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► **To cite this version:**

Yves Capdeboscq, Eleanor Gemida. A CONSTRUCTIBLE CONDUCTIVITY CLOAK VIA HOMOGENISATION. 2024. hal-04671822

HAL Id: hal-04671822

<https://hal.science/hal-04671822v1>

Preprint submitted on 16 Aug 2024

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A CONSTRUCTIBLE CONDUCTIVITY CLOAK VIA HOMOGENISATION

YVES CAPDEBOSCQ¹ AND ELEANOR GEMIDA^{1,2}

ABSTRACT. We show that approximate cloaks can be constructed by a finite number of layers of isotropic materials. We detail a construction method, and we provide an upper bound on the number of layers required. Our approach is to view anisotropic matrix-valued materials obtained by the cloaking-by-mapping method as the effective limit of a multiple materials rank-one laminate, and to quantify precisely the approximation error. We illustrate our result and a variant in dimension two with an example.

1. INTRODUCTION

In the field of Electrical Impedance Tomography, given $\rho > 0$, a passive cloak of order of approximation ρ is a coating material surrounding a (large) object of arbitrary conductivity, such that the Dirichlet-to-Neumann data measured on the outer coating is similar to the data one would obtain if the object had a constant conductivity, save for a (small) defect in a ball of radius ρ (see Definition 3 for a precise statement). This cloaking material “hides” the object making it akin to a small defect. Candidates for such approximate cloaking coatings have been derived by the cloaking-by-mapping method [GLU03, KSVW08, MN06, PSS06]. The key observation, that boundary measurements are impervious to interior transformations by a diffeomorphism which preserves the outer boundary, was originally made by Luc Tartar in connection with discussions about the so-called Calderón problem, see [KV84, KV87] for more details.

The materials thus exhibited are anisotropic metamaterials, with very contrasted radial and tangential conductivities. Such cloaks were subsequently enhanced, by additionally requiring minimal anisotropy in appropriate norms [GV14, CV22].

In [AKLL13, AGJ⁺12] an alternative construction, composed of an infinite sequence of isotropic layers of two materials was introduced. Building upon this approach, a variant of that scheme with three materials was proposed in [HV14], and it was shown to be asymptotically an approximation of a cloaking-by-mapping type material.

In this paper, we construct a passive cloak of order of approximation ρ made of a finite number of isotropic layers of three or more different isotropic homogeneous materials. In such a case, the difference between maximal and minimal constant conductivities is a measure of the distance to the identity map, and plays the same role as the anisotropy indicators used in [GV14, CV22]. In this sense, the construction we propose has the same level of anisotropy as the optimal cloaks proposed in [CV22]. Furthermore, we specify the maximum number of layers required as a function of ρ . Because it is constituted of finitely many layers, of a finite number of prescribed materials, such a structure can be built by additive manufacturing : it is a radial laminate.

Our approach is based on homogenisation theory, which we apply backwards compared to the purpose for which it was developed. A key insight from homogenisation theory is that mixing

2000 *Mathematics Subject Classification.* 35J25,35B27.

Key words and phrases. Approximate cloaking, homogenisation, laminate composites.

This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 945332.

This study contributes to the IdEx Université de Paris ANR-18-IDEX-0001.

isotropic materials can result in an anisotropic effective medium. We view a given anisotropic approximate radial cloak obtained via cloaking-by-mapping as an effective material. We then construct a sequence of in-homogeneous locally isotropic media with finite scale microstructures converging to this limit. We then determine a lower bound on the required scale that guarantees that this heterogeneous material enjoys the same properties as the limit homogenised one.

In this article, the domain is the unit ball $\Omega = B_1 \subset \mathbb{R}^d$, with $d \geq 2$. Given $\rho \in (0, \frac{1}{4})$ and two constant conductivities positive α and γ such that

$$(1.1) \quad \gamma > 3^{d-1} \max(1, \rho^{d-3}) \quad \text{and} \quad \alpha < \left(\frac{\rho}{3}\right)^{d-1},$$

and given an integer $N \geq 1$, we set

$$A_N : x \rightarrow \sum_{k=0}^{N-1} \mathbf{1}_{[\frac{k}{N}, \frac{k+1}{N}]}(|x|) \mathbf{1}_{[\frac{1}{4}, 1]}(|x|) a_k^N(N|x|),$$

where for each k the piecewise constant function $y \rightarrow a_k^N(y)$ is a 1-periodic laminate made of three materials, one of conductivity γ , one of conductivity 1, and one of conductivity α , of proportions θ_γ^k , θ_1^k and θ_α^k , so that $\theta_\alpha^k + \theta_\beta^k + \theta_\gamma^k = 1$. Our main result is the following.

Theorem 1. *Provided*

$$N \geq \begin{cases} \rho^{-3} |\ln \rho|^2 \left(\frac{\rho}{\alpha} + \rho\gamma + 1\right) & \text{when } d = 2 \\ \rho^{1-2d} \left(\frac{\rho^{d-1}}{\alpha} + \rho\gamma + 1\right) & \text{when } d \geq 3 \end{cases},$$

the conductivity A_N with proportions θ_γ^k , θ_1^k and θ_α^k as given in (2.22) and (2.20) is an approximate cloak of order ρ . In other words, given any $B \in L^\infty(B_{\frac{1}{4}})$ a symmetric positive definite matrix, let

$$A_N^B(x) = \begin{cases} B & \text{if } |x| < \frac{1}{4} \\ A_N(|x|) & \text{if } |x| \geq \frac{1}{4} \end{cases}.$$

The Dirichlet-to-Neumann map

$$\Lambda_{A_N^B} : H^{\frac{1}{2}}(\partial B_1) \rightarrow H^{-\frac{1}{2}}(\partial B_1)$$

$$\phi \rightarrow \left(A_N^B \nabla u \cdot \frac{x}{|x|} \right) \quad \text{where} \quad \begin{cases} \operatorname{div}(A_N^B \nabla u) = 0 & \text{in } B_1, \\ u = \phi & \text{on } \partial B_1 \end{cases}$$

satisfies

$$\left\| \Lambda_{A_N^B} - \Lambda_{I_d} \right\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial B_1), H^{-\frac{1}{2}}(\partial B_1))} \leq C \rho^d,$$

where C depends only on the dimension d , and the map Λ_{I_d} is the Dirichlet-to-Neumann map of the Laplacian on B_1 .

Remark 2. To minimise N , and taking into account (1.1), one could choose $\gamma = \frac{1}{\alpha} = \frac{4}{\rho}$ when $d = 2$. In that case, a possible choice is $N \propto \rho^{-3} |\ln \rho|^2$. We do not know that the power of ρ in the lower bound for N is sharp. The constraints on the conductivity α and γ imply very insulating materials and very conductive materials in dimension 2. While such a contrast is necessary for our approach, it isn't the case that materials with such high specifications should be used throughout the cloak. For a discussion on strategies to reduce the number of layers, see Section 5, where we allow for coarser scales and lower contrast materials in some parts of the cloak. Building on this, one could then optimise the structure to minimise a cost functional dependent of the amount of α and γ material used, for example.

Our main result presents a cloak where the scale of the microstructure, that is, $1/N$, is constant throughout the cloak. In the last section, we propose a variant where the scale of the microstructure varies, allowing wider layers of homogeneous materials in some parts of the cloak.

Our paper is organised as follows. In Section 2, we remind the readers of the cloaking-by-mapping method, and we exhibit a bijective diffeomorphism $\bar{B}_1 \setminus B_\rho \rightarrow \bar{B}_1 \setminus B_{\frac{1}{2}}$ which will be convenient for our purpose, because it is the identity map on a neighbourhood of ∂B_1 . We then choose a cloaking-by-mapping matrix based on this diffeomorphism, tweaked so that near $\partial B_{1/4}$ it is also the identity.

In Section 2.3, we detail how a piecewise constant and isotropic cloaking material can be obtained by radial lamination. While a cloak cannot be realised as the rank-one lamination of two materials, it can be realised as a rank-one lamination of three materials (or more) provided that the maximal conductivity is large enough, and the minimal conductivity is small enough - this will lead to the bounds (1.1).

In Section 3 we prove Theorem 1. It amounts to establishing a regularity estimate, and proving a quantitative convergence result.

In Section 2 we detail what properties the diffeomorphism show enjoy. The proof of Theorem 1 applies with any diffeomorphism and extension satisfying these specifications. In Section 4 we write down a diffeomorphism that fits all the requirements.

In Section 5 we discuss a variant of our main result in dimension two, involving less contrasted materials and variable layer widths, and end the paper with concluding remarks.

2. CONSTRUCTION OF A CLOAKING MATERIAL

2.1. Cloaking by change of variables. The domain is $\Omega = B_1$. Let σ uniformly elliptic symmetric matrix-valued function, that is, there exists $0 < \eta < \infty$ such that $\forall \xi \in \mathbb{R}^d$ there holds

$$(2.1) \quad \eta |\xi|^2 \leq \sigma(x) \xi \cdot \xi \text{ and } \eta |\xi|^2 \leq \sigma^{-1}(x) \xi \cdot \xi, \text{ a.e. } x \in \Omega.$$

Let $u \in H^1(\Omega)$ be the solution of

$$(2.2) \quad \begin{cases} \operatorname{div}(\sigma \nabla u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

for some given $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$. Boundary value problem (2.2) gives rise to a Dirichlet-to-Neumann map Λ_σ defined as

$$(2.3) \quad \Lambda_\sigma : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$$

$$(2.4) \quad u|_{\partial\Omega} \mapsto (\sigma \nabla u) \cdot \nu|_{\partial\Omega}.$$

We note Λ_{I_d} the Dirichlet-to-Neumann map when σ is the identity matrix.

