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► **To cite this version:**

Long Li, Etienne Mémin, Bertrand Chapron. A generalized stochastic formulation of the Ekman-Stokes model with statistical analyses. 2024. hal-04672188

HAL Id: hal-04672188

<https://hal.science/hal-04672188v1>

Preprint submitted on 17 Aug 2024

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1 **A generalized stochastic formulation of the Ekman-Stokes model with**
2 **statistical analyses**

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9 ABSTRACT: In this study, a novel stochastic formulation for the upper ocean Ekman boundary
10 layer is derived from scaling the generalized stochastic Craik-Leibovich equations. This formu-
11 lation encodes the interactions between wind, waves, and currents through the introduction of
12 the uncertainty of unresolved motions, using established physical parameterizations. Numerical
13 investigations of the time-dependent stochastic Ekman-Stokes model, incorporating wave mixing
14 and stochastic transport effects, reveal Ekman velocities with an increased uncertainty, a higher
15 kinetic energy and a stronger occurrence of extreme events, compared to traditional solutions.
16 Stronger correlations between zonal and meridional components are obtained, with more skewed
17 distributions and extreme values, particularly near the surface. A sensitivity analysis highlights
18 the impact of transient winds and surface waves on statistical moments. Transient winds reduce
19 vertical shear, deepen circulation, and increase uncertainty. Smaller surface waves also lead to
20 higher ensemble energy and stronger correlations. Energy and transport magnitude peak when
21 mean waves and wind are aligned, decreasing with rotation and redistributing current velocity
22 statistics. These findings underscore the model's enhanced ability to help capture complex ocean
23 dynamics with improved uncertainty representation.

24 SIGNIFICANCE STATEMENT: To describe the upper ocean Ekman boundary layer, a novel
25 stochastic model is derived to capture the interplay between random motions induced by wind,
26 waves, and currents. Accounting for the uncertainty of unresolved fluctuations, resulting Ekman
27 velocities display increased variability, higher kinetic energy, and more frequent extreme events
28 than traditional models. Near the surface, stronger correlations and skewed distributions are also
29 revealed. Sensitivity analyses highlight impacts of transient winds and surface waves, which
30 deepen circulation and increase uncertainty. Aligning wind and waves naturally maximizes energy
31 and transport. This model can offer new means to describe upper ocean dynamics, providing better
32 insights to anticipate ocean vertical fluxes and enhancing predictive capabilities.

33 1. Introduction

34 In the seminal work of Ekman (1905), an exact solution of simplified Navier–Stokes equations
35 was obtained, fundamental to the theory of ocean circulation and attractive for theoretical analysis.
36 The upper ocean boundary layer flow was initially conceived as steady, linear, of uniform density
37 and viscosity, and driven solely by a surface wind stress. Owing to a balance between Coriolis and
38 turbulent drag forces, the explicit solution displays the gradual rotation and decay of ageostrophic
39 velocity with depth, presenting a spiral vertical structure (Vallis 2017). Several limitations in this
40 early model have been identified, e.g. the assumption of vertically uniform eddy viscosity in the
41 ocean (Large et al. 1994) and the significant modifications to near-surface currents induced by
42 surface gravity waves (Huang 1979; Jenkins 1986; Xu and Bowen 1994; McWilliams et al. 1997,
43 2012). Buoyancy effects can also play a crucial role in the boundary layer dynamics, particularly at
44 fronts (Price and Sundermeyer 1999; McWilliams et al. 2009; Gula et al. 2014; McWilliams et al.
45 2015). Although incorporating these additional effects complicates analytical solutions, advanced
46 mathematical tools (Lewis and Belcher 2004; Wenegrat and McPhaden 2016b,a; Higgins et al.
47 2020) and accurate numerical methods offer viable approximations.

48 In this study, a stochastic approach is further considered. Building upon earlier works (Bauer
49 et al. 2020; Mémin 2014; Resseguier et al. 2017), a stochastic approach can help statistically
50 describe the ocean surface Ekman boundary layer. The location uncertainty framework can indeed
51 consistently integrate the compounding effects of random winds, modulated and modulating surface
52 waves, and turbulent mixing processes. Hereafter, we omit stratification effects, a facet to address

53 in future research once other interactions are clarified. By appropriately scaling the generalized
54 stochastic Craik-Leibovich equation already outlined in Bauer et al. (2020), both steady and time-
55 dependent stochastic Ekman-Stokes models are derived. The stochastic framework fully encodes
56 diffusive and advective mixing effects of random fluctuations, e.g. surface winds and waves, in
57 the vertical direction, alongside classic terms involved in the Ekman-Stokes model (Wenegrat
58 and McPhaden 2016b,a; Higgins et al. 2020). Notably, the uncertainty representation can be
59 consistently constrained with established physical parameterizations (McWilliams et al. 1997;
60 McWilliams and Huckle 2006). Through large ensemble simulations, uncertainties in the Ekman
61 boundary layer are then quantified, stemming from various random contributions (wind, waves,
62 and turbulence), also accounting for their intermittency. Noisy fluctuations are specified to follow
63 simple models. Specifically, the quadratic variation of these noise terms, corresponding to the
64 process defined by the limit in probability of the square of the noise increments, matches the ideal
65 Stokes drift of monochromatic linear waves and the K-profile-parameterization (KPP) for vertical
66 mixing (Large et al. 1994). Yet, the resulting impacts of the noise processes are ultimately much
67 more complex, and the system’s response is not at all trivial. In this study, statistical responses
68 are compared to those of a traditional benchmark model, followed by sensitivity analysis of the
69 proposed random model to various wind and wave parameters.

70 The paper is structured as follows: Section 2 presents the proposed stochastic Ekman-Stokes
71 models with detailed derivations and describes the uncertainty parameterization methods. Section 3
72 discusses the numerical results, including statistical diagnosis and sensitivity analysis of the random
73 models. Finally, in Section 4, we draw conclusions and provide an outlook for future research
74 endeavors.

75 **2. Stochastic formulation**

76 A brief review of the generalized stochastic Craik-Leibovich equations first reassesses our pre-
77 vious work. The stochastic Ekman-Stokes models is then derived by appropriately scaling the
78 nonlinear equations. Subsequently, we describe the uncertainty representation induced by estab-
79 lished physical parameterizations.

80 *a. Review of generalized stochastic Craik-Leibovich momentum equations*

81 The *Location Uncertainty* framework (Mémin 2014) emerges from a decomposition of the
 82 Lagrangian fluid flow into a resolved time-smooth flow component and an unresolved highly
 83 oscillating noise term:

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t \mathbf{v}(\mathbf{X}_s, s) ds + \int_0^t \boldsymbol{\sigma}(\mathbf{X}_s, s) dB_s, \quad (2.1)$$

84 where \mathbf{X} represents the stochastic Lagrangian particle trajectory, $\mathbf{v} = (u, v, w)^T$ denotes the resolved
 85 three-dimensional (3D) velocity, B is a cylindrical Brownian motion (see Da Prato and Zabczyk
 86 2014, chap. 2) defined on an infinite-dimensional Hilbert space (function space), $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$
 87 is a random correlation process defined on a space of Hilbert-Schmidt operators (mapping two
 88 function spaces). More detailed mathematical descriptions on this noise term definition can be
 89 found in Debussche et al. (2023); Li et al. (2023a,b).

90 Several systems of stochastic partial differential equations (SPDEs), describing the evolution of
 91 key oceanic variables (such as momentum, temperature, salinity, buoyancy, and sea-surface height)
 92 transported along the stochastic flow described by Eq. (2.1) under various regimes, have been
 93 derived by Brecht et al. (2021); Li et al. (2023a); Mémin (2014); Resseguier et al. (2017); Tucciarone
 94 et al. (2024). These SPDEs are established through the application of stochastic calculus rules
 95 and adherence to fundamental physical conservation laws accompanied with classical geophysical
 96 approximations. For the Navier-Stokes equations, such stochastic approximations converge (in 2D
 97 and 3D) toward the deterministic equations as the noise vanishes (Debussche et al. 2023), providing
 98 strong consistency to this large scale representation in a similar way as grid convergence for large
 99 eddy simulations. Hereafter, we exclusively concentrate on the stochastic momentum equations
 100 governing incompressible fluid motions under the Boussinesq approximation, namely:

$$d\mathbf{v} + ((\mathbf{v} - \mathbf{v}_s) \cdot \nabla \mathbf{v} - \nabla \cdot (\mathbf{a} \nabla \mathbf{v}) + f \hat{\mathbf{z}} \times \mathbf{v}) dt + (\boldsymbol{\sigma} \cdot \nabla \mathbf{v} + f \hat{\mathbf{z}} \times \boldsymbol{\sigma}) dB_t = (b \hat{\mathbf{z}} - \nabla p) dt - \nabla dP_t, \quad (2.2a)$$

$$\mathbf{a} := \frac{1}{2} \boldsymbol{\sigma} \boldsymbol{\sigma}^T, \quad \mathbf{v}_s := \nabla \cdot \mathbf{a}, \quad (2.2b)$$

$$\nabla \cdot (\mathbf{v} - \mathbf{v}_s) = 0, \quad \nabla \cdot \boldsymbol{\sigma} = 0. \quad (2.2c)$$

103 Here, dv_i denotes the temporal variation of v_i at a fixed point, f is the Coriolis frequency,
 104 $\hat{\mathbf{z}} = (0, 0, 1)^T$ denotes the vertical unit vector, $b = -g\rho'/\rho_o$ is the buoyancy variable under Boussinesq
 105 approximation (where g is the gravity constant, ρ_o is the background water density, and ρ' is the
 106 density anomaly), and p represent the resolved dynamic pressure (rescaled by ρ_o).

107 The SPDEs (2.2a) encodes physically meaningful terms. Specifically, the term $\boldsymbol{\sigma} \cdot \nabla v_i dB_t$
 108 represents the random advective processes induced by unresolved fluid motions. The diffusion
 109 term, $\nabla \cdot (\mathbf{a} \nabla v_i)$, depicts the unresolved diffusive mixing effects associated with the random
 110 symmetric non-negative diffusion tensor $\mathbf{a} = \boldsymbol{\sigma} \boldsymbol{\sigma}^T / 2$, also referred to the variance tensor as it
 111 corresponds to the one-point-two-times covariance matrix tensor. The nonlinear term $(\mathbf{v} - \mathbf{v}_s) \cdot \nabla v_i$
 112 involves an effective advection velocity, which adjusts the resolved velocity \mathbf{v} by removing the
 113 influence of a statistical drift $\mathbf{v}_s = \nabla \cdot \mathbf{a}$ that captures the inhomogeneity of the unresolved random
 114 field. This turbophoresis term, referred to *Itô-Stokes drift*, has been interpreted as a generalization
 115 of the Stokes drift in Bauer et al. (2020). Both the diffusion and corrective advection terms are
 116 rigorously derived by applying the generalized Itô formula (see Bauer et al. 2020, Appendix B).

117 The Coriolis terms $f \hat{\mathbf{z}} \times (\mathbf{v} dt + \boldsymbol{\sigma} dB_t)$ appear from a change of coordinates (from inertial to
 118 rotating) for the stochastic flow (2.1). The pressure noise dP_t/dt (in a distribution sense) is an
 119 additional Lagrange multiplier enforcing the incompressibility of unresolved fluid motions. Note
 120 that the continuity equations (2.2c) are derived from mass conservation in the stochastic framework
 121 (Mémin 2014). These non-divergent constraints also ensures energy conservation for the derived
 122 random systems (Brecht et al. 2021; Li et al. 2023a).

