and  $r \in \mathbb{R}_+$ .

By density of  $\{\langle x, v \rangle\}_{x \in \mathbb{Z}^d}$ , there exists  $x_1 \in \mathbb{Z}^d$  such that  $0 < \langle x_1, v \rangle \leq \varepsilon$ . If we take a smaller  $\varepsilon$ , we can assume without loss of generality  $\langle x_1, v \rangle = \varepsilon$ . In fact, if  $M$  is suitable for  $\varepsilon' < \varepsilon$  then  $M$  is suitable for  $\varepsilon$ . For  $i \in \llbracket 0, n \rrbracket$ , we pose  $x_i \stackrel{\text{def}}{=} i x_1$  with  $n \stackrel{\text{def}}{=} \lceil \frac{\omega}{\varepsilon} \rceil - 1$ . Then  $\langle x_i, v \rangle = i\varepsilon$ . The idea is to find a bound of paths for the finite set  $\{x_i\}_{i \in \llbracket 0, n \rrbracket}$  (which is relatively easy, by finiteness) and then to reduce to  $x_i$  for all  $x$  in the hyperplane.

Let us start by exhibiting an  $M$  bounding the paths for  $\{x_i\}_{i \in \llbracket 0, n \rrbracket}$ . Let  $i \in \llbracket 0, n \rrbracket$ . We pose  $A_i \stackrel{\text{def}}{=} \mathbb{P}(v, 0, \omega) \cap \mathcal{B}(x_i, r)$  which is finite. Let us take  $y \in A_i$ . By connectedness of  $\mathbb{P}(v, 0, \omega)$ , there exists a path from  $x_i$  to  $y$  in  $\mathbb{P}(v, 0, \omega)$ . Let  $l(y)$  be the length of this path. We pose  $M_i \stackrel{\text{def}}{=} \max_{y \in A_i} l(y)$  and  $M \stackrel{\text{def}}{=} \max_{i \in \llbracket 0, n \rrbracket} M_i$ .

Let us now show that  $M$  fits.

Let  $x, y \in \mathbb{P}(v, 0, \omega)$  such that  $\|y - x\|_2 \leq r$ . We pose  $i \stackrel{\text{def}}{=} \lfloor \frac{1}{\varepsilon} \langle x, v \rangle \rfloor \in \llbracket 0, n \rrbracket$ . Then  $\langle x_i, v \rangle \leq \langle x, v \rangle < \langle x_{i+1}, v \rangle = \langle x_i, v \rangle + \varepsilon$ .

We pose  $z \stackrel{\text{def}}{=} y - x$  and  $y_i \stackrel{\text{def}}{=} x_i + z$ . As a result,  $\|y_i - x_i\|_2 = \|y - x\|_2 \leq r$ . Thus, by definition of  $M$ , there exists a path from  $x_i$  to  $y_i$  with values in  $\mathbb{P}(v, 0, \omega)$  and of length less than  $M$ . Let us call  $u_1, \dots, u_k$  the sequence of neighbourhood vectors constituting this path (thus  $k \leq M$ ).

We show that this sequence  $u_1, \dots, u_k$  also forms a path from  $x$  to  $y$  in  $\mathbb{P}(v, 0, \omega + \varepsilon)$ .  $\sum_{j=1}^k u_j = y_i - x_i = z = y - x$  so the path is from  $x$  to  $y$ . Furthermore, for  $p \in \llbracket 0, k \rrbracket$ ,

$$\left\langle x + \sum_{j=1}^p u_j, v \right\rangle = \langle x, v \rangle + \left\langle \sum_{j=1}^p u_j, v \right\rangle \geq \langle x_i, v \rangle + \left\langle \sum_{j=1}^p u_j, v \right\rangle = \left\langle x_i + \sum_{j=1}^p u_j, v \right\rangle \geq 0$$

and

$$\left\langle x + \sum_{j=1}^p u_j, v \right\rangle = \langle x, v \rangle + \left\langle \sum_{j=1}^p u_j, v \right\rangle \leq \varepsilon + \langle x_i, v \rangle + \left\langle \sum_{j=1}^p u_j, v \right\rangle = \varepsilon + \left\langle x_i + \sum_{j=1}^p u_j, v \right\rangle < \varepsilon + \omega.$$

