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Averil Prost

► **To cite this version:**

Averil Prost. On the structure of the geometric tangent cone to the Wasserstein space. 2024. hal-04672554

**HAL Id: hal-04672554**

**<https://hal.science/hal-04672554v1>**

Preprint submitted on 19 Aug 2024

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# On the structure of the geometric tangent cone to the Wasserstein space

**Abstract.** This work aims at providing a simple characterization of the geometric tangent cone to the Wasserstein space. The canonical construction in positively curved spaces builds from the set of geodesics up to an abstract closure in an appropriate topology. However, in the particular case of the Wasserstein space, it is further known that this abstract closure lies in the larger set of measures over the underlying tangent space. It is shown in this work that each member of this larger set is equivalent near 0 to its projection over the geometric tangent cone, which allows to understand the latter as a quotient structure. The argument relies on a seemingly new Helmholtz-Hodge decomposition for measure fields.

**Keywords.** Wasserstein geometry, Tangent cone, Helmholtz-Hodge decomposition.

**MSC Codes.** 49Q15, 51Fxx, 35R06.

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## 1 Introduction

The space  $\mathcal{P}_2(\mathbb{R}^d)$  of Borel probability measures with finite second moment, endowed with the *Wasserstein distance*, focuses the interest from various mathematical communities ranging from metric geometry, PDEs, optimal transport to statistical physics, image processing and machine learning. In several respects,  $\mathcal{P}_2(\mathbb{R}^d)$  behaves like an infinite-dimensional manifold of nonnegative curvature. The analogy proved fruitful in the development of differential theories in  $\mathcal{P}_2(\mathbb{R}^d)$ , and lead for instance to the celebrated Otto calculus [Ott01, Vil09].

However, various notions of a tangent space to  $\mathcal{P}_2(\mathbb{R}^d)$  coexist in the literature. A very naive definition would be to consider the *measures over the tangent space of the base space*, that we denote  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$ . A member  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$  generalizes a vector field by allowing to distribute mass over several vectors, and may

be seen as a direction issued from its base measure. In effect, this set is so large that it is often impractical to deal with, and one would prefer to restrict to more amenable subsets.

For most applications in gradient flows or continuity equations, it is sufficient to consider a *regular tangent space*, morally built from gradients of smooth functions (see [AGS05, Chap. 8] or (4) below). Intuitively, the elements contained in this regular tangent space allow a measure  $\mu \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$  to move in all directions as soon as it does not split in the process: for instance, the regular tangent space to a Dirac mass points only towards other Dirac masses. Consequently, the regular tangent space is most suited for applications in some regular dense subset of  $\mathcal{P}_2(\mathbb{R}^d)$ , such as absolutely continuous measures with various regularity assumptions imposed on their density. Even smoother subsets have been considered to extend the computations of Riemannian geometry, as the dual theory in [GKP11] or the Christoffel symbols computed in [GRM24]. However, in some cases, the restriction to absolutely continuous measures becomes overly restrictive, and one would want to work in a more general setting [AF14, Pic19].

Another construction has been done in [Gig08, Chap. 4], by viewing  $\mathcal{P}_2(\mathbb{R}^d)$  as a metric space with curvature bounded from below. Such spaces enjoy a canonical abstract definition of a tangent cone, to be implemented in any particular case. The resulting *geometric tangent space* contains the regular one, and allows to split mass. Shortly, one starts from the set of *velocities of geodesics*, then builds up the positive cone over it, and takes the closure in  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$  with respect to an appropriate distance. However, it is not clear whether there would not remain interesting directions in  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$ .

The purpose of this paper is to give a negative answer to this question, in the following sense: any straight line directed by an element of  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$  is equivalent near 0 to a straight line directed by an element of the geometric tangent space (see Lemma 5.1 for a precise statement). In other words, the geometric tangent space is isometric to the quotient of  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$  by an equivalence relation, as in manifolds (see Theorem 5.2).

The equivalence near 0 between a measure field  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$  and its projection is already known in the case where  $\xi$  is supported on the graph of a vector field, as shown by Nicola Gigli in his PhD thesis [Gig08, Theorem 4.41]. In the general case, our strategy is to decompose an element  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$  in its *tangent* component, and a divergence-free – or *solenoidal* – component. The latter does not influence the behavior of the exponential of  $\xi$  near 0, and gets eliminated by the quotient. In the case where  $\xi$  is induced by a vector field, this decomposition exactly corresponds to the Helmholtz-Hodge theorem [Lad87, Chap.1, Section 2]. Let us point that it is distinct from the non-linear Brenier factorization [Bre91], which expresses  $\xi$  as a composition between a gradient-like map and a permutation-like map  $g$  such that  $\mu = g\#\mu$ .

The structure of the paper is the following. In Section 2, we collect definitions and useful results from the literature. Section 3 introduces solenoidal measure fields and their first properties, including upper semicontinuity with respect to the base measure. Section 4 is devoted to a more involved characterization of these measure fields, as the ones that let their base measure infinitesimally invariant. This is then used in Section 5, which contains our main results. Finally, Section 6 focuses on the reconstruction of measure fields with prescribed tangent and solenoidal components.

## 2 Notations and preliminaries

If  $X, Y$  are topological spaces and  $f : X \rightarrow Y$  is measurable, we denote by  $\#$  the pushforward of measures, that allows to transport a measure  $\mu \in \mathcal{P}(X)$  to a measure  $f\#\mu \in \mathcal{P}(Y)$  by  $(f\#\mu)(A) := \mu(f^{-1}(A))$  for all measurable  $A \subset Y$ . Let  $\mathcal{P}_2(\mathbb{R}^d)$  be the set of Borel probability measures over  $\mathbb{R}^d$  that has finite second moment, i.e. satisfying  $\int_{x \in \mathbb{R}^d} |x|^2 d\mu(x) < \infty$ . If  $\pi_x, \pi_y$  are the canonical projections from the product space  $(\mathbb{R}^d)^2$  to the first and second component, we denote by  $\Gamma(\mu, \nu)$  the set of *transport plans*  $\alpha$  such that  $\pi_x\#\alpha = \mu$  and  $\pi_y\#\alpha = \nu$ .

**Definition 2.1** (Wasserstein distance). *The Wasserstein distance between two given measures  $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$*

is defined as

$$d_{\mathcal{W}}^2(\mu, \nu) := \inf_{\eta \in \Gamma(\mu, \nu)} \int_{(x, y) \in (\mathbb{R}^d)^2} |y - x|^2 d\eta(x, y).$$

The application  $d_{\mathcal{W}}$  induces a complete distance on  $\mathcal{P}_2(\mathbb{R}^d)$ , and the infimum is attained in a set denoted  $\Gamma_o(\mu, \nu)$  [Vil09]. The convergence with respect to  $d_{\mathcal{W}}$  can be characterized as the combination of *narrow convergence*, in duality with continuous and bounded functions, and the convergence of the second moments. We will extensively use the fact that the application  $(\mu, \nu) \mapsto \Gamma_o(\mu, \nu)$  of optimal transport plans is sequentially upper semicontinuous and has compact images in the Wasserstein topology (see [Vil09, Lemma 4.4 and Theorem 5.20] and [AGS05, Proposition 7.1.5]).

**Measure fields** Denote by  $T\mathbb{R}^d := \bigcup_{x \in \mathbb{R}^d} \{x\} \times T_x \mathbb{R}^d$  the tangent bundle of  $\mathbb{R}^d$ , where here  $T_x \mathbb{R}^d$  can be identified to  $\mathbb{R}^d$ . It is a metric space when endowed with  $|(x, v) - (y, w)|^2 := |x - y|^2 + |v - w|^2$ . Similarly, we define

$$T^m \mathbb{R}^d := \bigcup_{x \in \mathbb{R}^d} \{x\} \times (T_x \mathbb{R}^d)^m$$

for any  $m \in \mathbb{N}$ , and endow it with the Euclidean distance. Given  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , the set  $\mathcal{P}_2(T\mathbb{R}^d)_\mu$  denotes the subset of  $\xi \in \mathcal{P}_2(T\mathbb{R}^d)$  such that  $\pi_x \# \xi = \mu$ . By a change of variable, any plan  $\eta = \eta(x, y)$  between  $\mu$  and  $\nu \in \mathcal{P}_2(\mathbb{R}^d)$  is associated to a unique *measure field*  $\xi \in \mathcal{P}_2(T\mathbb{R}^d)_\mu$  by

$$\xi = (\pi_x, \pi_y - \pi_x) \# \eta, \quad \eta = (\pi_x, \pi_x + \pi_v) \# \xi.$$

We say that the plan  $\eta$  is *induced* by the measure field  $\xi$ , or that  $\xi$  induces  $\eta$ . The terminology of *measure fields* is inspired from [Pic19], and enhances the connection with *vector fields*  $f \in L^2_\mu(\mathbb{R}^d; T\mathbb{R}^d)$ , that measure fields generalize by allowing to distribute mass along several vectors for each point  $x \in \mathbb{R}^d$ .

**Definition 2.2** (Rescaling). *For any  $\lambda \in \mathbb{R}$ , the rescaled measure field  $\lambda \cdot \xi$  is defined as  $(\pi_x, \lambda \pi_v) \# \xi$ . For convenience,  $-1 \cdot \xi$  is shortened in  $-\xi$ , and  $0 \cdot \xi$  in  $0_\mu$ .*

Each  $\xi \in \mathcal{P}_2(T\mathbb{R}^d)_\mu$  can be seen as the initial velocity of the “straight line” parameterized by

$$[0, 1] \ni s \mapsto \exp_\mu(s \cdot \xi) := (\pi_x + s\pi_v) \# \xi.$$

If  $\xi$  induced an optimal plan, i.e. if  $(\pi_x, \pi_x + \pi_v) \# \xi$  is optimal between its marginals, then  $s \mapsto \exp_\mu(s \cdot \xi)$  describes a geodesic in the Wasserstein space. At the opposite, to recover the initial velocities of geodesics, we follow [Gig08] in denoting

$$\exp_\mu^{-1}(\nu) := \{\xi \in \mathcal{P}_2(T\mathbb{R}^d)_\mu \mid (\pi_x, \pi_x + \pi_v) \# \xi \in \Gamma_o(\mu, \nu)\} \quad \forall \nu \in \mathcal{P}_2(\mathbb{R}^d).$$

The application  $\nu \mapsto \exp_\mu^{-1}(\nu)$  inherits the properties of optimal transport plans, in that it has nonempty and compact images in the Wasserstein topology over  $\mathcal{P}_2(T\mathbb{R}^d)$ , and is sequentially upper semicontinuous.

