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María ANGUIANO* and Francisco Javier SUÁREZ-GRAU†

Abstract

In this paper, we study the asymptotic behavior of the micropolar fluid flow through a thin domain assuming zero Dirichlet boundary condition on the top boundary, which is rapidly oscillating, and non-standard boundary conditions on the flat bottom. Assuming "Reynolds roughness regime", in which the thickness of the domain is very small compared to the wavelenth of the roughness (i.e. a very slight roughness), we rigorously derive a generalized Reynolds equation for pressure clearly showing the roughness-induced effects. Moreover, we give expressions for the average velocity and microrotation.

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Keywords: micropolar fluid; thin-film flow; rapidly oscillating boundary; nonzero boundary conditions; homogenization.

1 Introduction

Eringen [23] in 60's proposed the model of micropolar fluid which has been extensively studied both in the engineering and mathematical literature, due to its practical importance. This model takes into consideration the microstructure of the fluid particles and captures the effects of its rotation, so the micropolar fluid model describes the motion of a lot of real fluids in a better way than the classical (Newtonian) model. Some examples are liquid crystals, animal blood, muddy fluids, certain polymeric fluids or even water in models with small scales. The rotation of the fluid particles is mathematically described by introducing the microrotation field, together with the standard velocity and pressure fields. Accordingly, a new governing equation coming from the conservation of angular momentum. Thus, the stationary and linearized micropolar equations in dimensionless form is given by (see Lukaszewicz [25])

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = 2N^2 \mathrm{rot}(\mathbf{w}), \\ \mathrm{div}(\mathbf{u}) = 0, \\ -R_M \Delta \mathbf{w} + 4N^2 \mathbf{w} = 2N^2 \mathrm{rot}(\mathbf{w}), \end{cases} \tag{1.1}$$

where $\mathbf{u}$ denotes the velocity vector field, p the pressure and $\mathbf{w}$ the microrotation field. Dimensionless (non-Newtonian) parameter N^2 characterizes the coupling between the equations for the velocity and microrotation, and the second dimensionless parameter denoted by R_M is, in fact, related to the characteristic length of the microrotation effects.

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An important problem in lubrication theory is to derive lower-dimensional models to describe the flow of fluid in a narrow space between two surfaces in relative motion. To describe the classical setting, we can define a thin domain by

$$\widehat{\Omega}^\epsilon = \omega \times (0, \epsilon h(x')), \quad x' = (x_1, x_2) \in \omega \subset \mathbb{R}^2, \quad 0 < \epsilon \ll 1, \tag{1.2}$$

where ϵh represents the thickness of the domain, with h a smooth function and the small parameter ϵ is devoted to tend to zero. Thus, prescribing velocity on the bottom $\widehat{\Gamma}_0 = \omega \times \{0\}$ and zero on the top $\widehat{\Gamma}_1^\epsilon = \omega \times \{\epsilon h(x')\}$ (we don't care about the lateral boundary conditions here), i.e.

$$\mathbf{u} = \mathbf{s} = (\mathbf{s}', 0) = (s_1, s_2, 0) \quad \text{on } \widehat{\Gamma}_0, \qquad \mathbf{u} = 0 \quad \text{on } \widehat{\Gamma}_1^\epsilon, \tag{1.3}$$

then Bayada & Lukaszewicz [9] gave a mathematical proof of the transition from the micropolar equations (1.1) to a 2D Reynolds-like equation by using homogenization techiniques when $\epsilon \to 0$. This equation is given by

$$\mathrm{div}_{x'} \left(-\frac{h^3(x')}{1-N^2} \Phi(h(x'), N) \nabla_{x'} p(x') + \frac{h(x')}{2} \mathbf{s}' \right) = 0 \quad \text{in} \quad \omega, \tag{1.4}$$

where $\Phi(h(x'), N) = \frac{1}{12} + \frac{1}{4h^2(x')(1-N^2)} - \frac{1}{4h(x')} \sqrt{\frac{N^2}{1-N^2}} \coth\left(Nh(x')\sqrt{1-N^2}\right)$.

Throughout the mathematical literature, micropolar fluid models have been extensively studied in recent years by deriving different asymptotic Reynolds-like models taking into account that a surface presents micro-roughness, depending on the ratio between the size of the roughness and the thickness of the domain and on the boundary conditions considered, see for instance Anguiano & Suárez-Grau [2], Bayada *et al.* [7, 8], Boukrouche [14], Boukrouche & Paoli [15], Boukrouche *et al.* [16, 17], Dupuy *et al.* [21, 22], Marusic-Paloka [26, 27, 28], Pažanin [31], Pažanin & Radulović [33], Pažanin & Suárez-Grau [34], Suárez-Grau [38] and references therein.

One important problem in the lubrication framework concerns to the case of a thin domain with rapidly oscillating boundary given by

$$\Omega^\epsilon = \omega \times (0, h_\epsilon(x')) \quad \text{with} \quad h_\epsilon(x') = \epsilon h(x'/\epsilon^\ell) \quad \forall\, x' \in \omega \subset \mathbb{R}^2, \tag{1.5}$$

where h_ϵ represents the thickness of the domain, which is rapidly oscillating. The thickness is given by the small parameter $0 < \epsilon \ll 1$, and the top boundary is rough with small roughness of wavelength described by $0 < \epsilon^\ell \ll 1$, with $\ell \in (0, +\infty)$. Thus, depending on the limit of $\lambda = \lim_{\epsilon \to 0} \epsilon^{1-\ell}$, there exist three characteristic regimes:

- Reynolds roughness ($\lambda = 0$), where the thickness of the domain is smaller than the wavelenth of the roughness, i.e. a very slight roughness.
- Stokes roughness ($\lambda = 1$), where the thickness of the domain and the wavelenth of the roughness are proportional.
- High-frequency roughness ($\lambda = +\infty$), where the thickness of the domain is greater than the wavelenth of the roughness, i.e. a very strong roughness.

These characteristic regimes have been studied in Bayada and Chambat [5, 6] Benhaboucha *et. al.* [11], Fabricius *et al.* [24] and Mikelić [29] for Newtonian fluids, in Pazanin & Suárez-Grau [35] and Suárez-Grau [37] for micropolar fluids, and in Anguiano & Suárez-Grau [1] and Nakasato & Pažanin [30] for generalized Newtonian with power law viscosity law.

Note that previous results consider Dirichlet boundary condition for microrotation. However, Bayada *et al.* [3, 4] introduced a new boundary condition more general (and physically justified) at the fluid-solid interface. This boundary condition, called non-standard or non-zero boundary condition in the literature, links the value of

the micorotation with the rotation of the velocity by way of a coefficient α that characterizes the microrotation on the solid surfaces as follows

$$\mathbf{w} \times \mathbf{n} = \frac{\alpha}{2}\text{rot}(\mathbf{u}) \times \mathbf{n}, \quad \mathbf{w} \cdot \mathbf{n} = 0, \tag{1.6}$$

where $\mathbf{n}$ is the normal unit vector to the boundary. In addition, this condition allows a slippage in the tangential direction and retains a non-penetration condition in the normal direction on the boundary as follows

$$\beta(\mathbf{u} - \mathbf{s}) \times \mathbf{n} = \frac{R_M}{2N^2}\text{rot}(\mathbf{w}) \times \mathbf{n}, \quad \mathbf{u} \cdot \mathbf{n} = 0, \tag{1.7}$$

where β allows the control of the slippage at the wall when the value $\mathbf{u} - \mathbf{s}$ is not zero. Recently, this type of boundary conditions is attracting a lot of attention, as can be seen in various studies, see for instance Beneš *et al.* [10], Bonnivard *et al.* [12, 13], Pažanin [32] and Suárez-Grau [39].

Coming back to the classic setting in lubrication, starting from the micropolar equations (1.1) in the thin domain $\widehat{\Omega}^\epsilon$ with non-standard boundary conditions on the bottom $\widehat{\Gamma}_0$ and zero boundary conditions on the top $\widehat{\Gamma}_1^\epsilon$, after the homogenization process, a generalized Reynolds equation was derived in Bayada *et al.* [3, 4] of the form

$$-\text{div}_{x'}\Big(\Theta_1(x')\nabla_{x'}p(x') + \Theta_2(x')\mathbf{s}'\Big) = 0 \quad \text{in } \omega, \tag{1.8}$$

for certain explicit functions Θ_1 and Θ_2. Moreover, explicit expressions for the limit velocity and microrotation were given.

The purpose of this paper is to generalize the results in Bayada *et al.* [3, 4] to the case of rapidly oscillating domain Ω^ϵ given by (1.5) and derive a generalized lower-dimensional model influenced by the non-standard boundary conditions and the roughness of the top boundary $\Gamma_1^\epsilon = \omega \times \{\epsilon h(x'/\epsilon^\ell)\}$. We resctrict ourselves to the case of "Reynolds roughness", i.e. the case of a rapidly oscillating boundary described by (1.5) with $\lim_{\epsilon\to 0}\epsilon^{1-\ell} = 0$, which will allows us to derive an explicit equation. Accordingly, we consider the micropolar equations (1.1) assuming non-standard boundary conditions (1.6)–(1.7) on the bottom Γ_0 and zero boundary conditions on the top boundary Γ_1. As far as the authors know, the flow of a micropolar fluids with non-standard boundary conditions has not been yet considered in the above described setting. By applying reduction of dimension techniques together with an adaptation of the unfolding method to capture the micro-geometry of the roughness (see Section 5), we rigorously derive a generalized micropolar Reynolds equation given by

$$-\text{div}_{x'}\left(K\nabla_{x'}p(x') + L\mathbf{s}'\right) = 0 \quad \text{in } \omega,$$

with flow factors $K \in \mathbb{R}^{2\times 2}$ and $L \in \mathbb{R}$ respectively given by

$$K_{ij} = \int_{Z'} \Theta_1(z')\left(\partial_{z_i}q^j(z') + \delta_{ij}\right)dz', \quad i,j = 1,2, \qquad L = \int_{Z'}\Theta_2(z')\,dz',$$

where the function q_i, $i = 1, 2$ satisfies the following local periodic problem

$$-\text{div}_{z'}\left(\Theta_1(z')(\nabla_{z'}q_i(z') + \mathbf{e}_i) + \Theta_2(z')\mathbf{s}'\cdot\mathbf{e}_i\right) = 0 \quad \text{in } Z', \quad i,j = 1,2,$$

with Z' is the periodic cell in $\mathbb{R}^2$ and functions Θ_1 and Θ_2 are the same functions as in (1.8). In addition, we give the expressions for the average of the velocity and microrotation (see Theorem 6.4). Since the obtained findings are amenable for the numerical simulations, we believe that it could prove useful in the engineering practice as well.

The paper is structured as follows. In Section 2, we introduce the position of the problem and the existence and uniqueness results. To pass to the limit, it is necessary a rescaling of the problem, which is introduced in Section 3. In Section 4, we get the *a priori* estimates for velocity, microrotation and pressure, extended to a ϵ-independent domain. To capture the micro-geometry of the roughness, we introduce in Section 5 the adaptation of the unfolding method, give some properties and derive the limit problem. Finally, we deduce the generalize Reynolds equation in Section 6. We finish the paper with a list of references.

2 Position of the problem

In this section, we first define the rough thin domain, the necessary sets to study the asymptotic behavior of the solutions and some notation. Next, we introduce the micropolar equations and the boundary conditions in the thin domain. Finally, we give the variational formulation and the condition for existence and uniqueness of solution of the problem according to this setting.

2.1 Definition of the domain and some notation.

We consider ω a smooth and connected subset of $\mathbb{R}^2$. We define the thin domain with a rapidly varying thickness by

$$\Omega^\epsilon = \{x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega,\ 0 < x_3 < h_\epsilon(x')\}, \tag{2.9}$$

where the function $h_\epsilon(x') = \epsilon h\left(x'/\epsilon^\ell\right)$ represents the real distance between the two surfaces, see Figure 1. The small parameter $0 < \epsilon \ll 1$ is related to the film thickness and the small parameter $0 < \epsilon^\ell \ll 1$ is the wavelength of the roughness, see Figure 2. As said in the introduction, we restrict ourselves to the "Reynolds roughness regime", so we consider that ϵ^ℓ is of order greater or equal than ϵ, i.e. $0 < \ell < 1$, which implies

$$\lim_{\epsilon \to 0} \epsilon^{1-\ell} = 0. \tag{2.10}$$

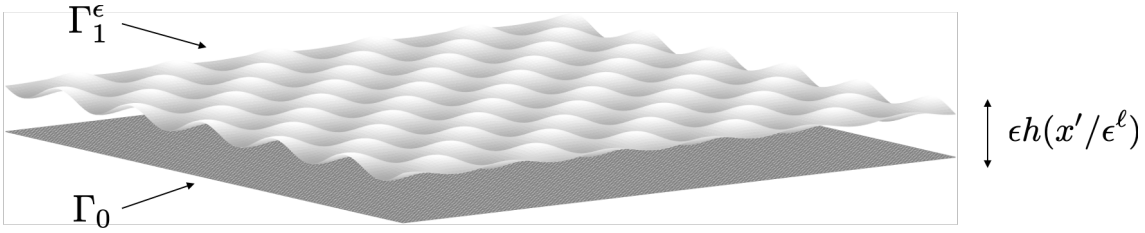

Figure 1: Domain Ω^ϵ, bottom flat boundary Γ_0 and top oscillating boundary Γ_1^ϵ

Function h is a positive and smooth function defined for z' in $\mathbb{R}^2$, Z'-periodic with $Z' = (-1/2, 1/2)^2$ the cell of periodicity in $\mathbb{R}^2$, and there exist $h_{\min}$ and $h_{\max}$ such that

$$0 < h_{\min} = \min_{z' \in Z'} h(z'), \quad h_{\max} = \max_{z' \in Z'} h(z')\,.$$

We define the boundaries of Ω^ϵ as follows

$$\Gamma_0 = \omega \times \{0\}, \quad \Gamma_1^\epsilon = \left\{(x', x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega,\ x_3 = h_\epsilon(x')\right\}, \quad \Gamma_{\text{lat}}^\epsilon = \partial\Omega_\epsilon \setminus (\Gamma_0 \cup \Gamma_1^\epsilon).$$

To define the microstructure of the periodicity of the boundary, we assume that the domain ω is divided by a mesh of size ϵ^ℓ: for $k' \in \mathbb{Z}^2$, each cell $Z'_{k',\epsilon^\ell} = \epsilon^\ell k' + \epsilon^\ell Z'$. We define $T_\epsilon = \{k' \in \mathbb{Z} : Z'_{k',\epsilon^\ell} \cap \omega \neq \emptyset\}$. In this setting, there exists an exact finite number of periodic sets Z'_{k',ϵ^ℓ} such that $k' \in T_\epsilon$. Also, we define $Z_{k',\epsilon^\ell} = Z'_{k',\epsilon^\ell} \times (0, h(z'))$ and $Z = Z' \times (0, h(z'))$, which is the reference cell in $\mathbb{R}^3$, see Figure 2. We define the boundaries $\widehat{\Gamma}_0 = Z' \times \{0\}$, $\widehat{\Gamma}_1 = Z' \times \{h(z')\}$.

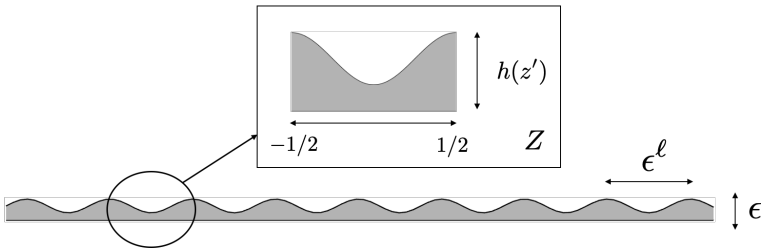

Figure 2: Domain Ω^ϵ in 2D and the reference cell Z in 2D

Applying a dilatation in the vertical variable, i.e. $z_3 = x_3/\epsilon$, we define the following rescaled sets

$$\begin{gathered}\widetilde{\Omega}^\epsilon = \{(x', z_3) \in \mathbb{R}^2 \times \mathbb{R} \,:\, x' \in \omega,\ 0 < z_3 < h(x'/\epsilon^\ell)\},\\ \widetilde{\Gamma}_1^\epsilon = \{(x', z_3) \in \mathbb{R}^2 \times \mathbb{R} \,:\, x' \in \omega,\ z_3 = h(x'/\epsilon^\ell)\} \quad \widetilde{\Gamma}_{\text{lat}}^\epsilon = \partial\widetilde{\Omega}_\epsilon \setminus (\widetilde{\Gamma}_0 \cup \widetilde{\Gamma}_1^\epsilon).\end{gathered} \tag{2.11}$$

The quantity $h_{\max}$ allows us to define:

- The extended sets $Q_\epsilon = \omega \times (0, \epsilon h_{\max})$, $\Omega = \omega \times (0, h_{\max})$ and $\Gamma_1 = \omega \times \{h_{\max}\}$.
- The extended cube $\widetilde{Q}_{k',\varepsilon} = Z'_{k',\varepsilon} \times (0, h_{\max})$ for $k' \in \mathbb{Z}^2$.
- The extended and restricted basic cell $\Pi = Z' \times (0, h_{\max})$.

We denote by C a generic constant which can change from line to line. Moreover, O_ϵ denotes a generic quantity, which can change from line to line, devoted to tend to zero when $\epsilon \to 0$.

We use the following notation for the partial differential operators

$$\Delta\varphi = \Delta_{x'}\varphi + \partial^2_{x_3}\varphi, \quad \text{div}(\varphi) = \text{div}_{x'}(\varphi') + \partial_{x_3}\varphi_3,$$

$$\text{rot}(\varphi) = (\partial_{x_2}\varphi_3 - \partial_{x_3}\varphi_2, -\partial_{x_1}\varphi_3 + \partial_{x_3}\varphi_1, \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1)^t,$$

where $\varphi = (\varphi', \varphi_3)$ with $\varphi' = (\varphi_1, \varphi_2)$, is a vector function defined in Ω^ϵ.

Finally, we introduce some functional spaces. L^2_0 is the space of functions of L^2 with zero mean value. Let $C^\infty_\#(Z)$ be the space of infinitely differentiable functions in $\mathbb{R}^3$ that are Z'-periodic. By $L^2_\#(Z)$ (resp. $H^1_\#(Z)$), we denote its completion in the norm $L^2(Z)$ (resp. $H^1(Z)$) and by $L^2_{0,\#}(Z)$ the space of functions in $L^2_\#(Z)$ with zero mean value.

2.2 Micropolar equations with non-standard boundary conditions

In this paper, we consider the 3D stationary linearized micropolar equations in dimensionless form by setting

$$\mathbf{u}_\epsilon = (\mathbf{u}'_\epsilon(x), u_{3,\epsilon}(x)), \quad \mathbf{w}_\epsilon = (\mathbf{w}'_\epsilon(x), w_{3,\epsilon}(x)), \quad p_\epsilon = p_\epsilon(x),$$

at a point $x \in \Omega_\epsilon$, which is given by

$$\begin{cases} -\Delta\mathbf{u}_\epsilon + \nabla p_\epsilon = 2N^2\text{rot}(\mathbf{w}_\epsilon) & \text{in } \Omega^\epsilon,\\ \text{div}(\mathbf{u}^\epsilon) = 0 & \text{in } \Omega^\epsilon,\\ -R_M\Delta\mathbf{w}_\epsilon + 4N^2\mathbf{w}_\epsilon = 2N^2\text{rot}(\mathbf{u}_\epsilon) & \text{in } \Omega^\epsilon. \end{cases} \tag{2.12}$$

We complete the above system with the following boundary conditions on the top

$$\mathbf{u}_\epsilon = 0, \quad \mathbf{w}_\epsilon = 0 \quad \text{on } \Gamma_1^\epsilon, \tag{2.13}$$

the following boundary condition on the lateral boundary

$$\mathbf{u}_\epsilon = \mathbf{g}_\epsilon, \quad \mathbf{w}_\epsilon = 0 \quad \text{on } \Gamma_{\text{lat}}^\epsilon, \quad \text{with} \quad \int_{\Gamma_{\text{lat}}^\epsilon} \mathbf{g}_\epsilon \cdot \mathbf{n}\, d\sigma = 0, \tag{2.14}$$

and the non-standard boundary conditions on the bottom

$$\begin{aligned} \mathbf{w}_\epsilon \times \mathbf{n} &= \frac{\alpha}{2}\text{rot}(\mathbf{u}_\epsilon) \times \mathbf{n} \quad \text{on } \Gamma_0, \\ R_M \text{rot}(\mathbf{w}_\epsilon) \times \mathbf{n} &= 2N^2\beta(\mathbf{u}_\epsilon - \mathbf{s}) \times \mathbf{n} \quad \text{on } \Gamma_0, \\ \mathbf{u}_\epsilon \cdot \mathbf{n} = 0, &\quad \mathbf{w}_\epsilon \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_0. \end{aligned} \tag{2.15}$$

Here, $\mathbf{s} = (\mathbf{s}', 0)$ with $\mathbf{s}' = (s_1, s_2)$, stands for the characteristic velocity of the moving surface. Dimensionless (non-Newtonian) parameter N^2 characterizes the coupling between the equations for the velocity and microrotation and it is of order $O(1)$. The second dimensionless parameter, denoted by R_M is, in fact, related to the characteristic length of the microrotation effects and will be compared with small parameter ϵ (see condition 2.20). Parameter α characterizes the microrotation on the solid surfaces (see [3] for more details). The coefficient β allows the control of the slippage at the wall when the value $\mathbf{u}_\epsilon - \mathbf{s}$ is not zero. Finally, the $\mathbf{n}$ is the outside normal vector to the boundary (observe that in the case of Γ_0 it holds $\mathbf{n} = -\mathbf{e}_3$).