Definition 3 ([KSVW08]). Let $D \subset \Omega$ be a measurable set. Let σ be a non-negative matrix-valued function defined on $\Omega \setminus D$. For any $\rho \in (0, \frac{1}{2})$, we say σ is an approximate cloak of the region D of order ρ if there exists a constant C such that for any symmetric positive definite extensions $B \in L^\infty(\Omega)$ across D ,

$$\sigma^B(x) = \begin{cases} B(x) & \text{for } x \in D \\ \sigma(x) & \text{for } x \in \Omega \setminus D \end{cases}$$

there holds

$$\|\Lambda_{\sigma^B} - \Lambda_{I_d}\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial B_1), H^{-\frac{1}{2}}(\partial B_1))} \leq C\rho^d.$$

We now consider the case when $D = B_{\frac{1}{2}}$ and we explain how an approximate cloak of order ρ for $B_{\frac{1}{2}}$ is obtained via the cloaking-by-mapping approach.

For a small parameter $0 < \rho < \frac{1}{2}$, let F be a C^2 bijective diffeomorphism from B_1 onto B_1 , such that $F(\bar{B}_1 \setminus B_\rho) = \bar{B}_1 \setminus B_{\frac{1}{2}}$. Suppose that $F|_{r=1} = I_d$ and $F(\partial B_\rho) = \partial B_{1/2}$. The associated near-cloak is the push-forward of the identity matrix by F given by

$$(2.5) \quad F_*[I_d] = \frac{DFDF^T}{|\det DF|} \circ F^{-1}$$

restricted to $\bar{B}_1 \setminus B_{\frac{1}{2}}$. Indeed, let B be a scalar or matrix-valued coefficient in $B_{\frac{1}{2}}$ and assume that it is positive definite and satisfies (2.1), for some η (which depends on B). If we consider the coefficient of the form

$$(2.6) \quad \sigma^B(y) = \begin{cases} B(y) & \text{in } B_{\frac{1}{2}} \\ F_*[I_d](y) & \text{in } B_1 \setminus B_{\frac{1}{2}} \end{cases}$$

then by a pull-back to $B_1 \setminus B_\rho$ (change of variables by F^{-1}), we obtain the coefficient

$$(2.7) \quad F_*^{-1}[\sigma^B](x) = \begin{cases} F_*^{-1}[B](x) & \text{in } B_\rho \\ I_d & \text{in } B_1 \setminus B_\rho \end{cases}.$$

We have the identity $\Lambda_{\sigma^B} = \Lambda_{F_*^{-1}[\sigma^B]}$ because F is the identity on the boundary. In other words, the boundary measurements associated with σ^B are identical to that obtained in a homogeneous region with a perturbation in the ball B_ρ , possibly very heterogeneous, described by $F_*^{-1}[B]$. It turns out that positive definite perturbations contained in a region of radius ρ well within the domain, contribute at most to a perturbation of the Dirichlet-to-Neumann map of order ρ^d , independently of the positive definite coefficient in the ball [NV09, theorem 1].

Theorem 4 ([KSVW08]). *Suppose the annulus $B_1 \setminus B_{\frac{1}{2}}$ has coefficient σ^B given in (2.6). There holds*

$$(2.8) \quad \|\Lambda_{\sigma^B} - \Lambda_{I_d}\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial B_1), H^{-\frac{1}{2}}(\partial B_1))} \leq C\rho^d$$

where C is an absolute constant. Equivalently,

$$(2.9) \quad \left| \int_{B_1} \sigma^B \nabla u_\rho \cdot \nabla u_\rho - \int_{B_1} \nabla U \cdot \nabla U \right| \leq C\rho^d \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2 \text{ for all } \varphi \in H^{\frac{1}{2}}(\partial B_1).$$

u_ρ and U are the solutions of (2.2) for $\sigma = \sigma^B$ and $\sigma = 1$, respectively.

Remark 5. When $d = 2$, a possible value for the constant C in (2.9) is $C = 16 \frac{2J'_{0,0} + 4}{J'_{0,0}} < 50$. We refer to [FV89, KSVW08] for the details. Note that the expression of F_*^{-1} within B_ρ is irrelevant.

Remark. It is well known that (2.8) and (2.9) are equivalent (see e.g. [FV89]). For any $\varphi \in H^{\frac{1}{2}}(\partial B_1)$, there holds

$$\begin{aligned} \int_{B_1} \sigma^B \nabla u_\rho \cdot \nabla u_\rho - \int_{B_1} \nabla U \cdot \nabla U &= \int_{\partial B_1} (\sigma^B \nabla u) \cdot n \varphi d\sigma - \int_{\partial B_1} \partial_n U \varphi d\sigma \\ &= \langle (\Lambda_{\sigma^B} - \Lambda_{I_d}) \varphi, \varphi \rangle_{H^{-\frac{1}{2}}(\partial B_1), H^{\frac{1}{2}}(\partial B_1)}, \end{aligned}$$

thus if (2.8) holds then

$$\left| \int_{B_1} \sigma^B \nabla u_\rho \cdot \nabla u_\rho - \int_{B_1} \nabla U \cdot \nabla U \right| \leq \|\Lambda_{\sigma^B} - \Lambda_{I_d}\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial B_1), H^{-\frac{1}{2}}(\partial B_1))} \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2 \leq C\rho^d \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2.$$

Conversely, by the polarisation identity, for any non zero $\varphi, \psi \in H^{\frac{1}{2}}(\partial B_1)$ and any $t > 0$

$$\begin{aligned} \langle (\Lambda_{\sigma^B} - \Lambda_{I_d}) \varphi, \psi \rangle_{H^{-\frac{1}{2}}(\partial B_1), H^{\frac{1}{2}}(\partial B_1)} &= \frac{1}{4} \left\langle (\Lambda_{\sigma^B} - \Lambda_{I_d}) \left(t\psi + \frac{1}{t}\varphi \right), t\psi + \frac{1}{t}\varphi \right\rangle_{H^{-\frac{1}{2}}(\partial B_1), H^{\frac{1}{2}}(\partial B_1)} \\ &\quad - \frac{1}{4} \left\langle (\Lambda_{\sigma^B} - \Lambda_{I_d}) \left(t\psi - \frac{1}{t}\varphi \right), t\psi - \frac{1}{t}\varphi \right\rangle_{H^{-\frac{1}{2}}(\partial B_1), H^{\frac{1}{2}}(\partial B_1)} \end{aligned}$$

Thus when (2.9) holds, and choosing $t^2 = \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)} / \|\psi\|_{H^{\frac{1}{2}}(\partial B_1)}$ yields

$$\begin{aligned} &\left| \langle (\Lambda_{\sigma^B} - \Lambda_{I_d}) \varphi, \psi \rangle_{H^{-\frac{1}{2}}(\partial B_1), H^{\frac{1}{2}}(\partial B_1)} \right| \\ &\leq C\rho^d \left(\left\| t\psi - \frac{1}{t}\varphi \right\|_{H^{\frac{1}{2}}(\partial B_1)}^2 + \left\| t\psi + \frac{1}{t}\varphi \right\|_{H^{\frac{1}{2}}(\partial B_1)}^2 \right) \\ &= C\rho^d \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)} \|\psi\|_{H^{\frac{1}{2}}(\partial B_1)}, \end{aligned}$$

therefore (2.8) holds. \square

2.2. A radial transformation . Given $\delta \in (0, \frac{1}{8}]$, let $g \in C^2([0, 1], [0, 1])$ be a C^2 diffeomorphism such that $g|_{[\rho, 1]} \in C^2([\rho, 1], [\frac{1}{2}, 1])$ is also a C^2 diffeomorphism, with $g(x) = x$ on $[0, \delta] \cup [1 - \delta, 1]$, $g(\rho) = \frac{1}{2}$. We further require that $t \rightarrow \frac{t(g^{-1})'(t)}{g^{-1}(t)}$ is non-increasing on $[\frac{1}{2}, 1]$. We finally impose that $\max_{[\frac{1}{2}, 1]} \frac{t(g^{-1})'(t)}{g^{-1}(t)} < \frac{2}{\rho}$. An example is given in Section 4. The radial transformation given by

$$\begin{aligned} F : \bar{B}_1 &\rightarrow \bar{B}_1 \\ x &\rightarrow g(|x|) \frac{x}{|x|} \end{aligned}$$

is a diffeomorphism. The approximate cloak associated to F on $\bar{B}_1 \setminus B_{\frac{1}{2}}$ is given by

$$(2.10) \quad F_* [I_d] = \left(\frac{g^{-1}(|x|)}{|x|} \right)^{d-2} \left(\frac{1}{\lambda(|x|)} \frac{x}{|x|} \otimes \frac{x}{|x|} + \lambda(|x|) \left(I_d - \frac{x}{|x|} \otimes \frac{x}{|x|} \right) \right),$$

where

$$\lambda(t) = \left| \left(\frac{1}{g'} \left(\frac{g}{t} \right) \right) \circ g^{-1}(t) \right| = \left| \frac{t(g^{-1})'(t)}{g^{-1}(t)} \right| \text{ for } t \in \left[\frac{1}{2}, 1 \right]$$

is a non-increasing function. On the inner boundary of the cloak, $r = \frac{1}{2}$, the eigenvalue in the tangential direction is maximal. This poses challenges for the quantitative analysis of the approximation scheme we propose, when it comes to obtaining estimates on the boundary. For this reason, we smoothly extend $\lambda(r)$ to small region to the left of $\frac{1}{2}$.