**Metric structure on  $\mathcal{P}_2(T\mathbb{R}^d)_\mu$**  For any fixed  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , it is convenient to define an appropriate set of plans between  $\xi, \zeta \in \mathcal{P}_2(T\mathbb{R}^d)_\mu$  by

$$\Gamma_\mu(\xi, \zeta) := \{\alpha = \alpha(x, v, w) \in \mathcal{P}_2(T^2 \mathbb{R}^d) \mid (\pi_x, \pi_v) \# \alpha = \xi, (\pi_x, \pi_w) \# \alpha = \zeta\}.$$

This set differs from  $\Gamma(\xi, \zeta)$  in that it prevents mass to be transferred from  $(x, v)$  to  $(y, w)$  if  $x \neq y$ .

**Definition 2.3** (Cone distance [Gig08]). *Given  $\xi, \zeta \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ , denote*

$$W_\mu^2(\xi, \zeta) := \inf_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d} |v - w|^2 d\alpha(x, v, w).$$

Then  $W_\mu$  induces a complete distance over  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ , and the infimum is attained over a nonempty set denoted  $\Gamma_{\mu, o}(\xi, \zeta)$ . We shorten  $W_\mu(\xi, 0_\mu)$  in  $\|\xi\|_\mu$ , and notice that  $W_\mu(\lambda \cdot \xi, \lambda \cdot \zeta) = |\lambda| W_\mu(\xi, \zeta)$  for any  $\lambda \in \mathbb{R}$ .

The definition of the distance comes jointly with the following *metric scalar product*.

**Definition 2.4** (Metric scalar product). *Given  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ , define  $\langle \cdot, \cdot \rangle_\mu : (\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu)^2 \rightarrow \mathbb{R}$  by*

$$\langle \xi, \zeta \rangle_\mu := \frac{1}{2} [\|\xi\|_\mu^2 + \|\zeta\|_\mu^2 - W_\mu^2(\xi, \zeta)].$$

By the definition of  $W_\mu$ , there holds

$$\langle \xi, \zeta \rangle_\mu = \sup_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v, w \rangle d\alpha = \int_{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v, w \rangle d\beta \quad \forall \beta \in \Gamma_{\mu, o}(\xi, \zeta). \quad (1)$$

The metric scalar product is positively homogeneous, in the sense that  $\langle \lambda \cdot \xi, \zeta \rangle_\mu = \langle \xi, \lambda \cdot \zeta \rangle_\mu = \lambda \langle \xi, \zeta \rangle_\mu$  for all  $\lambda \geq 0$ , but not homogeneous nor linear in any sense. Indeed, for any  $\xi, \zeta \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ ,

$$\langle \xi, -\zeta \rangle_\mu = \sup_{\alpha \in \Gamma_\mu(\xi, -\zeta)} \int_{(x, v, w)} \langle v, w \rangle d\alpha = - \inf_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x, v, w)} \langle v, w \rangle d\alpha \geq - \langle \xi, \zeta \rangle_\mu. \quad (2)$$

It however retains the link with the directional derivative of the squared Wasserstein distance through the crucial identity [AGS05, Section 7.3]

$$D_\mu d_{\mathcal{W}}^2(\cdot, \nu)(\xi) := \lim_{h \searrow 0} \frac{d_{\mathcal{W}}^2(\exp_\mu(h \cdot \xi), \nu)}{h} = \inf_{\eta \in \exp_\mu^{-1}(\nu)} -2 \langle \xi, \eta \rangle_\mu. \quad (3)$$

**Tangent spaces** Two sets are usually considered in the literature as *tangent spaces* to  $\mathcal{P}_2(\mathbb{R}^d)$  at a measure  $\mu$ , namely the *regular tangent space*

$$\mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d) := \overline{\{\nabla \varphi \mid \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d; \mathbb{R})\}}^{L_\mu^2(\mathbb{R}^d; \mathbb{T}\mathbb{R}^d)}, \quad (4)$$

and the *geometric tangent space*

$$\mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d) := \overline{\{\lambda \cdot \xi \mid \lambda \in \mathbb{R}^+ \text{ and } \xi \in \exp_\mu^{-1}(\nu) \text{ for some } \nu \in \mathcal{P}_2(\mathbb{R}^d)\}}^{W_\mu}. \quad (5)$$

The regular tangent space may be seen as a subset of the geometric one through the injection  $f \mapsto (id, f) \# \mu$ . Modulo identification, they coincide on “sufficiently regular” measures, characterized in [Gig11]. In the sequel, we drop the term *geometric* and refer to  $\mathbf{Tan}_\mu$  as the tangent space. The set  $\mathbf{Tan}_\mu$  is closed in  $(\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu, W_\mu)$ , and enjoys a well-defined projection

$$\pi_T^\mu : \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu \rightarrow \mathbf{Tan}_\mu$$

such that  $\pi_T^\mu \xi$  minimizes  $W_\mu(\xi, \zeta)$  among all  $\zeta \in \mathbf{Tan}_\mu$  [Gig08, Proposition 4.30]. The notation  $\pi_T^\mu$  differs from the usual notation  $\pi^\mu$  in order to highlight the projection on the *tangent* component, as opposite to the projection  $\pi_S^\mu$  on the *solenoidal* component to come in the following sections.

Let us gather some useful properties from Propositions 4.25, 4.29, 4.30 and 4.33 in [Gig08].

**Proposition 2.1** (Properties of the tangent space). *Let  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ . The set  $\mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$*

- *is stable by horizontal interpolation, in the sense that for any  $\xi, \zeta \in \mathbf{Tan}_\mu$ , any  $\alpha \in \Gamma_\mu(\xi, \zeta)$  and  $t \in [0, 1]$ , the measure field  $(\pi_x, (1-t)\pi_v + t\pi_w)\#\alpha$  also belongs to  $\mathbf{Tan}_\mu$ ,*
- *is stable by rescaling, i.e.  $\lambda \cdot \xi \in \mathbf{Tan}_\mu$  whenever  $\xi \in \mathbf{Tan}_\mu$  and  $\lambda \in \mathbb{R}$ ,*
- *enjoys a metric projection, in the sense that for any  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ , there exists a unique minimizer  $\pi_T^\mu \xi$  of  $W_\mu(\xi, \zeta)$  among all  $\zeta \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ ,*
- *satisfies  $\langle \xi, \zeta \rangle_\mu = \langle \pi_T^\mu \xi, \zeta \rangle_\mu$  for any  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$  and  $\zeta \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ .*

**Results of the paper** Our main result, stated in Lemma 5.1 below, lies in the equality

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \xi), \exp_\mu(h \cdot \pi_T^\mu \xi))}{h} = 0$$

for any  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ . If one sees  $h \mapsto \exp_\mu(h \cdot \xi)$  and  $h \mapsto \exp_\mu(h \cdot \pi_T^\mu \xi)$  as two curves issued from  $\mu$ , then the above inequality means that they are equivalent near 0 at order 1. Denoting this equivalence by  $\sim_\mu$ , we further show that  $\mathbf{Tan}_\mu$  is isometric to the quotient  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu / \sim_\mu$ .

Let us give a brief intuition of the argument. The key tool is to decompose  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$  as a “sum” of a tangent component and another component, that we call *solenoidal*, generalizing divergence-free vector fields. This decomposition allows to write  $\exp_\mu(h \cdot \xi)$  as the composition of two exponentials: one following the solenoidal component, that essentially *turns around* the base measure  $\mu$ , and a second one that morally follows the tangent component  $\pi_T^\mu \xi$ . The *curling* or *winding* behavior of solenoidal fields is made precise in Lemma 4.1, in which it is shown that such solenoidal measure fields  $\xi$ ,

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \xi), \mu)}{h} = 0.$$

The reader that would like a mental picture can think about the measure field  $\xi := (id, f)\#\mu$ , where  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is given by the rotation  $f(x, y) = (-y, x)$  and  $\mu$  is the normalized Lebesgue measure on the unit circle  $\mathbb{S}^1$ . In this example,  $\exp_\mu(h \cdot \xi)$  is concentrated on the circle of radius  $\sqrt{1+h^2}$ , and is at Wasserstein distance  $\sqrt{1+h^2} - 1 = o(h)$  of  $\mu$ .

### 3 Solenoidal fields

In the literature of fluid dynamics, divergence-free vector fields are termed *solenoidals* in reference to their swirling flows. This visual behavior has been precisely described by Smirnov in [Smi94], where it is shown that the flow of a smooth divergence-free vector field follows “almost-closed” paths. The picture is even clearer in dimension 2, where flow lines follow closed loops with constant velocity [BG22]. Consequently, the  $L^2$  scalar product against a potential vector field will vanish. This provides the basis for a geometrical definition of solenoidal fields.

**Definition 3.1** (Solenoidal measure fields). *Let  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ . An element  $\zeta \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$  is solenoidal, or divergence-free, if*

$$\langle \zeta, \xi \rangle_\mu = 0 \quad \forall \xi \in \mathbf{Tan}_\mu.$$

*The set of solenoidal elements is denoted  $\mathbf{Sol}_\mu \mathcal{P}_2(\mathbb{R}^d)$ .*

This definition is coherent with the solenoidal fields in  $L_\mu^2$ , since for any  $f, g \in L_\mu^2(\mathbb{R}^d; \mathbb{T}\mathbb{R}^d)$ , there holds  $\langle f\#\mu, g\#\mu \rangle_\mu = \langle f, g \rangle_{L_\mu^2}$ .