In order to study the asymptotic behavior of the solution of system (2.12)–(2.15), we also need to assume a certain regularity on the boundary data g_ϵ, and uniform estimates of relevant norms. A very general way of stating those properties is the following: there exists a sequence of lift functions $\mathbf{J}_\epsilon \in H^1(\Omega_\epsilon)^2$ satisfying $\text{div}(\mathbf{J}_\epsilon) = 0$ in Ω_ϵ, the boundary conditions

$$\mathbf{J}_\epsilon = 0 \text{ on } \Gamma_1^\epsilon, \quad \mathrm{J}_\epsilon = \mathbf{g}_\epsilon \text{ on } \Gamma_{\text{lat}}^\epsilon, \quad \mathbf{J}_\epsilon \cdot \mathbf{n} = 0 \text{ on } \Gamma_0, \tag{2.16}$$

and the estimates for every $\epsilon > 0$

$$\|\mathbf{J}_\epsilon\|_{L^2(\Omega_\epsilon)^3} \leq C\epsilon^{\frac{1}{2}}, \quad \|D\mathbf{J}_\epsilon\|_{L^2(\Omega_\epsilon)^{3\times 3}} \leq C\epsilon^{-\frac{1}{2}}, \quad \|\mathbf{J}_\epsilon\|_{L^2(\Gamma_0)^3} \leq C, \tag{2.17}$$

where $C > 0$ is a universal constant independent of ϵ.

Remark 2.1. *One typical construction of a boundary data g_ϵ and the associate lift function $\mathbf{J}_\epsilon$ is the following, see [3]. Consider a regular vector field $\widetilde{\mathbf{J}} \in H^1(\widetilde{\Omega}^\epsilon)^3$, satisfying*

$$\text{div}(\widetilde{\mathbf{J}}) = 0 \ \text{ in } \ \widetilde{\Omega}_\epsilon, \quad \widetilde{\mathbf{J}} = 0 \ \text{ on } \ \widetilde{\Gamma}_1^\epsilon, \quad \widetilde{\mathbf{J}} \cdot \mathbf{n} = 0 \ \text{ on } \ \Gamma_0.$$

Extending $\widetilde{\mathbf{J}}$ by zero outside of $\widetilde{\Omega}^\epsilon$, we can define $\mathbf{J}_\epsilon \in H^1(\Omega^\epsilon)^3$ by

$$\mathbf{J}_\epsilon(x) = (\widetilde{\mathbf{J}}'(x', x_3/\epsilon)), \epsilon\widetilde{J}_3(x', x_3/\epsilon)) \quad \forall\, x \in \widetilde{\Omega}_\epsilon.$$

Note that only normal component of the velocity is known on Γ_0 and is zero, while the tangential velocity is not given. Nevertheless, we can choose an artificial value $\mathbf{g} = (\mathbf{g}', 0)$, with $\mathbf{g}' = (g_1, g_2)$, of the velocity on Γ_0. We choose it in a way that function $\xi \in H^{\frac{1}{2}}(\partial\Omega^\epsilon)^3$ defined by

$$\xi = \begin{cases} 0 & \text{on } \Gamma_1^\epsilon, \\ \mathbf{g}_\epsilon & \text{on } \Gamma_{\text{lat}}^\epsilon, \\ (\mathbf{g}', 0) & \text{on } \Gamma_0, \end{cases}$$

satisfying $\int_{\partial\Omega^\epsilon} \xi \cdot \mathbf{n}\, d\sigma = 0$.

Then, by the change of variables $(x', x_3) = (x', \epsilon z_3)$*, it holds*

$$\int_{\Omega^\epsilon} |D\mathbf{J}_\epsilon|^2\, dx = \epsilon \int_{\widetilde{\Omega}^\epsilon} \left(|D_{x'}\widetilde{\mathbf{J}}'|^2 + \frac{1}{\epsilon^2}|\partial_{z_3}\widetilde{\mathbf{J}}'|^2 + \epsilon^2|\nabla_{x'}\widetilde{J}_3|^2 + |\partial_{z_3}\widetilde{J}_3|^2 \right) dx' dz_3,$$
$$\int_{\Omega^\epsilon} |\mathbf{J}_\epsilon|^2\, dx = \epsilon \int_{\widetilde{\Omega}^\epsilon} (|\widetilde{\mathbf{J}}'|^2 + \epsilon^2|\widetilde{J}_3|^2)\, dx' dz_3,$$

so that $\mathbf{J}_\epsilon$ *satisfies all the required properties (2.16) and (2.17).*

Next, we are able to give the problem with homogeneous boundary conditions on the lateral and top boundaries. Introducing

$$\mathbf{v}_\epsilon = \mathbf{u}_\epsilon - \mathbf{J}_\epsilon,$$

system (2.12)–(2.15) can be rewritten as follows

$$\begin{cases} -\Delta(\mathbf{v}_\epsilon + \mathbf{J}_\epsilon) + \nabla p_\epsilon = 2N^2 \mathrm{rot}(\mathbf{w}_\epsilon) & \text{in } \Omega^\epsilon, \\ \mathrm{div}(\mathbf{v}^\epsilon) = 0 & \text{in } \Omega^\epsilon, \\ -R_M \Delta \mathbf{w}_\epsilon + 4N^2 \mathbf{w}_\epsilon = 2N^2 \left(\mathrm{rot}(\mathbf{v}_\epsilon) + \mathrm{rot}(\mathbf{J}_\epsilon)\right) & \text{in } \Omega^\epsilon, \end{cases} \tag{2.18}$$

with the following boundary conditions

$$\begin{aligned} \mathbf{v}_\epsilon = 0, \quad \mathbf{w}_\epsilon = 0 \quad & \text{on } \Gamma_1^\epsilon \cup \Gamma_{\mathrm{lat}}^\epsilon, \\ \mathbf{w}_\epsilon \times \mathbf{n} = \frac{\alpha}{2} \left(\mathrm{rot}(\mathbf{v}_\epsilon) + \mathrm{rot}(\mathbf{J}_\epsilon)\right) \times \mathbf{n} \quad & \text{on } \Gamma_0, \\ R_M \mathrm{rot}(\mathbf{w}_\epsilon) \times \mathbf{n} = 2N^2 \beta (\mathbf{v}_\epsilon + \mathbf{g} - \mathbf{s}) \times \mathbf{n} \quad & \text{on } \Gamma_0, \\ \mathbf{v}_\epsilon \cdot \mathbf{n} = 0, \quad \mathbf{w}_\epsilon \cdot \mathbf{n} = 0 \quad & \text{on } \Gamma_0. \end{aligned} \tag{2.19}$$

In the next subsection, we recall the existence and uniqueness result of weak solution of problem (2.18)–(2.19).

2.3 Mathematical study

In this paper, the goal is to derive an effective model describing by using rigorous asymptotic analysis with respect to the small parameter ϵ. In particular, we will focus on detecting the effects of the roughness together with the effects of non-standard boundary conditions. We remark that different asymptotic behaviors of the flow can be deduced depending on the order of magnitude of the dimensionless parameter. Indeed, if we compare the characteristic number R_M with small parameter ϵ, three different asymptotic situations can be formally identifed (see e.g. [7, 9]). We consider the most interesting one, which leads to a strong coupling at main order, namely the regime

$$R_M = \epsilon^2 R_c, \quad R_c > 0. \tag{2.20}$$

Let us start by defining the notion of weak solution to system (2.18)–(2.19). Due to the boundary conditions, we introduce the following functional spaces

$$V^\epsilon = \{\varphi \in H^1(\Omega^\epsilon)^3 \,:\, \varphi = 0 \quad \text{in } \Gamma_1^\epsilon \cup \Gamma_{\mathrm{lat}}^\epsilon, \quad \varphi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_0\}, \quad V_0^\epsilon = \{\varphi \in V^\epsilon \,:\, \mathrm{div}(\varphi) = 0\},$$

equipped with the norm of $\|D\varphi\|_{L^2(\Omega^\epsilon)^{3\times 3}}$, and the space

$$L_0^2(\Omega^\epsilon) = \left\{ q \in L^2(\Omega^\epsilon) \,:\, \int_{\Omega_\epsilon} q\, dx = 0 \right\},$$

equipped with the norm of $L^2(\Omega^\epsilon)$. Observe that it holds the following identities, which will be important in the following:

$$-\Delta\varphi = \mathrm{rot}(\mathrm{rot}(\varphi)) - \nabla\mathrm{div}(\varphi) \quad \forall\varphi \in \mathcal{D}(\Omega^\epsilon)^3, \tag{2.21}$$

$$\int_{\Omega^\epsilon} \mathrm{rot}(\psi)\cdot\varphi\,dx = \int_{\Omega^\epsilon} \mathrm{rot}(\varphi)\cdot\psi\,dx - \int_{\Gamma_0} (\psi\times\mathbf{n})\cdot\varphi\,d\sigma(x') \quad \forall(\varphi,\psi)\in H^1(\Omega^\epsilon)^3\times H^1(\Omega^\epsilon)^3. \tag{2.22}$$

Proposition 2.2 (Theorem 2.1 in [3])**.** *Sufficiently regular solutions of (2.18)–(2.19) satisfy the weak formulation:*

Find $(\mathbf{v}_\epsilon, w_\epsilon, p_\epsilon)\in V_0^\epsilon\times V^\epsilon\times L^2_0(\Omega^\epsilon)$ *such that*

$$\int_{\Omega_\epsilon}\mathrm{rot}(\mathbf{v}_\epsilon)\cdot\mathrm{rot}(\varphi)\,dx - \int_\Omega p_\epsilon\,\mathrm{div}(\varphi)\,dx - 2N^2\int_{\Omega_\epsilon}\mathrm{rot}(\varphi)\cdot\mathbf{w}_\epsilon\,dx \tag{2.23}$$

$$+2\left(N^2-\frac{1}{\alpha}\right)\int_{\Gamma_0}(\mathbf{w}_\epsilon\times\mathbf{n})\cdot\varphi\,d\sigma(x') = -\int_{\Omega_\epsilon}\mathrm{rot}(\mathbf{J}_\epsilon)\cdot\mathrm{rot}(\varphi)\,dx \qquad \forall\varphi\in V^\epsilon,$$

$$\epsilon^2R_c\int_{\Omega_\epsilon}\mathrm{rot}(\mathbf{w}_\epsilon)\cdot\mathrm{rot}(\psi)\,dx + \epsilon^2R_c\int_{\Omega_\epsilon}\mathrm{div}(\mathbf{w}_\epsilon)\cdot\mathrm{div}(\psi)\,dx + 4N^2\int_{\Omega_\epsilon}\mathbf{w}_\epsilon\cdot\psi\,dx$$

$$-2N^2\int_{\Omega_\epsilon}\mathbf{v}_\epsilon\cdot\mathrm{rot}(\psi)\,dx - 2N^2(\beta-1)\int_{\Gamma_0}(\mathbf{v}_\epsilon\times\mathbf{n})\cdot\psi\,d\sigma(x') = 2N^2\int_{\Omega_\epsilon}\mathbf{J}_\epsilon\cdot\mathrm{rot}(\psi)\,dx \tag{2.24}$$

$$+2N^2\beta\int_{\Gamma_0}(\mathbf{g}-\mathbf{s})\times\mathbf{n}\cdot\psi\,d\sigma(x') - 2N^2\int_{\Gamma_0}(\mathbf{g}\times\mathbf{n})\cdot\psi\,d\sigma(x') \qquad \forall\psi\in V^\epsilon.$$

Proof. The derivation of the variational formulations (2.23) and (2.24) is given in [3, Theorem 2.1]. Here, we reproduce some steps:

– Multiplying $(2.18)_1$ by $\varphi\in V^\epsilon$, using (2.21) and (2.22), we deduce

$$\int_{\Omega_\epsilon}\mathrm{rot}(\mathbf{v}_\epsilon)\cdot\mathrm{rot}(\varphi)\,dx - \int_\Omega p_\epsilon\,\mathrm{div}(\varphi)\,dx - 2N^2\int_{\Omega_\epsilon}\mathrm{rot}(\varphi)\cdot\mathbf{w}_\epsilon\,dx \tag{2.25}$$

$$-\int_{\Gamma_0}(\mathrm{rot}(\mathbf{v}_\epsilon)\times\mathbf{n})\cdot\varphi\,d\sigma(x') - \int_{\Gamma_0}(\mathrm{rot}(\mathbf{J}_\epsilon)\times\mathbf{n})\cdot\varphi\,d\sigma(x')$$

$$= -2N^2\int_{\Omega^\epsilon}(\mathbf{w}_\epsilon\times\mathbf{n})\cdot\varphi\,d\sigma(x') - \int_{\Omega^\epsilon}\mathrm{rot}(\mathbf{J}_\epsilon)\cdot\mathrm{rot}(\varphi)\,dx.$$

Taking into account the boundary condition $(2.19)_2$, we obtain (2.23).

– Multiplying $(2.18)_3$ by $\psi\in V^\epsilon$, using (2.21) and (2.22), we get

$$\epsilon^2R_c\int_{\Omega_\epsilon}\mathrm{rot}(\mathbf{w}_\epsilon)\cdot\mathrm{rot}(\psi)\,dx + \epsilon^2R_c\int_{\Omega_\epsilon}\mathrm{div}(\mathbf{w}_\epsilon)\cdot\mathrm{div}(\psi)\,dx + 4N^2\int_{\Omega_\epsilon}\mathbf{w}_\epsilon\cdot\psi\,dx$$

$$-\epsilon^2R_c\int_{\Gamma_0}((\mathrm{rot}(\mathbf{w}_\epsilon)\times\mathbf{n})\cdot\psi\,d\sigma(x') - 2N^2\int_{\Omega_\epsilon}\mathbf{v}_\epsilon\cdot\mathrm{rot}(\psi)\,dx + 2N^2\int_{\Gamma_0}(\mathbf{v}_\epsilon\times\mathbf{n})\cdot\psi\,d\sigma(x') \tag{2.26}$$

$$= 2N^2\int_{\Omega_\epsilon}\mathbf{J}_\epsilon\cdot\mathrm{rot}(\psi)\,dx - 2N^2\int_{\Gamma_0}(\mathbf{g}\times\mathbf{n})\cdot\psi\,d\sigma(x').$$

Taking into account the boundary condition $(2.19)_3$, we obtain (2.24).

□

To finish this section, we give the existence and uniqueness result, whos proof can be found in [3, Theorem 2.2] taking into account the rescaling of $R_M = \epsilon^2 R_c$ and that the maximum of the function h_ϵ is $\epsilon^2 h_{\max}$.

Theorem 2.3. *Under the previous assumptions and assuming that α and β satisfy condition*

$$\gamma^2 = \left|\frac{1}{\alpha} - N^2 - N^2\beta\right|^2 < \frac{R_c}{h_{\max}^2}(1-N^2), \tag{2.27}$$

then, for every $\epsilon > 0$, there exists a unique weak solution $(\mathbf{v}_\epsilon, \mathbf{w}_\epsilon, p_\epsilon) \in V_0^\epsilon \times V^\epsilon \times L_0^2(\Omega^\epsilon)$ of (2.23)–(2.24).

3 Rescaling

To study the asymptotic behavior of the solutions $\mathbf{u}_\epsilon$, $\mathbf{w}_\epsilon$ and p_ϵ when ϵ tends to zero, we need to have the solutions in a domain with fixed height. To do this, we use the dilatation in the variable x_3 given by

$$z_3 = \frac{x_3}{\epsilon}\,, \tag{3.28}$$

in order to have the functions defined in $\widetilde{\Omega}_\epsilon$ and boundaries $\widetilde{\Gamma}_1^\epsilon$ and $\widetilde{\Gamma}_{\text{lat}}^\epsilon$ given in (2.11).

Using the change of variables (3.28) in (2.18)–(2.19), we obtain the rescaled system

$$\begin{cases} -\Delta_\epsilon(\widetilde{\mathbf{v}}_\epsilon + \widetilde{\mathbf{J}}_\epsilon) + \nabla_\epsilon \widetilde{p}_\epsilon = 2N^2 \text{rot}_\epsilon(\widetilde{\mathbf{w}}_\epsilon) & \text{in } \widetilde{\Omega}_\epsilon, \\ \text{div}(\widetilde{\mathbf{v}}^\epsilon) = 0 & \text{in } \widetilde{\Omega}_\epsilon, \\ -\epsilon^2 R_c \Delta_\epsilon \widetilde{\mathbf{w}}_\epsilon + 4N^2 \widetilde{\mathbf{w}}_\epsilon = 2N^2 \left(\text{rot}_\epsilon(\widetilde{\mathbf{u}}_\epsilon) + \text{rot}_\epsilon(\widetilde{\mathbf{J}}_\epsilon)\right) & \text{in } \widetilde{\Omega}_\epsilon, \end{cases} \tag{3.29}$$

with the rescaled boundary conditions

$$\begin{aligned} \widetilde{\mathbf{v}}_\epsilon = 0, \quad \widetilde{\mathbf{w}}_\epsilon = 0 \quad & \text{on } \widetilde{\Gamma}_1^\epsilon \cup \widetilde{\Gamma}_{\text{lat}}^\epsilon, \\ \widetilde{\mathbf{w}}_\epsilon \times \mathbf{n} = \frac{\alpha}{2}\left(\text{rot}_\epsilon(\widetilde{\mathbf{v}}_\epsilon) + \text{rot}(\widetilde{\mathbf{J}}_\epsilon)\right) \times \mathbf{n} \quad & \text{on } \Gamma_0, \\ \epsilon^2 R_c \,\text{rot}_\epsilon(\widetilde{\mathbf{w}}_\epsilon) \times \mathbf{n} = 2N^2\beta(\widetilde{\mathbf{v}}_\epsilon + \mathbf{g} - \mathbf{s}) \times \mathbf{n} \quad & \text{on } \Gamma_0, \\ \widetilde{\mathbf{v}}_\epsilon \cdot \mathbf{n} = 0, \quad \widetilde{\mathbf{w}}_\epsilon \cdot \mathbf{n} = 0 \quad & \text{on } \Gamma_0, \end{aligned} \tag{3.30}$$

where the unknown functions in the above system are given by $\widetilde{\mathbf{u}}_\epsilon(x', z_3) = \mathbf{u}^\epsilon(x', \epsilon z_3)$, $\widetilde{p}_\epsilon(x', z_3) = p_\epsilon(x', \epsilon z_3)$, $\widetilde{\mathbf{w}}_\epsilon(x', z_3) = \mathbf{w}_\epsilon(x', \epsilon z_3)$ and $\widetilde{\mathbf{J}}_\epsilon(x', z_3) = (\widetilde{\mathbf{J}}'(x', z_3), \epsilon J_3(x', z_3))$ for a.e. $(x', z_3) \in \widetilde{\Omega}^\epsilon$. Here, for a vectorial function $\widetilde{\varphi} = (\widetilde{\varphi}', \widetilde{\varphi}_3)$ defined in $\widetilde{\Omega}_\epsilon$, the rescaled operators are given by

$$\begin{aligned} \Delta_\epsilon \widetilde{\varphi} = \Delta_{x'}\widetilde{\varphi} + \frac{1}{\epsilon^2}\partial_{z_3}^2 \widetilde{\varphi}, \quad \text{div}_\epsilon(\widetilde{\varphi}) = \text{div}_{x'}(\widetilde{\varphi}') + \frac{1}{\epsilon}\partial_{z_3}\widetilde{\varphi}_3, \\ \text{rot}_\epsilon(\widetilde{\varphi}) = \left(\text{rot}_{x'}(\widetilde{\varphi}_3) + \frac{1}{\epsilon}\text{rot}_{z_3}(\widetilde{\varphi}'), \text{Rot}_{x'}(\widetilde{\varphi}')\right)^t, \end{aligned} \tag{3.31}$$

where, denoting by

$$(\widetilde{\varphi}')^\perp = (-\widetilde{\varphi}_2, \widetilde{\varphi}_1)^t, \tag{3.32}$$

we define

$$\text{rot}_{x'}(\widetilde{\varphi}_3) = (\partial_{x_2}\widetilde{\varphi}_3, -\partial_{x_1}\widetilde{\varphi}_3)^t, \quad \text{rot}_{z_3}(\widetilde{\varphi}') = \partial_{z_3}(\widetilde{\varphi}')^\perp, \quad \text{Rot}_{x'}(\widetilde{\varphi}') = \partial_{x_1}\widetilde{\varphi}_2 - \partial_{x_2}\widetilde{\varphi}_1. \tag{3.33}$$