Proposition 6. *There exists a function $\lambda_{ext} \in C^1([0, 1])$, non-decreasing then non-increasing (one bump) such that*

$$(2.11) \quad \lambda_{ext} = 1 \text{ on } \left[0, \frac{1}{4} + \delta \right], \quad \lambda_{ext} = \lambda \text{ on } \left[\frac{1}{2}, 1 \right],$$

$$(2.12) \quad 1 \leq \lambda_{ext} \leq \frac{3}{2} \lambda \left(\frac{1}{2} \right) < \frac{3}{\rho}.$$

Furthermore, there holds

$$(2.13) \quad \|\lambda_{ext}\|_{L^1(0,1)} \leq |\ln \rho| \quad \text{and} \quad \|\lambda_{ext}\|_{L^n(0,1)}^n \leq \frac{1}{\rho^{n-1}} \quad \text{for } n \geq 2.$$

When $d = 2$, set $h_{ext} = 1$. When $d \geq 3$, there exists a positive function $h_{ext} \in C^1([0,1])$ such that

$$(2.14) \quad h_{ext} = 1 \quad \text{on} \quad \left[0, \frac{1}{4} + \delta\right], \quad h_{ext}(t) = \left(\frac{g^{-1}(t)}{t}\right)^{d-2} \quad \text{on} \quad \left[\frac{1}{2}, 1\right],$$

$$(2.15) \quad h_{ext} \leq 1 \quad \text{and} \quad 1 \leq (\lambda_{ext})^{d-2} h_{ext} \leq 2^{d-2}.$$

As a result, the matrix-valued radial function σ given by

$$(2.16) \quad \sigma(x) = h_{ext}(|x|) \left(\frac{1}{\lambda_{ext}(|x|)} \frac{x}{|x|} \otimes \frac{x}{|x|} + \lambda_{ext}(|x|) \left(I_d - \frac{x}{|x|} \otimes \frac{x}{|x|} \right) \right), \quad \text{for } x \in \overline{B_1} \setminus B_{\frac{1}{4}}$$

is such that $\sigma(x) = I_d$ for $x \in B_{\frac{1}{4}+\delta} \setminus B_{\frac{1}{4}}$ and $x \in B_1 \setminus B_{1-\delta}$.

Proof. An example is written down in Section 4. \square

Proposition 7. Given $B \in L^\infty\left(B_{\frac{1}{4}}; \mathbb{R}^{d \times d}\right)$ a symmetric definite positive matrix-valued function satisfying (2.1) for some $\eta > 0$, let

$$\sigma^B(x) = \begin{cases} B(x) & \text{if } x \in B_{\frac{1}{4}}, \\ \sigma(x) & \text{if } x \in \overline{B_1} \setminus B_{\frac{1}{4}}, \end{cases}$$

where σ is given by (2.16). Then,

$$\|\Lambda_{\sigma^B} - \Lambda_{I_d}\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial B_1), H^{-\frac{1}{2}}(\partial B_1))} \leq C\rho^d$$

where C depends only on d .

Given any $\phi \in H^{\frac{1}{2}}(\partial B_1)$, we refer to $u \in H^1(B_1)$ as the function satisfying $u|_{\partial B_1} = \phi$ in the sense of traces and such that

$$\int_{B_1} \sigma^B \nabla u \cdot \nabla v dx = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

There holds

$$(2.17) \quad \langle \Lambda_{\sigma^B}(\phi), \phi \rangle_{H^{-\frac{1}{2}}(\partial B_1), H^{\frac{1}{2}}(\partial B_1)} = \int_{B_1} \sigma^B \nabla u \cdot \nabla u dx.$$

Proof of Proposition 7. While it follows from Theorem 4, we include a proof for completeness. Setting $x = F_\sigma(y)$, we obtain that

$$\begin{aligned} \langle \Lambda_{\sigma^B}(\phi), \phi \rangle_{H^{-\frac{1}{2}}(\partial B_1), H^{\frac{1}{2}}(\partial B_1)} &= \left\langle \Lambda_{I_d + 1_{B_\rho}}(\tilde{B} - I_d)(\phi), \phi \right\rangle_{H^{-\frac{1}{2}}(\partial B_1), H^{\frac{1}{2}}(\partial B_1)} \\ &= \int_{B_1 \setminus B_\rho} \nabla \tilde{u} \cdot \nabla \tilde{u} dx + \int_{B_\rho} \tilde{B} \nabla \tilde{u} \cdot \nabla \tilde{u} dx, \end{aligned}$$

where $\tilde{u} \in H^1(B_1)$ satisfies $\tilde{u}|_{\partial B_1} = \phi$ in the sense of traces and

$$\int_{B_1 \setminus B_\rho} \nabla \tilde{u} \cdot \nabla v dx + \int_{B_\rho} \tilde{B} \nabla \tilde{u} \cdot \nabla v dx = 0 \quad \text{for all } v \in H_0^1(\Omega),$$

where $\tilde{B} \in L^\infty(B_\rho)$ is a symmetric positive definite matrix. Thanks to the uniform boundedness result [NV09, theorem 1], there holds

$$\|\Lambda_{I_d} - \Lambda_{I_d+1_{B_\rho}}(\tilde{B}-I_d)\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial B_1), H^{-\frac{1}{2}}(\partial B_1))} \leq C\rho^d.$$

We used the notation that for set D its indicator function is noted 1_D . Since $\Lambda_{I_d+1_{B_\rho}}(\tilde{B}-I_d) = \Lambda_{\sigma^B}$, the conclusion follows. \square

2.3. A piecewise isotropic cloak by lamination. The matrix σ given by (2.16) is anisotropic and smooth, whereas our purpose is to produce a cloak that is piecewise constant and scalar. We make use of the fact that σ depends only on the radial variable to un-laminate σ , that is, to introduce a sequence of laminated structures whose effective limit (the limit in the sense of H -convergence see e.g. [BLP11, CD99, Mil02, All02, Tar09]) is σ . Given $B \in L^\infty(B_{\frac{1}{4}}; \mathbb{R}^{d \times d})$ a symmetric definite positive matrix, for $n \geq 1$, let

$$(2.18) \quad A_n^B(x) = \begin{cases} B(x) & \text{if } x \in B_{\frac{1}{4}}, \\ a_k(n|x|) & \text{if } x \in [\frac{k}{n}, \frac{k+1}{n}] \cap [\frac{1}{4}, 1], \quad k \in \{0, \dots, n-1\}, \end{cases}$$

where, for each $k \in \{0, \dots, n-1\}$ a_k is a 1-periodic function, such that

$$a_k(y) = \begin{cases} \alpha & \text{if } 0 \leq y < \theta_\alpha(\frac{k}{n}) \\ 1 & \text{if } \theta_\alpha(\frac{k}{n}) \leq y < \theta_\alpha(\frac{k}{n}) + \theta_1(\frac{k}{n}) \text{ for all } y \in [0, 1). \\ \gamma & \text{if } \theta_\alpha(\frac{k}{n}) + \theta_1(\frac{k}{n}) \leq y < 1, \end{cases}$$

and the functions θ_α , θ_1 , and $\theta_\gamma = 1 - \theta_\alpha - \theta_1$ are chosen so that

$$(2.19) \quad \begin{cases} \theta_\alpha(r)\alpha + \theta_1(r) + \theta_\gamma(r)\gamma = \lambda_{\text{ext}}(r)h_{\text{ext}}(r) \\ \frac{\theta_\alpha(r)}{\alpha} + \theta_1(r) + \frac{\theta_\gamma(r)}{\gamma} = \frac{\lambda_{\text{ext}}(r)}{h_{\text{ext}}(r)} \end{cases}.$$

In other words, the mean value of a_k equals $\lambda_{\text{ext}}(\frac{k}{n})h_{\text{ext}}(\frac{k}{n})$, whereas its harmonic mean equals $\lambda_{\text{ext}}(\frac{k}{n})^{-1}h_{\text{ext}}(\frac{k}{n})$. Explicitly, this means

$$(2.20) \quad \begin{cases} \theta_\alpha(r) = \frac{\alpha(\lambda_{\text{ext}}(r)(h_{\text{ext}}^2(r)+\gamma) - h_{\text{ext}}(r)(\gamma+1))}{h_{\text{ext}}(r)(1-\alpha)(\gamma-\alpha)} \\ \theta_1(r) = 1 - \frac{\lambda_{\text{ext}}(r)(h_{\text{ext}}^2(r)+\alpha\gamma) - h_{\text{ext}}(r)(\alpha\gamma+1)}{h_{\text{ext}}(r)(\gamma-1)(1-\alpha)} \\ \theta_\gamma(r) = \frac{\gamma(\lambda_{\text{ext}}(r)(h_{\text{ext}}^2(r)+\alpha) - h_{\text{ext}}(r)(\alpha+1))}{h_{\text{ext}}(r)(\gamma-\alpha)(\gamma-1)} \end{cases}.$$

It is a straightforward calculation to verify that provided

$$(2.21) \quad \begin{cases} \left(\max \frac{\lambda_{\text{ext}}}{h_{\text{ext}}}\right)\alpha < 1 \\ \gamma \geq \max \left(\frac{\lambda_{\text{ext}}h_{\text{ext}} - \alpha}{1 - \alpha \frac{\lambda_{\text{ext}}}{h_{\text{ext}}}}\right) \end{cases}$$

then θ_1 , θ_α and θ_γ are in $C^1([\frac{1}{4}, 1], [0, 1])$. Condition (2.21) holds under assumption (1.1).