**Remark 3.1** (Interpretation with plans). *Using the formula (1) for the metric scalar product, we may give an interpretation of Definition 3.1 in terms of transport plans. Indeed, if  $\xi \in \mathbf{Tan}_\mu$ , then  $-\xi \in \mathbf{Tan}_\mu$  by Proposition 2.1, and*

$$\langle \zeta, \xi \rangle_\mu = \sup_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v, w \rangle d\alpha = 0 = \langle \zeta, -\xi \rangle_\mu = - \inf_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v, w \rangle d\alpha.$$

Consequently, a measure field  $\zeta$  is solenoidal if for any  $\xi \in \mathbf{Tan}_\mu$  and any transport plan  $\alpha \in \Gamma_\mu(\xi, \zeta)$ , the integral of  $(x, v, w) \mapsto \langle v, w \rangle$  with respect to  $\alpha$  vanishes.

The following properties are deduced immediately.

**Lemma 3.1** (First characterization of solenoidal measure fields). *Let  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ . The following propositions are equivalent:*

- a.  $\xi \in \mathbf{Sol}_\mu$ ,
- b.  $D_\mu d_{\mathcal{W}}^2(\cdot, \nu)(\xi) = 0$  for all  $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ ,
- c.  $\pi_T^\mu \xi = 0_\mu$ .

*Proof.* Recalling that  $\langle \xi, \zeta \rangle_\mu = \langle \pi_T^\mu \xi, \zeta \rangle_\mu$  for any  $\zeta \in \mathbf{Tan}_\mu$ , the implication (c)  $\Rightarrow$  (a) holds trivially. Assume now that (a) holds. By the formula (3) of the directional derivative of the squared distance, and using that  $-2 \cdot \zeta \in \mathbf{Tan}_\mu$  as soon as  $\zeta \in \exp_\mu^{-1}(\nu)$  (see Proposition 2.1),

$$D_\mu d_{\mathcal{W}}^2(\cdot, \nu)(\xi) = \inf_{\zeta \in \exp_\mu^{-1}(\nu)} -2 \langle \xi, \zeta \rangle_\mu = 0.$$

Assume now that (b) holds for some  $\xi \in \mathcal{P}_2(\mathbb{T} \mathbb{R}^d)_\mu$ . By the construction of the tangent cone, there exists  $(a_n, \eta_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+ \times \mathbf{Tan}_\mu$  such that  $\eta_n \in \exp_\mu^{-1}(\exp_\mu(\eta_n))$ , and  $\lim_{n \rightarrow \infty} W_\mu(\pi_T^\mu \xi, a_n \cdot \eta_n) = 0$ . By [AGS05, Lemma 7.2.1], the unique optimal transport plan between  $\mu$  and  $\exp_\mu(1/2 \cdot \eta_n)$  is induced by  $1/2 \cdot \eta_n$ . Then

$$\langle \pi_T^\mu \xi, \eta_n \rangle_\mu = 2 \langle \pi_T^\mu \xi, \frac{1}{2} \cdot \eta_n \rangle_\mu = -D_\mu d_{\mathcal{W}}^2(\cdot, \exp_\mu(1/2 \cdot \eta_n))(\xi) = 0.$$

Notice that by definition, the metric scalar product is continuous with respect to  $W_\mu$ . Multiplying by  $a_n \geq 0$ , we get  $0 = \langle \pi_T^\mu \xi, a_n \cdot \eta_n \rangle_\mu \rightarrow_n \langle \pi_T^\mu \xi, \pi_T^\mu \xi \rangle_\mu = \|\pi_T^\mu \xi\|_\mu^2$ . Thus (c) holds.  $\square$

The tangent space enjoys good properties of stability by scaling and interpolation, as mentioned in Proposition 2.1. As one could guess, solenoidal measure fields share the same ‘‘algebraic’’ properties, that will be used in the sequel.

**Lemma 3.2** (Properties of the solenoidal space). *Let  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ . The set  $\mathbf{Sol}_\mu \mathcal{P}_2(\mathbb{R}^d)$*

- *is stable by horizontal interpolation, in the sense that for any  $\xi, \zeta \in \mathbf{Sol}_\mu$ , any  $\alpha \in \Gamma_\mu(\xi, \zeta)$  and  $t \in [0, 1]$ , the measure field  $(\pi_x, (1-t)\pi_v + t\pi_w) \# \alpha$  also belongs to  $\mathbf{Sol}_\mu$ ,*
- *is stable by rescaling, i.e.  $\lambda \cdot \xi \in \mathbf{Sol}_\mu$  whenever  $\xi \in \mathbf{Sol}_\mu$  and  $\lambda \in \mathbb{R}$ .*

*Proof.* By [Gig08, Proposition 4.27], the metric scalar product is convex in the sense that

$$\langle (\pi_x, (1-t)\pi_v + t\pi_w) \# \alpha, \eta \rangle_\mu \leq (1-t) \langle (\pi_x, \pi_v) \# \alpha, \eta \rangle_\mu + t \langle (\pi_x, \pi_w) \# \alpha, \eta \rangle_\mu$$

for any  $t \in [0, 1]$ ,  $\alpha \in \mathcal{P}_2(\mathbb{T}^2 \mathbb{R}^d)_\mu$  and  $\eta \in \mathcal{P}_2(\mathbb{T} \mathbb{R}^d)_\mu$ . Choosing now  $\alpha$  as a transport plan between two solenoidal measure fields  $\xi, \zeta$ , and  $\eta \in \mathbf{Tan}_\mu$ , we get

$$\langle (\pi_x, (1-t)\pi_v + t\pi_w) \# \alpha, \eta \rangle_\mu \leq 0.$$

Taking now  $-\eta$  in place of  $\eta$  and recalling from (2) that  $\langle \zeta, -\eta \rangle_\mu \geq -\langle \zeta, \eta \rangle_\mu$ , we obtain that the interpolation  $(\pi_x, (1-t)\pi_v + t\pi_w) \# \alpha$  is solenoidal. Similarly, if  $\lambda \in \mathbb{R}$  and  $\xi \in \mathbf{Sol}_\mu$ , then  $\langle \lambda \cdot \xi, \eta \rangle_\mu = |\lambda| \langle \xi, \text{sign}(\lambda) \cdot \eta \rangle_\mu = 0$  for any  $\eta \in \mathbf{Tan}_\mu$ .  $\square$

One may wonder if solenoidal measure fields are stable with respect to  $\mu$  in some sense, i.e. if the set-valued map  $\mu \mapsto \mathbf{Sol}_\mu \mathcal{P}_2(\mathbb{R}^d)$  has any continuity property. We first treat the case of  $\mu \mapsto \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ , which happens to be sequentially lower semicontinuous. Although we could not find any clear statement in the literature, the argument is well-known in the community of optimal transport, and Lemma 3.3 seems to belong to folklore knowledge.

**Lemma 3.3** (Sequential lower semicontinuity of **Tan**). *Let  $(\mu, \xi) \in \mathbf{Tan} \mathcal{P}_2(\mathbb{R}^d)$  and  $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}_2(\mathbb{R}^d)$  be a sequence converging towards  $\mu$  in the Wasserstein topology. There exists  $(\xi_n)_{n \in \mathbb{N}}$  with  $\xi_n \in \mathbf{Tan}_{\mu_n} \mathcal{P}_2(\mathbb{R}^d)$  such that  $\xi_n \rightarrow \xi$  with respect to  $d_{\mathcal{W}, \mathbb{T}\mathbb{R}^d}$ .*

*Proof.* First assume that  $\xi \in \exp_\mu^{-1}(\sigma)$  for some  $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$ . Denote  $\sigma_t := \exp_\mu((1-t) \cdot \xi)$  for some  $t \in (0, 1)$ . By [AGS05, Lemma 7.2.1], the unique optimal transport plan between  $\mu$  and  $\sigma_t$  is induced by the velocity  $(1-t) \cdot \xi$ . For each  $n \in \mathbb{N}$ , pick  $\xi_n^t \in \exp_{\mu_n}^{-1}(\sigma_t)$ , which belongs to  $\mathbf{Tan}_{\mu_n} \mathcal{P}_2(\mathbb{R}^d)$ . Since the application  $\nu \mapsto \exp_\nu^{-1}(\nu)$  is upper semicontinuous with nonempty compact images in  $(\mathcal{P}_2(\mathbb{T}\mathbb{R}^d), d_{\mathcal{W}, \mathbb{T}\mathbb{R}^d})$ , the union  $\bigcup_{n \in \mathbb{N}} \exp_{\mu_n}^{-1}(\nu)$  is itself relatively compact (see [Ber59, Theorem VI.3]). Thus so is the sequence  $(\xi_n^t)_{n \in \mathbb{N}}$ . Since Wasserstein limits of optimal plans are optimal between their marginals [Vil09, Theorem 5.20], each of its limit point lies in the singleton  $\exp_\mu^{-1}(\sigma_t) = \{(1-t) \cdot \xi\}$ . Hence for any fixed  $t > 0$ , the whole sequence  $(\xi_n^t)_{n \in \mathbb{N}}$  converges towards  $(1-t) \cdot \xi$ . Let now  $(t_m)_{m \in \mathbb{N}}$  be a vanishing sequence, and for each  $m$ , let  $n_m$  be large enough so that  $d_{\mathcal{W}, \mathbb{T}\mathbb{R}^d}((1-t_m) \cdot \xi, \xi_n^{t_m}) \leq 2^{-m}$  for all  $n \geq n_m$ . Define  $\xi_n := \xi_n^{t_m}$  if  $n_m \leq n < n_{m+1}$ . Then

$$d_{\mathcal{W}, \mathbb{T}\mathbb{R}^d}(\xi, \xi_n) \leq d_{\mathcal{W}, \mathbb{T}\mathbb{R}^d}(\xi, (1-t_m) \cdot \xi) + d_{\mathcal{W}}((1-t_m) \cdot \xi, \xi_n^{t_m}) \leq t_m \|\xi\|_\mu + 2^{-m} \xrightarrow{m \rightarrow \infty} 0.$$

This provides the desired sequence  $(\xi_n)_{n \in \mathbb{N}}$  in the case where  $\xi$  induces a geodesic. If now  $\xi = \alpha \cdot \zeta$  for some  $\alpha > 0$  and  $\zeta$  inducing a geodesic, the sequence of renormalized plans  $(\alpha \cdot \zeta_n)_{n \in \mathbb{N}}$  fills the claim. Finally, if  $\xi = \lim_{k \rightarrow \infty} \alpha^k \zeta^k$  with respect to  $W_\mu$ , the same diagonal argument applies.  $\square$

The relation between solenoidal measure fields and tangent measure fields reverses the continuity of **Sol** with respect to that of **Tan**.