We have therefore found a path from  $x$  to  $y$  in  $\mathbb{P}(v, 0, \omega + \varepsilon)$  of length less than  $M$ .  $\square$

## B.2 Path bounding: the rational case

**Proposition 7.5:** Let  $v$  be a rational vector and  $\omega > 0$  such that  $\mathbb{P}(v, 0, \omega)$  is connected. Let  $r \in \mathbb{R}_+$ . Then there exists  $M$  bounding the paths for  $v, 0, \omega, r, 0$ .

*Proof.* The proof is very similar to the irrational path bounding and is also inspired by the figure. Let  $v, \omega$  be such that  $\mathbb{P}(v, 0, \omega)$  is connected. We can suppose without loss of generality  $\gcd(v) = 1$ . We can also assume  $\omega \in \mathbb{N}$ . In fact, as  $v \in \mathbb{Z}^d$ ,  $\mathbb{P}(v, 0, \omega) = \mathbb{P}(v, 0, \lceil \omega \rceil)$ . Let  $r \in \mathbb{R}_+$ . By Bézout's theorem, there exists  $x_1 \in \mathbb{Z}^d$  such that  $\langle x_1, v \rangle = 1$ . We pose, for  $i \in \llbracket 0, n \rrbracket$ ,  $x_i \stackrel{\text{def}}{=} i x_1$  with  $n \stackrel{\text{def}}{=} \omega - 1$ . Thus  $\langle x_i, v \rangle = i$ . The idea is to find a bound of paths for the finite set  $\{x_i\}_{i \in \llbracket 0, n \rrbracket}$  (which is relatively easy, by finiteness) and then to reduce to  $x_i$  for all  $x$  in the hyperplane.

Let us start by exhibiting an  $M$  bounding the paths for  $\{x_i\}_{i \in \llbracket 0, n \rrbracket}$ . Let  $i \in \llbracket 0, n \rrbracket$ . Let  $A_i \stackrel{\text{def}}{=} P(v, 0, \omega) \cap \mathcal{B}(x_i, r)$  which is finite. Let us take  $y \in A_i$ . By connectedness of  $\mathbb{P}(v, 0, \omega)$ , there exists a path from  $x_i$  to  $y$  in  $\mathbb{P}(v, 0, \omega)$ . Let  $l(y)$  be the length of this path. We pose  $M_i \stackrel{\text{def}}{=} \max_{y \in A_i} l(y)$  and  $M \stackrel{\text{def}}{=} \max_{i \in \llbracket 0, n \rrbracket} M_i$ . Let us now show that  $M$  fits.

Let  $x, y \in \mathbb{P}(v, 0, \omega)$  such that  $\|y - x\|_2 \leq r$ . We pose  $i \stackrel{\text{def}}{=} \langle x, v \rangle \in \llbracket 0, n \rrbracket$ . Thus  $\langle x_i, v \rangle = \langle x, v \rangle$ . We pose  $z \stackrel{\text{def}}{=} y - x$  and  $y_i \stackrel{\text{def}}{=} x_i + z$ . As a result,  $\|y_i - x_i\|_2 = \|y - x\|_2 \leq r$ . Therefore, by definition of  $M$ , there exists a path from  $x_i$  to  $y_i$  with values in  $\mathbb{P}(v, 0, \omega)$  and of length less than  $M$ . Let us call  $u_1, \dots, u_k$  the sequence of neighbourhood vectors constituting this path (thus  $k \leq M$ ).