On $B_1 \setminus B_{\frac{1}{4}}$, $A_n^B(x) = A_n(|x|)$, where for all $t \in [0, 1]$

$$(2.22) \quad A_n(t) = \sum_{k=0}^{n-1} \left(\alpha 1_{[0, \theta_\alpha(\frac{k}{n})]}(nt-k) + 1_{[\theta_\alpha(\frac{k}{n}), \theta_\alpha(\frac{k}{n}) + \theta_1(\frac{k}{n})]}(nt-k) + \gamma 1_{[\theta_\alpha(\frac{k}{n}) + \theta_1(\frac{k}{n}), 1]}(nt-k) \right).$$

Remark 8. When $r > 1 - \delta$, or when $r < \frac{1}{4} + \delta$ then $\theta_\gamma(r) = \theta_\alpha(r) = 0$, and $\theta_1(r) = 1$. Thus if $t \in [0, \frac{1}{4} + \delta] \cup [1 - \delta, 1]$, then $A_n(t) = 1$.

Remark 9. Note that if only two materials were present instead of three (setting any of $\theta_1, \theta_\gamma, \theta_\alpha$ constant equal to zero), then system (2.19) could be solved for at most one value of λ_{ext} . A simple laminate of two isotropic materials cannot cloak $B_{\frac{1}{4}}$. It is possible to cloak with two isotropic materials using rank-2 sequential lamination (laminates of laminates in orthogonal directions, see e.g. [All02]), but this results in the introduction of much finer scales. The fact that layering using only two materials isn't enough was already observed in [HV14, AKLL13]: here, in context of effective media, it is the result of a direct calculation.

Note also that the fact that exactly three materials were used is not important: we only use that (2.19) is satisfied.

Lemma 10. *Let λ_{ext} be given by Proposition 6, and A_n be given by (2.22). Set*

$$(2.23) \quad \eta_1(n, \rho) = \begin{cases} |\ln \rho| + \frac{1}{n\rho} & \text{when } d = 2 \\ 1 & \text{when } d \geq 3 \end{cases}, \quad \text{and } \eta_2(n, \rho) = \begin{cases} |\ln \rho| + \frac{1}{n\rho} & \text{when } d = 2 \\ \rho^{2-d} \left(1 + \frac{1}{n\rho}\right) & \text{when } d \geq 3 \end{cases}.$$

There holds

$$\|A_n\|_{L^1(0,1)} \leq C\eta_1(n, \rho), \quad \text{and} \quad \left\| \frac{1}{A_n} \right\|_{L^1(0,1)} \leq C\eta_2(n, \rho).$$

For any $f \in W^{1,\infty}(\Omega_\delta)$,

$$\begin{aligned} \left| \int_{\Omega_\delta} (A_n(|x|) - h_{\text{ext}}(|x|) \lambda_{\text{ext}}(|x|)) f(x) dx \right| &\leq \frac{C}{n} \eta_1(n, \rho) \left(\|\partial_r f\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega_\delta)} \right) \\ \left| \int_{\Omega_\delta} \left(\frac{1}{A_n(|x|)} - \frac{\lambda_{\text{ext}}(|x|)}{h_{\text{ext}}(|x|)} \right) f(x) dx \right| &\leq \frac{C}{n} \eta_2(n, \rho) \|\partial_r f\|_{L^\infty(\Omega)} + \frac{C}{n\rho^{d-1}} \|f\|_{L^\infty(\Omega_\delta)} \end{aligned}$$

Proof. Consider first the case when $d = 2$, with $h_{\text{ext}} = 1$. From formula (2.22) we obtain

$$\begin{aligned} \int_0^1 A_n(t) dt &= \frac{1}{n} \sum_{k=0}^{n-1} \int_0^1 \alpha 1_{[0, \theta_\alpha(\frac{k}{n})]}(s) + 1_{[\theta_\alpha(\frac{k}{n}), \theta_\alpha(\frac{k}{n}) + \theta_1(\frac{k}{n})]}(s) + \gamma 1_{[\theta_\alpha(\frac{k}{n}) + \theta_1(\frac{k}{n}), 1]}(s) ds \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \lambda_{\text{ext}}\left(\frac{k}{n}\right). \end{aligned}$$

the last identity being due to (2.19). Similarly thanks to (2.19) there holds

$$\int_0^1 \frac{1}{A_n(t)} dt = \frac{1}{n} \sum_{k=0}^{n-1} \lambda_{\text{ext}}\left(\frac{k}{n}\right).$$

The function λ_{ext} being one bump, from the upper bound on λ_{ext} (2.12) we obtain

$$\begin{aligned} \left| \sum_{k=0}^{n-1} \frac{1}{n} \lambda_{\text{ext}}\left(\frac{k}{n}\right) - \int_0^1 \lambda_{\text{ext}}(t) dt \right| &\leq \left\| \sum_{k=0}^{n-1} 1_{[\frac{k}{n}, \frac{k+1}{n})} \lambda_{\text{ext}}\left(\frac{k}{n}\right) - \lambda_{\text{ext}}(\cdot) \right\|_{L^1(0,1)} \\ &\leq \frac{2}{n} (\max \lambda_{\text{ext}} - 1) \leq \frac{C}{n\rho}, \end{aligned}$$

and thanks to (2.13) and the triangle inequality,

$$\|A_n\|_{L^1(0,1)} = \left\| \frac{1}{A_n} \right\|_{L^1(0,1)} = \frac{1}{n} \sum_{k=0}^{n-1} \lambda_{\text{ext}}\left(\frac{k}{n}\right) \leq C \left(|\ln \rho| + \frac{1}{n\rho} \right).$$

We have

$$\int_{\Omega_\delta} (A_n(|x|) - \lambda_{\text{ext}}(|x|)) f(x) dx = \sum_{k=0}^{n-1} \int_{B_{\frac{k+1}{n}} \setminus B_{\frac{k}{n}}} \left(A_n - \lambda_{\text{ext}}\left(\frac{k}{n}\right) \right) f(x) dx + r_1,$$

with

$$|r_1| \leq C \left\| \sum_{k=0}^{n-1} 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right)} \lambda_{\text{ext}} \left(\frac{k}{n} \right) - \lambda_{\text{ext}}(\cdot) \right\|_{L^1(0,1)} \|f\|_{L^\infty(\Omega_\delta)} \leq \frac{C}{n\rho} \|f\|_{L^\infty(\Omega_\delta)}.$$

In a second step, because the average of A_n is given by (2.19),

$$\begin{aligned} & \sum_{k=0}^{n-1} \int_{B_{\frac{k+1}{n}} \setminus B_{\frac{k}{n}}} \left(A_n - \lambda_{\text{ext}} \left(\frac{k}{n} \right) \right) f(x) \, dx \\ &= \sum_{k=0}^{n-1} \int_{B_{\frac{k+1}{n}} \setminus B_{\frac{k}{n}}} \left(A_n - \lambda_{\text{ext}} \left(\frac{k}{n} \right) \right) f(x) \Big|_{|x|=\frac{k}{n}} \, dx + r_2 \\ &= r_2, \end{aligned}$$

with

$$\begin{aligned} r_2 &\leq C \left\| \sum_{k=0}^{n-1} 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right)} \lambda_{\text{ext}} \left(\frac{k}{n} \right) - A_n \right\|_{L^1(0,1)} \frac{1}{n} \|\partial_r f\|_{L^\infty(\Omega)} \\ &\leq \frac{C}{n} \left(\left\| \sum_{k=0}^{n-1} 1_{\left[\frac{k}{n}, \frac{k+1}{n}\right)} \lambda_{\text{ext}} \left(\frac{k}{n} \right) \right\|_{L^1(0,1)} + \|A_n\|_{L^1(0,1)} \right) \|\partial_r f\|_{L^\infty(\Omega)} \\ &\leq \frac{C}{n} \left(|\ln \rho| + \frac{1}{n\rho} \right) \|\partial_r f\|_{L^\infty(\Omega)}, \end{aligned}$$

as announced. The proof for the second estimate is similar. Suppose now that $d \geq 3$. By the same method,

$$\int_0^1 A_n(t) \, dt = \frac{1}{n} \sum_{k=0}^{n-1} \lambda_{\text{ext}} \left(\frac{k}{n} \right) h_{\text{ext}} \left(\frac{k}{n} \right), \quad \text{and} \quad \int_0^1 \frac{1}{A_n(t)} \, dt = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\lambda_{\text{ext}} \left(\frac{k}{n} \right)}{h_{\text{ext}} \left(\frac{k}{n} \right)}.$$

Since $\lambda_{\text{ext}} h_{\text{ext}} \leq C \max(1, \lambda^{3-d}) \leq C$,

$$\|A_n\|_{L^1(0,1)} \leq C.$$

On the other hand, since $h_{\text{ext}} \geq \frac{1}{\lambda_{\text{ext}}}$, $\left\| \frac{1}{A_n} \right\|_{L^1(0,1)} \leq \frac{1}{n} \sum_{k=0}^{n-1} \lambda_{\text{ext}} \left(\frac{k}{n} \right)^{d-1}$. As above, this means

$$\begin{aligned} \left\| \frac{1}{A_n} \right\|_{L^1(0,1)} &\leq \int_0^1 \lambda_{\text{ext}}^{d-1}(t) \, dt + \frac{1}{n} (\max \lambda_{\text{ext}}^{d-1} - 1) \\ &\leq C \left(\frac{1}{\rho^{d-2}} + \frac{1}{n\rho^{d-1}} \right). \end{aligned}$$

The other results follow, by the same method. \square

3. PROOF OF THEOREM 1

This section is devoted to the proof of the following proposition.