123 To interpret the Craik-Leibovich momentum equations in the stochastic framework, Equation
 124 (2.2a) can be rewritten in an equivalent form outlining the contribution of the effective advection
 125 velocity, denoted by $\mathbf{v}^* := \mathbf{v} - \mathbf{v}_s$ hereafter. Applying a change of variable from \mathbf{v} to \mathbf{v}^* and assuming
 126 a quasi-stationary Itô-Stokes drift ($d\mathbf{v}_s \approx 0$), we have

$$\begin{aligned}
 d\mathbf{v}^* + (\mathbf{v}^* \cdot \nabla + f \hat{\mathbf{z}} \times) (\mathbf{v}^* + \mathbf{v}_s) dt + (\boldsymbol{\sigma} \cdot \nabla (\mathbf{v}^* + \mathbf{v}_s) + f \hat{\mathbf{z}} \times \boldsymbol{\sigma}) dB_t \\
 = \left(b \hat{\mathbf{z}} - \nabla p + \nabla \cdot (\mathbf{a} \nabla (\mathbf{v}^* + \mathbf{v}_s)) \right) dt - \nabla dP_t.
 \end{aligned} \tag{2.3}$$

127 Manipulating further the advection term $\mathbf{v}^* \cdot \nabla \mathbf{v}_s$ by a classical vector calculus identity, it comes
 128 the so-called *generalized stochastic Craik-Leibovich momentum equations*:

$$\begin{aligned}
 d\mathbf{v}^* + ((\mathbf{v}^* - \mathbf{v}_s) \cdot \nabla \mathbf{v}^* - \nabla \cdot (\mathbf{a} \nabla \mathbf{v}^*) + f \hat{\mathbf{z}} \times \mathbf{v}^* - b \hat{\mathbf{z}} + \nabla p) dt + (\boldsymbol{\sigma} \cdot \nabla \mathbf{v}^* + f \hat{\mathbf{z}} \times \boldsymbol{\sigma}) dB_t + \nabla dP_t \\
 = \underbrace{(-f \hat{\mathbf{z}} \times \mathbf{v}_s)}_{\text{Coriolis Stokes}} - \underbrace{\nabla (\mathbf{v}^* \cdot \mathbf{v}_s)}_{\text{pressure correction}} + \underbrace{\mathbf{v}_s \times (\nabla \times \mathbf{v}^*)}_{\text{Craik-Leibovich vortex force}} - \underbrace{(\nabla \times \mathbf{v}_s) \times \mathbf{v}^*}_{\text{It\hat{o}-Stokes force}} + \underbrace{\nabla \cdot (\mathbf{a} \nabla \mathbf{v}_s)}_{\text{It\hat{o}-Stokes diffusive mixing}} dt - \underbrace{\boldsymbol{\sigma} \cdot \nabla \mathbf{v}_s}_{\text{It\hat{o}-Stokes advective mixing}} dB_t. \quad (2.4)
 \end{aligned}$$

129 The first line mirrors the structure of the stochastic Boussinesq momentum equation (2.2a), with
 130 the only difference being the substitution of \mathbf{v}^* for \mathbf{v} . In contrast, the second line accentuates the
 131 supplementary contributions stemming from the It\hat{o}-Stokes drift acting on \mathbf{v}^* . The initial three
 132 terms share identical expressions with the Coriolis-Stokes force, Stokes-corrected pressure, and
 133 Craik-Leibovich vortex force present in the classical Craik-Leibovich momentum equation (Craik
 134 and Leibovich 1976; Leibovich 1980; McWilliams et al. 1997). However, in this formulation, the
 135 vortex force characterizes the statistical influence of the inhomogeneity carried by the diffusion
 136 tensor of the random field on the large-scale current. Therefore, this momentum equation can
 137 be viewed as a generalized stochastic representation of the Craik-Leibovich system, where the
 138 turbophoresis term supplants the Stokes drift associated with wave motion. This distinction prompts
 139 us to designate the term more broadly as the It\hat{o}-Stokes drift. It is indeed not anymore exclusively
 140 associated with waves motion but more generally with inhomogeneous unresolved small-scale
 141 fluctuations. Additionally, this stochastic formulation introduces another force, referred to herein
 142 as the *It\hat{o}-Stokes force*, which pertains to the interaction between the effective velocity and the
 143 vorticity of the It\hat{o}-Stokes drift. This force can be collectively considered with the Coriolis force
 144 to yield a corrective term. Moreover, coupling effects exist between the It\hat{o}-Stokes drift and the
 145 unresolved turbulent motions through both advective and diffusive processes, encoded by the final
 146 two terms. These terms capture surface wave mixing effects when the It\hat{o}-Stokes drift approximates
 147 a real Stokes drift.

148 *b. Nondimensional stochastic momentum equations and scaling*

149 The procedure is now outlined for deriving the stochastic Ekman-Stokes models by scaling the
 150 generalized stochastic Craik-Leibovich momentum equations (2.3). Specifically in this study, we

151 do not account for stratification effects within the boundary layer. Hence, buoyancy is assumed to
 152 remain constant (see Vallis 2017, chap. 5). Without loss of generality, we set $b = 0$ in the following.

153 The horizontal variables and Coriolis frequency are first scaled as $\mathbf{x} = (x, y)^T = L\widehat{\mathbf{x}}$, $\mathbf{u}^\star =$
 154 $(u^\star, v^\star)^T = U\widehat{\mathbf{u}}$, and $f = f_0\widehat{f}$. Here, the capital letters represent variable scales and hatted vari-
 155 ables denote nondimensional variables. Following Pedlosky (1990, chap. 4), we adopt a change
 156 of vertical coordinates to account for stretching in the boundary layer so that $z = H\widehat{z}$, $\widehat{\zeta} = \widehat{z}/\widehat{\delta}_e$,
 157 and $\delta_e = H\widehat{\delta}_e$. Here, $\widehat{z} \neq O(1)$, $\widehat{\zeta} = O(1)$ and δ_e denotes the Ekman boundary layer thickness.
 158 The scales of vertical velocity $w^\star = (UH\widehat{\delta}_e/L)\widehat{w}$ and dynamic pressure $p = f_0UL\widehat{p}$ are suggested
 159 respectively by the continuity equation (2.2c) and classical geostrophic balance.

160 The correlation operator is decomposed into horizontal and vertical components, $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_\mathbf{x}, \sigma_z)^T$
 161 with $\boldsymbol{\sigma}_\mathbf{x} = (\sigma_x, \sigma_y)^T$. The derived diffusion tensor can thus be decomposed into $\mathbf{a}_{\mathbf{xx}} = \frac{1}{2}\boldsymbol{\sigma}_\mathbf{x}\boldsymbol{\sigma}_\mathbf{x}^T$,
 162 $a_{zz} = \frac{1}{2}\sigma_z^2$, and $\mathbf{a}_{\mathbf{xz}} = \frac{1}{2}\boldsymbol{\sigma}_\mathbf{x}\sigma_z$. We propose scaling $\mathbf{a}_{\mathbf{xx}}$ and a_{zz} as horizontal and vertical eddy
 163 viscosity coefficients, as

$$\mathbf{a}_{\mathbf{xx}} = A_h\widehat{\mathbf{a}}_{\mathbf{xx}}, \quad a_{zz} = A_v\widehat{a}_{zz}, \quad (2.5a)$$

164 respectively. The scaling for the correlation operator and the cross component of the diffusion
 165 tensor follows:

$$\boldsymbol{\sigma}_\mathbf{x} = \sqrt{2}A_h^{1/2}\widehat{\boldsymbol{\sigma}}_\mathbf{x}, \quad \sigma_z = \sqrt{2}A_v^{1/2}\widehat{\sigma}_z, \quad \mathbf{a}_{\mathbf{xz}} = A_h^{1/2}A_v^{1/2}\widehat{\mathbf{a}}_{\mathbf{xz}}. \quad (2.5b)$$

166 We next recall the Rossby number (Ro), horizontal Ekman number (E_h), vertical Ekman number
 167 (E_v), and the aspect ratio (δ), which are defined by

$$\text{Ro} = \frac{U}{f_0L}, \quad E_h = \frac{A_h}{f_0L^2}, \quad E_v = \frac{A_v}{f_0H^2}, \quad \delta = \frac{H}{L}. \quad (2.6)$$

168 Using these dimensionless numbers, the Itô-Stokes drift \mathbf{v}_s normalized by the characteristic scales
 169 of \mathbf{v}^\star scales as

$$\widetilde{\mathbf{u}}_s := \frac{\mathbf{u}_s}{U} = \frac{E_h}{\text{Ro}}\widehat{\nabla} \cdot \widehat{\mathbf{a}}_{\mathbf{xx}} + \frac{E_h^{1/2}E_v^{1/2}}{\text{Ro}}\frac{1}{\widehat{\delta}_e}\partial_{\widehat{\zeta}}\widehat{\mathbf{a}}_{\mathbf{xz}}, \quad (2.7a)$$

$$\widetilde{w}_s := \frac{w_s}{W} = \frac{E_h^{1/2}}{\text{Ro}}E_v^{1/2}\widehat{\nabla} \cdot \widehat{\mathbf{a}}_{\mathbf{xz}} + \frac{1}{\text{Ro}}\frac{E_v}{\widehat{\delta}_e}\partial_{\widehat{\zeta}}\widehat{a}_{zz}. \quad (2.7b)$$

171 Note that $\nabla = (\partial_x, \partial_y)^T$ represents the horizontal gradient from here on.

172 The advection time scale $t = (L/U)\widehat{t}$ is imposed as the characteristic time scale. It is important
 173 to note that the time scale, \widehat{t} , is not necessarily of $O(1)$. Discussed later, two different time
 174 scales will indeed be used to distinctly capture a stationary and a non stationary stochastic Ekman
 175 model. The variance of a (cylindrical) Brownian motion also takes the scale of time, with
 176 $B_t = (L/U)^{1/2}\widehat{B}_t$. Moreover, the scale of unresolved pressure noise are suggested by the geostrophic
 177 balance $f\widehat{\mathbf{z}} \times \sigma d\widehat{B}_t \approx -\nabla dP_t$, hence $P_t = \sqrt{2}(E_h/\text{Ro})^{1/2}f_0L^2\widehat{P}_t$.

178 Substituting the above scalings in Eq. (2.3), the following nondimensional stochastic momentum
 179 equations is obtained:

$$\begin{aligned}
 & \text{Ro} \left(d\widehat{\mathbf{u}} + (\widehat{\mathbf{u}} \cdot \widehat{\nabla} + \widehat{w} \partial_{\widehat{\zeta}}) (\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s) d\widehat{t} \right) + \sqrt{2}\text{Ro}^{1/2} \left(E_h^{1/2} \widehat{\sigma}_{\mathbf{x}} \cdot \widehat{\nabla} + \frac{E_v^{1/2}}{\widehat{\delta}_e} \widehat{\sigma}_z \partial_{\widehat{\zeta}} \right) (\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s) d\widehat{B}_t \\
 & = - \left(\widehat{f}(\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s)^\perp + \widehat{\nabla} \widehat{p} \right) d\widehat{t} - \sqrt{2} \frac{E_h^{1/2}}{\text{Ro}^{1/2}} \left(\widehat{f} \widehat{\sigma}_{\mathbf{x}}^\perp d\widehat{B}_t + \widehat{\nabla} d\widehat{P}_t \right) + E_h \widehat{\nabla} \cdot \left(\widehat{\mathbf{a}}_{\mathbf{xx}} \widehat{\nabla} (\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s) \right) d\widehat{t} \\
 & \quad + \frac{E_v}{\widehat{\delta}_e^2} \partial_{\widehat{\zeta}} \left(\widehat{\mathbf{a}}_{zz} \partial_{\widehat{\zeta}} (\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s) \right) d\widehat{t} + E_h^{1/2} \frac{E_v^{1/2}}{\widehat{\delta}_e} \left(\widehat{\nabla} \cdot \left(\widehat{\mathbf{a}}_{\mathbf{xz}} \partial_{\widehat{\zeta}} (\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s) \right) + \partial_{\widehat{\zeta}} \left(\widehat{\mathbf{a}}_{\mathbf{xz}} \widehat{\nabla} (\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s) \right) \right) d\widehat{t}, \quad (2.8a)
 \end{aligned}$$

180

$$\begin{aligned}
 & \delta^2 \text{Ro} \left(d\widehat{w} + (\widehat{\mathbf{u}} \cdot \widehat{\nabla} + \widehat{w} \partial_{\widehat{\zeta}}) (\widehat{w} + \widehat{w}_s) d\widehat{t} \right) + \sqrt{2}\delta^2 \text{Ro}^{1/2} \left(E_h^{1/2} \widehat{\sigma}_{\mathbf{x}} \cdot \widehat{\nabla} + \frac{E_v^{1/2}}{\widehat{\delta}_e} \widehat{\sigma}_z \partial_{\widehat{\zeta}} \right) (\widehat{w} + \widehat{w}_s) d\widehat{B}_t \\
 & = - \frac{1}{\widehat{\delta}_e^2} \left(\partial_{\widehat{\zeta}} \widehat{p} d\widehat{t} + \sqrt{2} \frac{E_h^{1/2}}{\text{Ro}^{1/2}} \partial_{\widehat{\zeta}} d\widehat{P}_t \right) + \delta^2 \left(E_h \widehat{\nabla} \cdot \left(\widehat{\mathbf{a}}_{\mathbf{xx}} \widehat{\nabla} (\widehat{w} + \widehat{w}_s) \right) + \frac{E_v}{\widehat{\delta}_e^2} \partial_{\widehat{\zeta}} \left(\widehat{\mathbf{a}}_{zz} \partial_{\widehat{\zeta}} (\widehat{w} + \widehat{w}_s) \right) \right. \\
 & \quad \left. + E_h^{1/2} \frac{E_v^{1/2}}{\widehat{\delta}_e} \left(\widehat{\nabla} \cdot \left(\widehat{\mathbf{a}}_{\mathbf{xz}} \partial_{\widehat{\zeta}} (\widehat{w} + \widehat{w}_s) \right) + \partial_{\widehat{\zeta}} \left(\widehat{\mathbf{a}}_{\mathbf{xz}} \widehat{\nabla} (\widehat{w} + \widehat{w}_s) \right) \right) \right) d\widehat{t}, \quad (2.8b)
 \end{aligned}$$