**Proposition 3.1** (Sequential upper semicontinuity of **Sol**). *Let  $(\mu_n, \xi_n)_{n \in \mathbb{N}} \subset \mathbf{Sol} \mathcal{P}_2(\mathbb{R}^d)$  be a sequence such that  $\mu_n \rightarrow_n \mu \in \mathcal{P}_2(\mathbb{R}^d)$  and  $\xi_n \rightarrow_n \xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$  in the respective Wasserstein topologies. Then  $\xi \in \mathbf{Sol}_\mu \mathcal{P}_2(\mathbb{R}^d)$ .*

*Proof.* Let  $\eta \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ . By Lemma 3.3, there exists a sequence  $(\eta_n)_{n \in \mathbb{N}}$  such that  $\eta_n \rightarrow_n \eta$  in the Wasserstein topology, and  $\eta_n \in \mathbf{Tan}_{\mu_n} \mathcal{P}_2(\mathbb{R}^d)$ . For each  $n \in \mathbb{N}$ , pick  $\alpha_n \in \Gamma_{\mu_n, o}(\xi_n, \eta_n)$ . Since the latter set is compact in  $\mathcal{P}_2(\mathbb{T}^2 \mathbb{R}^d)$ , and using again the stability of optimality [Vil09, Theorem 5.20], we may extract a subsequence  $(\alpha_{n_k})_k$  that converges towards some  $\alpha \in \Gamma_{\mu, o}(\xi, \eta)$  with respect to  $d_{\mathcal{W}, \mathbb{T}^2 \mathbb{R}^d}$ . Thus

$$\langle \xi, \eta \rangle_\mu = \int_{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v, w \rangle d\alpha = \lim_{n \rightarrow \infty} \int_{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v, w \rangle d\alpha_n = \lim_{n \rightarrow \infty} \langle \xi_n, \eta_n \rangle_\mu = 0.$$

Here we used the continuity and quadratic growth of the integrand  $(x, v, w) \mapsto \langle v, w \rangle$  to ensure the convergence. Since  $\eta \in \mathbf{Tan}_\mu$  is arbitrary,  $\xi \in \mathbf{Sol}_\mu \mathcal{P}_2(\mathbb{R}^d)$ .  $\square$

To illustrate Proposition 3.1, let us consider the dimension  $d = 1$ . In this particular case, the set of divergence-free vector fields (i.e. induced by an application) is reduced to the null field: indeed, it is the closure in  $L_\mu^2$  of smooth and compactly supported divergence-free vector fields, which satisfy  $\operatorname{div} f(x) = f'(x) = 0$ , hence identically vanish. However, there are nontrivial *measure fields*, as for instance  $\xi := \frac{1}{2}(g\#\mu) + \frac{1}{2}(-g\#\mu)$ , where  $g(x) := \mathbb{1}_{\{0 \leq x \leq 1\}}$  and  $\mu$  is the Lebesgue measure supported on  $[0, 1]$ . The fact that  $\xi \in \mathbf{Sol}_\mu$  can be verified by direct computation using Lemma 4.1 below, but also in a more geometric way. If  $\gamma^n \in \mathcal{C}([0, 1]; \mathbb{R}^2)$  is a constant-speed parameterization of the ellipsoid

$$x^2 + 2^n y^2 = 1,$$

define  $\mu_n := \gamma^n \# \mathcal{L}_{[0,1]}$  and  $\xi_n := (\gamma^n, \dot{\gamma}^n) \# \mathcal{L}_{[0,1]}$ . Each  $\xi^n$  is an *elementary solenoid* in the terminology of [Smi94], and it is easy to prove that it belongs to  $\mathbf{Sol}_{\mu_n}$ . As  $\mu_n \rightarrow_n \mu$  and  $\xi_n \rightarrow_n \xi$  in the respective Wasserstein topology, using the stability of  $\mathbf{Sol}$ , we get that  $\xi$  is indeed solenoidal. This raises the question of the validity of a decomposition *à la Smirnov*, by means of a vertical superposition of possibly degenerate elementary solenoids, among which  $\xi$  would be. Although it may be possible to proceed by regularization, the author has currently no working proof of this fact.

## 4 Characterization as non-escaping directions

By [Gig08, Theorem 4.19], the tangent cone is characterized by

$$\xi \in \mathbf{Tan}_{\mu} \quad \iff \quad \lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, \exp_{\mu}(h \cdot \xi))}{h} = \|\xi\|_{\mu}. \quad (6)$$

Equation (6) means that the “norm” of tangent elements coincides with the speed with which the curve  $h \mapsto \exp_{\mu}(h \cdot \xi)$  escapes from  $\mu$ . At the opposite, solenoidal measure fields are characterized as the initial velocities of curves that *do not* leave  $\mu$  at the infinitesimal level.

**Lemma 4.1** (Second characterization of solenoidal measure fields). *A measure field  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_{\mu}$  is solenoidal if and only if*

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, \exp_{\mu}(h \cdot \xi))}{h} = 0. \quad (7)$$

To prepare for the proof of Lemma 4.1, define  $W_{\mu,\nu} : \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_{\mu} \times \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_{\nu} \rightarrow \mathbb{R}^+$  by

$$W_{\mu,\nu}^2(\xi, \zeta) := \inf \left\{ \int_{((x,v),(y,w)) \in (\mathbb{T}\mathbb{R}^d)^2} |v - w|^2 d\alpha \mid \alpha \in \Gamma(\xi, \zeta) \text{ and } (\pi_x, \pi_y) \# \alpha \in \Gamma_o(\mu, \nu) \right\}.$$

The application  $W_{\mu,\nu}(\cdot, \cdot)$  does not induce a distance, but naturally appears in estimates. We refer the reader to [Pic19, Definition 4.1] for further comments. As one could expect, it vanishes when  $\xi$  and  $\zeta$  are close, in the following sense.

**Lemma 4.2** (Lower semi-continuity with respect to  $d_{\mathcal{W}}$ ). *Let  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$  and  $(\xi_n)_{n \in \mathbb{N}} \subset \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)$  such that  $d_{\mathcal{W}, \mathbb{T}\mathbb{R}^d}(\xi, \xi_n) \rightarrow_n 0$ . Denote  $\mu := \pi_x \# \xi$  and  $\mu_n := \pi_x \# \xi_n$  their respective marginals. Then*

$$\lim_{n \rightarrow \infty} W_{\mu, \mu_n}(\xi, \xi_n) = 0.$$

*Proof.* Denote again  $(\mu_n, \xi_n)_{n \in \mathbb{N}}$  a maximizing subsequence such that

$$\overline{\lim}_{n \searrow 0} W_{\mu, \mu_n}^2(\xi, \xi_n) = \lim_{n \rightarrow \infty} W_{\mu, \mu_n}^2(\xi, \xi_n) = \lim_{n \rightarrow \infty} \int_{(x,v),(y,w) \in (\mathbb{T}\mathbb{R}^d)^2} |v - w|^2 d\alpha_n$$

for some  $\alpha_n \in \Gamma(\xi, \xi_n)$  with  $(\pi_x, \pi_y) \# \alpha_n \in \Gamma_o(\mu, \mu_n)$ . By assumption, the set  $\{\xi\} \cup \{\xi_n \mid n \in \mathbb{N}\}$  is compact in the Wasserstein topology over  $\mathbb{T}\mathbb{R}^d$ . As the application  $(\xi, \xi_n) \mapsto \Gamma(\xi, \xi_n)$  has nonempty compact images and is upper semicontinuous with respect to  $d_{\mathcal{W}, (\mathbb{T}\mathbb{R}^d)^2}$ , the set  $\Gamma(\xi, \xi) \cup \bigcup_{n \in \mathbb{N}} \Gamma(\xi, \xi_n)$  is compact in  $(\mathcal{P}_2((\mathbb{T}\mathbb{R}^d)^2), d_{\mathcal{W}, (\mathbb{T}\mathbb{R}^d)^2})$ . Hence up to a further subsequence, we may assume that  $(\alpha_n)_n \rightarrow \alpha^* \in \mathcal{P}_2((\mathbb{T}\mathbb{R}^d)^2)$  for some  $\alpha^* \in \Gamma(\xi, \xi)$ . By stability of optimality [Vil09, Theorem 5.20],  $(\pi_x, \pi_x) \# \alpha^* \in \Gamma_o(\mu, \mu)$  and  $\alpha^*$  reaches the infimum defining  $W_{\mu, \mu}(\xi, \xi)$ . Hence, as  $((x, v), (y, w)) \mapsto |v - w|^2$  has quadratic growth and is continuous,

$$\lim_{n \searrow 0} \int_{(x,v),(y,w) \in (\mathbb{T}\mathbb{R}^d)^2} |v - w|^2 d\alpha_n = \int_{(x,v),(y,w) \in (\mathbb{T}\mathbb{R}^d)^2} |v - w|^2 d\alpha^* = W_{\mu, \mu}^2(\xi, \xi) = 0. \quad \square$$

Lemma 4.2 is sharp in the sense that there is no way to bound  $W_{\cdot, \cdot}$  uniformly by  $d_{\mathcal{W}, \mathbb{T}\mathbb{R}^d}$ , as shown in [Pic19, Remark 6]. Recall that in  $\mathbb{R}^d$ , any  $(x, v), (y, w) \in \mathbb{T}\mathbb{R}^d$  satisfy

$$D_y | \cdot - x |^2 (w) + D_x |y - \cdot|^2 (v) = 2 \langle y - x, w \rangle + 2 \langle x - y, v \rangle \leq 2 |x - y| |v - w|.$$

This inequality generalizes in the Wasserstein space, in which the role of the term  $|v - w|$  is taken by  $W_{\mu, \nu}$ .