Moreover, from estimates (2.16), we also have

$$\|\widetilde{\mathbf{J}}_\epsilon\|_{L^2(\widetilde{\Omega}_\epsilon)^3}\leq C,\quad \|D_\epsilon\widetilde{\mathbf{J}}_\epsilon\|_{L^2(\widetilde{\Omega}_\epsilon)^{3\times 3}}\leq C\epsilon^{-1},\quad \|\widetilde{\mathbf{J}}_\epsilon\|_{L^2(\Gamma_0)^3}\leq C. \tag{3.34}$$

In addition, the rescaled variational formulation in $\widetilde{\Omega}_\epsilon$ is the following:

Find $(\widetilde{\mathbf{v}}_\epsilon,\widetilde{\mathbf{w}}_\epsilon,\widetilde{p}_\epsilon)\in\widetilde{V}_0^\epsilon\times\widetilde{V}^\epsilon\times L_0^2(\widetilde{\Omega}_\epsilon)$ such that

$$\begin{aligned}
&\epsilon\int_{\widetilde{\Omega}_\epsilon}\mathrm{rot}_\epsilon(\widetilde{\mathbf{v}}_\epsilon)\cdot\mathrm{rot}_\epsilon(\widetilde{\varphi})\,dx'dz_3-\epsilon\int_{\widetilde{\Omega}_\epsilon}\widetilde{p}_\epsilon\,\mathrm{div}_\epsilon(\widetilde{\varphi})\,dx'dz_3-2N^2\epsilon\int_{\widetilde{\Omega}_\epsilon}\mathrm{rot}_\epsilon(\widetilde{\varphi})\cdot\widetilde{\mathbf{w}}_\epsilon\,dx'dz_3\\
&+2\left(N^2-\frac{1}{\alpha}\right)\int_{\Gamma_0}(\widetilde{\mathbf{w}}_\epsilon\times\mathbf{n})\cdot\widetilde{\varphi}\,d\sigma(x')=-\epsilon\int_{\widetilde{\Omega}_\epsilon}\mathrm{rot}_\epsilon(\widetilde{\mathbf{J}}_\epsilon)\cdot\mathrm{rot}_\epsilon(\widetilde{\varphi})\,dx'dz_3,\\
&\epsilon^3R_c\int_{\widetilde{\Omega}_\epsilon}\mathrm{rot}_\epsilon(\widetilde{\mathbf{w}}_\epsilon)\cdot\mathrm{rot}_\epsilon(\widetilde{\psi})\,dx'dz_3+\epsilon^3R_c\int_{\widetilde{\Omega}_\epsilon}\mathrm{div}_\epsilon(\widetilde{\mathbf{w}}_\epsilon)\cdot\mathrm{div}_\epsilon(\widetilde{\psi})\,dx'dz_3+4N^2\epsilon\int_{\widetilde{\Omega}_\epsilon}\widetilde{\mathbf{w}}_\epsilon\cdot\widetilde{\psi}\,dx'dz_3\\
&-2N^2\epsilon\int_{\widetilde{\Omega}_\epsilon}\widetilde{\mathbf{v}}_\epsilon\cdot\mathrm{rot}_\epsilon(\widetilde{\psi})\,dx'dz_3-2N^2(\beta-1)\int_{\Gamma_0}(\widetilde{\mathbf{v}}_\epsilon\times\mathbf{n})\cdot\widetilde{\psi}\,d\sigma(x')\\
&=2N^2\epsilon\int_{\widetilde{\Omega}_\epsilon}\widetilde{\mathbf{J}}_\epsilon\cdot\mathrm{rot}_\epsilon(\widetilde{\psi})\,dx'dz_3+2N^2\beta\int_{\Gamma_0}(\mathbf{g}-\mathbf{s})\times\mathbf{n}\cdot\widetilde{\psi}\,d\sigma(x')-2N^2\int_{\Gamma_0}(\mathbf{g}\times\mathbf{n})\cdot\widetilde{\psi}\,d\sigma(x'),
\end{aligned}\tag{3.35}$$

for every $(\widetilde{\varphi},\widetilde{\psi})\in\widetilde{V}^\epsilon\times\widetilde{V}^\epsilon$ and $\widetilde{q}\in L^2(\widetilde{\Omega}_\epsilon)$ obtained from (φ,ψ,q) by using the change of variables (3.28), where

$$\widetilde{V}^\epsilon=\{\widetilde{\varphi}\in H^1(\widetilde{\Omega}_\epsilon)^2\,:\,\widetilde{\varphi}=0\quad\text{in }\widetilde{\Gamma}_1^\epsilon\cup\widetilde{\Gamma}_{\mathrm{lat}}^\epsilon,\quad\widetilde{\varphi}\cdot\mathbf{n}=0\quad\text{on }\Gamma_0\},\quad\widetilde{V}_0^\epsilon=\{\widetilde{\varphi}\in\widetilde{V}^\epsilon\,:\,\mathrm{div}_\epsilon(\widetilde{\varphi})=0\}.$$

In the next section, we establish the *a priori estimates* for the rescaled unknown in an ϵ-independent domain.

4 *A priori* estimates and convergences

This section is devoted to derive the *a priori* estimates of the unknowns and is divided in three parts. First, we deduce the *a priori* estimates for velocity and microrotation in an ϵ-independent domain, i.e. the domain Ω defined in Section 2. Next, we extend the pressure to Ω and derive its corresponding *a priori* estimates. Finally, from previos *a priori* estimates, we deduce some convergence results.

4.1 Estimates for velocity and microrororation

To derive the desired estimates, let us recall and proof some well-known technical results (see for instance [1]).

Lemma 4.1 (Poincaré's inequality)**.** *For all $\varphi\in V^\epsilon$, there holds the following inequality*

$$\|\varphi\|_{L^2(\Omega^\epsilon)^3}\leq C\epsilon\|D\varphi\|_{L^2(\Omega^\epsilon)^{3\times 3}}. \tag{4.36}$$

Moreover, from the change of variables (3.28), there holds for all $\widetilde{\varphi}\in\widetilde{V}^\epsilon$ the following rescaled estimate

$$\|\widetilde{\varphi}\|_{L^2(\widetilde{\Omega}^\epsilon)^3}\leq C\epsilon\|D_\epsilon\widetilde{\varphi}\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}. \tag{4.37}$$

Lemma 4.2. *For all $\varphi\in V^\epsilon$, it holds*

$$\|D\varphi\|^2_{L^2(\Omega^\epsilon)^{3\times 3}}=\|\mathrm{rot}(\varphi)\|^2_{L^2(\Omega^\epsilon)^3}+\|\mathrm{div}(\varphi)\|^2_{L^2(\Omega^\epsilon)},\quad\|\mathrm{rot}(\varphi)\|^2_{L^2(\Omega^\epsilon)^3}\leq\|D\varphi\|^2_{L^2(\Omega^\epsilon)^{3\times 3}}, \tag{4.38}$$

and, if moreover, $\mathrm{div}(\varphi)=0$ *in Ω^ϵ, then*

$$\|\mathrm{rot}(\varphi)\|^2_{L^2(\Omega_\epsilon)^3}=\|D\varphi\|^2_{L^2(\Omega_\epsilon)^{3\times 3}}. \tag{4.39}$$

Moreover, from the change of variables (3.28), we have for all $\widetilde{\varphi} \in \widetilde{V}^\epsilon$ *that*

$$\|D_\epsilon\widetilde{\varphi}\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} = \|\mathrm{rot}_\epsilon(\widetilde{\varphi})\|^2_{L^2(\widetilde{\Omega}^\epsilon)^3} + \|\mathrm{div}_\epsilon(\widetilde{\varphi})\|^2_{L^2(\widetilde{\Omega}^\epsilon)}, \quad \|\mathrm{rot}_\epsilon(\widetilde{\varphi})\|^2_{L^2(\widetilde{\Omega}^\epsilon)^3} \leq \|D_\epsilon\widetilde{\varphi}\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}, \tag{4.40}$$

and if moreover, $\mathrm{div}_\epsilon(\widetilde{\varphi}) = 0$ *in* $\widetilde{\Omega}^\epsilon$*, then*

$$\|\mathrm{rot}_\epsilon\widetilde{\varphi}\|^2_{L^2(\widetilde{\Omega}_\epsilon)^3} = \|D_\epsilon\widetilde{\varphi}\|^2_{L^2(\widetilde{\Omega}_\epsilon)^{3\times 3}}. \tag{4.41}$$

Proof. For all $\varphi \in V^\epsilon$, it holds (see for instance [18] formula (IV.23))

$$\int_{\Omega^\epsilon} (|\mathrm{div}(\varphi)|^2 + |\mathrm{rot}(\varphi)|^2)\, dx = \int_{\Omega^\epsilon} |D\varphi|^2\, dx + \int_{\Gamma_0} ((\varphi\cdot\nabla)\mathbf{n})\cdot\varphi\, d\sigma(x').$$

Since Γ_0 is flat, then the term $\int_{\Gamma_0}((\varphi\cdot\nabla)\mathbf{n})\cdot\varphi\, d\sigma(x')$ vanishes, ans so we get (4.38). Then, (4.39) is a consequence of the free divergence condition.

Finally, the change of variables (3.28) applied to (4.38) and (4.39) implies (4.40) and (4.41), respectively, for any $\widetilde{\varphi}$ obtained from φ by the change of variables (3.28).

□

Lemma 4.3 (Trace estimates)**.** *For all* $\widetilde{\varphi} \in \widetilde{V}^\epsilon$*, the following inequalities hold*

$$\|\widetilde{\varphi}'\|_{L^2(\Gamma_0)^2} \leq \epsilon h_{\max}^{\frac{1}{2}} \|D_\epsilon\widetilde{\varphi}\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}, \quad \|\widetilde{\varphi}\times\mathbf{n}\|_{L^2(\Gamma_0)^2} \leq \epsilon h_{\max}^{\frac{1}{2}} \|D_\epsilon\widetilde{\varphi}\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}. \tag{4.42}$$

Proof. Thank to $\widetilde{\varphi}(x', h(x'/\epsilon^\ell)) = 0$ in ω, we have that

$$\int_{\Gamma_0} |\widetilde{\varphi}|^2 d\sigma = \int_\omega |\widetilde{\varphi}(x',0)|^2\, dx' = \int_\omega \left| \int_0^{h(x'/\epsilon^\ell)} \partial_{z_3}\widetilde{\varphi}(x', z_3)\, dz_3 \right|^2 dx' \leq h_{\max} \int_{\widetilde{\Omega}^\epsilon} |\partial_{z_3}\widetilde{\varphi}|^2\, dx' dz_3,$$

that is,

$$\|\widetilde{\varphi}'\|_{L^2(\Gamma_0)^2} \leq h_{\max}^{\frac{1}{2}} \|\partial_{z_3}\widetilde{\varphi}'\|_{L^2(\widetilde{\Omega}^\epsilon)^2},$$

which implies $(4.42)_1$. Since

$$\|\widetilde{\varphi}\times\mathbf{n}\|_{L^2(\Gamma_0)^3} = \|(-\widetilde{\varphi}_2, \widetilde{\varphi}_1, 0)\|_{L^2(\Gamma_0)^3},$$

from $(4.42)_1$, we deduce $(4.42)_2$.

□

Lemma 4.4 (A priori estimates)**.** *Under hypothesis (2.27), the following estimates hold for the solution* $(\widetilde{\mathbf{v}}_\epsilon, \widetilde{\mathbf{w}}_\epsilon)$ *of problem (3.35):*

$$\|\widetilde{\mathbf{v}}^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^3} \leq C, \quad \|D_\epsilon\widetilde{\mathbf{v}}^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} \leq C\epsilon^{-1}, \tag{4.43}$$

$$\|\widetilde{\mathbf{w}}^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^3} \leq C\epsilon^{-1}, \quad \|D_\epsilon\widetilde{\mathbf{w}}^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} \leq C\epsilon^{-2}. \tag{4.44}$$

As consequence, it also holds

$$\|\widetilde{\mathbf{u}}^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^3} \leq C, \quad \|D_\epsilon\widetilde{\mathbf{u}}^\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} \leq C\epsilon^{-1}. \tag{4.45}$$

Proof. The proof of (4.43) and (4.44) is similar to the one given in [3, Lemma 3.1], but here we are in the 3D case, so we will give some remarks of the development. We remark that, once proved estimates (4.43), the proof of (4.45) is straightforward from

$$\widetilde{\mathbf{u}}_\epsilon = \widetilde{\mathbf{v}}_\epsilon + \widetilde{\mathbf{J}}_\epsilon,$$

and estimates of $\widetilde{\mathbf{J}}_\epsilon$ given by (3.34). We will prove (4.43) and (4.44) in 4 steps.

Step 1. Mixed variational formulation. Problem (3.35) can be written in the following mixed variational form:

Find $(\widetilde{\mathbf{v}}_\epsilon, \widetilde{\mathbf{w}}_\epsilon, \widetilde{p}_\epsilon) \in \widetilde{V}^\epsilon \times \widetilde{V}^\epsilon \times L^2_0(\widetilde{\Omega}^\epsilon)$, such that

$$\mathcal{A}^\epsilon((\widetilde{\mathbf{v}}_\epsilon, \widetilde{\mathbf{w}}_\epsilon), (\widetilde{\varphi}, \widetilde{\psi})) + \mathcal{B}^\epsilon((\widetilde{\varphi}, \widetilde{\psi}), \widetilde{p}_\epsilon) = \mathcal{L}(\widetilde{\varphi}, \widetilde{\psi}) \quad \forall\, (\widetilde{\varphi}, \widetilde{\psi}) \in \widetilde{V}^\epsilon \times \widetilde{V}^\epsilon, \tag{4.46}$$

$$\mathcal{B}^\epsilon((\widetilde{\mathbf{v}}_\epsilon, \widetilde{\mathbf{w}}_\epsilon), \widetilde{q}) = 0 \qquad \forall\, \widetilde{q} \in L^2_0(\widetilde{\Omega}^\epsilon), \tag{4.47}$$

with $\mathcal{A}^\epsilon$ defined by

$$\begin{aligned}
\mathcal{A}^\epsilon((\widetilde{\mathbf{v}}_\epsilon, \widetilde{\mathbf{w}}_\epsilon), (\widetilde{\varphi}, \widetilde{\psi})) = \quad & \epsilon \int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{v}}_\epsilon) \cdot \mathrm{rot}_\epsilon(\widetilde{\varphi})\, dx' dz_3 - 2N^2\epsilon \int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\varphi}) \cdot \widetilde{\mathbf{w}}_\epsilon\, dx' dz_3 \\
& + \epsilon^3 R_c \int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{w}}_\epsilon) \cdot \mathrm{rot}_\epsilon(\widetilde{\psi})\, dx' dz_3 + \epsilon^3 R_c \int_{\widetilde{\Omega}_\epsilon} \mathrm{div}_\epsilon(\widetilde{\mathbf{w}}_\epsilon) \cdot \mathrm{div}_\epsilon(\widetilde{\psi})\, dx' dz_3 \\
& + 4N^2\epsilon \int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{w}}_\epsilon \cdot \widetilde{\psi}\, dx' dz_3 - 2N^2\epsilon \int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{v}}_\epsilon \cdot \mathrm{rot}_\epsilon(\widetilde{\psi})\, dx' dz_3 \\
& + 2\left(N^2 - \frac{1}{\alpha}\right) \int_{\Gamma_0} (\widetilde{\mathbf{w}}_\epsilon \times \mathbf{n}) \cdot \widetilde{\varphi}\, d\sigma(x') - 2N^2(\beta - 1) \int_{\Gamma_0} (\widetilde{\mathbf{v}}_\epsilon \times \mathbf{n}) \cdot \widetilde{\psi}\, d\sigma(x'),
\end{aligned} \tag{4.48}$$

with $\mathcal{B}^\epsilon$ given by

$$\mathcal{B}^\epsilon((\widetilde{\varphi}, \widetilde{\psi}), \widetilde{p}_\epsilon) = -\epsilon \int_{\widetilde{\Omega}^\epsilon} \widetilde{p}_\epsilon\, \mathrm{div}_\epsilon(\widetilde{\varphi})\, dx' dz_3, \tag{4.49}$$

and $\mathcal{L}^\epsilon$ given by

$$\mathcal{L}^\epsilon(\widetilde{\varphi}, \widetilde{\psi}) = 2N^2\epsilon \int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{J}}_\epsilon) \cdot \widetilde{\psi}\, dx' dz_3 - \epsilon \int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{J}}_\epsilon) \cdot \mathrm{rot}_\epsilon(\widetilde{\varphi})\, dx' dz_3 + 2N^2\beta \int_{\Gamma_0} (\mathbf{g} - \mathbf{s}) \times \mathbf{n} \cdot \widetilde{\psi}\, d\sigma(x'). \tag{4.50}$$

Taking into account that

$$2\left(N^2 - \frac{1}{\alpha}\right) \int_{\Gamma_0} (\widetilde{\mathbf{w}}_\epsilon \times \mathbf{n}) \cdot \widetilde{\varphi}\, d\sigma = -2\left(N^2 - \frac{1}{\alpha}\right) \int_{\Gamma_0} (\widetilde{\varphi} \times \mathbf{n}) \cdot \widetilde{\mathbf{w}}_\epsilon\, d\sigma(x'),$$

and

$$\begin{aligned}
-2N^2\epsilon \int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{v}}_\epsilon \cdot \mathrm{rot}_\epsilon(\widetilde{\psi})\, dx' dz_3 = \quad & -2N^2\epsilon \int_{\widetilde{\Omega}_\epsilon} \widetilde{\psi} \cdot \mathrm{rot}_\epsilon(\widetilde{\mathbf{v}}_\epsilon)\, dx' dz_3 + 2N^2 \int_{\Gamma_0} (\widetilde{\psi} \times \mathbf{n}) \cdot \widetilde{\mathbf{v}}_\epsilon\, d\sigma(x') \\
= \quad & -2N^2\epsilon \int_{\widetilde{\Omega}_\epsilon} \widetilde{\psi} \cdot \mathrm{rot}_\epsilon(\widetilde{\mathbf{v}}_\epsilon)\, dx' dz_3 - 2N^2 \int_{\Gamma_0} (\widetilde{\mathbf{v}}_\epsilon \times \mathbf{n}) \cdot \widetilde{\psi}\, d\sigma(x'),
\end{aligned}$$

we then give another expression for $\mathcal{A}^\epsilon$ by

$$\begin{aligned}
\mathcal{A}^\epsilon((\widetilde{\mathbf{v}}_\epsilon, \widetilde{\mathbf{w}}_\epsilon), (\widetilde{\varphi}, \widetilde{\psi})) = \quad & \epsilon \int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{v}}_\epsilon) \cdot \mathrm{rot}_\epsilon(\widetilde{\varphi})\, dx' dz_3 - 2N^2\epsilon \int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\varphi}) \cdot \widetilde{\mathbf{w}}_\epsilon\, dx' dz_3 \\
& + \epsilon^3 R_c \int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{w}}_\epsilon) \cdot \mathrm{rot}_\epsilon(\widetilde{\psi})\, dx' dz_3 + \epsilon^3 R_c \int_{\widetilde{\Omega}_\epsilon} \mathrm{div}_\epsilon(\widetilde{\mathbf{w}}_\epsilon) \cdot \mathrm{div}_\epsilon(\widetilde{\psi})\, dx' dz_3 \\
& + 4N^2\epsilon \int_{\widetilde{\Omega}_\epsilon} \widetilde{\mathbf{w}}_\epsilon \cdot \widetilde{\psi}\, dx' dz_3 - 2N^2\epsilon \int_{\widetilde{\Omega}_\epsilon} \widetilde{\psi} \cdot \mathrm{rot}_\epsilon(\widetilde{\mathbf{v}}_\epsilon)\, dx' dz_3 \\
& - 2\left(N^2 - \frac{1}{\alpha}\right) \int_{\Gamma_0} (\widetilde{\varphi} \times \mathbf{n}) \cdot \widetilde{\mathbf{w}}_\epsilon\, d\sigma(x') - 2N^2\beta \int_{\Gamma_0} (\widetilde{\mathbf{v}}_\epsilon \times \mathbf{n}) \cdot \widetilde{\psi}\, d\sigma(x').
\end{aligned} \tag{4.51}$$