181 where $\mathbf{u}^\perp = (-v, u)^T$. To further derive the equation of motions in the Ekman boundary layer, we
 182 make the following assumptions regarding the dimensionless numbers:

$$\text{Ro} \ll 1, \quad \delta \ll 1, \quad \widehat{\delta}_e \sim E_v^{1/2}, \quad E_h^{1/2} \sim \text{Ro}. \quad (2.9)$$

183 A small Rossby number mainly neglects the nonlinear advection terms, and a small aspect ratio
 184 ensures hydrostatic balance in the vertical. The third assumption sets the vertical friction to be as
 185 large as the Coriolis force in the boundary layer, even though the Ekman number remains small in
 186 general. The last assumption emerges from the ratio of horizontal friction to inertial acceleration
 187 $E_h/\text{Ro} = 1/\text{Re}$, where $\text{Re} = UL/A_h$ is the Reynolds number of the interior flow. In most cases of

188 geophysical interest the Reynolds number is quite large (Pedlosky 1990, chap. 4). In particular, we
 189 choose $1/\text{Re} \sim \text{Ro}$ here. Using the above assumptions, the previous nondimensional momentum
 190 equations reduce to

$$\begin{aligned} \text{Ro} \, d\widehat{\mathbf{u}} = & \left(-\widehat{f}(\widehat{\mathbf{u}} + \partial_{\zeta}\widehat{\mathbf{a}}_{\mathbf{x}z})^{\perp} - \widehat{\nabla}\widehat{p} + \partial_{\zeta}(\widehat{a}_{zz}\partial_{\zeta}(\widehat{\mathbf{u}} + \partial_{\zeta}\widehat{\mathbf{a}}_{\mathbf{x}z})) + O(\text{Ro}) \right) d\widehat{t} \\ & - \sqrt{2}\text{Ro}^{1/2} \left((\widehat{\sigma}_z\partial_{\zeta}(\widehat{\mathbf{u}} + \partial_{\zeta}\widehat{\mathbf{a}}_{\mathbf{x}z}) + \widehat{f}\widehat{\sigma}_{\mathbf{x}}^{\perp}) d\widehat{B}_t + \widehat{\nabla}d\widehat{P}_t \right) + O(\text{Ro})d\widehat{B}_t, \end{aligned} \quad (2.10a)$$

$$\delta^2 \text{Ro} \, d\widehat{w} = \left(-\frac{1}{\widehat{\delta}_e^2} \partial_{\zeta}\widehat{p} + O(\delta^2) \right) d\widehat{t} - \sqrt{2} \frac{\text{Ro}^{1/2}}{\widehat{\delta}_e^2} \partial_{\zeta}d\widehat{P}_t + O(\delta^2)d\widehat{B}_t. \quad (2.10b)$$

191
 192 In the following sections, a steady model with random coefficients is first derived and then a
 193 time-dependent random model, according to two different time scales.

194 *c. Steady Ekman-Stokes model with random coefficients*

195 To derive a steady solution, an intermediate time scale $\widehat{t} = O(1)$ is considered. The temporal
 196 variation of horizontal momentum is neglected, Eq. (2.10a) can thus be split into a prognostic
 197 equation of $O(1)$ in terms of “ $d\widehat{t}$ ” and a diagnostic equation of $O(\text{Ro}^{1/2})$ in terms of “ $d\widehat{B}_t$ ”. This
 198 splitting, based on physical scaling arguments, is also ensured from a mathematical point of view,
 199 by the canonical decomposition of semi-martingale (Le Gall 2016, chap. 4). Similarly, the vertical
 200 momentum (2.10b) reduces to two hydrostatic balances for the resolved and unresolved scales. It
 201 comes:

$$\widehat{f}(\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s)^{\perp} = -\widehat{\nabla}\widehat{p} + \partial_{\zeta}(\widehat{a}_{zz}\partial_{\zeta}(\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s)), \quad \widehat{\mathbf{u}}_s = \partial_{\zeta}\widehat{\mathbf{a}}_{\mathbf{x}z}, \quad (2.11a)$$

$$(\widehat{f}\widehat{\sigma}_{\mathbf{x}}^{\perp} - \widehat{\sigma}_z\partial_{\zeta}(\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s))d\widehat{B}_t = -\widehat{\nabla}d\widehat{P}_t, \quad (2.11b)$$

$$\partial_{\zeta}\widehat{p} = 0, \quad \partial_{\zeta}d\widehat{P}_t = 0. \quad (2.11c)$$

204 Following Vallis (2017, chap. 5), considering \widehat{p} independent of z facilitates the separation between
 205 pressure-driven interior geostrophic motions and boundary layer ageostrophic motions. Let $\widehat{\mathbf{u}} = \widehat{\mathbf{u}}_g +$
 206 $\widehat{\mathbf{u}}_e$ and $\widehat{p} = \widehat{p}_g + \widehat{p}_e$, where the Ekman layer corrections, denoted with a subscript e , are negligible
 207 away from the boundary layer. Combining with the hydrostasy, we have $\widehat{p}_e = 0$ everywhere, hence
 208 there is no boundary layer in the pressure field. Restoring the dimensions, the ageostrophic velocity

209 in the Ekman layer thus satisfies

$$f(\mathbf{u}_e + \mathbf{u}_s)^\perp = \partial_z(a_{zz}\partial_z(\mathbf{u}_e + \mathbf{u}_s)), \quad \mathbf{u}_s = \partial_z \mathbf{a}_{xz}. \quad (2.12)$$

210 The resulting horizontal Itô-Stokes drift in this Ekman boundary layer is given by the vertical
 211 derivative of the co-variation process (\mathbf{a}_{xz}) between the horizontal and vertical components of the
 212 unresolved motions. The previous vectorial equations can be rewritten as a scalar equation for the
 213 complex velocity $\mathbf{u} = u + iv$. Together with classical boundary conditions for the momentum flux,
 214 the steady Ekman-Stokes model reads:

$$\partial_z(a_{zz}\partial_z \mathbf{u}_e) - i f \mathbf{u}_e = i f \mathbf{u}_s - \partial_z(a_{zz}\partial_z \mathbf{u}_s), \quad (2.13a)$$

$$\rho_o a_{zz} \partial_z \mathbf{u}_e|_{z=0} = \boldsymbol{\tau}_w, \quad \rho_o a_{zz} \partial_z \mathbf{u}_e|_{z=-H} = 0, \quad (2.13b)$$

216 where $\boldsymbol{\tau}_w$ denotes the surface wind stress. In the general case, this elliptic partial differential
 217 equation (PDE) includes random coefficients such as the diffusion tensor \mathbf{a} , the Itô-Stokes drift
 218 \mathbf{u}_s , and the wind stress $\boldsymbol{\tau}_w$. It is important to note that the derived formulation (2.13) incorporates
 219 the vertical diffusive mixing effect of the Itô-Stokes drift, and represents a generalization of the
 220 classical steady Ekman model with Coriolis-Stokes force.

221 Following the method proposed by Wenegrat and McPhaden (2016b), an approximated analytical
 222 solution can be proposed for (2.13). To that purpose, we first vertically differentiate (2.13a) and
 223 then multiply by $\rho_o a_{zz}$ to form an equation for the Ekman current stress $\boldsymbol{\tau}_e = \rho_o a_{zz} \partial_z \mathbf{u}_e$, which
 224 reads:

$$(a_{zz} \partial_{zz}^2 - i f) \boldsymbol{\tau}_e = -(a_{zz} \partial_{zz}^2 - i f) \boldsymbol{\tau}_s, \quad (2.14a)$$

$$\boldsymbol{\tau}_e(0) = \boldsymbol{\tau}_w, \quad \boldsymbol{\tau}_e(-H) = 0. \quad (2.14b)$$

226 Similarly, $\boldsymbol{\tau}_s = \rho_o a_{zz} \partial_z \mathbf{u}_s$ represents the derived Itô-Stokes stress. Wenegrat and McPhaden (2016b)
 227 propose solving their generalized Ekman model, which shares the same left-hand side differential
 228 operator with Eq. (2.14a) but with different right-hand side forces, by initially approximating a
 229 solution to the homogeneous equation using the Wentzel-Kramer-Brillouin (WKB) method. Then,
 230 the authors solve for the inhomogeneous solution using the method of variation of parameters.

231 Precisely following their detailed derivation to solve our boundary value problem (2.14), the full
 232 solution of Ekman stress is given by

$$\tau_e(z) = \tau_w \left(\frac{a_{zz}(z)}{a_{zz}(0)} \right)^{1/4} \frac{\sinh(\theta(z))}{\sinh(\theta(0))} + \int_{-H}^0 G(z, \zeta) \left((if a_{zz}^{-1} - \partial_{zz}^2) \tau_s \right) (\zeta) d\zeta, \quad (2.15a)$$

$$\theta(z) = \sqrt{if} \int_{-H}^z a_{zz}^{-1/2}(\zeta) d\zeta. \quad (2.15b)$$

234 Here, G denotes a symmetric Green function defined by

$$G(z, \zeta) = \frac{a_{zz}^{1/4}(z) a_{zz}^{1/4}(\zeta)}{\sqrt{if} \sinh(\theta(0))} g(z, \zeta), \quad g(z, \zeta) = \begin{cases} \sinh(\theta(z)) \sinh(\theta(\zeta) - \theta(0)) & \text{if } \zeta > z \\ \sinh(\theta(\zeta)) \sinh(\theta(z) - \theta(0)) & \text{if } \zeta < z \end{cases}. \quad (2.15c)$$

235 The accuracy of the WKB approximate solution has been further discussed in Wenegrat and
 236 McPhaden (2016b), to generally show small errors for small vertical Ekman number (E_V). Once
 237 this Ekman stress is found, the Ekman current solution satisfying (2.13) can be recovered by

$$\mathbf{u}_e = -\frac{i}{\rho_0 f} \partial_z (\tau_e + \tau_s) - \mathbf{u}_s. \quad (2.16)$$

238 Integrating this relation vertically, a modified relation for the Ekman transport, $\mathbf{T}_e = \int_{-H}^0 \mathbf{u}_e(z) dz$
 239 reads:

$$\mathbf{T}_e = -\frac{i}{\rho_0 f} (\tau_w + \tau_s^0) - \mathbf{T}_s. \quad (2.17)$$

240 It shows that the Ekman transport \mathbf{T}_e is modified by both the surface Itô-Stokes stress $\tau_s^0 := \tau_s(0)$
 241 and the Itô-Stokes transport $\mathbf{T}_s = \int_{-H}^0 \mathbf{u}_s(z) dz$. Nevertheless, the Ekman-Stokes transport $\mathbf{T}_e + \mathbf{T}_s$ is
 242 rotated 90° from the left of the effective surface stress $\tau_w + \tau_s^0$, hence satisfies the classical integral
 243 relation. Moreover, the vertical velocity induced by the Ekman layer, usually referred to as the
 244 *Ekman pumping*, can be derived by integrating the continuity equation (2.2c) for the ageostrophic
 245 component, namely

$$w_e(-H) = \frac{1}{\rho_0 f} \nabla \times (\tau_w + \tau_s^0) - \nabla \cdot \mathbf{T}_s. \quad (2.18)$$

246 Accordingly, the Ekman pumping is now modified by both the curl of the surface Itô-Stokes
 247 stress and the horizontal divergence of the Itô-Stokes transport. Through the Itô-Stokes drift, it

248 incorporates the statistical effect of the inhomogeneity of the small-scale velocity component. It
 249 absorbs wavy motions, but also other physical processes affecting small-scale velocity fluctuations,
 250 as well as the effect of its vertical mixing. This relation can be beneficial for numerical ocean
 251 models at coarse resolution, providing a means to parameterize the Ekman-Stokes layer without
 252 explicitly solving the Ekman current (2.14) due to limited vertical resolution.