**Lemma 4.3** (First-order estimate). *For all  $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ ,  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$  and  $\zeta \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\nu$ , there holds*

$$D_\nu d_{\mathcal{W}}^2(\mu, \cdot)(\zeta) + D_\mu d_{\mathcal{W}}^2(\cdot, \nu)(\xi) \leq 2d_{\mathcal{W}}(\mu, \nu)W_{\mu, \nu}(\xi, \zeta). \quad (8)$$

*Proof.* Indeed, by the formula (3) of the directional derivative of  $d_{\mathcal{W}}^2$ , the left hand-side of (8) writes as

$$\begin{aligned} I &:= \inf_{\gamma \in \exp_\nu^{-1}(\mu)} -2 \langle \zeta, \gamma \rangle_\nu + \inf_{\eta \in \exp_\mu^{-1}(\nu)} -2 \langle \xi, \eta \rangle_\mu \\ &\leq \inf_{\omega \in \Gamma_o(\mu, \nu)} -2 \langle \zeta, (\pi_y, \pi_x - \pi_y) \# \omega \rangle_\nu - 2 \langle \xi, (\pi_x, \pi_y - \pi_x) \# \omega \rangle_\mu \\ &\leq \inf_{\omega \in \Gamma_o(\mu, \nu)} \inf_{\substack{\beta \in \Gamma_\nu(\zeta, (\pi_y, \pi_x - \pi_y) \# \omega) \\ \alpha \in \Gamma_\mu(\xi, (\pi_x, \pi_y - \pi_x) \# \omega)}} -2 \int_{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v, w \rangle d\beta - 2 \int_{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v, w \rangle d\alpha. \end{aligned} \quad (9)$$

Any couple  $(\alpha, \beta)$  in (9) gives rise to at least one plan  $\varpi = \varpi_{\alpha, \beta} \in \Gamma(\xi, \zeta)$  such that

$$(\pi_x, \pi_v, \pi_y - \pi_x) \# \varpi = \alpha \quad \text{and} \quad (\pi_y, \pi_w, \pi_x - \pi_y) \# \varpi = \beta.$$

Indeed, the applications  $(x, v, w) \mapsto (x, x + w)$  and  $(x, v, w) \mapsto (x + w, x)$  are continuous from  $\mathbb{T}^2 \mathbb{R}^d$  to  $(\mathbb{R}^d)^2$ . Then we may disintegrate

$$\alpha(x, v, w) = \int_{(a, b) \in (\mathbb{R}^d)^2} \alpha_{(a, b)}(x, v, w) d\omega(a, b) \quad \text{and} \quad \beta(x, v, w) = \int_{(a, b) \in (\mathbb{R}^d)^2} \beta_{(a, b)}(x, v, w) d\omega(a, b),$$

with  $\text{supp } \alpha_{(a, b)} \subset \{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d \mid x = a \text{ and } x + w = b\}$ , and in a symmetrical way,  $\text{supp } \beta_{(a, b)} \subset \{(x, v, w) \in \mathbb{T}^2 \mathbb{R}^d \mid x + w = a \text{ and } x = b\}$ . Consequently, there exists  $\sigma_{(a, b)} \in \mathcal{P}_2(\mathbb{T}_a \mathbb{R}^d)$  such that  $\alpha_{(a, b)} = \delta_a \times \sigma_{(a, b)} \times \delta_{b-a}$ , and  $\tau_{(a, b)} \in \mathcal{P}_2(\mathbb{T}_b \mathbb{R}^d)$  such that  $\beta_{(a, b)} = \delta_b \times \tau_{(a, b)} \times \delta_{a-b}$ . Define now  $\varpi \in \Gamma(\xi, \zeta)$  by

$$\begin{aligned} &\int_{((x, v), (y, w)) \in (\mathbb{T}\mathbb{R}^d)^2} \varphi(x, v, y, w) d\varpi \\ &= \int_{(a, b) \in (\mathbb{R}^d)^2} \int_{(v, w) \in \mathbb{T}_a \mathbb{R}^d \times \mathbb{T}_b \mathbb{R}^d} \varphi(a, v, b, w) d[\sigma_{(a, b)} \otimes \tau_{(a, b)}](v, w) d\omega(a, b). \end{aligned}$$

It is easy to verify that  $\varpi$  has the right marginals, and that  $(\pi_x, \pi_y) \# \varpi = \omega$  is optimal between  $\mu$  and  $\nu$ . This makes it an admissible plan for the definition of  $W_{\mu, \nu}(\xi, \zeta)$ . Denoting  $A_\omega$  the set of all  $\varpi_{\alpha, \beta}$  when  $\alpha, \beta$  range in their respective sets, (9) rewrites as

$$\begin{aligned} I &\leq \inf_{\omega \in \Gamma_o(\mu, \nu)} \inf_{\varpi \in A_\omega} \int_{((x, v), (y, w)) \in (\mathbb{T}\mathbb{R}^d)^2} -2 \langle w, x - y \rangle - 2 \langle v, y - x \rangle d\varpi \\ &\leq 2 \inf_{\omega \in \Gamma_o(\mu, \nu)} \inf_{\varpi \in A_\omega} \left( \int_{(x, y) \in (\mathbb{R}^d)^2} |y - x|^2 d\omega \right)^{1/2} \left( \int_{((x, v), (y, w)) \in (\mathbb{T}\mathbb{R}^d)^2} |w - v|^2 d\varpi \right)^{1/2} \\ &= 2d_{\mathcal{W}}(\mu, \nu)W_{\mu, \nu}(\xi, \zeta). \end{aligned} \quad \square$$

We are now in position to prove the second characterization of solenoidal fields.

*Proof of Lemma 4.1.* If (7) holds for some  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ , then for any  $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ ,

$$\begin{aligned} |D_\mu d_{\mathcal{W}}^2(\cdot, \nu)(\xi)| &= \lim_{h \searrow 0} \frac{|d_{\mathcal{W}}^2(\exp_\mu(h \cdot \xi), \nu) - d_{\mathcal{W}}^2(\mu, \nu)|}{h} \\ &\leq \lim_{h \searrow 0} (d_{\mathcal{W}}(\exp_\mu(h \cdot \xi), \nu) + d_{\mathcal{W}}(\mu, \nu)) \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \xi), \mu)}{h} = 0. \end{aligned}$$

The first characterization given by Lemma 3.1 then implies  $\xi \in \mathbf{Sol}_\mu$ . Reciprocally, assume that  $\xi$  is solenoidal. The function  $\varphi : s \mapsto d_{\mathcal{W}}^2(\mu, \exp_\mu(s \cdot \xi))$  being Lipschitz-continuous over  $[0, 1]$ , there holds

$$\varphi(h) = \int_{s=0}^h \frac{d}{ds} \varphi(s) ds. \quad (10)$$

Notice that for any  $s, \varepsilon \geq 0$ , there holds

$$\exp_\mu((s + \varepsilon) \cdot \xi) = (\pi_x + s\pi_v + \varepsilon\pi_v)\#\xi = (\pi_x + \varepsilon\pi_v)\#\xi_s = \exp_{\exp_\mu(s \cdot \xi)}(\varepsilon \cdot \xi_s),$$

where  $\xi_s := (\pi_x + s\pi_v, \pi_v)\#\xi$ . Then for any point  $s \in [0, 1]$  of differentiability of  $\varphi$ ,

$$\frac{d}{ds} \varphi(s) = \lim_{\varepsilon \searrow 0} \frac{d_{\mathcal{W}}^2(\mu, \exp_\mu((s + \varepsilon) \cdot \xi)) - d_{\mathcal{W}}^2(\mu, \exp_\mu(s \cdot \xi))}{\varepsilon} = D_{\exp_\mu(s \cdot \xi)} d_{\mathcal{W}}^2(\mu, \cdot)(\xi_s).$$

Applying the inequality (8) with  $\nu = \exp_\mu(s \cdot \xi)$  and  $\zeta = \xi_s$ ,

$$D_{\exp_\mu(s \cdot \xi)} d_{\mathcal{W}}^2(\mu, \cdot)(\xi_s) + D_\mu d_{\mathcal{W}}^2(\cdot, \exp_\mu(s \cdot \xi))(\xi) \leq 2d_{\mathcal{W}}(\mu, \exp_\mu(s \cdot \xi))W_{\mu, \exp_\mu(s \cdot \xi)}(\xi, \xi_s).$$

Since  $\xi \in \mathbf{Sol}_\mu$ , Lemma 3.1 implies that the second term of the left hand-side vanishes. Plugging this into (10) yields

$$\varphi(h) \leq \int_{s=0}^h 2d_{\mathcal{W}}(\mu, \exp_\mu(s \cdot \xi))W_{\mu, \exp_\mu(s \cdot \xi)}(\xi, \xi_s) ds.$$

Using  $2ab \leq a^2/h + hb^2$  for any  $h > 0$ ,

$$\varphi(h) \leq \int_{s=0}^h \left[ \frac{\varphi(s)}{h} + hW_{\mu, \exp_\mu(s \cdot \xi)}^2(\xi, \xi_s) \right] ds,$$

so that with a Grönwall Lemma,

$$\varphi(h) = d_{\mathcal{W}}^2(\mu, \exp_\mu(h \cdot \xi)) \leq h^2 e^1 \sup_{s \in [0, h]} W_{\mu, \exp_\mu(s \cdot \xi)}^2(\xi, \xi_s).$$

Since  $d_{\mathcal{W}, \mathbb{T}\mathbb{R}^d}(\xi, \xi_s) \leq d_{\mathcal{W}}(\mu, \exp_\mu(s \cdot \xi)) \leq s\|\xi\|$  goes to 0 with  $s$ , we may apply Lemma 4.2 to obtain that  $d_{\mathcal{W}}(\mu, \exp_\mu(h \cdot \xi)) = \sqrt{\varphi(h)} = o(h)$ .  $\square$

## 5 Structure of the tangent space

In this Section, we identify the solenoidal component of any  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ , and use it in Theorem 5.2 to prove the quotient structure of the tangent space. To this aim, we heavily rely on the following fine analysis of the projection over the tangent space.