We show that this sequence  $u_1, \dots, u_k$  also forms a path from  $x$  to  $y \in \mathbb{P}(v, 0, \omega + \varepsilon)$ .  $\sum_{j=1}^k u_j = y_i - x_i = z = y - x$  so the path is from  $x$  to  $y$ . In addition, for  $p \in \llbracket 0, k \rrbracket$ ,

$$\left\langle x + \sum_{j=1}^p u_j, v \right\rangle = \langle x, v \rangle + \left\langle \sum_{j=1}^p u_j, v \right\rangle = \langle x_i, v \rangle + \left\langle \sum_{j=1}^p u_j, v \right\rangle = \left\langle x_i + \sum_{j=1}^p u_j, v \right\rangle \in [0, n] \subset [0, \omega[.$$

We have therefore found a path from  $x$  to  $y$  in  $\mathbb{P}(v, 0, \omega)$  of length less than  $M$ .  $\square$

## B.3 Proof of semi-continuity 7.8

The proof of the semi-continuity theorem begins with a technical lemma characterising the transition from non-connectedness to connectedness by varying the thickness. The following lemma shows that, in a non-connected plane, there are 2 points  $x, y$  close in distance in  $\mathbb{Z}^d$  but in 2 different connected components.

**Lemma B.3:** Let  $v' \in \mathbb{R}^d$  and  $\varepsilon, \omega_0 > 0$  such that  $\mathbb{P}(v', 0, \omega_0)$  is not connected and not empty and  $\mathbb{P}(v', 0, \omega_0 + \varepsilon)$  is connected. It is further assumed that  $\varepsilon < \lambda(v')$ . Then there exists  $x, y \in \mathbb{Z}^d$  such that:

1.  $x, y \in \mathbb{P}(v', 0, \omega_0)$  (we even have  $x, y \in \mathbb{P}(v', 0, \omega_0 + \varepsilon - \lambda(v'))$ ).
2.  $x$  and  $y$  are not connected in  $\mathbb{P}(v', 0, \omega_0)$ .
3.  $x$  and  $y$  are at distance 2 in  $\mathbb{P}(v', 0, \omega_0 + \varepsilon)$ .

*Proof.* As  $\mathbb{P}(v', 0, \omega_0)$  is not connected and not empty, there are 2 vectors  $x'$  and  $y'$  not connected in  $\mathbb{P}(v', 0, \omega_0)$ . Let  $C$  be the connected component of  $x'$  in  $\mathbb{P}(v', 0, \omega_0)$ . Since  $\mathbb{P}(v', 0, \omega_0 + \varepsilon)$  is connected, there is a path  $x' = x_0 \leftrightarrow x_1 \dots \leftrightarrow x_k = y'$  in  $\mathbb{P}(v', 0, \omega_0 + \varepsilon)$ . As the vectors of  $\mathcal{V} \cap v'^\perp$  could later lead to some difficulties in the proof, we assume that these vectors are all located at the beginning of the path from  $x'$  to  $y'$ . This is permissible because moving  $u \in \mathcal{V} \cap v'^\perp$  does not change the scalar product. We can therefore place the vectors of  $\mathcal{V} \cap v'^\perp$  anywhere in the path.

Let us consider the greatest  $i$  such that  $x_{i-1} \in C$  and  $\forall j \geq i, x_j \notin C$ . Since  $x_{i-1} \in C$  and  $x_{i-1} \leftrightarrow x_i$ , we necessarily have  $x_i \notin \mathbb{P}(v', 0, \omega_0)$  (otherwise, we can deduce  $x_i \in C$ ). It can then be deduced that  $0 < i < k$ .