253 *d. Time-dependent stochastic Ekman-Stokes model*

254 In this case, a fast time scale is considered by applying a change of time coordinates (Crowe
 255 and Taylor 2018), such that $\widehat{\tau} = \widehat{t}/\text{Ro}$, and hence $\widehat{B}_\tau = \widehat{B}_t/\text{Ro}^{1/2}$. Therefore, equations (2.10a) and
 256 (2.10b) boils down to

$$\begin{aligned} d\widehat{\mathbf{u}} = & \left(-\widehat{f}(\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s)^\perp - \widehat{\nabla}\widehat{p} + \partial_{\widehat{z}}(\widehat{a}_{zz}\partial_{\widehat{z}}(\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s)) \right) d\widehat{\tau} \\ & - \sqrt{2} \left(\widehat{f}\widehat{\boldsymbol{\sigma}}_x^\perp + \widehat{\sigma}_z\partial_{\widehat{z}}(\widehat{\mathbf{u}} + \widehat{\mathbf{u}}_s) \right) d\widehat{B}_\tau - \sqrt{2}\widehat{\nabla}d\widehat{P}_\tau, \end{aligned} \quad (2.19a)$$

$$\partial_{\widehat{z}}\widehat{p} = 0, \quad \partial_{\widehat{z}}d\widehat{P}_\tau = 0. \quad (2.19b)$$

258 Unlike the steady case, the contributions of the “ $d\widehat{\tau}$ ” and “ $d\widehat{B}_t$ ” terms on the momentum evolution
 259 can no longer be separated. Both terms have exactly the same scales. Again separating the
 260 pressure-driven geostrophic motions, the wind-driven ageostrophic motions in the Ekman layer
 261 satisfy the following dimensional SPDE under complex notations:

$$d\mathbf{u}_e = \left(-if(\mathbf{u}_e + \mathbf{u}_s) + \partial_z(a_{zz}\partial_z(\mathbf{u}_e + \mathbf{u}_s)) \right) dt - (if\boldsymbol{\sigma}_x + \sigma_z\partial_z(\mathbf{u}_e + \mathbf{u}_s)) dB_t, \quad (2.20a)$$

$$\rho_o a_{zz}\partial_z\mathbf{u}_e|_{z=0} = \boldsymbol{\tau}_w, \quad \rho_o a_{zz}\partial_z\mathbf{u}_e|_{z=-H} = 0. \quad (2.20b)$$

263 This stochastic formulation extends the classical time-dependent Ekman-Stokes model
 264 (McWilliams et al. 1997), incorporating the vertical unresolved advective process of Ekman cur-
 265 rent, the random unresolved Coriolis force, and both the diffusive and advective vertical mixing
 266 effects of the Itô-Stokes drift.

267 Finding analytical expression of strong (pathwise) solutions to a SPDE with multiplicative
 268 noises is extremely challenging. A general representation of solutions can be used to study the
 269 well-posedness of the proposed SPDE. For instance, the notion of mild solutions (see Da Prato and

270 Zabczyk 2014, chap. 6) can be defined as a temporal convolution of nonlinear/force and noise terms
 271 by an analytical semigroup generated by the linear operator of the SPDE. Additionally, a general
 272 representation for the statistical moments of a solution to the SPDE can be formulated based on
 273 Wiener chaos expansion (Mikulevicius and Rozovskii 1998, 2004). However, analytically solving
 274 the expansion coefficients corresponding to the solutions of a very high-dimensional system of
 275 deterministic PDEs remains difficult. Instead, in Section 3, the statistical moments for solutions
 276 of the SPDE (2.20), incorporating other random parameters are investigated through numerical
 277 simulations, using the Monte Carlo method.

278 First, let us exemplify important integral properties of the conditional mean solution. Assuming
 279 the random correlation operator σ and the cylindrical Brownian motion B are independent, and
 280 taking the expectation of Eq. (2.20a) with respect to the conditional probability distribution of σ ,
 281 we deduce the following PDE for $\mathbb{E}^\sigma[\mathbf{u}_e] := \mathbb{E}[\mathbf{u}_e|\sigma]$:

$$\partial_t \mathbb{E}^\sigma[\mathbf{u}_e] = -if(\mathbb{E}^\sigma[\mathbf{u}_e] + \mathbf{u}_s) + \partial_z(a_{zz}\partial_z(\mathbb{E}^\sigma[\mathbf{u}_e] + \mathbf{u}_s)). \quad (2.21)$$

282 This resulting PDE remains random, with $\mathbb{E}^\sigma[\mathbf{u}]$ a function of σ which is random. Integrating
 283 vertically, it comes an initial value problem with random parameters for the conditional mean
 284 Ekman transport, $\mathbb{E}^\sigma[\mathbf{T}_e](t) = \int_0^{-H} \mathbb{E}^\sigma[\mathbf{u}_e](z, t) dz$, which reads

$$\frac{d}{dt} \mathbb{E}^\sigma[\mathbf{T}_e] = -if \mathbb{E}^\sigma[\mathbf{T}_e] + \frac{1}{\rho_o} (\tau_w + \tau_s^0) - if \mathbf{T}_s. \quad (2.22a)$$

285 Its solution is given by

$$\mathbb{E}^\sigma[\mathbf{T}_e](t) = e^{-ift} \mathbb{E}^\sigma[\mathbf{T}_e](0) + \frac{1}{\rho_o} \int_0^t e^{-if(t-r)} (\tau_w + \tau_s^0)(r) dr - \mathbf{T}_s (1 - e^{-ift}). \quad (2.22b)$$

286 We recall that the Itô-Stokes drift \mathbf{u}_s is assumed to be quasi-stationary (hence \mathbf{T}_s is also), whereas
 287 the wind stress τ_w and the vertical diffusion coefficient a_{zz} are possibly time-dependent (and
 288 consequently $\tau_s^0 = a_{zz}\partial_z \mathbf{u}_s|_{z=0}$ as well). Taking the divergence of the previous equation, we obtain

289 the following solution for the conditional mean Ekman pumping at the lower boundary:

$$\begin{aligned} \mathbb{E}^\sigma [w_e](t) = & \cos(ft)\mathbb{E}^\sigma [w_e](0) + (\cos(ft) + 1)\nabla \cdot \mathbf{T}_s + \frac{1}{\rho_o} \int_0^t \sin(f(t-r))\nabla \times (\boldsymbol{\tau}_w + \boldsymbol{\tau}_s^0)(r) dr \\ & + \cos(ft)\nabla \times (\mathbb{E}^\sigma [\mathbf{T}_e](0) + \mathbf{T}_s) + \frac{1}{\rho_o} \int_0^t \sin(f(t-r))\nabla \cdot (\boldsymbol{\tau}_w + \boldsymbol{\tau}_s^0)(r) dr. \end{aligned} \quad (2.23)$$

290 The Ekman pumping resulting from the unsteady Ekman-Stokes boundary layer depends on both
 291 the divergence and curl components of the initial Ekman transport, the steady Itô-Stokes transport,
 292 and the time-dependent effective surface stress. Note again that the additional terms compared to
 293 the classical case are related to the surface Itô-Stokes stress $\boldsymbol{\tau}_s^0$.

294 *e. Consistent physical parameterization for uncertainty representation*

295 In most of our previous works (Bauer et al. 2020; Resseguier et al. 2021; Brecht et al. 2021; Li
 296 et al. 2023a; Tucciarone et al. 2024), the noise was parameterized through a spectral decomposition
 297 of the correlation operator $\boldsymbol{\sigma}$. Stationary as well as time-evolving representations, performed
 298 with proper orthogonal decomposition, dynamic mode decomposition, wavelet basis, or auto-
 299 similarity assumptions, have been used for the definition of data-driven or model-based noises.
 300 The corresponding diffusion tensor \boldsymbol{a} and the resulting Itô-Stokes drift \mathbf{v}_s , defined in (2.2b),
 301 are then directly obtained from the noise basis function definition. In the present work, an
 302 opposite approach is somehow considered: given specific physical parameterizations for the vertical
 303 diffusion coefficient a_{zz} and the horizontal Stokes drift \mathbf{u}_s , objectives are to derive the corresponding
 304 vertical and horizontal components of the unresolved random flow component $\boldsymbol{\sigma}dB_t$. Besides, the
 305 randomness of the correlation operator $\boldsymbol{\sigma}$, which acts as an additional random source, is induced
 306 by a parameterized wind process and a steady distribution of the surface wave direction. For the
 307 sake of simplicity and without loss of generality, we present the formulation for one-dimensional
 308 boundary layer models in the following sections.

309 1) RANDOM WIND AND WIND STRESS

310 We adopt here the random wind parameterization proposed by McWilliams and Huckle (2006).
 311 The total wind \mathbf{u}_a is decomposed into its time average $\overline{\mathbf{u}_a}$ and fluctuations \mathbf{u}'_a . Each of the fluctuating

312 wind components is modeled as an independent Ornstein-Uhlenbeck (OU) process,

$$d\mathbf{u}'_a = -\frac{1}{T_a}\mathbf{u}'_a dt + \sqrt{\frac{2}{T_a}}\Sigma_a d\mathbf{W}_t, \quad (2.24)$$

313 where T_a denotes the memory time, Σ_a stands for the standard deviation of \mathbf{u}'_a , both conditioning the
 314 transient behavior of the OU process, and $\mathbf{W}_t = W_t^x + iW_t^y$ with W_t^x and W_t^y being two independent
 315 standard Brownian motions. These two Brownian motions are assumed mutually independent from
 316 the cylindrical Brownian motion involved in the definition of the unresolved current component
 317 σdB_t , as they are attached to processes originating from different physical medias (atmosphere
 318 and ocean). This independence also ensures that the conditional mean Ekman transport solution
 319 discussed in the previous section remains valid.

320 The ocean surface wind stress τ_w follows the classical bulk formula, quadratic with respect to
 321 the total wind \mathbf{u}_a :

$$\tau_w = C_D \rho_a |\mathbf{u}_a| \mathbf{u}_a = \rho_o u_*^2 e^{i\theta_*}, \quad (2.25)$$

322 where C_D is the air-sea drag coefficient, ρ_a is the air density, u_* is the friction velocity, and θ_*
 323 denotes the angular direction of the wind stress. It is important to note that the resulting parameters
 324 τ_w , u_* , and θ_* are random, leading to additional randomness in other parameters as discussed
 325 subsequently.

326 2) VERTICAL DIFFUSION COEFFICIENT

327 For the vertical diffusion, the well-established *K-Profile Parameterization* (KPP) model (Large
 328 et al. 1994; McWilliams and Huckle 2006) is considered to represent the vertical eddy viscosity
 329 a_{zz} within the turbulent surface boundary layer. This parameterization is nonlocal and adheres the
 330 predictions of Monin-Obukhov similarity theory along with other extensions (Fox-Kemper et al.
 331 2022). It is formulated as follows:

$$a_{zz}(z) = c_1 u_* h G(\zeta), \quad \zeta = -z/h, \quad h = c_2 \frac{u_*}{f}, \quad (2.26a)$$

332 where c_1 and c_2 are constants that determine the amplitude and shear of a_{zz} , h is the boundary
 333 layer depth, and G is a smooth function of the normalized depth ζ . We adopt a specific version of

334 G proposed by McWilliams and Huckle (2006), which includes an additional regularization term
 335 very near the ocean surface:

$$G(\zeta) = \zeta(1 - \zeta)^2 + H(\zeta_0 - \zeta) \frac{(\zeta - \zeta_0)^2}{2\zeta_0}, \quad (2.26b)$$

336 where H denotes the Heaviside step function. This KPP model is illustrated in Fig. 1 (a). Analyzed
 337 by McWilliams and Huckle (2006) in physical terms, the final regularization term represents
 338 an extra mixing phenomenon occurring within the oceanic boundary layer, possibly taking into
 339 account surface gravity wave breaking and mixing confined within a shallow layer with a thickness
 340 of $\zeta_0 h$ (where ζ_0 is assumed to be sufficiently small). Since u_* is random, both the diffusion
 341 coefficient a_{zz} and the boundary layer depth h are also random. Note that more complex schemes
 342 based on second-moment closure (Mellor and Yamada 1982; Harcourt 2013, 2015) could be further
 343 investigated following a similar construction.

344 3) STOKES DRIFT AND WAVE STRESS

345 Consider a steady, monochromatic, deep-water wave with surface elevation, to leading order in
 346 wave steepness, expressed as $\eta = \alpha \cos(kx - \omega t)$, where α is the wave amplitude, k is the horizontal
 347 wavenumber, $\omega = (gk)^{1/2}$ is the angular frequency satisfying the deep-water dispersion relation.
 348 The corresponding horizontal components of the Stokes drift are approximately given by (Phillips
 349 1977, chap. 4)

$$\mathbf{u}_s(z) = U_0 e^{2kz} e^{i\theta_s}, \quad (2.27a)$$

350 where $U_0 = \omega k \alpha^2$ represents the Stokes drift magnitude, and θ_s is the wave propagation direction.
 351 This Stokes drift velocity is nonlinear with respect to the wave amplitude and decays exponentially
 352 with depth from the ocean surface, Fig. 1 (b).

353 In reality, surface gravity waves exhibit a broad-band spectrum, leading to a more complex
 354 vertical profile for the Stokes drift (Huang 1971; Jenkins 1989). The randomness inherent in the
 355 KPP model results in the derived wave stress amplitude $\tau_s = a_{zz} \partial_z \mathbf{u}_s$ being equally random. To
 356 capture the uncertainty in the wave propagation direction, we parameterize it using a Gaussian
 357 distribution:

$$\theta_s \sim \mathcal{N}(\Theta_s, \Sigma_s^2), \quad (2.27b)$$

358 where Θ_s and Σ_s denote its mean and standard deviation, respectively. Note that the wave stress τ_s
 359 is consequently random but non-Gaussian.