**Proposition 5.1** (Optimal plans towards the projection [Gig08, Proposition 4.32]). *Let  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ ,  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ , and  $\pi_T^\mu \xi \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$  be the unique projection of  $\xi$  on  $\mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ . Then there exists an unique element  $\gamma \in \Gamma_{\mu,o}(\xi, \pi_T^\mu \xi)$ , that is induced by a map in the following sense: there exists a measurable map  $T : \mathbb{T}\mathbb{R}^d \rightarrow \mathbb{T}\mathbb{R}^d$  such that  $T(x, v) = (x, T_2(x, v))$ , and*

$$\gamma = (\pi_x, \pi_v, \pi_v + T_2) \# \xi.$$

Proposition 5.1 allows us to define a ‘‘difference’’ between  $\xi$  and its projection  $\pi_T^\mu \xi$  unambiguously. We now show that this difference is solenoidal. This generalizes the usual Helmholtz-Hodge decomposition to the set  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$  of measure fields. Indeed, whenever  $\xi = f \# \mu$  for some  $f \in L_\mu^2(\mathbb{R}^d; \mathbb{T}\mathbb{R}^d)$ , then the projection  $\pi_T^\mu \xi$  is precisely given by  $g \# \mu$ , where  $g \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$  is the projection of  $f$  over the closure of smooth gradients in  $L_\mu^2$  [Gig08, Proposition 4.38]. The difference  $f - g$  is then the divergence-free component of  $f$  in the classical theory.

**Theorem 5.1** (Helmholtz-Hodge decomposition). *Let  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$  and  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ . Denote  $\pi_T^\mu \xi$  the projection of  $\xi$  over  $\mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ , and  $\gamma$  the unique element of  $\Gamma_{\mu,o}(\xi, \pi_T^\mu \xi)$ , given by Proposition 5.1. Then*

$$\omega := (\pi_x, \pi_v - \pi_w) \# \gamma \tag{11}$$

is a solenoidal measure field which satisfies

$$\langle \xi, \zeta \rangle_\mu = \langle \omega, \zeta \rangle_\mu \quad \text{and} \quad W_\mu^2(\xi, \zeta) = W_\mu^2(\xi, \omega) + W_\mu^2(\omega, \zeta) \quad \forall \zeta \in \mathbf{Sol}_\mu \mathcal{P}_2(\mathbb{R}^d). \tag{12}$$

Consequently,  $\omega = \pi_S^\mu \xi$  is the metric projection of  $\xi$  over the set  $\mathbf{Sol}_\mu \mathcal{P}_2(\mathbb{R}^d)$ , i.e. minimizes  $W_\mu(\xi, \zeta)$  among all  $\zeta \in \mathbf{Sol}_\mu \mathcal{P}_2(\mathbb{R}^d)$ .

*Proof.* According to Proposition 5.1,  $\omega$  is of the form  $\omega = (-T) \# \xi$  for some measurable map  $T : \mathbb{T}\mathbb{R}^d \rightarrow \mathbb{T}\mathbb{R}^d$ . We first show that  $\omega \in \mathbf{Sol}_\mu$ , then establish (12).

Let  $\eta \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$  be arbitrary, and denote  $\Gamma_\mu(\gamma, \eta)$  the set of  $\beta = \beta(x, v, w_0, w_1) \in \mathcal{P}_2(\mathbb{T}^3\mathbb{R}^d)$  such that  $(\pi_x, \pi_v, \pi_{w_0}) \# \beta = \gamma$ , and  $(\pi_x, \pi_{w_1}) \# \beta = \eta$ . As the tangent space  $\mathbf{Tan}_\mu$  is stable by scaling and interpolation using any transport plan (see Proposition 2.1), the measure field  $\zeta := (\pi_x, \pi_{w_0} - h\pi_{w_1}) \# \beta$  belongs to  $\mathbf{Tan}_\mu$  for any  $\beta \in \Gamma_\mu(\gamma, \zeta)$  and  $h \in [0, 1]$ . Then

$$\begin{aligned} W_\mu^2(\xi, \pi_T^\mu \xi) &\leq W_\mu^2(\xi, \zeta) = \inf_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x,v,w) \in \mathbb{T}^2\mathbb{R}^d} |v - w|^2 d\alpha(x, v, w) \\ &= \inf_{\beta \in \Gamma_\mu(\gamma, \eta)} \int_{(x,v,w_0,w_1) \in \mathbb{T}^3\mathbb{R}^d} |v - w_0 + hw_1|^2 d\beta(x, v, w_0, w_1). \end{aligned}$$

Developing and using the optimality of  $\gamma$ , there holds

$$W_\mu^2(\xi, \pi_T^\mu \xi) \leq W_\mu^2(\xi, \pi_T^\mu \xi) + 2h \inf_{\beta \in \Gamma_\mu(\gamma, \eta)} \int_{(x,v,w_0,w_1) \in \mathbb{T}^3\mathbb{R}^d} \langle v - w_0, w_1 \rangle d\beta + h^2 \|\eta\|_\mu^2,$$

so that dividing by  $h > 0$  and sending  $h$  to 0,

$$\begin{aligned} 0 &\leq \inf_{\beta \in \Gamma_\mu(\gamma, \eta)} \int_{(x,v,w_0,w_1) \in \mathbb{T}^3\mathbb{R}^d} \langle v - w_0, w_1 \rangle d\beta \\ &= \inf \left\{ \int_{(x,v,w) \in \mathbb{T}^2\mathbb{R}^d} \langle v, w \rangle d\alpha \mid \exists \beta \in \Gamma_\mu(\gamma, \eta) \text{ such that } \alpha = (\pi_x, \pi_v - \pi_{w_0}, \pi_{w_1}) \# \beta \in \Gamma_\mu(\omega, \eta) \right\}. \end{aligned}$$

Let us show that the set of  $\alpha$  that may be attained as  $(\pi_x, \pi_v - \pi_{w_0}, \pi_{w_1}) \# \beta$  for some  $\beta \in \Gamma_\mu(\gamma, \eta)$  covers all  $\Gamma_\mu(\omega, \eta)$ . First disintegrate  $\omega = \omega_x \otimes \mu$ ,  $\eta = \eta_x \otimes \mu$  and  $\gamma = \gamma_x \otimes \mu$ . As the map  $(v, w_0) \rightarrow v - w_0 =: w$

is continuous, we may further disintegrate  $\gamma_x = (\gamma_x)_w \otimes \varpi_x$ , where  $\varpi_x$  and  $\omega_x$  coincide for  $\mu$ -almost every  $x \in \mathbb{R}^d$ . Now, pick any  $\alpha \in \Gamma_\mu(\omega, \eta)$  and disintegrate it into  $\alpha = \alpha_x \otimes \mu$ , where  $\alpha_x \in \Gamma(\varpi_x, \eta_x)$  for  $\mu$ -almost every  $x \in \mathbb{R}^d$ . The transport plan  $\beta \in \mathcal{P}_2(\mathbb{T}^3 \mathbb{R}^d)$  given by

$$\begin{aligned} & \int_{(x,v,w_0,w_1) \in \mathbb{T}^3 \mathbb{R}^d} \varphi(x,v,w_0,w_1) d\beta \\ &= \int_{(x,w) \in \mathbb{T} \mathbb{R}^d} \int_{(v,w_0,w_1) \in \mathbb{T}_x^3 \mathbb{R}^d} \varphi(x,v,w_0,w_1) d[(\gamma_x)_w \otimes \alpha_x](v,w_0,w_1) d\omega(x,w). \end{aligned}$$

indeed satisfies

$$(\pi_x, \pi_v, \pi_{w_0}) \# \beta = \gamma, \quad (\pi_x, \pi_{w_1}) \# \beta = \eta, \quad (\pi_x, \pi_v - \pi_{w_0}, \pi_{w_1}) \# \beta = \alpha.$$

Consequently, using the equivalent formulation (1) of the metric scalar product,

$$0 \leq \inf_{\alpha \in \Gamma_\mu(\omega, \eta)} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v, w \rangle d\alpha = -\langle \omega, -\eta \rangle_\mu.$$

Taking  $-\eta$  in place of  $\eta$ , there also holds  $0 \leq -\langle \omega, \eta \rangle_\mu$ , so that  $\langle \omega, \eta \rangle_\mu = 0$  for any  $\eta \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ . This shows that  $\omega$  is solenoidal.

On the other hand, let  $\zeta \in \mathbf{Sol}_\mu$ . From the particular structure of  $\omega = (-T) \# \xi$ , there holds

$$\begin{aligned} \langle \xi, \zeta \rangle_\mu &= \sup_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v + T_2(x,v), w \rangle d\alpha + \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} \langle -T_2(x,v), w \rangle d\alpha \\ &= \sup_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v, w \rangle d(\pi_x, \pi_v + T_2, \pi_w) \# \alpha + \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} \langle -T_2(x,v), w \rangle d\alpha. \end{aligned}$$

As  $(\pi_x, \pi_v + T_2, \pi_w) \# \alpha \in \Gamma_\mu(\pi_T^\mu \xi, \zeta)$  is a transport plan between  $\pi_T^\mu \xi \in \mathbf{Tan}_\mu$  and  $\zeta \in \mathbf{Sol}_\mu$ , the first summand vanishes by Remark 3.1. Then

$$\langle \xi, \zeta \rangle_\mu = \sup_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} \langle -T_2(x,v), w \rangle d\alpha = \langle \omega, \zeta \rangle_\mu.$$

Taking in particular  $\zeta = \omega$  yields  $\langle \xi, \omega \rangle_\mu = \|\omega\|_\mu^2$ . Consequently, for any  $\zeta \in \mathbf{Sol}_\mu \mathcal{P}_2(\mathbb{R}^d)$ ,

$$\begin{aligned} W_\mu^2(\xi, \zeta) &= \|\xi\|_\mu^2 - 2\langle \xi, \zeta \rangle_\mu + \|\zeta\|_\mu^2 = \|\xi\|_\mu^2 - 2\langle \xi, \omega \rangle_\mu + 2\|\omega\|_\mu^2 - 2\langle \omega, \zeta \rangle_\mu + \|\zeta\|_\mu^2 \\ &= W_\mu^2(\xi, \omega) + W_\mu^2(\omega, \zeta). \end{aligned} \quad \square$$