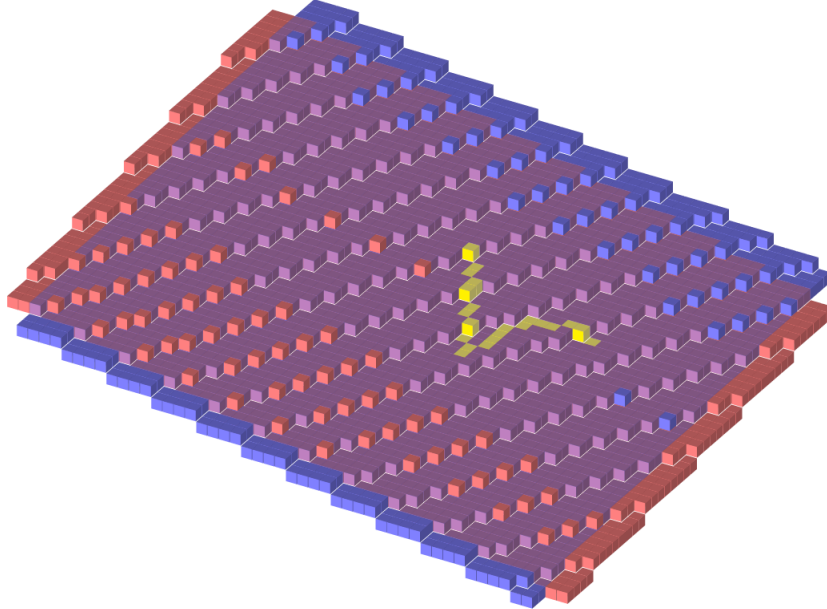
We can therefore pose  $x \stackrel{\text{def}}{=} x_{i-1}$  and  $y \stackrel{\text{def}}{=} x_{i+1}$ . Let us check that  $x$  and  $y$  agree:

- 2)  $x$  and  $y$  are not connected in  $\mathbb{P}(v', 0, \omega_0)$  because  $y = x_{i+1} \notin C$  which is the connected component of  $x$  and  $x'$  in  $\mathbb{P}(v', 0, \omega_0)$ .
- 3)  $x$  and  $y$  are at distance 2 in  $\mathbb{P}(v', 0, \omega_0 + \varepsilon)$  because  $x = x_{i-1} \leftrightarrow x_i \leftrightarrow x_{i+1} = y$ .

1) Let us show the result for  $y$ , the proof will be the same for  $x$ . To show  $y \in \mathbb{P}(v', 0, \omega_0)$ , we use the fact that  $y$  is of the form  $x_i + u$  for  $u$  in  $\mathcal{V}$ . We should also add that, as  $x_i - x_{i-1} \notin v'^\perp$ ,  $u \notin v'^\perp$ , the movements of  $v'^\perp$  having been placed at the beginning of the path. Thus  $\langle y, v' \rangle = \langle x_i, v' \rangle + \langle u, v' \rangle$ . However  $\langle x_i, v' \rangle \in [\omega_0, \omega_0 + \varepsilon[$ ,  $\langle y, v' \rangle \in [0, \omega_0 + \varepsilon[$  and  $|\langle u, v' \rangle| \geq \lambda(v') > \varepsilon$ , so  $\langle y, v' \rangle \in [0, \omega_0 + \varepsilon - \lambda(v')[ \subset [0, \omega_0[$ .  $\square$

**Theorem (semi-continuity) 7.8:** Let  $v \in \mathbb{R}^d$  with  $v \neq 0$  and  $\varepsilon > 0$ . Then there exists  $\eta > 0$  such that  $\forall v' \in \mathbb{R}^d, \|v - v'\|_2 \leq \eta \Rightarrow \Omega(v') \leq \Omega(v) + \gcd(v) + 16\varepsilon$ .

*Idea of proof:* We suppose by contradiction, for  $v'$  close to  $v$ , that  $\Omega(v') > \Omega(v) + 16\varepsilon + \gcd(v)$ . We place ourselves at a thickness  $\omega_0 \in ]\Omega(v) + \gcd(v), \Omega(v')[$ . As  $\mathbb{P}(v', 0, \omega_0)$  is not connected, we can (by the previous Lemma B.3) exhibit  $x$  and  $y$  in 2 different connected components of  $\mathbb{P}(v', 0, \omega_0)$  but close in distance in  $\mathbb{Z}^d$ . We will then consider  $\mathbb{P}(v, \mu, \omega_0)$ , for a well-chosen  $\mu$ , which is connected, to superpose it (like in Figure 17) on top of this zone in an intelligent way (i.e. by making  $\langle x, v \rangle + \mu \approx \langle x, v' \rangle$ ). This will give us a path between  $x$  and  $y$  of bounded length thanks to the path bounding theorem 7.7. The scalar products of the points traversed with  $v$  and  $v'$  are close, because  $\langle x, v \rangle + \mu \approx \langle x, v' \rangle$  and because the path has bounded length. This therefore ensures that there is a path from  $x$  to  $y$  in the plane  $\mathbb{P}(v', 0, \omega_0)$ .