360 4) UNRESOLVED RANDOM FLOW

361 Given a diffusion coefficient a_{zz} and a Stokes drift \mathbf{u}_s , the vertical and horizontal components of
 362 the unresolved noise flow are specified by the following projection formulation:

$$\tilde{\sigma}_z dB_t = \sqrt{2} \sum_n \langle a_{zz}^{1/2}, e_n \rangle e_n d\beta_n, \quad (2.28a)$$

$$\tilde{\sigma}_x dB_t = \sqrt{2} \sum_n \left(\langle a_{zz}^{-1/2} U_s, e_n \rangle + i \langle a_{zz}^{-1/2} V_s, e_n \rangle \right) e_n d\beta_n, \quad (2.28b)$$

$$\mathbf{U}_s = \int_{-H}^z \mathbf{u}_s(\zeta) d\zeta = U_s + iV_s, \quad (2.28c)$$

365 where $\{e_n\}$ is a set of orthogonal basis functions of the real-valued Hilbert space $L^2([-H, 0], \mathbb{R})$
 366 equipped with the inner product $\langle f, g \rangle = \int_{-H}^0 f(z)g(z) dz$, $\{\beta_n\}$ is a set of independent standard
 367 Brownian motions, and \mathbf{U}_s denotes antiderivative of Stokes drift. This latter reduces to $\mathbf{u}_s/(2k)$
 368 for the Stokes drift velocity corresponding to the monochromatic deep-water wave as defined in
 369 (2.27a). Better approximations of \mathbf{U}_s could be further explored for more accurate expression of \mathbf{u}_s .
 370 Note that Eq. (2.28b) is only defined within the support of the function a_{zz} , and we simply impose
 371 zero horizontal noise $\sigma_x dB_t$ outside the support.

372 The diffusion processes corresponding to the previous noise formulation are then given by

$$\tilde{a}_{zz} = \sum_n \langle a_{zz}^{1/2}, e_n \rangle^2 e_n^2, \quad \tilde{a}_{\mathbf{x}z} = \sum_n \langle a_{zz}^{1/2}, e_n \rangle \left(\langle a_{zz}^{-1/2} U_s, e_n \rangle + i \langle a_{zz}^{-1/2} V_s, e_n \rangle \right) e_n^2. \quad (2.29)$$

373 Parseval's theorem shows that the reconstructed diffusion coefficients are globally identified with
 374 the vertical viscosity and the antiderivative of Stokes drift:

$$\int_{-H}^0 \tilde{a}_{zz}(z) dz = \int_{-H}^0 a_{zz}(z) dz, \quad \int_{-H}^0 \tilde{a}_{\mathbf{x}z}(z) dz = \int_{-H}^0 \mathbf{U}_s(z) dz. \quad (2.30a)$$

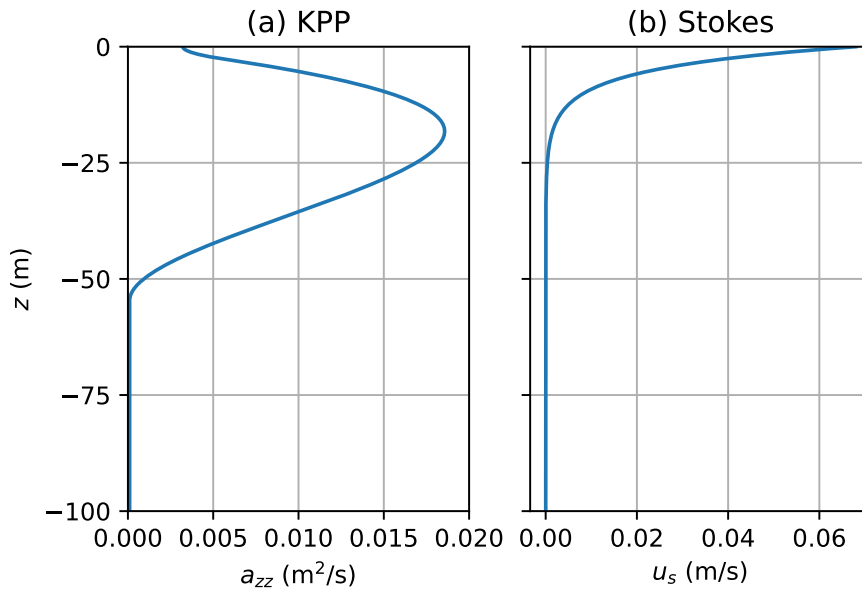
375 With the assumption that the basis functions $\{e_n\}$ are localized, meaning that each of them is
 376 significant only in a small localized region of the domain with negligible overlapping support,

377 Parseval's theorem is valid almost pointwise, and we obtain the approximation:

$$\tilde{a}_{zz}(z) \approx a_{zz}(z), \quad \tilde{\mathbf{a}}_{xz}(z) \approx \mathbf{U}_s(z). \quad (2.30b)$$

378 Consequently, the resulting Itô-Stokes drift $\tilde{\mathbf{u}}_s = \partial_z \tilde{\mathbf{a}}_{xz}$ approximates the given Stokes drift \mathbf{u}_s .

379 Although the noise is specified through simple models of diffusion (KPP) and Itô-Stokes statistical
 380 drift (Stokes drift of monochromatic waves), its normal distribution is not trivial to specify. It
 381 depends nonlinearly on random parameters. Its response to small variations of the parameters is
 382 not straightforward to infer. Furthermore, the resulting noise acts as both an additive noise and an
 383 advection process (transport noise), which is non-Gaussian.



384 FIG. 1. Illustration depicting the K-Profile Parameterization (KPP) viscosity coefficient alongside a zonal
 385 Stokes drift profile.

386 3. Statistical analyses

387 In this section, statistical properties of the proposed time-dependent stochastic Ekman-Stokes
 388 model (2.20) are investigated. Large-ensemble simulations are performed using a Monte Carlo
 389 method. The numerical schemes employed for simulating the SPDE, as well as for the Markovian
 390 wind stress process, are detailed in Appendix A.

391 The proposed model is first compared to a traditional benchmark model that does not account
 392 for stochastic transports and wave-mixing effects. Subsequently, we conduct comparative studies
 393 involving various wind and wave parameters.

394 *a. Comparison with a benchmark model*

395 The ensemble statistics derived from the proposed parameterized SPDE (2.20), referred to as
 396 *p-SPDE*, are first compared with those of a benchmark model (McWilliams et al. 1997) solely
 397 considering the Coriolis-Stokes force, driven by the parameterized PDE, designated as *p-PDE*:

$$\partial_t \mathbf{u}_e + if(\mathbf{u}_e + \mathbf{u}_s) = \partial_z(a_{zz}\partial_z \mathbf{u}_e). \quad (3.1)$$

398 This comparison is conducted under identical initial and boundary conditions. The common
 399 parameters between both simulations are listed in Table 1. Specifically, we assume the mean
 400 direction for both wind and wave to be zonal. It is noteworthy that the selected surface wave
 401 amplitude $\alpha = 0.8$ m and wavelength $\lambda = 60$ m (or wavenumber $k = 2\pi/\lambda \approx 1.05$ m⁻¹) proposed
 402 by McWilliams et al. (1997) imply a Stokes drift magnitude $U_0 \approx 0.068$ m s⁻¹ and a Stokes layer
 403 depth $h_s = 1/(2k) \approx 4.775$ m. Additionally, it is pertinent to mention that the transient wind level
 404 in this scenario is set to be equal to the mean wind level ($\Sigma_a = \bar{u}_a = 5$ m s⁻¹) to emphasize the
 405 rectification of the Ekman layer (McWilliams and Huckle 2006).

406 Under these parameters, the probability density functions (PDFs) for the surface forcings (wind,
 407 wind stress, and derived wave stress in the p-SPDE), after reaching the statistically stationary
 408 states, are shown Fig. 2. These bivariate PDFs are estimated from the ensemble vector fields by a
 409 non-parametric Kernel Density Estimation (KDE) method using Gaussian kernels. As anticipated,
 410 Fig.2 (a) illustrates the Gaussian nature of the wind vector \mathbf{u}_a with uncorrelated components,
 411 implying independence in the Gaussian case. However, the non-linearity of the bulk formula
 412 (2.25) results in a non-Gaussian distribution for the wind stress τ_w . Fig.2 (b) reveals that the
 413 distribution is skewed towards the left of the mean value. Additionally, Fig. 2 (c) highlights a
 414 significant bias in the distribution of the surface wave stress vector $\tau_s^0 = \rho_o a_{zz}\partial_z \mathbf{u}_s|_{z=0}$ derived in
 415 the p-SPDE (2.20).

419 With these parameterized distributions, we proceed to the analysis of the response of the two
 420 random models. Fig. 3 compares the ensemble mean and spread of the Ekman velocity components,

TABLE 1. Common parameters used for simulations.

Parameters	Value	Description
f	$0.73 \times 10^{-4} \text{ s}^{-1}$	Coriolis frequency
ρ_o	1000 kg m^{-3}	Water density
ρ_a	1 kg m^{-3}	Air density
C_D	1.3×10^{-3}	Air-sea drag coefficient
$\bar{\mathbf{u}}_a$	5 m s^{-1}	Mean wind speed
Σ_a	5 m s^{-1}	Standard deviation of transient wind
T_a	1 day	Memory time of fluctuation wind
c_1	0.4	von Kármán constant
c_2	0.7	Constant in boundary layer depth
ζ_0	0.05	Normalized depth of shallow layer
κ_b	$10^{-4} \text{ m}^2 \text{ s}^{-1}$	Background uniform viscosity
α	0.8 m	Surface wave amplitude
k	1.05 m^{-1}	Wavenumber of Stokes drift
Θ_s	0°	Stokes drift mean direction angle
Σ_s	5°	Standard deviation of wave angles
H	256 m	Vertical domain depth
N_z	512	Number of Chebyshev points
N_r	1000	Number of random realizations
Δt	30 min	Timestep

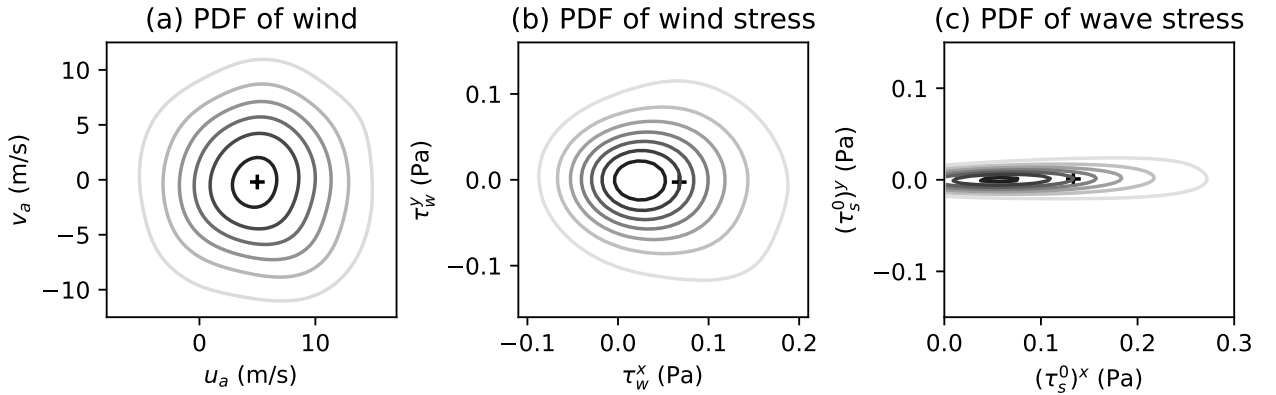
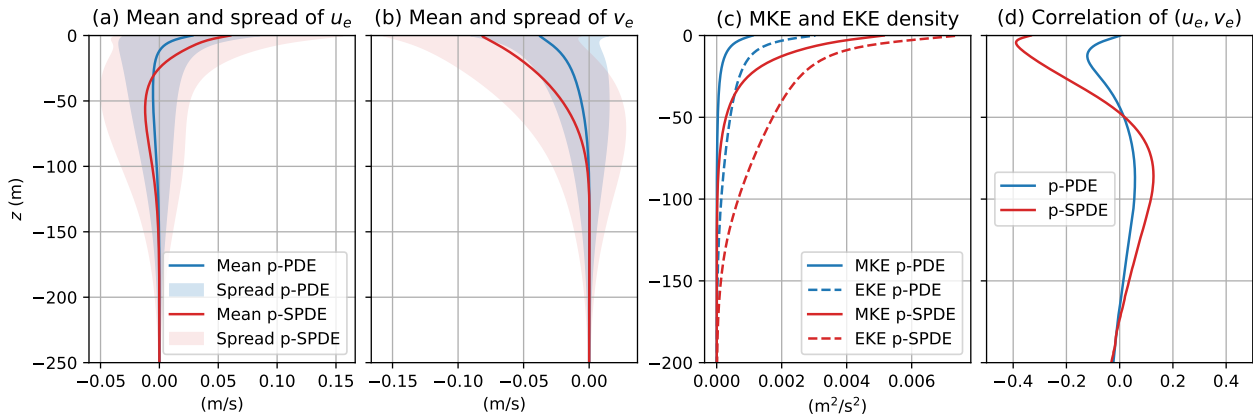


FIG. 2. Contour plots of probability density function for (a) the wind \mathbf{u}_a , (b) the wind stress $\boldsymbol{\tau}_w$ and (c) the derived surface wave stress $\boldsymbol{\tau}_s^0$ after 30 days. In each panel, darker-colored contours represent higher density and + marks indicate the ensemble mean.

along with the ensemble mean kinetic energy (MKE, defined as the energy of the mean solution) and eddy kinetic energy (EKE, defined as the sum of variance components) densities. Additionally, we examine the Pearson correlation coefficient (PCC) for the Ekman velocity components. Compared

424 to the p-PDE, the p-SPDE considering the additional wave mixing and stochastic transport effects
 425 produces a smoother profile of mean Ekman velocities with a higher spread over depth, indicating
 426 greater uncertainty (Fig. 3 (a,b)). This smoother mean profile is due to the diffusion brought by
 427 the Itô-Stokes drift.