Proposition 11. *Given $\varphi \in H^{\frac{1}{2}}(\partial B_1)$, let u be given by (2.17) and $u_n \in H^1(B_1)$ be the solution of*

$$(3.1) \quad \begin{cases} -\operatorname{div}(A_n^B \nabla u_n) = 0 & \text{in } \Omega, \\ u_n = \varphi & \text{on } \partial\Omega. \end{cases}$$

There exists a constant C , depending only on δ and the dimension, such that under assumption (1.1), and all $n \geq 1$,

$$\left| \int_{B_1} A_n^B \nabla u_n \cdot \nabla u_n dx - \int_{B_1} \sigma_B \nabla u \cdot \nabla u dx \right| \leq \frac{C}{n} \left(\frac{1}{\alpha} + \frac{1}{\rho^{d-1}} + \gamma \rho^{2-d} \right) \eta_1(n, \rho)^2 \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2,$$

where η_1 is given in Lemma 10.

We are now in position to conclude the proof of Theorem 1.

Proof of Theorem 1. Choose δ (say, $\delta = \frac{1}{8}$) and set $N > \rho^{-d} \left(\frac{1}{\alpha} + \frac{1}{\rho^{d-1}} + \gamma \rho^{2-d} \right)$ when $d \geq 3$, and $N > \rho^{-2} \left(\frac{1}{\alpha} + \frac{1}{\rho^{d-1}} + \gamma \right) |\ln \rho|^2$ when $d = 2$, then thanks to Proposition 11,

$$\|\Lambda_{A_N^B} - \Lambda_{\sigma^B}\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial B_1), H^{-\frac{1}{2}}(\partial B_1))} \leq C \rho^d \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2.$$

On the other hand, from Proposition 7 there holds

$$\|\Lambda_{\sigma^B} - \Lambda_{I_d}\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial B_1), H^{-\frac{1}{2}}(\partial B_1))} \leq C \rho^d \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2.$$

The conclusion follows from the triangle inequality. \square

We will localise all our estimates in the annulus Ω_δ given by

$$(3.2) \quad \Omega_\delta = B_{1-\frac{1}{2}\delta} \setminus B_{\frac{1}{4}+\frac{1}{2}\delta}.$$

To establish our result we use an estimate for solutions of elliptic boundary value problems such as (2.17) or (3.1), that is, with coefficients depending only on the radial variable, with two annuli where the coefficient is constant. Given $f \in C(\overline{\Omega_\delta})$ we denote by $\partial_r f$ its partial derivative in the radial direction when it exists. Given $f \in H^1(\Omega_\delta)$ we note

$$\nabla_\phi f := \left(I_d - \frac{x}{|x|} \otimes \frac{x}{|x|} \right) \nabla f$$

the gradient components orthogonal to the radial direction.

Lemma 12. *Let $M \in L^\infty(B_1, \mathbb{R}^{d \times d})$ be a positive definite matrix-valued function satisfying (2.1) for some $\eta > 0$, such that*

$M = I_d$ in $B_1 \setminus B_{1-\delta} \cup B_{\frac{1}{4}+\delta} \setminus B_{\frac{1}{4}}$, and for any $x \in B_1 \setminus B_{\frac{1}{4}}$, $M(x) = m_1(|x|) \frac{x}{|x|} \otimes \frac{x}{|x|} + m_2(|x|) \left(I_d - \frac{x}{|x|} \otimes \frac{x}{|x|} \right)$, with $m_2, m_3 \in L^\infty(\frac{1}{4}, 1)$. Suppose that $f \in H^1(B_1)$ is a weak solution of the boundary value problem

$$(3.3) \quad \begin{aligned} \operatorname{div}(M \nabla f) &= 0 \text{ in } B_1, \\ f &= \varphi \text{ on } \partial B_1, \end{aligned}$$

with $\varphi \in H^{\frac{1}{2}}(\partial B_1)$. Then, $f, \nabla_\phi f, m_1 \partial_r f, \Delta_\phi f \in C(\overline{\Omega_\delta}; \mathbb{R}^d)$, $m_1 \partial_r f \in W^{1,\infty}(\Omega_\delta)$, and $m_1 \partial_r(\nabla_\phi f) \in W^{1,\infty}(\Omega_\delta; \mathbb{R}^d)$. Furthermore,

$$\max_{\overline{\Omega_\delta}} (|f| + |\nabla_\phi f|) \leq C \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)},$$

$$\max_{\overline{\Omega_\delta}} (|m_1 \partial_r f| + |m_1 \partial_r(\nabla_\phi f)|) \leq C \left(1 + \|m_2\|_{L^1(\frac{1}{4}, 1)} \right) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}$$

and

$$|\partial_r(m_1 \partial_r f)(x)| + |\partial_r(m_1 \partial_r \nabla_\phi f)(x)| \leq C |m_2(x)| \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)} \text{ for a.e. } x \in \Omega_\delta.$$

Remark 13. The fact that laminate structures enjoy additional regularity is well-known [CKC86, LV00]. Lemma 12 is a variant of such results, where we derive precise estimates adapted to our needs. Its proof is at the end of this section.

Proof of Proposition 11. By construction, $\sigma^B = h_{\text{ext}}(|x|) \left(\frac{1}{\lambda_{\text{ext}}(|x|)} \frac{x}{|x|} \otimes \frac{x}{|x|} + \left(I_d - \frac{x}{|x|} \otimes \frac{x}{|x|} \right) \lambda_{\text{ext}}(|x|) \right)$ in $B_1 \setminus B_{\frac{1}{4}}$. Using the fact that $u_n - u \in H_0^1(B_1)$, we find

$$\begin{aligned} \int_{B_1} \sigma^B \nabla u \cdot \nabla u dx - \int_{B_1} A_n^B \nabla u_n \cdot \nabla u_n dx &= \int_{B_1} \sigma^B \nabla u \cdot \nabla u_n dx - \int_{B_1} A_n^B \nabla u_n \cdot \nabla u dx \\ &= \int_{\Omega_\delta} (A_n^B - \sigma^B) \nabla u_n \cdot \nabla u dx \\ &= A + B, \end{aligned}$$

where A is the radial component, and B is the tangential component, that is,

$$(3.4) \quad \begin{aligned} A &= \int_{\Omega_\delta} \left(\frac{h_{\text{ext}}}{\lambda_{\text{ext}}} - A_n \right) \partial_r u_n \partial_r u dx, \\ B &= \int_{\Omega_\delta} (\lambda_{\text{ext}} h_{\text{ext}} - A_n) \nabla_\phi u_n \cdot \nabla_\phi u dx. \end{aligned}$$

We rewrite the radial term to make the radial flux appear,

$$A = \int_{\Omega_\delta} \left(\frac{1}{A_n} - \frac{\lambda_{\text{ext}}}{h_{\text{ext}}} \right) A_n \partial_r u_n \frac{h_{\text{ext}}}{\lambda_{\text{ext}}} \partial_r u dx,$$

We have, by the product rule and Lemma 12 and Lemma 10

$$\begin{aligned} \left\| \partial_r \left(A_n \partial_r u_n \frac{h_{\text{ext}}}{\lambda_{\text{ext}}} \partial_r u \right) \right\|_{L^\infty(\Omega_\delta)} &\leq \left(\|h_{\text{ext}} \lambda_{\text{ext}}\|_{L^1(\Omega)} \|A_n\|_{L^\infty(\Omega)} + \|A_n\|_{L^1(\Omega)} \|\lambda_{\text{ext}} h_{\text{ext}}\|_{L^\infty(\Omega)} \right) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2 \\ &\leq \left(\|\lambda_{\text{ext}}^{3-d}\|_{L^1(\Omega)} \|A_n\|_{L^\infty(\Omega)} + \|A_n\|_{L^1(\Omega)} \|\lambda_{\text{ext}}^{3-d}\|_{L^\infty(\Omega)} \right) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2, \\ &\leq C \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2 \times \begin{cases} \left(\gamma + \frac{1}{\rho} \right) \left(|\ln \rho| + \frac{1}{n\rho} \right) & \text{when } d = 2 \\ \gamma & \text{when } d \geq 3 \end{cases} \\ &\leq C \eta_3(\gamma, \rho) \eta_1(n, \rho) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2 \end{aligned}$$

where $\eta_3(a, b) = \begin{cases} a + \frac{1}{b} & \text{when } d = 2 \\ a & \text{when } d \geq 3 \end{cases}$. We also have

$$\begin{aligned} \left\| A_n \partial_r u_n \frac{h_{\text{ext}}}{\lambda_{\text{ext}}} \partial_r u \right\|_{L^\infty(\Omega_\delta)} &\leq C \left(1 + \|A_n\|_{L^1(\Omega)} \right) \left(1 + \|\lambda_{\text{ext}} h_{\text{ext}}\|_{L^1(\Omega)} \right) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2 \\ &\leq C \eta_1(n, \rho)^2 \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2 \end{aligned}$$

thus applying with Lemma 10 with $f = A_n \partial_r u_n \frac{h_{\text{ext}}}{\lambda_{\text{ext}}} \partial_r u$, we obtain

$$A \leq \frac{C}{n} \eta_1(n, \rho) \left(\eta_2(n, \rho) \eta_3(\gamma, \rho) + \eta_1(n, \rho) \frac{1}{\rho^{d-1}} \right) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2.$$