**Remark 5.1** (Some Pythagoras identities). *Under the notations of the proof of Theorem 5.1, there holds*

$$\|\xi\|_\mu^2 = \|\pi_T^\mu \xi\|_\mu^2 + \|\pi_S^\mu \xi\|_\mu^2$$

for any  $\xi \in \mathcal{P}_2(\mathbb{T} \mathbb{R}^d)_\mu$ . Indeed, by definition,

$$\begin{aligned} W_\mu^2(\xi, \pi_S^\mu \xi) &= \inf_{\alpha \in \Gamma_\mu(\xi, \pi_S^\mu \xi)} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} |v + T_2(x,v) - T_2(x,v) - w|^2 d\alpha \\ &= \|\pi_T^\mu \xi\|_\mu^2 + \inf_{\alpha \in \Gamma_\mu(\xi, \pi_S^\mu \xi)} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v + T_2(x,v), -T_2(x,v) - w \rangle + |-T_2(x,v) - w|^2 d\alpha. \end{aligned}$$

As  $(\pi_x, \pi_v + T_2(\pi_x, \pi_v)) \# \alpha = \pi_T^\mu \xi$  is tangent, and  $(\pi_x, -T_2(\pi_x, \pi_v) - \pi_w) \# \alpha$  is an horizontal interpolation between two solenoidal fields, hence solenoidal by Lemma 3.2, the integral of the scalar product vanishes for any  $\alpha$  (see Remark 3.1). Hence, taking  $\alpha = (\pi_x, \pi_v, -T_2) \# \xi$ ,

$$W_\mu^2(\xi, \pi_S^\mu \xi) = \|\pi_T^\mu \xi\|_\mu^2 + \inf_{\alpha \in \Gamma_\mu(\xi, \pi_S^\mu \xi)} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} |-T_2(x,v) - w|^2 d\alpha = \|\pi_T^\mu \xi\|_\mu^2.$$

By a symmetric reasoning,  $W_\mu^2(\xi, \pi_T^\mu \xi) = \|\pi_S^\mu \xi\|_\mu^2$ . We deduce from (12) applied with  $\zeta = 0_\mu$  that

$$\|\xi\|_\mu^2 = W_\mu^2(\xi, \pi_S^\mu \xi) + W_\mu^2(\pi_S^\mu \xi, 0_\mu) = \|\pi_T^\mu \xi\|_\mu^2 + \|\pi_S^\mu \xi\|_\mu^2.$$

However, we warn the reader that the general Pythagoras formula does not hold, as explained in Section 6.

We may now extend to any measure field the identity of Lemma 4.1.

**Lemma 5.1** (Equivalence to the projection near 0). *Let  $\mu \in \mathcal{P}_2(\mathbb{R}^d)$  and  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ . There holds*

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \xi), \exp_\mu(h \cdot \pi_T^\mu \xi))}{h} = 0. \quad (13)$$

*Proof.* Under the notations of Theorem 5.1, let  $T : \mathbb{T}\mathbb{R}^d \rightarrow \mathbb{T}\mathbb{R}^d$  be the measurable map such that the projection of  $\xi$  on the tangent space is given by  $\pi_T^\mu \xi = (\pi_x, \pi_v + T_2)\#\xi$ , and the solenoidal component as  $\pi_S^\mu \xi = (-T)\#\xi$ . Notice that

$$\exp_\mu(h \cdot \xi) = (\pi_x + h\pi_v)\#\xi = (\pi_x + h(-T_2) + h(\pi_v + T_2))\#\xi = \exp_{\exp_\mu(h \cdot \pi_S^\mu \xi)}(h \cdot (\pi_T^\mu \xi)_h), \quad (14)$$

where  $(\pi_T^\mu \xi)_h := (\pi_x + h(-T_2), \pi_v + T_2)\#\xi$ . For each  $h > 0$ , let  $\alpha \in \Gamma(\pi_T^\mu \xi, (\pi_T^\mu \xi)_h)$  realize the infimum in  $W_{\mu, \exp_\mu(h \cdot \pi_S^\mu \xi)}(\pi_T^\mu \xi, (\pi_T^\mu \xi)_h)$ . By (14),

$$(\pi_x + h\pi_v, \pi_y + h\pi_w)\#\alpha \in \Gamma(\exp_\mu(h \cdot \pi_T^\mu \xi), \exp_{\exp_\mu(h \cdot \pi_S^\mu \xi)}(h \cdot (\pi_T^\mu \xi)_h)) = \Gamma(\exp_\mu(h \cdot \pi_T^\mu \xi), \exp_\mu(h \cdot \xi)).$$

Consequently, as the projection  $(\pi_x, \pi_y)\#\alpha$  is optimal between its marginals by definition of  $W_{\dots}$ ,

$$\begin{aligned} d_{\mathcal{W}}(\exp_\mu(h \cdot \pi_T^\mu \xi), \exp_\mu(h \cdot \xi)) &\leq \left( \int_{((x,v),(y,w)) \in (\mathbb{T}\mathbb{R}^d)^2} |x + hv - (y + hw)|^2 d\alpha \right)^{1/2} \\ &\leq \left( \int_{(\mathbb{R}^d)^2} |x - y|^2 d(\pi_x, \pi_y)\#\alpha \right)^{1/2} + h \left( \int_{(\mathbb{T}\mathbb{R}^d)^2} |v - w|^2 d\alpha \right)^{1/2} \\ &= d_{\mathcal{W}}(\mu, \exp_\mu(h \cdot \pi_S^\mu \xi)) + h W_{\mu, \exp_\mu(h \cdot \pi_S^\mu \xi)}(\pi_T^\mu \xi, (\pi_T^\mu \xi)_h). \end{aligned} \quad (15)$$

One easily gets that  $d_{\mathcal{W}, \mathbb{T}\mathbb{R}^d}(\pi_T^\mu \xi, (\pi_T^\mu \xi)_h)$  vanishes when  $h$  goes to 0. Using the characterization of solenoidal measure fields established in Lemma 4.1 and the estimate of Lemma 4.2, there holds

$$\begin{aligned} \overline{\lim}_{h \searrow 0} \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \pi_T^\mu \xi), \exp_\mu(h \cdot \xi))}{h} &\leq \overline{\lim}_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, \exp_\mu(h \cdot \pi_S^\mu \xi))}{h} + \overline{\lim}_{h \searrow 0} W_{\mu, \exp_\mu(h \cdot \pi_S^\mu \xi)}(\pi_T^\mu \xi, (\pi_T^\mu \xi)_h) \\ &= 0. \end{aligned} \quad \square$$

The estimate (13) allows us to recast the geometric tangent space as the quotient of the larger set  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ . The intuition is the same as in manifolds, where smooth curves are used to define a first space of possible directions, and then identified if they share the same germ. Here, we consider only exponential curves, but the mental picture remains the same. For convenience, denote

$$d_{\text{lim}}(\xi, \zeta) := \overline{\lim}_{h \searrow 0} \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \xi), \exp_\mu(h \cdot \zeta))}{h}.$$

Consider the equivalence relation  $\sim_\mu$  on  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$  given by  $\xi \sim_\mu \zeta$  if  $d_{\text{lim}}(\xi, \zeta) = 0$ , and let  $[\xi]$  be the equivalence class of a measure field  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ . With a slight abuse of notation, we denote

$$W_\mu([\xi], [\zeta]) := \inf \left\{ W_\mu(\xi', \zeta') \mid (\xi', \zeta') \in (\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu)^2, \xi' \sim_\mu \xi, \zeta' \sim_\mu \zeta \right\}.$$

**Theorem 5.2** (Quotient structure of the tangent space). *The metric spaces  $(\mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d), W_\mu)$  and  $(\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu / \sim_\mu, W_\mu)$  are isometric. The isometry associates to any  $\xi \in \mathbf{Tan}_\mu$  the equivalence class  $[\xi]$  for the relation  $\sim_\mu$ , and to any equivalence class in  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu / \sim_\mu$ , the common projection of its members on the tangent space.*

*Proof.* The crucial point is that  $\xi \sim_\mu \zeta$  if and only if  $\pi_T^\mu \xi = \pi_T^\mu \zeta$ . Indeed, if  $\xi$  and  $\zeta$  share the same projection, Lemma 5.1 implies that

$$d_{\text{lim}}(\xi, \zeta) \leq d_{\text{lim}}(\xi, \pi_T^\mu \xi) + d_{\text{lim}}(\pi_T^\mu \zeta, \zeta) = 0,$$

and  $\xi, \zeta$  are equivalent. On the other hand, if  $\xi \sim_\mu \zeta$ , then

$$W_\mu(\pi_T^\mu \xi, \pi_T^\mu \zeta) = d_{\text{lim}}(\pi_T^\mu \xi, \pi_T^\mu \zeta) \leq d_{\text{lim}}(\pi_T^\mu \xi, \xi) + d_{\text{lim}}(\xi, \zeta) + d_{\text{lim}}(\zeta, \pi_T^\mu \zeta) = 0.$$

Hence the quotient  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu / \sim_\mu$  is given by the collection of equivalence classes  $[\xi]$  for  $\xi \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ . Let us show that for any  $\xi, \zeta \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ , there holds  $W_\mu(\xi, \zeta) = W_\mu([\xi], [\zeta])$ : this would prove that  $W_\mu$  induces a distance on the quotient, and would provide the desired isometry. On the one hand, choosing an appropriate tangent plan, one gets that  $d_{\text{lim}}(\xi, \zeta) \leq W_\mu(\xi, \zeta)$  for any  $\xi, \zeta \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ . On the other hand, by [Gig08, Theorem 4.3], equality holds whenever  $\xi$  and  $\zeta$  are tangent. Hence for any  $\xi, \zeta \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ ,

$$W_\mu(\xi, \zeta) = d_{\text{lim}}(\xi, \zeta) = \inf_{\substack{\xi' \sim_\mu \xi \\ \zeta' \sim_\mu \zeta}} d_{\text{lim}}(\xi', \zeta') \leq \inf_{\substack{\xi' \sim_\mu \xi \\ \zeta' \sim_\mu \zeta}} W_\mu(\xi', \zeta') = W_\mu([\xi], [\zeta]) \leq W_\mu(\xi, \zeta).$$