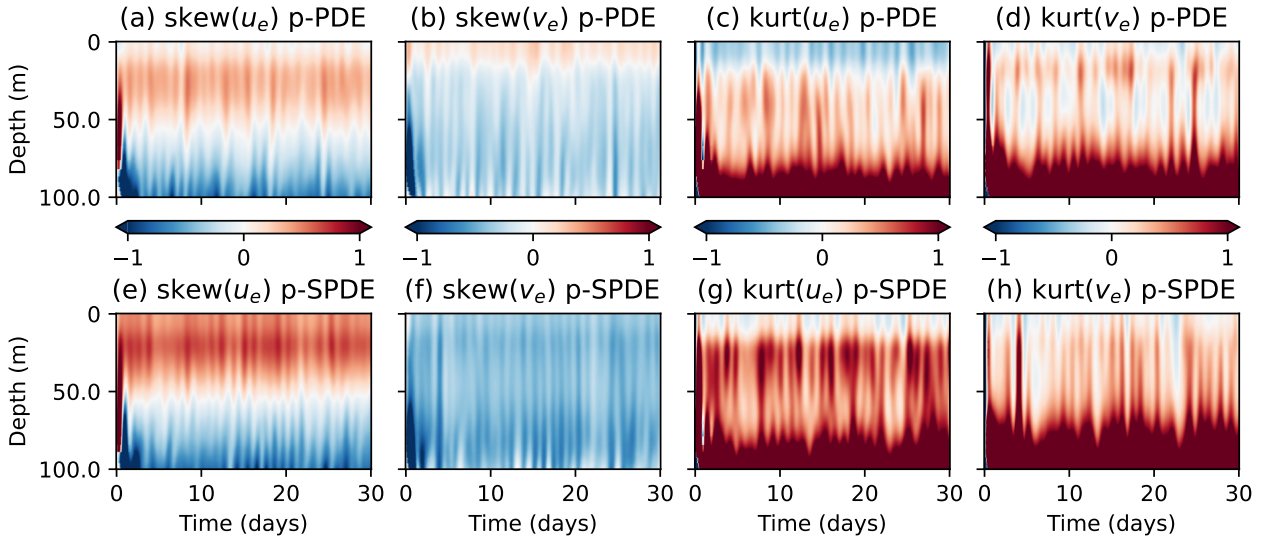
428 Additionally, the p-SPDE yields higher MKE and EKE with larger differences between them
 429 (Fig. 3 (c)), and stronger correlation between the zonal and meridional components (Fig. 3 (d))
 430 throughout the depth. These differences are particularly pronounced near the ocean surface. Note
 431 that both models exhibit higher EKE than MKE throughout the depth, a negative correlation in the
 432 upper 50 m and a positive correlation in the subsequent 100 m.



433 FIG. 3. Comparison of the ensemble mean (solid lines in (a) and (b)) and spread (shaded areas in (a) and
 434 (b)) for the Ekman velocity components, the ensemble mean kinetic energy density (solid lines in (c)) and eddy
 435 kinetic energy density (dashed lines in (c)), as well as the Pearson correlation coefficient for the Ekman velocity
 436 components (d), using different random models (represented by different colors). Note that these ensemble
 437 statistics are averaged over the last 20 days of a 30-day simulation.

438 Higher-order ensemble statistics of the random models can also be examined. The local skewness
 439 and kurtosis of the Ekman velocity components over time and depth are illustrated in Fig. 4. The
 440 skewness measures the asymmetry of distribution around its mean, and the (excess) kurtosis
 441 measures the “tailedness” of the distributions, i.e., the extremity of large values. It is common
 442 practice to include a -3 correction for the kurtosis estimator to provide a simple comparison
 443 to the (univariate) normal distribution, which has a zero value in that case. Distributions with
 444 positive (resp. negative) kurtosis produce more (resp. fewer) extreme events than the normal
 445 distribution. Skewness and kurtosis in the upper 100m-depth are shown in Fig. 4, noting that these

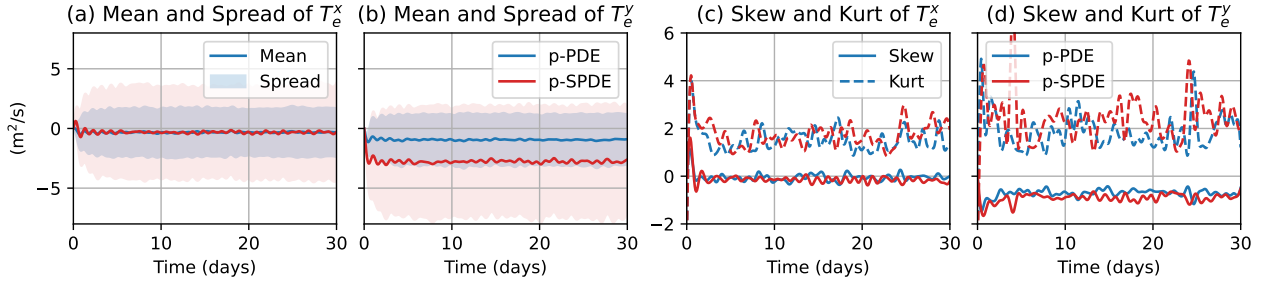
446 normalized statistics are not well-defined for the deeper regions with poor variance. Compared to
 447 the p-PDE, the p-SPDE frequently exhibits more positive skew for the zonal component (Fig. 4
 448 (a,e)), more negative skew and for the meridional component (Fig. 4 (b,f)), and more extremes for
 449 both components (Fig. 4 (c,d,g,h)), all predominantly occurring in the upper 50 m.



450 FIG. 4. Comparison of the ensemble skewness and kurtosis for the Ekman velocity components over time and
 451 depth using different random models (grouped by rows). A one-day low-pass filtering is applied to these time
 452 series of statistics at each point.

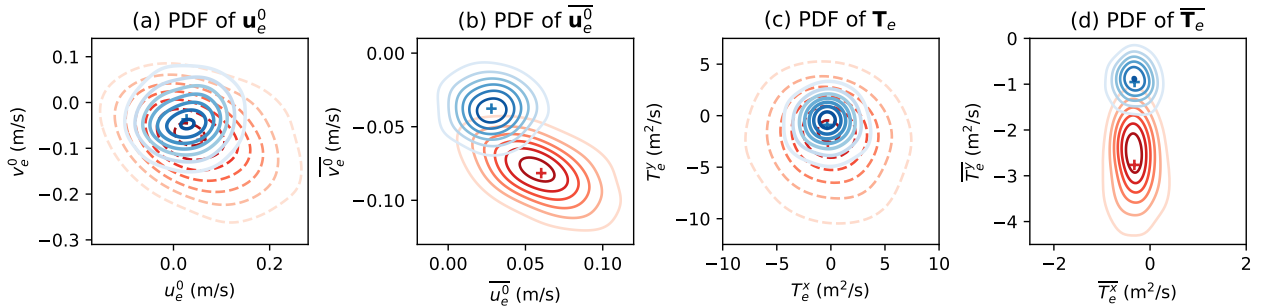
453 Statistical analyses are also conducted on the vertically integrated Ekman transport, Fig. 5.
 454 Compared to the p-PDE, the zonal transport component of the p-SPDE exhibits the same zero
 455 mean but higher uncertainties over time (Fig. 5 (a)), and almost the same zero skew but with
 456 slightly more extremes (Fig. 5 (c)). The meridional transport component of the p-SPDE shows
 457 a larger mean in magnitude with higher variance (Fig. 5 (b)), as well as more negative skew and
 458 more frequent extremes (Fig. 5 (d)).

463 The bi-variate PDFs for both instantaneous and time-averaged surface current velocity and
 464 Ekman transport are estimated and illustrated Fig. 6. Unsurprisingly, the p-SPDE exhibits higher
 465 variances than the p-PDE in both cases. Notably, the p-SPDE demonstrates a much stronger
 466 negative correlation for the surface current velocity compared to the p-PDE, which is consistent
 467 with the PCC shown in Fig.3 (d). For the proposed random model, the PDF of the surface current
 468 appears to be rotated approximately by 45 degrees to the left from the PDFs of the wind and wave



459 FIG. 5. Time evolution of the ensemble statistics for the Ekman transport components using various random
 460 models (distinguished by colors). The solid lines and shaded areas respectively represent the mean and spread in
 461 (a) and (b). The solid and dashed lines respectively depict the skewness and kurtosis in (c) and (d). A one-day
 462 low-pass filter is applied to these time series of statistics.

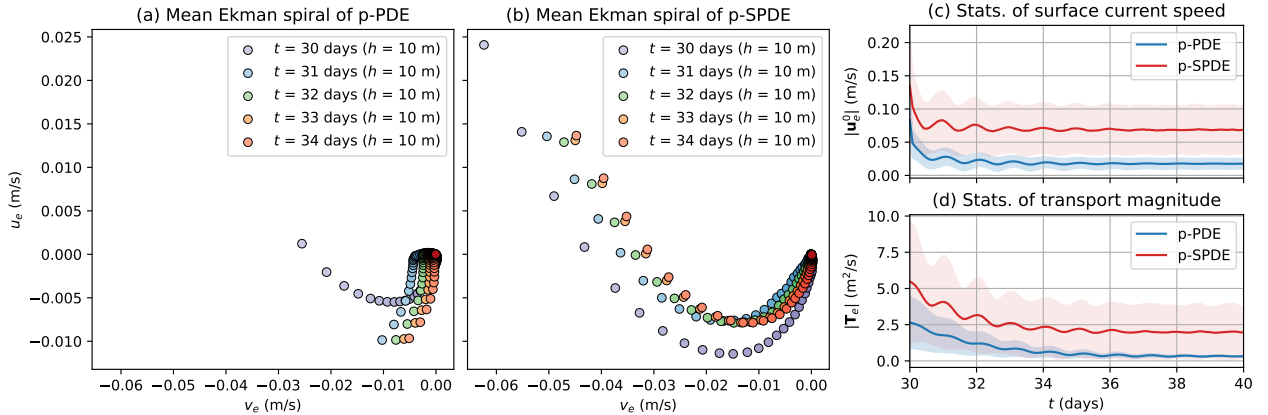
469 stresses (Fig. 2 (b,d)), while the PDF of Ekman transport seems to be rotated approximately by 90
 470 degrees to the left from those surface stresses.



471 FIG. 6. Comparison of PDFs for (a) instantaneous surface Ekman velocity (\mathbf{u}_e^0) after 30 days and (b) its time-
 472 average ($\overline{\mathbf{u}_e^0}$) over the last 20 days, as well as for (c) instantaneous Ekman transport (\mathbf{T}_e) and (d) its time-average
 473 ($\overline{\mathbf{T}_e}$), using different random models (p-PDE in blue and p-SPDE in red). In each panel, the colored signs “+”
 474 indicate the mean.

475 To further highlight the differences between the two schemes, the wind is stopped after 30 days of
 476 simulation, by setting $\tau_w = 0$, while maintaining the last day’s coefficient a_{zz} as a steady diffusion
 477 together with a stationary Stokes drift \mathbf{u}_s . The two random models are then run without surface
 478 momentum flux for several more days. Results, Fig. 7, demonstrate significant differences in their
 479 statistics. For instance, the mean Ekman spiral size of the p-PDE (Fig. 7a) quickly shrinks (the
 480 Ekman current profile becomes nearly centered at the origin) whereas the p-SPDE (Fig. 7b) almost
 481 preserves its mean spiral structure. Only its current speed decreases over time. Figures 7 (c,

482 d) show that both the mean and spread of the surface current speed $|\mathbf{u}_e^0|$, as well as the Ekman
 483 transport magnitude $|\mathbf{T}_e|$ for the two random models, rapidly decrease, reaching both steady states
 484 within one week. However, the residual mean values and uncertainties are significantly higher in
 485 the p-SPDE than in the p-PDE. This can be understood from equations (2.16), (2.17) and (2.22b)
 486 with $\tau_w = 0$. These results indicate that the p-SPDE transitions from a wind- and wave-driven
 487 model to a wave-driven model after the wind stops.