Similarly,

$$\begin{aligned} &\|\partial_r (\nabla_\phi u_n \cdot \nabla_\phi u)\|_{L^\infty(\Omega_\delta)} \\ &\leq C \left(\left\| \frac{1}{A_n} \right\|_{L^\infty(\Omega)} \left(1 + \|A_n\|_{L^1(\Omega)} \right) + \left\| \frac{\lambda_{\text{ext}}}{h_{\text{ext}}} \right\|_{L^\infty(\Omega)} \left(1 + \|\lambda_{\text{ext}} h_{\text{ext}}\|_{L^1(\Omega)} \right) \right) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2 \\ &\leq C \left(\frac{1}{\alpha} + \frac{1}{\rho^{d-1}} \right) \eta_1(n, \rho) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2, \end{aligned}$$

and

$$\begin{aligned} B &\leq \frac{C}{n} \left(\left(\frac{1}{\alpha} + \frac{1}{\rho^{d-1}} \right) \eta_1(n, \rho)^2 + \eta_3(1, \rho) \right) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2 \\ &\leq \frac{C}{n} \left(\frac{1}{\alpha} + \frac{1}{\rho^{d-1}} \right) \eta_1(n, \rho)^2 \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2, \end{aligned}$$

Thus when $d = 2$,

$$A + B \leq \frac{C}{n} \left(\frac{1}{\alpha} + \gamma + \frac{1}{\rho} \right) \left(|\ln \rho| + \frac{1}{n\rho} \right)^2 \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2,$$

and when $d \geq 3$,

$$A + B \leq \frac{C}{n} \left(\frac{1}{\alpha} + \frac{1}{\rho^{d-1}} + \gamma\rho^{2-d} \right) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2.$$

leading to the result. \square

Proof of Lemma 12. On $B_1 \setminus B_{1-\delta}$, f satisfies $\Delta f = 0$, and $f = \varphi$ on ∂B_1 . It follows that $f \in C^p(B_1 \setminus B_{1-\delta})$ for any $p \in \mathbb{N}$, and in particular

$$\|\nabla^p f\|_{L^\infty(\partial B_{1-\frac{1}{2}\delta})} \leq C(p) \|f\|_{L^2(B_1 \setminus B_{1-\delta})},$$

by Caccioppoli's inequality. Thanks to Poincaré's inequality, we have

$$\|f\|_{L^2(B_1 \setminus B_{1-\delta_1})} \leq C \left(\|\nabla f\|_{L^2(B_1 \setminus B_{1-\delta_1})} + \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)} \right).$$

Furthermore, from the fact that f is a minimiser of the Dirichlet energy

$$\int_{B_{1-\delta_1}} M \nabla \phi \cdot \nabla \phi dx + \int_{B_1 \setminus B_{1-\delta_1}} \nabla \phi \cdot \nabla \phi dx,$$

amongst all $\phi \in H^1(B_1)$ functions with trace φ , we deduce

$$(3.5) \quad \mathcal{E}_{B_1 \setminus B_{\frac{1}{4}}}(f) \leq C \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)},$$

and in particular $\|\nabla f\|_{L^2(B_1 \setminus B_{1-\delta_1})} \leq C \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}$. Altogether, we have obtained that

$$(3.6) \quad \|\nabla^p f\|_{L^\infty(\partial B_{1-\frac{1}{2}\delta_1})} \leq C(p) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)},$$

where C depends on p and δ . Note that f also solves $\operatorname{div}(M \nabla f) = 0$ in $B_{1-\frac{1}{2}\delta}$, and thus by the maximum principle and the previous bound, $\|f\|_{L^2(B_{\frac{1}{4}+\delta} \setminus B_{\frac{1}{4}})} \leq \|f\|_{L^\infty(\partial B_{1-\frac{1}{2}\delta})} \leq C \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}$. Noticing that f is also harmonic on $B_{\frac{1}{4}+\delta} \setminus B_{\frac{1}{4}}$, we obtain that by the same argument,

$$(3.7) \quad \|\nabla^p f\|_{L^\infty(B_{\frac{1}{4}+\frac{1}{2}\delta})} \leq C(p) \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}.$$

We now consider (3.3) as a boundary value problem posed on Ω_δ . Namely, f satisfies, in polar or (hyper)spherical coordinates

$$(3.8) \quad \begin{aligned} \partial_r (r^{d-1} m_1(r) \partial_r f) + r^{d-3} m_2(r) \Delta_S f &= 0 \text{ in } \Omega_\delta, \\ f &= g \text{ on } \partial B_{\frac{1}{4}+\frac{1}{2}\delta} \cup \partial B_{1-\frac{1}{2}\delta}, \end{aligned}$$

where $\Delta_S f$ is the Laplace–Beltrami operator ($\partial_{\theta\theta}$ when $d = 2$) and $g = f$. Remark that $\Delta_S f$ is solution of (3.8) with boundary value $g = \Delta_S f$. Thanks to (3.6) and (3.7) together with the maximum principle, and the energy estimate (3.5) applied on Ω_δ we deduce that

$$(3.9) \quad \|\Delta_S f\|_{L^\infty(\Omega_\delta)}^2 \leq C \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2.$$

Integrating (3.8) in r , we find for any r and r_0 in $(\frac{1}{4}, 1)$,

$$r^{d-1} m_1(r) \partial_r f(r, \phi) - r_0^{d-1} m_1(r_0) \partial_r f(r_0, \phi) = - \int_{r_0}^r s^{d-3} m_2(s) \Delta_S f(s, \phi) ds.$$

thus, with $r_0 = 1 - \frac{1}{2}\delta$, thanks to (3.6) and (3.9), we have obtained

$$\|m_1(r) \partial_r f(r, \phi)\|_{L^\infty(\Omega_\delta)} \leq C \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)} \left(1 + \|m_2\|_{L^1(\frac{1}{4}, 1)}\right).$$

Alternatively, we have directly

$$|\partial_r (r^{d-1} m_1(r) \partial_r f)| = |r^{d-3} m_2(r) \Delta_\phi f| \leq C \|m_2\|_{L^\infty(\frac{1}{4}, 1)} \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}.$$

The proof of the estimates for $\nabla_\phi f$ is similar. \square

4. AN EXPLICIT CLOAK CALCULATION

We introduce a radial transformation that is the identity on a neighbourhood near the boundary. A non-smooth piecewise linear candidate g_0 with $g_0(1) = 1$ and $g_0(\rho) = \frac{1}{2}$ is

$$(4.1) \quad g_0(t) = \begin{cases} \frac{1}{2} + \frac{\frac{1}{2}-2\delta}{1-2\delta-\rho} (t - \rho) & \text{for } t \in (\rho, 1 - 2\delta] \\ t & \text{for } t \in (1 - 2\delta, 1] \end{cases}.$$

A C^∞ variant is $g : t \rightarrow \frac{1}{2} + \int_\rho^t \tilde{g} \star \chi_{\frac{1}{2}\delta}(t) dt$, where χ is the standard mollifier, and

$$g(t) = \begin{cases} \frac{1}{2} + c_\delta & \text{for } t \leq 1 - 2\delta \\ 1 & \text{for } t > 1 - 2\delta \end{cases},$$

where $c_\delta > 0$ is such that $g(1) = 1$. In particular, g is affine near $r = \rho$, that is

$$g(r) = \frac{1}{2} + \left(\frac{1}{2} + c_\delta\right) (t - \rho),$$

and its inverse is also affine near $t = \frac{1}{2}$, with $g^{-1}(t) = \rho + \frac{2}{1+2c_\delta} (t - \frac{1}{2})$. The radial transformation is then given by

$$F : \bar{B}_1 \setminus B_\rho \rightarrow \bar{B}_1 \setminus B_{\frac{1}{2}} \\ x \rightarrow g(|x|) \frac{x}{|x|}.$$

The approximate cloak associated to F on $\bar{B}_1 \setminus B_{\frac{1}{2}}$ is given by

$$(4.2) \quad F_* [I_d] = \frac{1}{\lambda(|x|)} \frac{x}{|x|} \otimes \frac{x}{|x|} + \lambda(|x|) \left(I_d - \frac{x}{|x|} \otimes \frac{x}{|x|} \right),$$

where

$$\lambda(t) = \left(\frac{g}{rg'} \right) \circ g^{-1}(t) = \frac{t (g^{-1})'(t)}{g^{-1}(t)} \text{ for } t \in \left[\frac{1}{2}, 1 \right].$$

In particular near $t = \frac{1}{2}$, there holds

$$\lambda(t) = \frac{2}{1+2c_\delta} \frac{t}{\rho + \frac{2}{1+2c_\delta} \left(t - \frac{1}{2}\right)}.$$