Consequently, equality holds everywhere, which completes the proof.  $\square$

**Remark 5.2** (Quotient topology). *The distance  $[\xi], [\zeta] \mapsto W_\mu([\xi], [\zeta])$  metrizes the quotient topology on  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu / \sim_\mu$ , in which  $\mathcal{O} \subset \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu / \sim_\mu$  is open if  $\pi_T^{-1}(\mathcal{O}) \subset \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$  is open with respect to  $W_\mu$ . Assume on the one hand that  $\mathcal{O}$  is open for the quotient topology, and let  $[\xi] \in \mathcal{O}$  for some  $\xi \in \mathbf{Tan}_\mu$ . Then  $\xi \in (\pi_T^\mu)^{-1}(\mathcal{O})$ , which is open by definition. Let  $r > 0$  be the radius of a ball centered in  $\xi$  and contained in  $(\pi_T^\mu)^{-1}(\mathcal{O})$ . Then*

$$W_\mu([\xi], [\zeta]) \leq r \implies W_\mu(\xi, \zeta) \leq r \implies \zeta \in (\pi_T^\mu)^{-1}(\mathcal{O}) \implies [\zeta] \in \mathcal{O}.$$

*On the other hand, if  $\mathcal{O}$  is open for the topology induced by  $W_\mu$ , let  $\xi \in (\pi_T^\mu)^{-1}(\mathcal{O})$ . Denote  $r > 0$  a radius so that  $W_\mu([\xi], [\zeta]) \leq r$  implies  $[\zeta] \in \mathcal{O}$ . As  $W_\mu([\xi], [\zeta]) \leq W_\mu(\xi, \zeta)$ , there holds*

$$W_\mu(\xi, \zeta) \leq r \implies W_\mu([\xi], [\zeta]) \leq r \implies [\zeta] \in \mathcal{O} \implies \zeta \in (\pi_T^\mu)^{-1}(\mathcal{O}),$$

*and  $(\pi_T^\mu)^{-1}(\mathcal{O})$  is open in  $(\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu, W_\mu)$ . Hence the equivalence.*

## 6 Reconstruction from projections

In  $L_\mu^2(\mathbb{R}^d; \mathbb{T}\mathbb{R}^d)$ , it is trivial that vector fields are uniquely determined by their projections on tangent and solenoidal fields, since any  $f \in L_\mu^2$  can be reconstructed from its projections by sum. Curiously enough, this uniqueness does not stand in  $\mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ , and there might be several measure fields sharing the same projections on  $\mathbf{Tan}_\mu$  and  $\mathbf{Sol}_\mu$ . For instance, consider  $\mu$  to be the normalized Lebesgue measure on the circle  $\mathbb{S}^1 \subset \mathbb{R}^2$ . Let  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined as  $f(x, y) := (-y, x)$  and  $g(x, y) := (x, y)$ , and consider

$$\xi^0 := \frac{1}{2}((id, f + g)\#\mu) + \frac{1}{2}((id, -f - g)\#\mu), \quad \xi^1 := \frac{1}{2}((id, f - g)\#\mu) + \frac{1}{2}((id, -f + g)\#\mu).$$

Then  $\pi_T^\mu \xi^i = \frac{1}{2}(id, f)\#\mu + \frac{1}{2}(id, -f)\#\mu$  and  $\pi_S^\mu \xi^i = \frac{1}{2}(id, -g)\#\mu + \frac{1}{2}(id, g)\#\mu$  for  $i \in \{0, 1\}$ . In this example, the projections are arranged in such a way that the set  $\Gamma_{\mu, o}(\pi_T^\mu \xi^i, \pi_S^\mu \xi^i)$  of optimal transport plans for  $W_\mu$  contains more than one element. In the following result, we show that this situation is generic: one may build as many fields with prescribed projections  $\eta, \omega$  as there are ways to “sum”  $\eta$  and  $\omega$ , each sum being identified with an optimal transport plan in  $\Gamma_{\mu, o}(\eta, \omega)$ .

**Proposition 6.1** (Characterization by projections). *Let  $\xi \in \mathcal{P}_2(\mathbb{T}\mathbb{R}^d)_\mu$ ,  $\eta \in \mathbf{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$  and  $\omega \in \mathbf{Sol}_\mu \mathcal{P}_2(\mathbb{R}^d)$ . Then the following statements are equivalent:*

- a.  $\pi_T^\mu \xi = \eta$  and  $\pi_S^\mu \xi = \omega$ ,
- b. *there exists  $\alpha \in \Gamma_{\mu,o}(\eta, \omega)$  such that  $\xi = (\pi_x, \pi_v + \pi_w) \# \alpha$ .*

*Proof.* Assume that (a) holds, and denote again  $T : \mathbb{T}\mathbb{R}^d \rightarrow \mathbb{T}\mathbb{R}^d$  the measurable application such that  $T(x, v) = (x, T_2(x, v))$ ,  $\eta = (\pi_x, \pi_v + T_2) \# \xi$ , and  $\omega = (-T) \# \xi$ . On the one hand, there holds by definition of the metric scalar product that

$$W_\mu^2(\eta, \omega) = \|\eta\|_\mu^2 - 2 \langle \eta, \omega \rangle_\mu + \|\omega\|_\mu^2 = \|\eta\|_\mu^2 + \|\omega\|_\mu^2.$$

On the other hand, consider the plan  $\alpha := (\pi_x, \pi_v + T_2, -T_2) \# \xi \in \Gamma_\mu(\eta, \omega)$ . Then

$$\begin{aligned} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} |v - w|^2 d\alpha &= \int_{(x,v) \in \mathbb{T}\mathbb{R}^d} |(v + T_2(x, v)) - (-T_2(x, v))|^2 d\xi(x, v) \\ &= \|\eta\|_\mu^2 - 2 \int_{(x,v) \in \mathbb{T}\mathbb{R}^d} \langle v + T_2(x, v), -T_2(x, v) \rangle d\xi(x, v) + \|\omega\|_\mu^2, \end{aligned}$$

and the middle summand vanish by Remark 3.1. Consequently,  $\alpha$  is optimal between  $\eta$  and  $\omega$ , and as  $\xi = (\pi_x, \pi_v + \pi_w) \# \alpha$ , (b) holds.

Reciprocally, assume that (b) stands. To identify the tangent component of  $\xi$ , we use the same reasoning as in the proof of Lemma 5.1. For any  $h > 0$ , there holds

$$\exp_\mu(h \cdot \xi) = (\pi_x + h\pi_v) \# \xi = (\pi_x + h\pi_w + h\pi_v) \# \alpha = \exp_{\exp_\mu(h \cdot \omega)}(h \cdot \eta_h),$$

where  $\eta_h := (\pi_x + h\pi_w, \pi_v) \# \alpha$ . Let  $\beta \in \Gamma(\eta, \eta_h)$  realize the infimum in  $W_{\mu, \exp_\mu(h \cdot \omega)}(\eta, \eta_h)$ . Then  $(\pi_x + h\pi_v, \pi_y + h\pi_w) \# \beta \in \Gamma(\exp_\mu(h \cdot \xi), \exp_\mu(h \cdot \eta))$ , and replicating the computation of (15),

$$d_{\mathcal{W}}(\exp_\mu(h \cdot \xi), \exp_\mu(h \cdot \eta)) \leq d_{\mathcal{W}}(\exp_\mu(h \cdot \omega), \mu) + hW_{\exp_\mu(h \cdot \omega), \mu}(\eta_h, \eta).$$

As  $d_{\mathcal{W}, \mathbb{T}\mathbb{R}^d}(\eta_h, \eta) \leq d_{\mathcal{W}}(\exp_\mu(h \cdot \omega), \mu)$  goes to 0 when  $h$  does, using again the characterization of Lemma 4.1 and the estimate of Lemma 4.2, we get that  $d_{\mathcal{W}}(\exp_\mu(h \cdot \xi), \exp_\mu(h \cdot \eta)) = o(h)$ . Consequently,  $\pi_T^\mu \xi = \eta$ . Now, using the identity (13) with  $\zeta = \omega$  and  $\omega = \pi_S^\mu \xi$ , there holds

$$W_\mu^2(\omega, \pi_S^\mu \xi) = W_\mu^2(\xi, \omega) - W_\mu^2(\xi, \pi_S^\mu \xi) \leq \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} |(v + w) - w|^2 d\alpha - \|\eta\|_\mu^2 = 0.$$

This shows that  $\xi$  has the desired projections, and (a) holds.  $\square$

**Remark 6.1** (Pythagoras fails). *As a corollary of Proposition 6.1, the formula*

$$W_\mu^2(\xi, \zeta) = W_\mu^2(\pi_T^\mu \xi, \pi_T^\mu \zeta) + W_\mu^2(\pi_S^\mu \xi, \pi_S^\mu \zeta)$$

*does not hold. Indeed, there may be distinct  $\xi, \zeta$  sharing the same marginals, so that the left hand-side is positive while the right hand-side vanishes. However, the inequality  $\geq$  always holds: indeed, if  $T, S$  are the measurable applications given by Proposition 5.1 respectively associated to  $\xi$  and  $\zeta$ , then*

$$\begin{aligned} W_\mu^2(\xi, \zeta) &= \inf_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} |v - w|^2 d\alpha \\ &= \inf_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} |v + T - (w + S)|^2 + 2 \langle v + T_2 - (w + S_2), -T_2 - S_2 \rangle + |-T_2 - S_2|^2 d\alpha \\ &\geq W_\mu^2(\pi_T^\mu \xi, \pi_T^\mu \zeta) + W_\mu^2(\pi_S^\mu \xi, \pi_S^\mu \zeta) + 2 \inf_{\alpha \in \Gamma_\mu(\xi, \zeta)} \int_{(x,v,w) \in \mathbb{T}^2 \mathbb{R}^d} \langle v + T_2 - (w + S_2), -T_2 - S_2 \rangle d\alpha. \end{aligned}$$

*By Proposition 2.1 and Lemma 3.2, the measure field  $(\pi_x, \pi_v + T_2(\pi_x, \pi_v) - (\pi_w + S_2(\pi_x, \pi_w))) \# \alpha$  is tangent while  $(\pi_x, -T_2(\pi_x, \pi_v) - S_2(\pi_x, \pi_w)) \# \alpha$  is solenoidal, and the last term vanishes.*

**Acknowledgments** This research was funded in part by l'Agence Nationale de la Recherche (ANR), project ANR-22-CE40-0010. For the purpose of open access, the author has applied a CC-BY public copyright licence to any Author Accepted Manuscript (AAM) version arising from this submission. Data availability is not applicable to this article as no new data were created or analysed in this study.

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