Step 2. Estimate $\mathcal{A}^\epsilon((\widetilde{\mathbf{v}}_\epsilon,\widetilde{\mathbf{w}}_\epsilon),(\widetilde{\mathbf{v}}_\epsilon,\widetilde{\mathbf{w}}_\epsilon))$ *from below.* From identities (4.40) applied to $\widetilde{\mathbf{v}}_\epsilon$ and (4.41) applied to $\widetilde{\mathbf{w}}_\epsilon$, applying Cauchy-Schwarz's inequality and using estimates (4.42), we deduce

$$\begin{aligned}\mathcal{A}^\epsilon((\widetilde{\mathbf{v}}_\epsilon,\widetilde{\mathbf{w}}_\epsilon),(\widetilde{\mathbf{v}}_\epsilon,\widetilde{\mathbf{w}}_\epsilon)) = \;& \epsilon\int_{\widetilde{\Omega}_\epsilon}|\mathrm{rot}_\epsilon(\widetilde{\mathbf{v}}_\epsilon)|^2\,dx'dz_3 - 4N^2\epsilon\int_{\widetilde{\Omega}_\epsilon}\mathrm{rot}_\epsilon(\widetilde{\mathbf{v}}_\epsilon)\cdot\widetilde{\mathbf{w}}_\epsilon\,dx'dz_3\\ &+\epsilon^3R_c\int_{\widetilde{\Omega}_\epsilon}|\mathrm{rot}_\epsilon(\widetilde{\mathbf{w}}_\epsilon)|^2\,dx'dz_3+\epsilon^3R_c\int_{\widetilde{\Omega}_\epsilon}|\mathrm{div}_\epsilon(\widetilde{\mathbf{w}}_\epsilon)|^2\,dx'dz_3\\ &+4N^2\epsilon\int_{\widetilde{\Omega}_\epsilon}|\widetilde{\mathbf{w}}_\epsilon|^2\,dx'dz_3+2\gamma\int_{\Gamma_0}(\widetilde{\mathbf{v}}_\epsilon\times\mathbf{n})\cdot\widetilde{\mathbf{w}}_\epsilon\,d\sigma\\ \geq\;& \epsilon\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}+\epsilon^3R_c\|D_\epsilon\widetilde{\mathbf{w}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}+4N^2\epsilon\|\widetilde{\mathbf{w}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^3}\\ &-4N^2\epsilon\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}\|\widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^3}-2|\gamma|\epsilon^2h_{\max}\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}\|D_\epsilon\widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}.\end{aligned}$$

From condition (2.27), there exists $c_1>0$ such that

$$\frac{\gamma h_{\max}}{R_c}<c_1<\frac{1-N^2}{\gamma h_{\max}}. \tag{4.52}$$

Applying Young's inequality

$$\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|_{L^2(\widetilde{\Omega}_\epsilon)^{3\times3}}\|D_\epsilon\widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}_\epsilon)^{3\times3}}\leq\frac{c_1}{2\epsilon}\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|^2_{L^2(\widetilde{\Omega}_\epsilon)^{3\times3}}+\frac{\epsilon}{2c_1}\|D_\epsilon\widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}_\epsilon)^{3\times3}}. \tag{4.53}$$

By continuity, there exists a real number c_2 satisfying $0<c_2<c_1$, and such that

$$c_1<\frac{1-\frac{N^2}{c_2}}{\gamma h_{\max}}, \tag{4.54}$$

and applying again Young's inequality, we also have

$$\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|_{L^2(\widetilde{\Omega}_\epsilon)^{3\times3}}\|\widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}_\epsilon)^3}\leq\frac{1}{4c_2}\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|^2_{L^2(\widetilde{\Omega}_\epsilon)^{3\times3}}+c_2\|\widetilde{\mathbf{w}}_\epsilon\|^2_{L^2(\widetilde{\Omega}_\epsilon)^3}. \tag{4.55}$$

Then, we deduce

$$\begin{aligned}\mathcal{A}^\epsilon((\widetilde{\mathbf{v}}_\epsilon,\widetilde{\mathbf{w}}_\epsilon),(\widetilde{\mathbf{v}}_\epsilon,\widetilde{\mathbf{w}}_\epsilon))\geq\;& \epsilon\left(1-\frac{N^2}{c_2}-|\gamma|h_{\max}c_1\right)\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}\\ &+\epsilon^3\left(R_c-\frac{|\gamma|h_{\max}}{c_1}\right)\|D_\epsilon\widetilde{\mathbf{w}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}+4N^2(1-c_2)\epsilon\|\widetilde{\mathbf{w}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^3}\\ \geq\;& \epsilon A\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}+\epsilon^3B\|D_\epsilon\widetilde{\mathbf{w}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}},\end{aligned} \tag{4.56}$$

with

$$A=\left(1-\frac{N^2}{c_2}-|\gamma|h_{\max}c_1\right),\quad B=\left(R_c-\frac{|\gamma|h_{\max}}{c_1}\right).$$

Using conditions (4.52) and (4.54), A and B are positive.

Step 3. Estimate $\mathcal{L}^\epsilon(\widetilde{\mathbf{v}}_\epsilon,\widetilde{\mathbf{w}}_\epsilon)$. From Cauchy-Schwarz's inequality and (4.38), we get

$$\begin{aligned}|\mathcal{L}^\epsilon(\widetilde{\mathbf{v}}_\epsilon,\widetilde{\mathbf{w}}_\epsilon)| = \;& \left|2N^2\epsilon\int_{\widetilde{\Omega}^\epsilon}\mathrm{rot}_\epsilon(\widetilde{\mathbf{J}}_\epsilon)\cdot\widetilde{\mathbf{w}}_\epsilon\,dx'dz_3-\epsilon\int_{\widetilde{\Omega}_\epsilon}\mathrm{rot}_\epsilon(\widetilde{\mathbf{J}}_\epsilon)\cdot\mathrm{rot}_\epsilon(\widetilde{\mathbf{v}}_\epsilon)\,dx'dz_3+2N^2\beta\int_{\Gamma_0}(\mathbf{g}-\mathbf{s})\times\mathbf{n}\cdot\widetilde{\mathbf{w}}_\epsilon\,d\sigma\right|\\ \leq\;& 2N^2\epsilon^2\|\mathrm{rot}_\epsilon\widetilde{\mathbf{J}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}\|D_\epsilon\widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}+\epsilon\|D_\epsilon\widetilde{\mathbf{J}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}\\ &+2N^2\beta\epsilon h_{\max}^{\frac{1}{2}}\|\mathbf{g}-\mathbf{s}\|_{L^2(\Gamma_0)^2}\|D_\epsilon\widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}\\ =\;& \epsilon\|D_\epsilon\widetilde{\mathbf{J}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}\|D_\epsilon\widetilde{\mathbf{v}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}}+2N^2\epsilon C_\epsilon\|D_\epsilon\widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times3}},\end{aligned} \tag{4.57}$$

where $C_\epsilon = \epsilon\|\mathrm{rot}_\epsilon \widetilde{\mathbf{J}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} + \beta h_{\max}^{\frac{1}{2}}\|\mathbf{g}-\mathbf{s}\|_{L^2(\Gamma_0)^2}$. Observe that from properties (3.34) satisfied by $\mathbf{J}_\epsilon$, there exists a a constant $C>0$ such that $C_\epsilon \leq C$ for any $\epsilon>0$.

Step 4. Derivation of the a priori estimates. To obtain estimates (4.43) and (4.44), we use inequalities (4.56) and (4.57) to get

$$\begin{aligned}&\epsilon A\|D_\epsilon \widetilde{\mathbf{v}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} + \epsilon^3 B\|D_\epsilon \widetilde{\mathbf{w}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}\\ &\leq \epsilon\|D_\epsilon \widetilde{\mathbf{J}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}\|D_\epsilon \widetilde{\mathbf{v}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} + 2N^2\epsilon C_\epsilon\|D_\epsilon \widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}},\end{aligned} \tag{4.58}$$

which is equivalent to the following inequality

$$\begin{aligned}&\epsilon A\|D_\epsilon \widetilde{\mathbf{v}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} - \epsilon\|D_\epsilon \widetilde{\mathbf{J}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}\|D_\epsilon \widetilde{\mathbf{v}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}\\ &+\epsilon^3 B\|D_\epsilon \widetilde{\mathbf{w}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} - 2N^2\epsilon C_\epsilon\|D_\epsilon \widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} \leq 0.\end{aligned} \tag{4.59}$$

Summing in both sides $\frac{\epsilon}{4A}\|D_\epsilon \widetilde{\mathbf{J}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}$ and $\frac{N^2}{\epsilon B}C_\epsilon^2$, we observe that last inequality can be written equivalently as

$$\begin{aligned}&\left(\epsilon^{\frac{1}{2}}A^{\frac{1}{2}}\|D_\epsilon \widetilde{\mathbf{v}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} - \frac{\epsilon^{\frac{1}{2}}}{2A^{\frac{1}{2}}}\|D_\epsilon \widetilde{\mathbf{J}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}\right)^2 + \left(\epsilon^{\frac{3}{2}}B^{\frac{1}{2}}\|D_\epsilon \widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} - \frac{N^2}{\epsilon^{\frac{1}{2}}B^{\frac{1}{2}}}C_\epsilon\right)^2\\ &\leq \frac{\epsilon}{4A}\|D_\epsilon \widetilde{\mathbf{J}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} + \frac{N^2}{\epsilon B}C_\epsilon^2.\end{aligned} \tag{4.60}$$

which is written as follows

$$\begin{aligned}&\epsilon A\left(\|D_\epsilon \widetilde{\mathbf{v}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} - \frac{1}{2A}\|D_\epsilon \widetilde{\mathbf{J}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}\right)^2 + \epsilon^3 B\left(\|D_\epsilon \widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} - \frac{N^2}{\epsilon^2 B}C_\epsilon\right)^2\\ &\leq \frac{\epsilon}{4A}\|D_\epsilon \widetilde{\mathbf{J}}_\epsilon\|^2_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}} + \frac{N^4}{\epsilon B}C_\epsilon^2.\end{aligned} \tag{4.61}$$

Since C_ϵ is uniformly bounded, using assumptions (3.34), we deduce than the right hand side of the previous inequality is bounded by C/ϵ for a certain constant $C>0$. This implies $(4.43)_2$ and $(4.44)_2$. Finally, from the Poincaré inequality (4.37), we get estimates $(4.43)_1$ and $(4.44)_1$, respectively. This finishes the proof.

□

The extension of $(\widetilde{\mathbf{u}}^\epsilon, \widetilde{\mathbf{w}}^\epsilon)$ to the whole domain. The sequence of solutions $(\widetilde{\mathbf{u}}^\epsilon, \widetilde{\mathbf{w}}^\epsilon)$ is defined in a varying set $\widetilde{\Omega}^\epsilon$, but not defined in the fixed domain Ω independent of ϵ. Thus, to pass to the limit if ϵ tends to zero, we need convergences in fixed Sobolev spaces (defined in Ω), so we need to extend $(\widetilde{\mathbf{u}}^\epsilon, \widetilde{\mathbf{w}}^\epsilon)$ to the whole domain Ω. From the boundary conditions satisfied by $\widetilde{\mathbf{u}}^\epsilon$ and $\widetilde{\mathbf{w}}^\epsilon$ on the top boundary $\widetilde{\Gamma}_1^\epsilon$, we can extend them by zero in $\Omega\setminus\widetilde{\Omega}^\epsilon$ and we denote the extensions by the same symbol.

Lemma 4.5 (Estimates of extended unknowns)**.** *The extended functions $(\widetilde{\mathbf{u}}^\epsilon, \widetilde{\mathbf{w}}^\epsilon)$ satisfy the following estimates*

$$\|\widetilde{\mathbf{u}}_\epsilon\|_{L^2(\Omega)^3} \leq C, \quad \|D_\epsilon \widetilde{\mathbf{u}}_\epsilon\|_{L^2(\Omega)^{3\times 3}} \leq C\epsilon^{-1}, \tag{4.62}$$

$$\|\widetilde{\mathbf{w}}_\epsilon\|_{L^2(\Omega)^3} \leq C\epsilon^{-1}, \quad \|D_\epsilon \widetilde{\mathbf{w}}_\epsilon\|_{L^2(\Omega)^{3\times 3}} \leq C\epsilon^{-2}. \tag{4.63}$$

Proof. Estimates for the extension of $\widetilde{\mathbf{u}}_\epsilon$ and $\widetilde{\mathbf{w}}_\epsilon$ are obtained straightforward from (4.43) and (4.44), respectively.

□

4.2 Estimates for pressure

Extending the pressure is a much more difficult task. A continuation of the pressure for a flow in a porous media was introduced in [40]. This construction applies to periodic holes in a domain Ω_ϵ when each hole is strictly contained into the periodic cell. In this context, we can not use directly this result because the "holes" are along the oscillating boundary Γ_1^ϵ of Ω_ϵ and moreover, the scale of the vertical direction is smaller than the scales of the horizontal directions. This fact will induce several limitations in the results obtained by using the method, especially in view of the convergence for the pressure. In this sense, for the case of Newtonian fluids, an operator R^ϵ generalizing the results of [40] to this context (extending the pressure from Ω_ϵ to Q_ϵ) was introduced in [6, 29], and later extended to the case of non-Newtonian (power law) fluids in [1] by defining an extension operator R_p^ϵ, for every flow index $p > 1$.

Then, to extend the pressure to the whole domain Ω, the mapping R^ϵ (defined in [1, Lemma 4.6] as R_2^ϵ) allows us to extend the pressure p_ϵ from Ω_ϵ to Q_ϵ by introducing F_ϵ in $H^{-1}(Q_\epsilon)^3$ as follows (brackets are for duality products between H^{-1} and H_0^1)

$$\langle F_\epsilon, \varphi\rangle_{Q_\epsilon} = \langle \nabla p_\epsilon, R^\epsilon(\varphi)\rangle_{\Omega_\epsilon} \quad \text{for any } \varphi \in H_0^1(Q_\epsilon)^3\,. \tag{4.64}$$

We compute the right-hand side of (4.64) by using the first equation of (2.23), and taking into account $\mathbf{u}_\epsilon = \mathbf{v}_\epsilon + \mathbf{J}_\epsilon$, we have

$$\langle F_\epsilon, \varphi\rangle_{Q_\epsilon} = \quad -\int_{\Omega_\epsilon} \mathrm{rot}(\mathbf{u}_\epsilon)\cdot \mathrm{rot}(R^\epsilon(\varphi))\,dx + 2N^2\int_{\Omega_\epsilon} \mathrm{rot}\, w_\epsilon \cdot R^\epsilon(\varphi)\,dx\,. \tag{4.65}$$

Using Lemma 4.4 for fixed ϵ, we see that it is a bounded functional on $H_0^1(Q_\epsilon)$ (see the proof of Lemma 4.6 below) and in fact $F_\epsilon \in H^{-1}(Q_\epsilon)^3$. Moreover, $\mathrm{div}\varphi = 0$ implies $\langle F_\epsilon, \varphi\rangle_{Q_\epsilon} = 0$, and the DeRham theorem gives the existence of P_ϵ in $L_0^2(Q_\epsilon)$ with $F_\epsilon = \nabla P_\epsilon$.

Defining the rescaled extended pressure $\widetilde{P}_\epsilon \in L_0^2(\Omega)$ by

$$\widetilde{P}_\epsilon(x', z_3) = P_\epsilon(x', \epsilon z_3), \quad \text{a.e. } (x', z_3) \in \Omega,$$

we get for any $\widetilde{\varphi} \in H_0^1(\Omega)^3$ where $\widetilde{\varphi}(x', z_3) = \varphi(x', \epsilon z_3)$ that

$$\langle \nabla_\epsilon \widetilde{P}_\epsilon, \widetilde{\varphi}\rangle_\Omega \quad = -\int_\Omega \widetilde{P}_\epsilon\, \mathrm{div}_\epsilon\, \widetilde{\varphi}\, dx' dz_3 = -\epsilon^{-1}\int_{Q_\epsilon} P_\epsilon\, \mathrm{div}\,\varphi\, dx = \epsilon^{-1}\langle \nabla P_\epsilon, \varphi\rangle_{Q_\epsilon}\,.$$

Then, using the identification (4.65) of F_ϵ, we get

$$\langle \nabla_\epsilon \widetilde{P}_\epsilon, \widetilde{\varphi}\rangle_\Omega \quad = \epsilon^{-1}\Big(-\int_{\Omega_\epsilon} \mathrm{rot}(\mathbf{u}_\epsilon) : \mathrm{rot}(R^\epsilon(\varphi))\,dx + 2N^2\int_{\Omega_\epsilon} \mathrm{rot}(\mathbf{w}_\epsilon)\cdot R^\epsilon(\varphi)\,dx\Big)\,,$$

and applying the change of variables (3.28), we obtain

$$\langle \nabla_\epsilon \widetilde{P}_\epsilon, \widetilde{\varphi}\rangle_\Omega = \quad -\int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{u}}_\epsilon) : \mathrm{rot}_\epsilon(\widetilde{R}^\epsilon(\widetilde{\varphi}))\,dx' dz_3 + 2N^2\int_{\widetilde{\Omega}^\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{w}}_\epsilon)\cdot \widetilde{R}^\epsilon(\widetilde{\varphi})\,dx' dz_3\,, \tag{4.66}$$

where $\widetilde{R}^\epsilon(\widetilde{\varphi}) = R^\epsilon(\varphi)$ for any $\varphi \in H_0^1(Q_\epsilon)^3$ where $\widetilde{\varphi}(x', z_3) = \varphi(x', \epsilon z_3)$.

Now, we estimate the right-hand side of (4.66) to obtain the *a priori* estimate of the pressure $\widetilde{P}_\epsilon$.