**Fig. 17.** Illustration of the superposition of hyperplanes  $\mathbb{P}(v, \mu, \omega_0)$  (in blue) and  $\mathbb{P}(v', 0, \omega_0)$  (in red) around  $x$  and  $y$  (in yellow) which are connected by a path (in yellow).

*Proof.* Let  $v \in \mathbb{R}^d$  with  $v \neq 0$ . Let  $\varepsilon > 0$ . We pose  $\omega \stackrel{\text{def}}{=} \Omega(v)$ . We pose  $r \stackrel{\text{def}}{=} \max_{u, u' \in \mathcal{V}} \|u + u'\|_2$ . We start by obtaining a path bounding for  $v$ , which we need in the definition of the continuity modulus  $\eta$ . As  $6\varepsilon < 8\varepsilon$ , by the path bounding theorem, there exists  $M$  (independent of  $\mu$ ) such that:

$\forall \mu, \forall x, y \in \mathbb{P}(v, \mu, \omega + 6\varepsilon)$ , if  $\|y - x\|_2 \leq r$  then there is a path from  $x$  to  $y$  in  $\mathbb{P}(v, \mu, \omega + \bar{\mu} + 8\varepsilon)$  of length shorter than  $M$ , where  $\bar{\mu} \stackrel{\text{def}}{=} \mu \bmod \gcd(v)$  if  $v$  is rational and 0 otherwise.

We pose  $\eta \stackrel{\text{def}}{=} \frac{\varepsilon}{M \max_{u \in \mathcal{V}} \|u\|_2}$ . Let  $v' \in \mathbb{R}^d$  such that  $\|v - v'\|_2 \leq \eta$ . By taking  $\eta$  smaller, we can also assume

that  $|\|v\|_{\mathcal{V}} - \|v'\|_{\mathcal{V}}| \leq \varepsilon$ . We pose  $\omega' \stackrel{\text{def}}{=} \Omega(v')$ . We also pose  $\lambda \stackrel{\text{def}}{=} \lambda(v)$  and  $\lambda' \stackrel{\text{def}}{=} \lambda(v')$ . As  $\omega \geq \|v\|_{\mathcal{V}}$  and  $\omega' \leq \|v'\|_{\mathcal{V}} + \lambda'$ , we have  $\omega' - \omega \leq \varepsilon + \lambda'$ .

Let us suppose by contradiction  $\omega' > \omega + \gcd(v) + 16\varepsilon$ . We are then looking for  $\omega_0$  which allows us to apply lemma B.3. To do this, we consider  $\omega_0 \stackrel{\text{def}}{=} \omega' - \varepsilon$ . Thus,  $\omega < \omega_0 < \omega' < \omega_0 + 2\varepsilon$ . So,  $\mathbb{P}(v', 0, \omega_0)$  is not connected and  $\mathbb{P}(v', 0, \omega_0 + 2\varepsilon), \mathbb{P}(v, 0, \omega_0)$  are connected. By the previous lemma B.3, there exists  $x, y \in \mathbb{Z}^d$  such that:

1.  $x, y \in \mathbb{P}(v', 0, \omega_0 + 2\varepsilon - \lambda')$ .
2.  $x$  and  $y$  are not connected in  $\mathbb{P}(v', 0, \omega_0)$ .
3.  $x$  and  $y$  are at distance 2 in  $\mathbb{P}(v', 0, \omega_0 + 2\varepsilon)$ .