We introduce the following C^1 extension of λ denoted by

$$(4.3) \quad \lambda_{\text{ext}} : r \rightarrow \begin{cases} \lambda(r) & \text{for } \frac{1}{2} \leq r \leq 1 \\ \lambda\left(\frac{1}{2}\right) - \lambda'\left(\frac{1}{2}\right) \int_r^{\frac{1}{2}} \chi\left(\frac{s - (\frac{1}{2} - \delta_2)}{\delta_2}\right) ds & \text{for } \frac{1}{2} - \delta_2 \leq r < \frac{1}{2} \\ (\lambda_{\text{ext}}\left(\frac{1}{2} - \delta_2\right) - 1) \chi\left(\frac{r - (\frac{1}{2} - \delta_2 - \delta_3)}{\delta_3}\right) + 1 & \text{for } r < \frac{1}{2} - \delta_2 \end{cases}$$

where χ is a smooth non-decreasing function such that $\chi(0) = 0$, $\chi(1) = 1$, $\chi^{(n)}(0) = \chi^{(n)}(1) = 0$ for all $n \geq 1$, and δ_2 and δ_3 are two small positive parameters. We choose δ_2 so that $-\lambda'\left(\frac{1}{2}\right) \left(\int_0^1 \chi(s) ds\right) \delta_2 = \frac{1}{2} \lambda\left(\frac{1}{2}\right)$, that is,

$$\delta_2 = \frac{\rho}{4} \frac{1+2c_\delta}{1-2\rho(c_\delta+1)} \frac{1}{\int_0^1 \chi(s) ds},$$

so that

$$\max_{[0,1]} \lambda_{\text{ext}}(r) = \lambda_{\text{ext}}\left(\frac{1}{2} - \delta_2\right) = \frac{3}{2} \lambda\left(\frac{1}{2}\right) < \frac{3}{2\rho}.$$

The maximal value of λ_{ext} is comparable to that of largest eigenvalue λ of the original diffeomorphism F . Both are, when it comes to the dependence on ρ , similar to the optimal L^1 solution given in [CV22]. Altogether, the approximate cloak coefficient we consider is

$$(4.4) \quad \sigma(x) = \frac{1}{\lambda_{\text{ext}}(|x|)} \frac{x}{|x|} \otimes \frac{x}{|x|} + \lambda_{\text{ext}}(|x|) \left(I_d - \frac{x}{|x|} \otimes \frac{x}{|x|} \right), \text{ for } x \in \overline{B_1} \setminus B_{\frac{1}{4}}.$$

Note that for $x \in B_{\frac{1}{2} - \delta_1 - \delta_2} \setminus B_{\frac{1}{4}} \cup B_1 \setminus B_{\delta_1/2}$ $\sigma(x) = I_d$.

Next, we estimate $\int_0^1 \lambda_{\text{ext}}^n(t) dt$, with $n \geq 1$. We have

$$\int_0^1 \lambda_{\text{ext}}^n(t) dt \leq \frac{1}{2} - 2\delta_2 + \delta_3 \max \lambda_{\text{ext}}^n + \int_{\frac{1}{2}}^{1-3\delta} \lambda_{\text{ext}}^n dt + \int_{1-3\delta}^1 \lambda_{\text{ext}}^n dt.$$

Choosing $\delta_3 = \frac{\rho}{|\ln \rho|}$, the leading term is $\int_{\frac{1}{2}}^{1-3\delta} \lambda_{\text{ext}}^n dt = \int_{\frac{1}{2}}^{1-3\delta} \left(\frac{2}{1+2c_\delta} \frac{t}{\rho + \frac{2}{1+2c_\delta} \left(t - \frac{1}{2}\right)} \right)^n dt$. Since g is linear on this interval, the formula for λ_{ext} is explicit. We obtain that

$$\begin{aligned} \int_{\frac{1}{2}}^{1-2\delta} \lambda_{\text{ext}}^n dt &= \frac{1}{2} \ln \frac{1}{\rho} + C \text{ when } n = 1, \\ &= \frac{1}{2} \rho^{1-n} \left(1 + \frac{C}{\rho} \right) \text{ when } n \geq 2. \end{aligned}$$

where C bounded uniformly with respect to ρ .

Finally, we turn to h_{ext} when $d \geq 2$. We set

$$h_{\text{ext}}(t) = \begin{cases} \frac{1}{\lambda_{\text{ext}}(t)} & \text{on } [0, \frac{1}{2} - \delta_2], \\ \frac{1}{\lambda_{\text{ext}}(t)} \left(1 + \left(\left((g^{-1})' \left(\frac{1}{2} \right) \right)^{d-2} - 1 \right) \chi \left(\frac{t - (\frac{1}{2} - \delta_2)}{\delta_2} \right) \right) & \text{on } (\frac{1}{2} - \delta_2, \frac{1}{2}), \\ \frac{1}{\lambda_{\text{ext}}(t)} \left((g^{-1})'(t) \right)^{d-2} & \text{on } [\frac{1}{2}, 1]. \end{cases}$$

Since $(g^{-1})'$ is constant in a right-hand side neighbourhood of $\frac{1}{2}$, $h_{\text{ext}}\lambda_{\text{ext}}(t)$ is smooth. Furthermore, $(g^{-1})'$ decreases from $\frac{2}{1+2c_\delta}$ to 1 on $[\frac{1}{2}, 1]$, therefore $2^{d-2} > h_{\text{ext}}\lambda_{\text{ext}} \geq 1$. The other properties are readily verified.

5. A VARIABLE SCALE VARIANT OF THEOREM 1 WHEN $d = 2$.

Given $x_0 = 0 < x_1 < \dots < x_p = 1$ a partition of $(0, 1)$, if we set instead that for all $t \in [0, 1]$,

$$\begin{aligned} A_p(t) &= \alpha \sum_{k=0}^{p-1} 1_{[0, \theta_\alpha(x_k))} \left(\frac{t - x_k}{x_{k+1} - x_k} \right) \\ &\quad + \sum_{k=0}^{p-1} 1_{[\theta_\alpha(x_k), \theta_\alpha(x_k) + \theta_1(x_k))} \left(\frac{t - x_k}{x_{k+1} - x_k} \right) \\ &\quad + \gamma \sum_{k=0}^{p-1} 1_{[\theta_\alpha(x_k) + \theta_1(x_k), 1)} \left(\frac{t - x_k}{x_{k+1} - x_k} \right), \end{aligned}$$

and

$$\lambda_{\text{ext}}^p = \sum_{k=0}^{p-1} 1_{[x_p, x_{p+1})} \lambda_{\text{ext}}(x_p),$$

then a variant of Lemma 10 shows that

$$\left\| \frac{1}{A_p} \right\|_{L^1(0,1)} = \|A_p\|_{L^1(0,1)} \leq C \left(|\ln \rho| + \|\lambda_{\text{ext}}^p - \lambda_{\text{ext}}\|_{L^1(0,1)} \right),$$

and for any $f \in W^{1,\infty}(\Omega_\delta)$,

$$\begin{aligned} \left| \int_{\Omega_\delta} (A_p(|x|) - \lambda_{\text{ext}}(|x|)) f(x) dx \right| &\leq \sum_{k=0}^{p-1} \|\lambda_{\text{ext}} - \lambda_{\text{ext}}^p\|_{L^1(x_k, x_{k+1})} \|f\|_{L^\infty(x_k, x_{k+1})} \\ &\quad + \sum_{k=0}^{p-1} \|A_p - \lambda_{\text{ext}}^p\|_{L^1(x_k, x_{k+1})} (x_{k+1} - x_k) \|\partial_r f\|_{L^\infty(x_k, x_{k+1})}. \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\Omega_\delta} (A_p^{-1}(|x|) - \lambda_{\text{ext}}(|x|)) f(x) dx \right| &\leq \sum_{k=0}^{p-1} \|\lambda_{\text{ext}} - \lambda_{\text{ext}}^p\|_{L^1(x_k, x_{k+1})} \|f\|_{L^\infty(x_k, x_{k+1})} \\ &\quad + \sum_{k=0}^{p-1} \|A_p^{-1} - \lambda_{\text{ext}}^p\|_{L^1(x_k, x_{k+1})} (x_{k+1} - x_k) \|\partial_r f\|_{L^\infty(x_k, x_{k+1})} \end{aligned}$$

As a result, the estimate in Proposition 11 is modified to

$$\left| \int_{B_1} A_p^B \nabla u_p \cdot \nabla u_p dx - \int_{B_1} \sigma_B \nabla u \cdot \nabla u dx \right| \leq C \epsilon_p \|\varphi\|_{H^{\frac{1}{2}}(\partial B_1)}^2,$$

where

$$\begin{aligned} \epsilon_p &= \eta_p (|\ln \rho| + \eta_p)^2 \\ &\quad + \sum_{k=0}^{p-1} \|A_p - \lambda_{\text{ext}}^p\|_{L^1(x_k, x_{k+1})} (x_{k+1} - x_k) \left(\|A_p\|_{L^\infty(x_k, x_{k+1})} + \|\lambda_{\text{ext}}\|_{L^\infty(x_k, x_{k+1})} \right) (|\ln \rho| + \eta_p), \end{aligned}$$

$$+ \sum_{k=0}^{p-1} \|A_p^{-1} - \lambda_{\text{ext}}^p\|_{L^1(x_k, x_{k+1})} (x_{k+1} - x_k) \left(\|A_p\|_{L^\infty(x_k, x_{k+1})} + \|\lambda_{\text{ext}}\|_{L^\infty(x_k, x_{k+1})} \right) (|\ln \rho| + \eta_p),$$

and

$$\eta_p = \|\lambda_{\text{ext}}^p - \lambda_{\text{ext}}\|_{L^1(0,1)} = \sum_{k=0}^{p-1} \|\lambda_{\text{ext}} - \lambda_{\text{ext}}^p\|_{L^1(x_k, x_{k+1})},$$

and Theorem 1 is established provided $\epsilon_p < \rho^d$. When only three materials are used, $\|A_p\|_{L^\infty(x_k, x_{k+1})} = \gamma$ is constant throughout the cloak, and $C\gamma > \|\lambda_{\text{ext}}\|_{L^\infty(x_k, x_{k+1})}$. Bounding η_p by

$$\eta_p \leq \sum_{k=0}^{p-1} |\lambda_{\text{ext}}(x_{k+1}) - \lambda_{\text{ext}}(x_k)| (x_{k+1} - x_k),$$