Lemma 4.6 (Estimates for extended pressure)**.** *Under hypothesis (2.27), the following estimates hold for the the extension $\widetilde{P}_\epsilon \in L_0^2(\Omega)$ of the pressure $\widetilde{p}_\epsilon$*

$$\|\widetilde{P}_\epsilon\|_{L^2(\Omega)} \le C\epsilon^{-2}, \quad \|\nabla_\epsilon \widetilde{P}_\epsilon\|_{H^{-1}(\Omega)^3} \le C\epsilon^{-2}. \tag{4.67}$$

Proof. From the proof of Lemma 4.7-(i) in [1], we have that $\widetilde{R}^\epsilon(\widetilde{\varphi})$ satisfies the following estimates

$$\begin{aligned}
&\|\widetilde{R}^\epsilon(\widetilde{\varphi})\|_{L^2(\widetilde{\Omega}_\epsilon)^3} \leq C\left(\|\widetilde{\varphi}\|_{L^2(\Omega)^3} + \epsilon^\ell \sum_{j=1}^{2} \|\partial_{x_i}\widetilde{\varphi}\|_{L^2(\Omega)^3} + \|\partial_{z_3}\widetilde{\varphi}\|_{L^2(\Omega)^3}\right),\\
&\|\partial_{x_i}\widetilde{R}^\epsilon(\widetilde{\varphi})\|_{L^2(\widetilde{\Omega}_\epsilon)^{3\times 2}} \leq C\left(\frac{1}{\epsilon^\ell}\|\widetilde{\varphi}\|_{L^2(\Omega)^3} + \sum_{j=1}^{2} \|\partial_{x_i}\widetilde{\varphi}\|_{L^2(\Omega)^3} + \frac{1}{\epsilon^\ell}\|\partial_{z_3}\widetilde{\varphi}\|_{L^2(\Omega)^3}\right), \quad i=1,2,\\
&\|\partial_{z_3}\widetilde{R}^\epsilon(\widetilde{\varphi})\|_{L^2(\widetilde{\Omega}_\epsilon)^3} \leq C\left(\|\widetilde{\varphi}\|_{L^2(\Omega)^3} + \epsilon^\ell \sum_{j=1}^{2} \|\partial_{x_i}\widetilde{\varphi}\|_{L^2(\Omega)^3} + \|\partial_{z_3}\widetilde{\varphi}\|_{L^2(\Omega)^3}\right).
\end{aligned} \tag{4.68}$$

Thus, since $0<\epsilon<\epsilon^\ell \ll 1$, we deduce

$$\|\widetilde{R}^\epsilon(\widetilde{\varphi})\|_{L^2(\widetilde{\Omega}_\epsilon)^3} \leq C\|\widetilde{\varphi}\|_{H_0^1(\Omega)^3}, \quad \|\mathrm{rot}_\epsilon(\widetilde{R}^\epsilon(\widetilde{\varphi}))\|_{L^2(\widetilde{\Omega}_\epsilon)^{3\times 3}} \leq \frac{C}{\epsilon}\|\widetilde{\varphi}\|_{H_0^1(\Omega)^3}. \tag{4.69}$$

By using Cauchy-Schwarz's inequality, estimate (4.38), estimates for $D_\epsilon\widetilde{\mathbf{u}}_\epsilon$ in (4.45) and for $D_\epsilon\widetilde{\mathbf{w}}_\epsilon$ in (4.44) together with (4.69), we obtain

$$\begin{aligned}
\left|\int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{u}}_\epsilon) : \mathrm{rot}_\epsilon(\widetilde{R}^\epsilon(\widetilde{\varphi}))\,dx'dz_3\right| &\leq C\epsilon^{-1}\|D_\epsilon\widetilde{R}^\epsilon(\widetilde{\varphi})\|_{L^2(\widetilde{\Omega}_\epsilon)^{3\times 3}} \leq C\epsilon^{-2}\|\widetilde{\varphi}\|_{H_0^1(\Omega)^3},\\
\left|\int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{w}}_\epsilon)\cdot\widetilde{R}^\epsilon(\widetilde{\varphi})\,dx'dz_3\right| &\leq \|D_\epsilon\widetilde{\mathbf{w}}_\epsilon\|_{L^2(\widetilde{\Omega}_\epsilon)^{3\times 3}}\|\widetilde{R}^\epsilon(\widetilde{\varphi})\|_{L^2(\widetilde{\Omega}_\epsilon)^3}\\
&\leq C\epsilon^{-2}\|\widetilde{R}^\epsilon(\widetilde{\varphi})\|_{L^2(\widetilde{\Omega}_\epsilon)^3} C\|\widetilde{\varphi}\|_{H_0^1(\Omega)^3},
\end{aligned} \tag{4.70}$$

which together with (4.66) gives $\|\nabla_\epsilon\widetilde{P}_\epsilon\|_{H^{-1}(\Omega)^3} \leq C\epsilon^{-2}$, i.e. estimate $(4.67)_2$. By using the Nečas inequality there exists a representative $\widetilde{P}_\epsilon \in L^2_0(\Omega)$ such that

$$\|\widetilde{P}_\epsilon\|_{L^2(\Omega)} \leq C\|\nabla\widetilde{P}_\epsilon\|_{H^{-1}(\Omega)^3} \leq C\|\nabla_\epsilon\widetilde{P}_\epsilon\|_{H^{-1}(\Omega)^3},$$

which implies $(4.67)_1$.

□

4.3 Convergences

We give a compactness result concerning the asymptotic behavior of the extended sequences $(\widetilde{\mathbf{u}}_\epsilon, \widetilde{\mathbf{w}}_\epsilon, \widetilde{P}_\epsilon)$ satisfying the *a priori* estimates given in Lemmas 4.5 and 4.6.

Lemma 4.7. *For a subsequence of ϵ still denote by ϵ, we have the following convergence results:*

(i) (Velocity) There exists $\widetilde{\mathbf{u}} = (\widetilde{\mathbf{u}}', \widetilde{u}_3) \in H^1(0,h_{\max};L^2(\omega)^3)$, with $\widetilde{u}_3 \equiv 0$ and $\widetilde{\mathbf{u}}' = 0$ on Γ_1, such that

$$\widetilde{\mathbf{u}}^\epsilon \rightharpoonup \widetilde{\mathbf{u}} \text{ in } H^1(0,h_{\max};L^2(\omega)^3), \tag{4.71}$$

$$\mathrm{div}_{x'}\left(\int_0^{h_{\max}} \widetilde{\mathbf{u}}'(x',z_3)\,dz_3\right) = 0 \quad \text{in } \omega, \tag{4.72}$$

$$\left(\int_0^{h_{\max}} \widetilde{\mathbf{u}}'(x',z_3)\,dz_3\right)\cdot\mathbf{n} = 0 \quad \text{on } \partial\omega.$$

(ii) (Microrotation) There exist $\widetilde{\mathbf{w}} = (\widetilde{\mathbf{w}}', \widetilde{w}_3) \in H^1(0, h_{\max}; L^2(\omega)^3)$*, with* $\widetilde{w}_3 \equiv 0$ *and* $\widetilde{\mathbf{w}}' = 0$ *on* Γ_1*, such that*

$$\epsilon \widetilde{\mathbf{w}}_\epsilon \rightharpoonup \widetilde{\mathbf{w}} \text{ in } H^1(0, h_{\max}; L^2(\omega)^3). \tag{4.73}$$

(iii) (Pressure) There exist a function $\widetilde{p} \in L^2_0(\omega) \cap H^1(\omega)$ *(i.e.* $\widetilde{p}$ *is independent of* z_3 *and has mean value zero in* ω*), such that*

$$\epsilon^2 \widetilde{P}_\epsilon \to \widetilde{p} \text{ in } L^2(\Omega). \tag{4.74}$$

Proof. We start proving (i). We will only give some remarks and, for more details, we refer the reader to Lemmas 5.2-i) and 5.4-i) in [1].

We start with the extension $\tilde{u}_\epsilon$. Estimates (4.62) imply the existence of $\tilde{u} \in H^1(0, h_{\max}; L^2(\omega)^3)$ such that convergence (4.71) holds, and the continuity of the trace applications from the space of $\tilde{u}$ such that $\|\widetilde{\mathbf{u}}\|_{L^2}$ and $\|\partial_{z_3}\widetilde{\mathbf{u}}\|_{L^2}$ are bounded to $L^2(\Gamma_1)$ implies $\tilde{u} = 0$ on Γ_1. It can also be deduced that $\widetilde{u}_3 = 0$ on Γ_0 according to boundary condition $\widetilde{\mathbf{u}}_\epsilon \cdot \mathbf{n} = \widetilde{u}_{3,\epsilon} = 0$ on Γ_0.

Next, from the free divergence condition $\mathrm{div}_\epsilon(\widetilde{\mathbf{u}}_\epsilon) = 0$, it can be deduced that $\widetilde{u}_3$ is independent of z_3, which together with the boundary conditions satisfied by $\widetilde{u}_3$ on Γ_1 implies that $\widetilde{u}_3 \equiv 0$. Finally, from the free divergence condition and the convergence (4.71) of $\widetilde{\mathbf{u}}_\epsilon$, it is straightforward the corresponding free divergence condition in a thin domain given in (4.72).

We continue proving (ii). From estimates (4.63), the convergence of (4.73) and that $\widetilde{\mathbf{w}} = 0$ on Γ_1 and $\widetilde{w}_3 = 0$ on Γ_0 are obtained straighfordward. It remains to prove that $\tilde{w}_3 \equiv 0$. To do this, we consider as test function $\widetilde{\psi}(x', z_3) = (0, 0, \widetilde{\psi}_3)$ with $\widetilde{\psi}_3 \in \mathcal{D}(\Omega)$, in the variational formulation (3.35) extended to Ω, taking into account $\widetilde{\mathbf{u}}_\epsilon = \widetilde{\mathbf{v}}_\epsilon + \widetilde{\mathbf{J}}_\epsilon$ and the definition of the rotational given in (3.33), we get

$$\epsilon^3 R_c \int_\Omega \mathrm{rot}_{x'}(\widetilde{w}_{3,\epsilon}) \cdot \mathrm{rot}_{x'}(\widetilde{\psi}_3)\, dx' dz_3 + \epsilon R_c \int_\Omega \partial_{z_3}\widetilde{w}_{3,\epsilon}\, \partial_{z_3}\widetilde{\psi}_3\, dx' dz_3 + 4N^2\epsilon \int_\Omega \widetilde{w}_{3,\epsilon}\, \widetilde{\psi}_3\, dx' dz_3$$
$$-2N^2\epsilon \int_\Omega \widetilde{\mathbf{u}}'_\epsilon \cdot \mathrm{rot}_{x'}(\widetilde{\psi}_3)\, dx' dz_3 = 0.$$

Passing to the limit by using convergences of $\widetilde{u}_\epsilon$ and $\widetilde{w}_\epsilon$ given respectively in (4.71) and (4.73), we get

$$R_c \int_\Omega \partial_{z_3}\widetilde{w}_3\, \partial_{z_3}\widetilde{\psi}_3\, dx' dz_3 + 4N^2 \int_\Omega \widetilde{w}_3\, \widetilde{\psi}_3\, dx' dz_3 = 0\,,$$

and taking into account that $\widetilde{w}_3 = 0$ on $\Gamma_1 \cup \Gamma_0$, it is easily deduced that $\widetilde{w}_3 \equiv 0$ in Ω.

We finish the proof with (iii). Estimate (5.86) implies, up to a subsequence, the existence of $\tilde{P} \in L^2_0(\Omega)$ such that

$$\epsilon^2 \widetilde{P}_\epsilon \rightharpoonup \tilde{p} \quad \text{in } L^2(\Omega). \tag{4.75}$$

Also, from $\|\nabla_\epsilon \widetilde{P}_\epsilon\|_{L^2(\Omega)^3} \le C\epsilon^2$, by noting that $\partial_{z_3}\widetilde{P}_\epsilon/\epsilon^3$ also converges weakly in $H^{-1}(\Omega)$, we obtain $\partial_{z_3}\widetilde{P} = 0$ and so $\widetilde{P}$ is independent of z_3. Since $\widetilde{P}_\epsilon$ has mean value zero in Ω, then $\widetilde{p}$ also has mean value zero. To prove strong convergence, we refer to [1, Lemma 7]. Moreover, since $\widetilde{P}_\epsilon$ has mean value zero in Ω and $\widetilde{p}$ does not depend on z_3, it holds

$$\int_\Omega \epsilon^2 \widetilde{P}_\epsilon\, dx' dz_3 \to \int_\Omega \widetilde{p}\, dx' dz_3 = h_{\max} \int_\omega \widetilde{p}(x')\, dx' = 0,$$

which implies $\widetilde{p} \in L^2_0(\omega)$. Finally, the proof of $\widetilde{p} \in H^1(\omega)$ can be found in [3, Theorem 3.1], so we omit it.

□

5 Adaptation of the unfolding method

5.1 Definition and properties

As seen in Lemma 4.7, the change of variables (3.28) does not capture the oscillations of the domain $\widetilde{\Omega}^\epsilon$. To capture them, we use an adaptation of the unfolding method (see [19, 20] for more details) introduced to this context in [1].

Remark 5.1. *For $k' \in \mathbb{Z}^2$, we define $\kappa : \mathbb{R}^2 \to \mathbb{Z}^2$ by*

$$\kappa(x') = k' \Longleftrightarrow x' \in Z_{k',1}\,. \tag{5.76}$$

Remark that κ is well defined up to a set of zero measure in $\mathbb{R}^2$ (the set $\cup_{k'\in\mathbb{Z}^2}\partial Z_{k',1}$). Moreover, for every $\epsilon > 0$, we have

$$\kappa\left(\frac{x'}{\epsilon^\ell}\right) = k' \Longleftrightarrow x' \in Z_{k',\epsilon}\,.$$

Let us recall that this adaptation of the unfolding method divides the domain $\widetilde{\Omega}_\epsilon$ in cubes of lateral length ϵ^ℓ and vertical length $h(z')$. Thus, given $\widetilde{\varphi} \in L^2(\widetilde{\Omega}^\epsilon)^3$ (assuming $\widetilde{\varphi}$ extended by zero outside of ω), we define $\widehat{\varphi}_\epsilon \in L^2(\mathbb{R}^2 \times Z)^3$ by

$$\widehat{\varphi}_\epsilon(x', z) = \widetilde{\varphi}\left(\epsilon^\ell \kappa\left(\frac{x'}{\epsilon^\ell}\right) + \epsilon^\ell z', z_3\right) \quad \text{a.e.} \quad (x', z) \in \omega \times Z, \tag{5.77}$$

where the function κ is defined by (5.76). Also, given $\widetilde{\psi} \in L^2(\Omega)$, we define $\widehat{\psi}_\epsilon \in L^2(\mathbb{R}^2 \times \Pi)$ by

$$\widehat{\psi}_\epsilon(x', z) = \widetilde{\psi}\left(\epsilon^\ell \kappa\left(\frac{x'}{\epsilon^\ell}\right) + \epsilon^\ell z', z_3\right) \quad \text{a.e.} \quad (x', z) \in \omega \times \Pi. \tag{5.78}$$

Remark 5.2. *For $k' \in T_\epsilon$, the restriction of $\widehat{\varphi}_\epsilon$ to $Z'_{k',\epsilon} \times Z$ and $\widehat{\psi}_\epsilon$ to $Z'_{k',\epsilon} \times \Pi$ do not depend on x', while as a function of z it is obtained from $(\widetilde{\varphi}, \widetilde{\psi})$ by using the change of variables*

$$z' = \frac{x' - \epsilon^\ell k'}{\epsilon^\ell},$$

which transform $Z_{k',\epsilon}$ into Z and $\widetilde{Q}_{k',\epsilon}$ into Π, respectively.

Below, we give some properties of the change of variables (5.77), whose can be found in [1, Lemma 4.9] in the case $p = 2$.

Proposition 5.3. *We have the following properties concerning the estimates of vectorial functions $\widetilde{\varphi}$ and scalar functions ψ and their respective unfolding function $\widehat{\varphi}_\epsilon$ and $\widehat{\psi}_\epsilon$ given by (5.77) and (5.78):*

i) For every $\widetilde{\varphi} \in L^2(\widetilde{\Omega}^\epsilon)^3$, we have

$$\|\widehat{\varphi}_\epsilon\|_{L^2(\omega\times Z)^3} = \|\widetilde{\varphi}\|_{L^2(\widetilde{\Omega}^\epsilon)^3}. \tag{5.79}$$

ii) For every $\widetilde{\varphi} \in H^1(\widetilde{\Omega}^\epsilon)^3$, we have

$$\|D_{z'}\widehat{\varphi}_\epsilon\|_{L^2(\omega\times Z)^{3\times 3}} = \epsilon^\ell \|D_{x'}\widetilde{\varphi}\|_{L^2(\widetilde{\Omega}^\epsilon)^{3\times 3}}, \quad \|\partial_{z_3}\widehat{\varphi}_\epsilon\|_{L^2(\omega\times Z)^3} = \|\partial_{z_3}\widetilde{\varphi}\|_{L^2(\widetilde{\Omega}^\epsilon)^3}. \tag{5.80}$$

iii) For every $\widetilde{\psi} \in L^2(\Omega)$, we have

$$\|\widehat{\psi}_\epsilon\|_{L^2(\omega\times\Pi)} = \|\widetilde{\psi}\|_{L^2(\Omega)}. \tag{5.81}$$

In a similar way, let us introduce the adaption of the unfolding method on the boundary Γ_0 (see Cioranescu et al. [20] for more details). For this purpose, given $\widetilde{\varphi} \in L^2(\Gamma_0)^3$, we define $\widehat{\varphi}_\epsilon^b \in L^2(\mathbb{R}^2 \times \widehat{\Gamma}_0)^3$ by

$$\widehat{\varphi}_\epsilon^b(x', z) = \widetilde{\varphi}\left(\epsilon^\ell \kappa\left(\frac{x'}{\epsilon^\ell}\right) + \epsilon^\ell z', z_3\right) \quad \text{a.e.} \quad (x', z) \in \omega \times \widehat{\Gamma}_0, \tag{5.82}$$

where the function κ is defined by (5.76).

Remark 5.4. *(i) Observe that from this definition, if we consider $\widetilde{\varphi} \in L^2(\Gamma_0)$, a Z'-periodic function, we define $\widetilde{\varphi}_\epsilon(x', z_3) = \widetilde{\varphi}(x'/\epsilon^\ell, z_3)$, it follows $\widehat{\varphi}_\epsilon^b(x', z) = \widetilde{\varphi}(z)$.*

(ii) Observe that for $\widetilde{\varphi} \in H^1(\Gamma_0)$, $\widehat{\varphi}_\epsilon^b$ is the trace on Γ_0 of $\widehat{\varphi}_\epsilon$. Therefore, $\widehat{\varphi}_\epsilon^b$ has similar properties as $\widehat{\varphi}_\epsilon$. So it holds the following property:

$$\|\widehat{\varphi}_\epsilon^b\|_{L^2(\omega \times \widehat{\Gamma}_0)^3} = \|\widetilde{\varphi}_\epsilon\|_{L^2(\Gamma_0)^3}. \tag{5.83}$$

Definition 5.5. *We define the unfolded unknowns as follows:*

- *The unfolded velocity $\widehat{\mathbf{u}}_\epsilon$ is defined by applying (5.77) for $\widetilde{\varphi} = \widetilde{\mathbf{u}}_\epsilon$.*
- *The unfolded microrotation $\widehat{\mathbf{w}}_\epsilon$ is defined by applying (5.77) for $\widetilde{\varphi} = \widetilde{\mathbf{w}}_\epsilon$.*
- *The unfolded pressure $\widehat{P}_\epsilon$ is defined by applying (5.78) for $\widetilde{\psi} = \widetilde{P}_\epsilon$.*

We are now in position to obtain estimates for the unfolded unknowns $(\widehat{\mathbf{u}}_\epsilon, \widehat{\mathbf{w}}_\epsilon, \widehat{P}_\epsilon)$.

Lemma 5.6. *There exists a constant $C > 0$ independent of ϵ, such that $\widehat{\mathbf{u}}_\epsilon$, $\widehat{\mathbf{w}}_\epsilon$ and $\widehat{P}_\epsilon$ satisfy*

$$\|\widehat{\mathbf{u}}_\epsilon\|_{L^2(\omega \times Z)^3} \leq C, \quad \|D_{z'}\widehat{\mathbf{u}}_\epsilon\|_{L^2(\omega \times Z)^{3\times 2}} \leq C\epsilon^{\ell-1}, \quad \|\partial_{z_3}\widehat{\mathbf{u}}_\epsilon\|_{L^2(\omega \times Z)^3} \leq C, \tag{5.84}$$

$$\|\widehat{\mathbf{w}}_\epsilon\|_{L^2(\omega \times Z)^3} \leq C\epsilon^{-1}, \quad \|D_{z'}\widehat{\mathbf{w}}_\epsilon\|_{L^2(\omega \times Z)^{3\times 2}} \leq C\epsilon^{\ell-2}, \quad \|\partial_{z_3}\widehat{\mathbf{w}}_\epsilon\|_{L^2(\omega \times Z)^3} \leq C\epsilon^{-1}, \tag{5.85}$$

$$\|\widehat{P}_\epsilon\|_{L^2(\omega \times \Pi)} \leq C\epsilon^{-2}. \tag{5.86}$$

Proof. The result is a consequence of the combination of estimates given in Proposition 5.3 with estimates for $\widetilde{\mathbf{u}}_\epsilon$, $\widetilde{\mathbf{w}}_\epsilon$, $\widetilde{P}_\epsilon$ given in Lemma 4.4 and 4.6.