We then consider  $u_1, u_2 \in \mathcal{V}$  such that  $y = x + u_1 + u_2$ .

We now want to overlay, for a well-chosen  $\mu$ , the hyperplane  $\mathbb{P}(v, \mu, \omega_0)$ , which is connected, to the hyperplane  $\mathbb{P}(v', 0, \omega_0)$  at  $x$  and  $y$ , in order to find a path from  $x$  to  $y$  (like in Figure 17). We take  $\mu$  such that  $\langle x, v \rangle + \mu - 2\varepsilon = \langle x, v' \rangle$ . Thus,  $x \in \mathbb{P}(v, \mu - 2\varepsilon, \omega_0 + 2\varepsilon - \lambda')$ .

Let us bound  $|\langle y, v \rangle + \mu - \langle y, v' \rangle|$  to show that the planes do overlap at  $y$ :

$$\begin{aligned} |(\langle y, v \rangle + \mu - \langle y, v' \rangle) - 2\varepsilon| &\leq |\langle x, v \rangle + \mu - 2\varepsilon - \langle x, v' \rangle| + |\langle u_1 + u_2, v \rangle - \langle u_1 + u_2, v' \rangle| \\ &= |\langle u_1 + u_2, v - v' \rangle| \\ &\leq \|u_1 + u_2\|_2 \eta \\ &\leq 2\eta \max_{u \in \mathcal{V}} \|u\|_2 \\ &= \frac{2\varepsilon}{M} \\ &\leq \varepsilon \text{ assuming } M \geq 2. \end{aligned}$$

The aim is now to use, for a good  $\alpha$ , the connectedness of the hyperplane  $\mathbb{P}(v, \mu, \omega + \alpha)$  to find a path from  $x$  to  $y$  of short length and reproduce the same path in  $\mathbb{P}(v', 0, \omega_0)$ . As  $y \in \mathbb{P}(v', 0, \omega_0 + 2\varepsilon - \lambda')$ ,  $y \in \mathbb{P}(v, \mu, \omega_0 + 5\varepsilon - \lambda')$ . We also note that  $x \in \mathbb{P}(v, \mu - 2\varepsilon, \omega_0 + 2\varepsilon - \lambda') \subset \mathbb{P}(v, \mu, \omega_0 + 5\varepsilon - \lambda')$ . In addition,  $\|y - x\|_2 = \|u_1 + u_2\|_2 \leq r$ . We note that  $\omega_0 + 5\varepsilon - \lambda' \leq \omega + 6\varepsilon$  because  $\omega_0 - \omega - \lambda' = (\omega' - \omega - \lambda') - \varepsilon \leq 0 \leq \varepsilon$ . So, as  $\mathbb{P}(v, \mu, \omega_0 + 5\varepsilon - \lambda') \subset \mathbb{P}(v, \mu, \omega + 6\varepsilon)$ , there exists a path from  $x$  to  $y$  in  $\mathbb{P}(v, \mu, \omega + \bar{\mu} + 8\varepsilon)$  shorter than  $M$ . We note  $x = x_0 \leftrightarrow x_1 \dots \leftrightarrow x_k = y$  this path ( $k \leq M$ ). and we pose  $u'_i \stackrel{\text{def}}{=} x_i - x_{i-1}$ . So

$$\begin{aligned} |(\langle x_i, v \rangle + \mu - \langle x_i, v' \rangle) - 2\varepsilon| &\leq |\langle x, v \rangle + \mu - 2\varepsilon - \langle x, v' \rangle| + \left| \left\langle \sum_{j=1}^i u'_j, v - v' \right\rangle \right| \\ &\leq \sum_{j=1}^i |\langle u'_j, v - v' \rangle| \\ &\leq \sum_{j=1}^i \|u'_j\|_2 \eta \\ &\leq \eta M \max_{u \in \mathcal{V}} \|u\|_2 \\ &\leq \varepsilon. \end{aligned}$$