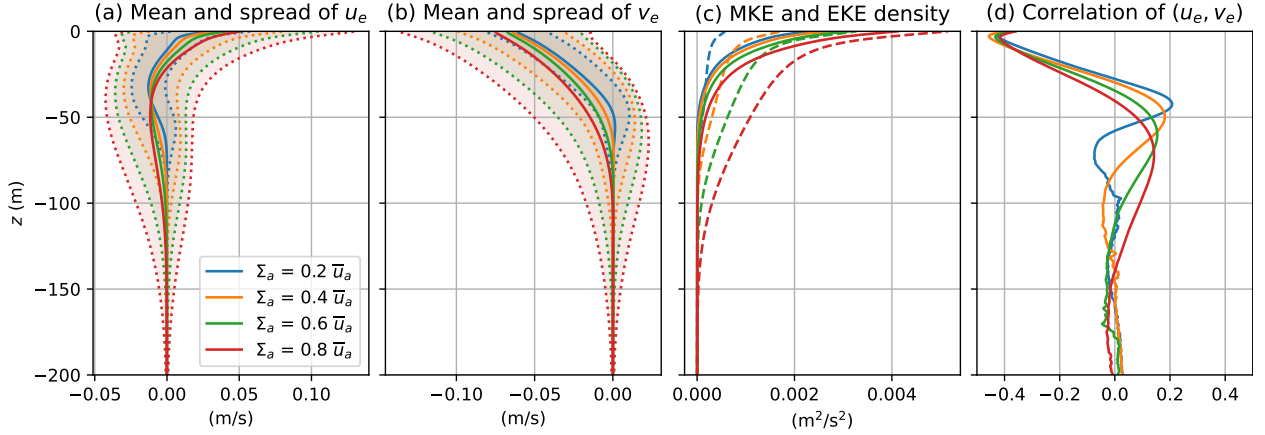


488 FIG. 7. Evolution comparison after wind stopped for (a, b) mean Ekman spiral, (c) mean and spread of surface
 489 current speed, and (d) mean and spread of Ekman transport magnitude, using different random models. The
 490 wind is stopped after 30 days of simulation. The mean Ekman spirals at different days in (a, b) are represented by
 491 various colors. In each spiral, a darker color indicates a deeper position, with a depth step of 5 meters between
 492 adjacent points.

493 *b. Comparison across transient wind levels*

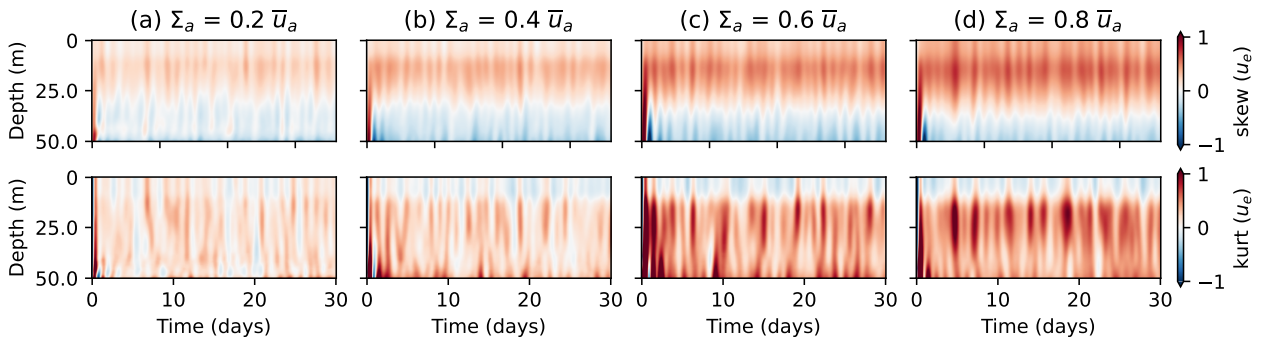
494 Sensitivity analyses to the wind gustiness are performed for the p-SPDE. For this purpose, various
 495 ensembles are simulated according to a range of Σ_a values while keeping the other parameters in
 496 Table 1 invariant. Illustrated in Fig. 8 (a,b), a higher variable wind reduces the vertical shear of the
 497 mean Ekman current, induces deeper circulation, and results in higher uncertainty. Figure 8 (c)
 498 demonstrates that increased wind variability leads to larger MKE and EKE throughout the depth,
 499 and in particular near the surface. Figure 8 (d) shows weaker negative correlation at the surface
 500 but with stronger variations. As described in Section 2e, increased gustiness results in a higher
 501 diffusion coefficient a_{zz} (and hence a higher magnitude of the correlation σ_z for the unresolved

502 motions). It also leads to a larger derived surface wave stress τ_s^0 , which amplifies both the wave
 503 mixing and stochastic transport effects in the p-SPDE.



504 FIG. 8. Comparison of the ensemble mean (solid lines in (a) and (b)) and spread (shaded areas in (a) and (b))
 505 for the Ekman velocity components, the MKE density (solid lines in (c)) and EKE density (dashed lines in (c)),
 506 and the correlation coefficient for the Ekman velocity components (d), with respect to different transient wind
 507 levels (represented by different colors).

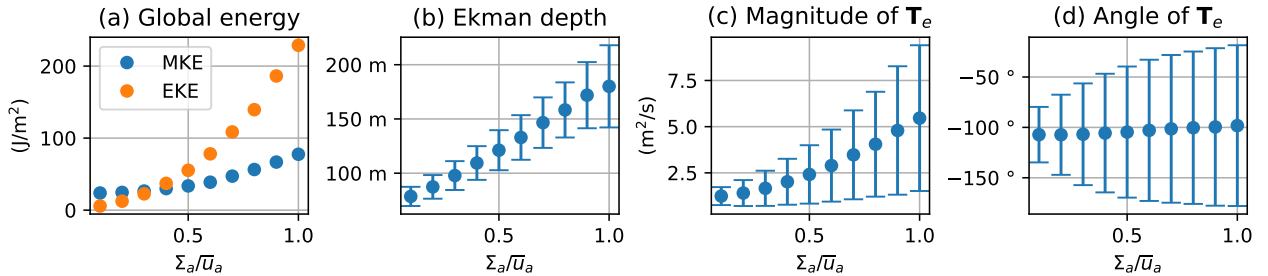
508 The ensemble skewness and kurtosis of the zonal Ekman velocity component with respect to
 509 different Σ_a values are illustrated Fig. 9. More skewed distributions with higher kurtosis are
 510 obtained, reflecting an increase in extreme events over time and near the surface. Not shown,
 511 similar results were observed for the meridional component.



512 FIG. 9. Comparison of the ensemble kurtosis for the zonal Ekman velocity component over time and depth
 513 with respect to different transient wind levels (grouped by columns).

514 To further quantify the sensitivity to a wider range of Σ_a values, we focus on the time-averaged
 515 ensemble statistics of diagnostic variables. Figure 10 (a) shows that both global MKE and EKE

516 increase with higher transient wind, with EKE rising more rapidly than MKE, especially when the
 517 transient wind exceeds half of the mean wind component. Figure 10 (b) demonstrates that both the
 518 mean and variance of the estimated boundary layer depth increase with transient wind. Figure 10
 519 (c) illustrates that the Ekman transport magnitude $|\mathbf{T}_e|$ significantly grows with wind. Figure 10
 520 (d) indicates that the mean angle of \mathbf{T}_e is around 100 degrees (relative to the left of the mean wind
 521 direction) with a slight increasing trend, while the uncertainty of this angular distribution increases
 522 rapidly with wind.



523 FIG. 10. Comparison of the ensemble statistics for diagnostic variables across various transient wind levels.
 524 The MKE and EKE are represented by different colors in (a). The mean and uncertainty are depicted using
 525 error bars in (b)-(d). These ensemble statistics are averaged over the last 20 days. It is noteworthy that circular
 526 statistics (Fisher 1993) are particularly computed in (d).

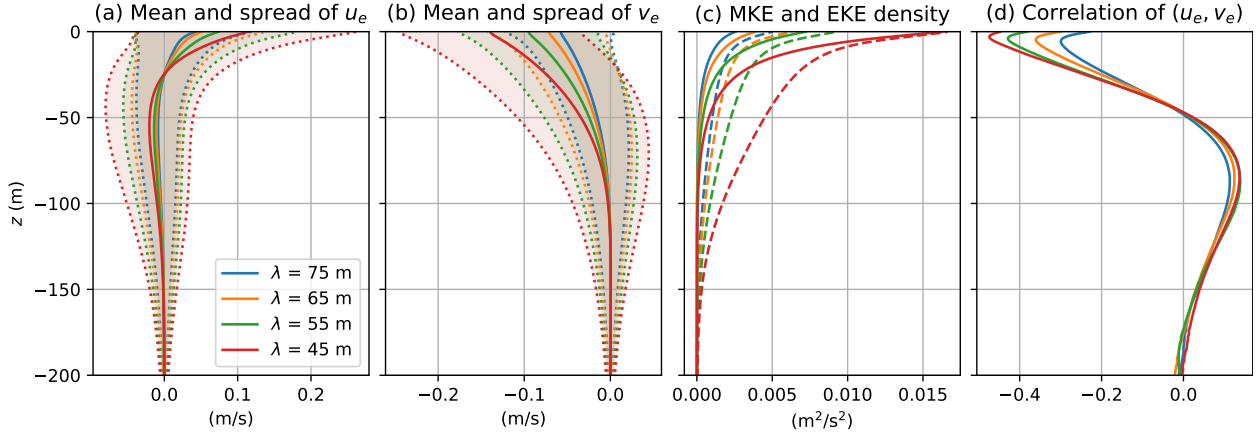
527 c. Comparison across wave parameters

528 Next, the sensitivity analyses is performed for the p-SPDE to the wavelength λ (or wavenumber
 529 k) and the mean angle Θ_s of the Stokes drift, respectively. The same metrics are used in the
 530 following analysis.

531 1) WAVELENGTH

532 Fig. 11 evidences that a smaller surface gravity wave results in smoother vertical profiles of the
 533 mean Ekman current with higher uncertainty (a,b), higher densities of MKE and EKE (c), and
 534 stronger correlation (d). These trends are particularly notable near the surface. Furthermore, as
 535 shown in Fig. 12 (a,c), both global MKE and EKE, as well as the mean and uncertainty of the
 536 transport magnitude, decrease with increasing wavelength. Fig. 12 (b) depicts a slight decreasing
 537 tendency for the mean and variance of the estimated Ekman layer depth with increasing wavelength.

538 Additionally, Fig. 12 (d) illustrates an even smaller decreasing trend for the Ekman transport angles.
 539 It is worth noting that a smaller wavelength leads to greater magnitude and vertical shear of the
 540 Stokes drift, Eq. (2.27a), thereby amplifying the wave mixing effects in the p-SPDE (2.20).



541 FIG. 11. Comparison of the ensemble mean (solid lines in (a) and (b)) and spread (shaded areas in (a) and (b))
 542 for the Ekman velocity components, the MKE density (solid lines in (c)) and EKE density (dashed lines in (c)),
 543 and the correlation coefficient for the Ekman velocity components (d), with respect to different wavelengths.