□

5.2 Equivalent weak variational formulation

We give the equivalent weak variational formulation of system (3.35), which will be useful in next sections in order to obtain the limit system taking into account the effects of the rough boundary.

We consider $\widetilde{\varphi}_\epsilon(x', z_3) = \epsilon\widehat{\varphi}(x', x'/\epsilon^\ell, z_3)$ and $\widetilde{\psi}_\epsilon = \widehat{\psi}(x', x'/\epsilon^\ell, z_3)$ as test function in (3.35), where $\widehat{\varphi}, \widehat{\psi} \in \mathcal{D}(\omega; C^\infty_\#(Z)^3)$, and taking into account the extension of the pressure, we have

$$\langle \nabla_\epsilon \widetilde{p}_\epsilon, \widetilde{\varphi}_\epsilon \rangle_{\widetilde{\Omega}^\epsilon} = \langle \nabla_\epsilon \widetilde{P}_\epsilon \cdot \widetilde{\varphi}_\epsilon \rangle_\Omega,$$

and taking into account $\widetilde{\mathbf{u}}_\epsilon = \widetilde{\mathbf{v}}_\epsilon + \widetilde{\mathbf{J}}_\epsilon$, we get

$$\epsilon \int_{\widetilde{\Omega}_\epsilon} \mathrm{rot}_\epsilon(\widetilde{\mathbf{u}}_\epsilon)\cdot \mathrm{rot}_\epsilon(\widetilde{\varphi}_\epsilon)\,dx'dz_3 - \epsilon\int_\Omega \widetilde{P}_\epsilon\,\mathrm{div}_\epsilon(\widetilde{\varphi}_\epsilon)\,dx'dz_3 - 2N^2\epsilon\int_{\widetilde{\Omega}_\epsilon}\mathrm{rot}_\epsilon(\widetilde{\varphi}_\epsilon)\cdot\widetilde{\mathbf{w}}_\epsilon\,dx'dz_3 \tag{5.87}$$

$$+2\left(N^2-\frac{1}{\alpha}\right)\int_{\Gamma_0}(\widetilde{\mathbf{w}}_\epsilon\times\mathbf{n})\cdot\widetilde{\varphi}\,d\sigma(x') = 0,$$

$$\epsilon^3 R_c\int_{\widetilde{\Omega}_\epsilon}\mathrm{rot}_\epsilon(\widetilde{\mathbf{w}}_\epsilon)\cdot\mathrm{rot}_\epsilon(\widetilde{\psi}_\epsilon)\,dx'dz_3 + \epsilon^3 R_c\int_{\widetilde{\Omega}_\epsilon}\mathrm{div}_\epsilon(\widetilde{\mathbf{w}}_\epsilon)\cdot\mathrm{div}_\epsilon(\widetilde{\psi}_\epsilon)\,dx'dz_3 + 4N^2\epsilon\int_{\widetilde{\Omega}_\epsilon}\widetilde{\mathbf{w}}_\epsilon\cdot\widetilde{\psi}_\epsilon\,dx'dz_3$$

$$-2N^2\epsilon\int_{\widetilde{\Omega}_\epsilon}\widetilde{\mathbf{u}}_\epsilon\cdot\mathrm{rot}_\epsilon(\widetilde{\psi}_\epsilon)\,dx'dz_3 - 2N^2\beta\int_{\Gamma_0}(\widetilde{\mathbf{u}}_\epsilon-\mathbf{s})\times\mathbf{n}\cdot\widetilde{\psi}_\epsilon\,d\sigma(x') \tag{5.88}$$

$$= -2N^2\int_{\Gamma_0}(\widetilde{\mathbf{u}}_\epsilon\times\mathbf{n})\cdot\widetilde{\psi}_\epsilon\,d\sigma(x').$$

Now, from the definition of rot_ϵ given in (3.31), and by the change of variables given in Remark 5.2 (see [1] for more details) applied to (5.87), we obtain

$$\begin{aligned}
&\frac{\epsilon^2}{\epsilon^{2\ell}}\int_{\omega\times Z}\mathrm{rot}_{z'}(\widehat{u}_{3,\epsilon})\cdot\mathrm{rot}_{z'}(\widehat{\varphi}_3)\,dx'dz + \frac{\epsilon^2}{\epsilon^2}\int_{\omega\times Z}\mathrm{rot}_{z_3}(\widehat{\mathbf{u}}'_\epsilon)\cdot\mathrm{rot}_{z_3}(\widehat{\varphi}')\,dx'dz + \frac{\epsilon^2}{\epsilon^{2\ell}}\int_{\omega\times Z}\mathrm{Rot}_{z'}(\widehat{\mathbf{u}}'_\epsilon)\cdot\mathrm{Rot}_{z'}(\widehat{\varphi})\,dx'dz\\
&-\epsilon^2\int_{\omega\times\Pi}\widehat{P}_\epsilon\,\mathrm{div}_{x'}(\widehat{\varphi})\,dx'dz - \frac{\epsilon^2}{\epsilon^\ell}\int_{\omega\times\Pi}\widehat{P}_\epsilon\,\mathrm{div}_{z'}(\widetilde{\varphi})\,dx'dz - \epsilon\int_{\omega\times\Pi}\widehat{P}_\epsilon\,\partial_{z_3}\widehat{\varphi}\,dx'dz\\
&-2N^2\frac{\epsilon^2}{\epsilon^\ell}\int_{\omega\times Z}\mathrm{rot}_{z'}(\widehat{\varphi}_3)\cdot\widehat{\mathbf{w}}'_\epsilon\,dx'dz - 2N^2\epsilon\int_{\omega\times Z}\mathrm{rot}_{z_3}(\widehat{\varphi}')\cdot\widehat{\mathbf{w}}'_\epsilon\,dx'dz - 2N^2\frac{\epsilon^2}{\epsilon^\ell}\int_{\omega\times Z}\mathrm{Rot}_{z'}(\widehat{\varphi}')\cdot\widetilde{w}_{3,\epsilon}\,dx'dz\\
&+2\left(N^2-\frac{1}{\alpha}\right)\epsilon\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{w}}_\epsilon\times\mathbf{n})\cdot\widehat{\varphi}^b_\epsilon\,dx'd\sigma(z') + O_\epsilon = 0,
\end{aligned} \tag{5.89}$$

where O_ϵ is devoted to tends to zero. To simplify the variational formulation, we observe that from estimates for $(\widehat{\mathbf{u}}_\epsilon,\widehat{\mathbf{w}}_\epsilon)$ given in Lemma 5.6 and convergence (2.10), we deduce

$$\left|\frac{\epsilon^2}{\epsilon^{2\ell}}\int_{\widetilde{\Omega}_\epsilon}\mathrm{rot}_{z'}(\widehat{u}_{3,\epsilon})\cdot\mathrm{rot}_{z'}(\widehat{\varphi}_3)\,dx'dz + \frac{\epsilon^2}{\epsilon^{2\ell}}\int_{\widetilde{\Omega}_\epsilon}\mathrm{Rot}_{z'}(\widehat{\mathbf{u}}'_\epsilon)\cdot\mathrm{Rot}_{z'}(\widehat{\varphi})\,dx'dz\right| \le C\epsilon^{1-\ell}\to 0,$$

$$\left|2N^2\frac{\epsilon^2}{\epsilon^\ell}\int_{\omega\times Z}\mathrm{rot}_{z'}(\widehat{\varphi}_3)\cdot\widehat{\mathbf{w}}'_\epsilon\,dx'dz + 2N^2\frac{\epsilon^2}{\epsilon^\ell}\int_{\omega\times Z}\mathrm{Rot}_{z'}(\widehat{\varphi}')\cdot\widetilde{w}_{3,\epsilon}\,dx'dz\right| \le C\epsilon^{1-\ell}\to 0.$$

Then, variational formulation (5.89) reads

$$\begin{aligned}
&\int_{\omega\times Z}\mathrm{rot}_{z_3}(\widehat{\mathbf{u}}'_\epsilon)\cdot\mathrm{rot}_{z_3}(\widehat{\varphi}')\,dx'dz - \epsilon^2\int_{\omega\times\Pi}\widehat{P}_\epsilon\,\mathrm{div}_{x'}(\widehat{\varphi})\,dx'dz\\
&-\epsilon^{2-\ell}\int_{\omega\times\Pi}\widehat{P}_\epsilon\,\mathrm{div}_{z'}(\widehat{\varphi})\,dx'dz - \epsilon\int_{\omega\times\Pi}\widehat{P}_\epsilon\,\partial_{z_3}\widehat{\varphi}\,dx'dz\\
&-2N^2\epsilon\int_{\omega\times Z}\mathrm{rot}_{z_3}(\widehat{\varphi}')\cdot\widehat{\mathbf{w}}'_\epsilon\,dx'dz + 2\left(N^2-\frac{1}{\alpha}\right)\epsilon\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{w}}_\epsilon\times\mathbf{n})\cdot\widehat{\varphi}\,dx'd\sigma(z') + O_\epsilon = 0,
\end{aligned} \tag{5.90}$$

To finish, proceeding similarly for the variational formulation (5.88), and using the estimates for $(\widehat{\mathbf{u}}_\epsilon,\widehat{\mathbf{w}}_\epsilon)$ given

in Lemma 5.6 and convergence (2.10) as in previous variational formulation, we deduce that (5.88) reads

$$\begin{aligned}
&\epsilon R_c \int_{\omega\times Z} \mathrm{rot}_{z_3}(\widehat{\mathbf{w}}'_\epsilon)\cdot \mathrm{rot}_{z_3}(\widehat{\psi}')\,dx'dz + \epsilon R_c \int_{\omega\times Z} \partial_{z_3}\widehat{w}_{3,\epsilon}\,\partial_{z_3}\widehat{\psi}_3\,dx'dz + 4N^2\epsilon \int_{\omega\times Z} \widehat{\mathbf{w}}_\epsilon\cdot\widehat{\psi}\,dx'dz\\
&-2N^2\int_{\omega\times Z}\widehat{\mathbf{u}}'_\epsilon\cdot\mathrm{rot}_{z_3}(\widehat{\psi}')\,dx'dz - 2N^2\beta\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}_\epsilon-\mathbf{s})\times\mathbf{n}\cdot\widehat{\psi}\,dx'd\sigma(z') \qquad (5.91)\\
&= -2N^2\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}_\epsilon\times\mathbf{n})\cdot\widehat{\psi}\,dx'd\sigma(z') + O_\epsilon,
\end{aligned}$$

where O_ϵ is devoted to tends to zero.

5.3 Convergences and limit problem

We give a compactness result concerning the related unfolding functions $(\widehat{\mathbf{u}}_\epsilon, \widehat{\mathbf{w}}_\epsilon, \widehat{P}_\epsilon)$ satisfying the *a priori* estimates given in Lemmas 5.6.

Lemma 5.7. *Consider the functions $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{w}}, \widetilde{p})$ obtained in Lemma 4.7. Then, for a subsequence of ϵ still denote by ϵ, we have the following convergence results:*

(i) (Velocity) There exists $\widehat{\mathbf{u}} = (\widehat{\mathbf{u}}', \widehat{u}_3) \in H^1(0,h(z');L^2_\#(\omega\times Z')^3)$, with $\widehat{u}_3\equiv 0$ and $\widehat{\mathbf{u}}'=0$ on $\widehat{\Gamma}_1$, such that it holds

$$\int_Z \widehat{\mathbf{u}}'(x',z)dz = \int_0^{h_{\max}} \widetilde{\mathbf{u}}'(x',z_3)\,dz_3\,, \qquad (5.92)$$

and moreover

$$\widehat{\mathbf{u}}_\epsilon \rightharpoonup \widehat{\mathbf{u}} \qquad \text{in } H^1(0,h(z');L^2(\omega\times Z')^3), \qquad (5.93)$$

$$\mathrm{div}_{z'}\widehat{\mathbf{u}}' = 0 \qquad \text{in } \omega\times Z, \qquad (5.94)$$

$$\mathrm{div}_{z'}\left(\int_0^{h(z')}\widehat{u}'\,dz_3\right) = 0 \qquad \text{in } \omega\times Z', \qquad (5.95)$$

$$\mathrm{div}_{x'}\left(\int_Z \widehat{u}'(x',z)\,dz\right) = 0 \qquad \text{in } \omega\,, \qquad (5.96)$$

$$\left(\int_Z \widehat{u}'(x',z)\,dz\right)\cdot\mathbf{n}' = 0 \qquad \text{on } \partial\omega\,.$$

(ii) (Microrotation) There exists $\widehat{\mathbf{w}}\in H^1(0,h(z');L^2_\#(\omega\times Z')^3)$, with $\widehat{w}_3\equiv 0$ and $\widehat{\mathbf{w}}'=0$ on $\widehat{\Gamma}_1$, such that it holds

$$\int_Z \widehat{\mathbf{w}}'(x',z)dz = \int_0^{h_{\max}} \widetilde{\mathbf{w}}'(x',z_3)\,dz_3, \qquad (5.97)$$

and moreover

$$\epsilon\widehat{\mathbf{w}}_\epsilon \rightharpoonup \widehat{\mathbf{w}} \quad \text{in } H^1(0,h(z');L^2(\omega\times Z')^3). \qquad (5.98)$$

(iii) (Pressure) The following convergence holds

$$\epsilon^2\widehat{P}_\epsilon \to \widetilde{p} \text{ in } L^2(\omega\times\Pi). \qquad (5.99)$$

Proof. The proof follows the lines of the proof of Lemma 4.7, so we will give some remarks. We start by (i). Estimates given in (5.84) imply the existence of $\hat{u} \in H^1(0, h(z'); L^2(\omega \times Z')^3)$ such that convergence (5.93) holds. As in Lemma 4.7, it holds that $\widehat{\mathbf{u}} = 0$ on $\omega \times \widehat{\Gamma}_1$ and $\widehat{u}_3 = 0$ on $\omega \times \widehat{\Gamma}_0$. It can also be proved the Z'-periodicity of $\hat{u}$. This can be obtained by proceeding as in [36, Lemma 5.4].

We have to take into account that applying the unfolded change of variables to the divergence condition $\mathrm{div}_\epsilon(\widetilde{u}_\epsilon) = 0$ and multiplying by ϵ, we get

$$\epsilon^{1-\ell}\mathrm{div}_{z'}(\widehat{\mathbf{u}}'_\epsilon) + \partial_{z_3}\widehat{u}_{3,\epsilon} = 0. \tag{5.100}$$

Passing to the limit, since relation (2.10), we get $\partial_{z_3}\widehat{u}_3 = 0$, which means that $\widehat{u}_3$ is independent of z_3. Due to the boundary conditions $\widehat{u}_3 = 0$ on $\omega \times (\widehat{\Gamma}_1 \cup \widehat{\Gamma}_0)$, it holds that $\widehat{u}_3 \equiv 0$.

Now, multiplying (5.100) by $\epsilon^{\ell-1}\widetilde{q}$ with $\widetilde{q}$ independent of z_3, after integrating by parts, we get

$$\int_{\omega\times Z} \widehat{\mathbf{u}}'_\epsilon \cdot \nabla_{z'}\widetilde{q}\, dx'dz = 0.$$

Passing to the limit and integrating by parts, we get (5.94). Observe that previous equality also can be written as follows

$$\int_{\omega\times Z'} \left(\int_0^{h(z')} \widehat{\mathbf{u}}'_\epsilon \, dz_3 \right) \nabla_{z'}\widetilde{q}\, dx'dz' = 0,$$

which passing to the limit and integrating by parts, gives (5.95).

For the proof of (5.92), we refer to [1, Lemma 5.4]. Next, putting relation (5.92) into the divergence condition (4.72), we get (5.96).

The proofs of (ii) is similar to the proof of (i) just taking into account the estimates of $\widehat{\mathbf{w}}_\epsilon$.

Finally, to prove (iii), we remark that the strong convergence of sequence $\epsilon^2\widehat{P}_\epsilon$ to $\widetilde{p}$ is a consequence of the strong convergence of $\epsilon^2\widetilde{P}_\epsilon$ to $\widetilde{p}$, see [20, Proposition 2.9].

□

Using convergences given in Lemma 4.7, we give the reduced two-pressured homogenized system satisfied by $(\widehat{\mathbf{u}}, \widehat{\mathbf{w}}, \widetilde{p})$.

Theorem 5.8 (Limit problem)**.** *The triplet of functions* $(\widehat{\mathbf{u}}, \widehat{\mathbf{w}}, \widetilde{p}) \in H^1(0, h(z'); L^2_\#(\omega\times Z')^3)\times H^1(0, h(z'); L^2_\#(\omega\times Z')^3) \times (L^2_0(\omega) \cap H^1(\omega))$*, with* $\widehat{u}_3 = \widehat{w}_3 \equiv 0$*, given in Lemma 4.7 is the unique solution of the two-pressure homogenized reduced micropolar problem*

$$\left\{ \begin{array}{rcll} -\partial^2_{z_3}\widehat{\mathbf{u}}' + \nabla_{z'}\widehat{\pi} - 2N^2\mathrm{rot}_{z_3}(\widehat{\mathbf{w}}') &=& -\nabla_{x'}\widetilde{p}(x') & \text{in } \omega \times Z, \\ -R_c\partial^2_{z_3}\widehat{\mathbf{w}}' + 4N^2\widehat{\mathbf{w}}' - 2N^2\mathrm{rot}_{z_3}(\widehat{\mathbf{u}}') &=& 0 & \text{in } \omega \times Z, \\ \widehat{\pi} \in L^2_\#(\omega \times Z'), & & & \end{array} \right. \tag{5.101}$$

with divergence conditions

$$\operatorname{div}_{z'}(\widehat{\mathbf{u}}') = 0 \qquad \text{in } \omega \times Z, \tag{5.102}$$

$$\operatorname{div}_{z'}\left(\int_0^{h(z')} \widehat{\mathbf{u}}'(x', z)\, dz_3\right) = 0 \qquad \text{in } \omega \times Z', \tag{5.103}$$

$$\operatorname{div}_{x'}\left(\int_Z \widehat{\mathbf{u}}'(x_1, z)\, dz\right) = 0 \qquad \text{in } \omega, \tag{5.104}$$

$$\left(\int_Z \widehat{u}'(x', z)\, dz\right) \cdot \mathbf{n}' = 0 \qquad \text{on } \partial\omega\,,$$

and boundary conditions

$$\widehat{\mathbf{u}}' = 0, \quad \widehat{\mathbf{w}}' = 0 \qquad \text{on } \omega \times \widehat{\Gamma}_1, \tag{5.105}$$

$$\partial_{z_3}\widehat{\mathbf{u}}' = -\frac{2}{\alpha}(\widehat{\mathbf{w}}')^{\perp} \qquad \text{on } \omega \times \widehat{\Gamma}_0, \tag{5.106}$$

$$\partial_{z_3}\widehat{\mathbf{w}}' = -2N\beta(\widehat{\mathbf{u}}' - \mathbf{s})^{\perp} \qquad \text{on } \omega \times \widehat{\Gamma}_0. \tag{5.107}$$

Proof. From Lemma 4.7, it remains to prove (5.101) and boundary conditions (5.106) and (5.107). We divide the proof in three steps.