Let us note that  $\omega + \bar{\mu} + 10\varepsilon \leq \omega + \gcd(v) + 15\varepsilon \leq \omega' - \varepsilon = \omega_0$ . As  $x_i \in \mathbb{P}(v, \mu, \omega + \bar{\mu} + 8\varepsilon)$ , we deduce that  $x_i \in \mathbb{P}(v', 0, \omega + \bar{\mu} + 10\varepsilon) \subset \mathbb{P}(v', 0, \omega_0)$ . Thus we have exposed a path from  $x$  to  $y$  in  $\mathbb{P}(v', 0, \omega_0)$ , which is a contradiction. This indeed proves  $\Omega(v') \leq \Omega(v) + \gcd(v) + \varepsilon$ .  $\square$

#### B.4 Uniform path bounding

**Theorem (Uniform path bounding) 7.14:** Let  $v$  be an irrational vector. Let  $\omega > 0$  such that  $\mathbb{P}(v, 0, \omega)$  is connected. Let  $r, \varepsilon > 0$ . Then there exist  $\eta > 0$  and  $M \in \mathbb{R}$  such that:  $\forall v' \in \mathbb{R}^d$ , if  $\|v - v'\|_2 \leq \eta$  and  $Q(v) \subset Q(v')$  then  $M$  bound the paths for  $v', (\mu = 0), \omega, r, \varepsilon$ .

*Proof.* We reuse the same scheme as in the path bounding theorem B.1, this time separating into layers of size  $\frac{\varepsilon}{2}$ . We can then find  $M$  in the same way as before. To show that  $M$  fits on an open set around  $v$ , we simply have to notice that the paths used from  $x_i$  (which are finitely many) pass through a point  $x$  of zero scalar product only for elements  $x$  of  $Q(v)$ . Therefore, by taking  $v'$  close to  $v$  such that  $Q(v) \subset Q(v')$ , these points will always remain in  $\mathbb{P}(v', 0, \omega)$ . Let  $v$  be an irrational vector and  $\omega, r, \varepsilon > 0$  such that  $\mathbb{P}(v, 0, \omega)$  is connected.

We start by dividing the plane into layers of size  $\frac{\varepsilon}{2}$  using vectors  $x_i$ . By density, there exists  $x_1 \in \mathbb{Z}^d$  such that  $\langle v, x_1 \rangle \leq \frac{\varepsilon}{2}$ . Assuming  $\varepsilon$  is small enough, we can suppose that  $\langle v, x_1 \rangle = \frac{\varepsilon}{2}$ . We pose, for  $i \in \llbracket 0, n \rrbracket$  with  $n \stackrel{\text{def}}{=} \left\lceil \frac{\omega}{\varepsilon/2} \right\rceil - 1$ ,  $x_i \stackrel{\text{def}}{=} i x_1$ .

Since the set  $\{x_i\}_{i \in \llbracket 0, n \rrbracket}$  is finite, we can bound the paths for this set. To do this, we consider  $A_i \stackrel{\text{def}}{=} \mathbb{P}(v, 0, \omega) \cap \mathcal{B}(x_i, r)$ . Let  $y \in A_i$ . Since  $\mathbb{P}(v, 0, \omega)$  is connected, there exists a path  $C(x_i, y)$  from  $x_i$  to  $y$  in  $\mathbb{P}(v, 0, \omega)$ . We pose  $M = \max_{i \in \llbracket 0, n \rrbracket} \max_{y \in A_i} l(C(x_i, y))$  where  $l$  is the length of the path.