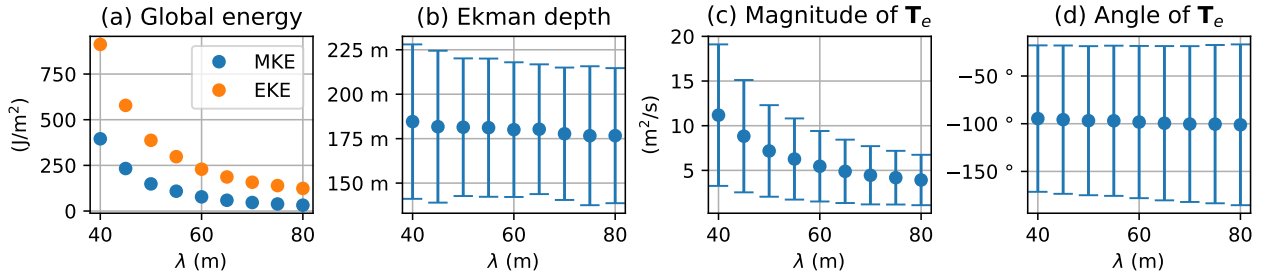
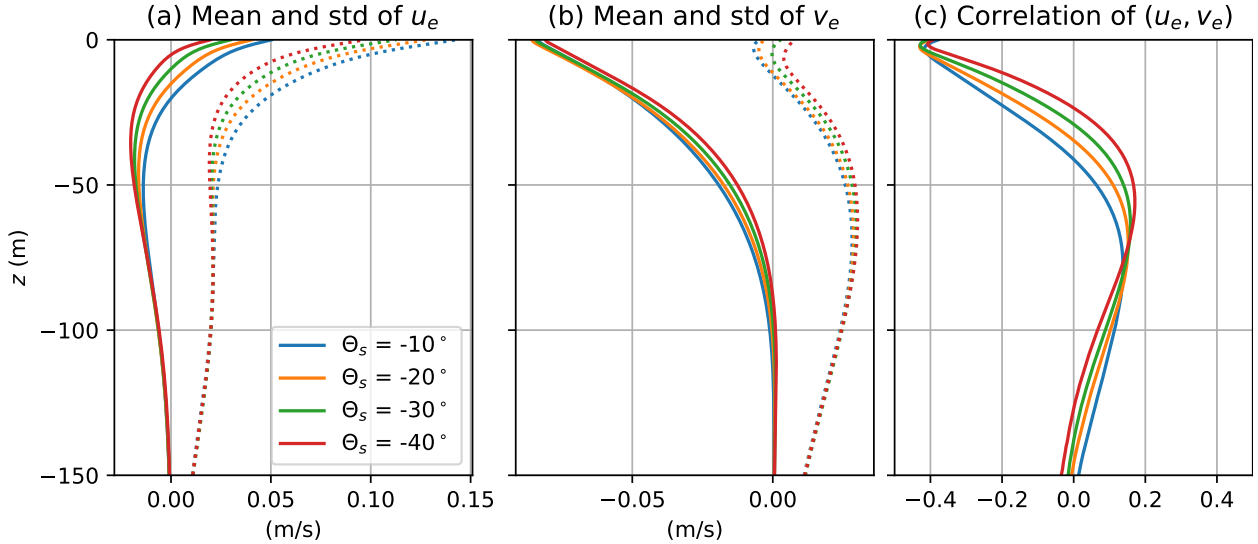


FIG. 12. Comparison of the ensemble statistics for diagnostic variables across various wavelengths.

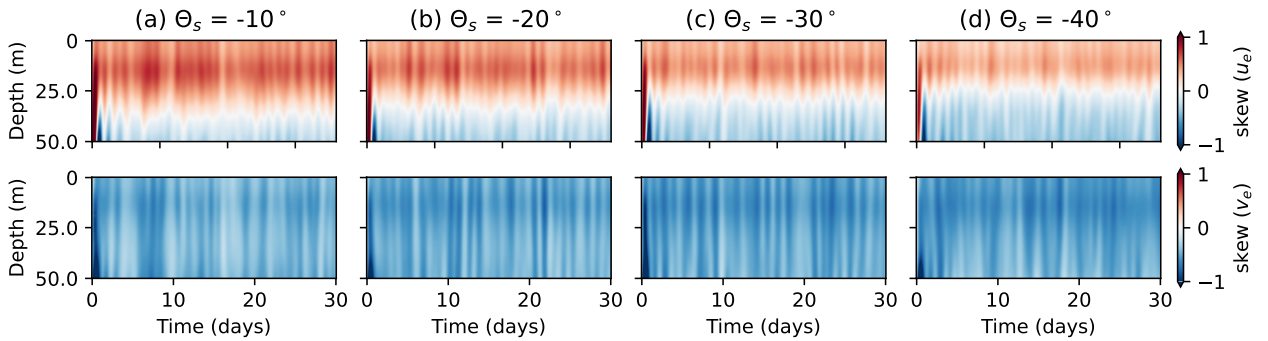
544 2) MEAN ANGLE

545 Fig.13 evidences that waves with a higher rotation from the left of the zonal wind increase both
 546 the mean and variance of the zonal Ekman velocity (a) while decreasing those of the meridional
 547 component (b) in the upper 100 meters. Additionally, they induce weaker correlation in the upper
 548 25 meters and stronger correlation in the subsequent 50 meters (c). In this scenario, the zonal
 549 component exhibits lower positive skew, whereas the meridional component shows higher negative

550 skew near the surface, Fig.14. This indicates a redistribution of the ensemble statistics between the
 551 two components.



552 FIG. 13. Comparison of the ensemble mean and std (respectively represented by solid lines in (a) and (b)) and
 553 the correlation coefficients (c) for the Ekman velocity components, with respect to different mean angles.



554 FIG. 14. Comparison of the ensemble skewness for the Ekman velocity components over time and depth with
 555 respect to different mean angles.

556 Global metrics, Fig.15 (a,c), further demonstrate that both MKE and EKE, as well as the mean
 557 and uncertainty of the transport magnitude, reach a maximum when the mean wave direction is
 558 aligned with the mean wind direction, decreasing symmetrically with higher rotations to either
 559 side. It is noteworthy that in the aligned case, the derived wave stress simply enhances the wind
 560 stress. Unsurprisingly, the response of the Ekman transport angle follows the variation of the mean

561 wave directions, as shown in Fig.15 (d). The difference between these output and input angles
 562 remains almost invariant.

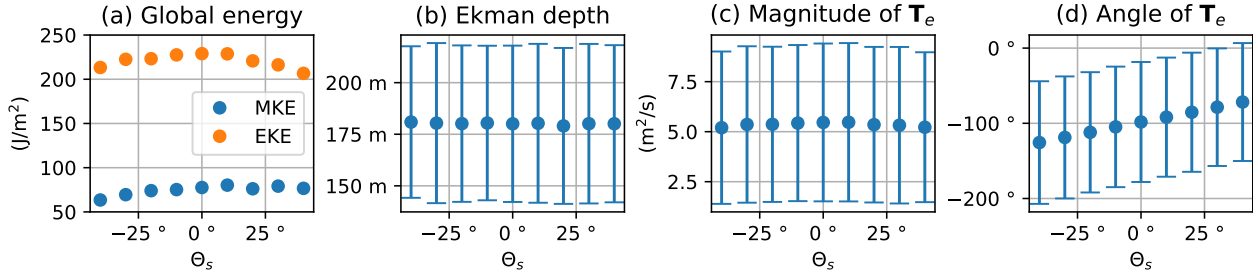


FIG. 15. Comparison of the ensemble statistics for diagnostic variables across various mean angles.

563 4. Conclusions

564 In this study, a novel stochastic formulation is developed to describe the upper ocean Ekman
 565 boundary layer. This formulation couples the contributions of random fluctuations with the bound-
 566 ary layer dynamics through an uncertainty representation of unresolved motions, fully consistent
 567 with established physical parameterizations.

568 Through numerical investigations, the statistical responses of the proposed time-dependent
 569 stochastic Ekman-Stokes model are analyzed. Comparisons are performed with a benchmark
 570 model solely driven by the Coriolis-Stokes force. Incorporating wave mixing and stochastic trans-
 571 port compound effects, the stochastic model exhibits smoother mean Ekman velocity profiles with
 572 increased uncertainty, along with higher mean kinetic energy (MKE) and eddy kinetic energy
 573 (EKE). Additionally, numerical results display stronger correlations between zonal and meridional
 574 components, with higher-order statistics indicating more skewed distributions and extreme value
 575 occurrences, particularly near the surface.

576 Sensitivity analyses revealed that increased wind variability reduces vertical shear, deepens
 577 circulation, and increases uncertainty and higher-order moments. Similarly, smaller surface waves
 578 would lead to higher MKE, EKE, and stronger correlations. Ensemble energy and Ekman transport
 579 magnitude peak when mean waves and wind were aligned, decrease with wind-wave misalignment,
 580 redistributing current velocity statistics. In terms of sensitivity, the primary source of uncertainty
 581 is associated with the wind, followed by the waves, with their direction being the final source of

582 uncertainty. These findings underscore the enhanced capability of the proposed model to possibly
583 capture and interpret the complex interactions and dynamics of Ekman currents.

584 Looking ahead, future research avenues include extending the stochastic Ekman-Stokes model to
585 incorporate stratification effects (Price and Sundermeyer 1999; McWilliams et al. 2009; Gula et al.
586 2014; McWilliams et al. 2015) and exploring the distinct impacts of wave mixing and stochastic
587 transport on thermal front evolution (Crowe and Taylor 2018, 2019). It is important to point out that
588 the retro-action of the waves, possibly modulated by upper ocean random currents, on the wind
589 stress, particularly on the atmospheric wave-induced turbulent components (Ayet and Chapron
590 2022), has not been taken into account in this study. Along the proposed stochastic framework,
591 developing more accurate noise term to better represent fully coupled ocean/atmosphere Ekman
592 models (Lewis and Belcher 2004) would be particularly interesting.

593 Additionally, further investigation of the nonlinear stochastic Craik-Leibovich equations through
594 numerical studies, particularly employing large-eddy simulations (LES) to include ocean Langmuir
595 circulation (McWilliams et al. 1997; Harcourt and D'Asaro 2008; McWilliams et al. 2012; Sullivan
596 and McWilliams 2019), offers promising directions for advancing our understanding of turbulent
597 ocean processes and enhancing predictive capabilities.

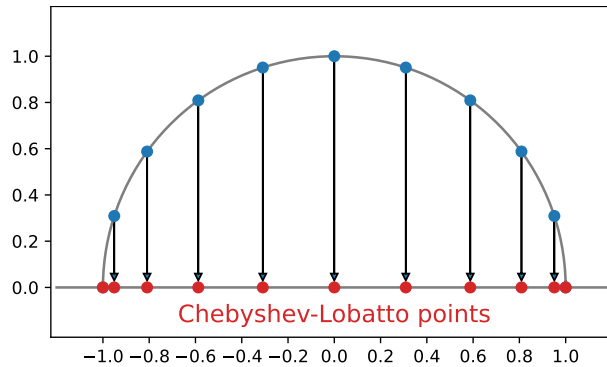
598 *Acknowledgments.* The authors acknowledge the support of the ERC EU project 856408-STUOD.

599 *Data availability statement.* The code to reproduce the simulation data, diagnostics, and figures
600 can be accessed at <https://github.com/matlong/SGESM>.

601 APPENDIX A

602 Numerical schemes

603 A pseudo-spectral Chebyshev method (Boyd 2001) is employed for the vertical discretization
604 of the boundary layer models. The so-called *Chebyshev points*, denoted as $\widehat{z}_j = \cos(j\pi/N_z)$, $j =$
605 $0, \dots, N_z$, are depicted in Fig. A1. Geometrically, these points represent the projections on $[-1, 1]$
606 of equispaced points on the upper half of the unit circle. Hence, the projection nodes are denser
607 near the two boundaries than in the mid-regions, which proves advantageous for boundary layer
608 problems and for representing the Stokes drift near the ocean surface. The grid points of the
609 vertical domain depth H are constructed using a linear transformation, $z_j = -H(\widehat{z}_j + 1)/2$. Then,
610 the Chebyshev spectral derivative and integration operators are built on these points to solve a
611 (S)PDE for non-periodic functions with high accuracy.



612 FIG. A1. Geometrical visualization of Chebyshev points (red points) as the projections on $[-1, 1]$ of equally
613 spaced points on the upper half of unit circle.

614 A linear-implicit Euler scheme (Jentzen and Kloeden 2011, chap. 8) is used for the time stepping
615 of the SPDE (2.20), namely

$$\mathbf{u}_j^{n+1} = (I_{j,k} - A_{j,k}\Delta t)^{-1} (\mathbf{u}_k^n + F_k^n(\mathbf{u}_s)\Delta t + G_k^n(\mathbf{u}^n, \mathbf{u}_s)\Delta B), \quad (\text{A1})$$

616 Here, \mathbf{u}_j^n denotes the discrete value of the Ekman current velocity \mathbf{u}_e on the node z_j at time t_n ,
617 $A = -ifI + Da_{zz}D$ represents the discrete linear operator of the SPDE, where I denotes the identity
618 matrix and D represents the Chebyshev differentiation matrix (Trefethen 2000, chap. 6). The
619 terms F and G correspond to the right-hand side forcing term of Eq. (2.20a).

620 Additionally, we consider an exact scheme for the wind using the Markovian property of the
621 Ornstein-Uhlenbeck process (2.24), expressed as:

$$\mathbf{u}_a^{n+1} = \bar{\mathbf{u}}_a + (\mathbf{u}_a^n - \bar{\mathbf{u}}_a)e^{-\Delta t/T_a} + \Sigma_a \sqrt{1 - e^{-2\Delta t/T_a}} \boldsymbol{\xi}^n, \quad (\text{A2})$$

622 Here, $\boldsymbol{\xi}^n = \xi_x^n + i\xi_y^n$ where ξ_x^n and ξ_y^n are independent random variables following a standard normal
623 distribution.

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