Step 1. We prove $(5.101)_1$ with boundary condition (5.106). According to Lemma 4.7, we consider in (5.90) where $\widehat{\varphi}(x', z) \in \mathcal{D}(\omega; C^{\infty}_{\#}(Z)^2)$ with $\widehat{\varphi}_3 \equiv 0$ in $\omega \times Z$ and $\operatorname{div}_{z'}(\widehat{\varphi}') = 0$ in $\omega \times Z$. This gives

$$\begin{aligned} &\int_{\omega\times Z} \operatorname{rot}_{z_3}(\widehat{\mathbf{u}}'_\epsilon) \cdot \operatorname{rot}_{z_3}(\widehat{\varphi}')\, dx'dz - \epsilon^2 \int_{\omega\times\Pi} \widehat{P}_\epsilon \operatorname{div}_{x'}(\widehat{\varphi}')\, dx'dz \\ &-2N^2\epsilon \int_{\omega\times Z} \operatorname{rot}_{z_3}(\widehat{\varphi}') \cdot \widehat{\mathbf{w}}'_\epsilon\, dx'dz + 2\left(N^2 - \frac{1}{\alpha}\right)\epsilon \int_{\omega\times\widehat{\Gamma}_0} (\widehat{\mathbf{w}}_\epsilon \times \mathbf{n}) \cdot \widehat{\varphi}^b_\epsilon\, dx'd\sigma(z') + O_\epsilon = 0, \end{aligned} \tag{5.108}$$

where O_ϵ is devoted to tends to zero when $\epsilon \to 0$. Below, let us pass to the limit when ϵ tends to zero in each term of the previous variational formulation:

- First term. Using convergence (5.93) and the definition of the operator rot_{z_3} given in (3.33), we get

$$\begin{aligned} \int_{\omega\times Z} \operatorname{rot}_{z_3}(\widehat{\mathbf{u}}'_\epsilon) \cdot \operatorname{rot}_{z_3}(\widehat{\varphi}')\, dx'dz &= \int_{\omega\times Z} \operatorname{rot}_{z_3}(\widehat{\mathbf{u}}') \cdot \operatorname{rot}_{z_3}(\widehat{\varphi}')\, dx'dz + O_\epsilon \\ &= \int_{\omega\times Z} \partial_{z_3}\widehat{\mathbf{u}}' \cdot \partial_{z_3}\widehat{\varphi}'\, dx'dz + O_\epsilon. \end{aligned}$$

- Second term. Using convergence (5.99), we get

$$\epsilon^2 \int_{\omega\times\Pi} \widehat{P}_\epsilon \operatorname{div}_{x'}(\widehat{\varphi}')\, dx'dz = \int_{\omega\times\Pi} \widetilde{p} \operatorname{div}_{x'}(\widehat{\varphi}')\, dx'dz + O_\epsilon.$$

- Third term. Using convergence (5.98) and integration by parts, we get

$$\begin{aligned} -2N^2\epsilon \int_{\omega\times Z} \operatorname{rot}_{z_3}(\widehat{\varphi}') \cdot \widehat{\mathbf{w}}'_\epsilon\, dx'dz &= -2N^2 \int_{\omega\times Z} \operatorname{rot}_{z_3}(\widehat{\varphi}') \cdot \widehat{\mathbf{w}}'\, dx'dz + O_\epsilon \\ &= -2N^2 \int_{\omega\times Z} \operatorname{rot}_{z_3}(\widehat{\mathbf{w}}') \cdot \widehat{\varphi}'\, dx'dz + 2N^2 \int_{\omega\times\widehat{\Gamma}_0} (\widehat{\varphi}')^{\perp} \cdot \widehat{\mathbf{w}}' dx'd\sigma(z') + O_\epsilon \\ &= -2N^2 \int_{\omega\times Z} \operatorname{rot}_{z_3}(\widehat{\mathbf{w}}') \cdot \widehat{\varphi}'\, dx'dz - 2N^2 \int_{\omega\times\widehat{\Gamma}_0} (\widehat{\mathbf{w}}')^{\perp} \cdot \widehat{\varphi}' dx'd\sigma(z') + O_\epsilon. \end{aligned}$$

- Fouth term. By continuity of the trace operator from $H^1(0,h(z');L^2(\omega\times Z')^3)$ into $L^2(\omega\times\widehat{\Gamma}_0)$ and using convergence (5.98), we get the convergence of $\epsilon\widehat{\mathbf{w}}_\epsilon(x',z'0)$ to $\widehat{\mathbf{w}}$, and so we get

$$\begin{aligned}2\left(N^2-\frac{1}{\alpha}\right)\epsilon\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{w}}_\epsilon\times\mathbf{n})\cdot\widehat{\varphi}\,dx'd\sigma(z') &= 2\left(N^2-\frac{1}{\alpha}\right)\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{w}}\times\mathbf{n})\cdot\widehat{\varphi}\,dx'd\sigma(z')+O_\epsilon\\ &= 2\left(N^2-\frac{1}{\alpha}\right)\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{w}}')^\perp\cdot\widehat{\varphi}'\,dx'd\sigma(z')+O_\epsilon.\end{aligned}$$

Therefore, by previous convergences, we deduce that the limit variational formulation is given by the following one

$$\begin{aligned}&\int_{\omega\times Z}\partial_{z_3}\widehat{\mathbf{u}}'\cdot\partial_{z_3}\widehat{\varphi}'\,dx'dz+\int_{\omega\times\Pi}\widetilde{p}\,\mathrm{div}_{x'}(\widehat{\varphi}')\,dx'dz\\ &-2N^2\int_{\omega\times Z}\mathrm{rot}_{z_3}(\widehat{\mathbf{w}}')\cdot\widehat{\varphi}'\,dx'dz-\frac{2}{\alpha}\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{w}}')^\perp\cdot\widehat{\varphi}'\,dx'd\sigma(z')=0,\end{aligned}\tag{5.109}$$

for every $\widehat{\varphi}'\in\mathcal{D}(\omega;C^\infty_\#(Z)^2)$ with $\mathrm{div}_{z'}(\widehat{\varphi}')=0$ in $\omega\times Z$. By density, (5.109) holds for every function φ' in the $H^1(0,h(z_1);L^2_\#(\omega\times Z'))$ with $\mathrm{div}_{z'}(\widehat{\varphi}')=0$ in $\omega\times Z$ and is equivalent to problem $(5.101)_1$ with boundary condition (5.106), and $\widehat{\pi}$ the Lagrange multiplier arising from the free z'-divergence condition.

Step 2. We prove $(5.101)_2$ with boundary condition (5.107). According to Lemma 4.7, we consider in (5.90) where $\widehat{\psi}(x',z)\in\mathcal{D}(\omega;C^\infty_\#(Z)^2)$ with $\widehat{\psi}_3\equiv 0$ in $\omega\times Z$. This gives

$$\begin{aligned}&\epsilon R_c\int_{\omega\times Z}\mathrm{rot}_{z_3}(\widehat{\mathbf{w}}'_\epsilon)\cdot\mathrm{rot}_{z_3}(\widehat{\psi}')\,dx'dz+4N^2\epsilon\int_{\omega\times Z}\widehat{\mathbf{w}}'_\epsilon\cdot\widehat{\psi}'\,dx'dz\\ &-2N^2\int_{\omega\times Z}\widehat{\mathbf{u}}'_\epsilon\cdot\mathrm{rot}_{z_3}(\widehat{\psi}')\,dx'dz-2N^2\beta\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}_\epsilon-\mathbf{s})\times\mathbf{n}\cdot\widehat{\psi}\,dx'd\sigma(z')\\ &=-2N^2\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}_\epsilon\times\mathbf{n})\cdot\widehat{\psi}\,dx'd\sigma(z')+O_\epsilon,\end{aligned}\tag{5.110}$$

where O_ϵ is devoted to tends to zero when $\epsilon\to 0$. Below, let us pass to the limit when ϵ tends to zero in each term of the previous variational formulation:

- First term. Using convergence (5.98) and the definition of the operator rot_{z_3} given in (3.33), we get

$$\begin{aligned}\epsilon R_c\int_{\omega\times Z}\mathrm{rot}_{z_3}(\widehat{\mathbf{w}}'_\epsilon)\cdot\mathrm{rot}_{z_3}(\widehat{\psi}')\,dx'dz &= R_c\int_{\omega\times Z}\mathrm{rot}_{z_3}(\widehat{\mathbf{w}}')\cdot\mathrm{rot}_{z_3}(\widehat{\psi}')\,dx'dz+O_\epsilon\\ &= R_c\int_{\omega\times Z}\partial_{z_3}\widehat{\mathbf{w}}'\cdot\partial_{z_3}\widehat{\psi}'\,dx'dz+O_\epsilon.\end{aligned}$$

- Second term. Using convergence (5.98), we get

$$4N^2\epsilon\int_{\omega\times Z}\widehat{\mathbf{w}}'_\epsilon\cdot\widehat{\psi}'\,dx'dz=\ 4N^2\int_{\omega\times Z}\widehat{\mathbf{w}}'\cdot\widehat{\psi}'\,dx'dz+O_\epsilon.$$

- Third term. Using convergence (5.93) and integration by parts, we get

$$\begin{aligned}-2N^2\int_{\omega\times Z}\widehat{\mathbf{u}}'_\epsilon\cdot\mathrm{rot}_{z_3}(\widehat{\psi}')\,dx'dz &= -2N^2\int_{\omega\times Z}\widehat{\mathbf{u}}'\cdot\mathrm{rot}_{z_3}(\widehat{\psi}')\,dx'dz+O_\epsilon\\ &= -2N^2\int_{\omega\times Z}\widehat{\psi}'\cdot\mathrm{rot}_{z_3}(\widehat{\mathbf{u}}')\,dx'dz+2N^2\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\psi}')^\perp\cdot\widehat{\mathbf{u}}'\,dx'd\sigma(z')+O_\epsilon\\ &= -2N^2\int_{\omega\times Z}\widehat{\psi}'\cdot\mathrm{rot}_{z_3}(\widehat{\mathbf{u}}')\,dx'dz-2N^2\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}')^\perp\cdot\widehat{\psi}'\,dx'd\sigma(z')+O_\epsilon.\end{aligned}$$

- Fourth term. By continuity of the trace operator from $H^1(0,h(z');L^2(\omega\times Z')^3)$ into $L^2(\omega\times\widehat{\Gamma}_0)$ and from convergence (5.93), we get the convergence of $\widehat{\mathbf{u}}_\epsilon(x',z',0)$ to $\widehat{\mathbf{u}}(x',z',0)$ and so

$$\begin{aligned}-2N^2\beta\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}'_\epsilon-\mathbf{s})\times\mathbf{n}\cdot\widehat{\psi}\,dx'd\sigma(z') = &\quad -2N^2\beta\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}-\mathbf{s})\times\mathbf{n}\cdot\widehat{\psi}\,dx'd\sigma(z')+O_\epsilon\\ = &\quad -2N^2\beta\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}'-\mathbf{s}')^\perp\cdot\widehat{\psi}'\,dx'd\sigma(z')+O_\epsilon.\end{aligned}$$

- Five term. Similar to the previous term, from convergence (5.93), we get

$$-2N^2\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}_\epsilon\times\mathbf{n})\cdot\widehat{\psi}\,dx'd\sigma(z') = \quad -2N^2\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}')^\perp\cdot\widehat{\psi}'\,dx'd\sigma(z')+O_\epsilon.$$

Therefore, by previous convergences, we deduce that the limit variational formulation is given by the following one

$$\begin{aligned}&R_c\int_{\omega\times Z}\partial_{z_3}\widehat{\mathbf{w}}'\cdot\partial_{z_3}\widehat{\psi}'\,dx'dz+4N^2\int_{\omega\times Z}\widehat{\mathbf{w}}'\cdot\widehat{\psi}'\,dx'dz\\ &-2N^2\int_{\omega\times Z}\widehat{\psi}'\cdot\mathrm{rot}_{z_3}(\widehat{\mathbf{u}}')\,dx'dz-2N^2\beta\int_{\omega\times\widehat{\Gamma}_0}(\widehat{\mathbf{u}}'-\mathbf{s})^\perp\cdot\widehat{\psi}'\,dx'd\sigma(z')=0,\end{aligned}\tag{5.111}$$

for every $\widehat{\psi}'\in\mathcal{D}(\omega;C^\infty_\#(Z)^2)$. By density, (5.111) holds for every function ψ' in the $H^1(0,h(z_1);L^2_\#(\omega\times Z'))$, and is equivalent to problem $(5.101)_2$ with boundary condition (5.107).

Step 3. Conclusion. Since $\widehat{\varphi}'$ and $\widehat{\psi}$ are arbitrary, we derive from (5.109) and (5.111) that $(\widehat{\mathbf{u}}',\widehat{\mathbf{w}}',\widetilde{p},\widehat{\pi})$ satisfies the system (5.101) with boundary conditions (5.105)–(5.107). To ensure that the whole sequence $(\widehat{\mathbf{u}}_\epsilon,\epsilon\widehat{\mathbf{w}}_\epsilon,\epsilon^2\widehat{P}_\epsilon)$ converges, it remains to prove the existence and uniqueness of weak solution of the effective system (5.101). This follows the lines of the proof of Theorem 2.27, so we omit it.

□

6 Reynolds equation

In this section, we give the main result of this paper, i.e. the Reynolds equation for the pressure $\widetilde{p}$ and the expressions for the average velocity and microrotation. We will proceed as follows. First, we give the expressions of $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{w}}$ by solving problem (5.101) with boundary conditions (5.105)–(5.107). Next, we integrate in Z the expression of $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{w}}$, which gives the expression for the average velocity and microrotation, which are given depending on local problems. Finally, putting the expression of the average velocity into the incompressibility condition (5.104), we deduce the Reynolds equation.

Lemma 6.1. *Assuming condition (2.27) and $\alpha\neq1$, the solutions of (5.101) with boundary conditions (5.105)–(5.107) are given by the following expressions*

$$\begin{aligned}\widehat{\mathbf{u}}'(x',z) = &\quad\Big[\Big(\frac{2N^2}{k}\left(\sinh(kz_3)-\sinh(kh(z'))+\gamma_\alpha(z_3-h(z'))\right)A_1(z')\\ &+\frac{2N^2}{k}\left(\cosh(kz_3)-\cosh(kh(z'))B_1(z')+\frac{z_3^2-h(z')^2}{2(1-N^2)}\right)\Big](\nabla_{x'}\widetilde{p}(x')+\nabla_{z'}\widehat{\pi}(x',z'))\\ &+\Big[\Big(\frac{2N^2}{k}\left(\sinh(kz_3)-\sinh(kh(z'))+\gamma_\alpha(z_3-h(z'))\right)A_2(z')\\ &+\frac{2N^2}{k}\left(\cosh(kz_3)-\cosh(kh(z'))\right)B_2(z')\Big]\mathbf{s}',\end{aligned}\tag{6.112}$$

$$\begin{aligned}\widehat{\mathbf{w}}'(x',z) = &\Big[\Big(\cosh(kz_3)+\frac{\gamma_\alpha}{2}\Big)A_1(z')+\sinh(kz_3)B_1(z')+\frac{z_3}{2(1-N^2)}\Big](\nabla_{x'}\widetilde{p}(x')+\nabla_{z'}\widehat{\pi}(x',z'))^\perp\\ &+\Big[\Big(\cosh(kz_3)-\frac{\gamma_\alpha}{2}\Big)A_2(z')+\sinh(kz_3)B_2(z')\Big](\mathbf{s}')^\perp,\end{aligned}\tag{6.113}$$

where

$$k=2N\sqrt{\frac{1-N^2}{R_c}},\qquad \frac{\gamma_\alpha}{2}=\frac{1-\alpha N^2}{\alpha-1},\tag{6.114}$$

and A_i, B_2, $i=1,2$, *are given by*

$$\begin{aligned}A_1(z') = &\frac{L(z')}{2(1-N^2)}\Big[h(z')\Big(4N^4(1-\cosh(kh(z')))+\frac{R_c}{\beta}k^2\Big)\\ &-k\sinh(kh(z'))\Big(\frac{R_c}{\beta}-2N^2h(z')^2\Big)\Big],\end{aligned}\tag{6.115}$$

$$A_2(z') = -2N^2kL(z')\sinh(kh(z')),\tag{6.116}$$

$$\begin{aligned}B_1(z') = &\frac{L(z')}{2(1-N^2)}\Big[2N^2h(z')\Big(2N^2\sinh(kh(z'))+\gamma_\alpha kh(z')\Big)\\ &+k(\frac{R_c}{\beta}-2N^2h(z')^2)\Big(\cosh(kh(z'))+\frac{\gamma_\alpha}{2}\Big)\Big],\end{aligned}\tag{6.117}$$

$$B_2(z') = -2N^2kL(z')\Big[\frac{\gamma_\alpha}{2}+\cosh(kh(z'))\Big],\tag{6.118}$$

and

$$\begin{aligned}L(z') = -\Big[&\Big(\tfrac{\gamma_\alpha}{2}+\cosh(kh(z'))\Big)\Big(4N^4[1-\cosh(kh(z'))]+\tfrac{R_c}{\beta}k^2\Big)\\ &+2N^2\sinh(kh(z'))\Big(\gamma_\alpha kh(z')+2N^2\sinh(kh(z'))\Big)\Big]^{-1}.\end{aligned}\tag{6.119}$$

Proof. The proof is obtaining by solving system (5.101) for $(\widehat{u}_1,\widehat{w}_2)$ with corresponding boundary conditions (5.105)–(5.107), by observing that problem (5.101) is an ordinary differential equation with respect to z_3 (i.e. considering variables x', z' as parameters). The proof can be found in [3, Lemma 3.5] for a system (5.101) defined only for variables (x', z_3), but the proof is the same just taking into account that here, two pressures appear. Following the proof, we observe that variable x' only appears in the pressures $\widetilde{p}(x')$ and $\widehat{\pi}(x', z')$.

We note that $(\widehat{u}_1,\widehat{w}_2)$ and $(\widehat{u}_2,-\widehat{w}_1)$ satisfy the same equations and boundary conditions by writting s_2 and z_2 instead of s_1 and z_1. So, expression for $(\widehat{u}_2,\widehat{w}_1)$ are obtained straightforward, from expression of $(\widehat{u}_1,\widehat{w}_2)$, see [4, Appendix 1].

□

Lemma 6.2. *Assuming condition (2.27) and* $\alpha=1$, *the solutions of (5.101) with boundary conditions (5.105)–(5.107) are given by the following expressions*

$$\begin{aligned}\widehat{\mathbf{u}}'(x',z) = &\Big[\frac{2N^2}{k}(\cosh(kz_3)-\cosh(kh(z'))B_1'(z')+\frac{z_3^2-h(z')}{2(1-N^2)}+\frac{A_1'(z')}{1-N^2}(z_3-h(z'))\Big](\nabla_{x'}\widetilde{p}(x')+\nabla_{z'}\widehat{\pi}(x',z'))\\ &+\Big[\frac{2N^2}{k}(\cosh(kz_3)-\cosh(kh(z'))B_2'(z')+\frac{A_2'(z')}{1-N^2}(z_3-h(z'))\Big]\mathbf{s}',\end{aligned}\tag{6.120}$$

$$\begin{aligned}\widehat{\mathbf{w}}(x',z) = &\Big[\sinh(kz_3)B_1'(z')+\frac{z_3}{2(1-N^2)}+\frac{A_1'(z')}{2(1-N^2)}\Big](\nabla_{x'}\widetilde{p}(x')+\nabla_{z'}\widehat{\pi}(x',z'))^\perp\\ &+\Big[\sinh(kz_3)B_2'(z')+\frac{A_2'(z')}{2(1-N^2)}\Big](\mathbf{s}')^\perp,\end{aligned}\tag{6.121}$$

where k is given by (6.114) and A_i', B_i', $i=1,2$, are given by

$$A_1'(z') = L'(z')\Big[h(z')\Big(4N^4(1-\cosh(kh(z'))) + \frac{R_c}{\beta}k^2\Big) - k\sinh(kh(z'))\Big(\frac{R_c}{\beta} - 2N^2h^2(z')\Big)\Big], \quad (6.122)$$

$$A_2'(z') = 4N^2k(1-N^2)L'(z')\sinh(kh(z')), \quad (6.123)$$

$$B_1'(z') = k\frac{L(z')}{2(1-N^2)}\Big[2N^2h(z') + \frac{R_c}{\beta}\Big], \quad (6.124)$$

$$B_2'(z') = -2N^2kL'(z'), \quad (6.125)$$

and

$$L(z') = -\Big[4N^4(1-\cosh(kh(z')) + \frac{R_c}{\beta}k^2 + 4N^2kh(z')\sinh(kh(z'))\Big]^{-1}. \quad (6.126)$$

Proof. The proof is similar to the case $\alpha \neq 1$, see [3, Lemma 3.6] and [4, Appendix 2].