An upper bound for ϵ_p is

$$\epsilon_p \leq C |\ln \rho| \sum_{k=0}^{p-1} (x_{k+1} - x_k) f_1 \left(x_k, x_{k+1}, \ln \frac{1}{\rho} \right),$$

with

$$f_1(a, b, c) = \gamma \|A_p - \lambda_{\text{ext}}^p\|_{L^1(a,b)} + c |\lambda_{\text{ext}}(a) - \lambda_{\text{ext}}(b)|.$$

On the other hand, given p , suppose given $1 < \gamma_1 < \dots < \gamma_p$ and $0 < \alpha_p < \dots < \alpha_1 < 1$, so that for each $k \in \{1, \dots, p\}$,

$$\begin{cases} \lambda_{\text{ext}}(x_p) \alpha_k < 1 \\ \gamma_k \geq \frac{\lambda_{\text{ext}}(x_k) - \alpha_k}{1 - \alpha_k \lambda_{\text{ext}}(x_k)}. \end{cases}$$

Then, there exists $\theta_\alpha^k, \theta_1^k, \theta_\gamma^k$ satisfying

$$\begin{cases} \theta_\alpha^k \alpha_k + \theta_1^k + \theta_\gamma^k \gamma_k &= \lambda_{\text{ext}}(x_k) \\ \frac{1}{\alpha_k} \theta_\alpha^k + \theta_1^k + \frac{1}{\gamma_k} \theta_\gamma^k &= \lambda_{\text{ext}}(x_k). \end{cases}$$

Suppose also that $\gamma_k + \frac{1}{\alpha_k} < C \lambda_{\text{ext}}(x_k)$. Then, laminating with these materials, since λ_{ext} is a one bump function, we obtain

$$\epsilon_p \leq C \sum_{k=0}^{p-1} |\ln \rho| f_2 \left(x_k, x_{k+1}, \ln \frac{1}{\rho} \right) (x_{k+1} - x_k),$$

with

$$f_2(a, b, c) = \max(\lambda_{\text{ext}}(a), \lambda_{\text{ext}}(b)) \|A_p - \lambda_{\text{ext}}^p\|_{L^1(a,b)} + c |\lambda_{\text{ext}}(a) - \lambda_{\text{ext}}(b)|.$$

We now describe a elementary method to coarsen the layers of the laminate. Denote x_{max} the point where λ_{ext} reaches its maximum, and set ϵ_0 the target error – for example, $\epsilon_0 = \frac{1}{|\ln \rho|} \rho^d$.

Start with $x_0 = \frac{1}{4}$, and $\lambda_{\text{ext}}(x_0) = 1$, and define x_n inductively. Set $z_0 = x_{\text{max}}$, and for all $m \geq 0$, $z_{m+1} = \frac{1}{2}(x_n + z_m)$. Let m_0 be the smallest integer for which f_1 or $f_2(z_{m_0}, x_n) \leq \epsilon_0$. We then define $x_{n+1} = z_{m_0}$. The maximum point x_{max} is reached after a finite number of steps, say p_0 . We then proceed similarly for the subsequent part. Given $n \geq p_0$, set with $z_0 = 1$, and for all $m \geq 0$, $z_{m+1} = \frac{1}{2}(x_n + z_m)$. Let m_0 be the smallest integer for which f_1 or $f_2(z_{m_0}, x_n) \leq \epsilon_0$. We then define $x_{n+1} = z_{m_0}$. The endpoint 1 is reached after a finite number of steps, p .

The resulting ϵ_p satisfies

$$\epsilon_p < C |\ln \rho| \sum_{k=0}^{p-1} \epsilon_0 (x_{k+1} - x_k) = C |\ln \rho| \epsilon_0.$$

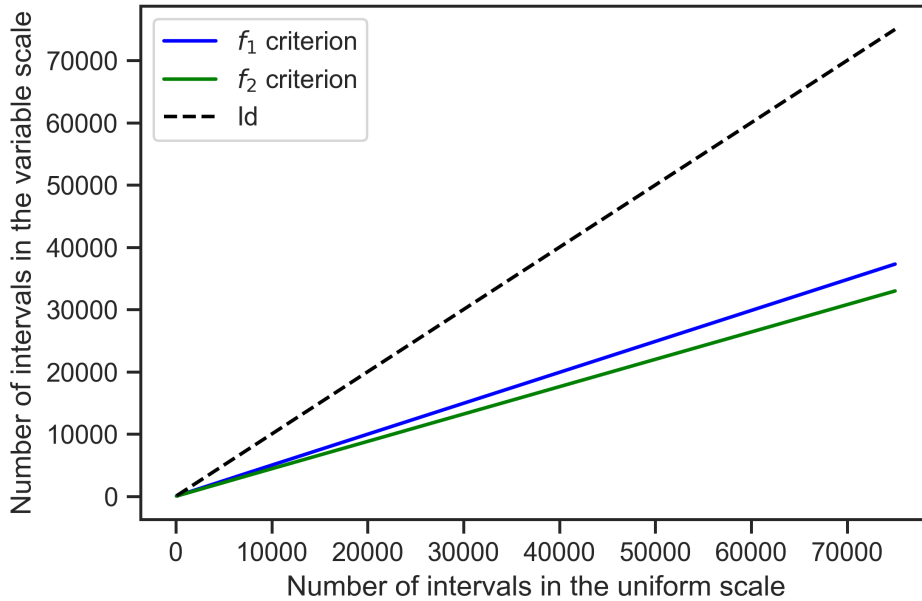


FIGURE 5.1. A comparison of the number of intervals needed when coarsening with three materials, corresponding to the error bound f_1 (blue), with multiple materials with the corresponding error bound f_2 (grid) or three materials and all layers having the same width (dashed).

We compare the different coarsening methods in Figure 5.1 on page 17. For the three materials coarsening scheme with the error function f_1 , we set $\epsilon_0 = \sum_{k=0}^{p-1} f_1\left(\frac{k}{n}, \frac{k+1}{n}, \max \lambda_{\text{ext}}\right) \frac{1}{n}$, replace $\ln \rho$ and plot the resulting maximal step number p_1 as a function of n , in blue. On the same plot, we represent the maximal step number p_2 obtained when using the many materials coarsening scheme with the function f_2 , with the same ϵ_0 , in green. the dashed line represents the uniform grid, which exactly attains ϵ_0 for f_1 . We observe that both scheme provide comparable results: the improvement obtained by allowing for many materials seems to scale linearly with the number of points.

In Figure 5.2 on page 18 we show the corresponding structure, corresponding to $n = 30$. As expected, the finer scales are required near $r = \frac{1}{2}$. Very thin layers of α and γ are required in the middle picture near $r = 1$. On the contrary, because we use less contrasted material in that region the right-most pictures shows large strips of different materials. Various nuances of orange/red correspond to γ_k , whereas various nuances of blue/cyan correspond to α_k . This allows for larger layers, which could be desirable for manufacturing reasons.

CONCLUSION

We have shown that three-materials radial laminates can be used to create approximate cloaks with equivalent properties to theoretical cloaks derived by analysis. The key is to vary the thickness of each laminated layer, Our result is uniform with respect to the boundary data: we have shown a property of the Dirichlet-to-Neumann operator. We illustrated our result on an example, and proposed a variant of our result with coarser layers, and substitute materials. Our result is quantitative : it guarantees an approximate cloak of order ρ for a given number of layers. We do not claim that this number is the lowest possible threshold. In the analysis,

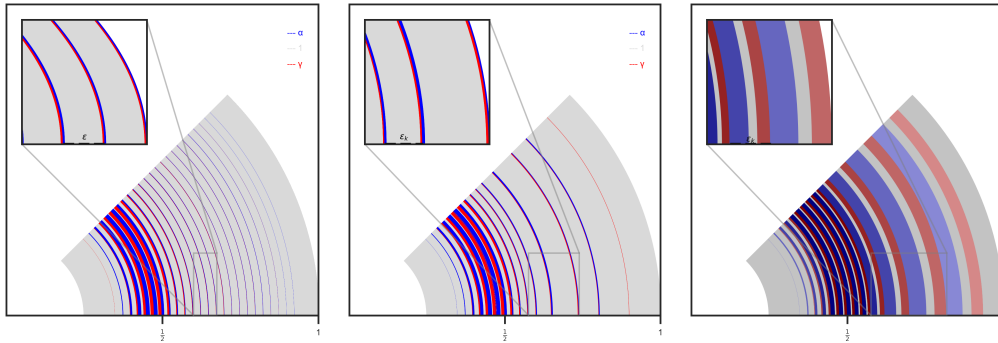


FIGURE 5.2. Three laminates with the same cloaking properties. From left to right, with a uniform scale and three materials, with three materials after coarsening, with multiple materials after coarsening

we made full use of the fact that because the cloak is a laminate, the gradient of the solution of the boundary value problem enjoys higher regularity. It is possible to prove a similar result with cloaks exhibiting more complex microstructures, but with the method we have used, the threshold scale will be lower, leading to finer structures. At the heart of this approach is numerical analysis, and precise error estimates. These estimates could be used in an optimisation problem, penalising too fine structures, or the total mass of highly conductive / highly insulating materials, should it be relevant in a cost-effective manufacturing process.

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