We now show that this path bounding is uniform on a compact ball of radius  $\eta$  around  $v$ . We define  $S \stackrel{\text{def}}{=} \bigcup_{i=0}^n \bigcup_{y \in A_i} C(x_i, y)$ , where we see the  $C(x_i, y)$  as sets. We set  $\alpha \stackrel{\text{def}}{=} \min_{x \in S \setminus Q(v)} \langle x, v \rangle$ .  $\alpha > 0$  by definition of  $Q(v)$ . We observe that  $\alpha \leq \langle x_1, v \rangle = \frac{\varepsilon}{2}$ . Finally, we set  $\eta \stackrel{\text{def}}{=} \frac{\alpha}{3 \max_{x \in S} \|x\|_2}$ . Let  $v' \in \mathbb{R}^d$  such that  $\|v - v'\|_2 \leq \eta$  and  $Q(v) \subset Q(v')$ . Let  $x \in S$ . By bounding  $|\langle x, v \rangle - \langle x, v' \rangle|$ , we can show a uniform path bounding from  $x$  for  $v'$ .

$$|\langle x, v \rangle - \langle x, v' \rangle| = |\langle x, v - v' \rangle| \leq \|x\|_2 \|v - v'\|_2 \leq \|x\|_2 \eta \leq \frac{\alpha}{3}.$$

Thus, if  $x \notin Q(v)$ ,  $0 \leq \langle x, v \rangle - \frac{\alpha}{3} \leq \langle x, v' \rangle \leq \langle x, v \rangle + \frac{\alpha}{3} < \omega + \frac{\alpha}{3}$ . So  $x \in \mathbb{P}(v', 0, \omega + \frac{\alpha}{3})$ . If  $x \in Q(v)$ ,  $\langle x, v' \rangle = 0$  because  $Q(v) \subset Q(v')$  thus  $x \in \mathbb{P}(v', 0, \omega + \frac{\alpha}{3})$ . So  $S \subset \mathbb{P}(v', 0, \omega + \frac{\alpha}{3})$ . In addition,  $\langle x_i, v' \rangle \leq \langle x_i, v \rangle + \frac{\alpha}{3} \leq \langle x_i, v \rangle + \frac{\varepsilon}{6} < \langle x_{i+1}, v \rangle - \frac{\varepsilon}{6} \leq \langle x_{i+1}, v \rangle - \frac{\alpha}{3} \leq \langle x_{i+1}, v' \rangle$ . Thus, the sequence  $(\langle x_i, v' \rangle)_{i \in \llbracket 0, n \rrbracket}$  is strictly increasing. Similarly, we obtain,  $\langle x_{i+1} - x_i, v' \rangle \leq \frac{\varepsilon}{2} + \frac{2\alpha}{3}$ .

Let us then show that  $M$  is suitable for  $v'$ . Let  $x, y \in \mathbb{P}(v', 0, \omega)$  such that  $\|x - y\|_2 \leq r$ . We consider  $i \in \llbracket 0, n \rrbracket$  such that  $\langle x_i, v' \rangle \leq \langle x, v' \rangle < \langle x_{i+1}, v' \rangle$ . We pose  $z \stackrel{\text{def}}{=} x - x_i$  and  $y_i \stackrel{\text{def}}{=} y - z$ . Thus,  $\langle z, v' \rangle = \langle x - x_i, v' \rangle \in [0, \frac{\varepsilon}{2} + \frac{2\alpha}{3}]$ .

As a result,  $\|y_i - x_i\|_2 = \|y - x\|_2 \leq r$ . Thus,  $C(x_i, y_i)$  is a path from  $x_i$  to  $y_i$  in  $\mathbb{P}(v, 0, \omega)$  with length less than  $M$ . This path is composed of elements of  $S$ , so it is also in  $\mathbb{P}(v', 0, \omega + \frac{\alpha}{3})$ . We denote by  $u_1, \dots, u_k$  the neighbouring vectors used in this path. Let  $p \in \llbracket 0, k \rrbracket$ .

$$\left\langle x + \sum_{j=1}^p u_j, v' \right\rangle = \left\langle x_i + \sum_{j=1}^p u_j, v' \right\rangle + \langle z, v' \rangle \leq \omega + \frac{\alpha}{3} + \langle z, v' \rangle < \omega + \frac{\varepsilon}{2} + \alpha \leq \omega + \varepsilon.$$

Therefore, we have found a path from  $x$  to  $y$  with length less than  $M$  in  $\mathbb{P}(v', 0, \omega + \varepsilon)$ .  $\square$