□

Lemma 6.3. *The z_3-average velocity and microrotation satisfy the following expressions:*

$$\widehat{\mathbf{U}}_{\rm av}'(x',z') = \int_0^{h(z')} \widehat{\mathbf{u}}'(x',z)\,dz_3 = -\Theta_1(z')(\nabla_{x'}\widetilde{p}(x') + \nabla_{z'}\widehat{\pi}(z')) + \Theta_2(z')\mathbf{s}', \qquad \widehat{U}_{\rm av,3} \equiv 0, \quad (6.127)$$

$$\widehat{\mathbf{W}}_{\rm av}'(x',z') = \int_0^{h(z')} \widehat{\mathbf{w}}'(x',z)\,dz_3 = \Phi_1(z')(\nabla_{x'}\widetilde{p}(x') + \nabla_{z'}\widehat{\pi}(z'))^{\perp} + \Phi_2(z')(\mathbf{s}')^{\perp}, \quad \widehat{W}_{\rm av,3} \equiv 0. \quad (6.128)$$

Here, functions Θ_i and Φ_i, $i=1,2$, are given depending on the case:

– *If $\alpha \neq 1$, then*

$$\begin{aligned}\Theta_1(z') = {} & \frac{h^3(z')}{3(1-N^2)} \\ & - \left[\frac{2N^2}{k}\Big(\frac{\cosh(kh(z'))-1}{k} - h(z')\sinh(kh(z'))\Big) - \frac{\gamma_\alpha}{2}h^2(z')\right]A_1(z') \\ & - \frac{2N^2}{k}\Big(\frac{\sinh(kh(z'))}{k} - h(z')\cosh(kh(z'))\Big)B_1(z'),\end{aligned} \quad (6.129)$$

$$\begin{aligned}\Theta_2(z') = {} & \left[\frac{2N^2}{k}\left(\frac{\cosh(kh(z'))-1}{k} - h(z')\sinh(kh(z'))\right) - \frac{\gamma_\alpha}{2}h^2(z')\right]A_2(z') \\ & + \frac{2N^2}{k}\left(\frac{\sinh(kh(z'))}{k} - h(z')\cosh(kh(z'))\right)B_2(z'),\end{aligned} \quad (6.130)$$

$$\Phi_1(z') = \frac{h(z')^2}{4(1-N^2)} + \Big(\frac{\sinh(kh(z'))}{k} + \frac{\gamma_\alpha}{2}\Big)A_1(z') + \frac{\cosh(kh(z'))-1}{k}B_1(z'), \quad (6.131)$$

$$\Phi_2(z') = \Big(\frac{\sinh(kh(z'))}{k} - \frac{\gamma_\alpha}{2}\Big)A_1(z') + \frac{\cosh(kh(z'))-1}{k}B_2(z'), \quad (6.132)$$

where A_i, B_i, $i=1,2$ are defined in Lemma 6.1.

– *If $\alpha = 1$, then*

$$\Theta_1(z') = \frac{h^3(z')}{3(1-N^2)} + \frac{h^2(z')}{2(1-N^2)} A_1'(z') - \frac{2N^2}{k}\left[\frac{\sinh(kh(z'))}{k} - h(z')\cosh(kh(z'))\right] B_1'(z'), \tag{6.133}$$

$$\Theta_2(z') = -\frac{h^2(z')}{2(1-N^2)} A_2'(z') + \frac{2N^2}{k}\left[\frac{\sinh(kh(z'))}{k} - h(z')\cosh(kh(z'))\right] B_2'(z'), \tag{6.134}$$

$$\Phi_1(z') = \frac{h^2(z')}{4(1-N^2)} + \frac{h(z')}{2(1-N^2)} A_1'(z') + \frac{\cosh(kh(z'))-1}{k} B_1'(z'), \tag{6.135}$$

$$\Phi_2(z') = \frac{h(z')}{2(1-N^2)} A_2'(z') + \frac{\cosh(kh(z'))-1}{k} B_2'(z'), \tag{6.136}$$

where A_i', B_i', $i = 1, 2$ are defined in Lemma 6.2.

Proof. Using divergence condition (5.103), we obtain for a.e. $x' \in \omega$ that

$$\int_{Z'} \left(\int_0^{h(z')} \widehat{\mathbf{u}}'(x', z)\, dz_3 \right) \nabla_{z'} \theta(z')\, dx' dz' = 0, \qquad \forall\, \theta \in H^1(Z').$$

From Lemmas 6.1 and 6.2, by averaging (6.112) or (6.120) with respect to z_3 between 0 and $h(z')$, we obtain

$$\widehat{\mathbf{U}}'_{\rm av}(x', z') = \int_0^{h(z')} \widehat{\mathbf{u}}'(x', z)\, dz_3 = -\Theta_1(z')(\nabla_{x'} \widetilde{p}(x') + \nabla_{z'} \widehat{\pi}(z')) + \mathbf{s}'\, \Theta_2(z'),$$

i.e. (6.127) with Θ_i, $i = 1, 2$ given by (6.129), (6.130) if $\alpha \neq 1$, and by (6.133), (6.134) if $\alpha = 1$. Similarly, we deduce (6.128) by averaging (6.113) or (6.121) with respect to z_3 between 0 and $h(z')$ as follows

$$\widehat{\mathbf{W}}'_{\rm av}(x', z') = \int_0^{h(z')} \widehat{\mathbf{w}}'(x', z)\, dz_3 = \Phi_1(z')(\nabla_{x'} \widetilde{p}(x') + \nabla_{z'} \widehat{\pi}(z'))^{\perp} + (\mathbf{s}')^{\perp} \Phi_2(z'),$$

with Φ_i, $i = 1, 2$ given by (6.131)–(6.132) if $\alpha \neq 1$, and by (6.135)–(6.136) if $\alpha = 1$.

□

Finally, we give the main result of this paper, i.e. the derivation of a Reynolds equation satisfied by $\widetilde{p}$ with the effects of the roughness of the top boundary and the non-standard boundary condition on the flat boundary of the original thin domain.

Theorem 6.4. *(Main result) Assuming condition (2.27) and considering the functions $(\widehat{\mathbf{u}}, \widehat{\mathbf{w}})$ given in Lemmas 6.1 and 6.2, we have that the z-average velocity and microrotation*

$$\widetilde{\mathbf{U}}_{\rm av}(x') = \int_Z \widehat{\mathbf{u}}(x', z)\, dz, \quad \widetilde{\mathbf{W}}_{\rm av}(x') = \int_Z \widehat{\mathbf{w}}(x', z)\, dz,$$

satisfy the following expressions

$$\begin{aligned} \widetilde{\mathbf{U}}_{\rm av}(x') &= K^{(1)} \nabla_{x'} \widetilde{p}(x') + L^{(1)} \mathbf{s}', \quad \widetilde{U}_{{\rm av},3} \equiv 0, \\ \widetilde{\mathbf{W}}_{\rm av}(x') &= K^{(2)} \nabla_{x'} \widetilde{p}(x')^{\perp} + L^{(2)} (\mathbf{s}')^{\perp}, \quad \widetilde{W}_{{\rm av},3} \equiv 0, \end{aligned} \tag{6.137}$$

for a.e. $x' \in \omega$. *Here,* $K^{(k)} \in \mathbb{R}^{2\times 2}$ *and* $L^{(k)} \in \mathbb{R}$, $k = 1, 2$, *are defined by*

$$\begin{aligned} K^{(1)} &= \int_{Z'} \Theta_1(z') \begin{pmatrix} \partial_{z_1} q^1(z') + 1 & \partial_{z_1} q^2(z') \\ \partial_{z_2} q^1(z') & \partial_{z_2} q^2(z') + 1 \end{pmatrix} dz', \quad L^{(1)} = \int_{Z'} \Theta_2(z')\, dz', \\ K^{(2)} &= \int_{Z'} \Phi_1(z') \begin{pmatrix} \partial_{z_2} q^2(z') + 1 & -\partial_{z_2} q^1(z') \\ -\partial_{z_1} q^2(z') & \partial_{z_1} q^1(z') + 1 \end{pmatrix} dz', \quad L^{(2)} = \int_{Z'} \Phi_2(z')\, dz', \end{aligned} \tag{6.138}$$

where function $q^i \in H^1_{\#}(Z')$, $i = 1, 2$, *satisfies the local problem*

$$\int_{Z'} \Theta_1(z')(\nabla_{z'} q^i(z') + \mathbf{e}_i) \cdot \nabla_{z'} \theta(z')\, dz' = \int_{Z'} \Theta_2(z') s_i \left(\mathbf{e}_i \cdot \nabla_{z'} \theta(z')\right) dz' \quad \forall\, \theta \in H^1(Z'). \tag{6.139}$$

Moreover, $\widetilde{p}$ *satisfies the following Reynolds equation*

$$\int_{\omega} K^{(1)} \nabla_{x'} \widetilde{p}(x') \cdot \nabla_{x'} \eta(x')\, dx' = \int_{\omega} L^{(1)} \mathbf{s}' \cdot \nabla_{x'} \eta(x')\, dx' \quad \forall\, \eta \in H^1(\omega). \tag{6.140}$$

Proof. From the expressions $\widehat{\mathbf{U}}'_{av}$ in (6.127) given by

$$\widehat{\mathbf{U}}'_{av}(x', z') = -\Theta_1(z')(\nabla_{x'} \widetilde{p}(x') + \nabla_{z'} \widehat{\pi}(z')) + \Theta_2(z')\mathbf{s}',$$

and divergence condition (5.103), i.e. $\mathrm{div}_{z'}(\widehat{\mathbf{U}}'_{av}) = 0$ in $\omega \times Z'$, we know that $\nabla_{z'} \widehat{\pi}$ holds the following problem

$$\int_{\omega \times Z'} \Theta_1(z')(\nabla_{x'} \widetilde{p}(x') + \nabla_{z'} \widehat{\pi}(z')) \cdot \nabla_{z'} \vartheta(x', z')\, dx' dz' = \int_{\omega \times Z'} \Theta_2(z') \mathbf{s}' \cdot \nabla_{z'} \vartheta(x', z')\, dx' dz' \quad \forall \vartheta \in H^1(\omega \times Z'). \tag{6.141}$$

Now, to find a problem for $\nabla_{x'} \widetilde{p}$ we proceed to eliminate the microscopic variable z'. To do this, we define

$$\widehat{\pi}(x', z') = \sum_{i=1}^{2} \partial_{x_i} \widetilde{p}(x') q^i(z') \quad \text{in } \omega \times Z',$$

and so, it holds

$$\nabla_{x'} \widetilde{p}(x') + \nabla_{z'} \widehat{\pi}(z') = \begin{pmatrix} \partial_{z_1} q^1(z') + 1 & \partial_{z_1} q^2(z') \\ \partial_{z_2} q^1(z') & \partial_{z_2} q^2(z') + 1 \end{pmatrix} \nabla_{x'} \widetilde{p}(x'),$$

and observe that this choice of $\widehat{\pi}$ and taking $\vartheta(x', z') = \eta(x')\theta(z')$ with $\eta \in H^1(\omega)$ and $\theta \in H^1(Z')$, then $\widehat{\pi}$ satisfies (6.141) by taking into account local problems (6.139) for q^i, $i = 1, 2$. Then, integrating $\widehat{\mathbf{U}}_{\mathrm{av}}(x', z')$ in Z', we deduce expression for $\widetilde{\mathbf{U}}_{\mathrm{av}}(x') = \int_{Z'} \widehat{\mathbf{U}}_{\mathrm{av}}(x', z')\, dz'$ given in (6.137).

Finally, putting expression of $\widetilde{\mathbf{U}}_{\mathrm{av}}$ in the divergence condition (5.104), i.e. $\mathrm{div}_{x'}(\widetilde{\mathbf{U}}_{\mathrm{av}}) = 0$ in ω, we deduce that $\widetilde{p}$ satisfies the Reynolds equation (6.140).

The derivation of the expression of $\widetilde{\mathbf{W}}'_{\mathrm{av}}$ given in $(6.137)_2$ is straightforward taking into account (6.128) and that

$$(\nabla_{x'} \widetilde{p}(x') + \nabla_{z'} \widehat{\pi}(z'))^{\perp} = \begin{pmatrix} \partial_{z_2} q^2(z') + 1 & -\partial_{z_2} q^1(z') \\ -\partial_{z_1} q^2(z') & \partial_{z_1} q^1(z') + 1 \end{pmatrix} (\nabla_{x'} \widetilde{p}(x'))^{\perp}.$$

Then, integrating $\widehat{\mathbf{W}}'_{\mathrm{av}}(x', z')$ in Z', we deduce expression for $\widetilde{\mathbf{W}}_{\mathrm{av}}(x') = \int_{Z'} \widehat{\mathbf{W}}_{\mathrm{av}}(x', z')\, dz'$ given in $(6.137)_2$.

□

References

[1] M. Anguiano, F.J. Suárez-Grau, Nonlinear Reynolds equations for non-Newtonian thin-film fluid flows over a rough boundary, IMA J. Appl. Math., **84** (2019) 63–95.

[2] M. Anguiano, F.J. Suárez-Grau, Mathematical derivation of a Reynolds equation for magneto-micropolar fluid flows through a thin domain, Z. Angew. Math. Phys., **75** (2024) 75: 28.

[3] G. Bayada, N. Benhaboucha and M. Chambat, New models in micropolar fluid and their application to lubrication, Math. Models Methods Appl. Sci., **15** (2005) 343–374.

[4] G. Bayada, N. Benhaboucha and M. Chambat, Wall slip induced by a micropolar fluid, J. Eng. Math., **60** (2008) 89–100.

[5] G. Bayada and M. Chambat, New models in the theory of the hydrodynamic lubrication of rough surfaces, J. Tribol., **110** (1988) 402–407.

[6] G. Bayada, M. Chambat, Homogenization of the Stokes system in a thin film flow with rapidly varying thickness, RAIRO Modél. Math. Anal. Numér. 23 (1989) 205–234.

[7] G. Bayada, M. Chambat and S.R. Gamouana, About thin film micropolar asymptotic equations, Quart. Appl. Math. **59** (2001) 413–439.

[8] G. Bayada, M. Chambat and S.R. Gamouana, Micropolar effects in the coupling of a thin film past a porous medium, **30** (2002) 187–216.

[9] G. Bayada and G. Lukaszewicz, On micropolar fluids in the theory of lubrication. Rigorous derivation of an analogue of the Reynolds equation. Internat. J. Engrg. Sci., **34** (1996) 1477–1490.

[10] M. Beneš, I. Pažanin, M. Radulović and B. Rukavina, Nonzero boundary conditions for the unsteady micropolar pipe flow: Well-posedness and asymptotics, Appl Math Comput., **427** (2022) 127184.

[11] N. Benhaboucha, M. Chambat and I. Ciuperca, Asymptotic behaviour of pressure and stresses in a thin film flow with a rough boundary, Quart. Appl. Math., **63** (2005) 369–400.

[12] M. Bonnivard, I. Pažanin and F.J. Suárez-Grau, Effects of rough boundary and nonzero boundary conditions on the lubrication process with micropolar fluid, Eur. J. Mech. B/Fluids, 72 (2018) 501–518.

[13] M. Bonnivard, I. Pažanin and F.J. Suárez-Grau, A generalized Reynolds equation for micropolar flows past a ribbed surface with nonzero boundary conditions, ESAIM: Math. Model. Numer. Anal. 56 (2022) 1255–1305.

[14] M. Boukrouche, A Reynolds equation rigorously derived from the micropolar Navier–Stokes system. Lecture Notes in Pure and Appl. Math., **223** (2001) 1–18.

[15] M. Boukrouche and L. Paoli, Asymptotic Analysis of a micropolar fluid flow in a thin domain with a free and rough boundary, SIAM J. Math. Anal., **44** (2012) 1211–1256.

[16] M. Boukrouche, L. Paoli and F. Ziane, Unsteady micropolar fluid flow in a thin domain with Tresca fluid–solid interface law, Comput. Math. Appl., **77** (2019) 2917–2932.

[17] M. Boukrouche, L. Paoli and F. Ziane, Micropolar fluid flow in a thick domain with multiscale oscillating roughness and friction boundary conditions, J. Math. Anal. Appl. **495** (2021) 124688.

[18] F. Boyer and P. Fabrie, Mathematical Tools for the Study of the Incompressible Navier?Stokes Equations and Related Models, Springer (2013).

[19] D. Cioranescu, A. Damlamian and G. Griso, Periodic unfolding and homogenization, C.R. Acad. Sci. Paris Ser. I 335 (2002) 99–104.

[20] D. Cioranescu, A. Damlamian and G. Griso, The periodic unfolding method in homogenization, SIAM J. Math. Anal. 40 (2008) 1585–1620.

[21] D. Dupuy, G. Panasenko and R. Stavre, Asymptotic methods for micropolar fluids in a tube structure, Math. Models Methods Appl. Sci. 14 (2004) 735–758.

[22] D. Dupuy, G. Panasenko and R. Stavre, Asymptotic solution for a micropolar flow in a curvilinear channel, ZAMM Z. Angew. Math. Mech. 88 (2008) 793–807.

[23] A.C. Eringen, Theory of micropolar fluids, J. Math. Mech. **16** (1966) 1–18.

[24] J. Fabricius, A. Tsandzana, F. Perez-Rafols and P. Wall, A Comparison of the Roughness Regimes in Hydrodynamic Lubrication, J. Tribol., **139** (2017) 051702.

[25] G. Lukaszewicz, Micropolar fluids, theory and applications, Modeling and Simulation in Science, Engineering and Technology (Birkhaüser, 1999).

[26] E. Marušić-Paloka, I. Pažanin and S. Marušić, An effective model for the lubrication with micropolar fluid, Mech. Res. Commun., 52 (2013) 69–73.

[27] E. Marušić-Paloka, I. Pažanin and M. Radulović, Flow of a micropolar fluid through a channel with small boundary perturbation Z. Naturforsch. A., **71** (2016) 607–619.

[28] E. Marušić-Paloka, I. Pažanin and M. Radulović, Justification of the higher order effective model describing the lubrication of a rotating shaft with micropolar fluid, Symmetry, **12** (2020) 334.

[29] A. Mikelić, Remark on the result on homogenization in hydrodynamical lubrication by G. Bayada and M. Chambat, RAIRO Modél. Math. Anal. Numér. 25 (1991) 363–370.

[30] J. C. Nakasato and I. Pažanin, Homogenization of the non-isothermal, non-Newtonian fluid flow in a thin domain with oscillating boundary, Z. Angew. Math. Phys., **74** (2023) 74: 211.

[31] I. Pažanin, Asymptotic behavior of micropolar fluid flow through a curved pipe, Acta Appl. Math., **116** (2011) 1–25.

[32] I. Pažanin, Asymptotic analysis of the lubrication problem with nonstandard boundary conditions for microrotation, Filomat, **30** (2016) 2233–2247.

[33] I. Pažanin and M. Radulović, Asymptotic analysis of the nonsteady micropolar fluid flow through a curved pipe, Appl. Anal., **99** (2020) 2045–2092.

[34] I. Pažanin and F.J. Suárez-Grau, Analysis of the thin film flow in a rough domain filled with micropolar fluid, Comput. Math. Appl. **68** (2014) 1915 –1932.

[35] I. Pažanin and F.J. Suárez-Grau, Roughness-induced effects on the thermomicropolar fluid flow through a thin domain, Stud. Appl. Math., **151** (2023) 716–751

[36] F.J. Suárez-Grau, Asymptotic behavior of a non-Newtonian flow in a thin domain with Navier law on a rough boundary, Nonlin. Anal. **117** (2015) 99–123.

[37] F.J. Suárez-Grau, Analysis of the roughness regime for micropolar fluids via homogenization, Bull. Malays. Math. Sci. Soc. **44** (2021) 1613–1652.

[38] F.J. Suárez-Grau, Mathematical modeling of micropolar fluid flows through a thin porous medium, J. Eng. Math., **126** (2021) 126: 7.

[39] F.J. Suárez-Grau, Homogenization of a micropolar fluid past a porous media with nonzero spin boundary condition, Math Meth Appl Sci., **44** (2021) 4835–4857.

[40] L. Tartar, Incompressible fluid flow in a porous medium convergence of the homogenization process. In: Appendix to Lecture Notes in Physics, 127. Berlin: Springer-Velag, 1980.